DIRAC AND MAJORANA FERMIONS

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References

As most directly related books to these notes, I refer to the book of Srednicki [1] and Ryder [2]. Other text books of Quantum Field Theory that are useful are given in refs [3-6]. The notes are part of the lecture notes for the course on Quantum Field Theory (next full course scheduled November 2012 through January 2013).

Chapter 1

Relativistic wave equations

1.1 The Klein-Gordon equation

We look at the Klein-Gordon (KG) equation to describe a free spinless particle with mass $M$ with a 'field' $\phi$,

$$\left(\partial_{\mu}\partial^{\mu} + M^2\right)\phi(r, t) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + M^2\right)\phi(r, t) = 0.$$  \hspace{1cm} (1.1)

Although it is straightforward to find the solutions of this equation, namely plane waves characterized by a wave number $k$,

$$\phi_k(r, t) = \exp(-ik^0t + ik \cdot r),$$  \hspace{1cm} (1.2)

with $(k^0)^2 = k^2 + M^2$, the interpretation of this equation as a single-particle equation in which $\phi$ is a complex wave function poses problems because the energy spectrum is not bounded from below and the probability is not positive definite.

The energy spectrum is not bounded from below: considering the above stationary plane wave solutions, one obtains

$$k^0 = \pm \sqrt{k^2 + M^2} = \pm E_k,$$  \hspace{1cm} (1.3)

i.e. there are solutions with negative energy.

Probability is not positive: in quantum mechanics one has the probability and probability current

$$\rho = \psi^*\psi$$  \hspace{1cm} (1.4)

$$j = -\frac{i}{2M} (\psi^*\nabla\psi - (\nabla\psi^*)\psi) \equiv -\frac{i}{2M} \psi^* \nabla \psi.$$  \hspace{1cm} (1.5)

They satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j,$$  \hspace{1cm} (1.6)

which follows directly from the Schrödinger equation. This continuity equation can be written down covariantly using the components $(\rho, j)$ of the four-current $j$,

$$\partial_{\mu}j^{\mu} = 0.$$  \hspace{1cm} (1.7)

Therefore, relativistically the density is not a scalar quantity, but rather the zero component of a four vector. The appropriate current corresponding to the KG equation (see Exercise 2.2) is

$$j^{\mu} = i \phi^* \partial^{\mu} \phi \quad \text{or} \quad (\rho, j) = \left(i \phi^* \partial_0 \phi, -i \phi^* \nabla \phi\right).$$  \hspace{1cm} (1.8)
It is easy to see that this current is conserved if $\phi$ (and $\phi^*$) satisfy the KG equation. The KG equation, however, is a second order equation and $\phi$ and $\partial \phi / \partial t$ can be fixed arbitrarily at a given time. This leads to the existence of negative densities.

These problems are related and have to do with the existence of particles and antiparticles, for which we need the interpretation of $\phi$ itself as an operator, rather than as a wave function. This operator has all possible solutions in it multiplied with creation (and annihilation) operators. At that point the dependence on position $\mathbf{r}$ and time $t$ is just a dependence on numbers/parameters on which the operator depends, just as the dependence on time was in ordinary quantum mechanics. Then, there are no longer fundamental objections to mix up space and time, which is what Lorentz transformations do. And, it is simply a matter of being careful to find a consistent (covariant) theory.

1.2 Mode expansion of solutions of the KG equation

Before quantizing fields, having the KG equation as a space-time symmetric (classical) equation, we want the most general solution. For this we note that an arbitrary solution for the field $\phi$ can always be written as a superposition of plane waves,

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3} \frac{2\pi}{2E_k} e^{-i k \cdot x} \tilde{\phi}(k)$$

with (in principle complex) coefficients $\tilde{\phi}(k)$. The integration over $k$-modes clearly is covariant and restricted to the ‘mass’-shell (as required by Eq. 1.1). It is possible to rewrite it as an integration over positive energies only but this gives two terms (use the result of exercise 2.3),

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{E_k}{2E_k} \left( e^{-i k \cdot x} a(E_k, k) + e^{i k \cdot x} b^*(E_k, -k) \right).$$

Introducing $\tilde{\phi}(E_k, k) \equiv a(k)$ and $\tilde{\phi}(-E_k, -k) \equiv b^*(k)$ one has

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{E_k}{2E_k} \left( e^{-i k \cdot x} a(k) + e^{i k \cdot x} b^*(k) \right) = \phi_+(x) + \phi_-(x).$$

In Eqs 1.10 and 1.11 one has eliminated $k^0$ and in both equations $k \cdot x = E_k t - k \cdot \mathbf{x}$. The coefficients $a(k)$ and $b^*(k)$ are the amplitudes of the two independent solutions (two, after restricting the energies to be positive). They are referred to as mode and anti-mode amplitudes (or because of their origin positive and negative energy modes). The choice of $a$ and $b^*$ allows an easier distinction between the cases that $\phi$ is real ($a = b$) or complex ($a$ and $b$ are independent amplitudes).

1.3 Symmetries of the Klein-Gordon equation

We will explicitly discuss the example of a discrete symmetry, for which we consider space inversion, i.e. changing the sign of the spatial coordinates, which implies

$$(x^\mu) = (t, \mathbf{x}) \rightarrow (t, -\mathbf{x}) \equiv (\tilde{x}^\mu).$$

Transforming everywhere in the KG equation $x \rightarrow \tilde{x}$ one obtains

$$\left( \tilde{\partial}_\mu \tilde{\partial}^\mu + M^2 \right) \phi(\tilde{x}) = 0.$$

Since $a \cdot b = \tilde{a} \cdot \tilde{b}$, it is easy to see that

$$\left( \tilde{\partial}_\mu \tilde{\partial}^\mu + M^2 \right) \phi(\tilde{x}) = 0,$$
implying that for each solution $\phi(x)$ there exists a corresponding solution with the same energy, $\phi^P(x) \equiv \phi(\tilde{x})$ (P for parity). It is easy to show that

$$\phi^P(x) = \phi(\tilde{x}) = \int \frac{d^3k}{(2\pi)^3 2E_k} (e^{-ik \cdot \tilde{x}} a(k) + e^{ik \cdot \tilde{x}} b^*(k))$$

or since one can define

$$\phi^P(x) \equiv \int \frac{d^3k}{(2\pi)^3 2E_k} (e^{-ik \cdot \tilde{x}} a(k) + e^{ik \cdot \tilde{x}} b^*(k)),$$

one has for the mode amplitudes $a^P(k) = a(-k)$ and $b^P(k) = b(-k)$. This shows how parity transforms $k$-modes into $-k$ modes.

Another symmetry is found by complex conjugating the KG equation. It is trivial to see that

$$(\partial_\mu \partial^\mu + M^2) \phi^*(x) = 0,$$

showing that with each solution there is a corresponding charge conjugated solution $\phi^C(x) = \phi^*(x)$. In terms of modes one has

$$\phi^C(x) = \phi^*(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} (e^{-ik \cdot x} a^P(k) + e^{ik \cdot x} b^{P*}(k))$$

i.e. for the mode amplitudes $a^C(k) = b(k)$ and $b^C(k) = a(k)$. For the real field one has $a^C(k) = a(k)$. This shows how charge conjugation transforms 'particle' modes into 'antiparticle' modes and vice versa.
Chapter 2

The Poincaré Group

2.1 The Lorentz group

Spin has been introduced as a representation of the rotation group $SU(2)$ without worrying much about the rest of the symmetries of the world. We considered the generators and looked for representations in finite dimensional spaces, e.g. $\sigma/2$ in a two-dimensional (spin 1/2) case. In this section we consider the Poincaré group, consisting of the Lorentz group and translations.

The Lorentz transformations are divided into rotations and boosts. Rotations around the z-axis are given by $\Lambda_R(\varphi, \hat{z}) = \exp(i \varphi J^3)$, infinitesimally given by $\Lambda_R(\varphi, \hat{z}) \approx I + i \varphi J^3$. Thus

$$\begin{pmatrix}
V^0 \\
V^1 \\
V^2 \\
V^3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \varphi & \sin \varphi & 0 \\
0 & -\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
V^0 \\
V^1 \\
V^2 \\
V^3
\end{pmatrix} \quad \longrightarrow \quad J^3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (2.1)$$

Boosts along the z-direction are given by $\Lambda_B(\eta, \hat{z}) = \exp(-i \eta K^3)$, infinitesimally given by $\Lambda_B(\eta, \hat{z}) \approx I - i \eta K^3$. Thus

$$\begin{pmatrix}
V^0 \\
V^1 \\
V^2 \\
V^3
\end{pmatrix} = \begin{pmatrix}
\cosh \eta & 0 & 0 & \sinh \eta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \eta & 0 & 0 & \cosh \eta
\end{pmatrix} \begin{pmatrix}
V^0 \\
V^1 \\
V^2 \\
V^3
\end{pmatrix} \quad \longrightarrow \quad K^3 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (2.2)$$

The parameter $\eta$ runs from $-\infty < \eta < \infty$. Note that the velocity $\beta = v = v/c$ and the Lorentz contraction factor $\gamma = (1 - \beta^2)^{-1/2}$ corresponding to the boost are related to $\eta$ as $\gamma = \cosh \eta$, $\beta \gamma = \sinh \eta$. Using these explicit transformations, we have found the generators of rotations, $J = (J^1, J^2, J^3)$, and those of the boosts, $K = (K^1, K^2, K^3)$, which satisfy the commutation relations (check!)

$$[J^i, J^j] = i \epsilon^{ijk} J^k,$$

$$[J^i, K^j] = i \epsilon^{ijk} K^k,$$

$$[K^i, K^j] = -i \epsilon^{ijk} J^k.$$

The first two sets of commutation relations exhibit the rotational behavior of $J$ and $K$ as vectors in $E(3)$ under rotations. From the commutation relations one sees that the boosts (pure Lorentz transformations) do not form a group, since the generators $K$ do not form a closed algebra. The commutator of two boosts in different directions (e.g. the difference of first performing a boost in the y-direction and thereafter in the x-direction and the boosts in reversed order) contains a rotation (in the example around the z-axis). This is the origin of the Thomas precession.
2.2 The generators of the Poincaré group

For the full Poincaré group, including the translations, writing the generator \( P = (H/c, \mathbf{P}) \) in terms of the Hamiltonian and the three-momentum operators, one obtains

\[
[P_i, P_j] = [P_i, H] = [J^i, H] = 0,
[J^i, J^j] = i \epsilon^{ijk} J^k,
[J^i, P_j] = i \epsilon^{ijk} P^k,
[J^i, K_j] = i \epsilon^{ijk} K^k,
[K^i, H] = i P^i,
[K^i, K^j] = -i \epsilon^{ijk} J^k/c^2,
[K^i, P_j] = i \delta^{ij} H/c^2.
\]

We have here reinstated \( c \), because one then sees that by letting \( c \to \infty \) the commutation relations of the Galilei group, known from non-relativistic quantum mechanics are obtained. In that case boosts and rotations decouple!

Exercises

Exercise 3.7 (optional)

One might wonder if it is actually possible to write down a set of operators that generate the Poincaré transformations, consistent with the (canonical) commutation relations of a quantum theory. This is possible for a single particle. Do this by showing that the set of operators,

\[
H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4},
\]

\[
P = \mathbf{p},
\]

\[
J = \mathbf{r} \times \mathbf{p} + \mathbf{s},
\]

\[
K = \frac{1}{2c^2} \left( c \mathbf{r} H + H \mathbf{r} \right) - \frac{\mathbf{p} \times \mathbf{s}}{H + mc^2}
\]

satisfy the commutation relations of the Poincaré group if the position, momentum and spin operators satisfy the canonical commutation relations, \([r^i, p^j] = i \delta^{ij}\) and \([s^i, s^j] = i \epsilon^{ijk} s^k\); the others vanish, \([r^i, r^j] = [p^i, p^j] = [r^i, s^j] = [p^i, s^j] = 0\).

Hint: for the Hamiltonian, show first the operator identity \([\mathbf{r}, f(\mathbf{p})] = i \nabla_p f(\mathbf{p})\); if you don’t want to do this in general, you might just check relations involving \( J \) or \( K \) by taking some (relevant) explicit components.

