# The Hierarchy Problem in the Standard Model

# and

# Little Higgs Theories

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# Abstract

The three couplings of the Standard Model are very different in size at low scales. However, when increasing the renormalization scale up to very high energies, they seem to converge to a single point. This hints at the appearance of a more fundamental theory at this scale, having only one coupling. We will review the calculation of the running of the Standard Model couplings, employing a very convenient method known as the Background Field Gauge, and thus showing the appearance of a very high 'unification' scale. Having established this scale, I will turn my attention to a *unified theory*, based on a gauge group $SU(5)$. We will show that the experimental constraint of having symmetry breaking at two widely separated scales requires very unnatural fine-tuning, constituting a so-called hierarchy problem.

The question arises if the hierarchy problem is a defect of $SU(5)$ theory, or a more general phenomenon. To answer this, we will consider the Standard Model as an effective theory of a more complete theory having some very heavy particles in there. Integrating out these heavy particles yields matching conditions which force the renormalized parameters of the high-energy theory to be unnaturally fine-tuned. One can trace this need for fine-tuning back to the quadratically divergent contributions to the Higgs boson mass.

Finally, we will consider a partial solution to the hierarchy problem which has only recently been discovered. In these 'Little Higgs' models, the Higgs boson is a pseudo-Goldstone bosons which transforms under a collection of symmetries, each of which is sufficient to prevent the generation of a Higgs potential. Only when all symmetries are explicitly broken by gauge and Yukawa interactions, a Higgs potential can arise. This effectively eliminates all quadratic divergences at one-loop level, and thereby pushes the scale to which the Higgs mass is natural up to about 10 TeV, which is well beyond the range of all current experiments.

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# Contents

# Chapter 1

# Introduction and Notation

The Standard Model describes the elementary particles of our world and their interactions with remarkable success. At first sight, there seems no reason why it should not be the ultimate theory of nature. However, a closer look reveals that there are good reasons for expecting physics beyond the Standard Model. One such motivation is the subject of this thesis, the so-called *hierarchy problem.*

Loosely speaking, the hierarchy problem is the statement that the mass of the Higgs boson acquires quadratically divergent quantum corrections. If one assumes the Standard Model to be valid up to very high energies, many orders of magnitude above the electroweak symmetry breaking scale $v$, the parameters in the theory need to be carefully fine-tuned to keep the Higgs mass at an acceptable value of at most a few hundred GeV. Since in the Standard Model there is no symmetry relating the various couplings, this situation is considered to be very unnatural.

In this thesis, we will investigate various concepts related to this problem. It is organized as follows: After giving an overview of general renormalization theory, we will calculate the running of the various Standard Model coupling constants to find that they suggest something interesting happening at the very high energy scale $M_X \approx 10^{15}$ GeV. In Chapter 4 we will consider unified theories, with a strong focus on $SU(5)$ unification. It is shown that the assumption of this particular model leads to a need for very careful fine-tuning of renormalized parameters. Next it is shown that the need for fine-tuning is not a consequence of the $SU(5)$ unified theory, but in fact a problem already present in the Standard Model. Traditionally, the arguments given are very qualitative in nature, based on a cut-off regularization method. We will work in dimensional regularization, which is the scheme that most particle physicists use when doing actual calculations.

After having considered the hierarchy problem from various points of view, chapter 6 will describe a recently discovered new (partial) solution to the hierarchy problem, the “Little Higgs” theory. Our focus will again be on a particular model, the so-called “Littlest Higgs”, which is the model with the lowest number of new degrees of freedom compared to the Standard Model. Finally we will have a brief look at possible traces of Little Higgs theories in future experiments.

## 1.1 Notation

- **Units** We will work in “natural” units, where $\hbar = c = 1$. In this system,

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}.$$

- **Metric** The flat space-time metric used is the one used in most recent field theory texts, $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

- **Dimensional regularization** We adopt a definition of $\epsilon$ such that $d = 4 - 2\epsilon$. Dimensional regularization (DR) is described in more detail in Appendix B

- **Group theory** Since we use the Cartan-Killing metric $\delta_{ab}$, there is no need to worry about placing indices of group elements up or down. For basic facts about Lie Groups, the reader is referred to Appendix C

# Chapter 2

# Renormalization and Symmetry

In this chapter, we take a cursory look at renormalization theory, including concepts as regularization, renormalization, and the renormalization group. We then turn to gauge theories, reviewing important concepts such as gauge invariance, quantization, and spontaneous symmetry breaking. We conclude the chapter by describing an extremely useful scheme when doing quantum calculations in a gauge theory, the so-called background field method.

## 2.1 General Renormalization Theory

The momentum integrals one encounters in a perturbative treatment of a quantum field theory are often divergent. To deal with the divergent integrals, one must give a mathematical prescription to exactly *define* this integral, that is, we have to *regularize* the integral. To obtain physically sensible (i.e. *finite*) results, one has to somehow 'absorb' these infinities into the original parameters of the theory. This is the subject of renormalization. We will look at each of these subjects in quite some detail below. In the analysis, we will only be concerned with so-called *ultraviolet (UV)* divergences, where the divergences comes from the large-momentum behaviour of the integrand. *Infrared (IR)* divergences, which usually arise when the fields involved are massless, will be ignored.

### Superficial Degree of Divergence

The *superficial degree of divergence* is the degree of divergence one would expect by naive power counting. A typical Feynman integral in $d$ dimensions is (after Wick rotation) of the form

$$\int d^d p_E \frac{p_E^{2\alpha}}{(p_E^2 - \Delta)^\beta} = \int d\Omega \int dr r^{d-1} \frac{r^\alpha}{(r-\Delta)^\beta} \sim \int dr r^{\alpha-\beta+d-1}, \tag{2.1}$$

where $p_E$ is the Euclidian momentum running through the loop. The integral in *convergent* if $\alpha - \beta < d$, *logarithmically divergent* is $\alpha - \beta = d$, etc. Now $D := \alpha - \beta - d$ is called the *superficial degree of divergence.*

It is not hard to derive a general formula for the superficial degree of divergence $D_\Gamma$ of a Feynman graph $\Gamma$, involving $n$ fields $\phi_i$ and $m$ vertex types. One finds (see e.g. de Wit & Vandoren (2002))

$$D_\Gamma = d - \sum_{\alpha=1}^{n} V_\alpha (d - \delta_\alpha) - \sum_{i=1}^{m} E_i d_i, \tag{2.2}$$

where

- $V_\alpha$ - number of vertices of type $\alpha$;
- $\delta_\alpha$ - dimension of the vertex $\alpha$;
- $E_i$ - number of external lines of field type $i$;
- $d_i$ - dimension of the field $i$.

Note that $d_\alpha := d - \delta_\alpha$ is the dimension of the coupling constant for vertex $\alpha$. Of course, being superficially convergent for a graph does not automatically mean that the graph actually converges; if a superficially convergent graph contains a divergent subgraph, it will still diverge. However, *Weinberg's theorem* (Weinberg, 1995a) states that if a graph, and all its subgraphs, have negative superficial degree of divergence, it will actually converge.

## Regularization

The divergent integrals need to be defined in a precise way to be able to perform algebraic manipulations on them. This is the purpose of a *regularization* method. There are various possible regularization methods; many of them are applicable only to a (small) subset of all possible quantum field theories. For our purposes, the two most important regularization schemes are

- **Cut-off regularization**: In this scheme, one only integrates momenta for which $|p_E| \leq \Lambda$; in the end one takes the limit $\Lambda \to \infty$. Alternatively, one can define the Feynman path integral with a *finite* grid size $a$ and in the end take the limit $a \to 0$. With this method, it's often problematic to preserve symmetries in the theory. Another drawback is that one has to evaluate integrals over finite domains, which is at the very best awkward. Despite these disadvantages, a cut-off like method is often employed in qualitative calculations, for it has a very clear and simple physical interpretation.

- **Dimensional regularization**: In this method, we calculate the integral as a function of the space-time dimension $d$; in the end one takes the limit $d \to 4$. The original divergences will appear as poles at $d = 4$. This method is widely applicable, as in most theories nothing refers explicitly to the dimension $d$. For an overview of some of the most common identities needed in dimensional regularization, see appendix B; for a proof of existence and uniqueness of integrals in an arbitrary (possibly complex) dimension, see Collins (1984).

## Renormalizability

We can distinguish three types of quantum field theories; a theory is called

- **super-renormalizable** by power counting if only a finite number of Feynman diagrams is superficially divergent;

- **renormalizable** (by power counting) if only a finite number of *amplitudes* (that is, Green's functions) superficially diverge; however, divergences occur at all orders in perturbation theory;

- **non-renormalizable** (by power counting) if all amplitudes are divergent at a sufficiently high order in perturbation theory.

From (2.2), one can see that these types correspond exactly to $d_\alpha > 0$, $d_\alpha = 0$, $d_\alpha < 0$, respectively.

It can be shown that this definition of renormalizability is equivalent to the definition already mentioned in the introduction of this chapter, namely that all all infinities in the theory can be absorbed into the parameters in the original Lagrangian. To precisely see what this means, we consider the method of *counterterms*.

## Counterterms

The method of counterterms is most easily explained by considering an example. Consider $\phi^4$ theory in $d$ dimensions. The Lagrangian is[1]

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_0\right)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4, \tag{2.3}$$

where the subscript 0 makes explicit the fact that we are dealing with the unrenormalized ('bare') parameters and fields. Now rescale the field $\phi_0$ according to

$$\phi_0 = Z^{1/2}\phi. \tag{2.4}$$

We call $\phi$ the *renormalized* field; $Z$ is commonly referred to as the *field strength renormalization*.

After rescaling, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} Z \left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_0^2 Z \phi^2 - \frac{\lambda_0}{4!} Z^2 \phi^4. \tag{2.5}$$

We eliminate the bare mass and coupling constant in favor of their *renormalized* counterparts by writing

$$Z = 1 + \delta_Z, \qquad m_0^2 Z = m^2 + \delta_m, \qquad \lambda_0 Z^2 = \lambda + \delta_\lambda. \tag{2.6}$$

[1]For some reason, the coupling constant in $\phi^3$-theory is always called $g$ but in $\phi^4$-theory it is known as $\lambda$.


Figure 2.1: Feynman rules for $\phi^4$ theory in renormalized perturbation theory.

With these definitions, the Lagrangian reads

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}\delta_Z\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4. \end{aligned} \tag{2.7}$$

The first part of (2.7) is identical to the original Lagrangian(2.3), except that bare quantities have been replaced by renormalized ones. The last three terms in (2.7) are the *counterterms*. In the literature, one often says we have "added" counterterms to the Lagrangian, but in fact we have merely split each term in the original Lagrangian.

What have we gained in this rewriting the Lagrangian like this? At first sight, splitting each term in the Lagrangian does not help in understanding, and removing, the divergences we encountered. But at least it gives us a method of organizing the infinities in a consistent way. We define the counterterms in such a way that all the renormalized parameters in the Lagrangian remain finite. Note that there is an intrinsic arbitrariness in this procedure. This arbitrariness turns out to have a deeper physical meaning and is the subject of the *renormalization group*, to be discussed shortly. But first we will explicitly show how the absorption of infinities occurs in $\phi^4$ theory. The reasons for this are twofold. First, it makes us acquainted with 'renormalized perturbation theory'. Secondly, it gives a first hint of the *hierarchy problem*, which will be discussed in much greater depth later on.

### Renormalized Perturbation Theory

In renormalized perturbation theory, we consider the counterterms in (2.7) as ordinary interactions. We can readily derive the Feynman rules for the theory (2.7), see Figure 2.1. Now, to calculate an amplitude, one proceeds as follows: sum over all possible diagrams that can be constructed from the propagators and vertices (including counterterms); the result will be a function of the (yet unknown) counterterms. Then choose a *renormalization scheme*, which fixes the counterterms. We will consider two broad classes of renormalization schemes:

- 'Physical', 'on-shell' or 'mass-dependent' renormalization. In this scheme one fixes the amplitudes to satisfy a set of 'renormalization conditions' at certain values of the external momenta. For example, in $\phi^4$ theory one can require that the renormalized mass of the scalar field is equal to the physical mass (that is, the location of the pole in the propagator).

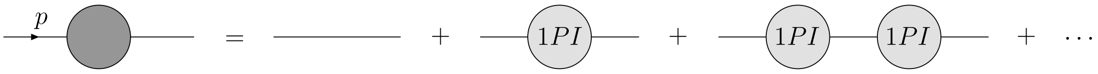
- 'Mass-independent' renormalization. In dimensional regularization, the divergences will appear as poles in the amplitude. One can then choose the counterterm to precisely cancel the pole. This is known as *minimal subtraction (MS)*. No more arbitrary is *modified minimal subtraction* ($\overline{MS}$, 'em-es-bar'), in which one replaces

$$\begin{aligned}\frac{\Gamma(\epsilon)}{\Delta^\epsilon} &= \frac{1}{\epsilon} - \gamma_E + \log(4\pi) - \log(\Delta) + \mathcal{O}(\epsilon) \\ &\longrightarrow -\log(\Delta) \end{aligned} \tag{2.8}$$

  (in other words, the counterterm is given by $\frac{1}{\epsilon} - \gamma_E + \log(4\pi)$ times the residue of the pole). Note that the right-hand side contains the logarithm of the dimensional quantity $\Delta$. This is a direct consequence of the fact that the dimension of the coupling constant is dependent on the space-time dimension. We can make the coupling constant dimensionless in *any* dimension by introducing a parameter $\mu$ with the dimension of mass; for example in $\phi^4$-theory we can replace $g$ by $\mu^{4-d}g$. $\mu$ is an *arbitrary mass scale*.

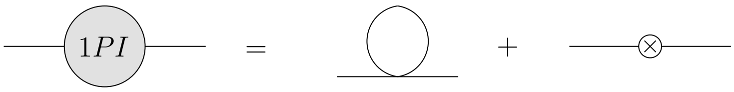
## An example in $\phi^4$ theory

As an example of the use of renormalized perturbation theory, we will work out the expressions for the amplitudes and counterterms in $\phi^4$ theory to the one-loop level. Let us for convenience define $-iM^2(p^2)$ as the sum of all one-particle irreducible diagrams. Then, the full propagator is given by



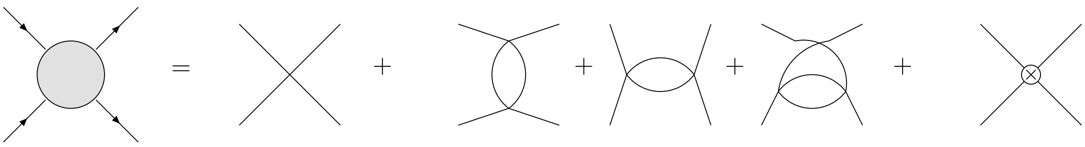

$$= \frac{i}{p^2 - m^2 - M^2(p^2)}. \tag{2.9}$$

At one-loop level, we have




$$\begin{aligned} &= -\frac{i\lambda}{2}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m) \\ &\xrightarrow[d\to 4]{} i\lambda\frac{m^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + 1 + \log(4\pi) - \log(m^2/\mu^2)\right) + i(p^2\delta_Z - \delta_m). \end{aligned} \tag{2.10}$$

The contributions to the two-particle scattering amplitude are, to one-loop level,




$$= -i\lambda + (-i\lambda)^2 \lim_{\epsilon\to 0}\left[iV(s,\epsilon) + iV(t,\epsilon) + iV(u,\epsilon)\right] - i\delta_\lambda, \tag{2.11}$$

where $d = 4 - 2\epsilon$, and $s$, $t$, $u$ are the usual Mandelstam variables, and

$$\begin{aligned} V(p^2,\epsilon) &= -\frac{1}{2(4\pi)^2}\int_0^1 dx\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{[m^2 - x(1-x)p^2]}\right)^\epsilon \\ &\xrightarrow[\epsilon\to 0]{} -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) - \log\left[(m^2 - x(1-x)p^2)/\mu^2\right]\right). \end{aligned} \tag{2.12}$$

Notice how we have introduced a mass scale $\mu$ to keep the coupling constant $\lambda$ dimensionless in any space-time dimension.

To identify the counterterms, we need to choose a renormalization scheme. In $MS$ and $\overline{MS}$ this is particularly simple; in $MS$ the counterterms are chosen to precisely cancel the poles in (2.10) and (2.12) and in $\overline{MS}$ we cancel an additional $\log(4\pi) - \gamma_E$. Thus we get in $\overline{MS}$:[2]

$$\begin{aligned} \delta_Z &= 0; \\ \delta_m &= -\lambda\frac{m^2}{32\pi^2}\frac{1}{\bar\epsilon}; \\ \delta_\lambda &= 3\frac{3\lambda^2}{32\pi^2}\frac{1}{\bar\epsilon}. \end{aligned} \tag{2.13}$$

As an example of a mass-dependent renormalization scheme, consider imposing the renormalization conditions of Figure (2.2). Here we have introduced a new parameter $\mu$ with the dimensions of mass, the *renormalization scale*. As we will see, it is convenient to choose $\mu$ of the same order as the typical momentum involved in the processes of interest. Now that we have identified the renormalization conditions, it

[2]To make our expressions look simpler, we introduce a variable $\bar\epsilon$ defined by $\frac{1}{\bar\epsilon} = \frac{1}{\epsilon} - \gamma + \log(4\pi)$

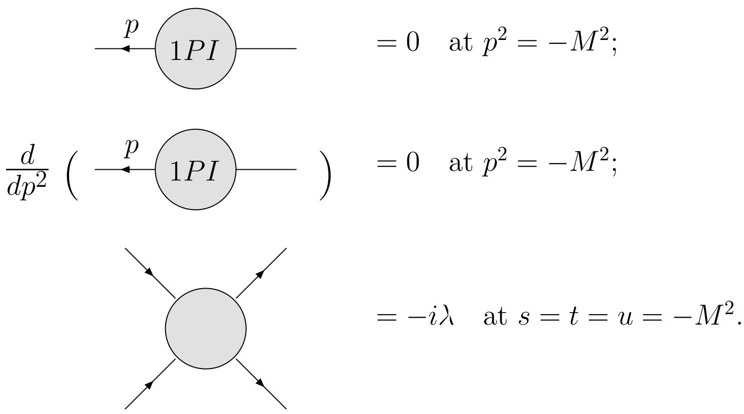



Figure 2.2: A possible set of renormalization conditions for $\phi^4$ theory. $\mu$ is the *renormalization scale.*

is a simple exercise to calculate the counterterms. The result is

$$\begin{aligned} \delta_Z &= 0; \\ \delta_m &= -\lambda \frac{m^2}{32\pi^2} \left( \frac{1}{\bar{\epsilon}} + 1 - \log m^2 \right); \\ \delta_\lambda &= \frac{3\lambda^2}{32\pi^2} \int_0^1 dx \left[ \frac{1}{\bar{\epsilon}} - \log \left( m^2 - x(1-x)\mu^2 \right) \right]. \end{aligned} \tag{2.14}$$

### Renormalization Group

We have seen that there is an intrinsic arbitrariness in the procedure of renormalization. This arbitrariness was reflected by the necessity to introduce the renormalization scale $\mu$. However, the physics should not depend on the choice of $\mu$; therefore a change in $\mu$ should be compensated by a change in the other parameters of the theory.

To see how this works and what the consequences are, we consider for simplicity a massless theory having a single dimensionless coupling constant $g$ (for example, $\phi^4$ theory in the massless limit). The renormalized $n$-point Green's functions of the theory $\Gamma^{(n)}$, are related to the bare Green's functions by the field strength renormalization $Z$:

$$\Gamma^{(n)}(p_i; g(\mu), \mu) = Z^{-n/2} \Gamma_0^{(n)}(p_i; g_0). \tag{2.15}$$

Notice in particular that the bare Green's functions (by definition) do not depend on the renormalization scale $\mu$. Thus,

$$\begin{aligned} 0 &= \frac{d}{d\mu} \Gamma_0^{(n)}(p_i; g_0) \\ &= Z^{n/2} \left( \mu \frac{\partial}{\partial \mu} + \beta(g, \mu) \frac{\partial}{\partial g} + \frac{1}{2} n \gamma(g, \mu) \right) \Gamma^{(n)}(p_i; g, \mu)., \end{aligned} \tag{2.16}$$

where

$$\beta(g, \mu) = \beta(g) = \mu \frac{\partial}{\partial \mu} g(\mu), \tag{2.17}$$

$$\gamma(g, \mu) = \gamma(g) = \mu \frac{\partial}{\partial \mu} \log Z. \tag{2.18}$$

Here we have used that $\beta$ and $\gamma$, by dimensional analysis, cannot depend on $\mu$ explicitly. We thus obtain the *Renormalization Group Equation*,

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} n \gamma(g) \right) \Gamma^{(n)}(p_i; g, \mu) = 0., \tag{2.19}$$

The important point is that the function $\beta$ and $\gamma$ are *universal*; that is, they are determined entirely by the theory and in particular do not depend on renormalization scale or the Green's function under consideration. In the next chapter we will see in detail how $\beta(g)$ ('the $\beta$ function') can be calculated. The $\gamma$ function, known as the *anomalous dimension* of the field $\phi$, is interesting as well, but will not be considered any further.

A while back we introduced the renormalization scale as a typical momentum scale involved in the experiments of interest. Now, (2.17) yields (once we have calculated $\beta$), a differential equation relating $\mu$ and $g(\mu)$. Thus, the coupling constant $g$ is in general no longer a constant, but something that changes with energy scale. Therefore, $g(\mu)$ is referred to as the *running coupling constant*. As an example, consider Quantum Electrodynamics (QED). One finds that $g(\mu)$ increases with energy, or, equivalently, decreases with distance scale. This has a very intuitive interpretation. Suppose we have an electric charge sitting somewhere. It will be surrounded by a cloud of virtual electron-positron pairs (remember we are doing quantum field theory here!). If we look from very close, there will not be many pairs in the way and we will see the full electric charge. However, the farther we move away, the more pairs will 'screen' our charge and the experienced strength will be diminished.

Remarkably, in Quantum Chromodynamics (QCD), exactly the opposite thing happens: the coupling constant *decreases* with energy. At low energies, the coupling is very large, binding quarks strongly. This makes hadronic matter possible.

## 2.2 Gauge theories

The nature of this section will be rather sketchy, the reader is assumed to be familiar with the concepts reviewed here. The material is merely included for completeness, and is by no means meant to be self-contained. A thorough treatment of gauge theories can be found in almost any textbook on (advanced) quantum field theory.

The quantum field theory describing interactions between electrons and photons, Quantum Electrodynamics (QED), is described by the Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{2.20}$$

where $\psi$ is the electron-positron field, and $A_\mu$ is the photon field. A crucial observation is that this Lagrangian is invariant under the transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x), \tag{2.21}$$

for an arbitrary function $\alpha(x)$. (2.21) Is called a *gauge transformation*. This symmetry is crucial in ensuring, among other things, the renormalizability of the theory. The question arises: can we generalize this 'principle of gauge invariance', and construct new, renormalizable, theories?

