University of Groningen

Formal aspects of cosmological models: higher derivatives and non-linear realisations

Klein, Remko

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2018

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Klein, R. (2018). Formal aspects of cosmological models: higher derivatives and non-linear realisations. [Groningen]: University of Groningen.

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 08-02-2019

university of groningen

Formal aspects of cosmological models: higher derivatives and non-linear realisations

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. E. Sterken
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

Thursday 13 December 2018 at 11.00 hours

by

Remko Klein

born on 19 March 1990
in Groningen

Supervisors

Prof. D. Roest

Prof. E.A. Bergshoeff

Assessment Committee

Prof. H. Waalkens

Prof. J.W. van Holten

Prof. D. Langlois

This work is part of the research program of the Foundation for Fundamental Research on Matter (FOM), which is part of the Netherlands Organisation for Scientific Research (NWO). The work described in this thesis was performed at the Van Swinderen Institute for Particle Physics and Gravity of the University of Groningen.

ISBN: 978-94-034-1242-9 (printed version)

ISBN: 978-94-034-1241-2 (electronic version)

Printed by Grafimedia-RUG

Copyright © 2018 Remko Klein

Contents

Chapter 1

Introduction

This thesis is a collection and adaptation of the original work portrayed in [I-VI]. This research was mostly done with applications to cosmology, i.e. to the physics of the universe as a whole, in mind. To get an appreciation for the reasons behind the research, we will first give a brief introduction to modern cosmology. Along the way we will introduce and comment on open questions and problems.

Introduction to cosmology

Modern cosmology deals with the evolution of the universe, be it in the (distant) past, the present or the (distant) future. In particular one studies the properties of space-time itself, as well as its matter and energy content and their behavior, on the largest scales. Prior to the 20th century this mostly entailed trying to explain the motion of the heavenly bodies that were known at those times. The first models were largely geocentric, whereas in the 16th century this view shifted when Kepler, Copernicus and Galilei each considered heliocentric models. Although the models kept improving, an underlying principle that could explain all the observed motions was lacking. Only when Newton formulated his theory of gravity in 1697, did we have an elegant and universally applicable theory at our disposal that could largely solve the problem of the motion of the heavenly bodies.

Within Newton's theory of gravity, space is considered to be a fixed background against which all motion occurs and time is considered to be absolute and universal. With the arrival of Einstein and his two theories of relativity this paradigm radically shifted. Firstly, the special theory of relativity (SR) states that time is relative and together with space forms a combined, fixed, Minkowski space-time. Combining the principles of relativity with gravity led Einstein to formulate the theory of General Relativity (GR) with the startling conclusion that space-time is not fixed but actually dynamical and interacts in a nontrivial way with matter: matter causes curvature of space and the curvature of space influences the motion of matter.

Interestingly, GR implies that space as a whole can, under certain conditions, expand or contract. Although Einstein at first did not take this possibility seriously (much in contrast to for instance Friedmann and Lemaître) and believed the universe to be static (apart from local changes due to matter), he turned out to be wrong upon Hubble's observation of the redshift of galaxies. The only feasible way to explain that all galaxies move away from us with a velocity proportional to their distance, is by an expansion of space itself. Going back in time this would mean that the visable universe should at some earlier moment have been very small, leading people to formulate the Hot Big Bang model (HBB) in 1948. The HBB model postulates a very dense and hot early universe consisting of unbound elementary particles, that underwent gradual expansion and corresponding cooling that allowed for the subsequent formation of nucleons, nuclei and atoms (this process is known as Big Bang Nucleosynthesis). Larger structures would then gradually form under the effect of gravity as predicted by GR.

A key prediction of the HBB model is the existence of the Cosmic Microwave Background radiation (CMB) that was emitted at the moment of recombination of nuclei and electrons (which is a misleading term as they had never been bound before this moment) which turned the universe from being opaque to transparant. Indeed, with the (accidental) discovery of the CMB by Wilson and Penzias in 1964 the HBB model became widely accepted. Although very succesful, ever improving observations of large scale structures (LSS) and the CMB pointed out several shortcomings of the HBB model. In particular, the detailed properties of the LSS and CMB cannot be explained by just the visual matter content combined with gravitational interactions as described by GR. If one asssumes GR there must be a large amount of unseen mass, dubbed dark matter [66].

Another puzzling fact came with the observation that the expansion of the universe at present time is accelerating [147, 150], which cannot be explained only with visible and/or dark matter as these generally have a halting effect on the expansion. Rather, this leads to the introduction of dark energy that in contrast to matter actually drives the expansion. Amending the HBB models with two particular instances of exotic components, namely cold dark matter and a cosmological constant, leads to the current standard model of cosmology. In this so called Λ-CDM model the contributions of the different components to the energy density of the universe are approximately 5% for visible matter, 27% for dark matter and 68% for dark energy [4].

Phenomenologically it does a great job, but on a more fundamental level the Λ-CDM model has quite some difficulties. Several of these pertain to the need for fine-tuning. Firstly, the present day universe is observed to be very nearly flat. Since a flat universe is actually unstable, this implies that in the past the universe must have been even flatter. This requires a relative fine-tuning of the energy density of the order of 10^{-62}. Secondly, there is the homogeneity and isotropy of the universe, as observed from large scale structures and in particular the CMB which has a temperature of about 2.725 Kelvin all along the sky, with only very small anisotropies of the order 10^{-4} Kelvin [74]. However, within the Λ-CDM, background radiation coming from directions on the sky more than 2 degrees apart have never been in causal contact

and as such it is puzzling as to why they would have such similar temperatures.

These problems can be tackled via the concept of inflation [5, 84, 122], which postulates a period in the very early universe, between around $10^{-36} - 10^{-32}$ seconds after the Big Bang, during which the universe expanded at an exponentially accelerating rate. The existence of such a period implies an increase of the size of the causal horizons of points in the universe, such that they can extend well beyond the visible horizon today. As such, the whole visible universe actually has been in causal contact in the past thus explaining its homogeneity and isotropy. In addition, any curvature of the universe prior to inflation gets pushed to scales beyond the boundary of the visible universe, leading to an observed flatness. In order to comfortably solve the above problems the duration of the period of inflation should be sufficiently large. One usually parametrises this via the number of e-folds N, which measures how many factors of e the universe expanded during inflation, and the minimum number is around $N = 50$ to $N = 60$ depending on the details of the model.

Given the idea of inflation one needs to construct actual theories that can produce this period. The simplest option turns out to minimally couple a canonical scalar field to GR. If its potential is chosen appropriately it can act as a driving force for a sufficiently long period of inflation in the very early universe. Even better, it can also account for the detailed features of the CMB anisotropies not explained in the standard Λ-CDM model: microscopic quantum fluctuations get blown up to macroscopic length scales by the exponentially accelerating expansion, eventually leading to the anisotropies in the CMB. This can actually be done in a quantitative manner by doing cosmological perturbation theory around the homogeneous and isotropic Friedman-Robertson-Walker (FRW) universe with a constant value for the scalar field [133]. Two out of three types of perturbations turn out to be relevant, namely scalar and tensor perturbations (vector perturbations decay in an expanding universe), and by quantizing them via standard canonical quantization one can derive very distinct predictions regarding the detailed properties of the CMB.

Indeed, by using an appropriate transfer function (which takes into account a whole host of intermediate physical effects) one can calculate the effect of the quantum fluctuations generated during inflation to the observed CMB anisotropies. In particular one can relate the power spectra of the fluctuations to the power spectra of the temperature and polarisation anisotropies. To lowest order there are, apart from the amplitude A_s of the scalar perturbations, two quantities one can extract from the CMB. Firstly, there is the spectral index, n_s, which gives a measure of scale-invariance of the scalar perturbation power spectrum. Secondly, there is the tensor-to-scalar ratio, r, defined as the relative power of the scalar and tensor perturbations. Other higher order parameters (such as those parametrising non-gaussianities) can in principle be extracted from higher order correlators, but so far these have not been observed (e.g. the CMB is highly Gaussian). The most recent Planck satellite data gives the following constraints: $n_s = 0.965 \pm 0.004$ and $r \leqslant 0.07$ at the one σ-level [4], meaning a slightly redshifted scalar power spectrum and at most a small amount of tensor perturbations. So far no actual detection of tensor perturbations has been made and this remains an active goal for future observations.

The Λ-CDM model augmented with inflation still has explaining to do concerning dark energy and dark matter. If dark energy is a cosmological constant this leads to the problem of the huge discrepancy of 60 orders of magnitude between its observed value and that predicted by taking quantum contributions to the vacuum energy into account [168]. Another origin of dark energy might be a scalar field, which as we already noted can drive the expansion of the universe. For example, if the potential of a scalar field is sufficiently small at present times it could explain the present day acceleration [149]. Another way to possibly explain cosmic acceleration is by direct modifications of the gravitational interaction itself, for example by allowing for non-minimal couplings or adding a mass to the graviton. In addition, such modified gravity theories can also be used to at least partially mimick the effects that dark matter has in standard GR, thus partly adressing the mystery of dark matter (see f.e. [36]).

Ideally one would like to be able to construct a physically well-motivated fundamental theory valid at all energy scales, called an ultraviolet (UV) complete theory, that succesfully describes the cosmological phenomena we discussed. This is the top-down approach in which one starts from a UV complete theory that presumably has highly complicated dynamics and from it derive an effective field theory that only describes the dynamics of the degrees of freedom that are relevant at the energy scales of interest. This can be done by integrating out the degrees of freedom above some cutoff scale Λ. To say that it has proven to be quite difficult to construct feasible UV complete theories (that necessarily include gravity) from which one can extract definite predictions, is an understatement. Countless physicists have worked on the problem for many decades now, and numerous ideas and theories, such as string theory, asymptotic safety, holography, loop quantum gravity, and so on, have been proposed, but as of yet the matter has not been settled.

Given the difficulty of constructing feasible UV complete theories, one usually takes the complementary bottom-up approach. Here one remains agnostic about the exact form of the UV theory but only assumes it to exhibit particular symmetries. These play an important role in the construction of theories: they offer protection against quantum corrections, can reduce the number of arbitrary coupling constants thereby increasing predictivity and can render small symmetry breaking parameters technically natural. For these reasons, amongst others, gauge and global symmetries often appear in cosmological (and other types of) model building. By writing down all terms compatible with a certain field content and the presumed symmetries of the UV theory one can construct the most general effective field theory that could possibly be obtained from the UV.

An effective field theory has a natural expansion in terms of the inverse of the cutoff scale Λ at which the theory no longer gives a consistent description and breaks down. As such, terms in the Lagrangian are ordered by their mass dimension: those with higher mass dimensions are more heavily suppressed in comparison to those with low mass dimensions. One thus effectively expands the theory in terms of the total number of fields and derivatives. Given a fixed order of fields, terms with more derivatives will be suppressed. The modern viewpoint is that all succesful field theories so far

are effective field theories of some more fundamental UV theory. For example, both GR and the Standard Model of particle physics are viewed as the leading terms of an EFT.

As reflected in the title of this thesis, we will be mainly interested in the formal structures underlying two general aspects that are relevant for model building via both the top-down as well as the bottum-up approach: higher derivatives and non-linearly realised symmetries.

Higher derivatives

When constructing cosmological models, via either of the two approaches, one might be tempted to add higher derivative terms to be able to explain a wider range of physical effects beyond those achievable by first derivative terms alone, or one might be forced to do so due to the symmetries one assumes the theory has. However, one has to be wary of such terms, involving second or higher order time derivatives of the fields, because they will generically introduce instabilities to the theory. This traces back to the old theorem of Ostrogradsky [143, 171, 172]. This theorem implies that, in the absence of any degeneracies, i.e. constraints, a higher derivative theory will have additional degrees of freedom that are ghost like, both in the classical as well as the quantum theory. Classically these ghosts lead to problematic runaway behavior in the solutions, whereas quantum mechanically they lead to an unstable vacuum. Therefore, healthy higher derivative theories are necessarily degenerate, i.e. they are constrained systems. Perhaps the best known higher derivative theory is GR itself: the Einstein-Hilbert term contains second derivatives of the metric. Nevertheless due to its many degeneracies it evades Ostrogradky's theorem and it is known to be healthy, but when considering additions and modifications one should be careful not to spoil the degeneracy and thereby introducing ghosts.

Given the problems Ostrogradsky ghosts introduce to a theory, any UV complete theory should be free of them. When dealing with an effective field theory this is not necessarily the case. The reason is that the ghost can be massive and thus only accessable from some energy scale onward. As long as this scale is beyond the intended range of validity of the EFT the eventual emergence of a ghost is in principle not problematic as the theory is expected to break down anyway. The viewpoint is then that this ghost is merely an artifact of dealing with an effective theory valid up to some finite scale, but the correct UV completion should be free of ghosts. In any case it is very interesting to investigate what, given a particular field content, the most general benign interactions including higher derivatives are that one can write down. Several aspects of healthy higher derivative theories are known. For example, in the simple example of a mechanical system with a single variable, it can be seen that any degenerate higher derivative theory amounts to an ordinary and thus healthy theory, with at most first derivatives in the action, up to an irrelevant total derivative. Such higher derivative theories are therefore trivial.

The first step beyond trivial higher derivatives regards field theories and a prime

example is (generalized) Galileon theories, consisting of a single scalar field with Lorentz invariant higher derivative interactions [54, 136]. The generalization for the spin-2 tensor to arbitrary dimensions leads to Lovelock gravity with specific R^n interactions [125], which in $D = 4$ corresponds to standard GR with a cosmological constant, i.e. $R + \Lambda$. In these examples the interactions have been chosen such that they still lead to second order field equations (as opposed to them being of the expected fourth order), meaning they are degenerate and evade the Ostrogradsky theorem. This can be understood by the observation that the higher derivative interactions can be packaged into a first order Lagrangian plus a total derivative, similar to the mechanics case; however, this ordinary Lagrangian cannot be written in a manifestly Lorentz invariant form. This trade off between manifest first order Lagrangians and manifestly Lorentz invariant Lagrangians (and the impossibility to have both) will be a recurring theme.

We note that in general one has to be careful: having second order field equations, following from a first order or higher order Lagrangian, does not guarantee the absence of additional ghosts and thus additional conditions might be necessary. In fact, in some cases such additional ghosts are actually interpretable as Ostrogradsky ghosts upon using a different field basis to describe the theory. Two well-known examples arise in the context of massive gravity [49,50] where generically the (in)famous Boulware-Deser ghost emerges [20], and vector theories [86] where the degree of freedom corresponding to the time component of the vector is a ghost; even though in their standard formulation the theories are first order, the Ostrogradsky nature of the ghosts becomes clear upon employing the Stückelberg mechanism.

A second generalization concerns coupled systems with multiple variables or fields, which as noted are particularly interesting with regards to model building for cosmology. Indeed, the last few years have seen a growing interest in such higher derivative theories with second or higher derivatives in the action. Similar to the case with a single variable, for many years the community only trusted a very special subset of these theories, namely the ones giving second order field equations while (erroneously) assuming that all the others are plagued by instabilities. For instance, the most general scalar-tensor theories with second order field equations are those of Horndeski [96], which coincide [116] with covariantized generalized Galileons [56, 57]. Similarly, covariant vector Galileons describe such couplings between a vector and tensor [86, 97, 164]. Very recently this was generalized to covariant tensor Galileons for the couplings between different tensors [32].

Only recently it has been realised that one can have healthy degenerate higher derivative theories even in the presence of higher order field equations, with the proposal of beyond Horndeski models [78,79,173]. These models have been further understood and generalised in [15,43,44,52,58,69,120,121] and now a complete classification for degenerate scalar-tensor theories within a certain Ansatz exists [14]. Analogously, similar constructions for vector interactions were introduced in [88] and a classification for degenerate vector-tensor theories (up to quadratic order) was given in [113]. A central theme of these constructions is the coupling between a higher derivative degree of freedom and a healthy first order one. In the above examples, these are a

scalar and a tensor or a vector and a tensor, respectively.

Although many examples of theories with such an interplay between higher derivative and healthy sectors have thus been constructed, a generally applicable analysis has been lacking so far. Furthermore, there is the question to what extent healthy higher derivative theories truly go beyond the first order ansatz. We have already noted that many can be rewritten in a manifestly first order form via a total derivative, albeit one possibly not compatible with manifest symmetries. A more complicated possibility is that theories are related via redefinitions of different types. Amongst these are the ordinary field redefinitions, but also the more general point transformations mixing fields and coordinates, and transformations that in addition involve derivatives of the fields. Indeed, the earliest examples of beyond Horndeski theories [173], are actually related to Horndeski via disformal transformations of the metric involving first derivatives of the scalar field.

It would be interesting to know what the more formal structures underlying the set of healthy higher derivative theories are. A better understanding can help one in constructing new and potentially interesting healthy higher derivative theories, be it with applications to cosmology or other areas of physics in mind. In this thesis we provide a first step in such an analysis, deriving general degeneracy conditions as well as examining the role of different types of redefinitions.

Non-linear realisations

It is natural to expect that somewhere along the line of going from high to low energies part of the symmetries of the UV theory are spontaneously broken because one will be effectively expanding around a solution that does not respect the symmetry. Such spontaneously broken symmetries, be them internal or space-time, are described by non-linear realisations; i.e. the transformation rules are non-linear. In relativistic theories, whenever a particular internal symmetry gets spontaneously broken, Goldstone's theorem states that an associated massless field emerges. These Goldstones are required for any non-linear realisation: any set of fields on which an internal symmetry is non-linearly realised must contain these Goldstones. Additionally they decouple from other fields in the low energy limit and as such the low energy effective theory will be dominated by the dynamics of these massless Goldstones. This is reflected group theoretically in the fact that the symmetry group can be consistently non-linearly realised purely on the Goldstones.

Given the above it is natural to consider effective field theories where the fields are interpreted as the Goldstone modes of spontaneously broken symmetries. For an internal symmetry group G which is spontaneously broken to a subgroup H, the tools to construct the non-linear realisation of the group G and accompanying invariant Lagrangians were developed by Callan, Coleman, Wess and Zumino (CCWZ) in the late 1960's [27, 38]. In this coset construction there is a single Goldstone boson for each broken generator and the dynamics of the Goldstones is dictated by the coset space G/H. Moreover, for compact, semi-simple groups, it has been proven that

all non-linear realisations of such a spontaneously broken symmetry are related by invertible field redefinitions, and as a consequence can be derived from the coset construction. This in turn guarantees, given a particular symmetry breaking pattern, the universality of all corresponding observables.

The generalisation of the coset construction of CCWZ to spontaneously broken spacetime symmetries came a few years later [99, 167] and has been used extensively in the context of constructing and understanding effective field theories used for model building in cosmology. Two notable examples are the scalar sector of the d-dimensional DBI Lagrangian which non-linearly realises the $(d + 1)$-dimensional Poincaré group, see e.g. [81], and the Volkov-Akulov Lagrangian which non-linearly realises supersymmetry with a single fermion [166]. Both of these theories, and their higher order corrections, can be derived using the coset construction. Complimentary methods include the study of hypersurfaces fluctuating in transverse directions, e.g. [53, 95, 100, 163], and the study of soft limits of general scattering amplitudes, e.g. [35, 101, 146]. See also [134] for a discussion on spontaneous breaking of spacetime symmetries in condensed matter systems, [60, 135] for a discussion on the coset construction for superfluids etc and [11, 33, 82, 93] for more examples related to cosmology and gravity.

The coset construction for spacetime symmetries involves added subtleties compared to the case of internal symmetries because Goldstone's theorem no longer applies. In many cases there is a distinction between the Goldstone modes corresponding to all broken generators: some Goldstones acquire a mass gap, whereas others remain massless. As a result the massive Goldstones can be integrated out of the EFT and in this way one obtains a new low energy EFT only involving the massless Goldstones which is valid up to the mass scale of the massive ones. In that sense the massive Goldstones are *inessential* to the particular symmetry breaking pattern; the massless Goldstones really are *essential*. In a restricted class of symmetry breaking patterns it can be shown that there is an induced consistent non-linear realisation of the symmetry group on the essential Goldstones alone, but in the general case this is not apparant (see also the next paragraph). A very clear example of the possible mismatch between broken generators and essential Goldstones is the conformal group in four dimensions spontaneously broken to its four dimensional Poincaré subgroup [98]. There are five broken generators yet a consistent non-linear realisation exists with a single Goldstone field, the dilaton, while the vector of the broken special conformal transformations is inessential.

Although in all scenarios with inessential modes one can integrate them out of the EFT, it is only in a particular class of theories one can potentially also eliminate them at the coset construction level by means of covariant constraints that allow one to algebraically express the inessentials in terms of the essential modes and their derivatives. This ensures a consistent non-linear realisation on the essentials alone, and allows one to systematically construct EFTs valid to all orders for the essentials. The canonical type constraints are the inverse Higgs constraints [99] that have a direct relation to the building blocks of the coset construction, but there is also the possibility of more general constraints that could for example arise as algebraic

equations of motion.

The existence of essential and inessential Goldstones complicates the universality question for space-time symmetries, already within the coset construction itself. Firstly, in many cases the different possibilities of elimination lead to equivalent EFTs for the essential Goldstones, but it is unclear whether this is always the case. Secondly, the possibility to inverse Higgs, at least via the canonical method, depends on the chosen coset parametrisation, i.e. on the chosen field basis. Now, it is often stated in the literature that one can inverse Higgs in the canonical way if a certain condition on the structure constants of the algebra is satisfied. However, it turns out that this is in fact not true and in general a series of conditions needs to be met rather than a single one. Importantly, this series of conditions depends on the chosen coset parameterisation. Indeed, already the very simple case of spontaneous breaking of the d-dimensional Poincaré group down to its $(d-1)$-dimensional subgroup illustrates this: the standard parametrisation considered in the original work [99, 167] is not the optimum one in this regard.

Also, prior to imposing inverse Higgs constraints, the relationship between different paramerisations is straightforward and involves transformations between the coset coordinates, which for spontaneously broken spacetime symmetries includes the spacetime coordinates and the fields. These are point transformations, and are the natural generalisation of field redefinitions in the internal case. However, as we will see, the construction of possible transformations becomes much more complicated after we impose inverse Higgs constraints, since the constraints are not necessarily mapped onto each other under the point transformations. This implies that there is not always a naturally induced mapping between two parametrisations after inverse Higgsing, and as a consequence it is unclear if equivalence is maintained.

