DESY 05-152, SFB/CPP-05-46
DCPT/05/110, IPPP/05/55
August 2005

hep-ph/0508265

Higher-Order Soft Corrections to Lepton Pair and Higgs Boson Production

S. Moch[a] and A. Vogt[b]

[a] *Deutsches Elektronensynchrotron DESY*
Platanenallee 6, D–15735 Zeuthen, Germany

[b] *IPPP, Department of Physics, University of Durham*
South Road, Durham DH1 3LE, United Kingdom

Abstract

Utilizing recent three-loop results on the quark and gluon splitting functions and form factors, we derive the complete threshold-enhanced third-order (N^3LO) QCD corrections to the total cross sections for the production of lepton pairs and the Higgs boson in hadron collisions. These results, for the latter case obtained in the heavy top-quark limit, are employed to extend the threshold resummation for these processes to the fourth logarithmic order. We investigate the numerical impact of the higher-order corrections for Higgs boson production at the TEVATRON and the LHC. Our results, suitably treated in Mellin N-space, provide a sufficiently accurate approximation to the full N^3LO contributions. Corrections of about 5% at the LHC and 10% at the TEVATRON are found for typical Higgs masses. The N^3LO predictions exhibit a considerably reduced dependence on the renormalization scale with, for the first time, stationary points close to the Higgs mass.

The production of lepton pairs and especially the Higgs boson H, see Refs. [1] for detailed reviews, are among the most important processes in high-energy proton collisions. The corresponding cross sections receive sizable higher-order QCD corrections, necessitating calculations beyond the standard next-to-leading order (NLO) of perturbative QCD. After the early calculation of the next-to-next-to-leading order (NNLO) corrections to the Drell-Yan process in Refs. [2, 3], considerable progress has been achieved in this field during the last five years. The NNLO corrections are now completely known also for Higgs production in the heavy top-quark limit [4–10], and the all-order resummation of the threshold logarithms [11, 12] has been extended to the next-to-next-to-leading logarithmic (NNLL) accuracy for both processes [13, 14]. Still, at colliders energies especially for Higgs production, corrections of yet higher order are not entirely negligible.

In this letter we derive the complete logarithmically enhanced soft-emission corrections to both lepton pair and Higgs boson production at the third order ($\mathrm{N^3LO}$) in the strong coupling constant α_s, and extend the corresponding threshold resummations to the $\mathrm{N^3LL}$ contributions. The first step is achieved by analysing the general mass-factorization structure of the third-order coefficient functions c_3 in terms of recent results on the three-loop splitting functions [15, 16] and the quark and gluon form factors [17, 18], analogous to the second-order procedure of Ref. [19]. For the second step we just have to determine one resummation coefficient, usually denoted by D_3, from these results, as the structure of the resummation exponent has already been derived in Ref. [20] to the required accuracy. As discussed below our results on c_3 for Higgs production, suitably treated in Mellin N-space, provide a very good approximation to the complete $\mathrm{N^3LO}$ corrections, thus facilitating improved predictions for the cross sections at the TEVATRON and the LHC.

In the soft limit, i.e., retaining only contributions of the forms

$$\mathcal{D}_k \;=\; \left[\frac{\ln^k(1-x)}{1-x}\right]_+ , \quad \delta(1-x) \tag{1}$$

to the coefficient functions, only the respective subprocesses $q\bar{q} \to \gamma^* \to l^+l^-$ and $gg \to H$ contribute to the Drell-Yan process and Higgs boson production. In the latter case, the Hgg vertex is an effective interaction in the limit of a heavy top quark,

$$\mathcal{L}_{\mathrm{eff}} \;=\; -\frac{1}{4} C_H \, H \, G^a_{\mu\nu} G^{a,\mu\nu} \;, \tag{2}$$

where $G^a_{\mu\nu}$ denotes the gluon field strength tensor, and the prefactor C_H includes all QCD corrections to the top-quark loop. This coefficient is of order α_s and known up to $\mathrm{N^3LO}$ (α_s^4) [21]. The analysis of the higher-order corrections in the heavy-top limit is justified by the agreement between this approximation and the full calculations at NLO [22–24].

The general structure of the expansion coefficients W_n^{b} of the bare partonic cross section,

$$W^{\mathrm{b}} \;=\; \sum_{n=0}^{\infty} \left(a_s^{\mathrm{b}}\right)^n W_n^{\mathrm{b}} \;, \quad a_s \;\equiv\; \frac{\alpha_s}{4\pi} \;, \tag{3}$$

is given by

$$
\begin{aligned}
W_0^{\rm b} &= \delta(1-x) \\
W_1^{\rm b} &= 2\,{\rm Re}\,\mathcal{F}_1\,\delta(1-x)+\mathcal{S}_1 \\
W_2^{\rm b} &= (2\,{\rm Re}\,\mathcal{F}_2+|\mathcal{F}_1|^2)\delta(1-x)+2\,{\rm Re}\,\mathcal{F}_1\mathcal{S}_1+\mathcal{S}_2 \\
W_3^{\rm b} &= (2\,{\rm Re}\,\mathcal{F}_3+2\,|\mathcal{F}_1\mathcal{F}_2|)\,\delta(1-x)+(2\,{\rm Re}\,\mathcal{F}_2+|\mathcal{F}_1|^2)\mathcal{S}_1+2\,{\rm Re}\,\mathcal{F}_1\mathcal{S}_2+\mathcal{S}_3\,. \qquad (4)
\end{aligned}
$$

Here $\mathcal{F}_n$ denotes the bare n-loop time-like quark or gluon form factor, calculated in dimensional regularization with $D = 4-2\varepsilon$ and, as all quantities, expanded in terms of $a_{\rm s} = \alpha_{\rm s}/(4\pi)$. The dependence of the pure real-emission contributions $\mathcal{S}_n$ on the scaling variable $x = M^2/s$ is given by the D-dimensional +-distributions $f_{2n,\varepsilon}$ defined by

$$
f_{k,\varepsilon}(x) \;=\; \varepsilon[\,(1-x)^{-1-k\varepsilon}\,]_+ \;=\; -\frac{1}{k}\,\delta(1-x)+\sum_{i=0}\frac{(-k\varepsilon)^i}{i\,!}\,\varepsilon\,\mathcal{D}_i \qquad (5)
$$

with $\mathcal{D}_k$ of Eq. (1). As appropriate for a parallel tretament of the two processes, the coefficient function for Higgs production is defined as in Ref. [8], i.e., C_H in Eq. (2) is kept as a prefactor.