The answer turns out to be a firm 'Yes'. A reformulation of the QED Lagrangian reveals that it is based on the requirement of a $U(1)$ gauge symmetry. With this we mean that the most general theory involving electrons and positrons, invariant under local phase transformations and having only renormalizable (that is, dimension $\leq 4$) terms in the Lagrangian, automatically yields (2.20). We do not need to put in the photon field by hand; the requirement of *local* invariance necessitates the introduction of a vector field $A_\mu$, the *gauge field*. By demanding symmetry, we automatically get the photon field!

The above symmetry can be generalized as follows: instead of basing the theory on the symmetry group $G = U(1)$, we choose another (Lie) group, for example $SU(2)$ or $SU(3)$. The choice $G = SU(2)$ was considered by Yang & Mills (1954), and the theory based on this group is therefore known as *Yang-Mills theory*. But, also the general case is referred to as 'Yang-Mills' theory.

We will now describe the Lagrangian of the *gauge theory* based on a group $G$. For simplicity, we assume that $G$ is a compact, simple Lie group. The requirement that $G$ is simple ensures that there is only a single coupling constant in the game; the demand of compactness frees us from worrying about all kinds of nasty mathematical details.

Let, then, $G$ be a compact, simple Lie group. An element of $G$ is called a *group operator*, and denoted $g$. The demand of gauge invariance then translates to the statement that the Lagrangian is invariant under the transformation $\phi(x) \to g(x)\phi(x)$ for any field $\phi$ in the theory. Notice the $x$ dependence of $g$, which makes the transformation local. This $x$ dependence of $g$ causes problems in derivatives terms; the invariance is lost because we get an additional term of the form $\phi(x)\partial_\mu g(x)$. The solution to this is to replace all ordinary derivatives by covariant ones, defined by

$$D_\mu = \partial_\mu - igA_\mu^a t_a. \tag{2.22}$$

Here $g$ is a coupling constant, $A_\mu^a$ the gauge field, and $t_a$ the generators of the group $G$ in a relevant representation (for more information about Lie groups and their representations, see ... ). The next step is to include a kinetic energy term for the new field $A_\mu^a$. To this end, we first define the *field strength* $F_{\mu\nu}^a$ by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \tag{2.23}$$

where $f^{abc}$ are the structure constants of $G$. The kinetic energy term is then

$$\mathcal{L}_{\mathrm{KIN}} = -\frac{1}{4}\left(F_{\mu\nu}^a\right)^2. \tag{2.24}$$

This is basically all there is to gauge theories at the classical level. However, to do quantum calculations, we need to quantize the theory. This is most conveniently done by the Faddeev-Popov method (Faddeev & Popov, 1987). To avoid counting physically equivalent field configurations twice, we need to fix a gauge. Unfortunately, we loose explicit gauge invariance, which makes calculations much harder. This problem will be dealt with in the next section, when we discuss the background field method.

### Spontaneous symmetry breaking

First however, we need to address another issue. Gauge theories as described above do not allow for massive gauge fields, since simply adding a mass term $m_A^2 A_\mu^a A_a^\mu$ as it stands destroys renormalizability of the theory. Fortunately, there is another way of giving mass to gauge fields, relying on the phenomenon of *spontaneous symmetry breaking*. It is known as the *Higgs mechanism*.

Suppose that the classical Lagrangian of some theory possesses a certain symmetry, either global or local. The vacuum state, the state of lowest energy, does not necessarily respect this symmetry. If it does, the theory is said to be in the *unbroken* phase; otherwise the symmetry is *spontaneously broken* ('spontaneous' because we did not explicitly break the symmetry by adding non-symmetric terms to the Lagrangian).

The original symmetry of the Lagrangian can be global or local, and spontaneous symmetry breaking has qualitatively different consequences. In the case of a global symmetry, the famous Goldstone theorem (Goldstone, 1961) applies: *Every broken symmetry generator of a (continuous) global symmetry corresponds to a massless particle with the same quantum numbers.* These massless particles are commonly called *Goldstone Bosons*. An important variation emerges when the original global symmetry is not exact, but *explicitly* broken by small symmetry-breaking terms in the classical Lagrangian. In this case, the broken symmetry generators correspond to new particles that are not exactly massless but have a small mass. These particles are now called 'Pseudo-Goldstone Bosons' (PSBs) . An example is chiral symmetry in QCD with two or three quark flavors. This symmetry is explicitly broken by quark mass terms, and the PSBs coming from the chiral symmetry breaking are identified with the pions.

A broken *local* symmetry would at first sight also give rise to Goldstone Bosons, but by means of a gauge transformation the Goldstone Boson fields can completely be eliminated from the theory. Hence these fields are unphysical. However the degrees of freedom in these fields are not lost: by the *Higgs mechanism*, the Goldstone Bosons get 'eaten' by the corresponding gauge fields, which acquire a mass. The Higgs mechanism is very important in the Standard Model, where is it *the* way to give mass to vector and fermion fields. The Standard Model is described in Appendix A.

## 2.3 The Background Field Method

A gauge field theory is, at the classical level, explicitly gauge invariant. However, to include quantum corrections, one needs to fix a gauge, and normally this destroys explicit gauge invariance. (Of course, physical quantities will turn out to be gauge-independent, but the Green functions in general are not.) It would be wonderful if one is able to fix a gauge without loosing explicit gauge invariance. In a sense, the *background field gauge* (BFG) does exactly this.

The basic idea of the background field method is to write the gauge field as $A + Q$, where $A$ is the (classical) *background field*, and $Q$ is the quantum field appearing in the functional integral. Then a special gauge is chosen, which breaks only the gauge invariance on the $Q$ field. Thus the background field remains explicitly gauge invariant. Of course, this would not help us much if we still needed reference to the quantum field Green functions. However, as we will see, to calculate physical quantities such as the $S$-matrix, we need the background field Green functions *only*. We will now consider the method in detail, closely following Abbott (1980).

The generating functional of (disconnected) Green functions is

$$Z[J] = \int \mathcal{D}Q \det\left[\frac{\delta G^a}{\delta\omega^b}\right] \exp\left\{i\int d^4x \left[\mathcal{L}(Q) - \frac{1}{2\alpha}(G^a)^2 + J^a_\mu Q^a_\mu\right]\right\}, \tag{2.25}$$

where $\mathcal{L}(Q)$ is the classical Lagrangian, and $G^a$ is a gauge-fixing term. The determinant can be exponentiated by introducing new anticommuting spin-0 fields $\eta$, $\bar\eta$. Since these fields have the wrong relation between spin and statistics, they cannot be real physical fields and are therefore called *ghosts*. The determinant becomes

$$\det\left[\frac{\delta G^a}{\delta\omega^b}\right] = \int \mathcal{D}\bar\eta\mathcal{D}\eta \exp\left\{\bar\eta^a\left(-\delta^2\delta^{ac} - g\delta^\mu f^{abc}A^b_\mu\right)\eta^c\right\}. \tag{2.26}$$

For clarity, we will consider only pure Yang-Mills theory,

$$\mathcal{L}(Q) = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2, \tag{2.27}$$

where

$$F^a_{\mu\nu} = \partial_\mu Q^a_\nu - \partial_\nu Q^a_\mu + gf^{abc}Q^b_\mu Q^c_\nu. \tag{2.28}$$

In the next chapter, where we put this method to use, we will include scalar and fermion fields.

The connected Green functions are generated by

$$W[J] = -i\ln Z[J], \tag{2.29}$$

and the one-particle irreducible Greens functions are generated by the effective action $\Gamma$, defined by

$$\Gamma[\bar Q] = W[J] - \int d^4x J^a_\mu \bar Q^a_\mu, \tag{2.30}$$

where

$$\bar Q^a_\mu = \frac{\delta W}{\delta J^a_\mu}. \tag{2.31}$$

In the background field method, we write the field in the classical Lagrangian as $A+Q$, where $A$ is the background field. Note however, that we do not couple the background field to the source $J$. Thus the generating functional now becomes

$$\tilde Z[J,A] = \int \mathcal{D}Q \det\left[\frac{\delta G^a}{\delta\omega^b}\right] \exp\left\{i\int d^4x \left[\mathcal{L}(A+Q) - \frac{1}{2\alpha}(G^a)^2 + J^a_\mu Q^a_\mu\right]\right\}, \tag{2.32}$$

and the quantities $\tilde W[J,A]$ and $\tilde\Gamma[\tilde Q, A]$ are defined in the obvious manner:

$$\tilde W[J,A] = -i\ln\tilde Z[J,A], \tag{2.33}$$

$$\tilde\Gamma[\tilde Q, A] = \tilde W[J,A] - \int d^4x J^a_\mu \tilde Q^a_\mu, \tag{2.34}$$

where $\tilde Q = \delta\tilde W/\delta J$.

We now choose the *background field gauge*, by taking

$$G^a = \partial_\mu Q^a_\mu + gf^{abc}A^b_\mu Q^c_\mu. \tag{2.35}$$

One can then show that $\tilde\Gamma[\tilde Q, A]$ is invariant under

$$\delta A^a_\mu = f^{abc}\omega^b A^a_\mu + \frac{1}{g}\partial_\mu\omega^a, \tag{2.36}$$

$$\delta\tilde Q^a_\mu = f^{abc}\omega^b\tilde Q^c_\mu, \tag{2.37}$$

where $\omega$ is a parameter specifying an infinitesimal gauge transformation $\delta Q^a_\mu = -f^{abc}\omega^b(A^c_\mu + Q^c_\mu) + (1/g)\partial_\mu\omega^a$. From this we see that $\tilde\Gamma[0,A]$ is an explicitly gauge invariant functional of $A$, since (2.36) is just an ordinary gauge transformation of the background field. Moreover, it is easily demonstrated that

$$\tilde\Gamma[0,A] = \Gamma\left[\bar Q\right]|_{\bar Q = A}. \tag{2.38}$$

Here $\tilde\Gamma$ is evaluated in the gauge (2.35), and $\Gamma$ in the gauge defined by $G^a = \partial_\mu Q^a_\mu - \partial_\mu A^a_\mu + gf^{abc}A^b_\mu Q^c_\mu$. Relation (2.38) ensures that physical quantities will come out the same no matter if we use $\Gamma$ or $\tilde\Gamma$.

## Computation of $\tilde{\Gamma}[0, A]$

The gauge-invariant effective action $\tilde{\Gamma}[0, A]$ is computed by summing all one-particle irreducible diagrams with $A$ fields on external legs and $Q$ fields inside loops. No $Q$ fields appear on external lines (since $\tilde{Q} = 0$), and no $A$ fields appear inside loops (since the functional integral is only over $Q$). One can now derive the Feynman rules. The Feynman rules relevant to our computations will be given in the next chapter; for a full set of Feynman rules in pure Yang-Mills theory, see Abbott (1980).

Of course, loop diagrams will often be divergent and need to be renormalized. Since quantum and ghost fields only appear inside loops, it is not necessary to renormalize them, since the propagator renormalization will exactly cancel against the vertex renormalizations at each end. However it is still necessary to renormalize the background field $A$, the coupling constant $g$, and the gauge-fixing parameter $\alpha$:

$$g_0 = Z_g g, \qquad A_0 = Z_A^{1/2} A, \qquad \alpha_0 = Z_\alpha \alpha. \tag{2.39}$$

In principle, it is possible to avoid the gauge-fixing parameter renormalization by retaining an arbitrary $\alpha$ during the calculations. However, in practice it is often easier to go to a Feynman-type gauge $\alpha = 1$.

The background field method retains explicit gauge invariance. This means that infinities appearing in the (gauge invariant) effective action $\tilde{\Gamma}[0, A]$ must take the (gauge invariant) form of a divergent constant times $\left(F^a_{\mu\nu}\right)^2$. But according to (2.39),

$$\left(F^a_{\mu\nu}\right)_0 = Z_A^{1/2} \left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g Z_g Z_A^{1/2} f^{abc} A^b_\mu A^c_\nu\right]. \tag{2.40}$$

This will only be of desired form if $Z_g Z_A^{1/2} = 1$, thus

$$Z_g = Z_A^{-1/2}. \tag{2.41}$$

It is hard to overestimate the convenience of this relationship. To calculate counterterms and renormalizations for the coupling constant, one does not need to consider any vertex diagram. A 'simple' calculation of the field strength renormalization is sufficient.

Having reviewed gauge theories and renormalization theory, we will now turn to the calculation of the running couplings of the Standard Model. It will turn out to be very convenient to do this calculation in the background field gauge. The running couplings will be seen to hint at a very large scale at which "something happens". The large difference between this scale and the electroweak scale suffers from a hierarchy problem.

# Chapter 3

# $\beta$ Functions in the Standard Model

In this chapter, we calculate the $\beta$-function for a general (non-abelian) gauge theory, based on a gauge group $G$. We shall use the results to establish the existence of a hierarchy problem.

Assume there are $n_f$ fermions in a representation $R$, and $n_s$ scalar fields in a representation $S$. The Lagrangian is thus given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}_i \left( i\not{D} - m_i \right) \psi_i + \left( D_\mu \phi_i \right)^\dagger \left( D^\mu \phi_i \right) - m_i^2 \phi_i^\dagger \phi_i, \tag{3.1}$$

where summation over the various indices is implied, and

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu Q^a_\nu - \partial_\nu Q^a_\mu + g f^{abc} Q^b_\mu Q^c_\nu; && (3.2) \\ D_\mu \psi_i &= \partial_\mu \psi_i - ig Q^a_\mu t^{(R)}_a \psi_i; && (3.3) \\ D_\mu \phi_i &= \partial_\mu \phi_i - ig Q^a_\mu t^{(S)}_a \phi_i. && (3.4) \end{aligned}$$

Here $t^{(R)}_a$ and $t^{(S)}_a$ are the representation matrices for the respective group representations.

As discussed in the previous chapter, in the background field method one replaces $Q^a_\mu \to A^a_\mu + Q^a_\mu$, where the background field $A^a_\mu$ does not appear inside loops and the new quantum field $Q^a_\mu$ does not appear on external lines.

## 3.1 Feynman rules

We will now derive the Feynman rules in the background field gauge. The Feynman rules for pure Yang-Mills theory are derived in Abbott (1980), and they are supplemented by fermion-gauge boson and scalar-gauge boson interactions. We start with the fermion-gauge boson vertices. Writing out the relevant term in $\mathcal{L}$, $\bar{\psi} g (A^a_\mu + Q^a_\mu) \gamma^\mu t_a \psi$, shows that the Feynman rules for the $Q\bar{\psi}\psi$ and $A\bar{\psi}\psi$ are identical and obviously equal to $ig\gamma^\mu t_a$.

Derivation of the rules for vertices involving scalar fields requires a little bit more algebra. The relevant term is $\left( D_\mu \phi \right)^\dagger \left( D^\mu \phi \right)$, which, written out fully, reads

$$\begin{aligned} \left( D_\mu \phi \right)^\dagger \left( D^\mu \phi \right) &= \left( \partial_\mu \phi^\dagger + ig (A^a_\mu + Q^a_\mu) t_a \phi^\dagger \right) \left( \partial^\mu \phi - ig (A^\mu_a + Q^\mu_a) t_a \phi \right) \\ &= (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - ig (A^a_\mu + Q^a_\mu) \left( (\partial_\mu \phi^\dagger) t_a \phi - \phi^\dagger t_a (\partial_\mu \phi) \right) + g^2 (A^a_\mu + Q^a_\mu)(A^\mu_b + Q^\mu_b) \phi^\dagger t_a t_b \phi. \end{aligned}$$

Once again, we see that the $A\phi\phi$ and $Q\phi\phi$ terms are identical. They are, in momentum space, found by Fourier transformation:

$$\phi(x) = \int dk \; \phi(k) e^{ik\cdot x}, \quad \phi^\dagger(x) = \int dl \; \phi^\dagger(l) e^{-il\cdot x}, \quad A^a_\mu = \int dm \; A^a_\mu(m) e^{im\cdot x}; \tag{3.5}$$

thus

$$\begin{aligned} &-ig \int dx A^a_\mu \left( (\partial_\mu \phi^\dagger) t_a \phi - \phi^\dagger t_a (\partial_\mu \phi) \right) \\ &\qquad = -ig \int dx \int dk \; dl \; dm \; A^a_\mu(m) e^{im\cdot x} \left[ -il_\mu \phi^\dagger(l) t_a \phi(k) - \phi^\dagger(l) t_a i k_\mu \phi(k) \right] e^{i(k-l)\cdot x} \\ &\qquad = -g \int dk \; dl \; dm \; A^a_\mu(m) \phi^\dagger(l) (k+l)_\mu t_a \phi(k) \delta(k - l + m), \end{aligned}$$

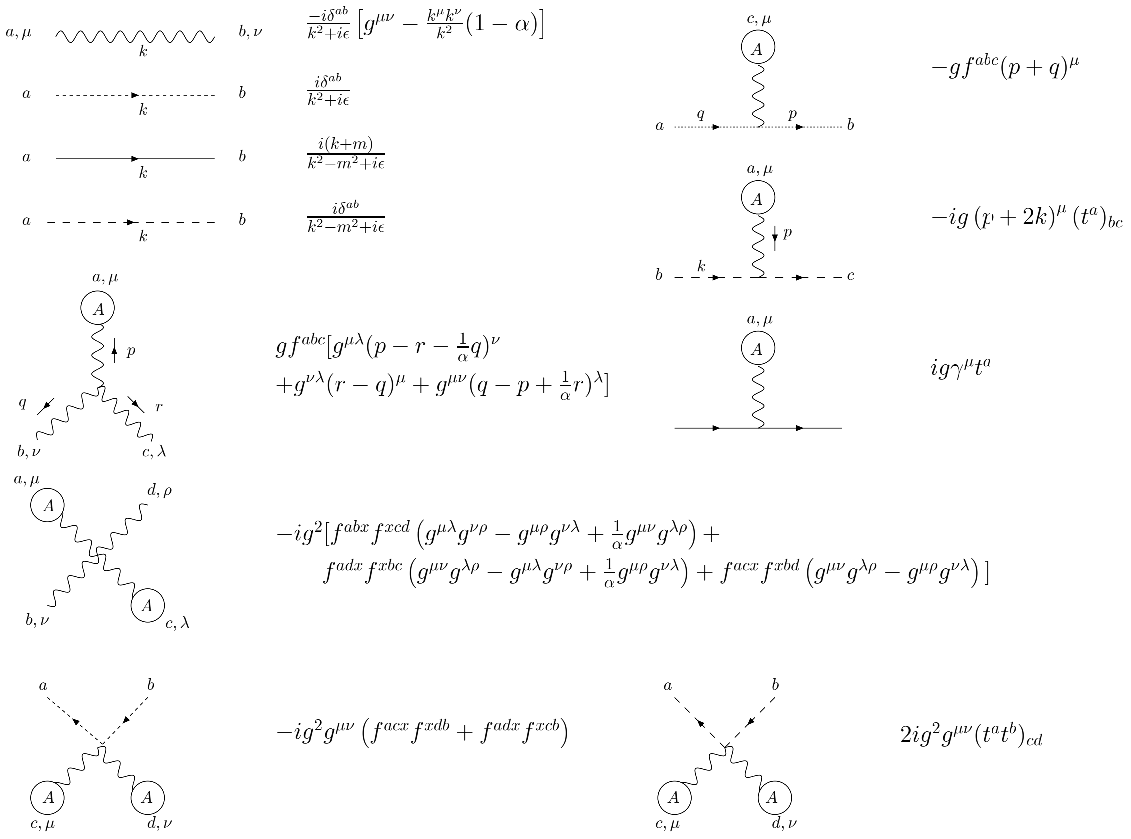



Figure 3.1: Some Feynman rules for gauge, ghost, fermion, and scalar fields and interactions in the background field gauge. Wiggly lines correspond to gauge fields, solid lines to fermions, dashed lines to scalar fields and dotted lines to ghosts. The parameter $\alpha$ is a gauge-fixing parameter; the limit $\alpha \to 1$ recovers a Feynman-'t Hooft type gauge.

from which we immediately read off the Feynman rules. We also have $AA\phi\phi$, $AQ\phi\phi$ and $QQ\phi\phi$ vertices, all of the form $g^2 A^a_\mu A^b_\nu g^{\mu\nu}\phi^\dagger t_a t_b \phi$.

The relevant Feynman rules for the vertices and propagators of gauge, ghost, fermion, and scalar fields are summarized in Figure 3.1. Not shown are vertices having an odd number of quantum field external lines, which cannot contribute at one-loop level.[1]

## 3.2 Calculation of the $\beta$ function

The $\beta$-function is defined by

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = -\mu g\frac{\partial}{\partial\mu}\ln Z_g, \tag{3.6}$$

where the last step follows from the fact that the renormalized coupling $g$ and the bare coupling $g_0$ are related by $g_0 = Z_g g$, where $g_0$ is independent of $\mu$:

$$\mu\frac{\partial g}{\partial\mu} = \mu g_0\frac{\partial}{\partial\mu}Z_g^{-1} = -\mu g_0 Z_g^{-2}\frac{\partial Z_g}{\partial\mu} = -\mu(g_0 Z^{-1})\frac{1}{Z_g}\frac{\partial}{\partial\mu}Z_g = -\mu g\frac{\partial}{\partial\mu}\ln Z_g.$$

Using the background field condition (2.41), $Z_g = Z_A^{-1/2}$, (3.6) can be rewritten as

$$\beta(g) = \frac{1}{2}g\mu\frac{1}{Z_A}\frac{\partial}{\partial\mu}Z_A = \frac{1}{2}g\mu\frac{\partial g}{\partial\mu}\frac{1}{Z_A}\frac{\partial}{\partial g}Z_A.$$

[1] This is not entirely obvious. Vertices with only one quantum line will never contribute because the effective action contains only 1PI diagrams. Three or more quantum lines can only be closed in pairs, and hence one needs at least two loops to create a valid diagram from those vertices.

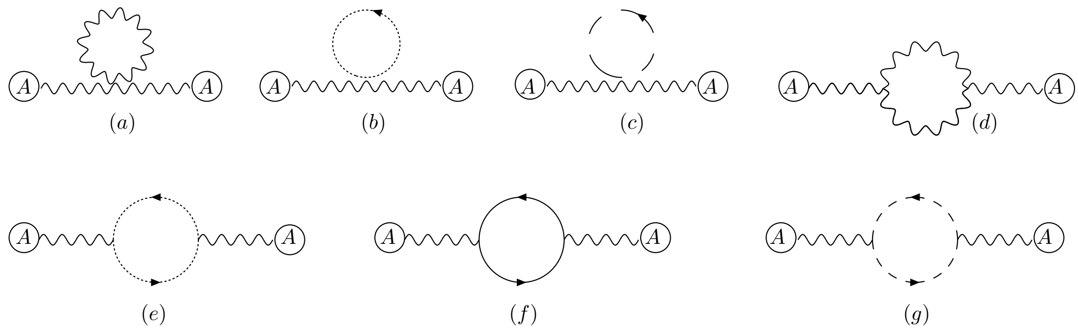



Figure 3.2: Graphs for a one-loop calculation of the $\beta$-function. All graphs have momentum $k$ flowing through, and the legs have Lorentz and group indices $\mu, a$ and $\nu, b$ respectively.