These open questions aside, the coset construction (for both internal and spacetime symmetries) is a very powerful tool in constructing interesting theories, cosmological and otherwise. One such application is in the construction of inflationary models with non-linearly realised symmetries in the kinetic sector but whose potential weakly break it so as to be able to realise an inflationary phase. As we shall see, this provides a useful way of characterising kinetic sectors for scalar field theories and we note that this has been considered before in the context of inflation in [26] and to classify condensed matter systems in e.g. [134]. The simplest example of such a scenario is realised by single field monomial inflation [123]. Here the scalar's canonical kinetic term is invariant under a shift symmetry which is broken by the potential energy $V = \lambda\phi^m$ with integer $m \geqslant 2$, providing a very simple realisation of inflation by a symmetry breaking potential. The symmetry breaking parameter λ is constrained to be very small, in Planck units, from the observed level of CMB temperature anisotropies and this is a technically natural scenario, meaning that this choice is not spoiled by (perturbative) quantum corrections thanks to the approximate shift symmetry [124].

However, these very simple inflationary models predict large values for the tensor-to-scalar ratio r and have been ruled out by CMB polarisation observations [2–4]. This

motivates one to investigate slightly more complicated inflationary models which can reduce the value of r without spoiling the radiative stability of the theory by allowing the scalar potential to break more complicated non-linear symmetries rather than a simple shift. This will require one to construct kinetic sectors with more scalar fields and, as we shall see, these can have interesting observational effects consistent with the current data.

Outline of the thesis

The first two chapters are introductory. In Chapter 2 we will give a thorough review of the basics of Lagrangian physics. In particular we focus on the general theory involving higher derivatives and discuss the relevance of redefinitions and symmetries in this context. In Chapter 3 we will discuss how to examine the dynamical content of theories in more detail and introduce the Hamiltonian formalism. We discuss the appearance of generic ghosts in non-degenerate higher derivative theories, and introduce two algorithms essential in examining degenerate theories. In Chapter 4 we apply these algorithms to very general classes of higher derivative theories (without gauge symmetries) and is largely original work [II,III]. This will result in general degeneracy conditions needed to ensure the absence of Ostrogradsky ghosts. Also a classification of healthy theories is given and their relation via different types of redefinitions is examined.

In Chapter 5 we switch gears and turn to non-linear realisations. It is mostly an adaption of [IV], and after giving a thorough review of the coset construction, we will examine the intricacies of non-linearly realised space-time symmetries as induced by the existence of inverse Higgs constraints. In particular we examine the universality question by considering different parametrisations and their possible relations, both prior and post inverse Higgs, as well as the role of different types of redefinitions. In Chapter 6, which is based on [V], we apply the coset construction to give a classification of inflationary models based on non-linearly realised symmetries of the kinetic sector. In particular we construct a novel class of models based on a Minkowski 3-brane fluctuating in an anti-de-Sitter ambient space, which gives universal predictions compatible with the current CMB data. We end with conclusions and an outlook.

Note: throughout this thesis we will use Planck units, i.e. we set $c = \hbar = G = 1$, unless stated otherwise.

Chapter 2

Lagrangian theory

In this chapter we will review some of the key aspects of classical Lagrangian physics. We set the stage by defining all the relevant objects such as the underlying space-time, the dynamical fields, as well as the action and corresponding Lagrangian. With applications to higher derivative theories in mind we consider arbitrary Lagrangians depending on derivatives of the fields up to some finite order n. We then discuss the principle of stationary action, show how to derive its dynamical consequence namely the equation of motion, and introduce the concept of degrees of freedom. We leave the discussion of analysing the dynamics of theories and their degrees of freedom in more detail to Chapter 3, where we will also introduce the complimentary Hamiltionian formalism.

We then introduce the concept of equivalence between different equations of motion as well as Lagrangians. We will in particular focus on the possibility of performing redefinitions of the variables without affecting the dynamical content of the theory. This includes the familiar and often used changes of space-time coordinates as well as standard field redefinitions. However, there is also the possibility to consider more general redefinitions mixing both space-time coordinates and the fields, as well as their derivatives, in a consistent manner. These so called contact, or more generally, Lie-Bäcklund transformations will be of particular interest for the rest of this thesis when examining the class of healthy higher derivative theories in Chapter 4 as well as the universality of non-linear realisations of space-time symmetries in Chapter 5.

We then discuss different types of variational symmetries, i.e. transformations of the space-time coordinates and fields that leave the action invariant up to some boundary term. Most well known in physics are the standard symmetry transformations only mixing coordinates and fields, which include standard space-time symmetries and internal symmetries. However, as in the case of redefinitions, one can consider more general transformations involving not only the coordinates and fields but also their derivatives up to some arbitrary order. These generalised/Lie-Bäcklund symmetries, like ordinary symmetries, will (assuming they are global and continuous) lead to

conserved currents and corresponding charges. We will also consider local and gauge symmetries and discuss their implications regarding the equations of motion. Finally, we discuss the breaking of symmetries, both explicitly and spontaneously, and their relation to non-linear realisations in detail.

Along the way we will introduce several interesting healthy higher derivative theories already mentioned in the introduction, including but not restricted to Galileons, Lovelock gravity and Horndeski's theory, and gradually discuss several of their formal properties.

Much of the basics on Lagrangian physics covered in this chapter can be found in for example [141, 142].

2.1 Fields, Lagrangians and equations of motion

Developing the Lagrangian formalism starts with picking a space-time, which one usually takes to be an arbitrary smooth D-dimensional manifold M of a certain signature depending on the case at hand. Now, throughout this thesis we will be mainly interested in local dynamics and will thus ignore the global topological structure of the space-time manifold. As such, for our purposes we can consider a local space-time with corresponding local coordinates:

$$M \simeq \mathbb{R}^d, \qquad x = (x_1, \ldots, x_d)\,. \tag{2.1}$$

For now we do not specify the signature since it will not be important for what is to follow in this chapter.

Next one defines the fields whose dynamics one wants to describe. To be able to properly do so we first define the space U of dependent variables, u, in which the dynamical fields will take values. Throughout this thesis we will be working with fields whose values can be real, complex or Grasmannian. For definiteness we will now consider the fields to be real valued, but the generalisation of what is to follow to other field values should be obvious. If the number of field components is m then the corresponding space is given by

$$U \simeq \mathbb{R}^m, \qquad u = (u_1, \ldots, u_m)\,, \tag{2.2}$$

and the fields are simply functions $\phi : M \to U$. An alternative and useful description of the fields is obtained by identifying them with graphs or sections $\Gamma_\phi = (x, \phi(x)) \subset J^{(0)} = M \times U$. Since we will also deal with the derivatives of the fields, it is worthwhile to consider the spaces in which the $m \times \binom{d+n-1}{n}$ partial derivatives of order n take value. For any order $n \geq 0$ they are given by

$$U^{(n)} \simeq \mathbb{R}^{m\binom{D+n-1}{n}}, \qquad u^{(n)} = (u_{x_{i_1}\ldots x_{i_n}})\,, \tag{2.3}$$

and the n-th order derivative of a field is then the corresponding graph $(x, \phi^{(n)}(x)) \subset M \times U^{(n)}$. However, rather than considering fields and derivatives separately it is

more convenient to describe them as combined graphs in the so called jet spaces. The n-th order jet space is defined as

$$J^{(n)} = M \times U \times U^{(1)} \times \ldots \times U^{(n)}, \qquad (x, u, u^{(1)}, \ldots, u^{(n)})\,, \tag{2.4}$$

and thus one can treat the space-time coordinates, fields and the derivatives up to order n in one go by considering graphs $\Gamma_\phi^{(n)} = (x, \phi(x), \ldots, \phi^{(n)}(x)) \subset J^{(n)}$. In this way one can define any n-th order Lagrangian density, i.e. one depending on at most n-th order derivatives, as a real valued function on this space, i.e.

$$\mathcal{L} : J^{(n)} \to \mathbb{R}, \qquad \mathcal{L}(x, u, u^{(1)}, \ldots, u^{(n)})\,. \tag{2.5}$$

Assuming a splitting of space and time, i.e. $D = d + 1$, one defines the Lagrangian as the space integral of the Lagrangian density

$$L = \int d^d x \mathcal{L}\,. \tag{2.6}$$

If one wants to describe all Lagrangians in one go, which we would like to do since we will be trying to relate Lagrangians of different orders to each other, one should make the further generalisation to the limiting infinite order jet space where $n \to \infty$:

$$J^{(\infty)} = M \times U \times U^{(1)} \times \ldots, \qquad (x, u, u^{(1)}, \ldots) \tag{2.7}$$

Any Lagrangian density of some finite order, as well as any depending on infinitely many derivatives, can be viewed as a function on this space:

$$\mathcal{L} : J^{(\infty)} \to \mathbb{R}, \qquad \mathcal{L}(x, u, u^{(1)}, \ldots) \tag{2.8}$$

From now on we will refer to Lagrangians of finite order as being *local*, and those of infinite order as being *non-local*. We will almost exclusively focus on local Lagrangians throughout this thesis.

Having defined all the relevant objects, we can suitably define the action corresponding to some Lagrangian of order n. To properly define the physical scenario one must choose a subspace $\Omega \subseteq M$, pick the class of fields one would like to consider on it as well as the boundary conditions the fields and their relevant derivatives should satisfy on $\partial\Omega$. The corresponding action is then defined as the functional

$$S[\phi] = \int_\Omega d^D x \mathcal{L}(x, \phi(x), ..., \phi^{(n)}(x)) = \int dt L\,, \tag{2.9}$$

where the Lagrangian is thus of course evaluated on the graphs of the infinite jet space corresponding to the chosen class of fields satisfying the boundary conditions. In practice one usually leaves these arbitrary and worries about them later. To derive the classical dynamics contained in this action one invokes the *principle of stationary action* which states that out of all configurations, those that will actually occur in nature are stationary points of the action. A configuration $\phi(x)$ is a stationary point

precisely when any perturbation around $\phi(x)$ gives a vanishing leading order contribution to the action. Thus consider some configuration $\phi(x)$ and consider perturbations around it, parametrised by some continuous infinitesimal parameter ϵ. Consistency with the setup demands compatibility with the chosen boundary conditions, i.e.

$$\phi(x,\epsilon) \equiv \phi(x) + \epsilon\delta\phi(x), \qquad \delta\phi|_{\partial\Omega} = 0\,. \tag{2.10}$$

It is easy to see that these variations induce corresponding variations in the derivatives

$$\begin{aligned}\phi^{(n)}(x,\epsilon) &\equiv (\phi(x,\epsilon))^{(n)} \\ &= \phi^{(n)}(x) + \epsilon(\delta\phi(x))^{(n)}, \qquad \text{i.e. } \delta\phi^{(n)} = (\delta\phi)^{(n)}\,,\end{aligned} \tag{2.11}$$

as well as any function $\mathcal{L}(x,\phi,\phi^{(1)},\ldots,\phi^{(n)})$:

$$\delta\mathcal{L} = \mathcal{L}_\phi\delta\phi + \ldots + \mathcal{L}_{\phi^{(n)}}\delta\phi^{(n)}\,. \tag{2.12}$$

Then, the leading order contribution to the action of this variation can be easily calculated:

$$\begin{aligned}\delta S &= \frac{d}{d\epsilon}S[x,\phi(x,\epsilon)]|_{\epsilon=0} = \int_\Omega \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(x,\phi(x,\epsilon),...,\phi^{(n)}(x,\epsilon))|_{\epsilon=0}dx \\ &= \int_\Omega \sum_{k=0}^{n} \mathcal{L}_{\phi^{(k)}}\delta\phi^{(k)}dx = \int_\Omega \Big(\sum_{k=0}^{n}(-\frac{d}{dx})^k\mathcal{L}_{\phi^{(k)}}\delta\phi + \nabla\cdot\mathcal{K}\Big)dx \\ &= \int_\Omega \sum_{k=0}^{n}(-\frac{d}{dx})^k\mathcal{L}_{\phi^{(k)}}\delta\phi dx + \int_{\partial\Omega}\mathcal{K}\cdot dA \qquad (dA = \text{ surface element}) \\ &= \int_\Omega \sum_{k=0}^{n}(-\frac{d}{dx})^k\mathcal{L}_{\phi^{(k)}}\delta\phi dx\,,\end{aligned} \tag{2.13}$$

where we defined

$$\mathcal{K} = \sum_{k=0}^{n}\sum_{i=0}^{k-1}(-\frac{d}{dx})^i\mathcal{L}_{\phi^{(k)}}\delta\phi^{(k-1-i)}\,, \tag{2.14}$$

and used that since it is linear in the variations which vanish at the boundary, integrating it over the boundary will yield zero. Since the variations are otherwise arbitrary, demanding that the variation of the action vanishes implies that for physical configurations the combination

$$E_L(\phi) = \sum_{k=0}^{n}(-\frac{d}{dx})^k L_{\phi^{(k)}}\,, \tag{2.15}$$

must vanish. These differential equation are the Euler-Lagrange equations also called the *equations of motion* of the theory and they are the direct dynamical consequence of the principle of stationary action. Thus any physical configuration should be a

solution to these equations of motion, which for an n-th order Lagrangian depending on m fields is a system of m partial differential equations of at most order $2n$:

$$\begin{aligned} E_L(\phi) &= (-1)^n L_{\phi^{(n)}\phi^{(n)}} \phi^{(2n)} + \dots \\ &= f(x, \phi, \dots, \phi^{(n)}) \phi^{(2n)} + g(x, \phi, \dots, \phi^{(2n-1)}) \,. \end{aligned} \tag{2.16}$$

Depending on the detailed properties of the Lagrangian, those of the equations of motion and thus also those of the corresponding physical solutions vary greatly. For example the solutions could be stable and well-behaved or, as we will see in the next chapter, unstable and problematic. Additionally, there can be a difference in the number of initial conditions (given consistent spatial boundary conditions) one needs to fix in order to uniquely specify a solution to the system, which is a direct measure of the amount of freedom in the theory. Usually one refers to half this number of initial conditions as the number of *degrees of freedom* in the theory, corresponding to the number of pairs of phase-space variables in the Hamiltonian description that we will discuss in the next chapter. The same definition holds for individual fields, i.e. if one needs to specify a number of initial conditions pertaining to a particular field it is said to describe half as many degrees of freedom. Generically the more fields and the higher the order of the Lagrangian, the more degrees of freedom are present in a theory. We note that the number of degrees of freedom in a theory is not necessarily an integer, although in large classes of theories such as mechanical systems and Lorentz invariant field theories not involving Grasmannian variables this is actually the case (see also Chapter 4).

There are two classes of Lagrangians that one can distinguish, namely the *non-degenerate* and *degenerate* ones (also called regular and singular respectively). The simplest Lagrangians are the non-degenerate ones which are precisely those theories for which the equations of motion are all fully independent and of maximal order in derivatives. In this case it is straightforward to determine the number of degrees of freedom which can then be directly read of from the order of the Lagrangian. However, many physically interesting theories are degenerate and in these cases there are combinations of equations of motion that are lower order, called *constraint equations*, or in the extreme case identically vanishing, called *gauge identities*. Their appearance signals relations between initial conditions or redundancies in the description respectively. In both cases the number of degrees of freedom is less compared to a non-degenerate theory of the same order and one has to do considerable work to determine the precise number. We will discuss the differences between non-degenerate and degenerate theories as well as the counting and generic properties of their degrees of freedom in much more detail in the next chapter.

Example: Galileons Consider a Lorentz invariant theory of a single scalar field. A generic second order theory of this kind will have a fourth order equation of motion,

$$E_\phi = \partial_\mu \partial_\nu \mathcal{L}_{\partial_\mu \partial_\nu \phi} - \partial_\mu \mathcal{L}_{\partial_\mu \phi} + \mathcal{L}_\phi \propto \mathcal{L}_{\partial_\mu \partial_\nu \phi \partial_\rho \partial_\sigma \phi} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi + \dots \,, \tag{2.17}$$

and describes two degrees of freedom and, as we will extensively discuss in the next chapter, one of these implies unstable behavior. This can be avoided if the theory is degenerate which opens up the possibility of an equation of motion that is actually second order such that only one degree of freedom is present. The most general class of manifestly Lorentz invariant theories of a single scalar field whose Lagrangian is second order and whose equation of motion is also second order is that of the *generalised Galileons* [57], which are a generalisation of the Galileons [136]. The set consists of several terms with an increasing number of second derivatives. The lowest order term is first order, i.e. $\mathcal{L}_0 = f_0(\phi, X)$, whereas the subsequent i-th order terms are given by:

$$\mathcal{L}_i = f_i(\phi, X)\delta^{\mu_1\cdots\mu_i}_{\nu_1\ldots\nu_i}\partial_{\mu_1}\partial^{\nu_1}\phi\ldots\partial_{\mu_i}\partial^{\nu_i}\phi\,, \tag{2.18}$$

where $f_i(\phi, X)$ are free functions and $\delta^{\mu_1\cdots\mu_i}_{\nu_1\ldots\nu_i} = i!\delta^{\nu_1}_{[\mu_1}\cdots\delta^{\nu_i}_{\mu_i]}$ is the i-th order generalised Kronecker delta symbol. In D dimensions only the first D terms are non-vanishing, and the D-th order term is actually equivalent to a linear sum of the lower order ones up to a total derivative. Thus in D dimensions the most general theory is given by:

$$\mathcal{L} = \sum_{i=0}^{D-1}\mathcal{L}_i\,, \tag{2.19}$$

and contains D freely specifiable functions. One can easily see that the equations of motion are second order due to the antisymmetric structure with which the second order derivatives enter the Lagrangian. Indeed, any potential third or fourth order derivative terms will come with either at least two μ or at least two ν indices contracted with the antisymmetric generalised Kronecker delta symbol and will thus vanish identically. We note that generically the second order derivatives enter the equations of motion nonlinearly, in contrast to the case of first order Lagrangians where they occur linearly. There are also generalisations to multiple coupled scalar fields called multi-Galileons [7, 55, 95, 144, 158] and these rely on the same antisymmetric structure:

$$\mathcal{L} = f_0(\phi_m, X_{mn}) + \sum_{i=1}^{D-1} f_i^{m_1\ldots m_i}(\phi_m, X_{mn})\delta^{\mu_1\cdots\mu_i}_{\nu_1\ldots\nu_i}\partial_{\mu_1}\partial^{\nu_1}\phi_{m_1}\ldots\partial_{\mu_i}\partial^{\nu_i}\phi_{m_i}\,, \tag{2.20}$$

where $X_{mn} = \partial_\mu\phi_m\partial^\mu f_n$. The fourth order derivatives drop out of the equations of motion directly due to the antisymmetric structure, whereas the third order derivatives vanish only if the functions $f_i^{m_1\ldots m_i}$ satisfy certain symmetry properties [7,158].

Example: GR and Lovelock. A generic diffeomorphism invariant gravity theory depending algebraically on the Riemann tensor, i.e. $\mathcal{L} = \sqrt{-g}f(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$, will have fourth order equations of motion,

$$E^{\mu\nu} = -2\nabla_\rho\nabla_\sigma\frac{\partial f}{\partial R_{\rho(\mu\nu)\sigma}} + R^{(\mu}_{\lambda\rho\sigma}\frac{\partial f}{\partial R_{\nu)\lambda\rho\sigma}} - \frac{1}{2}fg^{\mu\nu}\,, \tag{2.21}$$

again leading to instabilities. The most general subclass of such theories for which the third and fourth order terms drop out and thereby evades the instabilities, is that of Lovelock gravity [125]. Such theories have a similar structure to generalised Galileons but now the terms increase in the number of Riemann tensors involved:

$$\mathcal{L}_0 = \sqrt{-g}\Lambda_0, \qquad \mathcal{L}_i = \sqrt{-g}\Lambda_i \delta^{\mu_1 \dots \mu_{2i}}_{\nu_1 \dots \nu_{2i}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \dots R^{\nu_{2i-1} \nu_{2i}}_{\mu_{2i-1} \mu_{2i}} . \tag{2.22}$$

In D-dimensions only the first $[(D-1)/2]+1$ Lovelock terms contribute: in even dimensions the $([(D+1)/2]+1)$-th term is a total derivative and subsequent terms vanish identically, whereas in odd dimensions all higher order terms identically vanish. Therefore the most general Lovelock theory in D-dimensions is

$$\mathcal{L} = \sum_{i=0}^{[(D+1)/2]} \mathcal{L}_i \,, \tag{2.23}$$

which contains $[(D+1)/2]$ free parameters (but no free functions due to the nonexistence of diffeomorphism invariants involving at most first derivatives of the metric). The fact that one new term arises per two extra dimensions, in contrast to a new term for each extra dimension for the Galileons, is a direct consequence of the fact that the Riemann tensor is a four index object whereas the second derivative of the scalar is a two index object. Similarly to generalised Galileons, the antisymmetric structure is essential in ensuring the absence of higher than second order derivatives, i.e. derivatives of the Riemann tensor, in the equations of motion. In four dimensions the most general Lovelock theory is simply General Relativity with a cosmological constant, i.e. $\mathcal{L} = \sqrt{-g}(\Lambda + R)$ (and indeed the quadratic Gauss-Bonnet term is a total derivative in four dimensions).

Example: Horndeski and covariantised Galileons. As already noted in the introduction, scalar-tensor theories are often used in modelling cosmological phenomena, see f.e. [36]. A generic diffeomorphism invariant theory, i.e.

$$\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi) \,,$$

will have fourth order equations of motion for both the metric and the scalar, again resulting in problematic behavior. Therefore a lot of research has been done within the setup of Horndeski's theory [96], which is the most general diffeomorphism invariant second order theory in four dimensions involving a metric and a scalar, but nevertheless yielding second order equations of motion. This theory can be obtained from the generalised Galileons by covariantising them and adding suitable gravitational counterterms. These counterterms are necessary to ensure second order field equations because minimally covariantising introduces higher order terms. Any generalised Galileon term necessitates a string of counterterms. This process can actually be done in arbitrary dimension D and in general the i-th generalised Galileon term

correctly covariantised is [54]:

$$\mathcal{L}_i[f_i] = \sqrt{-g}\sum_{p=0}^{[i/2]} f_{i,p}(\phi, X)\delta^{\mu_1\ldots\mu_i}_{\nu_1\ldots\nu_i} R^{\nu_1\nu_2}_{\mu_1\mu_2}\ldots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} \nabla_{\mu_{2p+1}}\nabla^{\nu_{2p+1}}\phi\ldots\nabla_{\mu_i}\nabla^{\nu_i}\phi\,, \tag{2.24}$$

where $f_{i,p-1} \propto (f_{i,p})_X$. The most general theory is then a linear combination:

$$\mathcal{L} = \sqrt{-g} f_0(\phi, X) + \sum_{i=1}^{D-1} \mathcal{L}_i[f_i]\,. \tag{2.25}$$

In dimensions other than 4 it has not been shown that these are the most general theories leading to second order equations of motion, however the fact that generalised Galileons as well as Lovelock are the most general in any dimension combined with the universal construction of the covariantised Galileons suggest this to nevertheless be the case. Multi-Galileons have also been properly covariantised in a similar fashion [158]. Here a proof of generality is also lacking.