The expansion coefficients W_k obtained from Eq. (4) after renormalizing the coupling constant,

$$
a_{\rm s}^b \;=\; a_{\rm s}-\frac{\beta_0}{\varepsilon}a_{\rm s}^2+\left(\frac{\beta_0^2}{\varepsilon^2}-\frac{1}{2}\frac{\beta_1}{\varepsilon}\right)a_{\rm s}^3+\dots\;, \qquad (6)
$$

and multiplying, for Higgs production, with the square of the renormalization constant [25, 26]

$$
Z_{G^2} \;=\; [1-\beta(\alpha_{\rm s})/(a_{\rm s}\varepsilon)]^{-1}\;,\quad \beta(\alpha_{\rm s}) \;=\; -\beta_0\,a_{\rm s}^2-\beta_1\,a_{\rm s}^3-\beta_2\,a_{\rm s}^4-\dots\;, \qquad (7)
$$

of the operator $G^a_{\mu\nu}G^{a,\mu\nu}$ in Eq. (2) obey the Kinoshita-Lee-Nauenberg theorem [27, 28] and the mass-factorization relations (putting $\mu^2 = M^2_{\gamma^*,H}$)

$$
\begin{aligned}
W_1 &= \frac{2}{\varepsilon}\gamma_0+c_1+\varepsilon a_1+\varepsilon^2 b_1\,, \qquad (8)\\
W_2 &= \frac{1}{\varepsilon^2}\{(2\gamma_0-\beta_0)\gamma_0\}+\frac{1}{\varepsilon}\{\gamma_1+2c_1\gamma_0\}+c_2+2a_1\gamma_0+\varepsilon\{a_2+2b_1\gamma_0\}\,, \qquad (9)\\
W_3 &= \frac{1}{3\varepsilon^3}\{2\,(\gamma_0-\beta_0)\,(2\gamma_0-\beta_0)\,\gamma_0\}+\frac{1}{3\varepsilon^2}\{6\gamma_1\gamma_0-2\beta_0\gamma_1-2\beta_1\gamma_0\\
&\quad +3c_1\,(2\gamma_0-\beta_0)\,\gamma_0\}+\frac{1}{3\varepsilon}\{2\gamma_2+3c_1\,\gamma_1+6c_2\,\gamma_0+3a_1\,(2\gamma_0-\beta_0)\,\gamma_0\}\\
&\quad +c_3+a_1\gamma_1+2a_2\gamma_0+b_1\,(2\gamma_0-\beta_0)\,\gamma_0\;. \qquad (10)
\end{aligned}
$$

Here the anomalous dimensions γ_k are related by a conventional sign to the diagonal Altarelli-Parisi splitting functions P_k. In x-space, where these quantities (in the $\overline{\rm MS}$ scheme adopted throughout this article) have soft limits of the form [29]

$$
P_{k-1} \;=\; A_k\,\mathcal{D}_0+P_k^{\delta}\,\delta(1-x)\;, \qquad (11)
$$

the products in Eq. (8) – (10) have to be read as Mellin convolutions. To the required accuracy these convolutions can be readily carried out using, for example, the appendix of Ref. [30].

Eqs. (8) – (10) can be used to derive all $\mathcal{D}_k$ contributions to the $\overline{\rm MS}$ coefficient function c_n, once the coefficients A_n, P_n^δ are known together with all $1/\varepsilon$ pole terms of the n-loop form factor and suitable lower-order information. The salient point for this extraction is the structure (5) of the soft emissions linking the coefficients of $\varepsilon^{-1}\,\delta(1-x)$ to those of $\mathcal{D}_0$; thus a mass-factorization constraint on the former term fixes the latter coefficient. With the results of Refs. [2, 8, 9] and [15–18] the above conditions are fulfilled at $n = 3$ for both lepton pair and Higgs boson production.

Thus, treating Higgs boson production first, we insert the ε-expansion

$$\mathcal{S}_n \;=\; f_{2n,\varepsilon} \sum_{l=-2n}^{\infty} 2n\, s_{n,l}\, \varepsilon^l \tag{12}$$

into Eqs. (8) – (10) and recursively determine the coefficients $s_{n,l}$. The first-order result is known to all orders in ε. For later convenience, we here present its expansion up to order ε^3,

$$\mathcal{S}_1 \;=\; 2\, f_{2,\varepsilon}\, C_A \left\{ -\frac{4}{\varepsilon^2} + 6\,\zeta_2 + \frac{28}{3}\,\zeta_3\,\varepsilon + \frac{3}{2}\,\zeta_2^2\,\varepsilon^2 + \varepsilon^3 \left[-14\,\zeta_2\zeta_3 + \frac{124}{5}\,\zeta_5 \right] + \ldots \right\} . \tag{13}$$