In the dimensional regularization scheme in $d = 4 - 2\epsilon$ dimensions with minimal subtraction, $Z_A$ will be written as a series of poles in $\epsilon$,

$$Z_A = 1 + \sum_{i=1}^{\infty} \frac{Z_A^{(i)}}{\epsilon^i}. \tag{3.7}$$

In $MS$, $\mu \frac{\partial g}{\partial \mu} = -\epsilon g + \beta(g)$[2], which leads to

$$\beta(g) Z_A = -\frac{1}{2} g^2 \epsilon \frac{\partial}{\partial g} Z_A + \frac{1}{2} g \beta \frac{\partial}{\partial g} Z_A.$$

Comparing $\mathcal{O}\left(\frac{1}{\epsilon^0}\right)$-terms on either side, we obtain

$$\beta(g) = -\frac{1}{2} g^2 \frac{\partial}{\partial g} Z_A^{(1)}. \tag{3.8}$$

Let us pause a moment to stress the importance of this result. Since the gauge boson field strength renormalization $Z_A$ is given by $Z_A = 1 + \delta_A$, where $\delta_A$ is the counterterm corresponding to the gauge boson field strength, we see that the $\beta$-function can be entirely calculated from the $\frac{1}{\epsilon}$-term in the counterterm. Specifically, if $\delta_A = \frac{\beta_0}{\epsilon}\left(\frac{g}{4\pi}\right)^2$, then (3.8) shows that $\beta(g) = -\frac{\beta_0 g^3}{(4\pi)^2}$. Thus, *to calculate the $\beta$-function of the theory, it suffices to calculate the divergent contributions to the gauge boson propagator in the background field gauge*, that is, using the Feynman rules of Figure (3.1). This is exactly what we will do now. To evaluate the various diagrams we make use of standard techniques, described in appendices B and C.

The diagrams contributing to the (background) gauge boson propagator are summarized in Figure 3.2. Diagrams (a) and (b) involve scaleless integrals, which are zero in dimensional regularization[3]. The evaluation of diagram (c) is straightforward:

$$\begin{aligned}(c) &= 2ig^2 g^{\mu\nu} (t^a t^b)_{cd} \int \frac{d^d p}{(2\pi)^d} \frac{i\delta^{cd}}{p^2 - m^2 + i\epsilon} \\ &= -2g^2 g^{\mu\nu} \mathrm{tr}[t^a t^b] \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1 - \frac{d}{2})}{(m^2)^{1-d/2}} = -\frac{2ig^2 m^2}{(4\pi)^2} \frac{1}{\epsilon} C(S) \delta^{ab} g^{\mu\nu} + \text{ finite terms,}\end{aligned} \tag{3.9}$$

where in the last step we have used that the scalars are in the representation $S$.

The next graph, (d), is considerably more involved. Working in a Feynman-'t Hooft-type gauge $\alpha = 1$, we get

$$\begin{aligned}(d) &= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} (g f^{acd})(g f^{bef}) \frac{-ig_{\sigma\pi}\delta_{de}}{p^2 + i\epsilon} \frac{-ig_{\rho\tau}\delta_{cf}}{(p+k)^2 + i\epsilon} \\ &\quad \times \left[-2k^\sigma g^{\mu\rho} + (k+2p)^\mu g^{\sigma\rho} + 2k^\rho g^{\mu\sigma}\right] \left[2k^\pi g^{\nu\tau} - (k+2p)^\nu g^{\pi\tau} - 2k^\tau g^{\nu\pi}\right] \\ &= -\frac{g^2}{2} C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \int_0^1 dx \frac{N^{\mu\nu}}{(1-x)p^2 + x(p+k)^2},\end{aligned}$$

[2]The extra term $-\epsilon g$ when compared to (3.6) comes from the redefinition of the coupling $g \to \mu^\epsilon g$ to make it dimensionless in every dimension.

[3]This is a consequence of the convention to analytically continue $d$-dimensional integrals from low $d$ (where they are finite), to $d = 4$. In a sense, this prescription automatically provides counterterms to cancel poles at $d = 2$ and $d = 3$.

where

$$\begin{aligned}
N^{\mu\nu} &= \left[-2k^\sigma g^{\mu\rho} + (k+2p)^\mu g^{\sigma\rho} + 2k^\rho g^{\mu\sigma}\right]\left[2k^\sigma g^{\nu\rho} - (k+2p)^\nu g^{\sigma\rho} - 2k^\rho g^{\nu\sigma}\right] \\
&= -8k^2 g^{\mu\nu} + 8k^\mu 8^\nu - 4(k+2p)^\mu (k+2p)^\nu,
\end{aligned}$$

and

$$(1-x)p^2 + x(p+k)^2 = p^2 + 2xk\cdot p + xk^2 = (p+xk)^2 + x(1-x)k^2 \equiv \ell^2 - \Delta.$$

Then $k + 2p = 2\ell + k(1-2x)$, so that

$$N^{\mu\nu} = -8k^2 g^{\mu\nu} + (8 - 4(1-2x)^2)k^\mu k^\nu - 16\ell^\mu\ell^\nu + \text{terms linear in } \ell.$$

Hence

$$\begin{aligned}
(d) &= -\frac{g^2}{2}C_2(G)\delta^{ab}\int\frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\frac{-8k^2g^{\mu\nu} + (8-4(1-2x)^2)k^\mu k^\nu - 16\ell^\mu\ell^\nu}{(\ell^2-\Delta)^2} \\
&= -\frac{g^2}{2}C_2(G)\delta^{ab}\int_0^1 dx\frac{8i}{(4\pi)^2}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}\left(-k^2g^{\mu\nu} + (1 - \frac{1}{2}(1-2x)^2)k^\mu k^\nu + \frac{\Delta}{\epsilon-1}g^{\mu\nu}\right) \\
&= -\frac{g^2}{2}C_2(G)\delta^{ab}\frac{8i}{(4\pi)^2\epsilon}\left(-k^2g^{\mu\nu} + \frac{5}{6}k^\mu k^\nu + \frac{1}{6}k^2g^{\mu\nu}\right) + \text{finite terms} \\
&= \frac{10ig^2}{3(4\pi)^2}\frac{1}{\epsilon}C_2(G)\delta^{ab}\left(k^2g^{\mu\nu} - k^\mu k^\nu\right) + \text{finite terms} \qquad (3.10)
\end{aligned}$$

We now turn to graph (e). It is given by

$$\begin{aligned}
(e) &= -\int\frac{d^dp}{(2\pi)^d}(-gf^{dac})(-gf^{ebf})(k+2p)^\mu(k+2p)^\nu\frac{i\delta_{ce}}{p^2+i\epsilon}\frac{i\delta_{df}}{(p+k)^2+i\epsilon} \\
&= -g^2f^{acd}f^{bcd}\int\frac{d^dp}{(2\pi)^d}\frac{(k+2p)^\mu(k+2p)^\nu}{p^2(k+p)^2} \\
&= -g^2C_2(G)\delta^{ab}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{(1-2x)^2k^\mu k^\nu + 4\ell^\mu\ell^\nu}{(\ell^2-\Delta)^2}, \qquad \ell = p + xk, \quad \Delta = -x(1-x)k^2 \\
&= -g^2C_2(G)\delta^{ab}\int_0^1 dx\frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}\left((1-2x)^2k^\mu k^\nu - \frac{2\Delta}{\epsilon-1}g^{\mu\nu}\right) \\
&= \frac{ig^2}{3(4\pi)^2}\frac{1}{\epsilon}C_2(G)\delta^{ab}\left(k^2g^{\mu\nu} - k^\mu k^\nu\right) + \text{finite terms} \qquad (3.11)
\end{aligned}$$

For every fermion in the theory, we have a graph as shown in (f). It is equal to

$$\begin{aligned}
(f) &= -\int\frac{d^dp}{(2\pi)^d}\text{tr}\left[-ig\gamma^\mu t^a\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\cdot -ig\gamma^\nu t^b\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2+i\epsilon}\right] \\
&= -g^2\text{tr}\left[t^at^b\right]\int\frac{d^dp}{(2\pi)^d}\frac{\text{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+\not{p}+m)\right]}{(p^2-m^2)((k+p)^2-m^2)} \\
&= -g^2C(R)\delta^{ab}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{(\ell-xk)_\rho(\ell+(1-x)k)_\sigma\text{tr}\left[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\right] + m^2\text{tr}\left[\gamma^\mu\gamma^\nu\right]}{(\ell^2-\Delta)^2},
\end{aligned}$$

where $\ell = p + xk$ and $\Delta = m^2 - x(1-x)k^2$. The numerator can be rewritten by evaluating the traces over the $\gamma$-matrices, yielding

$$\begin{aligned}
(f) &= -4g^2C(R)\delta^{ab}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{2\ell^\mu\ell^\nu - 2x(1-x)k^\mu k^\nu - g^{\mu\nu}\left[\ell^2 - x(1-x)k^2 - m^2\right]}{(\ell^2-\Delta)^2} \\
&= -4g^2C(R)\delta^{ab}\int_0^1 dx\frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}\left(x(1-x)(k^2g^{\mu\nu} - 2k^\mu k^\nu - (\Delta+m^2)g^{\mu\nu}\right) \\
&= -\frac{4ig^2}{3(4\pi)^2}\frac{1}{\epsilon}C(R)\delta^{ab}\left(k^2g^{\mu\nu} - k^\mu k^\nu\right) + \text{finite terms.} \qquad (3.12)
\end{aligned}$$

The final contribution to the background field self energy is graph (g). Its evaluation is straightforward:

$$
\begin{aligned}
(g) &= \int \frac{d^dp}{(2\pi)^d}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{(k+p)^2-m^2+i\epsilon}(-ig)^2(k+2p)^\mu(k+2p)^\nu(t^a)_{cd}(t^b)_{dc} \\
&= g^2\mathrm{tr}\left[t^at^b\right]\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{4\ell^\mu\ell^\nu+(1-2x)^2k^\mu k^\nu}{(\ell^2-\Delta)^2}, \qquad \ell=p+xk, \quad \Delta=m^2-x(1-x)k^2 \\
&= g^2C(S)\delta^{ab}\int_0^1 dx\frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}\left((1-2x)^2k^\mu k^\nu-\frac{2\Delta}{\epsilon-1}g^{\mu\nu}\right) \\
&= \frac{ig^2}{(4\pi)^2\epsilon}C(S)\delta^{ab}\left(\frac{1}{3}k^\mu k^\nu+2g^{\mu\nu}(m^2-\frac{1}{6}k^2)\right)+\text{finite terms} \\
&= -\frac{ig^2}{3(4\pi)^2}\frac{1}{\epsilon}C(S)\delta^{ab}\left(k^2g^{\mu\nu}-k^\mu k^\nu-6m^2g^{\mu\nu}\right)+\text{finite terms} \qquad (3.13)
\end{aligned}
$$

The total divergent one-loop contribution to the background field self-energy is simply the sum of the diagrams in Figure 3.2, that is, expressions (3.9)–(3.13):

$$
\left(\Sigma^{ab}\right)^{\mu\nu}=\frac{ig^2\delta^{ab}}{(4\pi)^2}\frac{1}{\epsilon}\left(k^2g^{\mu\nu}-k^\mu k^\nu\right)\left[\frac{11}{3}C_2(G)-\frac{4}{3}n_fC(R)-\frac{1}{3}n_sC(S)\right]. \qquad (3.14)
$$

In minimal subtraction, the counterterm $\delta_A$ is defined such that $\left(\Sigma^{ab}\right)^{\mu\nu}-i\delta^{ab}\left(k^2g^{\mu\nu}-k^\mu k^\nu\right)\delta_A=0$, thus $\delta_A=\frac{1}{3\epsilon}\left(\frac{g}{4\pi}\right)^2\left[11C_2(G)-4n_fC(R)-n_sC(S)\right]$. It follows that

$$
\beta(g)=-\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G)-\frac{4}{3}n_fC(R)-\frac{1}{3}n_sC(S)\right). \qquad (3.15)
$$

## 3.3 $\beta$ Functions in the Standard Model

We can now use the result (3.15) for the $\beta$-function of a gauge theory to the specific case of the Standard Model. The Standard Model (see Appendix A) is based on a gauge group which is a direct product of the three simple groups $SU_C(3)$, $SU_{I_W}(2)$ and $U_Y(1)$. We can (and will) consider each part separately. In the process, we will relate the number of fermions $n_f$ and the number of scalars $n_s$ to the number of generations $n_g$. The LEP collider has shown that $n_g \geq 3$, and that, if there are more than three generations, they must be much heavier than the currently known generations.

- **QCD** In QCD, the gauge group is $G = SU_C(3)$. There are 2 fermions per generation (the quarks) in the fundamental representation, thus $n_f = 2n_g$. There are no scalar fields, hence $n_s = 0$. Using $C_2(G) = 3, C(N) = \frac{1}{2}$ (appendix C), we get

$$
\beta(g_s)=-\frac{g_s^3}{(4\pi)^2}\left(11-\frac{4}{3}n_g\right). \qquad (3.16)
$$

- **Elecroweak theory** In the electroweak theory, $SU_{I_W}(2)\times U_Y(1)$ is spontaneously broken to $U_{EM}(1)$. At first sight, this poses a problem when calculating the $\beta$-functions for $g$ and $g'$. However, it can be shown that the spontaneously broken theory can be renormalized using only *gauge symmetric* counterterms, and, moreover, the counterterms can be taken the same in the broken and unbroken theory (see e.g. Cheng & Li (1984) and references therein). Since the $\beta$-function is calculated entirely from the counterterms, therefore, we can simply do our calculations in the unbroken theory, and for our purposes ignore the spontaneous symmetry breaking. We can also ignore the mixing between the $SU_{I_W}(2)$ and $U_Y(1)$ gauge fields.

  **SU(2)** In the electroweak theory, each generation has one lepton and three colored quark $SU(2)$-doublets. Since only left-handed fermions run through the loop, we have to multiply the result (3.12) by an extra factor $\frac{1}{2}$, which effectively means $n_f = \frac{1}{2}\cdot 4\cdot n_g = 2n_g$. There is one scalar $SU(2)$ doublet, thus $n_s = 1, C(S) = \frac{1}{2}$, and we see that the $SU(2)$ coupling constant $g$ has $\beta$-function

$$
\beta(g)=-\frac{g_s^3}{(4\pi)^2}\left(\frac{22}{3}-\frac{4}{3}n_g-\frac{1}{6}\right). \qquad (3.17)
$$

  **U(1)** The only contributions to the $U(1)$ gauge boson propagator are the fermion and scalar loops. Using that the generator of $U(1)$ is $T = \frac{1}{2}Y$, and $Y = 1$ for the scalar doublet, we find that

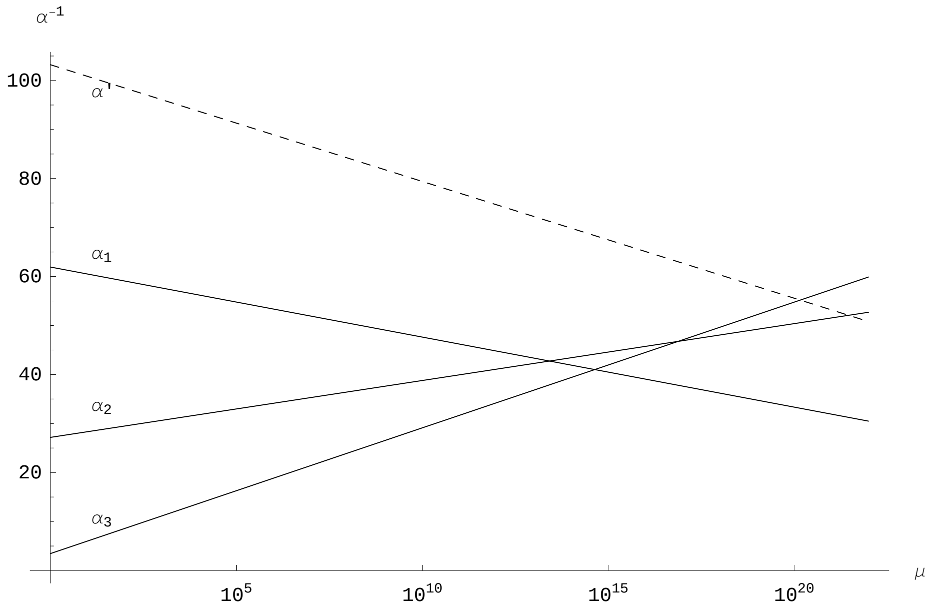



Figure 3.3: The running couplings of the Standard Model. It is seen that $\alpha_1$, $\alpha_2$, and $\alpha_3$ *almost* meet at the scale $M_X \approx 10^{15}$ GeV.

the scalar contribution is $-\frac{1}{6}$. To find the fermion contribution, we simply sum over the squared hypercharges of the quarks and leptons, and multiply by $\frac{1}{2}$ since we count only left-handed fermions. This gives $\frac{1}{2}\sum_f Y_f^2 = \frac{20}{3}n_g$, and thus

$$\beta(g') = -\frac{g'^3}{(4\pi)^2}\left(-\frac{20}{9}n_g - \frac{1}{6}\right). \tag{3.18}$$

Now that we have the $\beta$-functions for the three Standard Model couplings, we can calculate their dependence on the renormalization scale. This is done in the next section.

## 3.4 The running couplings

To one loop, the $\beta$-function is of the form $\beta(g) = -\beta_0 g^3/(4\pi)^2$. Using (3.6), we get

$$\mu\frac{\partial g}{\partial \mu} = \beta(g) = -\frac{\beta_0 g^3}{(4\pi)^2}. \tag{3.19}$$

This is easily solved to give

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} + \frac{\beta_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right), \tag{3.20}$$

where $\alpha = g^2/(4\pi)^2$ and $\mu_0$ is some reference scale. We choose $\mu_0 = M_Z = 90.117$ GeV. Experimental data shows that $\alpha_s(M_Z) = 0.118$ and $\alpha_e(M_Z) = \frac{1}{128}$. Using $e = g\sin\theta_w = g'\cos\theta_w$, where $\sin^2\theta_W(M_Z) \approx 0.23$, we obtain

$$\begin{aligned} \alpha_s(M_Z) &\simeq 0.118; \\ \alpha_g(M_Z) &\simeq 0.034; \\ \alpha_{g'}(M_Z) &\simeq 0.010. \end{aligned}$$

It is now possible to plot $1/\alpha$ as a function of $\log\mu$, as shown in Figure 3.3. Here we have taken the most likely choice for the number of generations, $n_g = 3$, and for convenience written $\alpha_g = \alpha_2$, $\alpha_s = \alpha_3$. The dashed line corresponds to the $U(1)$ coupling constant $g'$. Rather than this coupling constant, it is $g_1 \equiv (5/3)^{1/2}g'$ that is the fundamental coupling constant here. The reason for this will become clear in the next chapter, when we consider unified theories.

# Chapter 4

# Grand Unification

In the previous chapter, we saw that at a very large scale $M_X \approx 10^{15}$ GeV, the three Standard Model coupling constants become of comparable strength. This suggests that at scales above $M_X$, another symmetry emerges, that forces the couplings to be equal. In this view, the Standard Model is to be interpreted as a low-energy manifestation of a *unifying* theory, based on a (semi-)simple symmetry group $G$. Of course, the Standard Model is not invariant under $G$, so that this symmetry must be spontaneously broken to $SU_C(3) \times SU_{I_W}(2) \times U_Y(1)$.

The symmetry group $G$ must contain the Standard Model symmetry group as a subgroup. Since $SU(3)$ has rank 2, and $SU(2)$ and $U(1)$ have rank 1, it follows that $G$ must have rank at least 4. Let us try to keep things as simple as possible, and find a simple, compact, unifying group of rank 4. The classification theorem for compact Lie-groups tells us that there is not much to choose from; the only groups of rank 4 are $SO(8)$, $SO(9)$, $Sp(8)$, $SU(5)$, and the exceptional group $F_4$. However, in the Standard Model, fermion representations are not equivalent to their complex conjugates, thus the unifying group must allow for complex representations. This rules out everything except $G = SU(5)$. Of course it is possible to choose a unifying group of higher rank, but this will not be discussed.

Now let us give some additional motivation for unification. Although the Standard Model is remarkably successful in its predictions, it fails to answer many obvious questions. The most prominent of these are (Quigg, 1983):

- There are three independent gauge couplings, which are not related in any way. Is is possible to reduce the number of independent parameters?
- Are quarks and leptons related in any way? Why are fermions organized in the (rather ad hoc) pattern of right-handed singlets and left-handed doublets?
- The Standard Model suffers from anomalies which are by a very fortunate coincidence cancelled. Can this coincidence be explained using only group-theoretical properties of a bigger theory?
- The fermions seem to organize themselves into three generations. Why three?
- Why is electric charge quantized, and what explains the relations between the charges of fermions?

The Standard Model is incapable of answering these questions. Thus there seems plenty of logical evidence for a bigger theory, unifying the strong, weak, and electromagnetic forces. A theory in which these forces are unified is generally called a *grand unified theory.* We now turn our attention to the simplest grand unified theory, based on the gauge group $SU(5)$.[1]

## 4.1 $SU(5)$

$U(n)$ is the group of unitary $n \times n$ matrices, and $SU(n)$ is the subset of those matrices in $U(n)$ having determinant 1. We can write an arbitrary unitary matrix $\hat{U}$ as the exponent of a Hermitian matrix $\hat{H}$:

$$\hat{U} = \exp(i\hat{H}), \quad \hat{H}^\dagger = \hat{H}. \tag{4.1}$$

$\hat{H}$ is the *generating matrix* for $\hat{U}$. It is easy to show that $\det \hat{U} = 1$ if and only if $\mathrm{tr}\ \hat{H} = 0$. Every Hermitian $n \times n$ matrix can be written as a linear combination of $n^2$ Hermitian generators; these are

[1]Of course, a unified theory does not have to be a gauge theory at all; it might not even be a quantum field theory. We will ignore this possibility.

the generators for $U(n)$. The generators for $SU(n)$ satisfy the extra constraint of vanishing trace, hence there are $n^2 - 1$ of them.