Comment: extending this to more particles is a highly non-trivial procedure, but it can be done, although the presence of an interaction term \( V(\mathbf{r}_1, \mathbf{r}_2) \) inevitably leads to interaction terms in the boost operators. These do not matter in the non-relativistic limit \( (c \to \infty) \), that’s why many-particle non-relativistic quantum mechanics is ‘easy’.
Chapter 3

The Dirac equation

3.1 The Lorentz group and \( SL(2, \mathbb{C}) \)

Instead of the generators \( J \) and \( K \) of the homogeneous Lorentz transformations we can use the (hermitean) combinations

\[
A = \frac{1}{2} (J + i K), \quad (3.1)
\]
\[
B = \frac{1}{2} (J - i K), \quad (3.2)
\]

which satisfy the commutation relations

\[
[A^i, A^j] = i \epsilon^{ijk} A^k, \quad (3.3)
\]
\[
[B^i, B^j] = i \epsilon^{ijk} B^k, \quad (3.4)
\]
\[
[A^i, B^j] = 0. \quad (3.5)
\]

This shows that the Lie algebra of the Lorentz group is identical to that of \( SU(2) \otimes SU(2) \). This tells us how to find the representations of the group. They will be labeled by two angular momenta corresponding to the A and B groups, respectively, \((j, j')\). Special cases are the following representations:

- **Type I** (denoted \( M \)):
  \[
  J = \frac{\sigma}{2}, \quad K = -i \frac{\sigma}{2}, \quad (3.9)
  \]

- **Type II** (denoted \( \overline{M} \)):
  \[
  J = \frac{\sigma}{2}, \quad K = +i \frac{\sigma}{2}, \quad (3.10)
  \]

From the considerations above, it also follows directly that the Lorentz group is homeomorphic with the group \( SL(2, \mathbb{C}) \), similarly as the homeomorphism between \( SO(3) \) and \( SU(2) \). The group \( SL(2, \mathbb{C}) \) is the group of complex \( 2 \times 2 \) matrices with determinant one. It is simply connected and forms the covering group of \( L_+ \). The matrices in \( SL(2, \mathbb{C}) \) can be written as a product of a unitary matrix \( U \) and a hermitean matrix \( H \),

\[
M = \exp \left( \frac{i}{2} \varphi \cdot \sigma \right) \exp \left( \pm \frac{1}{2} \eta \cdot \sigma \right) = \begin{pmatrix} U(\varphi) H(\eta) \\ U(\varphi) \overline{H}(\eta) \end{pmatrix}, \quad (3.8)
\]

with \( \eta = \eta \hat{n} \) and \( \varphi = \varphi \hat{n} \), where we restrict (for fixed \( \hat{n} \)) the parameters \( 0 \leq \varphi \leq 2\pi \) and \( 0 \leq \eta < \infty \). With this choice of parameter-spaces the plus and minus signs are actually relevant. They precisely correspond to the two types of representations that we have seen before, becoming the defining representations of \( SL(2, \mathbb{C}) \):

- **Type I** (denoted \( M \)):
  \[
  J = \frac{\sigma}{2}, \quad K = -i \frac{\sigma}{2}, \quad (3.9)
  \]

- **Type II** (denoted \( \overline{M} \)):
  \[
  J = \frac{\sigma}{2}, \quad K = +i \frac{\sigma}{2}, \quad (3.10)
  \]

6
Let us investigate the defining (two-dimensional) representations of $SL(2, \mathbb{C})$. One defines spinors $\xi$ and $\eta$ transforming similarly under unitary rotations ($U^\dagger = U^{-1}$, $\overline{U} \equiv (U^\dagger)^{-1} = U$)

$$\xi \rightarrow U\xi, \quad \eta \rightarrow U\eta,$$

$$U(\varphi) = \exp(i\varphi \cdot \sigma/2)$$

but differently under hermitean boosts ($H^\dagger = H$, $\overline{H} \equiv (H^\dagger)^{-1} = H^{-1}$), namely

$$\xi \rightarrow H\xi, \quad \eta \rightarrow \overline{H}\eta,$$

$$H(\eta) = \exp(\eta \cdot \sigma/2),$$

$$\overline{H}(\eta) = \exp(-\eta \cdot \sigma/2).$$

Considering $\xi$ and $\eta$ as spin states in the rest-frame, one can use a boost to the frame with momentum $p$. Choosing the boost parameters such that $E = M\gamma = M \cosh(\eta)$ and $p = M\beta\hat{n} = M \sinh(\eta)\hat{n}$, the boost is given by

$$H(p) = \exp\left(\frac{\eta \cdot \sigma}{2}\right) = \cosh\left(\frac{\eta}{2}\right) + \sigma \cdot \hat{n} \sinh\left(\frac{\eta}{2}\right) = \frac{M + E + \sigma \cdot p}{\sqrt{2M(E + M)}}$$

(exercise 4.2). Also useful is the relation $H^2(p) = \tilde{\sigma}^\mu p_\mu/M = (E + \sigma \cdot p)/M$, where we used the sets of four operators defined by

$$\sigma^\mu \equiv (1, \sigma), \quad \tilde{\sigma}^\mu \equiv (1, -\sigma),$$

satisfying $\text{Tr}(\sigma^\mu \tilde{\sigma}^\nu) = -2g^{\mu\nu}$ and $\text{Tr}(\sigma^\mu \sigma^\nu) = 2g^{\mu\nu} = 2\delta^{\mu\nu}$ (the matrices, thus, are not covariant!).

### 3.2 Spin 1/2 representations of the Lorentz group

Both representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ of $SL(2, \mathbb{C})$ are suitable for representing spin 1/2 particles. The representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, furthermore, are inequivalent, i.e. they cannot be connected by a unitary transformation. Within the Lorentz group, they can be connected, but by a transformation belonging to the class $P^\dagger_1$. Under parity one has

$$(0, \frac{1}{2}) \longrightarrow (\frac{1}{2}, 0).$$

In nature parity turns (often) out to be a good quantum number for elementary particle states. For the spin 1/2 representations of the Poincaré group including parity we, therefore, must combine the representations, i.e. consider the four component spinor that transforms under a Lorentz transformation as

$$u = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} M(\Lambda) & 0 \\ 0 & \overline{M}(\Lambda) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

where $\overline{M}(\Lambda) = \epsilon M^*(\Lambda)\epsilon^{-1}$. For a particle at rest only angular momentum is important and we can choose $\xi(0, m) = \eta(0, m) = \chi_m$, the well-known two-component spinor for a spin 1/2 particle. Taking $M(\Lambda) = H(p)$, the boost in Eq. 3.13, we obtain for the two components of $u$ which we will refer to as chiral right and chiral left components,

$$u(p, m) = \begin{pmatrix} u_R \\ u_L \end{pmatrix} = \begin{pmatrix} H(p) & 0 \\ 0 & \overline{H}(p) \end{pmatrix} \begin{pmatrix} \chi_m \\ \chi_m \end{pmatrix},$$

with

$$H(p) = \frac{E + M + \sigma \cdot p}{\sqrt{2M(E + M)}},$$

$$\overline{H}(p) = \frac{E - M - \sigma \cdot p}{\sqrt{2M(E + M)}} = H^{-1}(p).$$
The Dirac equation

It is straightforward to eliminate \( \chi_m \) and obtain the following constraint on the components of \( u \),

\[
\begin{pmatrix}
0 & H^2(p) \\
H^{-2}(p) & 0
\end{pmatrix}
\begin{pmatrix}
u_R \\
u_L
\end{pmatrix} = \begin{pmatrix}
u_R \\
u_L
\end{pmatrix},
\]

(3.20)

or explicitly in the so-called Weyl representation

\[
\begin{pmatrix}
-M & E + \sigma \cdot p \\
E - \sigma \cdot p & -M
\end{pmatrix}
\begin{pmatrix}
u_R \\
u_L
\end{pmatrix} = 0,
\]

(3.21)

which is an explicit realization of the (momentum space) Dirac equation, which in general is a linear equation in \( p^\mu \),

\[
(p_\mu \gamma^\mu - M) u(p) = (\not{p} - M) u(p) = 0,
\]

(3.22)

where \( \gamma^\mu \) are \( 4 \times 4 \) matrices called the Dirac matrices\(^1\).

As in section 2 we can use \( P_\mu = i \partial_\mu \) as a representation for the momenta (translation operators) in function space. This leads to the Dirac equation for \( \psi(x) = u(p)e^{-i p \cdot x} \) in coordinate space,

\[
(i \gamma^\mu \partial_\mu - M) \psi(x) = 0,
\]

(3.23)

which is a covariant (linear) first order differential equation. It is of a form that we also played with in Exercise 2.7. The general requirements for the \( \gamma \) matrices are thus easily obtained. Applying \( (i \gamma^\mu \partial_\mu + M) \) from the left gives

\[
\left( \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + M^2 \right) \psi(x) = 0.
\]

(3.24)

Since \( \partial_\mu \partial_\nu \) is symmetric, this can be rewritten

\[
\left( \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + M^2 \right) \psi(x) = 0.
\]

(3.25)

To achieve also that the energy-momentum relation \( p^2 = M^2 \) is satisfied for \( u(p) \), one must require that for \( \psi(x) \) the Klein-Gordon relation \( (\Box + M^2) \psi(x) = 0 \) is valid for each component separately. From this one obtains the Clifford algebra for the Dirac matrices,

\[
\{ \gamma^\mu, \gamma^\nu \} = 2 \sigma^{\mu \nu},
\]

(3.26)

suppressing on the RHS the identity matrix in Dirac space. The explicit realization appearing in Eq. 3.21 is known as the Weyl representation. We will discuss another explicit realization of this algebra in the next section.

### 3.3 General representations of \( \gamma \) matrices and Dirac spinors

The general algebra for the Dirac matrices is

\[
\{ \gamma^\mu, \gamma^\nu \} = 2 \sigma^{\mu \nu}.
\]

(3.27)

Two often used explicit representations are the following\(^2\): The standard representation:

\[
\gamma^0 = \rho^3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = i \rho^2 \otimes \sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix};
\]

(3.28)

---

\(^1\)We define for a four vector \( a \) the contraction \( \not{a} = a^\mu \gamma^\mu \)

\(^2\)We use \( \rho^i \otimes \sigma^j \) with both \( \rho \) and \( \sigma \) being the standard \( 2 \times 2 \) Pauli matrices.
The Weyl (or chiral) representation:

$$\gamma^0 = \rho^1 \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = -i \rho^2 \otimes \sigma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}. \quad (3.29)$$

Different representations can be related to each other by unitary transformations,

$$\gamma_\mu \rightarrow S \gamma_\mu S^{-1}, \quad (3.30)$$

$$\psi \rightarrow S \psi. \quad (3.31)$$

We note that the explicit matrix taking us from the Weyl representation to the standard representation,

$$(\gamma_\mu)_{\text{S.R.}} = S R \gamma^\mu S^{-1},$$

is

$$(\gamma_\mu)_{\text{W.R.}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (3.32)$$

For all representations one has

$$\gamma^\dagger_\mu = \gamma^0 \gamma_\mu \gamma^0,$$ 

and an adjoint spinor defined by

$$\bar{\psi} = \psi^\dagger \gamma^0. \quad (3.34)$$

Another matrix which is often used is $\gamma_5$ defined as

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \frac{i}{4} \epsilon_{\mu \nu \rho \sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma. \quad (3.35)$$

It satisfies $\{\gamma_5, \gamma^\mu\} = 0$ and explicitly one has

$$(\gamma_5)_{\text{S.R.}} = \rho^1 \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_5)_{\text{W.R.}} = \rho^3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.36)$$

For instance in the Weyl representation (but valid generally), it is easy to see that

$$P_{R/L} = \frac{1}{2} (1 \pm \gamma_5) \quad (3.37)$$

are projection operators, that project out chiral right/left states, which in the case of the Weyl representation are just the upper and lower components.

**Lorentz invariance**

The Lorentz transformations can also be written in terms of Dirac matrices. For example, the rotation and boost generators in Weyl representation in Eqs 3.9 and 3.10 are represented by matrices $S^{\mu \sigma}$,

$$J^3 = S^{12} = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \frac{i}{4} [\gamma^1, \gamma^2] = \frac{i}{2} \gamma^1 \gamma^2,$$

$$K^3 = S^{30} = \frac{1}{2} \begin{pmatrix} 0 & -i \sigma^3 \\ -i \sigma^3 & 0 \end{pmatrix} = \frac{i}{4} [\gamma^3, \gamma^0] = \frac{i}{2} \gamma^3 \gamma^0,$$

and in general one has the transformation

$$\psi \rightarrow L \psi = \exp \left( -\frac{i}{2} \omega_{\rho \sigma} S^{\rho \sigma} \right) \psi, \quad (3.38)$$

with $S^{\rho \sigma} = \frac{i}{2} \sigma^{\rho \sigma} = \frac{i}{4} [\gamma^\rho, \gamma^\sigma]$. We note that $\psi \rightarrow L \psi$ and $\bar{\psi} \rightarrow \bar{\psi} L^{-1}$, while $L^{-1} \gamma^\mu L = \Lambda^\mu_\nu \gamma^\nu$. The latter assures Lorentz invariance of the Dirac equation (see Exercise 4.4)
The Dirac equation

Parity

There are a number of symmetries in the Dirac equation, e.g. parity. It is easy to convince oneself that if $\psi(x)$ is a solution of the Dirac equation,

$$ (i\not{\partial} - M) \psi(x) = 0, \quad \text{(3.39)} $$

one can apply space inversion, $x = (t, \mathbf{x}) \rightarrow \tilde{x} = (t, -\mathbf{x})$ and via a few manipulations obtain again the Dirac equation

$$ (i\not{\partial} - M) \psi^p(x) = 0, \quad \text{(3.40)} $$

but with $\psi^p(x) \equiv \eta_p \gamma_0 \psi(\tilde{x})$ (Exercise 4.7). Note that we have (as expected) explicitly in Weyl representation in Dirac space

$$ \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \overrightarrow{\psi}^p = \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad \text{(3.41)} $$

Charge conjugation

The existence of positive and negative energy solutions implies another symmetry in the Dirac equation. This symmetry does not change the spin 1/2 nature, but it does, for instance, reverse the charge of the particle. As with parity we look for a transformation, called charge conjugation, that brings $\psi \rightarrow \psi^c$, which is again a solution of the Dirac equation. Starting with $(i\not{\partial} - M) \psi(x) = 0$ we note that by hermitean conjugating and transposing the Dirac equation one obtains

$$ (i\gamma^\mu \partial_\mu + M) \overrightarrow{\psi^c}(x) = 0. \quad \text{(3.42)} $$

In any representation a matrix $C$ exist, such that

$$ C\gamma^\mu C^{-1} = -\gamma_\mu, \quad \text{(3.43)} $$
e.g.

$$ (C)_{S.R.} = -i\gamma^2 \gamma^0 = i\rho^1 \otimes \sigma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \quad \text{(3.44)} $$

$$ (C)_{W.R.} = -i\gamma^2 \gamma^0 = i\rho^3 \otimes \sigma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}. \quad \text{(3.45)} $$