The corresponding second-order coefficients read

$$\begin{aligned}
s_{2,-4} &= -8\,C_A^2 \\
s_{2,-3} &= -\frac{11}{3}\,C_A^2 + \frac{2}{3}\,C_A n_f \\
s_{2,-2} &= C_A^2 \left[-\frac{67}{9} + 58\,\zeta_2 \right] + \frac{10}{9}\,C_A n_f \\
s_{2,-1} &= C_A^2 \left[-\frac{404}{27} + \frac{77}{3}\,\zeta_2 + \frac{538}{3}\,\zeta_3 \right] + C_A n_f \left[\frac{56}{27} - \frac{14}{3}\,\zeta_2 \right] \\
s_{2,0} &= C_A^2 \left[-\frac{2428}{81} + \frac{469}{9}\,\zeta_2 + \frac{682}{9}\,\zeta_3 + \frac{16}{5}\,\zeta_2^2 \right] + C_A n_f \left[\frac{328}{81} - \frac{70}{9}\,\zeta_2 - \frac{124}{9}\,\zeta_3 \right] .
\end{aligned} \tag{14}$$

Here n_f denotes the number of effectively massless quark flavours, C_F and C_A are the usual SU(N) colour factors, with $C_F = 4/3$ and $C_A = 3$ for QCD, and ζ_n represents Riemann's ζ-function.

From $s_{2,-4} \ldots s_{2,-1}$ and the first-order results one recovers the $\mathcal{D}_k$ terms [4, 5] of the NNLO coefficient function at the renormalization and factorization scales $\mu_r^2 = \mu_f^2 = M_H^2$,

$$\begin{aligned}
c_2(x) \;=\;& 128\,C_A^2\,\mathcal{D}_3 - \left\{ \frac{176}{3}\,C_A^2 - \frac{32}{3}\,C_A n_f \right\} \mathcal{D}_2 + \left\{ C_A^2 \left[\frac{1072}{9} - 160\,\zeta_2 \right] - \frac{160}{9} C_A n_f \right\} \mathcal{D}_1 \\
&+ \left\{ C_A^2 \left[-\frac{1616}{27} + \frac{176}{3}\,\zeta_2 + 312\,\zeta_3 \right] + C_A n_f \left[\frac{224}{27} - \frac{32}{3}\,\zeta_2 \right] \right\} \mathcal{D}_0 \\
&+ \left\{ C_A^2 \left[93 + \frac{536}{9}\,\zeta_2 - \frac{220}{3}\,\zeta_3 - \frac{4}{5}\,\zeta_2^2 \right] - C_A n_f \left[\frac{80}{3} + \frac{80}{9}\,\zeta_2 + \frac{8}{3}\,\zeta_3 \right] \right. \\
&\left. - C_F n_f \left[\frac{67}{3} - 16\,\zeta_3 \right] \right\} \delta(1-x) \; .
\end{aligned} \tag{15}$$

The SU(N) result (14) for the coefficient $s_{2,0}$, on the other hand, is fixed by the corresponding $\delta(1-x)$ contribution to Eq. (15) derived in Ref. [9]. Actually this colour-factor decomposition is checked (and could have been predicted from the N = 3 QCD results of Refs. [4,5]) by the absence of a $C_F n_f$ term in $\mathcal{S}_2$ obvious from the colour structure of the contributing Feynman diagrams.

The quantity a_2 in Eq. (9) has not been computed so far, hence the coefficient $s_{2,1}$ is unknown at this point. This suggests a problem, as this coefficient enters the ε^0 part of Eq. (10). However, its contribution to the $\mathcal{D}_0$ term (but not the $\delta(1-x)$ piece) of c_3 is found to cancel in the end. Keeping $s_{2,1}$ as an unknown in the intermediate relations, the third-order coefficients in Eq. (13) are

$$\begin{aligned}
s_{3,-6} &= -\frac{32}{3} C_A^3 \\
s_{3,-5} &= -\frac{44}{3} C_A^3 + \frac{8}{3} C_A^2 n_f \\
s_{3,-4} &= C_A^3 \left[-\frac{2896}{81} + 184\,\zeta_2 \right] + \frac{536}{81} C_A^2 n_f - \frac{16}{81} C_A n_f^2 \\
s_{3,-3} &= C_A^3 \left[-\frac{21052}{243} + \frac{6710}{27}\,\zeta_2 + \frac{2440}{3}\,\zeta_3 \right] + C_A^2 n_f \left[\frac{4148}{243} - \frac{1220}{27}\,\zeta_2 \right] + \frac{4}{9} C_A C_F n_f - \frac{160}{243} C_A n_f^2 \\
s_{3,-2} &= C_A^3 \left[-\frac{51322}{243} + \frac{48856}{81}\,\zeta_2 + \frac{29876}{27}\,\zeta_3 - \frac{7592}{45}\,\zeta_2^2 \right] + C_A C_F n_f \left[\frac{110}{27} - \frac{32}{9}\,\zeta_3 \right] \\
&\quad + C_A^2 n_f \left[\frac{10468}{243} - \frac{9004}{81}\,\zeta_2 - \frac{5336}{27}\,\zeta_3 \right] + C_A n_f^2 \left[-\frac{16}{9} + \frac{88}{27}\,\zeta_2 \right] \\
s_{3,-1} &= C_A^3 \left[-\frac{617525}{2187} + \frac{251942}{243}\,\zeta_2 + \frac{56032}{27}\,\zeta_3 - \frac{1661}{10}\,\zeta_2^2 - \frac{79388}{9}\,\zeta_2\zeta_3 + \frac{49888}{5}\,\zeta_5 \right] \\
&\quad + C_A C_F n_f \left[\frac{1711}{81} - \frac{22}{3}\,\zeta_2 - \frac{304}{27}\,\zeta_3 - \frac{32}{15}\,\zeta_2^2 \right] + C_A n_f^2 \left[-\frac{9728}{2187} + \frac{880}{81}\,\zeta_2 + \frac{1040}{81}\,\zeta_3 \right] \\
&\quad + C_A^2 n_f \left[\frac{164194}{2187} - \frac{55154}{243}\,\zeta_2 - \frac{31520}{81}\,\zeta_3 + \frac{97}{3}\,\zeta_2^2 \right] + 4 C_A s_{2,1} \,. \qquad (16)
\end{aligned}$$

Analogous to $s_{2,0}$ discussed above, the coefficient $s_{3,0}$ cannot be derived by mass-factorization arguments, but requires a third-order calculation like in the case of deep-inelastic scattering [31].