2.2 Equivalence and redefinitions

An important aspect of Lagrangian field theory is that two seemingly different theories might actually describe the same dynamics because their equations of motion are *equivalent*. To properly discuss this equivalence, we note that the set of solutions to a given system of equations of motion is a particular subset of the sections of the infinite jet space, i.e

$$Sol(E_L(x, \phi)) \subset \Gamma^\infty \subset J^\infty\,. \tag{2.26}$$

Given a coordinate system $(x, \phi, \ldots)$ it is natural to call two equations of motion, as well as their respective Lagrangians L and $\bar{L}$, equivalent if their sets of solutions are the same:

$$Sol(E_L(x, \phi)) = Sol(E_{\bar{L}}(x, \phi)), \qquad \text{i.e.} \qquad E_L(x, \phi) = 0 \Leftrightarrow E_{\bar{L}}(x, \phi) = 0\,. \tag{2.27}$$

The simplest scenario is that two Lagrangian densities give *exactly* the same equations of motion

$$E_L(x, \phi) = E_{\bar{L}}(x, \phi)\,, \tag{2.28}$$

which is the case if and only if they differ by a total divergence, i.e $\mathcal{L}' = \mathcal{L} + \nabla\mathcal{M}$ (which follows directly from the fact that the only Lagrangians that give identically vanishing equations of motion are total divergences, i.e. $\mathcal{L} = \nabla\mathcal{M}$). More generally, two equations of motion can be different to each other, i.e. $E_L(x, \phi) \neq E_{\bar{L}}(x, \phi)$, but nevertheless share the same solutions. As a trivial example, consider two Lagrangians

that differ by some overall constant factor: their equations of motion also differ by a constant factor, but clearly they have the same solutions.

So far we have chosen a single coordinate system, $(x, \phi, \ldots)$, and written and compared two theories with respect to this system. That is, we have given the same interpretation to the coordinates of one Lagrangian to those of the other. However, in certain scenarios one should take into account that this might not be the case (whether one realises this a priori or not). In other words, it might be that the two Lagrangians are actually written in terms of different explicit coordinate systems for J^∞, $(x, \phi, \ldots)$ and $(\bar{x}, \bar{\phi}, \ldots)$ respectively, that have different interpretations or whose interpretation a priori is not fixed. In this scenario, the natural definition of equivalence involves the possibility of performing redefinitions of the variables, which are nothing but diffeomorphisms on the jet space:

$$\mathcal{F} : J^\infty \to J^\infty, \qquad \mathcal{F}(x, \phi, \ldots) = (\bar{x}, \bar{\phi}, \ldots) \,. \tag{2.29}$$

Armed with these, we call two sets of equations of motion, $E_L(x, \phi)$ and $E_{\bar{L}}(\bar{x}, \bar{\phi})$, equivalent if there exists a diffeomorphism as above such that the sets of solutions are mapped onto each other:

$$\mathcal{F}(Sol(E_L(x, \phi)) = Sol(E_{\bar{L}}(\bar{x}, \bar{\phi})), \qquad \text{i.e. } E_L(x, \phi) = 0 \Leftrightarrow E_{\bar{L}}(\bar{x}, \bar{\phi}) = 0 \,. \tag{2.30}$$

Again the simplest scenario is that $E_L(x, \phi) = E_{\bar{L}}(\bar{x}, \bar{\phi})$, but this is by no means necessary. Also, the redefinition that relates the solutions of the equations of motion does not automatically map the Lagrangians onto eachother. On the other hand, if two Lagrangians are related to each other under redefinitions, then their respective equations of motion are automatically equivalent. Indeed, starting from any Lagrangian one can perform redefinitions to generate differently looking but equivalent Lagrangians. This is often exploited in analysing physical theories and one often performs redefinitions to put a theory in a more manageable form that is better suited to the applications one has in mind. Before explicitly showing that one can indeed perform redefinitions at the level of the Lagrangian without affecting the dynamical content of the theory, let us first discuss the properties of general redefinitions in more detail.

2.2.1 Redefinitions: point, contact and beyond

Redefinitions of the jet space variables should of course respect the interpretation of the transformed coordinates as space-time coordinates, field values and derivatives. Thus if one has a section $(x, \phi(x), \phi^{(1)}(x), \ldots)$ then it must be transformed into another section $(\bar{x}, \bar{\phi}(\bar{x}), \bar{\phi}^{(1)}(\bar{x}), \ldots)$. In other words, the transformations of the derivatives must follow from those of the space-time coordinates and the fields alone. Thus any such diffeomorpishm takes the following form when evaluated on sections:

$$\bar{x} = f(x, \phi(x), \ldots), \qquad \bar{\phi}(\bar{x}) = g(x, \phi(x), \ldots) \,, \tag{2.31}$$

where the corresponding transformations of the derivatives straightforwardly, but in practice tediously, follow:

$$\begin{aligned}
\bar\phi^{(1)}(\bar x) &\equiv \frac{d}{d\bar x}\bar\phi(\bar x) = \Big(\frac{df}{dx}\Big)^{-1}\frac{dg}{dx} \\
\bar\phi^{(2)}(\bar x) &\equiv \frac{d^2}{d\bar x^2}\bar\phi(\bar x) = \Big(\frac{df}{dx}\Big)^{-1}\frac{d}{dx}\Big(\Big(\frac{df}{dx}\Big)^{-1}\frac{dg}{dx}\Big) \qquad (2.32)\\
&\dots \\
\bar\phi^{(n)}(\bar x) &\equiv \frac{d^n}{d\bar x^n}\bar\phi(\bar x) = \Big(\Big(\frac{df}{dx}\Big)^{-1}\frac{d}{dx}\Big)^n g(x,\phi(x),\dots)\,. \qquad (2.33)
\end{aligned}$$

Such redefinitions are also called *Lie-Bäcklund transformations* [8] and they are the most general redefinitions compatible with the derivative interpretation. They fall into two distinct classes: the non-local ones depending on derivatives of all orders up to infinity, and the local ones that depend on at most finitely many derivatives. Throughout this thesis we will be mainly interested in Lagrangians of finite order and hence will mostly consider the local Lie-Bäcklund transformations. All the often encountered types of redefinitions (and more) fall within this class of local transformations. Amongst these are changes in space-time coordinates, i.e. transformations of the form:

$$\bar x = f(x), \qquad \bar\phi(\bar x) = g(x,\phi(x))\,, \tag{2.34}$$

where the explicit form of g is determined by the tensorial nature of $\phi(x)$. These are often employed, for example by going from cartesian to polar coordinates. Another recurring type is the *field redefinition*:

$$\bar x = x, \qquad \bar\phi(\bar x) = g(\phi(x))\,. \tag{2.35}$$

More general is the set of *point transformations* which consists of the most general redefinitions not involving derivatives of the fields [1]:

$$\bar x = f(x,\phi(x)), \qquad \bar\phi(\bar x) = g(x,\phi(x))\,. \tag{2.36}$$

The point transformations are special in that they are well defined on any finite jet space J^n, $n \geq 0$ (meaning that they map all derivatives up to order n onto a new set of derivatives up to order n) and not just the infinite order one.

One could wonder whether there are also more general transformations with the similar property of being well defined on finite jet spaces from some order n onward. It turns out that the existence of such transformations is strongly constrained. Let us first introduce some terminology: any transformation preserving J^n is called an n-th order *contact transformation*. Clearly any $(n-1)$-th order contact transformation is

[1] Note that in mixing both space-time coordinates and fields one has to take care: the new field $\bar\phi(\bar x)$ is not always globally well-defined. This because if one wants to explicitly calculate it one must for a specific function $\phi(x)$ solve the first equation for x in terms of $\bar x$ which is only locally ensured to be possible. Hence the new function is in general only locally well-defined.

also an n-th order one, and we thus define an n-th order contact transformation to be *non trivial* if it is not an $(n-1)$-th order contact transformation. Interestingly it has been shown that very little non-trivial finite order contact transformations actually exist: if $\dim(U) > 1$ all contact transformations are point transformations, whereas if $\dim(U) = 1$ non trivial *first order* contact transformations do exist but no higher order ones. See f.e. [8] for more details. These non-trivial first order contact transformations are of course of the form:

$$\bar{x} = f(x, \phi(x), \phi^{(1)}(x)), \qquad \bar{\phi}(\bar{x}) = g(x, \phi(x), \phi^{(1)}(x))\,, \tag{2.37}$$

for which the induced transformation on the derivative is:

$$\bar{\phi}^{(1)}(\bar{x}) = h(x, \phi(x), \phi^{(1)}(x))\,, \tag{2.38}$$

and thus only exist if one considers a single component field (but in an arbitrary dimensional space-time). Any transformation which is not a point transformation or a first order contact transformation as above, so a generic Lie-Bäcklund transformation, is only well-defined on the infinite jet space. Let us distinguish two particular such classes of redefinitions. Firstly we call any redefinition of the form:

$$\bar{x} = x, \qquad \bar{\phi}(\bar{x}) = g(x, \phi(x), \ldots, \phi^{(n)}(x))\,, \tag{2.39}$$

an n-th order *derivative dependent* field redefinition. Secondly, we call redefinitions of the form

$$\bar{x} = f(x, \phi(x), \ldots, \phi^{(n)}(x)), \qquad \bar{\phi}(\bar{x}) = g(x, \phi(x), \ldots, \phi^{(n)}(x))\,, \tag{2.40}$$

that are not n-th order contact transformation, n-th order *extended* contact transformations.

2.2.2 Transformation of the equations of motion

Given the above redefinitions, let us confirm our intuition and explicitly show that they are admissable and that the resulting transformed Lagrangian (to be defined below) is indeed equivalent to the original. Starting from an explict action expressed in terms of the original coordinates, let us perform such a redefinition and define a new action as follows:

$$\bar{S}[\bar{\phi}(\bar{x})] \equiv S[\phi(x)]\,, \tag{2.41}$$

where thus by construction the new action evaluated on a certain configuration $\bar{\phi}(\bar{x})$ has the same value as the original action evaluated on the corresponding configuration $\phi(x)$. Upon writing this out in terms of the corresponding Lagrangians one finds

$$\int_{\bar{\Omega}} \bar{\mathcal{L}}(\bar{x}, \bar{\phi}(\bar{x}), \ldots \bar{\phi}^{(m)}(\bar{x}))d\bar{x} \equiv \int_{\Omega} \mathcal{L}(x, \phi(x), ..., \phi^{(n)}(x))dx\,, \tag{2.42}$$

and by transforming the integration domain one finds that the Lagrangians should be related as

$$\bar{\mathcal{L}}(\bar{x}, \bar{\phi}(\bar{x}), \dots \bar{\phi}^{(m)}(\bar{x})) \det\left(\frac{d\bar{x}}{dx}\right) \equiv \mathcal{L}(x, \phi(x), ..., \phi^{(n)}(x)) \,. \tag{2.43}$$

Thus we see that the Lagrangian does not transform as a scalar under redefinitions due to the transformation of the integration measure. Note that the order of the Lagrangians, m and n, need not be the same and in fact generically they will be different depending on the form of the redefinition. Point transformations do not change the order of Lagrangians of any order $n \geq 1$. Non-trivial first order contact transformations on the other hand generically transform first order Lagrangians to second order ones due to the Jacobian factor introducing second order derivatives; they do not raise the order of Lagrangians of order $n \geq 2$. A general Lie-Bäcklund transformation generically does not respect the order of any Lagrangian, and as such it is always possible to raise the order of a Lagrangian by performing a suitable redefinition. Now, for our purposes the converse question is much more interesting, especially with regards to healthy higher derivative theories: can one always lower the order of a healthy higher derivative theory via a suitable redefinition? In Chapter 4 we extensively examine this.

Given how the transformed action is defined, it is clear that if some original configuration $\phi(x)$ is a stationary point of S, then the corresponding transformed configuration $\bar{\phi}(\bar{x})$ is a stationary point of $\bar{S}$. Hence the principle of least action should be compatible with redefinitions and the dynamics contained in the two descriptions should be equivalent. As a consequence, we expect that the corresponding equations of motion have to be equivalent, i.e. $E_L(\phi) = 0 \Leftrightarrow E_{\bar{L}}(\bar{\phi}) = 0$. Indeed one can derive an explicit relation between the equations of motion from which this automatically follows. To this end consider again the variation of a configuration:

$$\phi(x, \epsilon) = \phi(x) + \epsilon\delta\phi \,. \tag{2.44}$$

This induces a corresponding variation in the transformed configuration (implicitly defined by the redefinition):

$$\bar{\phi}(\bar{x}, \epsilon) = \bar{\phi}(\bar{x}) + \epsilon\delta\bar{\phi} \,. \tag{2.45}$$

It is useful to express $\delta\bar{\phi}$ in terms of $\delta\phi$ (or vice-versa). To do this we first note that

$$\delta\bar{\phi} = \frac{d\bar{\phi}}{dx}\delta x + \frac{\partial\bar{\phi}}{\partial\phi^{(p)}}\delta\phi^{(p)} \,, \tag{2.46}$$

and subsequently that whilst varying $\bar{S}$, $\bar{x}$ is kept fixed and thus

$$0 = \frac{d\bar{x}}{dx}\delta x + \frac{\partial\bar{x}}{\partial\phi^{(p)}}\delta\phi^{(p)} \,. \tag{2.47}$$

Combining the expressions we get:

$$\begin{aligned} \delta\bar{\phi} &= \Big(\frac{\partial\bar{\phi}}{\partial\phi^{(p)}} - \frac{d\bar{\phi}}{dx}\Big(\frac{d\bar{x}}{dx}\Big)^{-1}\frac{\partial\bar{x}}{\partial\phi^{(p)}}\Big)\Big(\frac{d}{dx}\Big)^{p}\delta\phi \\ &\equiv \hat{P}\cdot\delta\phi \,. \end{aligned} \tag{2.48}$$

We also introduce the adjoint operator

$$\hat{P}^\dagger \cdot f = (-1)^p \Big(\frac{d}{dx}\Big)^p \Big(\Big(\frac{\partial\bar{\phi}}{\partial\phi^{(p)}} - \frac{d\bar{\phi}}{dx}\Big(\frac{d\bar{x}}{dx}\Big)^{-1}\frac{\partial\bar{x}}{\partial\phi^{(p)}}\Big) f\Big) . \tag{2.49}$$

Armed with these expressions the relation between the equations of motion can be easily derived. First we observe that

$$\begin{aligned}\int_{\bar{\Omega}} E_{\bar{L}}(\bar{\phi})\delta\bar{\phi}(\bar{x})d\bar{x} &= \frac{d}{d\epsilon}\bar{S}[\bar{\phi}(\bar{x},\epsilon)]|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}S[\phi(x,\epsilon)]|_{\epsilon=0} = \int_\Omega E_L(\phi)\delta\phi dx\, .\end{aligned} \tag{2.50}$$

Rewriting we then find:

$$\begin{aligned}\int_{\bar{\Omega}} E_{\bar{L}}(\bar{\phi})\delta\bar{\phi}(\bar{x})d\bar{x} &= \int_{\bar{\Omega}} E_{\bar{L}}(\bar{\phi})(\hat{P}\cdot\delta\phi)d\bar{x} \\ &= \int_\Omega E_{\bar{L}}(\bar{\phi})\det\Big(\frac{d\bar{x}}{dx}\Big)(\hat{P}\cdot\delta\phi)dx \\ &= \int_\Omega \hat{P}^\dagger\cdot\Big(\det\Big(\frac{d\bar{x}}{dx}\Big)E_{\bar{L}}(\bar{\phi})\Big)\delta\phi dx + \text{surface term}\, .\end{aligned} \tag{2.51}$$

As usual the surface term vanishes to due the variations vanishing on the boundary, and we conclude that

$$E_L(\phi) = \sum_{p=0}(-1)^p\Big(\frac{d}{dx}\Big)^p\Big(\det\Big(\frac{d\bar{x}}{dx}\Big)\Big(\frac{\partial\bar{\phi}}{\partial\phi^{(p)}} - \frac{d\bar{\phi}}{dx}\Big(\frac{d\bar{x}}{dx}\Big)^{-1}\frac{\partial\bar{x}}{\partial\phi^{(p)}}\Big)E_{\bar{L}}(\bar{\phi})\Big) . \tag{2.52}$$

From this it is easy to see that if $\bar{\phi}(\bar{x})$ solves $E_{\bar{L}}$ then $\phi(x)$ solves E_L, and vice versa due to the invertibility of the redefinition [2]. Thus we find that any local redefinition of coordinates on J^∞ can be performed at the level of the Lagrangian resulting in equivalent dynamics in the sense defined above. From this expression one can also explicitly see that, like the order of Lagrangians, the order of the equations of motion is not invariant under general redefinitions.

Example: Generalised Galileons. It is easy to see from the antisymmetric structure all generalised Galileons share that they are in fact linear in second order time derivatives. This together with again their antisymmetric structure allows one to add a suitable total derivative to write the theory in terms of first time derivatives only, although higher order spatial and mixed derivatives such as $\partial_i\dot{\phi}$, do generally occur in this formulation. Also, Lorentz covariance is generically not maintained in this

[2] If the redefinition is not invertible one can at most conclude that the dynamics of one is contained in the other but not the other way around and the theories are thus not equivalent. Such non-invertible transformations can nevertheless be useful and for example find their application in constructing so called mimetic gravity theories [31]. For a recent discussion on its status see for example [119].

rewriting. (See also Chapter 4.)
The fact that the set of generalised Galileons is the most general Lorentz invariant one with second order equations of motion, directly implies that it must be invariant under first order contact redefinitions of the form

$$\bar{x}^\mu = x^\mu + f(\phi, X)\partial^\mu\phi, \qquad \bar{\phi}(\bar{x}) = \phi(x) + g(\phi, X)\,, \tag{2.53}$$

where f and g are not arbitrary but must be chosen to ensure that the above is indeed an invertible contact transformation. These transformations will not raise the order of the Lagrangian or the equation of motion, will not introduce explicit coordinate dependence, and are Lorentz covariant. Therefore they will leave the set of generalised Galileons invariant. Such transformations will relate different generalised Galileons to each other that are therefore dynamically equivalent; in particular they relate generalised Galileon terms of different order in second derivatives to each other, as opposed to ordinary field redefinitions that can only relate terms of the same order. As an example of a nontrivial class of such transformations one can pick $f_\phi = g_\phi = 0$ and $2g_X = f + 2Xf_X$ (in turn implying $\bar{\partial}_\mu\bar{\phi} = \partial_\mu\phi$). Specific subclasses of such transformations leave invariant particular interesting subsets of the generalised Galileons; we will touch upon these so called Galileon dualities [48] in more detail later on. Note that the non-existence of first order contact transformations involving more than one field component implies that no such duality transformations involving derivatives exist for the set of generalised multi-Galileons: such transformations will generically introduce third order derivatives to the Lagrangians (see also [138]).

Example: GR and Lovelock. Like the generalised Galileons, General Relativity and more generally Lovelock theories can also be rewritten without second order time derivatives (though potentially with higher order mixed derivatives) by adding a suitable total derivative, which in this case breaks general covariance (or even Lorentz covariance). For example, one can rewrite GR in terms of the metric and connection only leading to:

$$\mathcal{L} = \sqrt{-g}g^{\mu\nu}(\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu})\,. \tag{2.54}$$

Apart from different Lagrangians all using the metric as a variable, an often used formulation of gravity theories actually uses different variables. These so called ADM variables [10] naturally arise when considering a particular foliation of space-time, resulting in a spatial metric h_{ij} on the space-like hypersurfaces, and the lapse function N and the shift vector N^i that descibe how the hypersurfaces are deformed along the time direction. These variables are related to the original metric components via an ordinary field redefintion:

$$g_{00} = -N^2, \qquad g_{i0} = h_{ij}N^j, \qquad g_{ij} = h_{ij}\,. \tag{2.55}$$

For example, using them allows one to rewrite GR in terms of intrinsic and extrinsic curvatures of the hypersurfaces, $\bar{R}_{ij}$ and K_{ij} respectively:

$$\mathcal{L} = \sqrt{h}N\big(\bar{R} + K^{ij}K_{ij} - K^2\big)\,. \tag{2.56}$$

In this formulation the detailed analysis of the dynamical properties of gravity theories is usually more transparant and one can formulate the equations of motion as a standard Cauchy problem and as such they are better suited to doing numerical calculations. In addition one usually uses these variables to perform the Hamiltonian analysis.

Example: Horndeski. The set of Horndeski theories as a whole is invariant [17] under disformal transformations of the form:

$$\bar{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\partial_\mu\phi\partial_\nu\phi\,. \tag{2.57}$$

In particular such a transformation performed on a Horndeski term of some order i will produce all terms of a lower or equal order, i.e. $\mathcal{L}_i[f] \to \sum_{j=0}^{i} \mathcal{L}_j[\bar{f}_j]$ where all the $\bar{f}_j$ are given in terms of f_i, $C(\phi)$ and $D(\phi)$. As such the subsets of Horndeski up to some order are separately invariant. Although this invariance has only been explicitly shown for Horndeski, it seems natural to expect similar results for covariantised Galileons in any dimension. By also allowing for dependence on the scalar field kinetic term, i.e. $C = C(\phi, X)$ and $D = D(\phi, X)$ one can generate theories that in form and structure go beyond Horndeski [173] but of course are equivalent. We will discuss such theories and generalisations thereof in more detail in the following chapters.

2.3 Variational symmetries

Let us now recall some basics on symmetries. To this end consider a system of differential equations

$$\triangle(x, \phi^{(n)}(x)) = 0\,. \tag{2.58}$$

Usually a symmetry of such a system of differential equations is defined as any invertible point transformation $F : J^0 \to J^0$, such that the induced transformation on sections, i.e. $(x', \phi'(x')) = F(x, \phi(x))$, satisfies

$$\triangle(x, \phi^{(n)}(x)) = 0 \qquad \Leftrightarrow \qquad \triangle(x', \phi'^{(n)}(x')) = 0\,. \tag{2.59}$$

Thus if $\phi(x)$ is a solution to the system then so is the transformed field $\phi'(x)$ (as is immediate from the above by renaming x' to x), i.e. symmetries map solutions to solutions. Invertibility of the transformation then implies that the full set of solutions, $Sol(\triangle(x, \phi^{(n)}(x)))$, gets mapped onto itself:

$$F(Sol(\triangle(x, \phi))) = Sol(\triangle(x', \phi'))\,. \tag{2.60}$$

We stress that this is different from performing redefinitions as these relate the solutions of one set of differential equations to solutions of a different, although equivalent, set of differental equations. Now, given our earlier discussion on more general transformations on the infinite jet space, it is natural to extend the considered symmetry

transformations to the more general Lie-Backlund transformations. Indeed this was already realised by Noether [137] and such symmetries are usually called *generalised* or *Lie-Bäcklund* symmetries [8,141,142]. These are thus diffeomorphisms $F : J^\infty \to J^\infty$ that map solutions to solutions. We will usually drop the characterization "generalised" and simply refer to symmetries.