Consider now $SU(5)$. Of course, there are many ways to choose the $5^2 - 1 = 24$ generators, but it is convenient to choose them such that any two generators $T^a, T^b$ satisfy

$$\mathrm{tr}\left[T^a T^b\right] = 2\delta^{ab}. \tag{4.2}$$

Moreover, we want to embed $SU_C(3) \times SU_{I_W}(2) \times U_Y(1)$ in $SU(5)$. From our experience with the Standard Model, we know that the color group $SU_C(3)$ is completely blind with respect to the electroweak group $SU_{I_W}(2) \times U_Y(1)$. This implies that the generators of $SU_{I_W}(2) \times U_Y(1)$ behave as unit matrices with respect to the $SU_C(3)$ generators, and vice versa. We can build this into the theory by reserving the first three rows and columns to $SU_C(3)$, and assign the last two rows and columns to $SU_{I_W}(2)$. This also completely specifies the embedding of $U_Y(1)$ into $SU(5)$: the generator corresponding to $U_Y(1)$ must, apart from being traceless, commute with both $SU_C(3)$ and $SU_{I_W}(2)$. There is only one way to achieve this (up to an overall factor): $\tilde{Y} = 1/\sqrt{15}\ \mathrm{diag}(-2, -2, -2, 3, 3)$.

In Greiner & Müller (1993), the explicit form of the generators of $SU(5)$ is derived. The result is

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \tilde{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 & & \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \tilde{\lambda}_3 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$\tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i & & \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ & & & & \\ & & & & \end{pmatrix},$$

$$\tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -i & & \\ 0 & i & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$\tilde{\lambda}_9 = \begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{10} = \begin{pmatrix} & & & -i & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{11} = \begin{pmatrix} & & & 0 & 0 \\ & & & 1 & 0 \\ & & & 0 & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix},$$

$$\tilde{\lambda}_{12} = \begin{pmatrix} & & & 0 & 0 \\ & & & -i & 0 \\ & & & 0 & 0 \\ 0 & i & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{13} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{14} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & -i & 0 \\ 0 & 0 & i & & \\ 0 & 0 & 0 & & \end{pmatrix}$$

$$\tilde{\lambda}_{15} = \begin{pmatrix} & & & 0 & 1 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{16} = \begin{pmatrix} & & & 0 & -i \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{17} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 1 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \end{pmatrix},$$

$$\tilde{\lambda}_{18} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & -i \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & i & 0 & & \end{pmatrix}, \quad \tilde{\lambda}_{19} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix}, \quad \tilde{\lambda}_{20} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & -i \\ 0 & 0 & 0 & & \\ 0 & 0 & i & & \end{pmatrix},$$

$$\tilde{\lambda}_{21} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}, \quad \tilde{\lambda}_{22} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & -i \\ & & & i & 0 \end{pmatrix}, \quad \tilde{\lambda}_{23} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix},$$

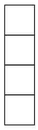
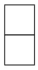
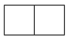
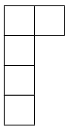
| Young tableaux | Dimension | $(SU(3), SU(2))_Y$ decomposition |
|---|---|---|
|  | **5** | $(\mathbf{3},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{2})_{1}$ |
|  | $\bar{\mathbf{5}}$ | $(\bar{\mathbf{3}},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{2})_{-1}$ |
|  | **10** | $(\bar{\mathbf{3}},\mathbf{1})_{-4/3} \oplus (\mathbf{3},\mathbf{2})_{1/3} \oplus (\mathbf{1},\mathbf{1})_{2}$ |
|  | **15** | $(\mathbf{6},\mathbf{1})_{-4/3} \oplus (\mathbf{3},\mathbf{2})_{1/3} \oplus (\mathbf{1},\mathbf{3})_{2}$ |
|  | **24** | $(\mathbf{8},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{2})_{-5/3} \oplus (\bar{\mathbf{3}},\mathbf{2})_{5/3} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{1},\mathbf{1})_{0}$ |

Table 4.1: Some irreducible $SU(5)$ representations and their dimensions. Also shown is the $SU(3) \times SU(2) \times U(1)$ decomposition of the $SU(5)$ multiplets.

and finally the generator of the $U(1)$ subgroup,

$$\tilde{\lambda}_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}.$$

The generators $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_8$ form an $SU(3)$ subalgebra, and $\tilde{\lambda}_{21}, \tilde{\lambda}_{22}, \tilde{\lambda}_{23}$ an $SU(2)$ subalgebra. $\tilde{\lambda}_{24}$ generates the $U(1)$ subgroup. Together, these generators describe the Standard Model gauge group $SU_C(3) \times SU_{I_W}(2) \times U_Y(1)$. The remaining generators $\tilde{\lambda}_9, \ldots, \tilde{\lambda}_{20}$ describe a coupling between the subgroups $SU_C(3)$ and $SU_{I_W}(2) \times U_Y(1)$. As we will see, these generators give rise to interactions absent in the standard model.

## Irreducible representations of $SU(5)$

The fermions of the Standard Model will be grouped in one or more irreducible representations (multiplets) of the gauge group $SU(5)$. This grouping will fix many properties of the unified theory. It is thus important to find the irreducible representations of $SU(5)$. The most convenient way to do this uses Young diagrams (Greiner & Müller, 1989). The irreducible representations of $SU(5)$ correspond to at most four rows of boxes, where the length of each row is no greater than the length of the preceding row. Rows correspond to symmetric products of one-particle states, and columns correspond to antisymmetric products. The dimension of the representation can be found as follows:. Some of the lowest-dimensional irreducible representations of $SU(5)$ are summarized in Table 4.1. From now on we will adopt the convention to denote an irreducible representation by its dimension, e.g.  $= \mathbf{5}$.[2]

In order to classify the elementary fermions in $SU(5)$ multiplets in a way compatible with the Standard Model, we need to investigate how the irreducible representations decompose in terms of $SU(3)$, $SU(2)$ and $U(1)$ representations. First ignore hypercharge. For the fundamental representation, the decomposition is obvious from the way we embedded $SU(3)$ and $SU(2)$ in $SU(5)$: we reserved the first three rows and columns to $SU(3)$, and the last two to $SU(2)$. This ensured that the $SU(3)$ and $SU(2)$ subgroups commuted with one another. This leads to the following trivial decomposition:

$$\mathbf{5} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}), \tag{4.3}$$

[2]Note that the dimension does, in general *not* uniquely define the representation. However, in the cases under consideration there will be no ambiguities.

where the first number in brackets represents the dimensionality of the $SU(3)$ multiplet, and the second number that of the $SU(2)$ multiplet. Then

$$\bar{\mathbf{5}} = (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}), \tag{4.4}$$

since in $SU(2)$, $\bar{\mathbf{2}} = \mathbf{2}$.

Higher-dimensional $SU(5)$ multiplets can be formed by taking products of the $\mathbf{5}$ and $\bar{\mathbf{5}}$. For example,

$$\mathbf{5} \times \mathbf{5} = \square \times \square = \square\square \oplus \begin{array}{c}\square\\\square\end{array} = \mathbf{15} \oplus \mathbf{10} \tag{4.5}$$

$$\bar{\mathbf{5}} \times \mathbf{5} = \begin{array}{c}\square\\\square\\\square\\\square\end{array} \times \square = \begin{array}{cc}\square & \square\\\square & \\\square & \\\square & \end{array} \oplus \begin{array}{c}\square\\\square\\\square\\\square\\\square\end{array} = \mathbf{24} \oplus \mathbf{1} \tag{4.6}$$

Now consider the $SU(3) \times SU(2)$ decomposition of these products:

$$\begin{aligned}
\mathbf{5} \times \mathbf{5} &= [(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})] \times [(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})] \\
&= (\mathbf{3} \times \mathbf{3}, \mathbf{1} \times \mathbf{1}) \oplus (\mathbf{3} \times \mathbf{1}, \mathbf{1} \times \mathbf{2}) \oplus (\mathbf{1} \times \mathbf{3}, \mathbf{2} \times \mathbf{1}) \oplus (\mathbf{1} \times \mathbf{1}, \mathbf{2} \times \mathbf{2}) \\
&= (\mathbf{6} \oplus \bar{\mathbf{3}}, \mathbf{1}) \oplus 2(\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1} \oplus \mathbf{3}) \\
&= (\mathbf{6}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus 2(\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}),
\end{aligned} \tag{4.7}$$

and similarly,

$$\begin{aligned}
\bar{\mathbf{5}} \times \mathbf{5} &= [(\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})] \times [(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})] \\
&= (\bar{\mathbf{3}} \times \mathbf{3}, \mathbf{1} \times \mathbf{1}) \oplus (\bar{\mathbf{3}} \times \mathbf{1}, \mathbf{1} \times \mathbf{2}) \oplus (\mathbf{1} \times \mathbf{3}, \mathbf{2} \times \mathbf{1}) \oplus (\mathbf{1} \times \mathbf{1}, \mathbf{2} \times \mathbf{2}) \\
&= (\mathbf{8} \oplus \mathbf{1}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1} \oplus \mathbf{3}) \\
&= (\mathbf{8}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}) \oplus 2(\mathbf{1}, \mathbf{1}).
\end{aligned} \tag{4.8}$$

Comparison of (4.6) and (4.8) immediately leads to the desired decomposition of the $\mathbf{24}$, see Table 4.1. To find the decomposition of $\mathbf{10}$ and $\mathbf{15}$, we compare (4.5) and (4.7). Using the symmetry properties of the various multiplets ($\mathbf{15}$ is symmetric, $\mathbf{10}$ is antisymmetric), we find also find the decomposition of these multiplets. The result is given in Table 4.1.

So far, we have ignored hypercharge. Let us correct this now. The hypercharge generator is (in the $\mathbf{5}$ representation) given by

$$\hat{Y} = \sqrt{\frac{5}{3}}\tilde{\lambda}_{24} = \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{4.9}$$

The normalization is unimportant; however the above choice of normalization is convenient because it reproduces, as we will see in a moment, the same values of hypercharge as in the Standard Model. It is then clear that $(\mathbf{3}, \mathbf{1})$ has hypercharge $-2/3$ and $(\mathbf{1}, \mathbf{2})$ has hypercharge 1.

In the $\bar{\mathbf{5}}$ representation, the hypercharge generator is given by $-\hat{Y}$. To see this, consider the corresponding group operators

$$\left(e^{i\alpha\hat{Y}}\right)^* = e^{-i\alpha\hat{Y}} = e^{i\alpha(-\hat{Y})}. \tag{4.10}$$

Also, the hypercharge of a product of two multiplets having hypercharges $y_1$ and $y_2$ is just $y_1 + y_2$. We now have all the necessary ingredients to keep track of the hypercharges in (4.7) and (4.8); the result is in Table 4.1.

## Fermion content

We have spend considerable time investigating the $SU_C(3) \times SU_{I_W}(2) \times U_Y(1)$ transformation properties of irreducible $SU(5)$ representations. The work has not be in vain: by comparison with the transformation properties of the fermions in the Standard Model we can immediately assign the fermions to representations of $SU(5)$. We consider the first generation, expressed entirely in left-handed particles and antiparticles. The transformation properties[3] are

$$\begin{array}{rcl} u_L, d_L & : & (\mathbf{3},\mathbf{2})_{1/3} \\ d_L^c & : & (\bar{\mathbf{3}},\mathbf{1})_{2/3} \\ u_L^c & : & (\bar{\mathbf{3}},\mathbf{1})_{-4/3} \\ \nu_L^e, e_L & : & (\mathbf{1},\mathbf{2})_{-1} \\ e_L^c & : & (\mathbf{1},\mathbf{1})_{2}. \end{array} \tag{4.11}$$

Comparison of (4.11) with Table 4.1 reveals that the fermions can be neatly distributed among the $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations. The $\bar{\mathbf{5}}$ can be represented by a five-dimensional vector,

$$\bar{\mathbf{5}} = \psi_i = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^- \\ \nu^e \end{pmatrix}_L . \tag{4.12}$$

The $\mathbf{10}$ can be represented by an antisymmetric $5\times 5$ matrix, having exactly 10 independent components. We organize the various $SU(3)\times SU(2)$ multiplets in this matrix as follows:

$$\mathbf{10} = \left( \begin{array}{c|c} (\bar{\mathbf{3}},\mathbf{1}) & (\mathbf{3},\mathbf{2}) \\ \hline (\mathbf{3},\mathbf{2}) & (\mathbf{1},\mathbf{1}) \end{array} \right).$$

Arranging the colour states $u_i^c$ in an antisymmetric matrix $\epsilon_{ijk}u_k^c$, we get

$$\mathbf{10} = \psi^{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L , \tag{4.13}$$

where the factor $1/\sqrt{2}$ is a convenient normalization, and many of the signs are a matter of convention.

## The $SU(5)$ gauge bosons

The gauge bosons transform under the adjoint representation, having dimension $5^2 - 1 = 24$. A glance at the decomposition of the $\mathbf{24}$ in Table (4.1) shows that we can make the following identifications:

$$\begin{array}{lcl} (\mathbf{8},\mathbf{1})_0 & \leftrightarrow & \text{an octet of gluons } G_\mu^a, \\ (\mathbf{1},\mathbf{3})_0 & \leftrightarrow & \text{an isovector of intermediate bosons } W_\mu^i, \\ (\mathbf{1},\mathbf{1})_0 & \leftrightarrow & \text{an isoscalar boson } B_\mu. \end{array}$$

The above account for $8+3+1 = 12$ of the 24 gauge bosons. There are 12 more gauge bosons, belonging to $(\mathbf{3},\mathbf{2})_{-5/3}$ and $(\bar{\mathbf{3}},\mathbf{2})_{5/3}$. Thus, the new gauge bosons form a coloured isospin doublet, which we denote by $X$ and $Y$, and their antiparticles:

$$(\bar{\mathbf{3}},\mathbf{2})_{5/3} = \begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}. \tag{4.14}$$

Using the relation $Q = T_3 + \frac{1}{2}Y$, we find the electric charges of the new gauge bosons:

$$Q(X) = \frac{1}{2} + \frac{1}{2}\cdot\frac{5}{3} = \frac{4}{3},$$
$$Q(Y) = -\frac{1}{2} + \frac{1}{2}\cdot\frac{5}{3} = \frac{1}{3}.$$

[3] We have omitted a possible right-handed neutrino, which would have to be a complete singlet under the Standard Model group: $v_L^{ec} : (\mathbf{1},\mathbf{1})_0$

Similar to what we did for the **10**, we can organize the 24 gauge bosons conveniently in a traceless $5\times 5$ matrix, having exactly 24 independent components:

$$\mathbf{24} = \left( \begin{array}{c|c} (\mathbf{8},\mathbf{1})_0 & (\mathbf{3},\mathbf{2})_{5/3} \\ ----- & ----- \\ (\bar{\mathbf{3}},\mathbf{2})_{-5/3} & (\mathbf{1},\mathbf{3})_0 \end{array} \right) + (\mathbf{1},\mathbf{1})_0.$$

Explicitly, suppressing the vector index $\mu$ for clarity,

$$\begin{aligned} \hat{A} \equiv \frac{1}{2}\sum_{a=1}^{24} A^a\tilde{\lambda}_a &= \frac{1}{2}\left[\sum_{a=1}^{8} G^a\tilde{\lambda}_a + \sum_{a=9}^{20} A^a\tilde{\lambda}_a + \sum_{a=21}^{23} W^a\tilde{\lambda}_a + B\tilde{\lambda}_{24}\right] \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} & & & X_1^c & Y_1^c \\ \frac{1}{\sqrt{2}}\sum_{a=1}^{8} G^a\lambda_a & & & X_2^c & Y_2^c \\ & & & X_3^c & Y_3^c \\ X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} \end{pmatrix} + \frac{B}{2\sqrt{15}}\begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}, \end{aligned} \tag{4.15}$$

where $\lambda_a$ are the familiar Gell-Mann matrices.

## The $SU(5)$ Lagrangian

We are now ready to construct the Lagrangian of the $SU(5)$ model. Starting point is the $SU(5)$ covariant derivative operating on the fundamental representation:

$$D_\mu\psi^{[5]} = \partial_\mu\psi^{[5]} - \frac{ig_5}{2}\sum_{a=1}^{24} A_\mu^a\tilde{\lambda}_a\psi^{[5]} = \partial_\mu\psi^{[5]} - ig_5\hat{A}_\mu\psi^{[5]}. \tag{4.16}$$

For the covariant derivative on the **10**, we need the generators of this representation. These are conveniently expressed as 10 antisymmetric $5\times 5$ matrices $\psi^{ij}$, $i,j = 1,\ldots,5$. Remembering that the **10** is the antisymmetric part of the product of two fundamental representations, we have

$$\psi^{ij} = \frac{1}{\sqrt{2}}\left(\psi^i\psi^j - \psi^j\psi^i\right), \tag{4.17}$$

where the product of two basis vectors of the **5** is understood to be a tensor product, yielding a $5\times 5$ matrix. To find the generators of **10**, we consider an infinitesimal transformation $\psi^i \to U^i_j\psi^j$, where $U^i_j = \delta^i_j - \frac{ig_5}{2}\xi^a(\tilde{\lambda}_a)^i_j$. Then

$$\begin{aligned} \psi^{ij} &\to U^i_{i'}\psi^{i'}U^j_{j'}\psi^{j'} - U^j_{i'}\psi^{i'}U^i_{j'}\psi^{j'} \\ &= U^i_{i'}U^j_{j'}\left(\psi^{i'}\psi^{j'} - \psi^{j'}\psi^{i'}\right) \\ &= U^i_{i'}U^j_{j'}\psi^{i'j'}. \end{aligned} \tag{4.18}$$

On the other hand, we know that $\psi^{ij} \to U^{ij}_{i'j'}\psi^{i'j'}$, where $U^{ij}_{i'j'} = \delta^i_{i'}\delta^j_{j'} - \frac{ig_5}{2}\xi^a(T_a)^{ij}_{i'j'}$. Thus (4.18) gives the following formula for the generators of the **10**:

$$(T_a)^{ij}_{i'j'} = \delta^i_{i'}(\tilde{\lambda}_a)^j_{j'} + \delta^j_{j'}(\tilde{\lambda}_a)^i_{i'}. \tag{4.19}$$

The covariant derivative is then

$$D_\mu\psi^{[10]} = \partial_\mu\psi^{[10]} - \frac{ig_5}{2}A^a_\mu T_a\psi^{[10]}. \tag{4.20}$$

Thus, the Lagrangian is

$$\begin{aligned} \mathcal{L}_{\rm int} &= -\frac{1}{4}\mathrm{Tr}\left(\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\right) + \bar{\psi}^{[5]}\not{D}\psi^{[5]} + \mathrm{Tr}\left(\bar{\psi}^{[10]}\not{D}\psi^{[10]}\right) \\ &= -\frac{1}{4}\mathrm{Tr}\left(\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\right) + \bar{\psi}^{[5]}\not{\partial}\psi^{[5]} - \mathrm{ig}_5\bar{\psi}^{[5]}\hat{\not{A}}\psi^{[5]} + \mathrm{Tr}\left(\bar{\psi}^{[10]}\not{\partial}\psi^{[10]} - 2\mathrm{ig}_5\bar{\psi}^{[10]}\hat{\not{A}}\psi^{[10]}\right), \end{aligned} \tag{4.21}$$

where the last step follows (4.16), (4.20), and the observation that, by antisymmetry of $\psi^{ij}$,

$$\begin{aligned}
-\mathrm{Tr}\left(\bar{\psi}^{[10]}\mathrm{D}_\mu\psi^{[10]}\right) &= \bar{\psi}^{ij}\left(\partial_\mu\psi^{ij} - \frac{ig_5}{2}A^a_\mu(T_a)^{ij}_{i'j'}\psi^{i'j'}\right)\\
&= \bar{\psi}^{ij}\partial_\mu\psi^{ij} - \frac{ig_5}{2}A^a_\mu\bar{\psi}^{ij}\left(\delta^i_{i'}(\tilde{\lambda}_a)^j_{j'} + \delta^j_{j'}(\tilde{\lambda}_a)^i_{i'}\right)\psi^{i'j'}\\
&= \bar{\psi}^{ij}\partial_\mu\psi^{ij} - \frac{ig_5}{2}A^a_\mu\left(\bar{\psi}^{ij}(\tilde{\lambda}_a)^j_{j'}\psi^{ij'} + \bar{\psi}^{ij}(\tilde{\lambda}_a)^i_{i'}\psi^{i'j}\right)\\
&= \bar{\psi}^{ij}\partial_\mu\psi^{ij} + \frac{ig_5}{2}A^a_\mu\cdot 2\bar{\psi}^{ij}(\tilde{\lambda}_a)^j_{j'}\psi^{j'i}\\
&= -\mathrm{Tr}\left(\bar{\psi}^{[10]}\partial_\mu\psi^{[10]} - 2\mathrm{i}\mathrm{g}_5\bar{\psi}^{[10]}\hat{\mathrm{A}}_\mu\psi^{[10]}\right).
\end{aligned}$$

## Predictions

It is remarkable that the elementary left-handed fermions of the Standard Model can be neatly fit into the fifteen states in the $\bar{\mathbf{5}}\oplus\mathbf{10}$ representation of $SU(5)$. But does this rearrangement help us in answering one or more of the questions posed in the introduction to this chapter? As a matter of fact, yes. For example, $SU(5)$ unification predicts quantization of charge. This is a direct consequence of the fact that the charge operator in the $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations is traceless, being a linear combination of the generators of the fundamental representation. Indeed, $Q = T_3 + \frac{1}{2}Y$. In the $\mathbf{5}$ we have $T_3 = \frac{1}{2}\tilde{\lambda}_{23}$ and $Y = \sqrt{\frac{5}{3}}\tilde{\lambda}_{23}$; in the $\mathbf{10}$, $T_3$ and $Y = \frac{5}{3}T_{24}$ are given by (4.19). Thus from (4.12) and (4.13) we have

$$\begin{aligned}
0 &= 3Q(d^c) + Q(e^-) + Q(\nu^e),\\
0 &= 3Q(u^c) + 3Q(u) + 3Q(d) + Q(e^+) = 3Q(d) - Q(e^-),
\end{aligned}$$

since $Q(\psi^c) = -Q(\psi)$. Thus we obtain

$$Q(\nu^e) = 0, \quad Q(d) = \frac{1}{3}Q(e^-). \tag{4.22}$$

From the above analysis we also draw the following interesting conclusion: quarks carry third integral charges precisely because they come in three colours!

Another successful feature of the $SU(5)$ is the prediction of the Weinberg angle $\theta_W$. From the covariant derivative (4.16), we recover the coupling of the $B$ boson as

$$-\frac{ig_5}{2}B_\mu\tilde{\lambda}_{24} = -\frac{ig_5}{2}\sqrt{\frac{3}{5}}B_\mu\hat{Y},$$

and the coupling of the $W$ bosons as

$$-\frac{ig_5}{2}\sum_{i=21}^{23}W^i_\mu\tilde{\lambda}_i = -\frac{ig_5}{2}W^i_\mu\tilde{\sigma}_i,$$

where the $\tilde{\sigma}_i$ are the usual Pauli $\sigma$-matrices embedded in the lower $2\times 2$ block of $SU(5)$.