The above results, after combination with the gluon splitting function [16] and form factor [18] according to Eqs. (3) and (4), lead to the following soft-emission contribution to the third-order (N^3LO) coefficient function for Higgs boson production at $\mu_r^2 = \mu_f^2 = M_H^2$:

$$c_3\Big|_{\mathcal{D}_5} = 512\, C_A^3 \,, \qquad (17)$$

$$c_3\Big|_{\mathcal{D}_4} = -\frac{7040}{9} C_A^3 + \frac{1280}{9} C_A^2 n_f \,, \qquad (18)$$

$$c_3\Big|_{\mathcal{D}_3} = C_A^3 \left[\frac{59200}{27} - 3584\,\zeta_2 \right] - \frac{10496}{27} C_A^2 n_f + \frac{256}{27} C_A n_f^2 \,, \qquad (19)$$

$$
\begin{aligned}
c_3\Big|_{\mathcal{D}_2} &= C_A^3\left[-\frac{67264}{27}+\frac{11968}{3}\zeta_2+11584\,\zeta_3\right]+C_A^2 n_f\left[\frac{14624}{27}-\frac{2176}{3}\zeta_2\right] \\
&\quad +32\,C_A C_F n_f-\frac{640}{27}C_A n_f^2\;, \qquad (20)\\
c_3\Big|_{\mathcal{D}_1} &= C_A^3\left[\frac{244552}{81}-\frac{9728}{3}\zeta_2-\frac{22528}{3}\zeta_3-\frac{9856}{5}\zeta_2^2\right]-C_A C_F n_f\left[504-384\,\zeta_3\right] \\
&\quad +C_A^2 n_f\left[-\frac{67376}{81}+\frac{6016}{9}\zeta_2+\frac{2944}{3}\zeta_3\right]+C_A n_f^2\left[\frac{1600}{81}-\frac{256}{9}\zeta_2\right]\;, \qquad (21)\\
c_3\Big|_{\mathcal{D}_0} &= C_A^3\left[-\frac{594058}{729}+\frac{137008}{81}\zeta_2+\frac{143056}{27}\zeta_3+\frac{4048}{15}\zeta_2^2-\frac{23200}{3}\zeta_2\zeta_3+11904\,\zeta_5\right] \\
&\quad +C_A^2 n_f\left[\frac{125252}{729}-\frac{34768}{81}\zeta_2-\frac{7600}{9}\zeta_3-\frac{544}{15}\zeta_2^2\right] \qquad (22)\\
&\quad +C_A C_F n_f\left[\frac{3422}{27}-32\,\zeta_2-\frac{608}{9}\zeta_3-\frac{64}{5}\zeta_2^2\right]-C_A n_f^2\left[\frac{3712}{729}-\frac{640}{27}\zeta_2-\frac{320}{27}\zeta_3\right]\;.
\end{aligned}
$$

Eqs. (17) – (21) agree with the results derived from the NNLL threshold resummation in Ref. [14]. Eq. (22) represents a new result of the present study.

We now turn to lepton pair production. The corresponding coefficients $s_{n,l}$ can be derived in the same manner, some being related to their Higgs counterparts by simple replacements of C_F by C_A. Specifically, $\mathcal{S}_1$ for the Drell-Yan case is obtained from Eq. (13) by a factor C_F/C_A, and $s_{n,-2n+k}$ for $n \geq 2$ and $k = 0$, 1 by multiplication with $(C_F/C_A)^{n-k}$ from Eqs. (14) and (16). Accordingly the coefficients of $\mathcal{D}_5$ and $\mathcal{D}_4$ of the third-order coefficient function for the Drell-Yan process are related to Eqs. (17) and (18), respectively, by factors C_F^3/C_A^3 and C_F^2/C_A^2. The remaining coefficients for the standard scale choice $\mu_r^2 = \mu_f^2 = M_{\gamma^*}^2$ are given by

$$
\begin{aligned}
c_3\Big|_{\mathcal{D}_3} &= \frac{7744}{27}C_A^2 C_F+C_A C_F^2\left[\frac{17152}{9}-512\,\zeta_2\right]-C_F^3\left[2048+3072\,\zeta_2\right]-\frac{2816}{27}C_A C_F n_f \\
&\quad -\frac{2560}{9}C_F^2 n_f+\frac{256}{27}C_F n_f^2\;, \qquad (23)\\
c_3\Big|_{\mathcal{D}_2} &= C_A^2 C_F\left[-\frac{28480}{27}+\frac{704}{3}\zeta_2\right]-C_A C_F^2\left[\frac{4480}{9}-\frac{11264}{3}\zeta_2-1344\,\zeta_3\right]+10240\,\zeta_3\,C_F^3 \\
&\quad +C_A C_F n_f\left[\frac{9248}{27}-\frac{128}{3}\zeta_2\right]+C_F^2 n_f\left[\frac{544}{9}-\frac{2048}{3}\zeta_2\right]-\frac{640}{27}C_F n_f^2\;, \qquad (24)\\
c_3\Big|_{\mathcal{D}_1} &= C_A^2 C_F\left[\frac{124024}{81}-\frac{12032}{9}\zeta_2-704\,\zeta_3+\frac{704}{5}\zeta_2^2\right] \\
&\quad -C_A C_F^2\left[\frac{35572}{9}+\frac{11648}{9}\zeta_2+5184\,\zeta_3-\frac{3648}{5}\zeta_2^2\right] \\
&\quad +C_F^3\left[2044+2976\,\zeta_2-960\,\zeta_3-\frac{14208}{5}\zeta_2^2\right]-C_A C_F n_f\left[\frac{32816}{81}-384\,\zeta_2\right]
\end{aligned}
$$