Thus we find back the Dirac equation,

$$ (i\not{\partial} - M) \psi^c(x) = 0. \quad \text{(3.46)} $$

with the solution

$$ \psi^c(x) = \eta_c C\overrightarrow{\psi}(x) = \eta_c C\gamma^0 \psi^*(x) \quad \text{(3.47)} $$

where $\eta_c$ is an arbitrary (unobservable) phase, usually to be taken unity. Note that the last step (relating $\psi^c$ and $\psi^*$ is valid in representations where $\gamma^0$ is real. Some properties of $C$ are $C^{-1} = C^\dagger$ and $C^T = -C$. One has $\overrightarrow{\psi^c} = -\overrightarrow{\psi^c} C^{-1}$. In S.R. (or W.R.) and all representations connected via a real (up to an overall phase) matrix $S$, $C$ is real and one has $C^{-1} = C^\dagger = C^T = -C$ and $[C, \gamma_5] = 0$. The latter implies that the conjugate of a right-handed spinor, $\psi_R^c$, is a left-handed spinor. Explicitly, in Weyl representation we find in Dirac space

$$ \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \overrightarrow{\psi}^c = C\overrightarrow{\psi^*} = \begin{pmatrix} \epsilon \eta^* \\ -\epsilon \xi^* \end{pmatrix}. \quad \text{(3.48)} $$
3.4 Plane wave solutions

For a free massive particle, the best representation to describe particles at rest is the standard representation, in which $\gamma^0$ is diagonal (see discussion of negative energy states in section 4.1). The explicit Dirac equation in the standard representation reads

$$\begin{pmatrix} i \frac{\partial}{\partial t} - M & i \sigma \cdot \nabla \\ -i \sigma \cdot \nabla & -i \frac{\partial}{\partial t} - M \end{pmatrix} \psi(x) = 0. \quad (3.49)$$

Looking for positive energy solutions $\propto e^{-iEt}$ (one finds two solutions), $\psi(x) = u^{\pm s}(p) e^{-i p \cdot x}$, with $E = E_p = +\sqrt{p^2 + M^2}$, where $u$ satisfies

$$\begin{pmatrix} E_p - M & -\sigma \cdot p \\ \sigma \cdot p & -(E_p + M) \end{pmatrix} u(p) = 0, \quad \iff \quad (\dot{p} - M) u(p) = 0. \quad (3.50)$$

There are also two negative energy solutions, $\psi(x) = v^{\pm s}(p) e^{i p \cdot x}$, where $v$ satisfies

$$\begin{pmatrix} -(E_p + M) & \sigma \cdot p \\ -\sigma \cdot p & (E_p - M) \end{pmatrix} v(p) = 0, \quad \iff \quad (\dot{p} + M) v(p) = 0. \quad (3.51)$$

Explicit solutions in the standard representation are

$$u(p, s) = \sqrt{E_p + M} \begin{pmatrix} \chi_s \\ \sigma \cdot p \frac{1}{E_p + M} \chi_s \end{pmatrix}, \quad v(p, s) = \sqrt{E_p + M} \begin{pmatrix} \sigma \cdot p \frac{1}{E_p + M} \chi_s \\ \bar{\chi}_s \end{pmatrix}, \quad (3.52)$$

where $\chi_s$ are two independent ($s = \pm$) two-component spinors. Note that the spinors in the negative energy modes (antiparticles) could be two different spinors. Choosing $\chi = \chi^*$ (the equivalent spin 1/2 conjugate representation), the spinors satisfy $C \bar{u}^T(p, s) = v(p, s)$ and $C \bar{v}^T(p, s) = u(p, s)$. The solutions are normalized to

$$\bar{u}(p, s) u(p, s') = 2M \delta_{ss'}, \quad \bar{v}(p, s) v(p, s') = -2M \delta_{ss'}, \quad (3.53)$$

$$\bar{u}(p, s) v(p, s') = \bar{v}(p, s) u(p, s') = 0, \quad (3.54)$$

$$u^T(p, s) u(p, s') = v^T(p, s) v(p, s') = 2E_p \delta_{ss'}. \quad (3.55)$$

An arbitrary spin 1/2 field can be expanded in the independent solutions. After separating positive and negative energy solutions as done in the case of the scalar field one has

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 2E_k} (u(k, s) e^{-i k \cdot x} b(k, s) + v(k, s) e^{i k \cdot x} d^* (k, s)). \quad (3.56)$$

It is straightforward to find projection operators for the positive and negative energy states

$$P_+ = \sum_s u(p, s) \bar{u}(p, s) \frac{\dot{p} + M}{2M}, \quad (3.57)$$

$$P_- = -\sum_s v(p, s) \bar{v}(p, s) \frac{-\dot{p} + M}{2M}. \quad (3.58)$$

In order to project out spin states, the spin polarization vector in the rest frame is the starting point. In that frame is a spacelike unit vector $s^\mu = (0, \hat{s})$. In an arbitrary frame one has $s \cdot p = 0$ and $s(p)$ can e.g. be obtained by a Lorentz transformation. It is easy to check that

$$P_s = \frac{1 \pm \gamma_5 \hat{s}}{2} = \frac{1}{2} \begin{pmatrix} 1 \pm \sigma \cdot \hat{s} & 0 \\ 0 & 1 \mp \sigma \cdot \hat{s} \end{pmatrix}. \quad (3.59)$$
(the last equality in the restframe and in standard representation) projects out spin \( \pm 1/2 \) states (check this in the restframe for \( \hat{s} = \hat{z} \)).

Note that for solutions of the massless Dirac equation \( \hat{p} \psi = 0 \). Therefore, \( \gamma_5 \hat{p} \psi = 0 \) but also \( \hat{\gamma}_5 \hat{p} \psi = 0 \). This means that in the solution space for massless fermions the chirality states, \( \psi_{R/L} \equiv P_{R/L} \psi \) are independent solutions. In principle massless fermions can be described by two-component spinors. The chirality projection operators in Eq. 3.37 replace the spin projection operators which are not defined (by lack of a rest frame).

Explicit examples of spinors are useful to illustrate spin eigenstates, helicity states, chirality, etc. For instance with the \( z \)-axis as spin quantization axis, one has in standard representation:

\[
\begin{align*}
\psi_R &= \frac{1}{\sqrt{E + M}} \begin{pmatrix} E + M \\ 0 \\ p^1 + i p^2 \\ -p^3 \\ \end{pmatrix}, \\
\psi_L &= \frac{1}{\sqrt{E + M}} \begin{pmatrix} E + M \\ 0 \\ -p^3 \\ p^1 + i p^2 \\ \end{pmatrix},
\end{align*}
\]

(3.60)

Helicity states (\( \hat{p} \) along \( \hat{z} \)) in Standard representation are:

\[
\begin{align*}
u(p, \lambda = +\frac{1}{2}) &= \begin{pmatrix} \sqrt{E + M} \\ 0 \\ \sqrt{E - M} \\ 0 \end{pmatrix}, & u(p, \lambda = -\frac{1}{2}) &= \begin{pmatrix} 0 \\ \sqrt{E + M} \\ 0 \\ -\sqrt{E - M} \end{pmatrix}, \\
v(p, \lambda = +\frac{1}{2}) &= \begin{pmatrix} 0 \\ \sqrt{E - M} \\ 0 \\ \sqrt{E + M} \end{pmatrix}, & v(p, \lambda = -\frac{1}{2}) &= \begin{pmatrix} 0 \\ 0 \\ \sqrt{E + M} \\ 0 \end{pmatrix}.
\end{align*}
\]

(3.62)

By writing the helicity states in Weyl representation it is easy to project out righthanded (upper components) and lefthanded (lower components). One finds for the helicity states \( u(p, \lambda) \) and \( v(p, \lambda) \) in Weyl representation:

\[
\begin{align*}
u(p, +) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E + M} + \sqrt{E - M} \\ 0 \\ \sqrt{E + M} - \sqrt{E - M} \\ 0 \end{pmatrix}, & u(p, -) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{E + M} - \sqrt{E - M} \\ 0 \\ \sqrt{E + M} + \sqrt{E - M} \end{pmatrix}, \\
v(p, +) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{E + M} - \sqrt{E - M} \\ \sqrt{E + M} + \sqrt{E - M} \\ 0 \end{pmatrix}, & v(p, -) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{E + M} + \sqrt{E - M} \\ 0 \end{pmatrix}.
\end{align*}
\]

(3.63)

Note that for helicity states \( C\bar{u}^T(p, \lambda) = -v(p, \lambda) \) and \( C\bar{v}^T(p, \lambda) = u(p, \lambda) \). Introducing the Weyl helicity spinors \( U_{R/L}(p, \lambda) \),

\[
\begin{align*}
U_R(p, +) &= \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, & U_R(p, -) &= \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & U_L(p, +) &= \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & U_L(p, -) &= \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]
we get
\[ u(p, \lambda = +1/2) = \sqrt{1 - \delta^2} \ U_R(p, +) + \delta \ U_L(p, +), \]
\[ u(p, \lambda = -1/2) = \sqrt{1 - \delta^2} \ U_L(p, -) + \delta \ U_R(p, -), \]
\[ v(p, \lambda = +1/2) = -\sqrt{1 - \delta^2} \ U_L(p, -) + \delta \ U_R(p, -), \]
\[ v(p, \lambda = -1/2) = \sqrt{1 - \delta^2} \ U_R(p, +) - \delta \ U_L(p, +), \]
where
\[ \delta = \frac{E + M - \sqrt{E^2 - M^2}}{2\sqrt{E} M} = \frac{M}{\sqrt{E(M + \sqrt{E^2 - M^2})}} \approx \frac{M}{2E}, \]
which vanishes in the ultra-relativistic limit \( E \gg M \) or in the massless case. At high energy positive helicity fermions (\( \lambda = +1/2 \)) are in essence righthanded, while the negative helicity fermions (\( \lambda = -1/2 \)) are mostly lefthanded. For a massless fermion right- and left-handed solutions coincide with helicity states.

\[ u(p, \lambda = +\frac{1}{2}) = v(p, \lambda = -\frac{1}{2}) = U_R(p, +) \quad \text{and} \quad u(p, \lambda = -\frac{1}{2}) = -v(p, \lambda = +\frac{1}{2}) = U_L(p, -). \]

**Exercise**

In this chapter we have used two representations (Standard and Weyl) for the gamma matrices, based on \( \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu
u} \) and anticommuting with \( \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \), thus \( \{\gamma_5, \gamma^\mu\} = 0 \). The hermitean conjugate matrices obey \( \gamma_\mu = \gamma^0\gamma_\mu\gamma^0 \) and the transposed matrices are found using \( C \) defined as \( C\gamma_\mu^T C^{-1} = -\gamma_\mu \). We have

**Standard Representation**

\[
\begin{align*}
\gamma^0 &= \rho^3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\gamma^i &= i\rho^2 \otimes \sigma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \\
\gamma_5 &= \rho^3 \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
C &= -i\gamma^0\gamma_5 = i\rho^3 \otimes \sigma^2 = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \\
C\gamma_0 &= \rho^2 \otimes \sigma^2 = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \\
K^j &= \frac{1}{2}\gamma^3\gamma^0 = -\frac{1}{2}\rho^1 \sigma^j = \frac{1}{2} \left( \begin{array}{c} 0 \\ i\sigma^j \end{array} \right) \\
K^j &= \frac{1}{2}\gamma^3\gamma^0 = -\frac{1}{2}\rho^3 \sigma^j = \frac{1}{2} \left( \begin{array}{c} -i\sigma^j \\ 0 \end{array} \right)
\end{align*}
\]

**Weyl Representation**

\[
\begin{align*}
\gamma^0 &= \rho^1 \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma^i &= -i\rho^2 \otimes \sigma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix} \\
\gamma_5 &= \rho^3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
C &= -i\gamma^0\gamma_5 = i\rho^3 \otimes \sigma^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \\
C\gamma_0 &= -\rho^2 \otimes \sigma^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \\
K^j &= \frac{1}{2}\gamma^3\gamma^0 = \frac{1}{2}\rho^1 \sigma^j = \frac{1}{2} \left( \begin{array}{c} 0 \\ -i\sigma^j \end{array} \right) \\
K^j &= \frac{1}{2}\gamma^3\gamma^0 = \frac{1}{2}\rho^3 \sigma^j = \frac{1}{2} \left( \begin{array}{c} i\sigma^j \\ 0 \end{array} \right)
\end{align*}
\]

Construct these same matrices in \( 1 + 1 \) dimension and discuss the implications for parity, chirality, particle-antiparticle, . . .

**solution**

The general relations are similar except for \( \gamma_5 \), which now is given by \( \gamma_5 = \gamma^0\gamma^1 \). Explicitly one has 2-dimensional matrices given by
The Dirac equation

Excercise

(a) Construct for both representations the explicit boost operators in 1+1 dimension and give the explicit spinors starting with Eq. 3.60.

(b) The full boost operator in Dirac space can be written as \( \exp(\phi \cdot \sigma) = \frac{M + E + \sigma \cdot p}{\sqrt{2M(E + M)}} \),

where \( E = M \gamma = M \cosh(\phi) \) and \( p = M \beta \gamma \hat{n} = M \sinh(\phi) \hat{n} \). For this you need to express \( \cosh(\eta/2) \) and \( \sinh(\eta/2) \) in terms of energy and momentum. Another simple check that you can perform is that the RHS indeed also is equal to

\[ H^2(p) = \exp(\phi \cdot \sigma) = (E + \sigma \cdot p)/M = \hat{\sigma}^\mu p_\mu/M. \]

(b) The full boost operator in Dirac space can be written as \( \exp(i\eta \cdot K) \). With the explicit matrices in Exercise 4.1 one e.g. immediately reproduces the result in Weyl representation (Eq. 3.17),

\[
\exp(i\eta \cdot K) = \frac{1}{\sqrt{2M(E + M)}} \begin{pmatrix} E + M + \sigma \cdot p & 0 \\ 0 & E + M - \sigma \cdot p \end{pmatrix}
\]

Give the full boost operator in Standard representation. Check that the explicit boost operators applied to a rest-frame spinor immediately give the explicit spinors starting with Eq. 3.60.