In the Standard Model, these couplings are $-ig'B_\mu\frac{1}{2}\hat{Y}$ and $-igW^a_\mu\frac{1}{2}\sigma_a$, respectively. Thus we identify the coupling constants $g$ and $g'$ as[4]

$$g = g_5, \quad g' = \sqrt{\frac{3}{5}}g_5. \tag{4.23}$$

This yields

$$\sin^2\theta_W = \frac{g'^2}{g^2+g'^2} = \frac{3/5g_5^2}{g_5^2+3/5g_5^2} = \frac{3}{8}. \tag{4.24}$$

Of course, this relation is only valid under the assumption of an unbroken $SU(5)$ gauge theory. We know that $SU(5)$ is not a good symmetry as low energy scales, thus we must assume that at a certain mass scale $M$, $SU(5)$ is spontaneously broken to $SU_C(3)\times SU_{I_W}(2)\times U_Y(1)$. Below $M$, it is necessary to include radiative corrections to the coupling constants. We have calculated the energy dependence of

[4]From these relations it is now clear why, in the previous chapter, we plotted $(5/3)^{1/2}g'$ rather than $g'$ itself; it is $(5/3)^{1/2}g'$ that is expected to be equal to $g$ in $SU(5)$.

the coupling constants in the previous chapter, (3.16)-(3.18). Under the *assumption* of unification at $M$, i.e. $\alpha_1(M) = \alpha_2(M) = \alpha_3(M) = \alpha_5(M)$, we have at one-loop order,

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_5(M)} - \frac{1}{6\pi}\left(4n_g + \frac{3}{10}\right)\log\left(\frac{\mu}{M}\right), \tag{4.25}$$

$$\frac{1}{\alpha_2(\mu)} = \frac{1}{\alpha_5(M)} + \frac{1}{6\pi}\left(21\frac{1}{2} - 4n_g\right)\log\left(\frac{\mu}{M}\right), \tag{4.26}$$

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_5(M)} + \frac{1}{6\pi}(33 - 4n_g)\log\left(\frac{\mu}{M}\right). \tag{4.27}$$

We can solve these equations for $M$ in terms of $\alpha_3$ and $\alpha_e = \left(\frac{1}{\alpha_2} + \frac{5}{3}\frac{1}{\alpha_1}\right)^{-1}$:

$$\frac{5}{3}\left(\frac{1}{\alpha_3(\mu)} - \frac{1}{\alpha_1(\mu)}\right) = \frac{37}{4\pi}\log\left(\frac{\mu}{M}\right),$$

$$\frac{1}{\alpha_3(\mu)} - \frac{1}{\alpha_2(\mu)} = \frac{23}{12\pi}\log\left(\frac{\mu}{M}\right),$$

thus

$$\frac{8}{3}\frac{1}{\alpha_3(\mu)} - \frac{1}{\alpha_e(\mu)} = \frac{67}{6\pi}\log\left(\frac{\mu}{M}\right).$$

Choosing $\mu = M_Z$, we obtain

$$M = M_Z \exp\left[\frac{6\pi}{67}\left(\frac{1}{\alpha_e(M_Z)} - \frac{8}{3}\frac{1}{\alpha_3(M_Z)}\right)\right] \approx 7\cdot 10^{14}\text{GeV}. \tag{4.28}$$

Moreover, since $\cos^2\theta_W = \frac{5}{3}\frac{\alpha_e}{\alpha_1}$, we have

$$\sin^2\theta_W = 1 - \frac{5}{3}\alpha_e(\mu)\left(\frac{1}{\alpha_5(M)} - \frac{1}{6\pi}\left(4n_h + \frac{3}{10}\right)\log\left(\frac{\mu}{M}\right)\right).$$

With

$$\frac{1}{\alpha_5(M)} = \frac{3}{8}\left(\frac{1}{\alpha_e(\mu)} - \frac{1}{6\pi}\left(21 - \frac{32}{3}n_g\right)\log\left(\frac{\mu}{M}\right)\right),$$

this yields, with $\mu = M_Z$

$$\sin^2\theta_W = \frac{3}{8}\left[1 - \frac{109}{18\pi}\alpha_e(M_Z)\log\left(\frac{M}{M_Z}\right)\right] \approx 0.21. \tag{4.29}$$

It goes without saying that this is a remarkable quantitative success of $SU(5)$ unification. As an aside, note that the results (4.28) and (4.29) do not depend on the number of fermion generations $n_g$.

An interesting prediction of the $SU(5)$ model is *proton decay*. The $X$ and $Y$ bosons transform leptons into quarks, making possible processes such as $uu \to e^+\bar{d}$ and $ud \to \bar{d}\bar{\nu}^e$. Therefore, the proton decays, among other possibilities, according to $p \to \pi^0 e^+, \pi^+\bar{\nu}^e$. Although proton decay is highly suppressed by the large masses of the $X$ and $Y$ bosons (see next section), the predicted lifetime of the proton, $10^{30}$ years, is too low compared with the experimental value. This effectively rules out the $SU(5)$ model in its simplest form, as described above. However, many of the properties of the model are so attractive that it is hard to imagine that grand unification, perhaps based on a different gauge group, does not have anything to do with reality at all.

## Breaking $SU(5)$

As we have seen, $SU(5)$ symmetry must be broken strongly, since at low energy the coupling constants are far from equal, and there is no hint of the predicted transformation of leptons into quarks. The latter means that the $X$ and $Y$ bosons must have huge masses. It is necessary to break in two stages,

$$SU(5) \xrightarrow{M_X} SU_C(3)\times SU_{I_W}(2)\times U_Y(1) \xrightarrow{M_W} SU_C(3)\times U_{EM}(1). \tag{4.30}$$

To achieve the first step in the symmetry breaking, we introduce a scalar field in the adjoint representation, $\hat{\Sigma} = \sum_{a=1}^{24}\Sigma^a\tilde{\lambda}_a$. To preserve gauge invariance (and hence renormalizability of the theory), we can only add $SU(5)$ symmetric terms to the Lagrangian. Imposing an extra symmetry $\Sigma \to -\Sigma$, we find that the most general renormalizable $SU(5)$ symmetric Lagrangian for the scalar field is

$$\begin{aligned} \mathcal{L}_\Sigma &= (D_\mu \Sigma_a)^\dagger (D^\mu \Sigma_a) + V(\Sigma), & (4.31)\\ V(\Sigma) &= -\frac{1}{2}\mu^2 \text{Tr } \Sigma^2 + \frac{a}{4}\left(\text{Tr } \Sigma^2\right)^2 + \frac{b}{2}\text{Tr } \Sigma^4. & (4.32)\end{aligned}$$

To find the covariant derivative in the adjoint representation, we follow the same procedure as we did for the **10** representation, (4.17)-(4.20). Since $\bar{\mathbf{5}} \times \mathbf{5} = \mathbf{24} \oplus \mathbf{1}$, we know that a state ${\psi_i}^j$ in the adjoint representation transforms as

$${\psi_i}^j \to {(U^*)_i}^{i'} {U^j}_{j'} {\psi_{i'}}^{j'}, \tag{4.33}$$

where ${U^i}_j = \delta^i_j - \frac{ig_5}{2}\xi^a {(\tilde{\lambda}_a)^i}_j$. We thus find the generators of the adjoint representation,

$$(T_a)^{i'j}_{ij'} = {(\tilde{\lambda}_a)_i}^{i'} \delta^{j'}_j - \delta^i_{i'} {(\tilde{\lambda}_a)^j}_{j'} \tag{4.34}$$

The covariant derivative in the adjoint representation then is

$$\begin{aligned} D_\mu {\Sigma_i}^j &= \partial_\mu {\Sigma_i}^j - \frac{ig_5}{2} A^a_\mu (T_a)^{i'j}_{ij'} {\Sigma_{i'}}^{j'} \\ &= \partial_\mu {\Sigma_i}^j - \frac{ig_5}{2} A^a_\mu \left( {(\tilde{\lambda}_a)_i}^{i'} \Sigma^j_{i'} - {\Sigma_i}^{j'} (\tilde{\lambda}_a)^j_{j'} \right) \\ &= \partial_\mu {\Sigma_i}^j - \frac{ig_5}{2} A^a_\mu {[\tilde{\lambda}_a, \Sigma]_i}^j, \end{aligned}$$

which can conveniently be written, with the help of definition (4.15), as

$$D_\mu \hat{\Sigma} = \partial_\mu \hat{\Sigma} - ig_5 [\hat{A}, \hat{\Sigma}]. \tag{4.35}$$

Let us now investigate the potential (4.32). We could in principle minimize the potential by first diagonalizing $\hat{\Sigma}$ by a unitary transformation, and then from a general ansatz for $\hat{\Sigma}$ find the minimum by differentiation (Greiner & Müller, 1993). We choose, instead, a simpler approach. After the first step of symmetry breaking, there should still be an $SU(3) \times SU(2) \times U(1)$ symmetry. Thus, only the $X$ and $Y$ bosons should acquire mass, while the remaining bosons should remain massless. If $\hat{\Sigma}$ acquires a vacuum expectation value $\hat{\Sigma}_0$, the gauge bosons acquire a mass

$$\frac{1}{2}\text{Tr}\left[(\text{D}_\mu \hat{\Sigma}_0)^\dagger (\text{D}^\mu \hat{\Sigma}_0)\right] = \frac{\text{g}_5^2}{2} \sum_{\text{a}=1}^{24} |\text{A}_\text{a}^\mu|^2 \sum_{\text{i,k}} \left| [\tilde{\lambda}, \hat{\Sigma}_0]_{\text{ik}} \right|^2 \tag{4.36}$$

If the $SU(3)$ and $SU(2)$ gauge bosons are to remain massless, $\Sigma_0$ should therefore commute with the corresponding generators $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_8$ and $\tilde{\lambda}_{20}, \ldots, \tilde{\lambda}_{24}$. Since $\Sigma_0$, being in the adjoint representation, should also have vanishing trace, it is restricted to be proportional to $\hat{Y}$:

$$\hat{\Sigma_0} = \begin{pmatrix} -v & & & & \\ & -v & & & \\ & & -v & & \\ & & & \frac{3}{2}v & \\ & & & & \frac{3}{2} \end{pmatrix}. \tag{4.37}$$

Explicit computation from (4.36) shows that

$$M_X^2 = M_Y^2 = \frac{25}{8} g_5^2 v^2, \tag{4.38}$$

while the other gauge bosons remain massless. The potential (4.32) becomes

$$V(\hat{\Sigma}_0) = -\frac{15}{4}\mu^2 v^2 + \frac{15}{16} v^4 \left(15a + 7b\right), \tag{4.39}$$

which has a minimum for

$$v = \sqrt{\frac{2\mu^2}{15a + 7b}} \tag{4.40}$$

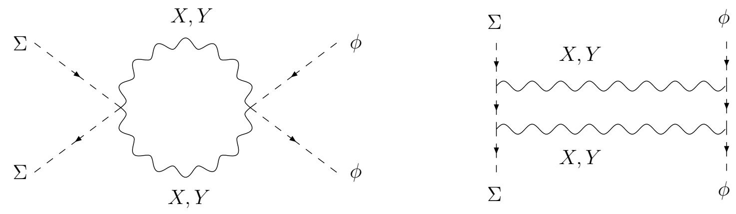



Figure 4.1: Example of graphs inducing couplings between the $\Sigma$ and $\phi$ scalars.

We will now examine the second stage of symmetry breaking. This can be accomplished by a scalar field $\hat{\phi}$ in the fundamental representation,

$$\phi = \begin{pmatrix} h^1 \\ h^2 \\ h^3 \\ h^+ \\ -h^0 \end{pmatrix}. \tag{4.41}$$

Choosing the potential

$$V(\phi) = -\frac{1}{2}\nu^2\phi^\dagger\phi + \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2, \tag{4.42}$$

we have the vacuum expectation value

$$v_0 = \frac{\nu}{\sqrt{\lambda}} \tag{4.43}$$

along *any* direction.

There are, however, some problems with the two-stage breaking as described above:

- The direction of the vacuum expectation value (4.43) is arbitrary. However, since $SU(5)$ is already broken to $SU(3) \times SU(2) \times U(1)$, different choices of the direction would lead to entirely different physical consequences. Taking the vacuum expectation value in the 4 or 5 direction breaks $SU(2)$ as desired, but choosing the 1,2 or 3 direction would break $SU(3)$, which is clearly undesirable. We need a way to incorporate the structure of the broken $SU(5)$ into the second stage of symmetry breaking.

- We have not included cross couplings between the $\Sigma$ and $\phi$ fields. However, such terms will be induced by diagrams as in Figure 4.1, which are divergent. Hence such terms need to be included in the Lagrangian to ensure renormalizability of the theory.

Fortunately, both deficiencies can be repaired by including the gauge invariant and renormalizable cross couplings between $\Sigma$ and $\phi$ through the potential

$$V(\hat{\Sigma}, \phi) = \alpha\phi^\dagger\phi \mathrm{Tr}\ \hat{\Sigma}^2 + \beta\phi^\dagger\hat{\Sigma}\phi. \tag{4.44}$$

Proper minimization of the total potential $V = V(\hat{\Sigma}) + V(\hat{\Sigma}, \phi) + V(\phi)$ is quite complicated. However we can simplify our task by making the following observations:

- The two symmetry breaking scales are about 12 orders of magnitude apart. We can therefore assume that the relative effect of the $\phi$ field on the vacuum expectation value of the $\hat{\Sigma}$ field is of order $M_X/M_W \approx 10^{-12}$, a very small number.

- The $\phi$ field is introduced to establish the breaking $SU_{I_W}(2) \times U_Y(1) \longrightarrow U_{EM}(1)$, but leaving the $SU_C(3)$ symmetry intact. It therefore makes sense to search for a $\phi$ vacuum expectation value entirely in the direction of the $SU(2)$ subgroup. By an $SU(2)$ gauge transformation we can then point $\langle\phi\rangle$ in the 5-direction:

$$\langle\phi\rangle = v_0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.45}$$

- The remarks that led to (4.37) are still valid, except that now also the $SU(2)$ symmetry is broken. This gives a new ansatz for the vacuum expectation value of $\hat{\Sigma}$,

$$\langle\Sigma\rangle = v \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \frac{3}{2}+\frac{1}{2}\epsilon & \\ & & & & \frac{3}{2}-\frac{1}{2}\epsilon \end{pmatrix}. \tag{4.46}$$

One can now easily work out the potential

$$V = \frac{1}{2}v^2(15+\epsilon^2)\left(-\frac{1}{2}\mu^2 + \alpha v_0^2 + \frac{a}{8}v^2(15+\epsilon^2)\right) + \frac{b}{4}v^4\left(\frac{105}{8}+18\epsilon^2\right)$$
$$+ \frac{\beta}{4}v_0^2v^2\,(-3+\epsilon)^2 - \frac{1}{2}\nu^2v_0^2 + \frac{\lambda}{4}v_0^4 + \mathcal{O}(\epsilon^3),$$

so that the minimum of $V$ is at those values of $v$,$v_0$ and $\epsilon$ for which

$$\begin{aligned} 0 = \frac{\partial V}{\partial v} &= -\frac{15}{2}\mu^2 v + \frac{225}{4}av^3 + \frac{105}{4}bv^3 + 15\alpha v_0^2 v + 9\beta v_0^2, \\ 0 = \frac{\partial V}{\partial v_0} &= 15\alpha v_0 v^2 + \left(\frac{9}{2} - 3\epsilon\right)\beta v_0 v^2 - \nu^2 v_0 + \lambda v_0^3, \\ 0 = \frac{\partial V}{\partial \epsilon} &= -\frac{1}{2}\mu^2 v^2\epsilon + \frac{15}{4}av^4\epsilon + 18bv^4\epsilon - \frac{3}{2}\beta v_0^2 v^2. \end{aligned} \tag{4.47}$$

Here we have ignored all terms of order $\epsilon^2$, which are extremely small. (4.47) can be rewritten as

$$\begin{aligned} \mu^2 &= \frac{15}{2}av^2 + \frac{7}{2}bv^2 + 2\alpha v_0^2 + \frac{9}{30}\beta v_0^2, \\ \nu^2 &= \lambda v_0^2 + 15\alpha v^2 + \left(\frac{9}{2} - 3\epsilon\right)\beta v^2, \end{aligned} \tag{4.48}$$

and

$$\epsilon \approx \frac{6\beta}{29b}\left(\frac{v_0}{v}\right)^2. \tag{4.49}$$

We have succeeded. The $\hat{\Sigma}$ field breaks $SU(5)$ to $SU(3)\times SU(2)\times U(1)$. From (4.46) we see that $SU(2)\times U(1)$ is broken, but since $\epsilon$ is of order $(v_0/v)^2 \approx 10^{-24}$, this breaking is quite negligible. Moreover, $\phi$ breaks $SU_{I_W}(2)\times U_Y(1)$ down to just $U_{EM}(1)$ as desired.

Is everything alright now? A closer look at (4.48) reveals that the answer is, again, negative. Since $v_0/v \approx 10^{-12}$, (4.48) gives (ignoring the term involving $\epsilon$)

$$\nu^2 - \left(15\alpha + \frac{9}{2}\beta\right)v^2 = \lambda v_0^2 = \mathcal{O}(10^{-24})v^2. \tag{4.50}$$

This is a problem because it requires a very precise fine-tuning of parameters, a property which is undesirable in any physical theory. The necessity for this unnatural fine-tuning is known as the *hierarchy problem.* Let us expose the necessity of fine-tuning more clearly by a numerical example. Taking $v = 10^{12}$, $\alpha = 0$, and $\nu = 10^{12}$, we need $\beta = \frac{2}{9}\left(1 - 10^{-24}\right) = 0.22\ldots2$ (24 2's) to satisfy (4.48). Suppose we change $\beta$ a tiny little bit, say one part in $10^{10}$, to $\beta = 0.2222222222$. Then

$$\nu^2 - \left(15\alpha + \frac{9}{2}\beta\right)v^2 = 10^{24} - \frac{9}{2}\beta\cdot 10^{24} = 10^{14}, \tag{4.51}$$

which is not close to $10^{-24}v^2 = 1$ at all!

In the next chapter, we will consider the hierarchy problem in great detail.

## 4.2 Other groups

As already mentioned in the introduction to this chapter, $SU(5)$ is the smallest, but not the only possible choice for a unifying gauge group. In fact, many more gauge groups have been considered, in an attempt to provide an answer to even more of the questions posed earlier. Proposed unifying groups include, among many others, $E_6$ and $SO(10)$.

$SO(10)$ has some very attractive properties. It has a 16-dimensional representation in which one can neatly fit the 15 fermions of a generation, leaving room for a right-handed neutrino. Recent experiments favor a massive neutrino, which necessitates the inclusion of a right-handed neutrino. Moreover, the 16-dimensional representation is automatically (that is, based on group theory alone) anomaly-free, thereby explaining the miraculous anomaly cancellation of the Standard Model.

Successful as it is, $SO(10)$ does not explain the problem of family replication. Attempts have been made to find groups which contain all known fermions of all generations in a single irreducible representation, so far with limited success. Still, many of the questions posed in the introduction to this chapter have been answered in the context of unified theories, which makes the concept of grand unification qualitatively and quantitatively a great success.

# Chapter 5

# The Hierarchy Problem

The hierarchy problem is the statement that the Higgs mass is unnaturally small. Already in the previous chapter we have seen how it arises if we assume that a grand unified theory exists. In this chapter, we will investigate the hierarchy problem further.

We start by reviewing the 'naturalness' argument of 't Hooft (1979). We consider (again) the case where the grand unified theory based on $SU(5)$ and ask ourselves the question what features of this model lead to the fine-tuning problems. We then have a look at the usual qualitative arguments that 'something is wrong', based on a cut-off of the theory. But cut-off regularization is not what people normally use when doing calculations; therefore we turn our attention to dimensional regularization. The assumption we make here is that the Standard Model is an effective field theory of a high energy theory with very heavy particles. This effective theory can be anything, in particular do we not assume a grand unified theory such as $SU(5)$.

## 5.1 Naturalness

In physics we expect macroscopic behaviour to follow from a microscopic theory. It is undesirable and indeed unlikely that the microscopic theory contains various free parameters that are carefully adjusted to give the macroscopic system some special properties. What we really want is that the effective theory at a low energy scale $\mu_1$ follows from the properties of a much higher energy scale $\mu_2$, without the need for fine-tuning the various parameters in the high-energy theory to an accuracy of order $\mu_1/\mu_2$. However, a parameter is allowed to be very small (of order $\mu_1/\mu_2$), provided that this property is not spoiled by higher order effects.

This observation lead 't Hooft (1979) to the following definition of naturalness: *at any energy scale $\mu$, a physical parameter or set of parameters $\alpha_i(\mu)$ is allowed to be very small only if the replacement $\alpha_i(\mu) = 0$ would increase the symmetry of the system.* We will now investigate the consequences of the requirement of naturalness in the Standard Model.

Difficulties with naturalness occur only in theories with scalar fields. In the Standard Model, the only scalar field is the Higgs field, with mass-squared $m_H^2$. First ignore the Higgs self-coupling $\lambda\phi^4$ and set the gauge couplings to zero (note that we are effectively considering a free scalar field theory now). At energies $\mu \gg m_H$, $m_H$ is small. Setting $m_H \to 0$ indeed increases the symmetry to

$$\phi(x) \to \phi(x) + \text{ const.} \tag{5.1}$$

Now consider the full theory. The symmetry (5.1) is at best broken by $g^2/4\pi = \mathcal{O}(1/137)$ effects due to the gauge interactions. The quartic Higgs interaction also breaks the symmetry, this time by $\mathcal{O}(\lambda)$ effects. By dimensional analysis, at the scale $\mu$ we have $m_H^2 = \mathcal{O}(\lambda\mu^2)$, so that

$$m_H^2/\mu^2 \sim \mathcal{O}(\lambda). \tag{5.2}$$

Using the expression for the Higgs mass, $m_H^2 = \lambda v^2$ (give or take a factor of $\sqrt{2}$), where $v$ is the vacuum expectation value of the Higgs field, $v \approx 174$ GeV. Hence,

$$\mu \sim \mathcal{O}(v) = \mathcal{O}(174\text{GeV}). \tag{5.3}$$

For energy scales much beyond $v$, the model becomes increasingly unnatural.