Specialising to Lagrangian theories, a transformation is a symmetry of the theory if it is a symmetry of the corresponding equations of motion. I.e. precisely when

$$E_L(x, \phi^{(n)}) = 0 \qquad \Leftrightarrow \qquad E_L(x', \phi'^{(n)}) = 0\,. \tag{2.61}$$

Such symmetries of the equations of motion of a theory can have different consequences depending on their origin. At the very least they can be used to generate more solutions out of one solution, which might be useful in certain situations. However, there is a class of symmetries that have more far reaching implications regarding the properties of the theory. These are the so called *variational symmetries* which find their origin at the level of the action. Variational symmetries are defined as transformations on J^∞ that leave the action invariant up to a boundary term, i.e

$$\begin{aligned} S[x,\phi] &= S'[x',\phi'] \\ &= S[x',\phi'] + \text{ boundary term}\,. \end{aligned} \tag{2.62}$$

Written out one gets:

$$\begin{aligned} \int_\Omega \mathcal{L}(x,\phi^{(n)})dx &= \int_{\Omega'} \mathcal{L}'(x',\phi'^{(n)})dx' \\ &= \int_{\Omega'} (\mathcal{L}(x',\phi'^{(n)}) + \nabla'\mathcal{M}')dx' \\ &= \int_\Omega \mathcal{L}(x',\phi'^{(n)}) \det\left(\frac{dx'}{dx}\right)dx + \int_\Omega \nabla\mathcal{M}dx\,, \end{aligned} \tag{2.63}$$

or, in terms of the Lagrangians:

$$\mathcal{L}(x',\phi'^{(n)}) \det\left(\frac{dx'}{dx}\right) = \mathcal{L}(x,\phi^{(n)}) - \nabla\mathcal{M}\,. \tag{2.64}$$

Given our discussion on the relation between equations of motion after performing an invertible transformation, one immediately sees that any transformation that only transforms the action up to a boundary term will yield a symmetry of the equations of motion:

$$E_L(x,\phi^{(n)}) = 0 \qquad \Leftrightarrow \qquad E_{L'}(x',\phi'^{(n)}) = E_L(x',\phi'^{(n)}) = 0\,, \tag{2.65}$$

where we used that the equations of motion following from $\mathcal{L}'$ and $\mathcal{L}$ are the same since they differ by a total divergence. Hence every variational symmetry is indeed a symmetry, whereas the converse is not true: not every symmetry of the equations of motion is a variational symmetry. Before discussing the implications of variational symmetries, let us first say a bit more about the symmetry transformations themselves.

2.3.1 Groups and group actions

It is immediate that the collection of all symmetry transformations of a certain system must realise a group structure: the composition of two such transformations again maps solutions to solutions, the transformations are invertible and clearly the identity transformation is a symmetry. Therefore they must form a realisation of some abstract group G on J^∞. Any realisation of a group G on the jet space coordinates can be written as

$$g \cdot (x, \phi, \ldots) = F_g(x, \phi, \ldots) = (x', \phi', \ldots), \qquad F_{g_1} \circ F_{g_2} = F_{g_1 g_2}\,, \tag{2.66}$$

where $g, g_1, g_2 \in G$. Of course the transformation rules should be compatible with the interpretation of the different coordinates, meaning that on sections we have:

$$\begin{aligned} x' &= F_g^{(x)}(x, \phi(x), \ldots) \\ \phi'(x') &= F_g^{(\phi)}(x, \phi(x), \ldots) \\ \phi'^{(1')}(x') &= F_g^{(\phi^{(1)})}(x, \phi(x), \ldots) = \left(\frac{dF_g^{(x)}}{dx}\right)^{-1} \frac{dF_g^{(\phi)}}{dx} && (2.67) \\ &\ldots && (2.68) \end{aligned}$$

The actual properties of F_g can be wildly different depending on the case at hand and one can classify groups and their realisations in different ways according to different properties. We will now discuss several of such classifications.

Linear and non-linear. Firstly one can make the distinction between *linear* and *non-linear* realisations via the corresponding dependence of F_g on the jet space variables. The simplest realisations are the well known linear representations of a group which are often encountered:

$$(x', \phi'(x), \ldots) = D_g \cdot (x, \phi(x), \ldots)\,, \tag{2.69}$$

where D_g is a jet space coordinate independent matrix. For a realisation to be linear on all the jetspace variables and consistent with the derivative interpretation one cannot allow for mixing between the space-time coordinates x and the other coordinates, i.e. one must have $x' = D_g^{(x)} \cdot x$ and $\phi'(x') = D_g^{(\phi)} \cdot (\phi(x), \phi^{(1)}(x), \ldots)$. Mixing introduces non-linear transformations of at least some of the derivative variables due to Jacobian factors. One can also consider realisations that are linear on a restricted subset of the jetspace variables, the most familiar ones being linear representations on the fields, i.e. $\phi'(x') = D_g \cdot \phi(x)$. The more general non-linear realisations naturally arise when one considers the concept of spontaneously broken symmetries. As mentioned in the introduction, we will extensively examine the properties and construction of non-linear realisations later in this thesis.

Point, contact and beyond. From our discussion concerning redefinitions it is clear that symmetry transformations can be similarly classified. The simplest transformations in that sense are thus the *point symmetries* where the group acts on the space $J^0 = M \times U$, inducing transformations (at least locally) on the corresponding graphs $\Gamma \subset J^0$:

$$x' = F_g^{(x)}(x, \phi(x)), \qquad \phi'(x') = F_g^{(\phi)}(x, \phi(x)) \,. \tag{2.70}$$

One can also have consistent *contact symmetries* that act as contact transformations, but non trivial ones thus only exist in the case of a single component field. A generic local generalised symmetry transformation takes the following form:

$$x' = F_g^{(x)}(x, \phi(x), \ldots, \phi^{(n)}(x)), \qquad \phi'(x') = F_g^{(\phi)}(x, \phi(x), \ldots, \phi^{(n)}(x)) \,. \tag{2.71}$$

whereas one can also consider the even more general non-local transformations:

$$x' = F_g^{(x)}(x, \phi(x), \ldots), \qquad \phi'(x') = F_g^{(\phi)}(x, \phi(x), \ldots) \,. \tag{2.72}$$

Internal and space-time. The metric of the space-time manifold M on which one has defined the theory might be invariant under certain coordinate transformations

$$x' = F_g^{(x)}(x) \tag{2.73}$$

These isometries of space-time are usually called *space-time symmetries*, and they are accompanied by transformations on the fields (and their derivatives), i.e. $\phi'(x') = F_g^{(\phi)}(x, \phi(x), \ldots)$, whose forms depend on the properties of the fields one considers. An entirely different class is that of *internal symmetries*. These are defined as transformations that commute with space-time symmetries and only act on the fields (and their derivatives):

$$x' = x, \qquad \phi'(x') = F_g^{(\phi)}(\phi(x), \ldots) \,. \tag{2.74}$$

A generic symmetry transformation will be neither of the two types above, i.e. it is not an isometry of space-time nor does it commute with them. Somewhat confusingly these more general symmetries are usually also refered to as space-time symmetries, even though they are not symmetries of space-time. We will follow this terminology throughout the thesis. The classic no-go theorem by Coleman and Mandula [37] restricts the appearance of such more general space-time symmetries in relativistic settings: if an interacting theory with a symmetry group that contains Poincaré and acts linearly and as an ordinatry symmetry on the fields is to have sensible scattering amplitudes between particle states, its symmetry group must necessarily be a direct product of Poincaré with some internal symmetry group. One way to circumvent this is by considering theories without particle states, such as theories with conformal symmetry. The other obvious way is to consider non-linearly realised symmetries.

Discrete and continuous. Group elements can be described by parameters, ϵ, that can take discrete values or run over a continuous range of values, depending on the group one considers. Groups falling in the former catagory are called discrete groups. Continuous groups on the other hand fall in the latter category and in that case that case the group elements, $g(\epsilon)$, are continuous functions of the parameters and this is respected by multiplication and taking inverses: if $g(\epsilon_1)g(\epsilon_2) = g(\epsilon_3)$ then $\epsilon_3(\epsilon_1, \epsilon_2)$ is continuous, and if $(g(\epsilon))^{-1} = g(\bar{\epsilon})$ then $\bar{\epsilon}(\epsilon)$ is continuous. A special subclass of the continuous groups is that of the smooth groups that also have a the structure of a smooth manifold. These are the well known Lie groups.

Global and local. Another classification based on the parameters of the group is that of global versus local. The simplest case is that of *global symmetries* for which the group elements are parametrised by global parameters ϵ. In that case the action of a group element is the same for each point in the jet space, i.e.

$$g \cdot (x, \phi(x), \ldots) = F(\epsilon; x, \phi(x), \ldots)\,. \tag{2.75}$$

More generally the group elements can be parametrised by parameters that depend on the space-time variables, i.e. $\epsilon(x)$, as well as the corresponding derivatives:

$$g \cdot (x, \phi(x), \ldots) = F(\epsilon(x), \epsilon^{(1)}(x), \ldots; x, \phi(x), \ldots)\,. \tag{2.76}$$

Depending on possible restrictions on the functions $\epsilon(x)$ subsets can be interpreted as (possibly an infinite number of) global symmetries. For example this is the case if they are Taylor expandable. However if there is true functional freedom, i.e. when $\epsilon(x)$ can be arbitrary, this is not the case and they truly go beyond global symmetries and have wildly different consequences, as we will discuss later. These are called *local symmetries.*[3]

Finite and infinitesimal. So far we have considered at least locally defined *finite* group transformations, i.e. where the parameters take finite values, which are usually difficult to work with. From now on we will be focusing on Lie groups, i.e. groups that are also smooth manifolds. If one restricts to the elements that are continuously connected to the identity element one can consider infinitesimal transformations, for which the parameters are taken to be infinitesimally small. In practice there is really no need to consider the finite transformations: it is these infinitesimal transformations that have the important implications in the form of Noether's theorems.

To properly discuss infinitesimal transformations we must introduce the *Lie algebra* corresponding to the Lie group. The Lie algebra is an algebraic structure that

[3]One can go even further by allowing the functions to depend on all jet space variables, i.e. $\epsilon(x, \phi, \ldots)$. As long as the transformation only depends on the total derivatives, as opposed to partial derivatives, of the parameter one can actually restrict to dependence on space-time coordinates alone without loss of generality: invariance of a theory under one implies invariance under the other, and they give rise to the same gauge identities. We will not consider the more general transformations involving isolated partial derivatives such as $\epsilon_\phi(x, \phi, \ldots)$ etc.

gives a local characterisation of the group around the identity element. Given any group element $g(\epsilon)$ connected to the identity element, the associated element of the Lie algebra, called a generator, is defined as $X \equiv \frac{dg}{d\epsilon}|_{\epsilon=0}$. If the group is parametrised by a set of I independent parameters ϵ^i, then the Lie algebra is a vector space spanned by the set of I generators $X_i = \frac{dg}{d\epsilon^i}|_{\epsilon=0}$, with the additional property (which makes it a Lie algebra) that the generators satisfy commutation relations and the Jacobi identity:

$$[X_i, X_j] = f_{ij}{}^k X_k , \tag{2.77}$$

$$[X_i, [X_j, X_k]] + [X_k, [X_i, X_j]] + [X_j, [X_k, X_i]] = 0 . \tag{2.78}$$

Here $f_{ij}{}^k$ are the structure constants of the algebra. Conversely, one can also start from a Lie algebra and reconstruct a corresponding group that is at least locally well defined by exponentiating: $g(\epsilon) = e^{\epsilon^i X_i}$. We note that different groups can share the same Lie algebra and as such in general the correspondence is not one-to-one, but when restricted to simply connected groups the correspondence is in fact bijective.

Now consider a Lie group with a global action thus parametrised by some parameters ϵ, and take these to be small. One can then expand the finite group transformation as

$$x' = x + \epsilon f(x, \phi(x), \ldots) + O(\epsilon^2) , \tag{2.79}$$

$$\phi'(x') = \phi(x) + \epsilon g(x, \phi(x), \ldots) + O(\epsilon^2) , \tag{2.80}$$

and similarly for the derivatives. The infinitesimal transformation in the direction of ϵ is then defined as

$$\delta_\epsilon(x, \phi, \ldots) = \frac{d}{d\epsilon}\big(x' - x, \phi'(x') - \phi(x), \ldots\big)|_{\epsilon=0} , \tag{2.81}$$

and the corresponding generator is:

$$\delta_\epsilon \equiv f(x, \phi(x), \ldots)\frac{\partial}{\partial x} + g(x, \phi(x), \ldots)\frac{\partial}{\partial \phi} + \ldots \tag{2.82}$$

One can show that these form a representation of the Lie algebra, i.e. one finds

$$[\delta_i, \delta_j](x, \phi(x), \ldots) = f_{ij}{}^k \delta_k(x, \phi(x), \ldots) . \tag{2.83}$$

Again, by starting from such infinitesimal transformation rules one can under suitable existence conditions construct the finite transformations by exponentiation (which are usually only locally well-defined). These conditions are always met when dealing with point transformations, but if one deals with infinitesimal generalised symmetry transformations it might very well happen that there is in fact no corresponding well-defined finite transformation (globally nor locally) depending on the solvability of a corresponding system of differential equations (see also [141, 142]). Whether or not this is the case is in practice not always so important because the main consequence of variational symmetries only requires the transformations to be defined infinitesimally.

If the group action is local, and thus parametrised by some space-time dependent parameters $\epsilon(x)$, one gets:

$$x' = x + f_0(x, \phi, \ldots)\epsilon(x) + \ldots + f_n(x, \phi, \ldots)\epsilon^{(n)}(x) + O(\epsilon^2)\,, \tag{2.84}$$

$$\phi'(x') = \phi(x) + g_0(x, \phi, \ldots)\epsilon(x) + \ldots + g_m(x, \phi, \ldots)\epsilon^{(m)}(x) + O(\epsilon^2)\,, \tag{2.85}$$

and the corresponding generator is given by

$$\delta_\epsilon = \big(\sum_{i=0}^{n} f_i(x, \phi, \ldots)\epsilon^{(i)}(x)\big)\frac{\partial}{\partial x} + \big(\sum_{i=0}^{m} g_i(x, \phi, \ldots)\epsilon^{(i)}(x)\big)\frac{\partial}{\partial \phi} + \ldots \tag{2.86}$$

Active and passive. When doing calculations with a generic transformation as above one often runs into the nuisance that if the space-time coordinates are also transformed, then taking derivatives and variations no longer commutes, i.e. $\delta\phi^{(n)} \neq (\delta\phi)^{(n)}$, since the derivative operator itself also transforms. Luckily, given any symmetry transformation that also affects the space-time coordinates, there is an alternative transformation that realises the same underlying group but leaves the space-time coordinates untouched. To see this, consider some arbitrary group action, then the variations are calculated by comparing the transformed jet space coordinates at different space-time points:

$$\delta x = \frac{d}{d\epsilon}(x' - x)|_{\epsilon=0}, \qquad \delta\phi(x) = \frac{d}{d\epsilon}(\phi'(x') - \phi(x))|_{\epsilon=0}\,, \tag{2.87}$$

we call this the *passive* transformation. However it is also consistent to calculate variations by comparing the transformed objects at the same space-time point, which we call the *active* transformation [4]:

$$\delta_A x = 0, \qquad \delta_A\phi(x) = \frac{d}{d\epsilon}(\phi'(x) - \phi(x))|_{\epsilon=0}\,. \tag{2.88}$$

One can relate the two transformations explicitly by noting that

$$\begin{aligned}\phi'(x') &= \phi'(x + \epsilon\delta x + O(\epsilon^2)) \\ &= \phi'(x) + \epsilon(\partial'\phi')(x)\delta x + O(\epsilon^2) \\ &= \phi'(x) + \epsilon\partial\phi(x)\delta x + O(\epsilon^2)\,,\end{aligned} \tag{2.89}$$

which yields

$$\delta_A\phi(x) = \delta\phi(x) - \partial\phi\delta x\,. \tag{2.90}$$

These active transformation rules form realise the same underlying algebra and as we will see in the next subsection they are equivalent in the sense that if one is symmetry then so is the other and additionally they have same implications. Thus for all practical purposes they are two guises of one and the same symmetry. The active

[4] In the mathematics literature the active transformations are called *evolutionary* [141, 142].

transformation is the most convenient when considering infinitesimal transformations, precisely because now taking variations and derivatives commutes, i.e. $\delta_A \phi^{(n)} = (\delta_A \phi)^{(n)}$. Additionally one has the very useful relation

$$\delta\mathcal{L} = \delta_A \mathcal{L} + \delta x \nabla \mathcal{L}\,, \tag{2.91}$$

for any Lagrangian. However, we do note that even though the passive transformation might be a point transformation, the active transformation involves derivatives and is in general a Lie-Bäcklund transformation. In particular the finite transformation will generically involve derivatives of all orders.

Let us end by discussing the interplay between symmetry transformations and redefinitions. Starting from some group G and corresponding action $(x', \phi', \ldots) = F_g(x, \phi, \ldots)$ and performing a redefinition $(\bar{x}, \bar{\phi}, \ldots) = \mathcal{F}(x, \phi, \ldots)$, one finds that there is a consistent action of G on the redefined variables given by

$$(\bar{x}', \bar{\phi}'(\bar{x}'), \ldots) = \bar{F}_g(\bar{x}, \bar{\phi}(\bar{x}), \ldots) \equiv (\mathcal{F} \circ F_g \circ \mathcal{F}^{-1})(\bar{x}, \bar{\phi}(\bar{x}), \ldots)\,. \tag{2.92}$$

From the above it is clear that the form of a group action can be wildly different depending on which coordinates one chooses. Given any class of symmetry transformations, the transformed transformation will belong to that same class if the redefinition one performs also belongs to that class. For example, a linear transformation will preserve the linearity of the representation. However, generically starting from the simplest realisation, i.e. a linear representation of the point type, a generic redefinition will turn this into a non-linear Lie-Backlund symmetry. Thus given a particular realisation one should always wonder whether its true nature is obscured by the particular choice of coordinates. We note that there do exist essential realisations of all the types we discussed in the sense that they cannot be put in a simpler form via a particular redefinition.

2.3.2 Global symmetries and conserved currents

Noether has shown [137] that continuous global variational symmetries have far reaching implications regarding the structure of the equations of motion, more so than non-variational symmetries. In particular, corresponding to any such symmetry transformation there is a combination of the equations of motion that is a total divergence. To see this consider some continuous group and consider an infinitesimal transformation parametrised by global parameters ϵ and calculate the variation of the Lagrangian to first order:

$$\begin{aligned}
\Delta\mathcal{L} &\equiv \frac{d}{d\epsilon}|_{\epsilon=0}\Big(\mathcal{L}(x', \phi', \ldots)\det\left(\frac{dx'}{dx}\right) - \mathcal{L}(x, \phi, \ldots)\Big) \\
&= \mathcal{L}\nabla\delta x + \delta\mathcal{L} \\
&= \delta_A\mathcal{L} + \nabla(\mathcal{L}\delta x) \\
&= \delta_A\phi E_L(\phi) + \nabla(\mathcal{L}\delta x + \mathcal{K})\,.
\end{aligned} \tag{2.93}$$

where $\mathcal{K}$ is given by (2.14) and we used (2.91). Given that the transformation is a symmetry we also know that $\Delta\mathcal{L} = \frac{d}{d\epsilon}|_{\epsilon=0}(\nabla\mathcal{M})$, which enables one to write:

$$\delta_A\phi E_L(\phi) = -\nabla J, \qquad J \equiv \mathcal{L}\delta x + \mathcal{K} - \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{M}\,. \tag{2.94}$$

Thus corresponding to every symmetry parameter ϵ there is a current J that is conserved on solutions, and there will be an associated conserved charge obtained by integrating the time component of the current over a space-like volume V (with unit normal n). That is:

$$Q \equiv \int_V J^0 dV \qquad \Rightarrow \qquad \frac{dQ}{dt} = \int_V \frac{d}{dt}J^0 dV = -\int_V \partial_i J^i dV = -\int_{\partial V} J^i n_i dA = 0\,, \tag{2.95}$$

assuming the spatial components of the current (so essentially the fields and derivatives upon which they depend) vanish on the boundary. Of course the charge is only well-defined if the integral converges.

Note that to any current one can always add a so called trivial current that does not change the corresponding conservation law and charge. Such a trivial current is at most the sum of two different parts, one which itself vanishes on solutions and one whose divergence identically vanishes for any field configuration, i.e.

$$J_{\text{trivial}} = J_1 + J_2, \qquad J_1 \approx 0, \qquad \nabla J_2 = 0\,. \tag{2.96}$$

Such a current does not give rise to a meaningful conservation law or conserved charge: an identically vanishing current is quite meaningless, and a current that is conserved independent of whether the configuration is a solution to the equations of motion has no dynamical relevance. It is natural to consider any two currents that differ by such a trivial piece as being equivalent. Now, two different symmetry transformations might actually give rise to equivalent currents and thus have the same physical implications and should therefore be considered equivalent. This is precisely what happens for the passive and active viewpoints we already mentioned. Firstly we observe that for any symmetry of the action, the corresponding active form is also a symmetry since their respective variations differ by a total derivative term, i.e.

$$\Delta\mathcal{L} - \Delta_A\mathcal{L} = \nabla(\mathcal{L}\delta x), \qquad (\Delta_A\mathcal{L} = \delta_A\mathcal{L})\,, \tag{2.97}$$

so if one is a total derivative then so is the other and therefore they are both symmetries. The corresponding currents differ by a piece that is identically conserved, i.e. $\nabla(J - J_A) \equiv 0$ and thus for all practical purposes they are two instances of one and the same symmetry of the theory.