$$+C_F^2 n_f \left[\frac{4288}{9}+\frac{2048}{9}\zeta_2+1280\,\zeta_3\right]+C_F n_f^2\left[\frac{1600}{81}-\frac{256}{9}\zeta_2\right]\,, \tag{25}$$

$$\begin{aligned} c_3\Big|_{\mathcal{D}_0} \;=\;& C_A^2 C_F\left[-\frac{594058}{729}+\frac{98224}{81}\zeta_2+\frac{40144}{27}\zeta_3-\frac{2992}{15}\zeta_2^2-\frac{352}{3}\zeta_2\zeta_3-384\,\zeta_5\right] \\ & +C_A C_F^2\left[\frac{25856}{27}-\frac{12416}{27}\zeta_2+\frac{26240}{9}\zeta_3+\frac{1408}{3}\zeta_2^2-1472\,\zeta_2\zeta_3\right] \\ & -C_F^3\left[4096\,\zeta_3+6144\,\zeta_2\zeta_3-12288\,\zeta_5\right]-C_F n_f^2\left[\frac{3712}{729}-\frac{640}{27}\zeta_2-\frac{320}{27}\zeta_3\right] \\ & +C_A C_F n_f\left[\frac{125252}{729}-\frac{29392}{81}\zeta_2-\frac{2480}{9}\zeta_3+\frac{736}{15}\zeta_2^2\right] \\ & -C_F^2 n_f\left[6-\frac{1952}{27}\zeta_2+\frac{5728}{9}\zeta_3+\frac{1472}{15}\zeta_2^2\right]\,. \end{aligned} \tag{26}$$

Also Eqs. (23) – (25) agree with the results derived from the NNLL threshold resummation [13], while Eq. (26) is a new result of this study. The additional coefficients for $\mu_f^2 \neq M_{\gamma^*}^2$ and $\mu_r^2 \neq \mu_f^2$ can be derived analogously to Eqs. (2.16) – (2.18) of Ref. [32] or using the N^{3}LL threshold resummation expression [20], but will be skipped here for brevity.

We note that the ζ-function terms of highest transcendentality n, i.e, the coefficients of ζ_n and $\zeta_i\,\zeta_j$ with $i+j=n$, in the ε^{-2l+n} contributions to the pure real-emission function $\mathcal{S}_l$ agree between Higgs production and the Drell-Yan process for the Super-Yang-Mills case $C_A = C_F = n_c$. The same holds for the quark and gluon form factors [17, 18] and, consequently, also for the ζ-function terms of weight n in the soft logarithms $\mathcal{D}_{2l-1-n}$ of the coefficient functions for Higgs boson and lepton pair production, see Eqs. (17)–(22) and (23)–(26). By construction, generalizing Eq. (4), this feature extends to all orders of perturbation theory.

We now turn to the threshold resummation. For the processes under consideration, the coefficient functions exponentiate after transformation to Mellin N-space [11, 12],

$$C^N \;=\; (1+a_s\,g_{01}+a_s^2\,g_{02}+\ldots)\cdot\exp\left(G^N\right)+\,O(N^{-1}\ln^n N)\,. \tag{27}$$

Here g_{0k} collects the N-independent contributions at the k-th order, and the resummation exponent G^N takes the form

$$G^N(Q^2) \;=\; \ln N\cdot g_1(\lambda)+g_2(\lambda)+a_s\,g_3(\lambda)+a_s^2\,g_4(\lambda)+\ldots \tag{28}$$

with $\lambda=\beta_0\,a_s\ln N$. The functions g_3 and g_4 have been determined in Refs. [13, 14] and [20], respectively. Besides the quantities A_k in Eq. (11) and lower-order coefficients, the functions g_k depend on one parameter, usually denoted by D_{k-1}.

Before we present our new results for the coefficient D_3, we recall, for the convenience of the reader, the N-independent first- and second-order contributions which enter its determination. For Higgs boson production these coefficients read

$$g_{01} \;=\; C_A(16\,\zeta_2+8\,\gamma_e^2)\,, \tag{29}$$

$$
\begin{aligned}
g_{02} &= C_A^2 \left[93 + \frac{1072}{9}\zeta_2 - \frac{308}{9}\zeta_3 + 92\zeta_2^2 + \frac{1616}{27}\gamma_e - 56\gamma_e\zeta_3 + \frac{536}{9}\gamma_e^2 + 112\gamma_e^2\zeta_2 \right. \\
&\quad \left. + \frac{176}{9}\gamma_e^3 + 32\gamma_e^4 \right] + C_A n_f \left[-\frac{80}{3} - \frac{160}{9}\zeta_2 - \frac{88}{9}\zeta_3 - \frac{224}{27}\gamma_e - \frac{80}{9}\gamma_e^2 - \frac{32}{9}\gamma_e^3 \right] \\
&\quad + C_F n_f \left[-\frac{67}{3} + 16\zeta_3 \right] \,. \qquad (30)
\end{aligned}
$$

The corresponding results for the Drell-Yan case are given by

$$
\begin{aligned}
g_{01} &= C_F(-16 + 16\zeta_2 + 8\gamma_e^2) \,, \qquad (31) \\
g_{02} &= C_F^2 \left[\frac{511}{4} - 198\zeta_2 - 60\zeta_3 + \frac{552}{5}\zeta_2^2 - 128\gamma_e^2 + 128\gamma_e^2\zeta_2 + 32\gamma_e^4 \right] \\
&\quad + C_A C_F \left[-\frac{1535}{12} + \frac{376}{3}\zeta_2 + \frac{604}{9}\zeta_3 - \frac{92}{5}\zeta_2^2 + \frac{1616}{27}\gamma_e - 56\gamma_e\zeta_3 + \frac{536}{9}\gamma_e^2 \right. \\
&\quad \left. - 16\gamma_e^2\zeta_2 + \frac{176}{9}\gamma_e^3 \right] + C_F n_f \left[\frac{127}{6} - \frac{64}{3}\zeta_2 + \frac{8}{9}\zeta_3 - \frac{224}{27}\gamma_e - \frac{80}{9}\gamma_e^2 - \frac{32}{9}\gamma_e^3 \right] \,. \qquad (32)
\end{aligned}
$$