## 5.2 The hierarchy problem in GUTs

In the context of a grand unified theory based on $SU(5)$ symmetry, the hierarchy problem arose in (4.48). The mass $\nu$ of the scalar field triggering electroweak symmetry breaking (EWSB) at the scale $v_0$ acquires contributions of the order of the GUT breaking scale $v$. To obtain an acceptable mass for the scalar doublet (which, after all, should be identified with the Higgs doublet of the Standard Model), this requires an extremely unnatural cancellation of one part in $10^{12}$. This is the hierarchy problem in its clearest form. Note that it does not expose a mathematical contradiction in the theory, but is merely an aesthetical problem. It is as if someone has carefully adjusted the fundamental parameters of our universe in such a way as to make those dramatic cancellations possible. A slight change of one parameter will drastically change the world we live in. Note that the relation (4.48) is a relation between *unrenormalized* parameters. This means that, even if we can adjust the parameters of the theory such that (4.48) is satisfied, quantum corrections will destroy the result and necessitate a new fine-tuning. In fact, Gildener (1976) has argued that *in perturbation theory*, it is impossible to satisfy a relation like (4.48). The argument is very simple: if the relation is satisfied at the classical level, the one-loop corrections will be so large as to completely destroy the relation. If we then adjust the parameters such that the relation is satisfied at the one-loop level, the two-loop corrections are too large to maintain it, *ad infinitum*. Of course, not being able to satisfy the relation in perturbation theory is mostly a statement of the weaknesses of perturbation theory, but it does signify that at least we have a *practical* problem.

The question arises if this need for fine-tuning is a nasty feature of the $SU(5)$ model, or a more general phenomenon. In this section I will argue any unified theory with a unification scale well beyond the TeV scale suffers from this problem. In the next section we will make things worse by arguing that the mere assumption that the Standard Model is just an effective field theory of some high energy theory gives rises to a hierarchy problem.

To answer the first question, we have a close look at the derivation of relation (4.48). What was essential in the derivation, and what not? The answer seems to be: *only the fact that there is a large hierarchy of scales is relevant.* In fact, if we build a theory on another gauge group, the general structure will remain the same. The only things that change in our equations are the dimensions of our matrices and vectors, and a few numerical constants. But the same couplings between scalar fields and gauge bosons exist, and the form of the scalar field potential remains the same.

A little bit more complex to analyze is the effect of more complex breaking patterns. Especially when considering unified gauge theories based on large groups like $SO(10)$ or $E_6$, many different symmetry breaking patterns exist. Besides breaking down to $SU_C(3) \times SU_{I_W}(2) \times U_Y(1)$ in one big step, one can break in several smaller steps. But the fact remains that as soon as the energy difference between two associated symmetry breaking scales becomes to large, the need for fine-tuning parameters at the larger scale reappears. It is unlikely that one will find a grand unified theory not suffering from the hierarchy problem without imposing some symmetry relating the various parameters in the high-energy theory. Such a symmetry could be of any nature, an intriguing possibility is *supersymmetry.* We will not consider supersymmetry any further, but Schmaltz (2003) gives a nice introduction to how supersymmetry can solve the hierarchy problem. Other proposed solutions include a low fundamental quantum gravity scale or theories involving extra dimensions, and a new class of models known as 'Little Higgs' theories. The former two will not be discussed, the latter is the subject of the next chapter.

## 5.3 The Standard Model as an effective field theory

When we view the Standard Model as an effective field theory of some higher-energy theory, a similar problem of naturalness arises. To investigate this, we leave for the moment the complexity of the Standard Model and focus on the by now familiar $\phi^4$ theory. We follow the treatment of Collins (1984).

Consider then a theory of two scalar fields, the light $\phi_l$ and the heavy $\phi_h$. Imposing an extra symmetries $\phi_l \to -\phi_l$ and $\phi_h \to -\phi_h$ for simplicity, the Lagrangian of the full theory is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_l)^2 - \frac{1}{2}m^2\phi_l^2 + \frac{1}{2}(\partial_\mu \phi_h)^2 - \frac{1}{2}M^2\phi_h^2 - \mu^{4-d}\left[\frac{\lambda_1}{4!}\phi_l^4 + \frac{\lambda_2}{2!2!}\phi_l^2\phi_h^2 + \frac{\lambda_3}{4!}\phi_h^4\right] + \text{ counterterms.} \tag{5.4}$$

Note that all terms that can be radiatively generated have been included to ensure renormalizability.

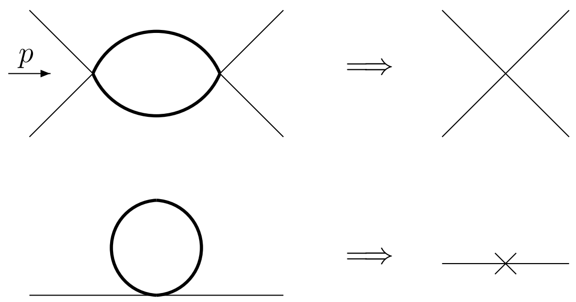


Figure 5.1: One-loop contributions in the high energy theory to the two- and four-point functions of the light scalar field. Thick lines correspond to the heavy field, and thin lines correspond to the light field.

Thus the counterterms have the same form as the terms in the Lagrangian, and can be put in the form

$$\begin{aligned}\mathcal{L}_{\rm ct} &= \frac{1}{2}(Z_l-1)(\partial_\mu\phi_l)^2 - \frac{1}{2}\left[m^2(Z_m-1)+M^2Z_{mM}\right]\phi_l^2\\ &+\frac{1}{2}(Z_h-1)(\partial_\mu\phi_h)^2 - \frac{1}{2}\left[M^2(Z_M-1)+m^2Z_{Mm}\right]\phi_h^2\\ &-\mu^{4-d}\left[\frac{\lambda_{1B}-\lambda_1}{4!}\phi_l^4+\frac{\lambda_{2B}-\lambda_2}{2!2!}\phi_l^2\phi_h^2+\frac{\lambda_{3B}-\lambda_3}{4!}\phi_h^4\right]\end{aligned} \tag{5.5}$$

According to the *decoupling theorem*, at energy scales much less than $M$, the physics is described by an *effective* field theory with Lagrangian of the form

$$\mathcal{L}_{\rm eff} = \frac{1}{2}(\partial_\mu^*)^2 - \frac{1}{2}m^{*2}\phi^{*2} - \mu^{4-d}\frac{\lambda^*}{4!}\phi^{*4} + \text{ counterterms.} \tag{5.6}$$

Here $\phi^* = z^{1/2}\phi_l$. The effects of the heavy particles have been included in the effective theory through changed parameters $z$, $g^*$ and $m^*$ that we will now calculate at the one-loop level. The relevant diagrams contributing to the two- and four-point functions in the high energy theory are shown in Figure 5.1. Assuming that the scale of the incoming momentum $p$ is much smaller than $M$, we can ignore terms proportional to powers of $p/M$ and obtain in dimensional regularization (where as usual $d = 4-2\epsilon$)

$$(a) = \frac{i\lambda_2^2}{256\pi^2}\left[\frac{1}{\epsilon}+\gamma_E+\ln\frac{M^2}{4\pi\mu^2}+\mathcal{O}(p^2/M^2)\right]; \tag{5.7}$$

$$(b) = \frac{i\lambda_2 M^2}{32\pi^2}\left[\frac{1}{\epsilon}+1-\gamma_E+\ln\frac{M^2}{4\pi\mu^2}\right]. \tag{5.8}$$

Diagram $(a)$ corresponds to the effective $\phi_l^4$ interaction. In minimal subtraction, the counterterm absorbs only the pole term of (5.7), so that the effective coupling in the low-energy theory is

$$z^2\lambda^* = \lambda_1 - \frac{\lambda_2^2}{256\pi^2}\left[\gamma_E+\ln\frac{M^2}{4\pi\mu^2}\right]. \tag{5.9}$$

Diagram $(b)$ will contribute to the effective propagator of the low-energy theory. The quadratic terms in the effective Lagrangian (5.6) have (in Fourier space) coefficient $(z+\delta_z)p^2-(m^{*2}z+\delta_m)$. The light field propagator in the full theory contributes a simple $p^2-m^2$. In minimal subtraction, the counterterms absorb exactly the pole terms of (5.8), so that we arrive at

$$zp^2 - zm^{*2} = p^2 - m^2 + \frac{\lambda_2M^2}{32\pi^2}\left[1-\gamma_E+\ln\frac{M^2}{4\pi\mu^2}\right].$$

This immediately gives

$$z = 1; \tag{5.10}$$

$$m^{*2} = m^2 - \frac{\lambda_2M^2}{32\pi^2}\left[1-\gamma_E+\ln\frac{M^2}{4\pi\mu^2}\right]. \tag{5.11}$$

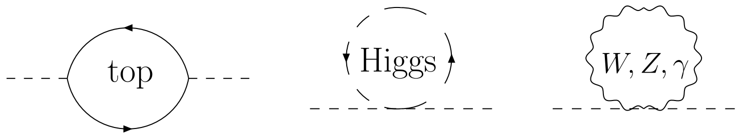



Figure 5.2: The most significant quadratically divergent contributions to the Higgs mass in the Standard Model.

The latter equation exposes the hierarchy problem. To keep the physical mass of the $\phi_l$ field in the effective theory $m^{*2}$ finite as $M \to \infty$, the renormalized mass parameter $m$ of the light field has to be fine-tuned within a relative accuracy of $m^{*2}/M^2$, see (5.11). The renormalized parameters $m$ and $M$ have to finely tuned to make the two terms in this equation cancel. This need for fine-tuning is clearly undesirable.

Essentially, not much changes in this situation when we consider the Standard Model. We will consider the Standard Model as an effective field theory of some high-energy theory, with new very heavy gauge, fermion and/or scalar fields. Ignoring the interactions with gauge and fermion fields, the Higgs boson is simply described by a $\phi^4$ theory. It can fulfill the role of the light field $\phi_l$ in the above derivation. The heavy field can be anything with a mass $M \gg v$, the electroweak symmetry breaking scale. We arrive immediately at (5.11), and conclude that the Standard Model, considered as an effective theory, also suffers from a hierarchy problem. However, in the Standard Model we have also gauge fields and fermions, each of which gives rise to another term proportional to a heavy mass-squared in (5.11). Of course one can now argue that these contributions are opposite in sign and cancel each other, thereby resolving the need of fine-tuning the Higgs mass parameter to its coupling. But, in order for this cancellation to be natural, we need a symmetry relating the various gauge, fermion, and scalar field couplings, a symmetry that is simply absent in the minimal version of the Standard Model. Thus the hierarchy problem remains.

## 5.4 Cut-off arguments and new physics

At high energies, heavy particles from a not yet fully known high-energy theory might well contribute to various processes. Therefore, assume that the Standard Model is valid up to some scale $\Lambda$, say $\Lambda = 10$ TeV. At higher energies we do not know how to compute loop diagrams, thus we will cut such loops off at $\Lambda$. The hierarchy problem arises because the contributions to the Higgs mass are quadratically divergent. The most important contributions are the one-loop diagrams involving the top quark, the $SU(2) \times U(1)$ gauge bosons, and the Higgs boson itself, see Figure 5.2. The contributions are, for $\Lambda = 10$ TeV (Schmaltz, 2003),

$$\text{top loop} \qquad -\frac{3}{8\pi^2}\lambda_t^2\Lambda^2 \quad \sim \; -(2\text{ TeV})^2, \tag{5.12}$$

$$\text{gauge loop} \qquad \frac{1}{16\pi^2}g^2\Lambda^2 \quad \sim \; (0.7\text{ TeV})^2, \tag{5.13}$$

$$\text{Higgs loop} \qquad \frac{1}{16\pi^2}\lambda^2\Lambda^2 \quad \sim \; (0.5\text{ TeV})^2. \tag{5.14}$$

The total Higgs mass at one-loop order is then approximately

$$m_H^2 = m_{\text{tree}}^2 - [100 - 10 - 5](200\text{ GeV})^2. \tag{5.15}$$

If the Higgs mass is two be only a few hundred GeV, a fine tuning of about one part in 100 among the tree-level parameters is required, see Figure 5.3. Thus we see again a manifestation of the hierarchy problem. If we want the Standard Model to be valid up to $\Lambda = 100$ TeV, the fine-tuning required is much greater, about one part in 10000; if on the other hand we expect new physics to take over at around $\Lambda = 1$ TeV, the need for fine-tuning disappears completely.

We now turn the argument around. Suppose we find a fine-tuning of at most 1 part in 10 acceptable. Up to what scale can we expect the Standard Model to be valid? The largest contribution should be no more than about 10 times the Higgs mass, and we find a cut-off $\Lambda \approx 2$ TeV. At this scale we expect to find new particles, whose presence is dictated by some symmetry, that *naturally* cancel the contribution from the top quark. Since the gauge- and Higgs loop contributions are smaller, the new particles cancelling

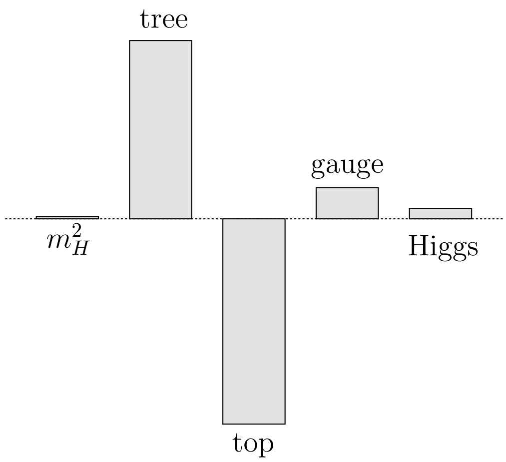



Figure 5.3: The fine-tuning required to obtain an acceptable Higgs mass in the Standard Model with cut-off $\Lambda = 10$ TeV.

them are allowed to be a bit heavier, about 5 TeV for new gauge bosons, and 10 TeV for a new Higgs-like particle. It is important to note that these are maximum masses, arising from the demand that fine-tuning remains at an acceptable level. One can also predict lower bounds, by calculating the effect of new particles on Standard Model processes, and comparing with precision electroweak data.

Recently a new type of models known as 'Little Higgs' models has emerged, which indeed introduce new particles at the TeV scale to *naturally* cancel the quadratically divergent contributions to the Higgs boson mass. The Little Higgs models are the subject of the next chapter.

# Chapter 6

# Little Higgs Theories

The hierarchy problem hints to new TeV scale physics protecting the Higgs from getting large radiative corrections. Classical candidates include supersymmetry, technicolor models, or a low fundamental quantum gravity scale. However in recent years a new type of models has emerged, which realizes the Higgs as a pseudo-Goldstone boson. These *Little Higgs* models are characterized by a set of global and gauge symmetries which are spontaneously broken to the Standard Model gauge group. The Higgs boson is then identified as a particular subset of the Goldstone bosons in the symmetry breaking. The symmetries are explicitly broken by gauge, Yukawa and scalar couplings, but in such a way that the Higgs remains protected from quadratic divergences. We will see exactly how this works in the minimal version of such model, appropriately dubbed "The Littlest Higgs" (Arkani-Hamed et al., 2002a), which is based on the symmetry-breaking $SU(5) \to SO(5)$. Various other models exist, including $SU(6) \to Sp(6)$ (Low et al., 2002), the 'minimal moose' $SU(3)^2 \to SU(3)$ (Arkani-Hamed et al., 2002b), and general mooses $SU(3)^n \to SU(3)^k$ (Gregoire & Wacker, 2002). For more information on those alternative models, the reader is referred to the literature.

## 6.1 The Littlest Higgs

### Symmetry structure

In the Littlest Higgs model, we start from a global symmetry based on a group $G$ which spontaneously breaks to a subgroup $H$, where the symmetry breaking occurs at a scale $f$ of the order of a TeV. Since the model should be an extension of the Standard Model, the unbroken symmetry group $H$ should contain $SU(2) \times U(1)$. Naively, the gauge interactions will induce one-loop quadratically divergent contributions to the Higgs mass, as in the Standard Model. To avoid this, we assume that $G$ contains a gauged subgroup consisting of *two copies* of $SU(2) \times U(1)$: $G \supset G_1 \times G_2 = SU(2)_1 \times U(1)_1 \times SU(2)_2 \times U(1)_2$. The trick is now to arrange this in such a way that each $G_i$ commutes with a different subgroup $X_i$ of $G$, and hence preserves a different global symmetry which is sufficient to forbid a Higgs mass term. Only when both gauge groups come into play, the symmetry is sufficiently broken to allow a mass term for the Higgs boson. We will see that at one-loop level one can therefore only get logarithmically divergent contributions, and quadratic divergences first appear at two-loop level.

In the Littlest Higgs model, we choose $G = SU(5)$ and $H = SO(5)$. The symmetry groups protecting the Higgs mass are chosen to be $X_i = SU(3)_i$. It is readily seen that these choices satisfy the above requirements. Also, each $X_i$ contains an $SU(2) \times U(1)$ subgroup under which some $X_i$ generators transform like doublets. It is important that the doublet generators of the $X_i$ should not lie entirely inside $H$, because they should explicitly break the symmetry protecting the Higgs mass.

### Symmetry breaking

The symmetry breaking $SU(5) \to SO(5)$ arises from a vacuum expectation value in the $\Sigma_0$ direction, where $\Sigma_0$ is the $5 \times 5$ symmetric matrix

$$\Sigma_0 = \begin{pmatrix} & & \mathbf{1}_{2\times 2} \\ & 1 & \\ \mathbf{1}_{2\times 2} & & \end{pmatrix}. \tag{6.1}$$

$\Sigma_0$ indeed induces the desired symmetry breaking. To see this, let $\tilde{\lambda}_a$ denote the standard generators of $SU(5)$, as in chapter 4. These are either symmetric or antisymmetric, where the 10 antisymmetric generators can be identified with the $SO(5)$ subgroup. Also note that $\Sigma_0 = A^2 = A^T A$, where $A$ is the symmetric, unitary matrix

$$A = \frac{1}{2}\begin{pmatrix} 1+i & 0 & 1-i \\ 0 & 2 & 0 \\ 1+i & 0 & 1-i \end{pmatrix}. \tag{6.2}$$

We define new generators by $X_a := A\tilde{\lambda}_a A^{-1}$, which clearly satisfy the $SU(5)$ Lie algebra. Then

$$X_a\Sigma_0 = (A\tilde{\lambda}_a A^{-1})(A^2) = A\tilde{\lambda}_a A = \pm(A\tilde{\lambda}_a A)^T = \pm(X_a\Sigma_0)^T = \pm\Sigma_0 X_a^T, \tag{6.3}$$

where the plus sign corresponds to symmetric $\tilde{\lambda}_a$ and the minus sign to antisymmetric $\tilde{\lambda}_a$. Now, an unbroken symmetry preserves the vacuum: $O\Sigma_0 O^T = \Sigma_0$, where $O = \exp(i\alpha^a X_a)$. This leads to

$$\Sigma_0 = (1 + i\alpha^a X_a)\Sigma_0(1 + i\alpha^a X_a^T) = \Sigma_0 + i\alpha^a(X_a\Sigma_0 + \Sigma_0 X_a^T) + \mathcal{O}(\alpha^2), \tag{6.4}$$

thus the unbroken generators $T_a$ satisfy $T_a\Sigma_0 + \Sigma_0 T_a^T = 0$, which by the above correspond precisely to the $SO(5)$. The 14 symmetric generators satisfy $X_a\Sigma_0 - \Sigma_0 X_a^T = 0$ and are broken.

Not only breaks $\Sigma_0$ the global symmetry group $G = SU(5)$ down to $SO(5)$; it also breaks the gauge group $SU(2)_1 \times U(1)_1 \times SU(2)_2 \times U(1)_2$ to its 'diagonal' subgroup $SU(2) \times U(1)$, which then is identified with the Standard Model electroweak gauge group. Again, this is most easily seen by considering the relevant generators.

The first $SU(2) \times U(1)$ subgroup is embedded in the $SU(5)$ in such a way as to preserve a global $SU(3)$ symmetry in the lower $3 \times 3$ block; the second $SU(2) \times U(1)$ preserves an $SU(3)$ symmetry in the upper $3 \times 3$ block. Thus we define the generators of $G_1 = SU(2)_1 \times U(1)_1$ as

$$Q_1^a = \begin{pmatrix} \frac{\sigma^a}{2} & \\ & \phantom{0} \end{pmatrix}, \qquad Y_1 = \frac{1}{10}\begin{pmatrix} -3 & & & & \\ & -3 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}, \tag{6.5}$$

and the $G_2 = SU(2)_2 \times U(1)_2$ as

$$Q_2^a = \begin{pmatrix} \phantom{0} & \\ & -\frac{\sigma^{a*}}{2} \end{pmatrix}, \qquad Y_2 = \frac{1}{10}\begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}, \tag{6.6}$$

The $SU(2) \times U(1)$ generators

$$Q^a = \frac{1}{\sqrt{2}}(Q_1^a + Q_2^a), \qquad Y = Y_1 + Y_2 \tag{6.7}$$

are unbroken, since

$$\begin{aligned} Q_a\Sigma_0 + \Sigma_0 Q_a^T &= \begin{pmatrix} \sigma_a/2 & & \\ & 0 & \\ & & -\sigma_a^*/2 \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} + \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} \sigma_a^T/2 & & \\ & 0 & \\ & & -\sigma_a^{*T}/2 \end{pmatrix} \\ &= \begin{pmatrix} & & \sigma_a/2 \\ & 0 & \\ -\sigma_a^*/2 & & \end{pmatrix} + \begin{pmatrix} & & -\sigma_a^{*T}/2 \\ & 0 & \\ \sigma_a^T/2 & & \end{pmatrix} = 0, \end{aligned} \tag{6.8}$$

where in the last step we used the hermiticity of the Pauli matrices. Similarly, $Y\Sigma_0 + \Sigma_0 Y^T = 0$.