(solution)

\[
\exp(i\eta \cdot K) = \frac{1}{\sqrt{2M(E + M)}} \begin{pmatrix} E + M & \sigma \cdot p \\ \sigma \cdot p & E + M \end{pmatrix}
\]

(b) Construct for both representations the explicit boost operators in 1+1 dimension and give the explicit spinors \( u(\pm |p|) \) and \( v(\pm |p|) \) with \( |p| = +\sqrt{E^2 - M^2} \), discussing their nature (such as parity, chirality, particle-antiparticle).

(solution)

The boost operator in Standard Representation

\[
\exp(i\eta K) = \frac{1}{\sqrt{2M(E + M)}} \begin{pmatrix} E + M & p \\ p & E + M \end{pmatrix}
\]
The Dirac equation

and in Weyl Representation

\[ \exp(i\eta K) = \frac{1}{\sqrt{2M(E + M)}} \begin{pmatrix} E + M + p & 0 \\ 0 & E + M - p \end{pmatrix} \]

giving Standard Representation spinors

\[ u(\pm |p|) = \begin{pmatrix} \sqrt{E + M} \\ \pm \sqrt{E - M} \end{pmatrix} \]

\[ v(\pm |p|) = \begin{pmatrix} \pm \sqrt{E - M} \\ \sqrt{E + M} \end{pmatrix} \]

and Weyl Representation spinors

\[ u(\pm |p|) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E + M} \pm \sqrt{E - M} \\ \sqrt{E + M} \mp \sqrt{E - M} \end{pmatrix} \]

\[ v(\pm |p|) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \sqrt{E - M} \mp \sqrt{E + M} \\ \pm \sqrt{E - M} \mp \sqrt{E + M} \end{pmatrix} \]

The Weyl Representation results show that in 1+1 dimension for \( M = 0 \), right-handed fermions are right-movers and left-handed fermions are left-movers (rather than specific helicity states in 3+1 dimensions).

**Excercise**

Apply space-inversion, \( x \rightarrow \tilde{x} \), to the Dirac equation and use this to show that the spinor \( \psi^P(x) = \gamma_0 \psi(\tilde{x}) \), where \( \tilde{x} = (t, -x) \) is also a solution of the Dirac equation.
Chapter 4

Classical lagrangian field theory

4.1 Euler-Lagrange equations

In classical field theory one proceeds in complete analogy to classical mechanics but using functions depending on space and time (classical fields, think for instance of a temperature or density distribution or of an electromagnetic field). Consider a lagrangian density \( L \) which depends on these functions, their derivatives and possibly on the position, \( L(\phi(x), \partial_\mu \phi(x), x) \) and an action

\[
S[\phi] = \int_{t_1}^{t_2} dt \ L(\phi(x), \partial_\mu \phi(x)) = \int_{R} d^4x \ L(\phi, \partial_\mu \phi, x). \tag{4.1}
\]

Here \( R \) indicates a space-time volume bounded by \( (R^3, t_1) \) and \( (R^3, t_2) \), also indicated by \( \partial R \) (a more general volume in four-dimensional space-time with some boundary \( \partial R \) can also be considered). Variations in the action can come from the coordinates or the fields, indicated as

\[
x'^\mu = x'^\mu + \delta x'^\mu, \tag{4.2}
\]

\[
\phi'(x) = \phi(x) + \delta \phi(x) \tag{4.3}
\]

or combined

\[
\phi'(x') = \phi(x) + \Delta \phi(x), \tag{4.4}
\]

with \( \Delta \phi(x) = \delta \phi(x) + (\partial_\mu \phi) \delta x'^\mu \). The resulting variation of the action is

\[
\delta S = \int_{R} d^4x' \ L(\phi', \partial_\mu \phi', x') - \int_{R} d^4x \ L(\phi, \partial_\mu \phi, x). \tag{4.5}
\]

The change in variables \( x \to x' \) in the integration volume involves a surface variation of the form

\[
\int_{\partial R} d\sigma_\mu \ L \delta x'^\mu. \tag{4.6}
\]

Note for the specific choice of the surface for constant times \( t_1 \) and \( t_2 \),

\[
\int_{\partial R} d\sigma_\mu \ ... = \int_{(R^3, t_1)} d^3x \ ... = \int_{(R^3, t_2)} d^3x \ ... \tag{4.6}
\]

Furthermore the variations \( \delta \phi \) and \( \delta \partial_\mu \phi \) contribute to \( \delta S \), giving

\[
\delta S = \int_{R} d^4x \left[ \frac{\delta L}{\delta (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\delta L}{\delta \phi} \delta \phi \right] + \int_{\partial R} d\sigma_\mu \ L \delta x'^\mu
\]

\[
= \int_{R} d^4x \left[ \frac{\delta L}{\delta \phi} - \partial_\mu \left( \frac{\delta L}{\delta (\partial_\mu \phi)} \right) \right] \delta \phi + \int_{\partial R} d\sigma_\mu \left[ \frac{\delta L}{\delta (\partial_\mu \phi)} \delta \phi + \mathcal{L} \delta x'^\mu \right]. \tag{4.7}
\]

\(^1\)Taking a functional derivative, indicated with \( \delta F[\phi]/\delta \phi \) should pose no problems. We will come back to it in a bit more formal way in section 9.2.
With for the situation of classical fields all variations of the fields and coordinates at the surface vanishing, the second term is irrelevant. The integrand of the first term must vanish, leading to the Euler-Lagrange equations,

\[ \partial_\mu \left( \frac{\delta L}{\delta (\partial_\mu \phi)} \right) - \frac{\delta L}{\delta \phi} = 0. \]  

(4.8)

### 4.2 Lagrangians for spin 0 and 1/2 fields

By an appropriate choice of lagrangian density the equations of motion discussed in previous chapters for the scalar field (spin 0), the Dirac field (spin 1/2) and the vector field (spin 1) can be found.

**The scalar field**

It is straightforward to derive the equations of motion for a real scalar field \( \phi \) from the lagrangian densities,

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 \]  

(4.9)

\[ = -\frac{1}{2} \phi \left( \partial_\mu \partial^\mu + M^2 \right) \phi, \]  

(4.10)

which differ only by surface terms, leading to

\[ (\Box + M^2)\phi(x) = 0. \]  

(4.11)

For the complex scalar field one conventionally uses

\[ L = \partial_\mu \phi^* \partial^\mu \phi - M^2 \phi^* \phi \]  

(4.12)

\[ = -\phi^* \left( \partial_\mu \partial^\mu + M^2 \right) \phi, \]  

(4.13)

which can be considered as the sum of the lagrangian densities for two real scalar fields \( \phi_1 \) and \( \phi_2 \) with \( \phi = (\phi_1 + i\phi_2)/\sqrt{2} \). One easily obtains

\[ (\Box + M^2)\phi(x) = 0, \]  

(4.14)

\[ (\Box + M^2)\phi^*(x) = 0. \]  

(4.15)

**The Dirac field**

The appropriate lagrangian from which to derive the equations of motion is

\[ \mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - M \bar{\psi} \psi = \frac{i}{2} \bar{\psi} \not\partial \psi - \frac{i}{2} \bar{\psi} \not\partial \psi - M \bar{\psi} \psi \]  

(4.16)

\[ = \overline{\psi} \left( i \not\partial - M \right) \psi, \]  

(4.17)

where the second line is not symmetric but in the action only differs from the symmetric version by a surface term (partial integration). Using the variations in \( \bar{\psi} \) (in the symmetric form),

\[ \frac{\delta \mathcal{L}}{\delta (\partial_\mu \bar{\psi})} = -\frac{i}{2} \gamma^\mu \bar{\psi} \]  

\[ \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = \frac{i}{2} \not\partial \psi - M \psi, \]  

one obtains immediately

\[ \left( i \not\partial - M \right) \psi = 0, \]  

(4.18)
and similarly from the variation with respect to \( \psi \)

\[
\bar{\psi} \left( i \not\! \partial + M \right) \psi = 0. \tag{4.19}
\]

It is often useful to link to the two-component spinors \( \xi \) and \( \eta \) which we started with in chapter 4, or equivalently separate the field into right- and lefthanded ones. In that case one finds trivially

\[
\mathcal{L} = \frac{1}{2} \bar{\psi}_R i \not\! \partial \psi_R + \frac{1}{2} \bar{\psi}_L i \not\! \partial \psi_L - M (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R), \tag{4.20}
\]

showing e.g. that the lagrangian separates into two independent parts for \( M = 0 \). Using the two-spinors \( \xi \) and \( \eta \), the mass term in the Dirac lagrangian 4.20 is given by

\[
\mathcal{L}_M (\text{Dirac}) = - M (\xi^\dagger \eta + \eta^\dagger \xi). \tag{4.21}
\]

There exists another possibility to write down a mass term with only one kind of fields, namely

\[
\mathcal{L}_M (\text{Majorana}) = \frac{1}{2} (M \eta^\dagger \eta^* - M^* \eta^T \eta). \tag{4.22}
\]

These two terms are each others conjugate\(^2\).

At the level of the equations of motion one has, using the Pauli matrices \( \sigma^\mu \) and \( \tilde{\sigma}^\mu \) in Eq. 3.14, for a Dirac fermion

\[
i(\sigma \cdot \partial) \xi = M_D \eta \quad \text{and} \quad i(\tilde{\sigma} \cdot \partial) \eta = M_D \xi. \tag{4.23}
\]

With the mass term in Eq. 4.22 this becomes for Majorana fermions

\[
i(\sigma \cdot \partial) \xi = M_\xi \epsilon \xi^* \quad \text{or} \quad i(\tilde{\sigma} \cdot \partial) \epsilon \xi^* = - M_\xi^* \xi, \quad \text{and} \quad i(\tilde{\sigma} \cdot \partial) \eta = M_\eta^* \eta^*, \tag{4.24}
\]

in which the mass can be complex. Squaring gives \((\partial^2 - |M_\xi|^2) \xi = (\partial^2 - |M_\eta|^2) \eta = 0\).

It is also possible to introduce a ‘real’ (four-component) spinor satisfying \( \Upsilon^c = \Upsilon \) of which the left part coincides with \( \psi_L \),

\[
\Upsilon_L = \psi_L = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \Rightarrow \quad \Upsilon = \begin{pmatrix} \epsilon \eta^* \\ \eta \end{pmatrix}, \tag{4.25}
\]

for which \( \eta_0 = \epsilon \eta_0^* \). We note that

\[
\psi_L^c \equiv (\psi_L)^c = C \bar{\psi}_L^T = \begin{pmatrix} \epsilon \eta^* \\ 0 \end{pmatrix} = \Upsilon_R. \tag{4.26}
\]

Since the kinetic term in \( \mathcal{L} \) separates naturally in left and right parts (or \( \xi \) and \( \eta \)), it is in the absence of a Dirac mass term possible to introduce a lagrangian in which only left fields \( \psi_L \) and \( \psi_L^c \) appear or one can work with ‘real’ spinors or Majorana spinors,

\[
\mathcal{L} = \frac{1}{2} \bar{\psi}_L i \not\! \partial \psi_L - \frac{1}{2} (M \bar{\psi}_L \psi_L + M^* \bar{\psi}_L \psi_L^c) \tag{4.27}
\]

\[
= \frac{1}{4} \Upsilon_L i \not\! \partial \Upsilon - \frac{1}{2} (M \Upsilon_R \Upsilon_L + M^* \Upsilon_L \Upsilon_R). \tag{4.28}
\]

This gives an expression with a mass term that is actually of the same form as the Dirac mass term, but note the factor \( 1/2 \) as compared to the Dirac lagrangian, which comes because we in essence use ‘real’ spinors. The Majorana case is in fact more general, since a lagrangian with both Dirac and Majorana mass terms can be rewritten as the sum of two Majorana lagrangians after redefining the fields (See e.g. Peshkin and Schroeder or Exercise 12.7).

\(^2\)This term is written down with \( \eta \) being an anticommuting Grassmann number for which

\[
\alpha \beta = - \beta \alpha, \quad (\alpha \beta)^* = \beta^* \alpha^* = - \alpha^* \beta^*,
\]

and thus \((\beta^* \alpha)^* = \alpha^* \beta = - \beta \alpha^* \). The reasons for Grassmann variables will become clear in the next chapter.
Chapter 5

Quantization of fields

5.1 The real scalar field

We have expanded the (classical) field in plane wave solutions, which we have split into positive and negative energy pieces with (complex) coefficients \(a(k)\) and \(a^*(k)\) multiplying them. The quantization of the field is achieved by quantizing the coefficients in the Fourier expansion, e.g. the real scalar field \(\phi(x)\) becomes

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[ a(k) e^{-i k \cdot x} + a^\dagger(k) e^{i k \cdot x} \right],
\]

(5.1)

where the Fourier coefficients \(a(k)\) and \(a^\dagger(k)\) are now operators. Note that we will often write \(a(k)\) or \(a^\dagger(k)\), but one needs to realize that in that case \(k_0^2 = E_k^2 = k^2 + M^2\). The canonical momentum becomes

\[
\Pi(x) = \dot{\phi}(x) = \frac{-i}{2} \int \frac{d^3k}{(2\pi)^3} \left[ a(k) e^{-i k \cdot x} - a^\dagger(k) e^{i k \cdot x} \right].
\]

(5.2)

It is easy to check that these equations can be inverted (see Exercise 2.4 for the classical field)

\[
a(k) = \int d^3x \ e^{i k \cdot x} i \partial_0 \phi(x),
\]

(5.3)

\[
a^\dagger(k') = \int d^3x \ \phi(x) i \partial_0 e^{-i k' \cdot x}
\]

(5.4)

It is straightforward to prove that the equal time commutation relations between \(\phi(x)\) and \(\Pi(x')\) are equivalent with 'harmonic oscillator - like' commutation relations between \(a(k)\) and \(a^\dagger(k')\), i.e.

\[
[a(k),\Pi(x')]_{x^0=x'^0} = i \delta^3(x-x') \quad \text{and}
\]

\[
[\phi(x),\phi(x')]_{x^0=x'^0} = [\Pi(x),\Pi(x')]_{x^0=x'^0} = 0,
\]

(5.5)

is equivalent with

\[
[a(k),a^\dagger(k')] = (2\pi)^3 2E_k \delta^3(k-k') \quad \text{and}
\]

\[
[a(k),a(k')] = [a^\dagger(k),a^\dagger(k')] = 0.
\]

(5.6)
The hamiltonian can be rewritten in terms of a number operator $N(k) = N(k) = a^\dagger(k) a(k)$, which represents the 'number of particles' with momentum $k$.

$$H = \int d^3x \left[ \frac{1}{2} (\partial_\phi \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} M^2 \phi^2 \right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} \left( a^\dagger(k) a(k) + a(k) a^\dagger(k) \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} E_k N(k) + E_{vac}, \quad (5.7)$$

where the necessity to commute $a(k) a^\dagger(k)$ (as in the case of the quantum mechanics case) leads to a zero-point energy, in field theory also referred to as vacuum energy

$$E_{vac} = \frac{1}{2} V \int \frac{d^3k}{(2\pi)^3} E_k,$$ \quad (5.8)

where the vacuum contribution disappears because of rotational symmetry. Just as in the case of the harmonic oscillator it is essential (axiom) that there exists a ground state $|0\rangle$ that is annihilated by $a(k), a(k)|0\rangle = 0$.