Inserting the above results into the explicit formulae for the resummation exponents [13,14,20], we recover the known coefficients (as always referring to the expansion parameter $a_s = \alpha_s/(4\pi)$)

$$
\begin{aligned}
D_1 &= 0 \,, \qquad (33) \\
D_2 &= C_I \left[C_A \left(-\frac{1616}{27} + \frac{176}{3}\zeta_2 + 56\zeta_3 \right) + n_f \left(\frac{224}{27} - \frac{32}{3}\zeta_2 \right) \right] \,, \qquad (34)
\end{aligned}
$$

and derive the new third-order contribution

$$
\begin{aligned}
D_3 &= C_I C_A^2 \left[-\frac{594058}{729} + \frac{98224}{81}\zeta_2 + \frac{40144}{27}\zeta_3 - \frac{2992}{15}\zeta_2^2 - \frac{352}{3}\zeta_2\zeta_3 - 384\zeta_5 \right] \\
&\quad + C_I C_A n_f \left[\frac{125252}{729} - \frac{29392}{81}\zeta_2 - \frac{2480}{9}\zeta_3 + \frac{736}{15}\zeta_2^2 \right] \qquad (35) \\
&\quad + C_I C_F n_f \left[\frac{3422}{27} - 32\zeta_2 - \frac{608}{9}\zeta_3 - \frac{64}{5}\zeta_2^2 \right] + C_I n_f^2 \left[-\frac{3712}{729} + \frac{640}{27}\zeta_2 + \frac{320}{27}\zeta_3 \right]
\end{aligned}
$$

with $C_I = C_F$ for the Drell-Yan case, and $C_I = C_A$ for Higgs production. Hence, not unexpectedly, we find that also D_3 is maximally non-abelian, with the quark and gluon cases related by an overall factor C_F/C_A. This is the same behaviour as shown by the cusp anomalous dimensions A_n in Eq. (11) [29] and by the form-factor resummation coefficients f_n known up to three loops [9,18]. In fact, there is a simple relation between the coefficients D_n and f_n (using the notation of Ref. [18]),

$$
\begin{aligned}
D_2 &= 2\beta_0 s_{1,0} - 2f_2 \\
D_3 &= 2\beta_1 s_{1,0} - 4\beta_0^2 s_{1,1} + 4\beta_0 \left(s_{2,0} - 36/5\,\zeta_2^2 C_I^2 \right) - 2f_3 \,, \qquad (36)
\end{aligned}
$$

of which the first line of has been derived before in Ref. [33] from an extension of the threshold resummation to N-independent contributions. In the present mass-factorization framework, the $s_{n,l}$ terms in Eqs. (36) can be traced back to the coupling-constant renormalization of Eqs. (4).

We now turn to the numerical impact of the N^{3}LO and resummation corrections to the coefficient functions, confining ourselves to Higgs boson production for brevity. All parameters are taken over from Ref. [8], i.e., we use the values $m_t = 173.4$ GeV (very close to the present world average) and $G_F = 4541.7$ pb for the top mass and Fermi constant in the prefactor C_H in Eq. (2), and the parton distributions of Refs. [34, 35] with their respective values of strong coupling constant at LO, NLO and NNLO, $\alpha_s(M_Z)$ = 0.130, 0.119 and 0.115. Anticipating a slight further reduction at N^{3}LO, we employ $\alpha_s(M_Z)$ = 0.114 at this order. The N^{3}LO corrections to C_H in the heavy-top limit are taken from Ref. [21]. All higher-order contributions are calculated in the heavy top-quark approximation, but normalized to the full lowest-order result.

As mentioned above, the $\delta(1-x)$ contributions to the N^{3}LO coefficient functions c_3 cannot be derived at this point. However, we note that the coefficients g_{0k} in Eqs. (29) – (32) are much larger than their $\delta(1-x)$ counterparts for c_1 and especially for c_2. We expect the same behaviour for c_3. Moreover, a good approximation (to about 10% or less) to the full double convolutions $g * g * [c_i(x)/x]$ is obtained at NLO and NNLO by transforming to N-space and keeping only the $\ln^k N$ and N^0 terms arising from the +-distributions (but not the δ-function) in c_1 and c_2. Consequently Eqs. (17) – (22) facilitate a sufficient approximation to the complete N^{3}LO correction, to which we assign a conservative 20% uncertainty, i.e., twice the offset found at NLO and NNLO.

The corresponding results are added in Fig. 1 to the total cross sections up to NNLO [6–8] at the TEVATRON and the LHC for the standard choice $\mu_r = \mu_f = M_H$ of the renormalization and factorization scales. The dependence of the cross sections on μ_r is illustrated in Fig. 2 for two representative values of the Higgs mass M_H. Also shown in Fig. 1 are the additional contributions of the N^{3}LL threshold resummation (28), see also Ref. [20], of the terms beyond N^{3}LO. In principle this resummation requires a second coefficient, the four-loop cusp anomalous dimension A_4, besides D_3 of Eq. (35). However, the effect of A_4 can safely be expected to be very small, as corroborated by the Padé estimate of Ref. [17] employed in our numerical analysis. The Mellin-inversion of the exponentiated result (which is entirely dominated by the next few orders in α_s in the present case) has been performed using the standard 'minimal prescription' contour [36].