It is important to note once more that the gauge couplings of $G_1$ leave a global $SU(3)_2$ symmetry in the lower $3\times3$ block. When the gauge couplings of $G_2$ are turned off, the enhanced symmetry forbids the radiative generation of a Higgs mass. Under an $SU(3)_1$ transformation, the Higgs (see next subsection) transforms according to $h_i \to h_i + f\epsilon_i + \mathcal{O}(\epsilon^2)$, and therefore a mass term $hh^\dagger$ is not invariant, and hence forbidden by the symmetry. Similar remarks can be made when the $G_1$ gauge couplings are turned off

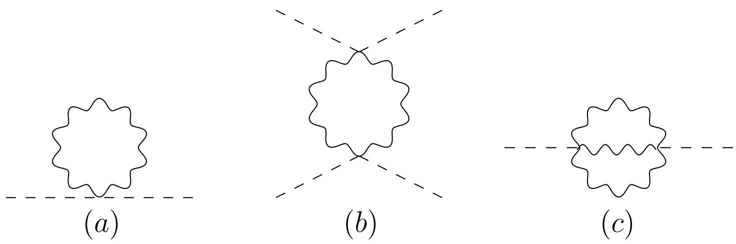



Figure 6.1: Diagrams contributing to the Higgs mass at one- and two-loop. Dashed lines correspond to the $\Sigma$ field, wiggly lines are gauge bosons. Diagram (a) is the quadratically divergent gauge boson loop responsible for the hierarchy problem in the Standard Model, (b) is a logarithmically divergent contribution to the Higgs mass, and (c) is a two-loop quadratically divergent contribution.

and only $G_2$ is active. Only when both gauge couplings come into play, the symmetry is broken enough to allow the appearance of a Higgs mass. *This eliminates any possible one-loop quadratic divergence (due to gauge bosons) from the Higgs mass!*. See Figure 6.1: The one-loop diagram (a) has a single gauge coupling in it, and hence cannot contribute to the Higgs mass. Diagram (b) is also a one-loop diagram, and uses both gauge couplings, but is only logarithmically divergent and therefore not generating large radiative corrections to the Higgs mass at the TeV scale. The two-loop diagram (c) is quadratically divergent, but its value is a loop factor of $1/(4\pi)$ smaller and thus sufficiently suppressed to prevent the hierarchy problem from being reintroduced at the TeV scale.

## Goldstone bosons

In the breaking $SU(5) \to SO(5)$, we get 14 Goldstone bosons corresponding to the 14 broken generators. Under the unbroken $SU(2) \times U(1)$, these transform as

$$\mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{2}_{\pm\frac{1}{2}} \oplus \mathbf{3}_{\pm 1}. \tag{6.9}$$

The first two sets are eaten by the gauge bosons corresponding to the broken $G_1 \times G_2$ generators, thereby giving them a TeV scale mass. The third set is a complex doublet, identified as the Higgs boson, and the last set is an additional complex triplet.

We can conveniently parameterize the Goldstone bosons by the non-linear sigma model field

$$\Sigma(x) = \mathrm{e}^{i\Pi/f}\Sigma_0 \mathrm{e}^{i\Pi^T/f} = \mathrm{e}^{2i\Pi/f}, \tag{6.10}$$

where $f$ is an order TeV decay constant, and $\Pi = \pi^a X_a$. The Goldstone boson matrix $\Pi$ can be written in terms of fields with definite electroweak quantum numbers, using (6.9) and ignoring the Goldstone bosons that are eaten by the heavy gauge bosons corresponding to the broken generators of $[SU(2) \times U(1)]^2$:

$$\Pi = \begin{pmatrix} & \frac{h^\dagger}{\sqrt{2}} & \phi^\dagger \\ \frac{h}{\sqrt{2}} & & \frac{h^*}{\sqrt{2}} \\ \phi & \frac{h^T}{\sqrt{2}} & \end{pmatrix} \tag{6.11}$$

Here $h$ is the Higgs doublet and $\phi$ is a complex $SU(2)$-weak triplet represented as a symmetric $2 \times 2$ matrix,

$$h = (h^+, h^0), \qquad \phi = \begin{pmatrix} \phi^{++} & \frac{\phi^+}{\sqrt{2}} \\ \frac{\phi^+}{\sqrt{2}} & \phi^0 \end{pmatrix}. \tag{6.12}$$

## The $\Sigma$ Lagrangian

The $\Sigma$ field is described by a non-linear $\sigma$-model, whose leading order term is (see e.g. Georgi (1984)):

$$\mathcal{L}_\Sigma = \frac{1}{2}\frac{f^2}{2}\mathrm{Tr}\left|D_\mu \Sigma\right|^2. \tag{6.13}$$

The covariant derivative is

$$D_\mu \Sigma = \partial_\mu \Sigma - i \sum_{j=1}^{2} \left\{ g_j W^a_{\mu j} (Q^a_j \Sigma + \Sigma Q^{aT}_j) + i g'_j B_{\mu j} (Y_j \Sigma + \Sigma Y^T_j) \right\}. \tag{6.14}$$

The $\Sigma$ field acquires a vacuum expectation value, and 4 of the 14 Goldstone bosons arising from the breaking $SU(5) \to SO(5)$ are eaten to give mass to 4 particular linear combinations of the gauge fields. To find these mass eigenstates, consider the terms in $\mathcal{L}_\Sigma$ quadratic in the gauge fields (by substituting $\Sigma = \Sigma_0$, as usual, and suppressing space-time indices for clarity):

$$\begin{aligned} \mathcal{L}_\Sigma(\Sigma = \Sigma_0) = &\frac{1}{2}\frac{f^2}{4}\left[g_1^2 W_1^a W_1^a + g_2^2 W_2^a W_2^a - 2 g_1 g_2 W_1^a W_2^a\right] \\ &+ \frac{1}{2}\frac{f^2}{4}\frac{1}{5}\left[g_1'^2 B_1 B_1 + g_2'^2 B_2 B_2 - 2 g_1' g_2' B_1 B_2\right]. \end{aligned} \tag{6.15}$$

The mass matrices for the $W$ and $B$ fields,

$$\frac{f^2}{4}\begin{pmatrix} g_1^2 & -g_1 g_2 \\ -g_1 g_2 & g_2^2 \end{pmatrix}, \qquad \frac{f^2}{4}\begin{pmatrix} g_1'^2 & -g_1' g_2' \\ -g_1' g_2' & g_2'^2 \end{pmatrix}, \tag{6.16}$$

can easily be diagonalized by the orthogonal transformations

$$\begin{aligned} W &= s W_1 + c W_2, & W' &= -c W_1 + s W_2, \\ B &= s' B_1 + c' B_2, & B' &= -c' B_1 + s' B_2, \end{aligned} \tag{6.17}$$

where the mixing angles are given by

$$\begin{aligned} s &= \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, & c &= \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \\ s' &= \frac{g_2'}{\sqrt{g_1'^2 + g_2'^2}}, & c' &= \frac{g_1'}{\sqrt{g_1'^2 + g_2'^2}}. \end{aligned} \tag{6.18}$$

The mass eigenvalues corresponding to $W'$ and $B'$ are

$$m^2_{W'} = \frac{f^2}{4}\left(g_1^2 + g_2^2\right), \qquad m^2_{B'} = \frac{f^2}{4}\frac{1}{5}\left(g_1'^2 + g_2'^2\right), \tag{6.19}$$

while the orthogonal combinations $W$ and $B$ remain massless. The $W$ and $B$ will be identified with the Standard Model gauge bosons, and will acquire mass from electroweak symmetry breaking. $W'$ and $B'$ are new, heavy, gauge bosons. Their masses are at the TeV scale.

We now reexpress $\mathcal{L}_\Sigma$ in terms of the mass eigenstates $W$, $W'$, $B$ and $B'$, paying particular attention to the couplings of the gauge bosons to two scalars. To this end, we expand $\Sigma$ in powers of $1/f$:

$$\Sigma = \Sigma_0 + \frac{2i}{f}\begin{pmatrix} \phi^\dagger & \frac{h^\dagger}{\sqrt{2}} & 0 \\ \frac{h^*}{\sqrt{2}} & 0 & \frac{h}{\sqrt{2}} \\ 0 & \frac{h^T}{\sqrt{2}} & \phi \end{pmatrix} - \frac{1}{f^2}\begin{pmatrix} h^\dagger h^* & \sqrt{2}\phi^\dagger h^T & h^\dagger h + 2\phi^\dagger\phi \\ \sqrt{2} h \phi^\dagger & 2 h h^\dagger & \sqrt{2} h^* \phi \\ h^T h * + 2\phi\phi^\dagger & \sqrt{2}\phi h^\dagger & h^T h \end{pmatrix} + \mathcal{O}(\frac{1}{f^3}). \tag{6.20}$$

Substituting this expression in $\mathcal{L}_\Sigma$ and using (6.17), we obtain the couplings of the gauge bosons to two scalars (see Han et al. (2003)):

$$\begin{aligned} \mathcal{L}_\Sigma(W \cdot W) = &\frac{g^2}{4}\left[W^a W^b - \frac{c^2 - s^2}{sc} W^a W'^b\right] \mathrm{Tr}\left[h^\dagger h \delta^{ab} + 2\phi^\dagger\phi\delta^{ab} + 2\sigma^a \phi^\dagger \sigma^{bT}\phi\right] \\ &- \frac{g^2}{4}\left[W'^a W'^a \mathrm{Tr}\left[h^\dagger h + 2\phi^\dagger\phi\right] - \frac{c^4 + s^4}{2 s^2 c^2} W'^a W'^b \mathrm{Tr}\left[2\sigma^a\phi^\dagger\sigma^{bT}\phi\right]\right], \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} \mathcal{L}_\Sigma(B \cdot B) = &g'^2\left[B^2 - \frac{c'^2 - s'^2}{s'c'} B B'\right] \mathrm{Tr}\left[\frac{1}{4} h^\dagger h + \phi^\dagger\phi\right] \\ &- g'^2\left[B'^2 \mathrm{Tr}\left[\frac{1}{4} h^\dagger h\right] - \frac{(c'^2 - s'^2)^2}{4 s'^2 c'^2} B'^2 \mathrm{Tr}\left[\phi^\dagger\phi\right]\right], \end{aligned} \tag{6.22}$$

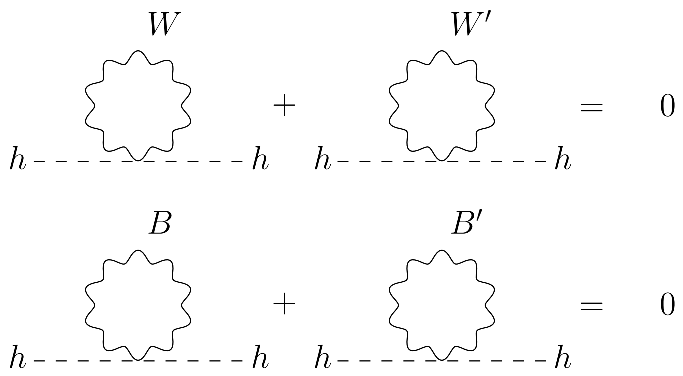



Figure 6.2: Cancellation of the quadratic divergences due to the gauge bosons.

where we have suppressed the vector index $\mu$ on the gauge fields.

Eqs. (6.21) and (6.22) explicitly shows the cancellation of quadratic divergences to the Higgs mass due to gauge bosons. The heavy gauge bosons $W'$ and $W$ have equal but opposite couplings to the $h\dagger h$ term and therefore the quadratic divergence cancels between the two seagull diagrams involving those gauge bosons, see Figure 6.2. Similarly, the quadratic divergences due to $B'$ and $B$ cancel each other. The first uncancelled contribution to the Higgs mass involves both heavy and light gauge bosons. The only diagram of this type is the logarithmically divergent of Figure 6.1(b).

## The Coleman-Weinberg Potential

The Coleman-Weinberg potential (Coleman & Weinberg, 1973) is a potential that is absent at tree level, but generated by quantum effects at one- and higher-loop level. It is such that the symmetry of the Lagrangian is spontaneously broken by the radiative corrections.

In the Littlest Higgs model, the global symmetries of the Littlest Higgs model prevent the appearance of a Higgs potential at tree level. However, since the fermion and gauge interactions explicitly break all the symmetry protecting the appearance of a Higgs potential, yielding a Coleman-Weinberg potential which can be conveniently parameterized as

$$V = \lambda_{\phi^2} f^2 \mathrm{tr}(\phi^\dagger \phi) + i\lambda_{h\phi h} f \left(h\phi^\dagger h^T - h * \phi h^\dagger\right) - \mu^2 h h^\dagger + \lambda_{h^4}(hh^\dagger)^2. \tag{6.23}$$

Here we have neglected terms involving $\phi^4$ and $h^2\phi^2$ since their contribution is small.

The quadratically divergent one-loop contributions to the Coleman-Weinberg potential due to gauge bosons are cut off at a scale $\Lambda \sim 4\pi f$ and is given by

$$\frac{\Lambda^2}{(4\pi)^2} \mathrm{tr} M_V^2(\Sigma), \tag{6.24}$$

where $M^2(\Sigma)$ is the gauge boson mass matrix in a background $\Sigma$. From the covariant derivative (6.14) one finds the potential

$$\mathcal{L}_c = \frac{1}{2} c f^4 \left\{ g_j^2 \sum_a \mathrm{tr}\left[\left(Q_j^a \Sigma\right)\left(Q_j^a \Sigma\right)^*\right] + g_j'^2 \mathrm{tr}\left[\left(Y_j \Sigma\right)\left(Y_j \Sigma\right)^*\right] \right\}. \tag{6.25}$$

Here $c$ is a $\mathcal{O}(1)$ coefficient parameterizing the unknown UV physics at the scale $\Lambda$. Upon expanding the $\Sigma$ field in terms of $h$ and $\phi$, we get a potential

$$\begin{aligned}\mathcal{L}_c =& \frac{1}{2} c(g_1^2 + g_1'^2)\left[f^2 \mathrm{tr}\left(\phi^\dagger\phi\right) - \frac{if}{2}\left(h\phi^\dagger h^T - h^*\phi h^\dagger\right) + \frac{1}{4}\left(hh^\dagger\right)^2 + \cdots\right] \\ &+ \frac{1}{2} c(g_2^2 + g_2'^2)\left[f^2 \mathrm{tr}\left(\phi^\dagger\phi\right) + \frac{if}{2}\left(h\phi^\dagger h^T - h^*\phi h^\dagger\right) + \frac{1}{4}\left(hh^\dagger\right)^2 + \cdots\right].\end{aligned} \tag{6.26}$$

We could also have arrived at this form from symmetry considerations. Under an $SU(3)_1$ transformation, the $h$ and $\phi$ fields transform according to

$$h_i \to h_i + f\epsilon_i + \cdots \tag{6.27}$$

$$\phi_{ij} \to \phi_{ij} - i(\epsilon_i h_j + \epsilon_j h_i) + \cdots, \tag{6.28}$$

while an $SU(3)_2$ transformation acts as

$$h_i \to h_i + f\eta_i + \cdots \tag{6.29}$$

$$\phi_{ij} \to \phi_{ij} + i(\eta_i h_j + \eta_j h_i) + \cdots . \tag{6.30}$$

The invariant quantities are

$$\left|\phi_{ij} \pm \frac{i}{2f}(h_i h_j + h_j h_i)\right|^2 , \tag{6.31}$$

which, upon expansion, yield the terms in square brackets in (6.26).

There is also a quadratically divergent contribution to the Coleman-Weinberg potential generated by fermions loops,

$$\mathcal{L}_{c'} = -\frac{1}{2}c'\lambda_1^2 f^4 \epsilon^{wx}\epsilon_{yz}\epsilon^{ijk}\epsilon_{kmn}\Sigma_{iw}\Sigma_{jx}\Sigma^{*my}\Sigma^{*nz} + \text{h.c.}, \tag{6.32}$$

which is $SU(3)_1$ symmetric and thus must have the the same form as the term proportional to $g_2^2 + g_2'^2$ in (6.26), with coefficient $-\frac{1}{2}c'\lambda_1^2$. As long as $c(g_1^2 + g_1'^2 + g_2^2 + g_2'^2) - c'\lambda_1^2 > 0$, the mass squared for the triplet remains positive. At energies below the triplet mass, we can integrate this particle out. That is, we calculate the equation of motion for $\phi$ and substitute its solution. This leads to a quartic potential for $h$,

$$\lambda\left(hh^\dagger\right)^2, \quad \text{where} \quad \lambda = c\frac{(g_1^2 + g_1'^2 - c'/c\lambda_1^2)(g_2^2 + g_2'^2)}{g_1^2 + g_1'^2 - c'/c\lambda_1^2 + g_2^2 + g_2'^2}. \tag{6.33}$$

Note that turning off the gauge couplings $g_2$ and $g_2'$, so that the $SU(3)_2$ symmetry is back in place, indeed yields $\lambda = 0$. Similarly, turning off the $SU(3)_1$ breaking terms $g_1$, $g_1'$ and $\lambda_1$ yields $\lambda = 0$ and a Higgs potential is indeed not generated.

We will now prove that $\lambda$ has the value given in (6.33). The total potential is

$$\begin{aligned}(\frac{1}{2}c(g_1^2 + g_1'^2) - \frac{1}{2}c'\lambda_1^2)f^2|\phi_{ij} + \frac{i}{2f}(h_i h_j + h_j h_i)|^2 \\ + \frac{1}{2}c(g_2^2 + g_2'^2)f^2|\phi_{ij} - \frac{i}{2f}(h_i h_j + h_j h_i)|^2.\end{aligned} \tag{6.34}$$

To find the equation of motion, we have to find the minimum of the potential by differentiating with respect to $\phi$ and equating to zero. This yields an equation for $\phi$,

$$\left[c(g_1^2 + g_1'^2) - c'\lambda_1^2\right]\left(\phi_{ij} + \frac{i}{f}h_i h_j\right) + c(g_2^2 + g_2'^2)\left(\phi_{ij} - \frac{i}{f}h_i h_j\right) = 0, \tag{6.35}$$

which can be solved for $\phi$. Substituting this solution in (6.34) we get

$$c\frac{(g_1^2 + g_1'^2 - c'/c\lambda_1^2)(g_2^2 + g_2'^2)}{g_1^2 + g_1'^2 - c'/c\lambda_1^2 + g_2^2 + g_2'^2}(hh^\dagger)^2, \tag{6.36}$$

as desired.

Note also that from (6.26) and (6.32), we can express the coefficients $\lambda_{h^4}$, $\lambda_{h\phi h}$ and $\lambda_{\phi^2}$ in terms of $c$, $c'$, $g$, $g'$, $\lambda_1$ and the mixing angles $s$ and $c$. In particular, we find the relation

$$\lambda_{h^4} = \frac{1}{4}\lambda_{\phi^2}. \tag{6.37}$$

## Electroweak symmetry breaking

For $\mu^2 > 0$, the Coleman-Weinberg potential (6.23) triggers electroweak symmetry breaking, resulting in vacuum expectation values for the $h$ and $\phi$ fields,

$$\langle h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & v \end{pmatrix}, \qquad \langle\phi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix}. \tag{6.38}$$

Phenomenological observations require that $v' \ll v$, since a large vev for the $\phi$ field would be in gross violation of the observed physics of the Standard Model. The vacuum expectation values trigger electroweak symmetry breaking, giving thereby masses to the $W$ and $B$ bosons. Also the heavy gauge bosons $W'$ and $B'$ get additional contributions of order $v^2/f^2$, but we will ignore this.

We can find the vacuum by substituting (6.38) in the potential (6.23) and minimizing with respect to $v$ and $v'$. The result is

$$v^2 = \frac{\mu^2}{\lambda_{h^4} - \lambda_{h\phi h}^2/\lambda_{\phi^2}}, \qquad v' = \frac{\lambda_{h\phi h}}{2\lambda_{\phi^2 2}}\frac{v^2}{f}. \tag{6.39}$$

Diagonalizing the Higgs mass matrix, we obtain the Higgs and triplet masses to leading order (Han et al., 2003)

$$m_\Phi^2 \simeq \lambda_{\phi^2} f^2, \qquad m_H^2 \simeq 2\left(\lambda_{h^4} - \lambda_{h\phi h}^2/\lambda_{\phi^2}\right) v^2 = 2\mu^2. \tag{6.40}$$

Here $\Phi$ and $H$ indicate the *mass* eigenstates of the scalars, as opposed to the gauge eigenstates $\phi$ and $h$.

A remark about the Higgs mass parameter $\mu$ is in place. As we argued before, there is no one-loop quadratic divergence to this parameter. However, a there is a logarithmically divergent contribution at the one-loop level, and a quadratically divergent two-loop contribution of order $\Lambda/(4\pi)^4 \sim f^2/(4\pi)^2$, which could be as large as the one-loop contribution. Instead of attempting to evaluate these two-loop contributions, we will simply treat the parameter $\mu$ as a new free parameter of order $f^2/(4\pi)^2$.

Let us now estimate a bound on the scale $f$ to keep the Higgs naturally light. Quantum effects give contributions to the parameter $\mu$ contained in two parameters $a_{1-\text{loop}}$ and $a_{2_\text{loop}}$, so that

$$m_H^2 = 2\mu^2 \simeq a_{1-\text{loop}}\frac{f^2}{(4\pi)^2} + a_{2-\text{loop}}\frac{f^2}{(4\pi)^2}. \tag{6.41}$$

If we define 'naturalness' by requiring that there is no large cancellation and $m_H^2$ is at least 10% of the largest contribution on the right-hand side, we obtain

$$f \leq \frac{4\pi m_H}{\sqrt{0.1 a_\text{max}}} \simeq \frac{8\text{ TeV}}{\sqrt{a_\text{max}}}\left(\frac{m_H}{200\text{ GeV}}\right). \tag{6.42}$$

For $a_\text{max}$ of order 10, and $m_H \simeq 200$ GeV, we get $f \sim 2.5$ TeV.

## Fermions

So far, we have ignored fermions completely. However, the top quark in the Standard Model constitutes a dangerously large quadratically divergent contribution to the Higgs mass. To cancel this contribution, we need to add new fermions in the Littlest Higgs model, to precisely cancel the top loop. We will not be concerned with quadratic divergences due to other quarks; their masses are so much lower than the mass of the top quark, that their contributions at a scale $\Lambda$ of order 1 TeV are quite negligible and do not necessitate fine-tuning.

The newly introduced fermions are a pair of colored Weyl fermions $\tilde{t}$ and $\tilde{t}'^c$ with quantum numbers $(\mathbf{3},\mathbf{1})$ and $(\bar{\mathbf{3}},\mathbf{1})$ under the two global $SU(3)$s. For convenience we form the row vector $\chi = (b_3, t_3, \tilde{t})$. Unlike the fermions in the Standard Model, we can form a bare mass term, which we *choose* of order $f$. The coupling of the Standard Model top quark and the new fermions to the Goldstone boson field $\Sigma$ is

$$\mathcal{L}_t = \frac{1}{2}\lambda_1 f \epsilon_{ijk}\epsilon_{xy}\chi_i\Sigma_{jx}\Sigma_{ky}u_3'^c + \lambda_2 f\tilde{t}\tilde{t}'^c + \text{h.c.}, \tag{6.43}$$

where $i,j,k$ are summed over $1,2,3$ and $x,y$ are summed over $4,5$. $u_3'$ is the right-handed top quark of the Standard Model. It is readily verified that the first term is $SU(3)_1$-invariant but breaks $SU(3)_2$, while the second term does the converse. Hence to generate a contribution to the Higgs mass parameter from the extended top sector, both $\lambda_1$ and $\lambda_2$ need to be turned on. From this it follows that a quadratic divergence cannot be generated at the one-loop level.