The fact that we can perform redefinitions without affecting the underlying dynamics, implies that if a theory has some symmetries, the transformed theory should exhibit these as well. Indeed, if a theory $\mathcal{L}(x, \phi, \ldots, \phi^{(n)})$ is invariant under the original group action, F_g, then the transformed Lagrangian $\bar{\mathcal{L}}(\bar{x}, \bar{\phi}, \ldots, \bar{\phi}^{(m)})$ is invariant

under the transformed group action. This can be quite simply shown explicitly:

$$
\begin{aligned}
\bar{\mathcal{L}}(\bar{x}', \bar{\phi}', \ldots) &= \mathcal{L}(x', \phi', \ldots) \det\left(\frac{d\bar{x}'}{dx'}\right)^{-1} \\
&= \left(\mathcal{L}(x, \phi, \ldots) \det\left(\frac{dx'}{dx}\right)^{-1} + \nabla'\mathcal{M}'\right) \det\left(\frac{d\bar{x}'}{dx'}\right)^{-1} \\
&= \left(\bar{\mathcal{L}}(\bar{x}, \bar{\phi}, \ldots) \det\left(\frac{d\bar{x}}{dx}\right) \det\left(\frac{dx'}{dx}\right)^{-1} + \nabla'\mathcal{M}'\right) \det\left(\frac{d\bar{x}'}{dx'}\right)^{-1} \\
&= \bar{\mathcal{L}}(\bar{x}, \bar{\phi}, \ldots) \det\left(\frac{d\bar{x}'}{d\bar{x}}\right)^{-1} + \bar{\nabla}'\bar{\mathcal{M}}' \,.
\end{aligned}
\tag{2.98}
$$

Here we used that $(\bar{x}', \bar{\phi}', \ldots) = \mathcal{F}(x', \phi', \ldots)$, the relation between $\mathcal{L}$ and $\bar{\mathcal{L}}$, and invariance of $\mathcal{L}$.

Example: Generalised Galileons. Generalised Galileons in $(d+1)$-dimensions have a linearly realised Poincaré symmetry, $ISO(1,d)$, with infinitesimal transformations:

$$
\delta x^\mu = \epsilon^\mu + \omega^\mu_\nu x^\nu , \qquad \delta\phi = 0 \qquad \Leftrightarrow \qquad \delta_A \phi = -(\epsilon^\mu + \omega^\mu_\nu x^\nu)\partial_\mu \phi \,. \tag{2.99}
$$

The corresponding conserved currents are the energy momentum tensor $T^{\mu\nu}$ for the translations and $M^{\mu,\rho\sigma} = T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho$ for the Lorentz transformations, yielding as conserved charges the relativistic linear and angular momentum respectively.

Particular subsets of the generalised Galileons have an interesting higher dimensional origin. For example, consider a Minkowski d-brane fluctuating in a $(d+2)$-dimensional ambient Minkowski or AdS space-time. The brane breaks the ambient space-time symmetries down to the $(d+1)$-dimensional Poincaré subgroup, resulting in a theory on the brane that non-linearly realises these broken symmetries. If the ambient space-time is chosen to be Minkowski, one gets the DBI galileons that thus non-linearly realise a $(d+2)$-dimensional Poincaré symmetry, i.e. $ISO(1, d+1)$, with transformation rules:

$$
\delta_A \phi = c + b_\mu (x^\mu + \phi \partial^\mu \phi) \tag{2.100}
$$

This corresponds to a particular choice of the free functions $f_i(\phi, X)$. Considering an ambient $d+2$ dimensional AdS space leads to the AdS/conformal Galileons that non-linearly realise the isometries of AdS space, i.e. the conformal group $SO(2, d+1)$:

$$
\delta_A \phi = c(1 + x^\mu \partial_\mu \phi) + b_\mu (2x^\mu + 2x^\mu x^\nu \partial_\nu \phi - x^2 \partial_\mu \phi) \,. \tag{2.101}
$$

There are additional interesting subsets one can consider. One of these corresponds to taking the non-relativistic limit of the DBI Galileons, leading to the standard Galileons that non-linearly realise the $(d+2)$-dimensional Galileon group $GAL(1, d+1)$:

$$
\delta_A \phi = c + b_\mu x^\mu \,. \tag{2.102}
$$

These Galileons correspond to picking the free functions to be $f_i(\phi, X) = \phi$. Furthermore there is a subset of these, dubbed the special Galileons, that have an enhanced symmetry:

$$\delta_A \phi = c + b_\mu x^\mu + s_{\mu\nu}(x^\mu x^\nu + \partial^\mu \phi \partial^\nu \phi), \qquad s_{\mu\nu} = s_{\nu\mu}, \qquad s^\mu_\mu = 0\,. \tag{2.103}$$

Let us also note that there are also subsets that have an infinite number of global symmetries. In particular, the free theory $\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi$ has the following tower [94]:

$$\delta_A \phi = a + a_\mu x^\mu + a_{\mu\nu} x^\mu x^\nu + \ldots + a_{\mu_1 \ldots \mu_n} x^{\mu_1} \cdots x^{\mu_n} + \ldots \tag{2.104}$$

where the parameters are all completely symmetric and traceless. One can actually introduce a source, $\rho(x)\phi$, without breaking the symmetry and one can see that in this case the conserved charges corresponding to these symmetries are nothing but the familiar multipole moments [94].

The fact that the set of generalised Galileons is invariant under duality transformations of the form (2.53), does not imply that the different subsets discussed above are as well. Indeed, a generic duality transformation will not leave the symmetry transformations invariant, leading to different transformation rules for the redefined coordinates and fields and therefore the transformed theory generically goes outside of the consider class. Of course, as discussed the resulting theories do realise the same underlying symmetry groups but in a different manner. Interestingly there do exist subsets of the duality transformations that leave particular subsets of the generalised Galileons invariant. For example, the set of Galileons is invariant under duality transformations of the form [48, 51]:

$$\bar{x}^\mu = x^\mu + \alpha \partial^\mu \phi, \qquad \bar{\phi}(\bar{x}) = \phi(x) + \frac{\alpha}{2}(\partial \phi(x))^2, \qquad \bar{\partial}\bar{\phi} = \partial \phi, \qquad \alpha = \text{constant}\,, \tag{2.105}$$

as can easily be checked directly at the level of the action, as well as from the transformation of the group action which is seen to be invariant (see [111] for a four parameter generalisation). Now, the set of DBI Galileons on the other hand is invariant under:

$$\bar{x}^\mu = x^\mu + \alpha \frac{\partial^\mu \phi}{\sqrt{1 + (\partial \phi)^2}}, \qquad \bar{\phi}(\bar{x}) = \phi(x) - \alpha \frac{1}{\sqrt{1 + (\partial \phi)}}, \qquad \bar{\partial}\bar{\phi} = \partial \phi\,, \tag{2.106}$$

which indeed reduces to the Galileon duality transformation in the non-relativistic limit. This transformation is actually the correct local form of a transformation considered in [30] where duality invariance of the set of DBI Galileons under transformations of the form $\bar{x}^\mu = x^\mu + \alpha \frac{\partial^\mu \phi}{\sqrt{1+(\partial\phi)^2}}$ such that $\bar{\partial}\bar{\phi} = \partial \phi$ was shown. However, there the transformation of the field $\phi(x)$ was not given in an explicit form (which was actually not needed to prove the result), but rather in an implicit non-local form. Again, there is a clear interpretation of the transformations now stemming from automorphisms of the higher dimensional Poincaré algebra. The existence of these duality transformations for both Galileons and DBI Galileons has a clear interpretation in terms of automorphisms of the underlying symmetry algebras as becomes

evident when considering the coset construction. We will discuss this in more detail in Chapter 5.

It is interesting to classify all the possible non-linearly realised symmetries subsets of the generalised Galileons can have. One brute force method to do so is to make a very general Ansatz for the symmetry transformations and determine invariant Lagrangians. Whilst doing so one should always keep redefinitions in mind that can relate seemingly different transformation rules to eachother. Of course, a first hint that two such transformations might be equivalent is if they realise the same underlying group. However, given that universality of non-linear realisations of space-time symmetries has not been shown, this is not sufficient. As an illustration of the complication of redefinitions, we note that in [139] a general analysis was done involving towers of higher order transformations that extend the standard and special Galileon transformations:

$$\delta\phi = \sum_{i=0}^{N} a_{\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_i} x^{\mu_1}\cdots x^{\mu_p}\partial^{\mu_{p+1}}\phi\cdots\partial^{\mu_i}\phi\,, \tag{2.107}$$

with the parameters being totally symmetric and traceless. Via this method additional theories with an infinite number of global symmetries were found. However, later it was realised that these are all dual, and thus equivalent, to the free theory under Galileon duality transformations. An alternative classification method is by searching for enhanced soft limits of scattering amplitudes of the corresponding quantum theory. Indeed a full classification of all theories (having a kinetic term and being expandable in powers of fields and derivatives) with extended shift symmetries has been made in this manner in the series of papers [34,35,146], leading only to theories already known (up to redefinitions): DBI Galileons, Galileons, special Galileons and the free theory. This was recently confirmed in [18, 19] based on a very efficient analysis purely in terms of algebras that are consistent with inessential Goldstones.

The set of generalised multi-Galileons also contains interesting subsets that non-linearly realise particular space-time symmetries, amongst which are the multi-field versions of the ordinary, DBI, and conformal/AdS Galileons that can be obtained by considering higher co-dimension branes. A classification scheme for multi-scalar theories based on an analysis of Lie algebras has also been put forward in [18, 19]. We will discuss the AdS multi-Galileons in much more detail in Chapter 5 where we show that they can give rise to very interesting inflationary scenarios.

2.3.3 Local symmetries and gauge redundancies

Let us now consider local symmetry transformations. An infinitesimal transformation of this type expressed in the active point of view takes the following form:

$$\delta_A\phi = f_0(x,\phi,\ldots)\epsilon(x) + \ldots + f_n(x,\phi,\ldots)\epsilon^{(n)}(x)\,. \tag{2.108}$$

The corresponding local symmetry groups are always infinite dimensional (but as already noted not every infinite dimensional symmetry is local) and they always consist

of two types of transformations: those that preserve the boundary conditions, i.e. for which $\epsilon(x)|_{\partial\Omega} = \ldots = \epsilon^{(n)}(x)|_{\partial\Omega} = 0$, and those that do not. The former are called *gauge symmetries* and they are not really true symmetries, but merely signal that one is using a redundant description. In particular they imply that the equations of motion are underdetermined and that solutions to a given boundary problem are not unique but rather that they contain arbitrary functions. This is immediate by noting that if $\phi(x)$ is a solution to a certain boundary problem and if the local transformation vanishes at the boundary, then the transformed configuration is also a solution to this particular boundary problem. The latter type of local transformations, also called large gauge transformations, do map physically different solutions to each other and in that sense resemble global symmetries.

Given the viewpoint that a physical theory should give unique predictions given specified boundary and initial conditions, one demands that physical observables should always correspond to quantities that are independent of these free functions. In other words: they should be *gauge invariant*. Thus, one can specify these arbitrary functions at will without changing physical observables and performing a gauge transformation simply amounts to changing this choice. As such, gauge transformations only map solutions within the same physical equivalence class to eachother, in contrast to true symmetries that map physically different solutions to eachother. Although gauge symmetries are not true symmetries they nevertheless appear in many physically interesting theories. The reason is that in many cases one is automatically led to such a redundant description if one enforces other properties on the theory, such as for example manifest Lorentz invariance or locality (see examples below).

The arbitrariness of systems with local symmetries is reflected in the existence of identically vanishing relations amongst the equations of motion and their derivatives [137]. Starting from a local symmetry and by performing a similar calculation as for global symmetries one finds

$$E_L(\phi) \sum_{k=0}^{n} f_k(x, \phi, \ldots) \epsilon^{(k)}(x) = \nabla J \,, \tag{2.109}$$

which in turn yields:

$$\epsilon(x) \sum_{k=0}^{n} (-1)^k f_k(x, \phi, \ldots) \frac{d^k}{dx^k} E_L(\phi) + \nabla \mathcal{G} = \nabla J \,. \tag{2.110}$$

Now we restrict to the case of a gauge symmetry. Integrating both expression over some arbitrary volume Ω and using that the total derivatives depend linearly on the arbitrary functions $\epsilon(x)$ and their derivatives (which vanish on the boundary), one gets

$$\int_\Omega \epsilon(x) \Big(\sum_{k=0}^{n} (-1)^k f_k(x, \phi, \ldots) \frac{d^k}{dx^k} E_L(\phi) \Big) dx = 0 \,. \tag{2.111}$$

Since the free functions are otherwise arbitrary one concludes that

$$\sum_{k=0}^{n}(-1)^k f_k(x,\phi,\ldots)\frac{d^k}{dx^k}E_L(\phi)=0\,. \tag{2.112}$$

We stress that nowhere we have used the equations of motion and therefore this combination vanishes identically and not merely on-shell. Such combinations are called gauge identities and they are the direct consequence of the presence of a local symmetry. The converse statement also holds: any identically vanishing relation between equations of motion and their derivatives is induced by a local symmetry.

As a consequence of the gauge identities, one can show that the conserved currents one would associate to local symmetries are actually trivial:

$$\nabla(J-\mathcal{G})\equiv 0 \qquad \Rightarrow \qquad J=\mathcal{G}+\mathcal{F}, \qquad \mathcal{G}\approx 0, \qquad \nabla\mathcal{F}\equiv 0\,, \tag{2.113}$$

where we used that $\mathcal{G}$ is a particular combination of the equations of motion and derivatives thereof and thus vanishes on-shell. Of course, a theory with local symmetries can additionally have global symmetries not contained in the local ones which do lead to non-trivial currents and charges.

Example: Maxwell. Consider pure Maxwell, i.e. $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with corresponding equation of motion $E^\mu=\partial_\nu F^{\nu\mu}$. It has a local abelian symmetry group acting as $A'_\mu(x)=A_\mu(x)+\partial_\mu\theta(x)$. Accordingly there is the following gauge identity:

$$\partial_\mu E^\mu=\partial_\mu\partial_\nu F^{\mu\nu}=0\,. \tag{2.114}$$

The conserved current is given by

$$J^\mu=\partial_\nu\alpha(x)F^{\mu\nu}=\partial_\nu(\alpha(x)F^{\mu\nu})-\alpha(x)\partial_\nu F^{\mu\nu}\,, \tag{2.115}$$

which is indeed seen to be of the trivial type: the divergence of the first term vanishes identically due to antisymmetry of the field strength, and the second term vanishes on-shell. The above conclusions hold for any Lorentz invariant vector theory constructed out of $F_{\mu\nu}$ and derivatives thereof, though the explicit expressions will of course be different.

Example: GR. General Relativity and Lovelock theories are invariant under diffeomorphisms which are local symmetry transformations. On the metric these act as:

$$\delta x^\mu=-\xi^\mu(x), \qquad \delta g_{\mu\nu}=2g_{\alpha(\mu}\partial_{\nu)}\xi^\alpha \qquad \Leftrightarrow \qquad \delta_A g_{\mu\nu}=2g_{\alpha(\mu}\partial_{\nu)}\xi^\alpha+\xi^\rho\partial_\rho g_{\mu\nu}\,, \tag{2.116}$$

The corresponding gauge identities are:

$$\nabla_\mu E^{\mu\nu}=0\,. \tag{2.117}$$

If there is matter coupled to gravity in a diffemorphism invariant manner these identities get modified. For example in the case of scalar-tensor theories such as covariant Galileons one has

$$\nabla_\mu E^{\mu\nu} = -\frac{1}{2}\partial^\nu \phi E_\phi \,, \tag{2.118}$$

and in fact Horndeski made extensive use of this property to derive his most general scalar-tensor theory in four dimensions. Note that global Poincaré transformations are a subset of the diffeomorphisms (simply take $\xi^\mu(x) = \epsilon^\mu + \omega^\mu_\nu x^\nu$) implying that the corresponding currents and charges are trivial, thus leading to for instance the well-known result that one cannot properly define energy in GR. Of course, all of the above holds for any diffeomorphism invariant theory.

Example: Stückelberg trick. Lastly let us discuss the so called Stückelberg trick [161] of introducing/restoring gauge invariance. If one adds a mass term to Maxwell or GR, their respective gauge symmetries are lost. For reasons we will discuss in the next chapter one would actually like to restore these gauge symmetries and one can do so by introducing Stückelberg fields. This trick can actually be applied to any theory, though its usefulness is very much theory dependent. The general idea is as follows: consider a Lagrangian depending on some fields ϕ, i.e. $\mathcal{L} = \mathcal{L}(\phi, \phi^{(1)}, \ldots)$. Now introduce some new fields χ. One can take the viewpoint that the Lagrangian in fact also depends on these new fields, but trivially so, i.e. $\mathcal{L} = \mathcal{L}(\phi, \ldots, \chi, \ldots)$ with $\mathcal{L}_\chi = \mathcal{L}_{\chi^{(1)}} = \ldots = 0$. From this point of view the theory has a trivial gauge symmetry

$$\delta\phi = 0, \qquad \delta\chi = \xi \,, \tag{2.119}$$

with corresponding gauge identities $E_\chi = 0$. In other words the fields χ are pure gauge as they should be given the way we introduced them. Equivalence of the two viewpoints is immediate. Next, one can consider an invertible redefinition

$$\bar{\phi} = f(\phi, \phi^{(1)}, \ldots, \chi, \chi^{(1)}, \ldots), \qquad \bar{\chi} = g(\phi, \phi^{(1)}, \ldots, \chi, \chi^{(1)}, \ldots) \,, \tag{2.120}$$

and obtain an equivalent Lagrangian:

$$\bar{\mathcal{L}}(\bar{\phi}, \ldots, \bar{\chi}, \ldots) = \mathcal{L}(\phi, \ldots, \chi, \ldots) \,. \tag{2.121}$$

This Lagrangian is also gauge invariant with transformation rules induced by the redefinition:

$$\delta\bar{\phi} = \delta f, \qquad \delta\bar{\chi} = \delta g \,, \tag{2.122}$$

and its triviality is somewhat obscured and the gauge identities take a more complicated form. In this theory some of the degrees of freedom originally carried by the fields ϕ can be viewed to have been transferred to the Stückelberg fields $\bar{\chi}$.

As an example consider pure Maxwell and add a mass term thus leading to the Proca Lagrangian, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu$. The gauge symmetry, $\delta A_\mu = \partial_\mu \xi$,

present in the massless case is lost. By introducing a redundant scalar field ϕ, and subsequently performing the redefinition $\bar{A}_\mu = A_\mu - \frac{1}{m}\partial_\mu\phi$ and $\bar{\phi} = \phi$, one can recover a gauge symmetry that acts in the same way on $\bar{A}_\mu$ as it would in the massless case:

$$\delta\bar{A}_\mu = \partial_\mu\xi, \qquad \delta\bar{\phi} = -m\xi \,. \tag{2.123}$$

The resulting theory is

$$\bar{\mathcal{L}} = -\frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{1}{2}m^2\bar{A}_\mu\bar{A}^\mu - m\partial_\mu\bar{\phi}\bar{A}^\mu - \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}\,, \tag{2.124}$$

and the extra scalar degree of freedom originally carried by the massive vector field is now transferred to the Stückelberg field. In this way the different degrees of freedom of a massive vector field are disentangled. We will come back to this in the next chapter.

2.4 Symmetry breaking

Apart from the importance of symmetries being present in a theory, another important aspect of physical theories is actually the breaking of such symmetries. There are roughly two ways to break a symmetry: explicitly or spontaneously. In the former case the theory as a whole is no longer invariant under the symmetry, whereas in the latter case it still is but the configuration one is expanding around is not invariant. Both scenarios are often encountered when dealing with physical theories and they can have very distinct features and implications. Let us review the basics.

2.4.1 Explicit symmetry breaking

Starting from an invariant theory one can explicitly add terms that do not respect the symmetry, leading to a Lagrangian that is no longer invariant. If one does this arbitrarily, all information of the original symmetry is lost and one might as well have started from a theory without any symmetries. The more interesting scenario is when the breaking is small (or weak), i.e. the term that breaks the symmetry is proportional to a small parameter λ, and taking the limit $\lambda \to 0$ one recovers the original invariant Lagrangian. If this limit is continuous not only at the level of the Lagrangian but also at the level of observables, then any deviation from the invariant observables will vanish in the limit. Thus, if the parameter is taken to be small, then all predictions of the broken theory are sufficiently close to that of the invariant theory and there are approximately conserved currents and charges. Additionally, quantum corrections will be proportional to higher powers of λ and as such the smallness of λ is generically not spoiled and this gives a consistent setup which is called technically natural. We will consider such setups in the final chapter where we construct inflationary models with symmetries in the kinetic sector but whose potential weakly breaks it.

2.4.2 Spontaneous symmetry breaking

Whenever one writes down a Lagrangian one has implicitly chosen a particular field basis and one is effectively expanding around the origin, $\phi(x) = 0$, with respect to that basis. Now assume the theory has a global symmetry group that respects this origin, i.e. it leaves it invariant, then the theory is said to be *manifestly* symmetric in that field basis. A particular configuration $\phi(x) = f(x)$ is said to be invariant if it coincides with the transformed configuration, $f'(x') \equiv (g \cdot \phi(x))|_{\phi(x)=f(x)}$, when compared at the same space-time point, i.e. when:

$$f'(x) - f(x) = 0 \,. \tag{2.125}$$

In other words, one should consider the active transformation rules to correctly asses whether a configuration is invariant, which is thus the case precisely when $(g_A \cdot \phi(x))|_{\phi(x)=f(x)} = f(x)$.

Now consider a theory with such a symmetry group that is manifest in the chosen field basis, such that under the active transformation we have:

$$(g_A \cdot \phi(x))|_{\phi(x)=0} = 0 \,. \tag{2.126}$$

Although this origin (and accompanying field basis) is a convenient one in relation to the symmetry, it might not be the correct configuration around which to expand from a dynamical point of view. This is the case if the origin is not a vacuum configuration, i.e. a configuration of minimal energy in which a theory will in principle spontaneously end up. It is around a suitable vacuum that one should really be expanding the theory.

Assume $\phi_0(x) \neq 0$ is such a vacuum configuration. To correctly describe deviations from this vacuum one should perform an invertible redefinition of the form

$$\bar{\phi}(\bar{x}) = \mathcal{F}(\phi(x), x) \,, \tag{2.127}$$

with the property that $\bar{\phi}(x) = 0$ if and only if $\phi(x) = \phi_0(x)$, such that the origin of this new field basis corresponds to the vacuum configuration. The corresponding induced group action is given by

$$g \cdot \bar{\phi}(\bar{x}) \equiv \mathcal{F}(g \cdot \phi(x), g \cdot x) \,. \tag{2.128}$$

First assume that the vacuum is invariant under the original group action, i.e. $(g_A \cdot \phi(x))|_{\phi=\phi_0} = \phi_0$. The new origin is then invariant under the induced action, i.e. $(g_A \cdot \bar{\phi}(\bar{x}))|_{\bar{\phi}=0} = 0$, and the theory is thus also manifestly symmetric when written in terms of the new basis.