The inclusion of our new result for the coefficient function c_3 effects an increase of the cross sections by about 10% at the TEVATRON and 5% at the LHC. The estimated uncertainties due to the approximate character of c_3 (see above) thus amount to 2% and 1%, respectively. Contributions of yet higher orders, as estimated by the threshold resummation, lead to a further increase by roughly half the N^{3}LO effect. Lacking N^{3}LO and threshold-resummed (see Ref. [37] for a first study) parton distributions, the NNLO gluon distribution of Ref. [35] has been employed for all results beyond the next-to-leading order. Based on the pattern of the available orders, one may expect slightly smaller (by about 2%) gluon-gluon luminosities at N^{3}LO.

The residual uncertainty due to uncalculated contributions of yet higher order is often estimated by varying μ_r and/or μ_f. At the LHC the representative variation of μ_r with fixed μ_f, illustrated in Fig. 2 for two Higgs masses, yields uncertainties of less than 4% for the conventional interval $1/2\,M_H \le \mu_r \le 2\,M_H$ at N^{3}LO, an improvement by almost a factor of three over the 9 to 11% stability of the NNLO cross sections. At the TEVATRON the corresponding μ_r dependence decreases

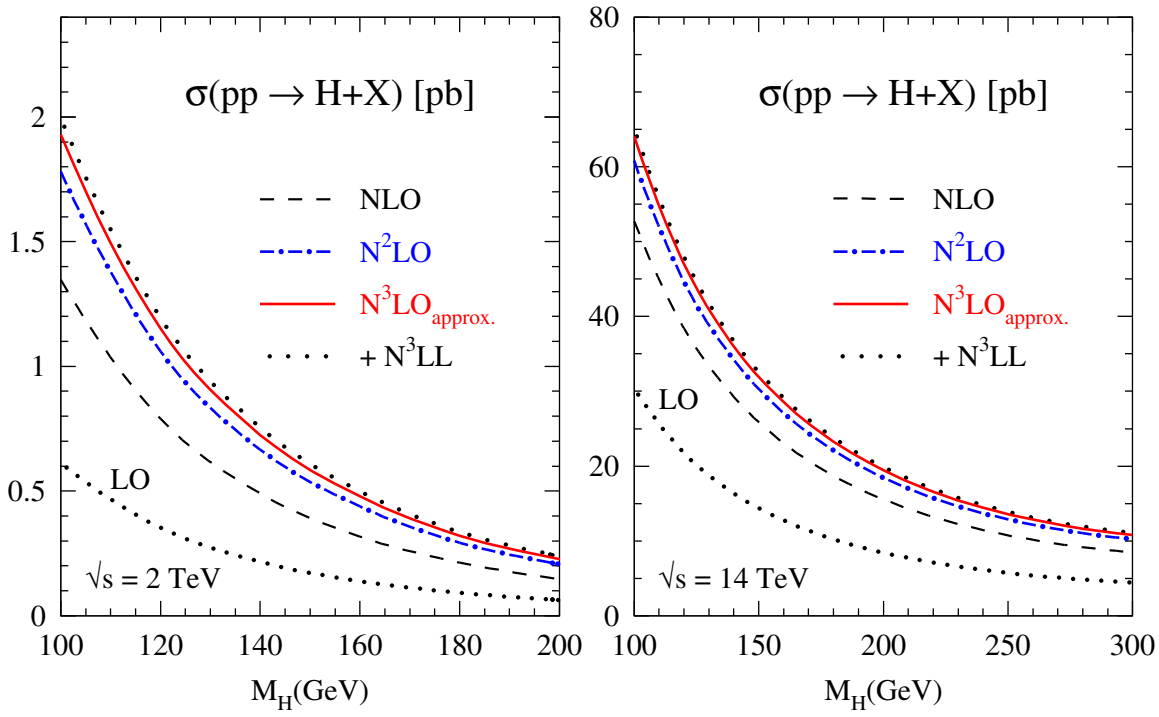

Figure 1: The perturbative expansion of the total cross section for Higgs boson production at the TEVATRON (left) and the LHC (right) for the standard scale choice $\mu_r = \mu_f = M_H$.