5.2 The complex scalar field

In spite of the similarity with the case of the real field, we will consider it as a repetition of the quantization procedure, extending it with the charge operator and the introduction of particle and antiparticle operators. The field satisfies the Klein-Gordon equation and the density current ($U(1)$ transformations) and the energy-momentum tensor are

$$j_\mu = i \phi^* \partial_\mu \phi,$$

$$\Theta_{\mu\nu} = \partial_{(\mu} \phi \partial_{\nu)} \phi - L g_{\mu\nu}. \quad (5.12)$$

The quantized fields are written as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[ a(k) e^{-i k \cdot x} + b^\dagger(k) e^{i k \cdot x} \right], \quad (5.13)$$

$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[ b(k) e^{-i k \cdot x} + a^\dagger(k) e^{i k \cdot x} \right], \quad (5.14)$$

and satisfy the equal time commutation relation (only nonzero ones)\n
$$[\phi(x), \partial_0 \phi^\dagger(y)]_{x^0 = y^0} = i \delta^3(x - y), \quad (5.15)$$

which is equivalent to the relations (only nonzero ones)\n
$$[a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = (2\pi)^3 2E_k \delta^3(k - k'). \quad (5.16)$$
The hamiltonian is as before given by the normal ordered expression

\[ H = \int d^3x : \Theta^{00}(x) : \]
\[ = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} E_k \left[ a^\dagger(k)a(k) + b(k)b^\dagger(k) \right] : \]
\[ = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} E_k \left[ a^\dagger(k)a(k) + b^\dagger(k)b(k) \right], \tag{5.17} \]

i.e. particles (created by \( a^\dagger \)) and antiparticles (created by \( b^\dagger \)) with the same momentum contribute equally to the energy. Also the charge operator requires normal ordering (in order to give the vacuum eigenvalue zero),

\[ Q = i \int d^3x : [\phi^\dagger(x) \partial_0 \phi - \partial_0 \phi^\dagger(x)] : \]
\[ = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} : [a^\dagger(k)a(k) - b(k)b^\dagger(k)] : \]
\[ = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} [a^\dagger(k)a(k) - b^\dagger(k)b(k)]. \tag{5.18} \]

The commutator of \( \phi \) and \( \phi^\dagger \) is as for the real field given by

\[ [\phi(x), \phi^\dagger(y)] = i \Delta(x - y). \tag{5.19} \]

## 5.3 The Dirac field

From the lagrangian density

\[ \mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - M \bar{\psi} \psi, \tag{5.20} \]

the conserved density and energy-momentum currents are easily obtained,

\[ j_\mu = \bar{\psi} \gamma_\mu \psi, \tag{5.21} \]
\[ \Theta_{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma_\mu \gamma^\nu \partial_\nu \psi - \left( \frac{i}{2} \bar{\psi} \gamma^\nu \partial_\nu \psi - M \bar{\psi} \psi \right) g_{\mu\nu}. \tag{5.22} \]

The canonical momentum and the hamiltonian are given by

\[ \Pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\psi}(x)} = i \psi^\dagger(x), \tag{5.23} \]
\[ \mathcal{H}(x) = \Theta^{00}(x) = -\frac{i}{2} \bar{\psi} \gamma^\dagger \partial_0 \psi + M \bar{\psi} \psi = i \bar{\psi} \gamma^0 \partial_0 \psi = i \psi^\dagger \partial_0 \psi, \tag{5.24} \]

where the last line is obtained by using the Dirac equation. The quantized fields are written

\[ \psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \left[ b(k,s)u(k,s)e^{-ik \cdot x} + d^\dagger(k,s)v(k,s)e^{ik \cdot x} \right], \tag{5.25} \]
\[ \bar{\psi}(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \left[ b^\dagger(k,s)\bar{u}(k,s)e^{ik \cdot x} + d(k,s)\bar{v}(k,s)e^{-ik \cdot x} \right]. \tag{5.26} \]
In terms of the operators for the $b$ and $d$ quanta the Hamiltonian and charge operators are (omitting mostly the spin summation in the rest of this section)

\[ H = \int d^3x \, : \psi^\dagger(x) i \partial_0 \psi(x) : \]  
\[ = \int \frac{d^3k}{(2\pi)^3 2E_k} \, E_k : [b^\dagger(k)b(k) - d(k)d^\dagger(k)] : \]  
\[ Q = \int d^3x \, : \psi^\dagger \psi : \]  
\[ = \int \frac{d^3k}{(2\pi)^3 2E_k} : [b^\dagger(k)b(k) + d(k)d^\dagger(k)] : , \]

which seems to cause problems as the antiparticles (d-quanta) contribute negatively to the energy and the charges of particles (b-quanta) and antiparticles (d-quanta) are the same.

The solution is the introduction of anticommutation relations,

\[ \{ b(k, s), b^\dagger(k', s') \} = \{ d(k, s), d^\dagger(k', s') \} = (2\pi)^3 2E_k \delta^3(k - k') \delta_{ss'} . \]

Note that achieving normal ordering, i.e. interchanging creation and annihilation operators, then leads to additional minus signs and

\[ H = \int \frac{d^3k}{(2\pi)^3 2E_k} \, E_k [b^\dagger(k)b(k) + d^\dagger(k)d(k)] \]  
\[ Q = \int \frac{d^3k}{(2\pi)^3 2E_k} \, E_k [b^\dagger(k)b(k) - d^\dagger(k)d(k)] . \]
Chapter 6

Discrete symmetries

In this chapter we discuss the discrete symmetries, parity (P), time reversal (T) and charge conjugation (C). The consequences of P, T and C for classical quantities is shown in the table 1.

6.1 Parity

The parity operator transforms

\[ x^\mu = (t, r) \longrightarrow \tilde{x}^\mu \equiv x^\mu = (t, -r). \]  

(6.1)

We will consider the transformation properties for a fermion field \( \psi(x) \), writing

\[ \psi(x) \longrightarrow P_{op}\psi(x)P_{op}^{-1} = \eta_P A \psi(\tilde{x}) \equiv \psi^p(x), \]

(6.2)

where \( \eta_P \) is the intrinsic parity of the field and \( A \) is a 4 \( \times \) 4 matrix acting in the spinor space. Both \( \psi^p \) and \( \psi \) satisfy the Dirac equation. We can determine \( A \), starting with the Dirac equation for \( \psi(x) \),

\[ (i\gamma^\mu \partial_\mu - M) \psi(x) = 0. \]

Table 6.1: The behavior of classical quantities under P, T, and C

<table>
<thead>
<tr>
<th>quantity</th>
<th>P</th>
<th>T</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>t</td>
<td>-t</td>
<td>t</td>
</tr>
<tr>
<td>r</td>
<td>-r</td>
<td>r</td>
<td>r</td>
</tr>
<tr>
<td>( x^\mu )</td>
<td>( \tilde{x}^\mu \equiv x^\mu )</td>
<td>( -\tilde{x}^\mu )</td>
<td>( x^\mu )</td>
</tr>
<tr>
<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
</tr>
<tr>
<td>( p )</td>
<td>( -p )</td>
<td>( -p )</td>
<td>( p )</td>
</tr>
<tr>
<td>( \bar{p} )</td>
<td>( \bar{p} )</td>
<td>( \bar{p} )</td>
<td>( p )</td>
</tr>
<tr>
<td>( L )</td>
<td>( L )</td>
<td>( -L )</td>
<td>( L )</td>
</tr>
<tr>
<td>( s )</td>
<td>( s )</td>
<td>( -s )</td>
<td>( s )</td>
</tr>
<tr>
<td>( \lambda = s \cdot \bar{p} )</td>
<td>( -\lambda )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
</tbody>
</table>
After parity transforming $x$ to $\tilde{x}$ the Dirac equation becomes after some manipulations

\[
\left( i\gamma^\mu \partial_\mu - M \right) \psi(\tilde{x}) = 0, \\
\left( i\gamma^\mu \partial_\mu - M \right) \psi(\tilde{x}) = 0, \\
\left( i\gamma^\mu \partial_\mu - M \right) \psi(\tilde{x}) = 0, \\
\left( i\gamma^\mu \partial_\mu - M \right) \gamma_0 \psi(\tilde{x}) = 0. \tag{6.3}
\]

Therefore $\gamma_0 \psi(\tilde{x})$ is again a solution of the Dirac equation and we have

\[
\psi^\dagger(x) = \gamma_0 \psi(\tilde{x}). \tag{6.4}
\]

It is straightforward to apply this to the explicit field operator $\psi(x)$ using

\[
\gamma_0 u(k, m) = u(\tilde{k}, m), \quad \gamma_0 u(k, \lambda) = u(\tilde{k}, -\lambda), \tag{6.5}
\]

\[
\gamma_0 v(k, m) = -v(\tilde{k}, m), \quad \gamma_0 v(k, \lambda) = -v(\tilde{k}, -\lambda). \tag{6.6}
\]

Check this for the standard representation; note that if helicity $\lambda$ is used instead of the $z$-component of the spin $m$, the above operation reverses the sign of $\lambda$ (which does depend on the sign of $p^3$, however!). The result is

\[
\psi^\dagger(x) = P_{op} \psi(x) P_{op}^{-1} = \eta P \gamma_0 \psi(\tilde{x}) \tag{6.7}
\]

\[
= \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \eta P \left[ b(k, \lambda) \gamma_0 u(k, \lambda) e^{-ik\cdot x} + d^\dagger(k, \lambda) \gamma_0 v(k, \lambda) e^{ik\cdot x} \right] \\
= \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \eta P \left[ b(k, \lambda) u(\tilde{k}, -\lambda) e^{-ik\cdot x} - d^\dagger(k, \lambda) v(\tilde{k}, -\lambda) e^{ik\cdot x} \right] \\
= \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \eta P \left[ b(\tilde{k}, -\lambda) u(k, \lambda) e^{-ik\cdot x} - d^\dagger(k, \lambda) v(\tilde{k}, -\lambda) e^{ik\cdot x} \right]. \tag{6.8}
\]

From this one sees immediately that

\[
P_{op} b(k, \lambda) P_{op}^{-1} = \eta P b(\tilde{k}, -\lambda), \tag{6.9}
\]

\[
P_{op} d(k, \lambda) P_{op}^{-1} = -\eta P d(\tilde{k}, -\lambda), \tag{6.10}
\]

i.e. choosing $\eta P$ real ($\eta P = \pm 1$) particle and antiparticle have opposite parity.

In the same way as the Fermion field, one can also consider the scalar field and vector fields. For the scalar field we have seen

\[
\phi(x) \rightarrow P_{op} \phi(x) P_{op}^{-1} = \eta P \phi(\tilde{x}), \tag{6.11}
\]

and for the vector field

\[
A^\mu(x) \rightarrow P_{op} A^\mu(x) P_{op}^{-1} = -A_\mu(\tilde{x}). \tag{6.12}
\]

The latter behavior of the vector field will be discussed further below.