However, even though the theory is invariant under G, there is no reason that $\phi_0(x)$ should be too. Indeed, symmetry transformations generically map a solution to a different solution; only in special cases is a solution itself invariant. Thus, assume the vacuum is not invariant, i.e. $(g_A \cdot \phi(x))|_{\phi=\phi_0} \neq \phi_0(x)$, then neither is the origin of the new basis, i.e. $(g_A \cdot \bar{\phi}(\bar{x}))|_{\bar{\phi}=0} \neq 0$. In terms of the field basis $\bar{\phi}(\bar{x})$ the theory is

thus not manifestly invariant and the induced group action is necessarily non-linear. If one starts from a theory that is manifestly invariant, but one is driven to a vacuum configuration that is not invariant, the vacuum is said to *spontaneously break* the symmetry (even though the theory is of course invariant in any field basis).

The takeaway point is that the spontaneous breaking of a group G to a subgroup H always leads to 1) the symmetries in G/H not respecting the origin and thus being non-linearly realised and 2) the symmetries in H respecting the origin (but possibly non-linearly realised). Thus if one wants to construct an effective field theory from the bottom up one must be able to construct realisations with these properties as well as theories invariant under them. In the case of ordinary symmetries, i.e. those acting as point transformations[5] in the passive formulation, this can be done in a systematic manner by using the *coset construction* first developed by CCWZ [27, 38] for broken internal symmetries and later generalised to broken space-time symmetries by [99, 167].

In Chapter 5 we will extensively discuss the details of the coset construction, but let us already adress several aspects. We focus on constructing relativistic theories, i.e. those that are manifestly symmetric under Poincaré, with possibly additional manifest and broken symmetries[6]. In the case where the full symmetry group takes the form $G = ISO(1, d) \times G_i$ where G_i is a compact and semi-simple internal group, which is then spontaneously broken to $H = ISO(1, d) \times H_i$ (such that the broken symmetries correspond to G_i/H_i), it has been proven [38] that any point transformation satisfying 1) and 2) can be put into the coset construction form via suitable invertible field redefinitions. In these cases the coset construction thus gives universal results and allows one to construct the most general effective field theory from the bottom up. If G_i is a more general internal group, f.e. non-compact, no proof of universality exists and as such one cannot be sure of whether using the coset construction to build effective theories from the bottom up is exhaustive. However, to our knowledge no examples have been constructed that are proven to not follow from the coset construction. Similarly, if G is a more general group that no longer takes a simple direct product form but contains additional (broken) space-time symmetries, no universality statements have been made. This also applies to theories with non-linearly realised supersymmetries.

In the case of broken internal symmetries, the coset constructions makes explicit that two types of fields are present in such a resulting non-linear realisation: a set of massless fields and a set of massive fields. The former are essential for any non-linear realisation in the sense that they are a necessary and sufficient ingredient to any non-linear realisation of a given group. There is one for each broken generator in G/H and they are necessarily derivatively coupled and thus massless and they are of course the famous *Goldstones* of Goldstone's theorem [80]. The other type of fields, also called *matter* fields, are not necessarily derivatively coupled and can acquire a

[5] As far as we are aware not much research has been done with regards to the systematic construction of non-linear realisations beyond those that are point transformations.

[6] One can also use the coset construction to construct non-relativistic theories. This has been done in the context of condensed matter systems in [134]

mass. These are inessential to the non-linear realisation and they can be integrated out, or simply omitted, to obtain an EFT for the Goldstones alone.

Turning to space-time symmetries one runs into subtleties. In particular, if a broken generator T_1 commutes into another broken generator T_2 under space-time translations, i.e. schematically $[P_\mu, T_1] \propto T_2$, one can show that its associated Goldstone field is actually massive, inessential for the non-linear realisation and in that sense more akin to a matter field. In practice one can either impose a so called inverse Higgs condition [99] to eliminate it (in terms of the Goldstone, and derivatives thereof, corresponding to T_2) or integrate it out. In this way one can end up with an EFT solely in terms of the essential Goldstones.[7] We will discuss these subtleties and their implications f.e. regarding the universality question in much more detail in Chapter 5.

[7]Interestingly the resulting realisations can, but not necessarily will, take the form of (extended) contact transformations because by eliminating the inessentials one can in certain cases introduce derivative dependence in the passive transformation rules.

Chapter 3

Hamiltonians, ghosts and constraints

In the previous chapter we have focused on the basics of Lagrangian physics, and introduced all the relevant objects and ideas in order to deal with such theories. So far we have mainly described the mathematical framework; in this chapter we will focus on how to analyse some of the physical properties of a given theory. In particular we will see that there are very general arguments that greatly restrict the class of potentially physically interesting Lagrangians.

Recall that the dynamics of a Lagrangian is contained in the equations of motion and that any physical configuration should be a solution thereof. If one needs to specify N initial conditions to solve the equations of motion, the theory is said to describe $N/2$ degrees of freedom. The same definition holds for individual fields, i.e. if one needs to specify N initial conditions pertaining to a particular field it is said to describe half as many degrees of freedom (although there might not always be a clear relation between a single particular field and an initial condition). Generically the more fields and the higher the order of the Lagrangian, the more degrees of freedom are present in the theory. If a theory is non-degenerate, i.e. it is unconstrained nor contains gauge redundancies, one can directly read off the number of degrees of freedom present in the theory. This because in this case the equations of motion are all independent and their orders are what one would expect from the order of the Lagrangian. If on the other hand the theory is degenerate, i.e. it is constrained and/or contains gauge redundancies, it propagates less than the naive number of degrees of freedom because now the equations of motion are of a lower order and might additionally not all be independent (leading to gauge identities).

Now, degrees of freedom can potentially have wildly varying properties and when constructing a physical model one usually has some desired properties in mind. For example one could demand that the theory is invariant under some symmetry, which

then limits the types of fields and corresponding degrees of freedom one can consider and their possible interactions as dictated by the transformation rules of the fields. For example considering Lorentz symmetry one can consider scalar fields, vector fields, etc. in principle all with different numbers of components and potential degrees of freedom (although that of course also depends on the actual dynamics, i.e. Lagrangian). Wanting to describe something invariant under some symmetry greatly narrows down the possible Lagrangians one can consider. We will touch upon this extensively in Chapter 5.

Apart from such considerations, one also wishes that the degrees of freedom present are dynamically well-behaved. This puts severe constraints on the Lagrangians one can consider. Indeed, by far most Lagrangians suffer from one or another type of pathological behavior usually in the form of different types of instabilities. Given a theory it is not always easy to spot whether they suffer from such pathologies, but there are some types that are quite generic and widespread that lead to very general conditions on Lagrangians to be at least potentially free of this behavior. We will focus on two types of such generic sick behaviors: one has to do with higher derivatives in the theory, which generically introduce so called Ostrogradsky ghosts both at the classical as well as the quantum level [143, 171]. These ghost degrees of freedom carry negative energy and render the theory at hand unstable which as a result is unfit as a UV complete theory. As an EFT the theory could still make sense as long as the ghost degree of freedom only becomes relevant at an energy scale beyond the intended validity of the EFT. The other type specifically has to do with fermions that are seen to generically introduce negative norms after quantization again leading to major problems. Both types are inevitable when considering non-degenerate theories: non-degenerate higher derivative theories always have Ostrogradsky ghosts, and any non-degenerate fermionic theory leads to negative norm states. The only possible way to avoid these instabilities is thus by considering degenerate theories. It might indeed happen that such theories have constraints that are precisely such that they get rid of the unwanted sick degrees of freedom, thus leading to a healthy higher derivative bosonic theory and/or a healthy fermionic theory.

In order to analyse the detailed dynamical properties of the degrees of freedom of a theory, one usually resorts to the *Hamiltonian formalism.* That is, rather than using the Lagrangian and its equations of motion to do the analysis, one constructs the Hamiltonian and proceeds from there. This because the nature of degrees of freedom is often more clear from the structure of the Hamiltonian, which of course corresponds to the energy of the system. Indeed, also the appearance and nature of the two types of generic ghosts just mentioned is most easily seen in the Hamiltonian picture. However for other purposes an analysis in the Lagrangian formalism is more suited.

In the first section of this chapter we consider non-degenerate theories and review the degree of freedom counting and the Hamiltonian formalism. We will also show how the generic pathological behaviors emerge, first for higher-derivative bosonic theories and later for fermionic theories. As discussed above, eventually one would like to consider degenerate theories, which are a bit more difficult to analyse. Already

just uncovering all the constraints and redunancies and the corresponding counting of degrees of freedom can require quite some work, both in the Lagrangian as well as the Hamiltonian picture. Luckily we have two algorithms at our disposal: the *Lagrangian constraint algorithm* [154,162] and the *Dirac-Bergmann constraint algorithm* [63,89]. In the second section we will discuss the general properties of degenerate theories and review both algorithms. Along the way we will discuss several well-known higher derivatives theories. In the next chapter we will apply them to quite general higher derivative theories and derive conditions in order for them to have sufficient degeneracies to be free of the ghost degrees of freedom.

3.1 Non-degenerate theories and ghosts

The distinction between non-degenerate and degenerate Lagrangians we already mentioned, can be made precise by considering the Hessian matrix of the Lagrangian density with respect to the highest time derivatives of the fields. So for a general Lagrangian depending on multiple fields ϕ_m, which enter at some highest order N_m, this Hessian is calculated with respect to the variables $(\frac{d}{dt})^{N_m}\phi_m$. If this matrix is invertible the theory is non-degenerate, if it is singular the theory is degenerate. As we will see invertibility directly implies that no constraints or redundancies are present in the theory, and in that sense the non-degenerate Lagrangians are mathematically the simplest theories one can consider. Indeed the analysis of their equations of motion and degrees of freedom is relatively easy and so is moving to the Hamiltonian picture. However upon going to the Hamiltonian picture it becomes clear that large classes of non-degenerate theories, namely the higher derivative ones as well as fermionic ones in general, are actually very poorly behaved since they lead to instabilities. In this section we will give an overview of the generic properties of non-degenerate theories, the counting of degrees of freedom and their associated ghosts. We will start off with purely bosonic theories before moving to theories involving fermions.

Note that although the Lagrangian formalism treats time and space on an equal footing, the very definition of degrees of freedom actually breaks this as it refers merely to initial conditions rather than also to spatial boundary conditions. This stems from the fact that we consider a system to be dynamical only if it has non-trivial time evolution and as such dynamical degrees of freedom should be defined with respect to time evolution. As a result one considers the time derivative of a field, equation of motion, etc. to be independent of the field, equation of motion, etc. itself, whereas spatial derivatives thereof are considered to be dependent. This viewpoint is directly reflected in the Hamiltonian formalism which actually manifestly treats space and time differently in accordance with the special role of time evolution. Given this fact it is convenient to split our space-time coordinates accordingly, i.e. $x^\mu = (t, x^i)$ and we will mostly denote time derivatives with dots and spatial derivatives as ∂_i.

3.1.1 Bosons

To introduce the Hamiltonian formalism we first consider standard first order theories before continuing to higher derivative theories. The following is mostly a field theoretic extension of the mechanics discussion of [171].

First order theories. Consider a standard first order Lagrangian depending on an M component field and calculate the equations of motion:

$$\begin{aligned} E_L(\phi) &= -\frac{d}{dx}\mathcal{L}_{\phi^{(1)}} + \mathcal{L}_\phi \\ &= -\mathcal{L}_{\dot{\phi}\dot{\phi}}\ddot{\phi} + f(\partial_i\phi^{(1)}, \phi^{(1)}, \phi)\,. \end{aligned} \tag{3.1}$$

Non-degeneracy amounts to the condition that $\mathcal{L}_{\dot{\phi}\dot{\phi}}$ is invertible, which implies that one can rewrite the equations of motion as

$$\ddot{\phi} = \tilde{f}(\partial_i\phi^{(1)}, \phi^{(1)}, \phi)\,, \tag{3.2}$$

and that all the equations of motion are independent. Existence theorems for systems of linear second order partial differential equations then tell us that given consistent boundary conditions, the system has $2M$ solutions fully specified by the $2M$ initial conditions for $\dot{\phi}$ and ϕ. The system thus describes M degrees of freedom.

To examine these degrees of freedom in more detail we now switch to the Hamiltonian formalism. The first step is defining the *canonical variables*: [1]

$$q \equiv \phi, \qquad p \equiv \frac{\delta L}{\delta\dot{\phi}}\,, \tag{3.3}$$

that together parametrise *phase space*, (q,p), whose dimension at a given space-time point is $2M$. Thus for a mechanical system it is simply $2M$ whereas for field theories the total dimension is infinite although we will usually just refer to the dimension per space-time point.

Next, the *Hamiltonian* H and corresponding density $\mathcal{H}$ are defined as follows

$$H \equiv \int d^{d-1}x\mathcal{H}, \qquad \mathcal{H} \equiv p\dot{q} - \mathcal{L}\,. \tag{3.4}$$

Since the theory at hand is non-degenerate, the mapping $(\phi,\dot{\phi}) \leftrightarrow (q,p)$ is invertible and one can uniquely write $\dot{q} = f(q, \partial_i q, p)$. Therefore the Hamiltonian (density) can be uniquely specified as a function of the canonical variables, i.e.

$$\mathcal{H}(q,\partial_i q, p) = pf(q,\partial_i q,p) - \mathcal{L}(q,\partial_i q, f(q,\partial_i q,p)), \tag{3.5}$$

[1] Here $\frac{\delta}{\delta f}$ denotes the functional derivative, so in this example we get $\frac{\delta L}{\delta\dot{\phi}} = \mathcal{L}_{\dot{\phi}}$ as the Lagrangian does not depend on any spatial derivatives of $\dot{\phi}$.

It is easy to see by simply using all the definitions that the original Lagrangian equations of motion can be reformulated as

$$\dot{p} = -\frac{\delta H}{\delta q}, \qquad \dot{q} = \frac{\delta H}{\delta p}\,, \tag{3.6}$$

which are called *Hamilton's equations of motion.* These equations also follow from the variational principle applied to the action

$$S = \int d^d x (p\dot{q} - \mathcal{H})\,, \tag{3.7}$$

where the canonical variables (q, p) are varied. One can write the time derivative of any functional F of the canonical variables (and their derivatives) in terms of the *Poisson bracket.* This bracket is defined between any two functionals F and G in the following way

$$\{F, G\} = \int d^{d-1} x \Big(\frac{\delta F}{\delta q}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta G}{\delta q} \Big)\,. \tag{3.8}$$

The Poisson bracket is antisymmetric and bilinear and satisfies Leibniz's rule and the Jacobi identity. Using this definition and Hamilton's equations of motion, time evolution can be written as:

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}\,. \tag{3.9}$$

So if the system has no explicit time dependence, time evolution is completely governed by the Hamiltonian, i.e. the Hamiltonian generates time translations. Due to antisymmetry of the Poisson bracket the Hamiltonian is in that case seen to be conserved, as it of course should given that it corresponds to the *energy* of the system. That is, the energy of a physical configuration is simply the Hamiltonian evaluated on this configuration, whereas the Hamiltonian density of course corresponds to its energy density. As we will see it is this property that allows us to quickly identify the generic ghosts of higher derivative theories. By looking at the form of the Hamiltonian density (3.5) it is clear that there aren't any generic properties shared among non-degenerate first order theories. Rather the dynamics of the theory strongly depends on the explicit form of L (also implicitly through f).

There is much more one can say about the Hamiltonian formalism, in particular regarding its geometric properties, but this would go beyond the scope of what we need for the remainder of this thesis.

Higher order theories. Let us now consider a second order theory of an M component field:

$$\mathcal{L} = \mathcal{L}(\phi, \phi^{(1)}, \phi^{(2)})\,, \tag{3.10}$$

with equations of motion:

$$
\begin{aligned}
E_{\mathcal{L}}(\phi) &= D^{(2)}\mathcal{L}_{\phi^{(2)}} - D^{(1)}\mathcal{L}_{\phi^{(1)}} + \mathcal{L}_{\phi} \\
&= \mathcal{L}_{\ddot{\phi}\ddot{\phi}} \frac{d^4\phi}{dt^4} + f(\partial_i\phi^{(3)}, \phi^{(3)}, \ldots, \phi) \,.
\end{aligned} \tag{3.11}
$$

Non-degeneracy now says $\mathcal{L}_{\ddot{\phi}\ddot{\phi}}$ is invertible, which implies that one can uniquely solve for the fourth order time derivatives, i.e.

$$
\frac{d^4\phi}{dt^4} = f(\partial_i\phi^{(3)}, \phi^{(3)}, \ldots, \phi) \,. \tag{3.12}
$$

From this one sees that in order to solve this sytems one needs to specify initial conditions for $(\phi, \dot{\phi}, \ddot{\phi}, \dddot{\phi})$. Since there are $4M$ of them, the theory propagates $2M$ degrees of freedom: 2 per component of the field. To examine the properties of these additional degrees of freedom we again go the the Hamiltonian formalism. For second order theories the canonical variables are defined as [2]:

$$
\begin{aligned}
q^2 &\equiv \dot{\phi}, \qquad p_2 \equiv \frac{\delta L}{\delta\ddot{\phi}} \,, \\
q^1 &\equiv \phi, \qquad p_1 \equiv \frac{\delta L}{\delta\dot{\phi}} - \frac{d}{dt}\frac{\delta L}{\delta\ddot{\phi}} \,,
\end{aligned} \tag{3.13}
$$

and phase space is $4M$ dimensional (at a given point in space). Non-degeneracy implies that one can invert these relations to solve for $\ddot{\phi}$ as follows:

$$
\ddot{\phi} = f(q^1, \partial_i q^1, q^2, \partial_i q^2, p_2) \,, \tag{3.14}
$$

where we note that there is no dependence on p_1, which is quite important indeed. The Hamiltonian density is given by:

$$
\begin{aligned}
\mathcal{H} &\equiv p_1\dot{\phi} + p_2\ddot{\phi} - \mathcal{L} \\
&= p_1 q^2 + p_2 f - \mathcal{L}(q^1, \partial_i q^1, q^2, \partial_i q^2, f) \,.
\end{aligned} \tag{3.15}
$$

The Lagrangian equation of motion can be reformulated in terms of the Hamiltonian equations of motion which in this case are:

$$
\begin{aligned}
\dot{p}_1 &= -\frac{\delta H}{\delta q^1}, \qquad \dot{q}^1 = \frac{\delta H}{\delta p_1} \\
\dot{p}_2 &= -\frac{\delta H}{\delta q^2}, \qquad \dot{q}^2 = \frac{\delta H}{\delta p_2} \,.
\end{aligned} \tag{3.16}
$$

Again, the Poisson bracket can be defined on phase space in the same way and the Hamiltonian can be used for time evolution and it can be interpreted as the energy of the system.

[2] There are other choices for canonical coordinates one can pick, but these ones introduced by Ostrogradsky are particularly suited for our discussion.

By looking at the form of the Hamiltonian we notice that the last two terms are dependent on the specific form of the Lagrangian, but the first term is not. This first term is present and the same for *every* non-degenerate second order theory. It is precisely this term that poses a problem for two reasons: firstly it is the only term where the variable p_2 occurs and secondly it is linear in p_2. This means that the Hamiltonian, in addition to not being bounded from above, is not bounded from below either thus leading to a linear instability in the direction of p_2. As a result one of the degrees of freedom can carry negative amounts of energy and thus corresponds to a *ghost* degree of freedom. If the two degrees of freedom did not interact with each other this wouldn't have posed a problem due to energy conservation. However, they do interact and as such the mode with negative energy can source the positive energy mode whilst conserving the total energy of the system. This translates into physically unacceptable runaway behavior of the solutions to the equations of motion. The conclusion is that classical non-degenerate second order theories necessarily contain particular degrees of freedom, called Ostrogradsky ghosts, that lead to instabilities in the theory.

So far the discussion has been fully classical, but in fact the problem persists after quantization. One can show that there will always be negative and positive energy carrying particles and as such they can be created out of the vacuum without violating conservation of energy. Since there are vastly more states with particles than without, entropy arguments actually render the vacuum highly unstable and one is instantaneously driven from the vacuum to higher and higher particle number states. So also quantum mechanically the theory is quite troublesome.

All these problems only get worse when going to higher than second order theories. To this end consider an n-th order theory, $L(\phi, \ldots, \phi^{(n)})$. Then non-degeneracy implies that the equations of motion can be rewritten as

$$\phi^{(2n)} = f(\partial_i \phi^{(2n-1)}, \phi^{(2n-1)}, \ldots, \phi)\,, \tag{3.17}$$

and one now needs to specify $n \times 2M$ initial conditions leading to $n \times M$ degrees of freedom. Thus, every order one goes up in time derivatives there is an additional degree of freedom. The canonical variables can defined inductively as:

$$\begin{aligned} q^n &\equiv d^{n-1}\phi/dt^{n-1}, \qquad & p_n &\equiv \frac{\delta L}{\delta(d^n\phi/dt^n)}\,, \\ q^{k-1} &\equiv d^{k-1}\phi/dt^{k-1}, \qquad & p_{k-1} &\equiv \frac{\delta L}{\delta(d^{k-1}\phi/dt^{k-1})} - \frac{d}{dt}p_k\,. \end{aligned} \tag{3.18}$$

Non-degeneracy now implies one can solve to find:

$$d^n\phi/dt^n = f(q^i, \partial_i q^i, \ldots, p_n)\,, \tag{3.19}$$

i.e. it the only momentum coordinate it depends on is p_n. Thus we are led to the following structure for the Hamiltonian density

$$\mathcal{H} = p_1 q^2 + \ldots + p_{n-1}q^n + p_n f(q^i, \partial_i q^i, p_n) - \mathcal{L}(q^i, \partial_i q^i, f(q^i, \partial_i q^i, p_n))\,, \tag{3.20}$$

and we conclude that each additional degree of freedom leads to an unstable direction in phase space and as such corresponds to an Ostrogradsky ghost. The generalisation to non-degenerate theories involving multiple fields that each appear at their own order in derivatives should be obvious. We thus conclude that *any* non-degenerate theory that is higher than first order leads to one or more ghost degrees of freedom and is hence unstable.