Expanding the $\Sigma$ field as usual, we obtain the following couplings of the Higgs doublet to the various fermions:

$$\mathcal{L}_t = \lambda_2 f\tilde{t}\tilde{t}'^c - \frac{\lambda_1}{f}\tilde{t}hh^\dagger u_3'^c + \lambda_1 f\tilde{t}u_3'^c - i\lambda_1\sqrt{2}q_3 h u_3'^c + \text{h.c.} + \cdots, \tag{6.44}$$

where the dots indicate terms involving the heavy scalar $\phi$, and $q_3 = (b_3, t_3)$. Although it may not be immediately obvious, (6.44) explicitly shows how the quadratically divergent top loop gets cancelled by the new heavy fermions, see Figure 6.3.

From the terms in (6.44) one can also see that the combination

$$\tilde{t}^c = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\left(\lambda_2\tilde{t}'c + \lambda_1 u_3'^c\right) \tag{6.45}$$

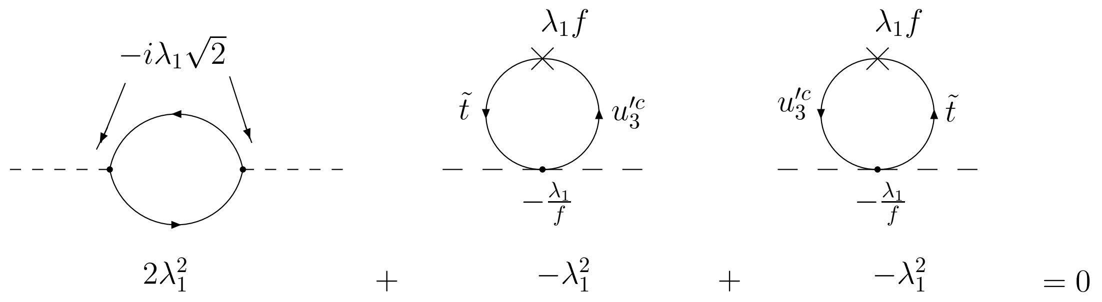



Figure 6.3: Cancellation of the quadratic divergences due to the top quark, from (6.44).

is a mass eigenstate, yielding a mass term for the heavy fermions

$$f\sqrt{\lambda_1^2+\lambda_2^2}\tilde{t}\tilde{t}^c = -m_{\tilde{t}}\tilde{t}\tilde{t}^c. \tag{6.46}$$

Integrating out this heavy quark from the Lagrangian (6.44) yields the desired Standard Model Yukawa coupling

$$\lambda_t q_3 h u_3^c, \quad \text{where} \quad \lambda_t = \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}. \tag{6.47}$$

We see that the Lagrangian (6.43) does exactly the desired things: the quadratic divergence due to the top loop is cancelled, and the Standard Model Yukawa coupling to the quark doublet is produced at energies below the masses of the new heavy fermions.

### UV Completion

The Littlest Higgs model (and other Little Higgs models) are an effective field theory description of a non-linear $\sigma$-model, valid up to a cutoff scale $\Lambda \sim 10$ TeV. There are various possible scenarios for the high-energy behaviour of the theory. One possibility is supersymmetry, broken at a scale $\sim 100$ TeV, which is sufficiently high to avoid conflict with current experimental data.

Another possibility is strong gauge dynamics at the scale $\Lambda$. One can imagine an $SO(N)$ gauge theory with 5 Weyl Fermions $\Psi_i$ in the fundamental representation. At the scale where the $SO(N)$ coupling becomes strong, the fermions will condense into a fermion-fermion condensate $\langle\Psi_i\Psi_j\rangle$ breaking $SU(5) \to SO(5)$. The reader is referred to the literature for more information.

## 6.2 Signatures

Ultimately, only experiment can decide whether Little Higgs theories in general, and the Littlest Higgs theory in particular, have their place in nature. In this section, we briefly consider a few traces one should expect to find in future experiments at particle accelerators such as the Large Hadron Collider (LHC), if the Littlest Higgs is part of a solution to the hierarchy problem.

The new particles in the Littlest Higgs theory are an scalar triplet $\phi$, new gauge bosons $W'$ and $B'$, and new charge-2/3 fermions $\tilde{t}, \tilde{t}^c$. What are the masses of these particles? Of course, lower bounds come from the fact that we these particles have not been produced in current experiments. But also the contributions of these particles to (electroweak) observables, which have been measured to great accuracy, provide lower bounds on these masses, of order a TeV. There are also upper bounds, generally coming from naturalness requirements. The logarithmic contributions to the Higgs mass-squared parameter should not exceed the Higgs mass by more that a factor of, say, 10 ($\sim 10\%$ fine-tuning). This led us to estimate (6.42) for $f$. Assuming that all physics is weakly coupled, and assuming $m_H \approx 200$ GeV , we get upper bounds of about 2 TeV for the new fermions, 6 TeV for the new vector bosons, and 10 TeV for the scalar triplet. These upper bounds are such that the new particles should be accessible for discovery at the LHC.

# Appendix A

# The Standard Model

The Standard Model is the widely accepted quantum field theory of fundamental particles and their interactions. Its basic components are:

- six *quarks*, which go under the names *up*, *down*, *strange*, *charm*, *top*, *bottom*;
- six *leptons*, the *electron*, *muon* and *tau*-lepton, and their corresponding neutrinos;
- interaction particles, including the *gluon*, the *W*- and *Z-bosons* and the *photon*.

## QCD

There is evidence that quarks possess an internal degree of freedom, which can take three values. It is commonly referred to as *color*. To describe the strong interaction one assigns to each quark a fundamental $SU(3)$ triplet and demands invariance under local $SU(3)$ transformations. Doing so gives rise to eight spin-1 gauge fields, the *gluons*. The interactions between quarks and gluons is known as *Quantum Chromodynamics*, or *QCD*. It is a simple unbroken $SU(3)$ gauge theory, and the Lagrangian can be written down immediately:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \sum_q \bar{q}(i\not{D} - m_q)q, \tag{A.1}$$

where

$$D_\mu = \partial_\mu - ig_s G^a_\mu \lambda_a; \tag{A.2}$$

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu. \tag{A.3}$$

Furthermore, the $\lambda_a$ are the Gell-Mann matrices which form a basis for the adjoint representation of $SU(3)$, $f^{abc}$ are the structure constants, and the $G^a_\mu$ are the eight gluon fields.

## The Glashow-Weinberg-Salam model

The Glashow-Weinberg-Salam (GWS) theory unifies the electromagnetic and weak interactions in a spontaneously broken $SU(2) \times U(1)$ gauge theory. In the GWS theory, we distinguish between the left- and right-handed spinors $\psi_L$ and $\psi_R$, defined by

$$\psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi, \qquad \psi_R = \left(\frac{1+\gamma_5}{2}\right)\psi,$$

and assign each of them the quantum numbers *hypercharge* $Y$ and *weak isospin* $I_W$, see Table A.1. The left-handed fermions form doublets of $SU(2)$, and the right-handed fermions are singlets, as in Table A.1.

We start with the Dirac Lagrangian for massless spinor fields,

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi, \tag{A.4}$$

and write this in terms of the above left- and right-handed spinors as

$$\mathcal{L} = \bar{E}^i_L(i\not{\partial})E^i_L + \bar{e}^i_R(i\not{\partial})e^i_R + \bar{Q}^i_L(i\not{\partial})Q^i_L + \bar{u}^i_R(i\not{\partial})u^i_R + \bar{d}^i_R(i\not{\partial})d^i_R, \tag{A.5}$$

Table A.1: Quantum numbers of fermions

| | | | | $I_3$ | $Y$ |
|---|---|---|---|---|---|
| $E^i_L =$ | $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ | $\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$ | $\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$ | $\frac{1}{2}$ | $-1$ |
| $Q^i_L =$ | $\begin{pmatrix} u \\ d \end{pmatrix}_L$ | $\begin{pmatrix} c \\ s \end{pmatrix}_L$ | $\begin{pmatrix} t \\ b \end{pmatrix}_L$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $e^i_R =$ | $e^-_R$ | $\mu^-_R$ | $\tau^-_R$ | 0 | $-2$ |
| $u^i_R =$ | $u_R$ | $c_R$ | $t_R$ | 0 | $\frac{4}{3}$ |
| $d^i_R =$ | $d_R$ | $s_R$ | $b_R$ | 0 | $-\frac{2}{3}$ |

where summation over the generation index $i = 1, 2, 3$ is understood as usual. This Lagrangian is invariant under global $SU(2) \times U(1)$ transformations.

To make the symmetry *local*, we construct the covariant derivative

$$D_\mu = \partial_\mu - igA^a_\mu T^a - i\frac{1}{2}g' B_\mu Y \tag{A.6}$$

(where $T^i$ and $Y$ are generators of the relevant representations of the Lie Algebras of $SU(2)$ and $U(1)$) and make the substitution $\partial_\mu \to D_\mu$ in (A.5).

Now define

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right); \tag{A.7}$$

$$Z^0_\mu = \cos\theta_w A^3_\mu - \sin\theta_w B_\mu; \tag{A.8}$$

$$A_\mu = \sin\theta_w A^3_\mu + \cos\theta_w B_\mu, \tag{A.9}$$

where $\theta_w$, the *weak mixing angle* is defined by

$$g = \frac{e}{\sin\theta_w}; \qquad e = \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{A.10}$$

With these definitions we can rewrite the covariant derivative (A.6) as

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) - \frac{ig}{\cos\theta_w}\left(T^3 - \sin^2\theta_w Q\right) - ieA_\mu Q. \tag{A.11}$$

Here $T^\pm = T^1 \pm iT^2$ and $Q = T^3 + \frac{1}{2}Y$. One sees that $A_\mu$ can be identified with the photon field, and $Q$ with electric charge.

The next step is to introduce a complex scalar $SU(2)$ doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, having quantum numbers $I_W = 1/2$ and $Y = 1$ by

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{A.12}$$

In the case that $\mu^2 < 0$, the lowest energy state is not at $\phi = 0$, but at

$$\phi^\dagger\phi = -\frac{\mu^2}{2\lambda}. \tag{A.13}$$

We can use the $SU(2)$ gauge invariance to write

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v^2 = -\frac{\mu^2}{\lambda}; \tag{A.14}$$

this is called the *unitary* gauge. We say that the symmetry is *spontaneously broken*; $v$ is called the *vacuum expectation value.*

To work out the consequences of this spontaneous symmetry breaking, we expand the kinetic term in the Higgs Lagrangian (A.12), using the definition of the covariant derivative (A.11). Some straightforward but tedious algebra yields

$$(D_\mu\langle\phi\rangle)^\dagger(D_\mu\langle\phi\rangle) = \frac{1}{2}\frac{v^2}{4}\left[g^2\left|W^+_\mu\right|^2 + g^2\left|W^-_\mu\right|^2 + \left(g^2 + g'^2\right)\left|Z^0_\mu\right|^2\right]. \tag{A.15}$$

We see that the gauge fields $W_\mu^\pm$ obtain a mass $M_W = gv/2$, and $Z_\mu^0$ acquires a mass $M_Z = \sqrt{g^2 + g'^2}v/2$, while $A_\mu$ remains massless, as it should be if $A_\mu$ is to be the photon field.

So far, the fermions are massless. We cannot simply add ordinary mass terms such as $\Delta\mathcal{L} = -m_e\left(\bar{e}_L e_R + \bar{e}_R e_L\right)$, since they violate gauge invariance. To give mass to the various fermions, we must again invoke the mechanism of spontaneous symmetry breaking. We add the following (gauge invariant) couplings to the Lagrangian:

$$\Delta\mathcal{L}_e = -\lambda_e^i \bar{E}_{La}^i \phi_a e_R^i + \text{h.c.}, \tag{A.16}$$

where h.c. indicates hermitian conjugate, and one sums over the $SU(2)$ index $a$ and the generation index $i$. If we now replace $\phi$ by its vacuum expectation value (A.14), we obtain an electron mass term with $m_e = \frac{1}{\sqrt{2}}\lambda_e v$.

Similarly, we can obtain quark mass terms by writing down the couplings

$$\Delta\mathcal{L}_q = -\lambda_d^i \bar{Q}_{La}^i \phi_a d_R^i - \lambda_u^i \epsilon^{ab} \bar{Q}_{La}^i \phi_b^\dagger u_R^i + \text{h.c.}, \tag{A.17}$$

giving $m_q = \frac{1}{\sqrt{2}}\lambda_q v$.

Finally, we add kinetic terms for the gauge fields,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a, \tag{A.18}$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$.

Collecting all parts of the Lagrangian, and writing the scalar doublet explicitly in terms of the *Higgs* field $h$,

$$\phi(x) = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{A.19}$$

we can write the total Lagrangian of the GWS theory in unitary gauge ($U(x) = \mathcal{I}$) as

$$\begin{aligned}
\mathcal{L}_{GWS} = &-\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \sum_f \bar{f}\left(i\partial\!\!\!/ + m_f\right)f \\
&+ g\left(W_\mu^+ J_W^{\mu+} + W_\mu^- J_W^{\mu-} + Z_\mu^0 J_Z^\mu\right) + eA_\mu J_{EM}^\mu \\
&+ \frac{1}{2}\left(\partial_\mu h\right)^2 + \left[m_W^2 W^{\mu+}W_\mu^- + \frac{1}{2}m_Z^2 Z^\mu Z_\mu\right]\cdot\left(1 + \frac{h}{v}\right)^2 \\
&- \frac{1}{2}m_h^2 h^2 - \frac{m_h^2}{2v}h^3 - \frac{m_h^2}{8v^2}h^4 - \sum_f \frac{m_f}{v}\bar{f}fh,
\end{aligned} \tag{A.20}$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu; \tag{A.21}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_n u A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c; \tag{A.22}$$

$$J_W^{\mu+} = \frac{1}{\sqrt{2}}\left(\bar{\nu}_L^i \gamma^\mu e_L^i + \bar{u}_L^i \gamma^\mu d_L^i\right); \tag{A.23}$$

$$J_W^{\mu-} = \frac{1}{\sqrt{2}}\left(\bar{e}_L^i \gamma^\mu \nu_L^i + \bar{d}_L^i \gamma^\mu u_L^i\right); \tag{A.24}$$

$$J_Z^\mu = \frac{1}{\cos\theta_w}\sum_f \bar{f}\gamma^\mu\left(T_f^3 - Q_f \sin^2\theta_w\right)f; \tag{A.25}$$

$$J_{EM}^\mu = \sum_f \bar{f}\gamma^\mu Q_f f, \tag{A.26}$$

along with $g = e/\sin\theta_w$, $m_W = gv/2$ and $m_Z = m_W/\cos\theta_w$. It is important that the sums over fermions include left- and right-handed spinors separately.

To actually be able to do calculations, we have to quantize the above theory. This is done most easily using the standard methods of Faddeev and Popov, and thus introducing ghost fields. It is convenient not to work in unitary gauge, but in a more general class of gauge condition, called the $R_\xi$ gauges. These are defined by the Faddeev-Popov gauge-fixing function

$$G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu - \xi e v \phi\right). \tag{A.27}$$

In the limit $\xi \to \infty$, we recover the unitary gauge. Moreover, to explicitly retain the Goldstone bosons in the GWS theory, we rewrite the scalar doublet $\phi$ in terms of four real scalar fields as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(\phi^1 - i\phi^2) \\ v + (h + i\phi^3) \end{pmatrix}. \tag{A.28}$$

The fields $\phi^i$ are the (massless) Goldstone bosons, and $h$ is the (massive) Higgs boson. For a complete set of Feynman rules, see Appendix B of Cheng & Li (1984).

# Appendix B

# Dimensional Regularization

In this appendix we summarize the method of dimensional regularization, some common loop integrals, and properties of the Γ-function.

The idea of dimensional regularization is very simple. Instead of evaluating the momentum integral in $d = 4$ dimensions, we calculate the integral as a function of the dimension $d$, and in the end take the limit $d \to 4$. The original divergences will appear as a pole at $d = 4$ (and at possible other values of $d$). Of course, one must define what is meant by a $d$-dimensional for $d$ non-integral. This is done in depth in Collins (1984).

## Loop integrals

To write the loop integrals in a convenient form, one often introduces so-called *Feynman parameters* to combine denominators:

$$\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \int_0^1 dx_1\dots dx_n\,\delta\left(1-\sum_i x_i\right)\frac{\prod x_i^{\alpha_i-1}}{\left[\sum_i x_iA_i\right]^{\sum_i\alpha_i}}\frac{\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}. \tag{B.1}$$

An important special case is

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{\left[xA+(1-x)B\right]^2}. \tag{B.2}$$

It is now straightforward to perform the $d$-dimensional momentum integral. The denominator will be a quadratic function in the momenta $p_i$. Completing the square by shifting the integration variable $p$ to a new variable $\ell$, we can rewrite the denominator in the form $(\ell^2-\Delta)^n$, where $\Delta$ does not depend on $\ell$. Then for reasons of symmetry, terms with an odd number of powers of $\ell$ in the numerator vanish. Also by symmetry, we can replace

$$\ell^\mu\ell^\nu \quad\to\quad \frac{1}{d}\ell^2 g^{\mu\nu} \tag{B.3}$$

$$\ell^\mu\ell^\nu\ell^\rho\ell^\sigma \quad\to\quad \frac{1}{d(d+2)}(\ell^2)^2\left(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right) \tag{B.4}$$

The most common $d$-dimensional momentum integrals (in Minkowski space) are summarized below:

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{B.5}$$

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{B.6}$$

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^2}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{d(d+2)}{4}\frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{B.7}$$

## The Γ-function

The Γ-function is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx. \tag{B.8}$$

Using integration by parts, one sees that $\Gamma(n) = (n-1)\Gamma(n-1)$. Thus, since $\Gamma(1) = 1$, it follows that for $n$ a positive integer, $\Gamma(n) = n!$. Moreover, using the fact that $\Gamma(n-1) = \Gamma(n)/(n-1)$, one sees that the $\Gamma$-function has simples poles at $n = 0, -1, -2, \ldots$.

Near $x = -n$, one has the approximation

$$\Gamma(x) = \frac{(-1)^n}{n!}\left(\frac{1}{x+n} - \gamma_E + 1 + \ldots + \frac{1}{n} + \mathcal{O}(x+n)\right), \tag{B.9}$$

where $\gamma_E$ is the Euler-Mascheroni constant, $\gamma_E \approx 0.5772$. Near $x = 0$ this takes the form

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(x). \tag{B.10}$$

# Appendix C

# Some Facts About Lie Groups

For the physicist, a *Lie group* $G$ is simply a continuously generated group. Every group element $g$ can be written

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2). \tag{C.1}$$

Here the coefficients $\alpha^a$ specify a parameterization of $g$ in terms of the *generators* $T^a$, which are linear operators on the tangent space to $G$. The generators completely determine the local structure of the Lie group. They satisfy the commutation relations

$$\left[T^a, T^b\right] = if^{abc}T^c, \tag{C.2}$$

where $f^{abc}$ are the *structure constants*. The generators are normally conveniently chosen such that the structure constants are completely antisymmetric. The $f^{abc}$ satisfy the *Jacobi identity*,

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \tag{C.3}$$

The vector space spanned by the $T^a$, along with the operation of commutation, is the *Lie algebra* of $G$, denoted $\mathfrak{g}$ . In the following, we will assume that the number of generators is finite. Such Lie algebras are called *compact*.

A subset $\mathfrak{l}$ of generators is an *invariant subalgebra* if for any $X \in \mathfrak{l}$ and any $Y \in \mathfrak{g}$, $[Y, X] \in \mathfrak{l}$. The whole algebra and 0 are trivial invariant subalgebras. If a Lie algebra has no nontrivial invariant subalgebras, it is called *simple*. If the elements of $\mathfrak{l}$ commute, $\mathfrak{l}$ is called *Abelian*. A Lie algebra that contains no Abelian invariant subalgebras is called *semi-simple*. Note that a simple algebra is also semi-simple, but the converse is not true.

## Representations

Given the (Lie) group $G$, a $d$-dimensional (unitary) representation of the corresponding Lie algebra is a set of $d \times d$ Hermitian matrices $t^a$ satisfying the commutation relations (C.2). It is called *irreducible* if it cannot be written as a direct sum of other representations. We denote the representation matrices of an irreducible representation $r$ by $t^a_r$.

If the Lie algebra is semi-simple, the $t^a_r$ are traceless. However, the trace of the product of two generators is not zero, and we can choose the generators $T^a$ of the Lie algebra such that it is proportional to the identity matrix for every representation:

$$\text{tr}\left[t^a_r t^b_r\right] = C(r)\delta^{ab}. \tag{C.4}$$

The operator $T^2 = T^a T^a$ is constant on irreducible representations:

$$t^a_r t^a_r = C_2(r) \cdot \mathbb{I}_{d(r)\times d(r)}, \tag{C.5}$$

where $d(r)$ is the dimension of the representation. $C_2(r)$ is called the *quadratic Casimir operator*. Applying (C.5) to the adjoint representation $A$, we find

$$f^{acd}f^{bcd} = C_2(A)\delta^{ab}. \tag{C.6}$$

## Classification of simple, compact Lie algebras

Surprisingly, there are not many different simple, compact Lie algebras. There are four infinite families,

- $A_n = \mathfrak{su}(n+1)$, the algebra of the special unitary group of rank $n$;
- $B_n = \mathfrak{so}(2n+1)$, the algebra of the group of orthogonal $(2n+1) \times (2n+1)$ matrices with unit determinant;
- $C_n = \mathfrak{sp}(n+1)$, the algebra of the symplectic group of rank $n$;
- $D_n = \mathfrak{so}(2n)$, the algebra of the group of orthogonal $2n \times 2n$ matrices with unit determinant.

Furthermore, there are five *exceptional* algebras, $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$. The subscript denotes the *rank* of the algebra, that is, the maximum number of commuting generators.

The *Classification Theorem* of Wilhelm Killing and Elie Cartan states that the above list is exhaustive. The theorem is considered to be one of the big triumphs of nineteenth-century mathematics. For more information on this intriguing subject, consult any textbook on Lie groups. A good start is *Lie Algebras in Particle Physics* by Georgi (1999).

## Special Unitary groups

The for our purposes most important Lie groups are $SU(2)$, $SU(3)$ and $SU(5)$. $SU(N)$ groups have an $N$-dimensional representation, the *fundamental* representation, and of course an adjoint representation, denoted by $G$. We have

$$C(N) = \frac{1}{2}, \qquad C_2(N) = \frac{N^2-1}{2N}, \qquad C(G) = C_2(G) = N. \tag{C.7}$$

The $SU(N)$ groups have many more irreducible representations, but their dimensions depend on the dimension of the group. They can be found by the method of *Young tableaux.* One represents the fundamental representation by a box, and constructs other representations by building diagrams from these boxes according to certain rules. The dimension of the representation can be found from a counting rule. For more details, see e.g. Greiner & Müller (1989).

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