### 6.2 Charge conjugation

We have already seen the particle-antiparticle symmetry with under what we will call charge conjugation the behavior

\[
\psi(x) \rightarrow \psi^c(x) = \eta C \overline{\psi}(x), \tag{6.13}
\]
the latter being also a solution of the Dirac equation. The action on the spinors gives
\begin{align}
C \bar{u}^T(k, \lambda) &= v(k, \lambda), \\
C \bar{v}^T(k, \lambda) &= -u(k, \lambda),
\end{align}
(6.14) (6.15)

Therefore
\begin{align}
\psi^f(x) &= C_{op} \psi(x) C_{op}^{-1} = \eta_C C \psi^T(x) \\
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{\eta_C}{2E_k} \left[ d(k, \lambda)C \bar{u}^T(k, \lambda) e^{-ik \cdot x} + b^\dagger(k, \lambda)C \bar{v}^T(k, \lambda) e^{ik \cdot x} \right] \\
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{\eta_C}{2E_k} \left[ d(k, \lambda)u(k, \lambda) e^{-ik \cdot x} - b^\dagger(k, \lambda)v(k, \lambda) e^{ik \cdot x} \right].
\end{align}
(6.16) (6.17)

This shows that
\begin{align}
C_{op} b(k, \lambda) C_{op}^{-1} &= \eta_C d(k, \lambda), \\
C_{op} d(k, \lambda) C_{op}^{-1} &= -\eta_C^* b(k, \lambda).
\end{align}
(6.18) (6.19)

### 6.3 Time reversal

The time reversal operator transforms
\[ x^\mu = (t, \mathbf{r}) \rightarrow -\tilde{x}^\mu \equiv -x_\mu = (-t, \mathbf{r}). \]
(6.20)

We will again consider the transformation properties for a fermion field \( \psi(x) \), writing
\[ \psi(x) \rightarrow T_{op} \psi(x) T_{op}^{-1} = \eta_T A \psi(-\tilde{x}) \equiv \psi^f(-\tilde{x}), \]
(6.21)

where \( A \) is a 4 \times 4 matrix acting in the spinor space. As time reversal will transform 'bra' into 'ket', \( T_{op} \phi = \langle \phi^f \rangle = (|\phi^f\rangle)^* \), it is antilinear\(^1\). Norm conservation requires \( T_{op} \) to be anti-unitary\(^2\). For a quantized field one has
\[ T_{op} f_k(x) b_k T_{op}^{-1} = f_k^*(-\tilde{x}) T_{op} b_k T_{op}^{-1}, \]
i.e. to find \( \psi^f(-\tilde{x}) \) that is a solution of the Dirac equation, we start with the complex conjugated Dirac equation for \( \psi \),
\[ ((\gamma^\mu)^\ast \partial_\mu - M) \psi(x) = 0. \]
The (time-reversed) Dirac equation becomes,
\begin{align}
\left(-((\gamma^\mu)^\ast \partial_\mu - M)\right) \psi(-\tilde{x}) &= 0, \\
(\gamma^\mu \partial_\mu - M) \psi(-\tilde{x}) &= 0, \\
(\gamma^\mu \partial_\mu - M) \psi(-\tilde{x}) &= 0, \\
\gamma^\mu C^{-1} \gamma^\mu C \partial_\mu - M) \psi(-\tilde{x}) &= 0, \\
(\gamma^\mu \partial_\mu - M) \gamma_5 C \psi(-\tilde{x}) &= 0.
\end{align}
(6.22)

Therefore \( \gamma_5 C \psi(-\tilde{x}) \) is again a solution of the (ordinary) Dirac equation and we can choose (phase is convention)
\[ \psi^f(x) = \gamma_5 C \psi(-\tilde{x}). \]
(6.23)

---

\(^1\)A is antilinear if \( A(\lambda \phi + \mu \psi) = \lambda^* A \phi + \mu^* A \psi \).

\(^2\)An antilinear operator is anti-unitary if \( A^\dagger = A^{-1} \). One has \( \langle A\phi|A\psi \rangle = \langle \phi|\psi \rangle^* = \langle A\psi|A\phi \rangle^* = \langle \phi|A^\dagger A \phi \rangle = \langle \phi|\phi \rangle. \)
Table 6.2: The transformation properties of physical states for particles \( \langle a; p, \lambda \rangle \) and antiparticles \( \langle \bar{a}; p, \lambda \rangle \).

<table>
<thead>
<tr>
<th>state</th>
<th>( P )</th>
<th>( T )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle a; p, \lambda \rangle )</td>
<td>( \langle a; -p, -\lambda \rangle )</td>
<td>( \langle a; -p, \lambda \rangle )</td>
<td>( \langle a; p, \lambda \rangle )</td>
</tr>
<tr>
<td>( \langle \bar{a}; p, \lambda \rangle )</td>
<td>( \langle \bar{a}; -p, -\lambda \rangle )</td>
<td>( \langle \bar{a}; -p, \lambda \rangle )</td>
<td>( \langle \bar{a}; p, \lambda \rangle )</td>
</tr>
</tbody>
</table>

In the standard representation \( i \gamma_5 C = i \sigma_2 \) and it is straightforward to apply this to the explicit field operator \( \psi(x) \) using

\[
\gamma_5 C \, u(k, \lambda) = u^*(\tilde{k}, \lambda),
\]

(6.24)

\[
\gamma_5 C \, v(k, \lambda) = v^*(\tilde{k}, \lambda),
\]

(6.25)

(check this for the standard representation and make proper use of helicity states for \(-p\)). The result is

\[
\psi^d(x) = T_{op} \psi(x) T_{op}^{-1} = i \eta \gamma_5 C \psi(-\tilde{x})
\]

(6.26)

\[
= \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_k} \eta \left[ b(k, \lambda) i \gamma_5 C u(k, \lambda) e^{i k \cdot \tilde{x}} + d^\dagger(k, \lambda) i \gamma_5 C v(k, \lambda) e^{-i k \cdot \tilde{x}} \right]
\]

\[
= \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_k} \eta \left[ b(k, \lambda) u^*(\tilde{k}, \lambda) e^{i k \cdot x} + d^\dagger(k, \lambda) v^*(\tilde{k}, \lambda) e^{-i k \cdot x} \right]
\]

\[
= \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_k} \eta \left[ b^\dagger(\tilde{k}, \lambda) u^*(k, \lambda) e^{ik \cdot x} + d^\dagger(k, \lambda) v^*(\tilde{k}, \lambda) e^{-ik \cdot x} \right]
\]

(6.27)

From this one obtains

\[
T_{op} \, b(k, \lambda) \, T_{op}^{-1} = \eta \gamma_5 \, b(\tilde{k}, \lambda),
\]

(6.28)

\[
T_{op} \, d(k, \lambda) \, T_{op}^{-1} = \eta \gamma_5 \, d(\tilde{k}, \lambda).
\]

(6.29)

In Table 2 the behavior of particle states under the various transformations has been summarized. Note that applying an anti-unitary transformation such as \( T_{op} \) one must take for the matrix element the complex conjugate. Therefore one has \( \langle A|k \rangle = \langle A|b^\dagger(k)|0 \rangle = \langle A|T^d T^\dagger b^\dagger(k) T^d T^\dagger|0 \rangle^* = \langle A'|b^\dagger(\tilde{k})|0 \rangle^* = \langle 0|b(k)|A' \rangle = \langle \tilde{k}|A' \rangle \).

Exercise 8.4

Start with the expression for the fermion quantum field \( \psi(x) \). One can rewrite the wave functions in terms of the real Weyl helicity spinors \( U_{R/L}(k, \pm) \). Show that one can express \( \psi \) in terms of real spinors

\[
\psi(x) = \Upsilon_1(x) + i \Upsilon_2(x),
\]

of the form

\[
\Upsilon_i(x) = \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_k} u_i(k, \lambda) \left[ b_i(k, \lambda) e^{-ik \cdot x} + b_i^\dagger(k, \lambda) e^{ik \cdot x} \right].
\]

Give the expressions for the creation operators \( b(k, \lambda) \) and \( d^\dagger(k, \lambda) \) in terms of \( b_1(k, \lambda) \) and \( b_2(k, \lambda) \) and the two spinors \( u(k, \lambda) \) in terms of the (real) Weyl helicity spinors (see Chapter 4).
(solution)
We can express the field in terms of the Weyl spinors as

\[ \psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left\{ U_R(k, +) b(k, +) e^{-ik \cdot x} + U_L(k, -) b(k, -) e^{-ik \cdot x} ight. \\
- U_L(k, -) d^\dagger(k, +) e^{ik \cdot x} + U_R(k, +) d^\dagger(k, -) e^{ik \cdot x} \\
+ \delta \left\{ U_L(k, +) b(k, +) e^{-ik \cdot x} + U_R(k, -) b(k, -) e^{-ik \cdot x} \\
+ U_R(k, -) d^\dagger(k, +) e^{ik \cdot x} + U_L(k, +) d^\dagger(k, -) e^{ik \cdot x} \right\} \]

re-ordered as

\[ \psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left\{ U_R(k, +) \left[ b(k, +) e^{-ik \cdot x} + d^\dagger(k, -) e^{ik \cdot x} \right] \\
+ U_L(k, -) \left[ b(k, -) e^{-ik \cdot x} - d^\dagger(k, +) e^{ik \cdot x} \right] \right\} \\
+ \delta \left\{ U_L(k, +) \left[ b(k, +) e^{-ik \cdot x} + d^\dagger(k, -) e^{ik \cdot x} \right] \\
+ U_R(k, -) \left[ b(k, -) e^{-ik \cdot x} - d^\dagger(k, +) e^{ik \cdot x} \right] \right\} \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left\{ u(k, +) \left[ b(k, +) e^{-ik \cdot x} + d^\dagger(k, -) e^{ik \cdot x} \right] \\
+ u(k, -) \left[ b(k, -) e^{-ik \cdot x} - d^\dagger(k, +) e^{ik \cdot x} \right] \right\} \]

with

\[ u(k, +) = \sqrt{1 - \delta^2} U_R(k, +) + \delta U_L(k, +), \]
\[ u(k, -) = \sqrt{1 - \delta^2} U_L(k, -) + \delta U_R(k, -). \]

Introducing the creation and annihilation operators

\[ b(k, +) = b_1(k, +) + i b_2(k, +), \]
\[ b(k, -) = b_1(k, -) + i b_2(k, -), \]
\[ d(k, -) = b_1(k, +) - i b_2(k, +), \]
\[ d(k, +) = -b_1(k, -) + i b_2(k, -), \]

one finds the requested form.
Chapter 7

The standard model

7.1 The starting point: $SU(2)_W \otimes U(1)_Y$

Symmetry plays an essential role in the standard model that describes the elementary particles, the quarks (up, down, etc.), the leptons (elektrons, muons, neutrinos, etc.) and the gauge bosons responsible for the strong, electromagnetic and weak forces. In the standard model one starts with a very simple basic lagrangian for (massless) fermions which exhibits more symmetry than observed in nature. By introducing gauge fields and breaking the symmetry a more complex lagrangian is obtained, that gives a good description of the physical world. The procedure, however, implies certain nontrivial relations between masses and mixing angles that can be tested experimentally and sofar are in excellent agreement with experiment.

The lagrangian for the leptons consists of three families each containing an elementary fermion (electron $e^-$, muon $\mu^-$ or tau $\tau^-$), its corresponding neutrino ($\nu_e$, $\nu_\mu$ and $\nu_\tau$) and their antiparticles. As they are massless, left- and righthanded particles, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5) \psi$ decouple. For the neutrino only a lefthanded particle (and righthanded antiparticle) exist. Thus

$$L(f) = i \bar{e}_{R/L} \gamma^\mu \partial_\mu e_{R/L} + i \nu_{R/L} \gamma^\mu \partial_\mu \nu_{R/L} + (\mu, \tau).$$ (7.1)

One introduces a (weak) $SU(2)_W$ symmetry under which $e_R$ forms a singlet, while the lefthanded particles form a doublet, i.e.

$$L = \begin{pmatrix} L^0 \\ L^- \end{pmatrix} = \begin{pmatrix} \nu_e \\ \nu_L \end{pmatrix} \quad \text{with } T_W = \frac{1}{2} \text{ and } T^3_W = \begin{cases} +1/2 \\ -1/2 \end{cases}$$

and

$$R = R^- = e_R \quad \text{with } T_W = 0 \text{ and } T^3_W = 0.$$ (7.4)

Thus the basic lagrangian density is a Dirac lagrangian with massless (independent left- and righthanded species)

$$\mathcal{L}^f = i \bar{L} \gamma^\mu \partial_\mu L + i \bar{R} \gamma^\mu \partial_\mu R,$$ (7.2)

which has an $SU(2)_W$ symmetry under transformations $e^{i \alpha} \bar{T}_W$, explicitly

$$L \overset{SU(2)_W}{\rightarrow} e^{i \alpha/2} \bar{T}_W L,$$ (7.3)

$$R \overset{SU(2)_W}{\rightarrow} R.$$ (7.4)

One notes that the charges of the leptons can be obtained as $Q = T^3_W - 1/2$ for lefthanded particles and $Q = T^3_W + 1$ for righthanded particles. The difference between charge and 3-component of isospin is called weak hypercharge and one writes

$$Q = T^3_W + \frac{Y_W}{2}.$$ (7.5)
The weak hypercharge $Y_W$ is an operator that generates a $U(1)_Y$ symmetry with for the lefthanded and righthanded particles different hypercharges, $Y_W(L) = -1$ and $Y_W(R) = -2$. The particles transform according to $e^{i\theta Y_W/2}$, explicitly

$$L \xrightarrow{U(1)_Y} e^{-i\beta/2}L,$$

$$R \xrightarrow{U(1)_Y} e^{-i\beta}R.$$  

(7.6)  

(7.7)

Next the $SU(2)_W \otimes U(1)_Y$ symmetry is made into a local symmetry introducing gauge fields $\tilde{W}_\mu$ and $B_\mu$ in the covariant derivative $D_\mu = \partial_\mu - ig\tilde{W}_\mu \cdot \tilde{T}_W - ig' B_\mu Y_W/2$, explicitly

$$D_\mu L = \partial_\mu L - \frac{i}{2} g \tilde{W}_\mu \cdot \tau L + \frac{i}{2} g' B_\mu L,$$

$$D_\mu R = \partial_\mu R + ig' B_\mu R,$$

(7.8)  

(7.9)

where $\tilde{W}_\mu$ is a triplet of gauge bosons with $T_W = 1$, $T_W^3 = \pm 1$ or 0 and $Y_W = 0$ (thus $Q = T_W^3$) and $B_\mu$ is a singlet under $SU(2)_W$ ($T_W = T_W^3 = 0$) and also has $Y_W = 0$. Putting this in leads to

$$L^{(f)} = L^{(f1)} + L^{(f2)},$$

(7.10)

$$L^{(f1)} = i\bar{L}\gamma^\mu(\partial_\mu + ig' B_\mu)R + i\bar{L}\gamma^\mu(\partial_\mu + ig' B_\mu - \frac{i}{2} g\tilde{W}_\mu \cdot \tau)L,$$

$$L^{(f2)} = -\frac{1}{4}(\partial_\mu \tilde{W}_\mu - \partial_\nu \tilde{W}_\nu + g\tilde{W}_\mu \times \tilde{W}_\nu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2.$$  