In terms of the Lagrangian equations of motion Ostrogradsky's theorem reads as follows: a non-degenerate higher order theory implies higher than second order equations of motion that describe additional degrees of freedom and these are always ghosts. Now, one might be tempted to interpret this as meaning that if a theory leads to second order equations of motion it is free of ghosts, and if it leads to higher order equations of motion ghosts always appear. However, this would be a mistake as Ostrogradskys theorem does not imply the ghostfreeness of any non-degenerate first order theory (even though it has second order equations of motion), nor the appearance or absence of ghosts in any degenerate theory be it with higher order equations of motion or not. Indeed, there are numerous counterexamples to this overzealous interpretation and we will now discuss several of them.

Example: Derivative dependent field redefinitions. Consider a generic first order theory involving two fields

$$\mathcal{L}(\phi, \partial\phi, \chi, \partial\chi)\,, \tag{3.21}$$

and assume it to be healthy. Generic invertible redefinitions including derivatives will introduce higher order derivatives to the Lagrangian and the equations of motion. For example, one can perform an invertible redefinition of the form

$$\bar{\chi} = \chi, \qquad \bar{\phi} = \phi + f(\chi, \partial\chi)\,, \tag{3.22}$$

to obtain an equivalent but higher order Lagrangian

$$\bar{\mathcal{L}}(\bar{\phi}, \partial\bar{\phi}, \bar{\chi}, \partial\bar{\chi}, \partial\partial\bar{\chi}) \equiv \mathcal{L}(\phi, \partial\phi, \chi, \partial\chi)\,, \tag{3.23}$$

with, generically, higher than second order equations of motion. However since we have an invertible redefinition relating the two theories we know that $\bar{\mathcal{L}}$ does not propagate additional (ghost) degrees of freedom and we have thus constructed a healthy theory with higher order equations of motion. It evades Ostrogradsky's theorem because it is degenerate, as can be easily checked by calculating the appropriate Hessian and observing that it is singular:

$$\begin{pmatrix} \bar{\mathcal{L}}_{\ddot{\bar{\chi}}\ddot{\bar{\chi}}} & \bar{\mathcal{L}}_{\ddot{\bar{\chi}}\dot{\bar{\phi}}} \\ \bar{\mathcal{L}}_{\dot{\bar{\phi}}\ddot{\bar{\chi}}} & \bar{\mathcal{L}}_{\dot{\bar{\phi}}\dot{\bar{\phi}}} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{\dot{\phi}\dot{\phi}}(f_{\dot{\chi}})^2 & \mathcal{L}_{\dot{\phi}\dot{\phi}} f_{\dot{\chi}} \\ \mathcal{L}_{\dot{\phi}\dot{\phi}} f_{\dot{\chi}} & \mathcal{L}_{\dot{\phi}\dot{\phi}} \end{pmatrix}\,. \tag{3.24}$$

Thus it is trivial to generate healthy theories with higher order equations of motion. This does raise the question whether starting from a healthy theory with higher derivative equations of motion one can always put it in a first order form by performing a suitable redefinition. We will thoroughly adress this question in the next chapter where we perform a general analysis of healthy higher derivative theories.

Example: $f(R_{\mu\nu\rho\sigma})$ and $f(R)$ gravity. As already noted the equations of motion for a generic $f(R_{\mu\nu\rho\sigma})$ theory are fourth order. This introduces $5+1$ additional degrees of freedom, corresponding to massive spin 2 and massive spin 0 excitations respectively (see also the next sections). Since these degrees of freedom arise due to the emergence of additional higher order derivatives in the Lagrangian one would expect them to be ghosts. It turns out that generically the massive spin 2 degree of freedom is indeed a ghost, however interestingly the massive scalar is not. The problematic ghost degrees of freedom can be evicted by only allowing dependence on the Ricci scalar, i.e. by considering $f(R)$ gravity (see for a review [160]). The equations of motion are then given by:

$$E^{\mu\nu} = (g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu)f_R(R) + f_R(R)R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}f(R)\,. \tag{3.25}$$

As long as $f_{RR} \neq 0$ these contain fourth order time derivatives stemming from the second order time derivative of the Ricci scalar. However, only one particular combination of the metric components enter in such a way (fully specified by the form of the Ricci scalar) and as such only one particular combination of the components is actually higher derivative and only one extra higher derivative degree of freedom is present. The fact that this scalar degree of freedom is not a ghost, contrary to what one would naively expect, is most easily seen by putting the theory in an equivalent form by introducing a scalar field:

$$\mathcal{L} = \sqrt{-g}(f(\chi) + f'(\chi)(R-\chi))\,. \tag{3.26}$$

Assuming that $f_{RR} \neq 0$ (i.e. we are excluding GR), then indeed the equation of motion for χ simply gives the algebraic relation $\chi = R$, which plugged in the action yields the original action, and the two Lagrangians are thus equivalent. By doing the redefinitions $\phi = \log f'(\chi)$ and $\bar{g}_{\mu\nu} = f'(\chi)g_{\mu\nu}$ (which are invertible precisely under our assumption that $f_{RR} \neq 0$) one can put this in a more familiar form:

$$\bar{\mathcal{L}} = \sqrt{-\bar{g}}(\bar{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi))\,, \tag{3.27}$$

where $V(\phi)$ is some particular potential whose form depends on that of function f. Thus one ends up with simple GR minimally coupled to a massive scalar field, which is known to generically be ghost free. Even though in the pure metric formulation one cannot get rid of the fourth order terms, it still does not lead to a ghost. Of course the loophole is degeneracy in the form of gauge invariance, which in this particular case is sufficient to conclude that the extra degree of freedom is not an Ostrogradsky ghost. We note that this is quite exceptional: generically if an extra degree of freedom due to higher derivatives is present it is a ghost regardless of whether there is any gauge symmetry present. Indeed as already noted the gauge invariance of general $f(R_{\mu\nu\rho\sigma})$ theories is not sufficient to also make the massive spin 2 degrees of freedom healthy. In most cases the only way to avoid ghosts is to have degeneracy in such a way as to remove those degrees of freedom altogether.

Example: Vector theories. Consider a general first order theory of a vector field, i.e.

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu)\,. \tag{3.28}$$

A non-degenerate first order theory will contain D degrees of freedom, one for each component of the vector field. It turns out that one of these is actually an Ostrogradsky ghost in disguise even though the theory is first order and non-degenerate. To see this more clearly we make use of the Stückelberg trick by performing the field redefinition $\bar{A}_\mu = A_\mu - \partial_\mu \phi$ for some scalar field ϕ such that we can work with the equivalent Lagrangian (see also [92]):

$$\bar{\mathcal{L}}(\bar{A}_\mu, \partial_\mu \bar{A}_\nu, \phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) \equiv \mathcal{L}(A_\mu, \partial_\mu A_\nu)\,, \tag{3.29}$$

which has a gauge symmetry acting as $\delta \bar{A}_\mu = \partial_\mu \xi$ and $\delta\phi = \xi$. The ghost problem already emerges when considering a free theory for a massless vector, i.e. one with only quadratic kinetic terms. There are two terms one can write down up to total derivatives:

$$\mathcal{L} = \alpha F_{\mu\nu}F^{\mu\nu} + \beta S_{\mu\nu}S^{\mu\nu}, \qquad F_{\mu\nu} = \partial_{[\mu}A_{\nu]}, \qquad S_{\mu\nu} = \partial_{(\mu}A_{\nu)}\,, \tag{3.30}$$

which after Stückelberging yields:

$$\bar{\mathcal{L}} = \alpha \bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \beta \bar{S}_{\mu\nu}\bar{S}^{\mu\nu} + 2\beta \bar{S}_{\mu\nu}\partial^\mu\partial^\nu\phi + \beta\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\,. \tag{3.31}$$

Of course, due to the gauge symmetry this theory is degenerate so strictly we cannot apply Ostrogradski's theorem, but nevertheless it turns out that the degeneracy is not sufficient to evade the ghost degree of freedom associated to the second order derivatives of the scalar field. As a result one has to set $\beta = 0$ to eliminate the ghost degree of freedom. This eliminates all dependence on the scalar field and simply leads to pure Maxwell as the only healthy kinetic structure at the quadratic level. In terms of the original formulation this means that the time component A_0 does not enter with derivatives (due to the antisymmetry of the field strength) and the corresponding degree of freedom does not propagate. The ghost degree of freedom is thus seen to be absent in any theory that does not depend on $S_{\mu\nu}$, and these theories all describe massive or massless spin 1 excitations, i.e. $D-1$ and $D-2$ degrees of freedom respectively, depending on whether they depend non-derivatively on A_μ or not. Interestingly, there are also theories that depend on $S_{\mu\nu}$ that are nevertheless free of the Ostrogradsky ghosts. These so called generalised Proca theories [86] all have a specific form very much reminiscent of the Galileon structure and we will discuss them in more detail later.

Example: Tensor theories. Next consider a general theory of a symmetric tensor field $h_{\mu\nu}$, i.e. $\mathcal{L}(\partial_\rho h_{\mu\nu}, h_{\mu\nu})$.[3] A non-degenerate first order theory will describe

[3] Most of the information of this example can be found in the reviews [92, 155] and references therein.

$\frac{1}{2}D(D+1)$ degrees of freedom, one corresponding to each independent component, of which multiple ones are interpretable as Ostrogradsky ghosts. Let us focus on four space-time dimensions. To disentangle the degrees of freedom we again Stückelberg now by introducing a vector field:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}A_{\nu)}\,. \tag{3.32}$$

This introduces a gauge symmetry:

$$\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}, \qquad \delta A_\mu = -\xi_\mu\,. \tag{3.33}$$

There are four different kinetic terms one can construct up to total derivatives:

$$\alpha_1\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \alpha_2\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \alpha_3\partial_\mu h^{\mu\nu}\partial_\nu h + \alpha_4\partial_\mu h\partial^\mu h, \qquad h = h^\mu_\mu\,. \tag{3.34}$$

Introducing the vector field will generically yield problematic second order derivatives. To evade these one must set $-2\alpha_1 = \alpha_2 = -\alpha_3 = 2\alpha_4$ which actually eliminates all dependence on the vector field. Additionally ensuring that the kinetic terms have the correct sign results in the Fierz-Pauli Lagrangian for a massless symmetric tensor:

$$\mathcal{L}_{FP}^{m=0} = -\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h\partial^\mu h\,. \tag{3.35}$$

It is immediate that the theory still has gauge symmetries, called linearised diffeomorphisms, which purely act on the tensor:

$$\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}\,. \tag{3.36}$$

The theory describes the 2 degrees of freedom of a massles spin 2 particle and it corresponds to linearised General Relativity that one can obtain by letting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding to quadratic order. Indeed as the name suggests the gauge transformations are the corresponding linearisations of the diffeomorphisms of GR. There is also the well-known converse statement that in four dimensions GR is the unique non-linear completion of the Fierz-Pauli Lagrangian that does not introduce additional degrees of freedom [62].

Next one can try to add a mass term to this theory. There are two possibilities:

$$\beta_1 h^2 + \beta_2 h_{\mu\nu}h^{\mu\nu}\,. \tag{3.37}$$

Adding this to the FP theory breaks the linearised diffeomorphism invariance, which one can restore by Stückelberging. This reintroduces dependence on the first derivative of the vector field, and as we have learned from vector theories one has to be careful not to introduce ghosts even in this case. Therefore we Stückelberg once more and introduce a scalar field, i.e. $\bar{A}_\mu = A_\mu + \partial_\mu\phi$. To ensure that the resulting second order derivatives of the scalar are benign one must set $\beta_1 = -\beta_2$ leading to the quadratic Galileon structure as well as non-problematic couplings to the vector. Taking the correct sign for the mass term yields the massive Fierz-Pauli theory

$$\mathcal{L}_{FP}^{m} = \mathcal{L}_{FP}^{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\,, \tag{3.38}$$

which describes five degrees of freedom: the ± 2, ± 1 and 0 helicity states of a massive spin 2 particle in four dimensions.

One can wonder whether a non-linear completion of the massive Fierz-Pauli theory without introducing additional degrees of freedom can be found, as is the massless case. Indeed, without too much effort one can promote the mass term to a fully non-linear potential whilst keeping the kinetic term linear. For instance one can consider the potential $-2\big(\det(\delta^\mu_\nu + h^\mu_\nu) - h\big)$. More interesting would be to also introduce higher order derivative interactions, which is significantly harder. In particular one would like to find a non-linear theory that reduces to GR in the (correct) massless limit, i.e. a theory of massive gravity. Note that such a theory necessarily breaks diffeomorphism invariance since there is no covariant mass term one can construct out of the metric only. Rather one must introduce some fixed background tensor $f_{\mu\nu}$ and use it to construct mass terms by contracting it with the metric, i.e. $V = V(f_{\mu\nu}, g_{\mu\nu})$. The simplest choice is $f_{\mu\nu} = \eta_{\mu\nu}$ which preserves Lorentz invariance, but one can also pick other backgrounds with different symmetries. Since we know that diffeomorphism invariance is going to be broken, there is apriori no reason to not also consider completions of the kinetic structure that do not respect diffeomorphism invariance, but the simplest option is to promote the kinetic structure to that of GR. Doing so whilst keeping the mass term untouched actually introduces an additional scalar degree of freedom, the Boulware-Deser ghost [20], whose Ostrogradsky nature can again be made explicit by Stückelberging (this time to reinstate full diffeomorphism invariance). One is then left with trying to suitably modify the mass term to a full non-linear potential in the hope of avoiding this degree of freedom. A natural guess would be the addition of the higher order terms that reduce to Galileon invariants for the Stückelberg scalar, i.e.

$$\mathcal{L}_n = \sqrt{-g}\,\delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n} h^{\mu_1}_{\nu_1}\cdots h^{\mu_n}_{\nu_n}\,, \qquad h^\mu_\nu = \delta^\mu_\nu - g^{\mu\rho} f_{\rho\nu}\,. \tag{3.39}$$

However, this is not sufficient and problematic couplings between higher derivatives of the scalar to the metric persist for $n > 2$. Rather the unique non-linear completion, with $D-2$ free parameters, is given by dRGT massive gravity [49]. It still uses the antisymmetric Galileon structure but with respect to a more complicated combination of the dynamical and background metrics:

$$\mathcal{L} = \sqrt{-g}\Big(R + \Lambda + \sum_{i=1}^{D-1} \alpha_i \delta^{\nu_1\ldots\nu_i}_{\mu_1\ldots\mu_i} \mathcal{K}^{\mu_1}_{\nu_1}\cdots\mathcal{K}^{\mu_i}_{\nu_i}\Big)\,, \qquad \mathcal{K}^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1}f})^\mu_\nu\,. \tag{3.40}$$

One can also promote the background metric to a dynamical one by adding a kinetic term. This leads to a class of healthy bimetric theories [85]:

$$\mathcal{L} = \alpha\sqrt{-g}R[g] + \beta\sqrt{-f}R[f] + \sqrt{-g}V(f,g)\,, \tag{3.41}$$

describing $5+2$ degrees of freedom. To both dRGT gravity and the bimetric theory one can add the higher order Lovelock terms relevant to the dimension of interest without reintroducing the Boulware-Deser ghost.

3.1.2 Fermions

So far we have considered bosonic theories, where the fields take real (or complex) values. Of course many interesting theories also contain fermionic fields. If one wishes to classicaly describe fermions one must use Grassmann valued fields to capture their anti-commutation relations. All the concepts developed so far are easily generalised to fermions, as long as one keeps in mind that they anticommute and the order of products thus matters. For simplicity consider a generic first order non-degenerate mechanical system involving a fermion ψ^{α}:

$$L(\psi, \dot{\psi})\,. \tag{3.42}$$

The canonical variables are defined as (suppressing the fermion indices):

$$q \equiv \psi, \qquad p \equiv \frac{\delta L}{\delta \dot{\psi}}\,, \tag{3.43}$$

and one can invert the relations to give $\dot{\psi} = f(q,p)$. The Hamiltonian is given by:

$$H = \dot{q}p - L(\psi, f(q,p))\,. \tag{3.44}$$

As opposed to the case of higher derivative bosonic theories, there is nothing obviously wrong with the Hamiltonian. However, after quantisation it becomes clear that the theory contains negative norm states (see [89]). To see this, let us canonically quantise by promoting the phase space variables to operators and the fermionic Poisson brackets to anticommutator brackets. Upon doing so we see that the operators satisfy the following nonvanishing relations

$$\{q,q\} = \{p,p\} = 0, \qquad \{q,p\} = -i\,. \tag{3.45}$$

Given their classical properties the operator q should be hermitian, whereas p should be anti-hermitian. Making use of this we see that we can introduce new hermitian operators

$$a = \frac{1}{\sqrt{2}}(q - ip), \qquad b = \frac{1}{\sqrt{2}}(q + ip)\,, \tag{3.46}$$

which obey

$$\{a,a\} = -1, \qquad \{b,b\} = 1\,. \tag{3.47}$$

Since a is hermitian and it squares to -1 the Hilbert space must contain states with negative norms, implying that probabilities no longer sum to unity thus leading to the breakdown of the quantum theory. The analysis easily extends to non-degenerate theories of multiple fermions possibly coupled to bosons, as well as field theories. One concludes that all non-degenerate theories that contain first order fermions suffer the same problems as non-degenerate higher order bosonic theories.[4]. Thus if one wishes

[4] However, they do pop up in the quantisation of gauge theories as *unphysical* Faddeev-Popov ghosts, which can be appropriately eliminated using BRST conditions.

to have a healthy theory involving fermions one must necessarily consider degenerate systems and indeed familiar fermionic theories such as Diracs theory are precisely that (see also next section). Not much research has been done regarding higher than first order fermions.

3.2 Degenerate theories and constraint analyses

The fact that non-degenerate higher order/fermionic theories are plagued by ghosts naturally leads one to consider degenerate ones, i.e. those for which the Hessian with respect to the highest order time derivatives is singular. Indeed the arguments of the previous section crucially depend on the invertibility of the Hessian and no general argument exists for degenerate theories. As such ghosts are not inevitably present and degeneracy opens up the possibility to consider non-trivial fermionic theories as well as higher derivative theories for bosons and fermions alike. Indeed, we have already discussed many such examples in the previous chapters and sections. As already mentioned degeneracy implies that constraints and/or gauge redundancies are present in the theory and as a result such a theory propagates less degrees of freedom than one would naively expect. Thus given a degenerate theory one could hope that its degeneracy is precisely such that it eliminates the would be ghost degrees of freedom. However, let us stress again that generically this is not the case: degeneracy is a necessary condition for the absence of the generic ghosts, but it is by no means a sufficient one. For example, it might just remove some of the would be healthy degrees of freedom rather than the ghosts, which isn't very helpful.

To illustrate this, start from a non-degenerate higher derivative theory and introduce auxiliary fields to get rid of the higher order derivatives. The end result is a degenerate first order theory that propagates Ostrogradski ghosts in disguise. Thus given a degenerate theory, higher derivative or first order, one must be careful to conclude that no generic ghosts (as well as non-generic ones of course) are present and one should do a thorough analysis to examine the precise dynamical content of the theory. This can be done in the Lagrangian picture by following the *Lagrangian constraint analysis* which is an algorithm that allows one to uncover all the constraints and gauge redundancies in a system and subsequently studying the stability properties of the equations of motion. In practice this stability analysis might be quite difficult, and at times it is more convenient to resort to the Hamiltonian picture to examine the dynamics. For degenerate theories this necesitates the use of the well known *Dirac-Bergmann constraint algorithm* to uncover the constraint structure of a theory and be able to properly construct the corresponding Hamiltonian formulation.

In this section we will give an overview of both algorithms, whereas in the next chapter we will specifically apply these algorithms to quite general higher order theories to derive conditions to ensure ghostfreeness.

3.2.1 Lagrangian constraint analysis

In the Lagrangian picture degeneracy has direct implications regarding the structure of the equations of motion. By taking specific combinations of the equations of motion corresponding to the null vectors of the kinetic matrix, one finds that these are of a lower order than one would naively expect from the order of the Lagrangian. Now, there are two distinct scenarios that can occur. Firstly such a combination can be a true, not identically vanishing, dynamical *constraint equation* that relates the different lower order derivatives and fields to each other and as such reduces the space of consistent initial conditions. Any such constraint equation imposes one condition on the initial conditions and thus eliminates half a degree of freedom.[5] The second possibility is that a particular combination actually vanishes identically. As we have seen in the previous chapter where we discussed Noether's second theorem, these *gauge identities* signal the presence of gauge redundancies and they reduce the number of degrees of freedom by an amount depending on the number of effective parameters in the corresponding gauge transformations.

In addition to these *primary* constraint equations and gauge identities following directly from the null vectors of the Hessian, there might be additional ones that are not so obvious. In order to spot these one should take into account the dynamical consequences of the constraint equations (i.e. their time derivatives), which could lead to additional *secondary*, *tertiary*, etc. constraint equations as well as gauge identities. Any n-ary constraint equation and/or gauge identity is a combination of derivatives of order n and lower of the equations of motion. The way to systematically uncover all the constraint equations is via the *constraint algorithm*. By going through this algorithm one can disentangle the equations of motion thereby enabling one to do a proper counting of the degrees of freedom, which will be the main use of the algorithm in this thesis. In addition the end result provides a good starting point to do a more detailed analysis of the dynamical properties of the equations of motion, such as finding solutions and their stability properties. However, we will refrain from such analyses and solely focus on the degree of freedom counting.

We will now describe the algorithm in detail [154,162].[6] For simplicity we will first focus on the case of first order Lagrangians of (bosonic and/or fermionic) mechanical systems as the analysis is less involved and all the essential elements are present; we will later on generalise to field theories and higher order Lagrangians without much effort. Thus consider some first order Lagrangian depending on M variables q_m:

$$L(\dot{q}_m, q_m)\,. \tag{3.48}$$

[5]For bosonic mechanical systems these constraints always come in pairs, thus always ensuring an integer number of degrees of freedom. Generalising to field theory and/or fermionic variables/fields however this is no longer the case and the peculiarity of non-integer number of degrees of freedom can occur. We will comment more on this in the next chapter.

[6]We note that not all Lagrangians one writes down are actually consistent, meaning that the equations of motion might not have any solutions. A trivial example is $L = q$ yielding $0 = 1$ as an equation of motion, which is clearly inconsistent. We will not consider such pathological cases and from now on we assume consistency throughout.