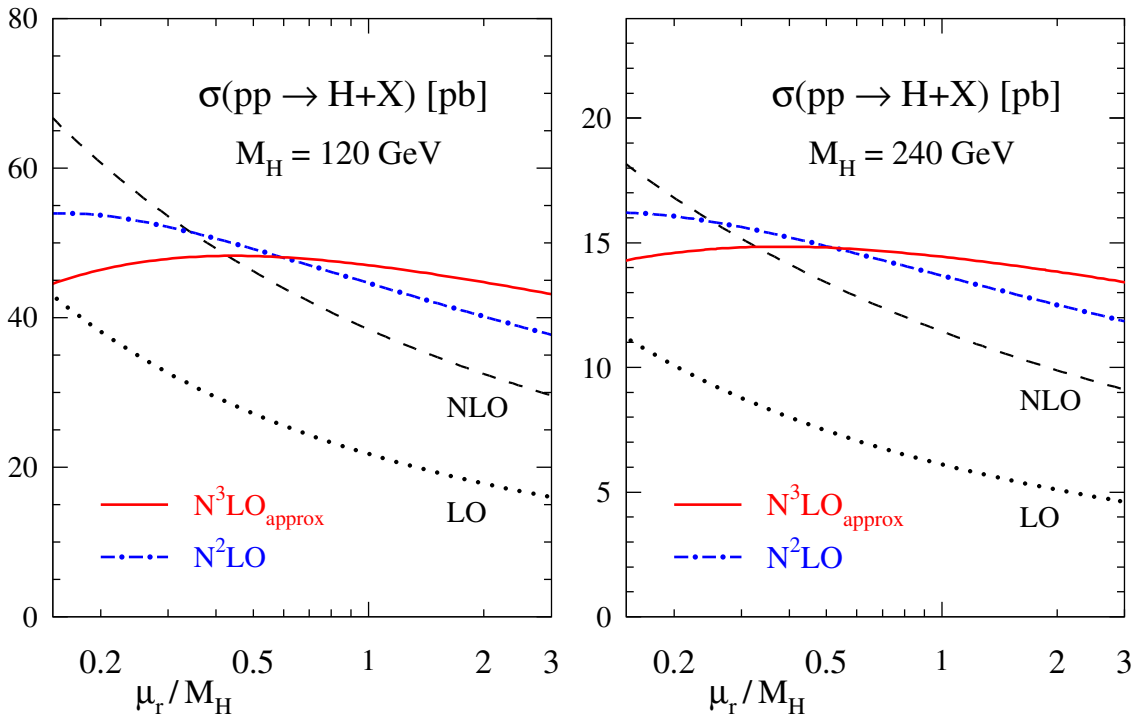

Figure 2: The dependence of the fixed-order predictions for the LHC cross section on the renormalization scale μ_r at $\mu_f = M_H$ for two representative values of the Higgs boson mass M_H.

from about 11% at NNLO to 5% at N^{3}LO for $M_H = 120$ GeV where, as in Fig. 2, the N^{3}LO cross section exhibits a stationary point close to $\mu_r = 1/2\,M_H$. Considering these and the above results, 5% at the LHC and 7% at the TEVATRON appear to represent conservative estimates of the improved cross-section uncertainties due to the truncation of the perturbation series at N^{3}LO.

Acknowledgments: We are grateful to E. Laenen for a brief, but very stimulating discussion, and to J. Smith, J. Blümlein and V. Ravindran for providing the FORTRAN codes of Refs. [8] and [38]. The symbolic manipulations for this study have been performed in FORM [39]. The work of S.M. has been supported in part by the Helmholtz Gemeinschaft under contract VH-NG-105 and by the Deutsche Forschungsgemeinschaft in Sonderforschungsbereich/Transregio 9.

Note added: Shortly after the completion of this letter, Ref. [40] appeared, which addresses the threshold resummation especially for lepton-pair production in the approach of Ref. [33]. In particular, our result (35) for the coefficient D_3 for the Drell-Yan process is confirmed by this research.

References

[1] A. Djouadi, hep-ph/0503172; hep-ph/0503173

[2] T. Matsuura, S.C. van der Marck and W.L. van Neerven, Nucl. Phys. B319 (1989) 570

[3] R. Hamberg, W. van Neerven and T. Matsuura, Nucl. Phys. B359 (1991) 343, [Erratum ibid. B644 (2002) 403]

[4] R.V. Harlander and W.B. Kilgore, Phys. Rev. D64 (2001) 013015, hep-ph/0102241

[5] S. Catani, D. de Florian and M. Grazzini, JHEP 0105 (2001) 025, hep-ph/0102227

[6] R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. 88 (2002) 201801, hep-ph/0201206

[7] C. Anastasiou and K. Melnikov, Nucl. Phys. B646 (2002) 220, hep-ph/0207004

[8] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B665 (2003) 325, hep-ph/0302135

[9] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B704 (2005) 332, hep-ph/0408315

[10] C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0501130 (Nucl. Phys. B, in press)

[11] G. Sterman, Nucl. Phys. B281 (1987) 310

[12] S. Catani and L. Trentadue, Nucl. Phys. B327 (1989) 323

[13] A. Vogt, Phys. Lett. B497 (2001) 228, hep-ph/0010146

[14] S. Catani, D. de Florian, M. Grazzini and P. Nason, JHEP 07 (2003) 028, hep-ph/0306211

[15] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192

[16] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111

[17] S. Moch, J.A.M. Vermaseren and A. Vogt, JHEP 0508 (2005) 049, hep-ph/0507039

[18] S. Moch, J.A.M. Vermaseren and A. Vogt, hep-ph/0508055 (Phys. Lett. B, in press)

[19] T. Matsuura and W.L. van Neerven, Z. Phys. C38 (1988) 623

[20] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B726 (2005) 317, hep-ph/0506288

[21] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Nucl. Phys. B510 (1998) 61, hep-ph/9708255

[22] S. Dawson, Nucl. Phys. B359 (1991) 283

[23] A. Djouadi, M. Spira and P.M. Zerwas, Phys. Lett. B264 (1991) 440

[24] M. Spira, A. Djouadi, D. Graudenz and P.M. Zerwas, Nucl. Phys. B453 (1995) 17, hep-ph/9504378

[25] H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12 (1975) 467

[26] J.C. Collins, A. Duncan and S.D. Joglekar, Phys. Rev. D16 (1977) 438

[27] T. Kinoshita, J. Math Phys. 3 (1962) 650

[28] T.D. Lee and M. Nauenberg, Phys. Rev. B133 (1964) 1549

[29] G.P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257

[30] W.L. van Neerven and A. Vogt, Nucl. Phys. B603 (2001) 42, hep-ph/0103123

[31] J.A.M. Vermaseren, A. Vogt, and S. Moch, Nucl. Phys. B724 (2005) 3,hep-ph/0504242

[32] W.L. van Neerven and A. Vogt, Nucl. Phys. B588 (2000) 345, hep-ph/0006154

[33] T.O. Eynck, E. Laenen and L. Magnea, JHEP 06 (2003) 057, hep-ph/0305179

[34] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, Eur. Phys. J. C23 (2002) 73, hep-ph/0110215

[35] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, Phys. Lett. B531 (2002) 216 hep-ph/0201127

[36] S. Catani, M.L. Mangano, P. Nason and L. Trentadue, Nucl. Phys. B478 (1996) 273, hep-ph/9604351

[37] G. Corcella and L. Magnea, hep-ph/0506278

[38] J. Blümlein and V. Ravindran, Nucl. Phys. B716 (2005) 128, hep-ph/0501178

[39] J.A.M. Vermaseren, math-ph/0010025

[40] E. Laenen and L. Magnea, hep-ph/0508284