In order to break the symmetry to the symmetry of the physical world, the $U(1)_Q$ symmetry (generated by the charge operator), a complex Higgs field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\theta_2 + i\theta_1) \\ \sqrt{2}(\theta_4 - i\theta_3) \end{pmatrix}$$

(7.11)

with $T_W = 1/2$ and $Y_W = 1$ is introduced, with the following lagrangian density consisting of a symmetry breaking piece and a coupling to the fermions,

$$L^{(h)} = L^{(h1)} + L^{(h2)},$$

(7.12)

where

$$L^{(h1)} = (D_\mu \phi)\dagger(D^\mu \phi) - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2,$$

and

$$L^{(h2)} = -Ge(T_\phi R + \bar{T}_\phi L),$$

$$D_\mu \phi = (\partial_\mu - \frac{i}{2} g\tilde{W}_\mu \cdot \tau - \frac{i}{2} g' B_\mu)\phi.$$  

(7.13)

The Higgs potential $V(\phi)$ is choosen such that it gives rise to spontaneous symmetry breaking with $\phi^\dagger \phi = -m^2/2\lambda \equiv v^2/2$. For the classical field the choice $\theta_i \equiv v$ is made, which assures with the choice of $Y_W$ for the Higgs field that $Q$ generates the remaining $U(1)$ symmetry. Using local gauge invariance $\theta_i$ for $i = 1, 2$ and 3 may be eliminated (the necessary $SU(2)_W$ rotation is precisely $e^{-i\vec{\theta}(x) \cdot \vec{\tau}}$, leading to the parametrization

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

(7.14)
and

\[
D_\mu \phi = \begin{cases} 
- i g_2 \left( \frac{W^1_\mu - i W^2_\mu}{\sqrt{2}} \right) (v + h) \\
\frac{1}{\sqrt{2}} \partial_\mu h + i \left( \frac{g W^2_\mu - g'B_\mu}{\sqrt{2}} \right) (v + h)
\end{cases}.
\] (7.15)

Up to cubic terms, this leads to the lagrangian

\[
\mathcal{L}^{(h)} = \frac{1}{2} (\partial_\mu h)^2 + m^2 h^2 + \frac{g^2 v^2}{8} \left[ (W^1_\mu)^2 + (W^2_\mu)^2 \right]
+ \frac{v^2}{8} \left( gW^3_\mu - g'B_\mu \right)^2 + \ldots
\] (7.16)

\[
= \frac{1}{2} (\partial_\mu h)^2 + m^2 h^2 + \frac{g^2 v^2}{8} \left[ (W^+)^2 + (W^-)^2 \right]
+ \frac{(g^2 + g'^2) v^2}{8} (Z_\mu)^2 + \ldots,
\] (7.17)

where the quadratically appearing gauge fields that are furthermore eigenstates of the charge operator are

\[
W^\pm_\mu = \frac{1}{\sqrt{2}} \left( W^1_\mu \pm i W^2_\mu \right),
\] (7.18)

\[
Z_\mu = g W^3_\mu - g'B_\mu \sqrt{g^2 + g'^2}
\] (7.19)

\[
A_\mu = g' W^3_\mu + g B_\mu \sqrt{g^2 + g'^2}
\] (7.20)

and correspond to three massive particle fields (\(W^\pm\) and \(Z^0\)) and one massless field (photon \(\gamma\)) with

\[
M^2_W = \frac{g^2 v^2}{4},
\] (7.21)

\[
M^2_Z = \frac{g^2 v^2}{4 \cos^2 \theta_W} = M^2_W \frac{\cos^2 \theta_W}{\cos^2 \theta_W},
\] (7.22)

\[
M^2_\gamma = 0.
\] (7.23)

The weak mixing angle is related to the ratio of coupling constants, \(g'/g = \tan \theta_W\).

The coupling of the fermions to the physical gauge bosons are contained in \(\mathcal{L}^{(f)}\) giving

\[
\mathcal{L}^{(f)} = i \bar{\tau} \gamma^\mu \partial_\mu \tau + i \bar{\tau} \gamma^\mu \partial_\mu \nu_e - g \sin \theta_W \bar{\tau} \gamma^\mu \nu_e A_\mu \\
+ \frac{g}{\cos \theta_W} \left( \sin^2 \theta_W \bar{\tau} \gamma^\mu e_R - \frac{1}{2} \cos 2\theta_W \bar{\tau} \gamma^\mu e_L + \frac{1}{2} \bar{\tau} \gamma^\mu e_L \right) Z_\mu \\
+ \frac{g}{\sqrt{2}} (\bar{\tau} \gamma^\mu e_L W^-_\mu + \bar{\tau} \gamma^\mu e_L W^+_\mu).
\] (7.24)

From the coupling to the photon, we can read off

\[
e = g' \cos \theta_W.
\] (7.25)

The coupling of electrons or muons to their respective neutrinos, for instance in the amplitude for the decay of the muon

\[
\mu^- \rightarrow e^- \nu_e, \quad \mu^- \rightarrow e^- \bar{\nu}_e.
\]
is given by

\[-i \mathcal{M} = -\frac{g^2}{2} (\bar{\psi}_\nu \gamma^\mu \psi) \frac{-i g \rho_\sigma + \ldots}{k^2 + M_W^2} (\bar{\nu}_\nu \gamma^\mu \nu)\]

\[\approx i \frac{g^2}{8 M_W^2} (\bar{\psi}_\nu \gamma^\mu (1 - \gamma_5) \psi) (\bar{\nu}_\nu \gamma^\mu (1 - \gamma_5) \nu)\]

\[\equiv i \frac{G_F}{\sqrt{2}} (j_L^{\mu \dagger})_\rho (j_L^{\rho})_\mu,\]  

(7.26)

the good old four-point interaction introduced by Fermi to explain the weak interactions, i.e. one has the relation

\[\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} = \frac{e^2}{8 M_W^2 \sin^2 \theta_W} = \frac{1}{2 \alpha^2}.\]  

(7.28)

In this way the parameters \(g, g', \) and \(v\) determine a number of experimentally measurable quantities, such as

\[\alpha = e^2/4\pi \approx 1/127,\]

(7.29)

\[G_F = 1.1664 \times 10^{-5} \text{ GeV}^{-2},\]

(7.30)

\[\sin^2 \theta_W = 0.2312,\]

(7.31)

\[M_W = 80.40 \text{ GeV},\]

(7.32)

\[M_Z = 91.19 \text{ GeV}.\]

(7.33)

The coupling of the \(Z^0\) to fermions is given by \((g/\cos \theta_W) \gamma^\mu\) multiplied with

\[T_W^3 \frac{1}{2} (1 - \gamma_5) - \sin^2 \theta_W Q = \frac{1}{2} C_V - \frac{1}{2} C_A \gamma_5,\]

(7.34)

with

\[C_V = T_W^3 - 2 \sin^2 \theta_W Q,\]

\[C_A = T_W^3.\]

(7.35)

(7.36)

From this coupling it is straightforward to calculate the partial width for \(Z^0\) into a fermion-antifermion pair,

\[\Gamma(Z^0 \to f \bar{f}) = \frac{M_Z}{48\pi} \frac{g^2}{\cos^2 \theta_W} (C_V^2 + C_A^2).\]

(7.37)

For the electron, muon or tau, leptons with \(C_V = -1/2 + 2 \sin^2 \theta_W \approx -0.05\) and \(C_A = -1/2\) we calculate \(\Gamma(e^+ e^-) \approx 78.5\) MeV (exp. \(\Gamma_e \approx 8\) MeV). For each neutrino species (with \(C_V = 1/2\) and \(C_A = 1/2\)) one expects \(\Gamma(\nu_e) \approx 155\) MeV. Comparing this with the total width into (invisible!) channels, \(\Gamma_{\text{invisible}} = 480\) MeV one sees that three families of (light) neutrinos are allowed. Actually including corrections corresponding to higher order diagrams the agreement for the decay width into electrons can be calculated much more accurately and the number of allowed (light) neutrinos turns to be even closer to three.

The masses of the fermions and the coupling to the Higgs particle are contained in \(\mathcal{L}^{(h)}\). With the chosen vacuum expectation value for the Higgs field, one obtains

\[\mathcal{L}^{(h)} = -\frac{G_F v}{\sqrt{2}} (\bar{\nu}_L e_R + \bar{e}_R \nu_L) - \frac{G_F}{\sqrt{2}} (\bar{\nu}_L e_R + \bar{e}_R \nu_L) h\]

\[= -m_e \bar{\nu}_e e_R - \frac{m_e}{v} \nu_R e_L.\]  

(7.38)

\footnote{The value of \(\alpha\) deviates from the known value \(\alpha \approx 1/137\) because of higher order contributions, giving rise to a \textit{running coupling constant} after renormalization of the field theory.}
First, the mass of the electron comes from the spontaneous symmetry breaking but is not predicted (it is in the coupling $G_e$). The coupling to the Higgs particle is weak as the value for $v$ calculated e.g. from the $M_W$ mass is about 250 GeV, i.e. $m_e/v$ is extremely small.

Finally we want to say something about the weak properties of the quarks, as appear for instance in the decay of the neutron or the decay of the $\Lambda$ (quark content $uds$),

\begin{align*}
    n &\rightarrow pe^-\bar{\nu}_e \quad \leftrightarrow \quad d \rightarrow we^-\bar{\nu}_e, \\
    \Lambda &\rightarrow pe^-\bar{\nu}_e \quad \leftrightarrow \quad s \rightarrow we^-\bar{\nu}_e.
\end{align*}

The quarks also turn out to fit into doublets of $SU(2)_W$ for the lefthanded species and into singlets for the righthanded quarks. As shown in Fig. 7.1, this requires particular $Y_W-T^3_W$ assignments to get the charges right.

A complication arises for quarks (and as we will discuss in the next section in more detail also for leptons) as it are not the 'mass' eigenstates that appear in the weak isospin doublets but linear combinations of them,

\begin{align*}
    \begin{pmatrix}
        u \\
        d' \\
        s' \\
        b'
    \end{pmatrix}_L &\rightarrow
    \begin{pmatrix}
        c \\
        s \\
        t
    \end{pmatrix}_L,
\end{align*}

where

\begin{equation}
    \begin{pmatrix}
        d' \\
        s' \\
        b'
    \end{pmatrix}_L =
    \begin{pmatrix}
        V_{ud} & V_{us} & V_{ub} \\
        V_{cd} & V_{cs} & V_{cb} \\
        V_{td} & V_{ts} & V_{tb}
    \end{pmatrix}
    \begin{pmatrix}
        d \\
        s \\
        b
    \end{pmatrix}_L. \quad (7.39)
\end{equation}

This mixing allows all quarks with $T^3_W = -1/2$ to decay into an up quark, but with different strength. Comparing neutron decay and $\Lambda$ decay one can get an estimate of the mixing parameter $V_{us}$ in the
The standard model

7.2 Family mixing in the Higgs sector and neutrino masses

The quark sector

Allowing for the most general (Dirac) mass generating term in the lagrangian one starts with

$$ \mathcal{L}^{(h_{2,q})} = -\overline{Q}_L \phi A_d D_R - D_R^\dagger A_d^\dagger \phi^1 Q_L - \overline{Q}_L \phi A_u U_R - U_R^\dagger A_u^\dagger \phi^1 \gamma \phi Q_L $$

(7.40)

where we include now the three lefthanded quark doublets in $Q_L$, the three righthanded quarks with charge +2/3 in $U_R$, and the three righthanded quarks with charges −1/3 in $D_R$, each of these containing the three families, e.g. $U_R = \begin{pmatrix} u_R & c_R & t_R \end{pmatrix}$. The $A_u$ and $A_d$ are complex matrices in the $3 \times 3$ family space. The Higgs field is still limited to one complex doublet. Note that we need the conjugate Higgs singlet in the case of the charge +2/3 quarks, for which we need the appropriate weak isospin doublet

$$ \phi^c = \begin{pmatrix} \phi^0 \phi^- \end{pmatrix} = \begin{pmatrix} v + h \end{pmatrix}. $$

For the (squared) complex matrices we can find positive eigenvalues,

$$ A_u ^\dagger A_u = V_u G_u^2 \gamma_\mu^{\dagger} \gamma_\mu , \quad \text{and} \quad A_d ^\dagger A_d = V_d G_d^2 \gamma_\mu^{\dagger} \gamma_\mu, $$

(7.41)

where $V_u$ and $V_d$ are unitary matrices, allowing us to write

$$ A_u = V_u G_u W_u^{\dagger} \quad \text{and} \quad A_d = V_d G_d W_d^{\dagger}, $$

(7.42)

with $G_u$ and $G_d$ being real and positive and $W_u$ and $W_d$ being different unitary matrices. Thus one has

$$ \mathcal{L}^{(h_{2,q})} \rightarrow -D_R^\dagger V_d M_d V_d^{\dagger} D_R - D_R^\dagger W_d M_d W_d^{\dagger} D_R - \overline{U}_L^\dagger V_u M_u W_u^{\dagger} U_R - U_R^\dagger W_u M_u W_u^{\dagger} U_L $$

(7.43)

with $M_u = G_u v / \sqrt{2}$ (diagonal matrix containing $m_u$, $m_c$, and $m_t$) and $M_d = G_d v / \sqrt{2}$ (diagonal matrix containing $m_d$, $m_s$, and $m_b$). One then reads off that starting with the family basis as defined via the left doublets that the mass eigenstates (and states coupling to the Higgs field) involve the righthanded states $U_R^{\text{mass}} = V_u^\dagger U_R$ and $D_R^{\text{mass}} = V_d^\dagger D_R$ and the lefthanded states $U_L^{\text{mass}} = V_d^\dagger U_L$ and $D_L^{\text{mass}} = V_u^\dagger D_L$. Working with the mass eigenstates one simply sees that the weak current coupling to the $W^\pm$ becomes $\overline{U}_L \gamma^\mu D_L = U_L^{\text{mass}} \gamma^\mu V_d^\dagger V_d^\dagger D_L^{\text{mass}}$, i.e. the weak mass eigenstates are

$$ D_L' = D_L^{\text{weak}} = V_u^\dagger V_d D_L^{\text{mass}} = V_{\text{CKM}} D_L^{\text{mass}}, $$

(7.44)

the unitary CKM-matrix introduced above in an ad hoc way.
The lepton sector (massless neutrinos)

For a lepton sector with a lagrangian density of the form

\[
\mathcal{L}^{(b2, \ell)} = -\bar{L}_\phi \Lambda_\ell E_R - \bar{E}_R \Lambda_\phi^\dagger L,
\]

in which

\[
L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}
\]

is a weak doublet containing the three families of neutrinos \((N_L)\) and charged leptons \((E_L)\) and \(E_R\) is a three-family weak singlet, we find massless neutrinos. As before, one can write \(\Lambda_\ell = V_\ell G_\ell W_\ell^\dagger\) and we find

\[
\mathcal{L}^{(b2, \ell)} \implies -M_\ell \left( \bar{E}_L V_\ell^\dagger W_\ell^\dagger E_R - \bar{E}_R W_\ell V_\ell^\dagger E_L \right),
\]

with \(M_\ell = G_\ell v/\sqrt{2}\) the diagonal mass matrix with masses \(m_e\), \(m_\mu\) and \(m_\tau\). The mass fields \(E_R^\text{mass} = W_\ell^\dagger E_R, E_L^\text{mass} = V_\ell^\dagger E_L\). For the (massless) neutrino fields we just can redefine fields into \(N_L^\text{mass} = V_\ell^\dagger N_L\), since the weak current is the only place where they show up. The \(W\)-current then becomes \(\bar{E}_L \gamma^\mu N_L = \bar{E}_L^\text{mass} \gamma^\mu N_L^\text{mass}\), i.e. there is no family mixing for massless neutrinos.

The lepton sector (massive Dirac neutrinos)

In principle a massive Dirac neutrino could be accounted for by a lagrangian of the type

\[
\mathcal{L}^{(b2, \ell)} = -\bar{L}_\phi \Lambda_\ell E_R - \bar{E}_R \Lambda_\phi^\dagger L - \bar{E}_R \Lambda_n N_R - \bar{N}_R \Lambda_n^\dagger \phi^\dagger L,
\]

with three righthanded neutrinos added to the previous case, decoupling from all known interactions. Again we continue as before now with matrices \(\Lambda_\ell = V_\ell G_\ell W_\ell^\dagger\) and \(\Lambda_n = V_n G_n W_n^\dagger\), and obtain

\[
\mathcal{L}^{(b2, \ell)} \implies -\bar{E}_L V_\ell M_\ell W_\ell^\dagger E_R - \bar{E}_R W_\ell M_\ell V_\ell^\dagger E_L - \bar{N}_L V_n M_n W_n^\dagger N_R - \bar{N}_R W_n M_n V_n^\dagger N_L.
\]

We note that there are mass fields \(E_R^\text{mass} = W_\ell^\dagger E_R, E_L^\text{mass} = V_\ell^\dagger E_L\) and the weak current becomes \(\bar{E}_L \gamma^\mu N_L = \bar{E}_L^\text{mass} \gamma^\mu V_\ell^\dagger N_L^\text{mass}\). Working with the mass eigenstates for the charged leptons we see that the weak eigenstates for the neutrinos are \(N_L^\text{weak} = V_\ell^\dagger N_L\) with the relation to the mass eigenstates for the lefthanded neutrinos given by

\[
N_L^\text{mass} = U_{\text{PMNS}}^\dagger N_L^\text{weak} = V_\ell^\dagger V_n N_L^\text{mass} = U_{\text{PMNS}}^\dagger N_L^\text{mass},
\]

with \(U_{\text{PMNS}} = V_\ell^\dagger V_n\) known as the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix.

For neutrino’s this matrix is parametrized in terms of three angles \(\theta_{ij}\) with \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\) and one angle \(\delta\),

\[
U_{\text{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

a parametrization that in principle also could have been used for quarks. In this case, it is particularly useful because \(\theta_{12}\) is essentially determined by solar neutrino oscillations requiring \(\Delta m^2_{12} \approx 8 \times 10^{-5}\) eV\(^2\) (convention \(m_2 > m_1\)), while \(\theta_{23}\) then is determined by atmospheric neutrino oscillations requiring \(\Delta m^2_{23} \approx 2.5 \times 10^{-3}\) eV\(^2\). The mixing is intriguingly close to the Harrison-Perkins-Scott tri-bimaximal mixing matrix

\[
U_{\text{HPS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2/3} & \sqrt{1/3} \\ 0 & -\sqrt{1/3} & \sqrt{2/3} \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/3} & \sqrt{2/3} & 0 \\ 0 & -\sqrt{1/3} & \sqrt{2/3} \end{pmatrix}.
\]

(7.51)
The lepton sector (massive Majorana fields)

An even simpler option than sterile right-handed Dirac neutrinos, is to add in Eq. 7.46 a Majorana mass term for the (leptihanded) neutrino mass eigenstates,

$$\mathcal{L}^{\text{mass},\nu} = -\frac{1}{2} \left( M_L \overline{N}_L^c N_L + M_L^* \overline{N}_L^c N_L^c \right),$$

(7.52)

although this option is not attractive as it violates the electroweak symmetry. The way to circumvent this is to introduce as in the previous section righthanded neutrinos, with for the righthanded sector a mass term $M_R$,

$$\mathcal{L}^{\text{mass},\nu} = -\frac{1}{2} \left( M_R \overline{N}_R^c N_R + M_R^* \overline{N}_R^c N_R^c \right).$$

(7.53)

In order to have more than a completely decoupled sector, one must for the neutrinos as well as charged leptons, couple the right- and lefthanded species through Dirac mass terms coming from the coupling to the Higgs sector as in the previous section. Thus (disregarding family structure) one has two Majorana neutrinos, one being massive. For the charged leptons there cannot exist a Majorana mass term as this would break the U(1) electromagnetic symmetry. For the leptons, the left- and righthanded species then just form a Dirac fermion.

For the neutrino sector, the massless and massive Majorana neutrinos, coupled by a Dirac mass term, are equivalent to two decoupled Majorana neutrinos (see below). If the Majorana mass $M_R \gg M_D$ one actually obtains in a natural way one Majorana neutrino with a very small mass. This is called the see-saw mechanism (outlined below).

For these light Majorana neutrinos one has, as above, a unitary matrix relating them to the weak eigenstates. Absorption of phases in the states is not possible for Majorana neutrinos, however, hence the mixing matrix becomes

$$V_{\text{PMNS}} = U_{\text{PMNS}} K \quad \text{with} \quad K = \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(7.54)

containing three (CP-violating) phases ($\alpha_1, \alpha_2$ and $\delta$).

The see-saw mechanism

Consider (for one family $N = n$) the most general Lorentz invariant mass term for two independent Majorana spinors, $\Upsilon'_1$ and $\Upsilon'_2$ (satisfying $\Upsilon'^c = \Upsilon'$ and as discussed in chapter 6, $\Upsilon'_L \equiv (\Upsilon'_L)^c = \Upsilon'_R$ and $\Upsilon'_R = \Upsilon'_L$). We use here the primes starting with the weak eigenstates. Actually, it is easy to see that this incorporates the Dirac case by considering the lefthanded part of $\Upsilon'_1$ and the righthanded part of $\Upsilon'_2$ as a Dirac spinor $\psi$. Thus

$$\Upsilon'_1 = n_L^c + n_L, \quad \Upsilon'_2 = n_R + n_R^c, \quad \psi = n_R + n_L.$$  

(7.55)

As the most general mass term in the lagrangian density we have

$$\mathcal{L}^{\text{mass}} = -\frac{1}{2} \left( M_L \overline{\Upsilon}_L^c \Upsilon_L + M_L^* \overline{\Upsilon}_L^c \Upsilon_L^c \right) - \frac{1}{2} \left( M_R \overline{\Upsilon}_R^c \Upsilon_R + M_R^* \overline{\Upsilon}_R^c \Upsilon_R^c \right) - \frac{1}{2} \left( M_D \overline{n}_L n_R + M_D^* \overline{n}_L^c n_R^c \right) - \frac{1}{2} \left( M_D \overline{n}_R n_L + M_D^* \overline{n}_R^c n_L^c \right)$$

(7.56)

$$= -\frac{1}{2} \left( \overline{n}_L \begin{pmatrix} M_L & M_D \\ M_D & M_R \end{pmatrix} \begin{pmatrix} n_L \\ n_R \end{pmatrix} \right) + \text{h.c.}$$

(7.57)

which for $M_D = 0$ is a pure Majorana lagrangian and for $M_L = M_R = 0$ and real $M_D$ represents the Dirac case. The mass matrix can be written as

$$M = \begin{pmatrix} M_L & e^{i\theta} \sqrt{|M_D|} \\ |M_D| e^{-i\theta} & M_R \end{pmatrix}.$$  

(7.58)
taking $M_L$ and $M_R$ real and non-negative. This choice is possible without loss of generality because the phases can be absorbed into $\Upsilon_1'$ and $\Upsilon_2'$ (real must be replaced by hermitean if one includes families). This is a mixing problem with a symmetric (complex) mass matrix leading to two (real) mass eigenstates. The diagonalization is analogous to what was done for the $A$-matrices and one finds $U M U^\dagger = M_0$ with a (unitary) matrix $U$, which implies $U^* M^\dagger U^\dagger = U^* M^* U^\dagger = M_0$ and a 'normal’ diagonalization of the (hermitean) matrix $MM^\dagger$,

$$U (MM^\dagger) U^\dagger = M_0^2, \quad (7.59)$$

Thus one obtains from

$$MM^\dagger = \begin{pmatrix} M_L^2 + |M_D|^2 & |M_D| (M_L e^{-i\phi} + M_R e^{+i\phi}) \\ |M_D| (M_L e^{+i\phi} + M_R e^{-i\phi}) & M_R^2 + |M_D|^2 \end{pmatrix}, \quad (7.60)$$

the eigenvalues

$$M_{L,R}^2 = \frac{1}{2} \left[ M_L^2 + M_R^2 + 2|M_D|^2 \right] \pm \sqrt{(M_L^2 - M_R^2)^2 + 4|M_D|^2 (M_L^2 + M_R^2 + 2M_L M_R \cos(2\phi))}, \quad (7.61)$$

and we are left with two decoupled Majorana fields $\Upsilon_1$ and $\Upsilon_2$, related via

$$\begin{pmatrix} \Upsilon_{1L} \\ \Upsilon_{2L} \end{pmatrix} = U^* \begin{pmatrix} n_L \\ n_R \end{pmatrix}, \quad \begin{pmatrix} \Upsilon_{1R} \\ \Upsilon_{2R} \end{pmatrix} = U \begin{pmatrix} n_L^* \\ n_R \end{pmatrix}. \quad (7.62)$$

for each of which one finds the lagrangians

$$\mathcal{L} = \frac{1}{4} \overleftrightarrow{T} \gamma^\mu \gamma^\nu \Upsilon_i - \frac{1}{2} M_i \overleftrightarrow{T} \Upsilon_i$$

for $i = 1, 2$ with real masses $M_i$. For the situation $M_L = 0$ and $M_R \gg M_D$ (taking $M_D$ real) one finds $M_1 \approx M_D^2/M_R$ and $M_2 \approx M_R$.

**Exercise**

In this exercise two limits are investigated for the two-Majorana case.

(a) Calculate for the special choice $M_L = M_R = 0$ and $M_D$ real, the mass eigenvalues and show that the mixing matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

which enables one to rewrite the Dirac field in terms of Majorana spinors. Give the explicit expressions that relate $\psi$ and $\psi^c$ with $\Upsilon_1$ and $\Upsilon_2$.

**Solution**

One finds $M_1 = M_2 = M_D$. For both left- and righthanded fields the relations between $\psi$, $\psi^c$ and $\Upsilon_1$ and $\Upsilon_2$ are the same,

$$\psi = \frac{1}{\sqrt{2}} (\Upsilon_1 + i \Upsilon_2), \quad \psi^c = \frac{1}{\sqrt{2}} (\Upsilon_1 - i \Upsilon_2).$$

(b) A more interesting situation is $0 = M_L < |M_D| \ll M_R$, which leads to the so-called seesaw mechanism. Calculate the eigenvalues $M_L = 0$ and $M_R = M_X$. Given that neutrino masses are of the order of $1/20$ eV, what is the mass $M_X$ if we take for $M_D$ the electroweak symmetry
breaking scale \( v \) (about 250 GeV).

(solution)

The eigenvalues are \( M_1 \approx M_2 D / \sqrt{2} \) and \( M_2 \approx M \). For a neutrino mass of the order of \( 1/20 \) eV, and a fermion mass of the order of the electroweak breaking scaling 250 GeV, this leads to \( M_X \sim 10^{15} \) GeV. The recoupling matrix in this case is

\[
U = \begin{pmatrix}
    i \cos \theta_S & -i \sin \theta_S \\
    \sin \theta_S & \cos \theta_S
\end{pmatrix},
\]

with \( \sin \theta_S \approx M_D / M_X \). The weak current couples to \( n_L = \sin \theta_S \Upsilon_2 - i \cos \theta_S \Upsilon_1 \), where \( \Upsilon_1 \) is the light neutrino (mass) eigenstate.