The basic structure of the algorithm consists of a number of identical steps. The starting point of the analysis performed in each step is formed by a set of 'equations of motion'. In the first step these are the actual equations of motion following from the Lagrangian. We have already seen that if the Lagrangian is non-degenerate one can uniquely solve for all the second derivatives in terms of the velocities and positions and in order to solve this system one needs to specify $2M$ initial conditions, $(\dot{q}_m, q_m)_0$, and the theory thus propagates M degrees of freedom. Let us now assume the theory is degenerate such that rank $L_{\dot{q}_m\dot{q}_n} = R < M$. Then, since this Hessian is symmetric there are $M - R$ independent null vectors, v^r. Thus there are equally many combinations of the equation of motion that are lower than second order; i.e. the combinations

$$C^r \equiv E_{q_m} v^r_m = f^r(\dot{q}_n, q_n)\,, \tag{3.49}$$

are at most first order. This means one cannot solve for all the accelerations but merely R linear combinations of them. In general one can thus take suitable linear combinations of the equations of motions to split them into

- R independent second-order dynamical equations,
- a_1 independent algebraic constraint equations,
- d_1 independent first-order differential constraint equations,
- g_1 independent gauge identities

with $a_1 + d_1 + g_1 = M - R$. One might think that this concludes the first step of the analysis, but one should keep in mind that the first derivatives of algebraic constraint equations are in fact differential constraint equations. The original d_1 differential constraint equations might not all be independent of these dynamical consequences, implying that particular combinations of these identically vanish, thus leading to additional gauge identities involving first derivatives of the equations of motion. Taking this into account one finds that there are actually $d'_1 \leq d_1$ true differential constraints, $g'_1 \geq g_1$ gauge identities but still $a_1 + d'_1 + g'_1 = M - R$. Having performed the analysis leading to a number of truly independent constraints and identities concludes the first step of the algorithm.

It is now also clear why more steps are present in the algorithm: the second and first derivatives of respectively the algebraic and differential constraints are second-order differential equations with whom the R dynamical equations of the first step might be degenerate. Thus the way to proceed is to augment the original equations of motion with the corresponding dynamical consequences of the constraint equations. This larger set of 'equations of motion' can then be analysed in precisely the same manner as done in the previous step, leading to $l_2 = a_2 + d_2$ secondary constraints and g_2 additional gauge identities all involving derivatives of the original equations of motion. In turn one must consider the derivatives of these additional constraint equations and proceed to step 3, etc. At some finite step no new constraints are found

and the algorithm terminates. One has then uncovered the full set of $l = l_1+l_2+l_3+\dots$ independent constraints and $g = g_1 + g_2 + g_3 + \dots$ independent gauge identities.

After the termination of the algorithm, the number of degrees of freedom present in the theory can be quite easily computed. Each constraint equation gives a relation that the velocities and positions should satisfy and hence constrains the initial conditions. As such each constraint equation removes half a degree of freedom. The effect of gauge identities is slightly more involved. In order to correctly asess their effect, one should construct the corresponding gauge transformations via Noether's second theorem and count the number of effectively independent gauge parameters that parametrise it, which is directly related to whether the gauge identity is primary, secondary, etc. Remember that a gauge parameter and its time derivatives are all considered to be independent. It can be shown that the total number of degrees of freedom is then given by [65, 83, 90, 148]

$$M - \frac{1}{2}(l + g + e)\,. \tag{3.50}$$

where e is the total number of effective gauge parameters.[7]

Now let us generalise the above by considering first order field theories. The algorithm is in essence the same, but one has to take into account the following additions:

- During any given step of the algorithm, spatial derivatives (of any order) of the 'equations of motion' of that given step are also allowed in forming possible new constraint equations. This ties in with the viewpoint that spatial derivatives of an object are not dynamically independent of that object.
- At any step of the algorithm the 'equations of motion' might contain, in addition to purely second order time derivatives, problematic terms involving spatial derivatives of second order time derivatives. Any constraint equation must of course be free of both types of problematic terms. The spatial derivatives of the 'equations of motion' play a key role in being able to achieve this.

We will see in the next chapter that indeed these differences make an appearance when we analyse general field theories. The counting of degrees of freedom is the same as in the mechanics case bearing in mind that one calculates the number of degrees of freedom per space-time point rather than the total amount which is in principle infinite. Additionally when doing the counting one has to take into acount that a gauge parameter and any of its spatial derivatives are considered to be dependent.

The generalisation to arbitrary finite order Lagrangians is quite simple by noting that one can put any higher order theory in an equivalent first order form by introducing auxiliary fields and corresponding Lagrange multipliers. To see this consider

[7] We should note that the above degree of freedom counting is strictly only applicable if the theory at hand satisfies Diracs conjecture [63], i.e. for which each first class constraint in the Hamiltonian picture generates a gauge transformation. Counterexamples to this conjecture exist but they are quite contrived and so far every physically interesting system satifies the conjecture [89, 154].

some Lagrangian $L(\phi, \phi^{(1)}, \phi^{(2)})$, then the following Lagrangian is equivalent:

$$\bar{L}(\phi, \phi^{(1)}, \lambda) = L(\phi, A, A^{(1)}) + \lambda(\phi^{(1)} - A)\,, \tag{3.51}$$

as one can see by calculating its equations of motion:

$$\begin{aligned} E^{\phi}_{\bar{L}} &= L_{\phi} - \lambda^{(1)}\,, \\ E^{A}_{\bar{L}} &= L_A - DL_{A^{(1)}} - \lambda\,, \\ E^{l}_{\bar{L}} &= \phi^{(1)} - A\,. \end{aligned} \tag{3.52}$$

By plugging the unique solutions for the multiplier λ and auxiliary field A in the remaining equations of motion yield the original equation of motion. Thus if one wants to analyse a general higher order theory one can simply put it in first order form, do the analysis as above and then translate back to the original formulation. Although convenient, it is not necessary to resort to a first order formulation to be able to perform a Lagrangian constraint analysis. One could quite easily modify the algorithm to also be directly applicable to higher order Lagrangians. However, in practice the first order formulation is more uniform and systematic since one doesn't have to worry about different variables entering the Lagrangian with different order derivatives, which actually determines what one calls a constraint equation and what not. Of course, the first order formulation introduces additional degeneracies (by introducing auxiliary fields and multipliers) that potentially complicate the analysis, but these are actually quite benign and easy to handle. One can also take intermediate routes by only introducing auxiliary fields for some of the higher derivative fields, for example only the higher order time derivatives and not the spatial derivatives, etc.

Example: Vector theories. Consider first pure Maxwell theory, i.e. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The equations of motion are given by

$$E^{\mu} = \partial_{\nu}F^{\nu\mu}\,, \tag{3.53}$$

and from the antisymmetric structure of the field strength it is immediate that E^0 is a constraint equation. Time evolving and proceeding to the next step yields the gauge identity we already encountered, i.e. $\partial_{\mu}E^{\mu} = \partial_{\mu}\partial_{\nu}F^{\mu\nu} = 0$. No secondary constraint equations are present and the algorithm terminates. We thus have one constraint equation, one gauge identity and two effective gauge parameters since the gauge transformation involves $\partial_i\epsilon(x)$ (which are equivalent to $\epsilon(x)$ itself), as well as the time derivative $\dot{e}(x)$. We conclude that we have

$$N = D - \frac{1}{2}(1 + 1 + 2) = D - 2 \tag{3.54}$$

degrees of freedom and these correspond to the states of a spin 1 particle in D dimensions. The above is easily extended to any Lorentz invariant Lagrangian of the form $\mathcal{L}(F_{\mu\nu})$.

Next consider a massive vector as described by the Proca Lagrangian, i.e. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu$. The theory is no longer gauge invariant but E^0 is still a constraint equation and time evolving it now leads to a secondary constraint equation:

$$\partial_\mu E^\mu = \partial_\mu \partial_\nu F^{\mu\nu} + m^2 \partial_\mu A^\mu \,. \tag{3.55}$$

In other words, the gauge identity has become an ordinary constraint equation in comparison to the massless case. No further constraints are present and one finds $N = D - \frac{1}{2}(2) = D - 1$ degrees of freedom as appropriate for a massive spin 1 particle. One can easily generalise this an arbitrary theory of the form $\mathcal{L}(A_\mu, F_{\mu\nu})$.

Now consider the generalised Proca theories [86]. These depend non-trivially on the symmetric combination $S_{\mu\nu} = \partial_{(\mu}A_{\nu)}$. As we have already seen, dependence on $S_{\mu\nu}$ is generically problematic because this makes A_0 propagate a degree of freedom which is a ghost. In order to ensure that the degree of freedom corresponding to A_0 does not propagate one can consider theories with the anti-symmetric Galileon structure multiplying the symmetric pieces, i.e. consider terms like

$$\mathcal{L}_n = f_n(A_\mu A^\mu)\delta^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_n} S^{\mu_1}_{\nu_1} \cdots S^{\mu_n}_{\nu_n} \,. \tag{3.56}$$

This structure directly implies that $\mathcal{L}_{\dot{A}_0 \dot{A}_0} = \mathcal{L}_{\dot{A}_0 \dot{A}_i} = 0$ for these theories, and one can immediately conclude that E^0 is a primary constraint equation (and one can show that there is also an accompanying secondary constraint equation). Thus the problematic degree of freedom does not propagate. In the Stückelberg formulation this means that the second derivatives of the scalar field precisely come in the benign Galileon combinations (as well as non-problematically coupled to the healthy sector), thus leading to second order equations of motion.

Example: Gravity theories. Consider any Lovelock theory in D dimensions. We already noted that there are four gauge identities $\nabla_\mu E^{\mu\nu} = 0$. Since the equations of motion are at most second order in time derivatives, these directly imply that $E^{0\mu}$ are in fact at most first order in time derivatives and are thus constraint equations. Thus by going through the algorithm we will firstly find these D constraint equations, and time evolving (and of course considering relevant spatial derivatives) will then precisely lead to the D gauge identities but no further Lagrangian constraints. Thus the algorithm terminates at the second step. Since the symmetric tensor in D-dimensions contains $\frac{1}{2}D(D+1)$ components, and the gauge transformation is parametrised by $D + D$ effective parameters, ξ^μ and $\dot{\xi}^\mu$, one concludes that the number of propagating degrees of freedom is given by

$$N = \frac{1}{2}D(D+1) - \frac{1}{2}(D + D + 2D) = \frac{1}{2}D(D-3) \tag{3.57}$$

which around flat space-time correspond to the polarisations of a massless spin 2 particle in D dimensions. If one goes beyond the set of Lovelock theories the analysis becomes more involved because the equations of motion will no longer (all) be at most

second order and one should introduce auxiliary fields such that one can apply the standard algorithm. In any case, going beyond Lovelock will introduce one or more additional degrees of freedom [45]. A generic theory of the form $\mathcal{L} = \sqrt{-g}f(R_{\mu\nu\rho\sigma})$ contains $1 + \frac{1}{2}(D+1)(D-2)$ additional degrees of freedom corresponding to a scalar and a massive spin 2 respectively, leading to a total number of $D(D-2)$ degrees of freedom [61]. As already noted in the previous section, the scalar degree of freedom can be healthy but those corresponding to the massive spin 2 are ghosts.

Example: Generalised covariant Galileons. As noted if one considers a generally covariant scalar-tensor theory the gauge identities are $\nabla_\mu E^{\mu\nu} = -\frac{1}{2}\phi^\nu E_\phi$. Since in the case of generalised Galileons all equations of motion are at most second order, one can again conclude that the equations of motion $E^{0\mu}$ are constraint equations. Generically no further constraints are found and one thus concludes that there is just one additional scalar degree of freedom due to the presence of the (higher derivative) scalar field.

Example: Dirac and Volkov-Akulov. Consider the famous Dirac Lagrangian, $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial\psi - m\bar{\psi}\psi$, describing the dynamics of a four component complex fermion ψ^α. It is immediate that the theory is degenerate, as it should be, from its linear dependence on the first derivative. As a result the equations of motion for all four components are first order and hence constraint equations:

$$E^\alpha = (i(\gamma^\mu)^\alpha_\beta\partial_\mu - m\delta^\alpha_\beta)\psi^\beta \,. \tag{3.58}$$

No secondary constraints are present and as such the theory propagates $N = 4 - \frac{1}{2}(4) = 2$ complex degrees of freedom and is free from the problematic modes.

Another particularly interesting purely fermionic theory is the Volkov-Akulov Lagrangian [166] for a two-component complex fermion λ^α:

$$\mathcal{L} = -\det(M^\mu_\nu) = -\delta^{\nu_1\cdots\nu_4}_{\mu_1\cdots\mu_4} M^{\mu_1}_{\nu_1}\cdots M^{\mu_4}_{\nu_4}\,, \qquad M^\mu_\nu = \delta^\mu_\nu + i\lambda\sigma^\mu\partial_\nu\bar{\lambda} - i\partial_\nu\lambda\sigma^\mu\bar{\lambda}\,. \tag{3.59}$$

This theory has the very special property that it non-linearly realises $N = 1$ supersymmetry, i.e. λ is the corresponding Goldstino, and it does so without the need for an accompanying bosonic field as is necessary for linear supersymmetry. One immediately recognises the familiar anti-symmetric structure so often encounter for healthy higher derivative bosonic theories. Also in this different setting this structure implies degeneracy: only one ν index can be zero at a time and thus the theory is at most linear in the first time derivative. As a result the equations of motion are again all constraint equations and the theory propagates $N = 2 - \frac{1}{2}(1) = 1$ complex degree of freedom and is free of the negative-norm state.

3.2.2 Hamiltonian constraint analysis

In the Hamiltonian picture degeneracy manifests itself as the non invertibility of the definition of the canonical coordinates, as well as the occurance of relations between the canonical coordinates. As a result one cannot immediately define a corresponding Hamiltonian and one has to resort to the well-known Dirac-Bergman theory of constrained Hamiltonians which we will now discuss [63, 89]. Again we will first discuss the mechanics case and adress the generalisation to field theory later. In addition, we just consider first order Lagrangians since we have seen that this can be done without loss of generality. Thus consider a degenerate Lagrangian

$$L(\phi_m, \dot{\phi}_m)\,, \tag{3.60}$$

then rank $L_{\dot{\phi}_m \dot{\phi}_n} = R < M$, which implies that the first derivatives $L_{\dot{\phi}}$ are functionally dependent. In fact there exist $M - R$ relations $\Phi(\phi, L_{\dot{\phi}}) = 0$. Thus upon introducing the standard canonical variables

$$q \equiv \phi, \qquad p \equiv L_{\dot{\phi}}\,, \tag{3.61}$$

we see that the momenta are not all independent and there exist relations between them,

$$\Phi(q,p) = 0\,, \tag{3.62}$$

which we call the *primary constraints.* Together these define a subset of phase space called the *primary constraint surface.* As a result the definition of the canonical variables is not invertible and one cannot uniquely write $\dot{\phi} = g(q,p)$. This is clear from the fact that the mapping is between an $2M$ and an $M + R < 2M$ dimensional space respectively. So due to the presence of the primary constraints there is a family of such functions that will do the job and the standard Hamiltonian $H = p\dot{q} - L$ is not uniquely defined as a function of the canonical variables. Since a proper Hamiltonian need only be well-defined on the constraint surface one can always add terms that vanish by virtue of the constraints. Thus one could, or rather should, consider the Hamiltonian:

$$H_T = H + u^m \Phi_m\,. \tag{3.63}$$

Here we introduced the multipliers u thereby defining an extended phase space (q, p, u) keeping in mind the constraints $\Phi = 0$. Indeed considering H_T and applying the variational principle to the corresponding action we find the Hamiltonian equations of motion:

$$\dot{p} = -\frac{\partial H}{\partial q} - u\frac{\partial \Phi}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p} + u\frac{\partial \Phi}{\partial p}, \qquad \Phi(p,q) = 0\,, \tag{3.64}$$

which are equivalent to the equations of motion following from the Lagrangian. The total Hamiltonian can also be used for time evolution as one can use the Hamiltonian equations of motion to show that:

$$\frac{d}{dt} f(p,q) = \{f, H_T\} = \{f, H\} + u\{f, \Phi\}\,. \tag{3.65}$$

From the equations $\dot{q} = \frac{\partial H}{\partial p} + u\frac{\partial \Phi}{\partial p}$, and the fact that the derivatives $\frac{\partial \Phi}{\partial p}$ are independent whenever the constraints are, it follows that one can in principle uniquely solve for u in terms of $(\phi, \dot{\phi})$. This leads to the following definitions:

$$q \equiv \phi, \qquad p \equiv L_{\dot{\phi}}, \qquad u \equiv u(\phi, \dot{\phi})\,, \tag{3.66}$$

whose inverse is given by

$$\phi = q, \qquad \dot{\phi} = \dot{q} = \frac{\partial H}{\partial p} + u\frac{\partial \Phi}{\partial p}, \qquad \Phi(q,p) = 0\,. \tag{3.67}$$

Therefore we have a well defined invertible mapping taking us from the Lagrangian to the Hamiltonian picture. This identification of the primary constraints and constructing the corresponding total Hamiltonian constitutes the first step of the algorithm.

The necessity of additional steps follows in a similar manner as for the Lagrangian algorithm, namely from the possibility that time evolving constraints might lead to additional constraints. Indeed, assuming that the theory at hand is consistent, the constraint surface of the theory should be consistent in that all motion takes place on this surface. Thus there are two scenarios left: either the constraint surface set by the primary constraints is invariant under time evolution and thus consistent by itself, or the theory actually possesses more constraints and only the corresponding, more restrictive, constraint surface is consistent with time evolution. To examine whether more constraints are present in the theory one must thus evolve the primary constraints. If these derivatives vanish on the primary constraint surface no further constraints beyond the primary ones are present. This is the case if and only if

$$\dot{\Phi}_m = \{\Phi_m, H\} + u^n\{\Phi_m, \Phi_n\} = a^n_m \Phi_n \approx 0\,, \tag{3.68}$$

such that indeed the derivatives vanish on the primary constraint surface. Here we use '$\approx$' to denote *weak equality*, meaning equality on the constraint surface. The other possibility is that the result is not a linear combination of the primary constraints and thus doesn't vanish on the primary constraint surface:

$$\dot{\Phi}_m = \{\Phi_m, H\} + u^n\{\Phi_m, \Phi_n\} \neq a^n_m \Phi_n\,. \tag{3.69}$$

Consistency then leads us to conclude that in fact the true constraint surface is more restrictive than the one set by the primary constraints alone and we must add the constraints set by *demanding* $\dot{\Phi}_m \approx 0$. These conditions can have two different implications. In order to see this we pick a particularly convenient basis for the primary constraints, denoted by (γ_α, χ_a), such that

$$\{\gamma_\alpha, \gamma_\beta\} \approx \{\gamma_\alpha, \chi_a\} \approx 0, \qquad \{\chi_a, \chi_b\} \approx C_{ab}, \qquad \det C \not\approx 0\,. \tag{3.70}$$

This can always be done. Using this basis one finds that demanding $\dot{\Phi}_m \approx 0$ leads to two distinct cases. First consider the constraints χ_a, then

$$\dot{\chi}_a = \{\chi_a, H\} + u^b C_{ab}\,, \tag{3.71}$$

and demanding $\dot{\chi}_a \approx 0$ doesn't set any more constraints on the canonical variables p and q, but rather determines a subset of the Lagrange multipliers:

$$u^a \approx -\left(C^{-1}\right)^{ab}\{\chi_b, H\}\,. \tag{3.72}$$

Considering γ_α however, the u^n dependence drops out and one finds an additional, *secondary*, constraint for the canonical variables set by

$$0 \approx \{\gamma_\alpha, H\}\,. \tag{3.73}$$

That is, the constraint surface of the theory is actually smaller (and thus stronger) than the one set by the primary constraints alone. Having identified all the secondary constraints and the corresponding more restricted constraint surface concludes step 2. Subsequent steps follow by similarly analysing this constraint surface set by the primary and secondary constraints. This could then lead to tertiary, quarternary, etc. constraints and additional expressions involving some of the multipliers u. Finally, after some finite number of steps (assuming one starts from a Lagrangian with a finite number of fields and derivatives) this process will terminate and one encounters no new constraints at that particular iteration of the procedure. One has then uncovered all the constraints in the theory.

We have seen that in Lagrangian picture there were two types of degeneracies, those leading to constraint equations only and those associated to gauge redundancies leading to gauge identities. The constraints encountered in the Hamiltonian analysis can be split in a similar manner, namely into *first class* and *second class* constraints (not to be confused with primary and secondary). The first class constraints have vanishing Poisson bracket with all constraints, whereas second class constraints have at least one non-vanishing bracket. Thus the splitting is very much like the one used in each step of the algorithm, but now done with respect to the full set of all constraints of the theory. An important point is that the multiplier corresponding to the primary second class constraints are determined, whereas those of the primary first class ones are not. Thus the total Hamiltonian contains arbitrary functions. This signals the presence of gauge redundancies and indeed one can show that all *primary first class* constraints can be used to generate gauge transformations, whereas this is never the case for second class constraints. Dirac conjectured [63] that in fact *all* first class constraints generate gauge transformations, but it turns out there are counterexamples. However, these counterexamples are quite contrived and so far all physically interesting theories have been found to actually satisfy the conjecture [89, 154]

Assuming the theory satisfies this conjecture one can count the degrees of freedom as follows. Let there be M first order fields, s second class constraints and f first class constraints, then the number of degrees of freedom is given by

$$M - \frac{1}{2}(s + 2f)\,. \tag{3.74}$$

This can be derived as follows. Firstly, any constraint (first or second class) gives a relation between the canonical variables that needs to be specified and therefore

each removes one phase space direction and thus half a degree of freedom. Secondly, for each first class constraint there is some arbitrary function. These one can fix by introducing gauge conditions, each of which gives an additional relation between the canonical variables, removing an additional phase space direction. The relation with the degree of freedom counting in the Hamiltonian formalism can be seen by noting that the number of second class Hamiltonian constraints is given by $l + g - e$, the number of first class Hamiltonian constraints equals e and the total number of gauge identities g equals the number of primary first class constraints (see [65,83,90,148]). Using this recovers the Lagrangian degree of freedom formula.

Like for the Lagrangian algorithm, generalising the Hamiltonian algorithm to field theories is quite simple. One should just keep in mind that at any step of the algorithm one can take spatial derivatives of the constraints at will and one has to consider those to properly determine whether the constraint surface at that step is consistent with time-evolution.

Example: Vector theories. First consider again pure Maxwell theory. The conjugate momenta are

$$p_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = \dot{A}_i, \qquad p_0(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0\,, \tag{3.75}$$

and we thus have one primary constraint $\Phi \equiv p_0$ yielding the Hamiltonian density

$$\mathcal{H}_T = p_i A^i - \mathcal{L} = \frac{1}{2} p_i p^i + \frac{1}{2} \partial_i A_j \partial^i A^j + u\Phi\,. \tag{3.76}$$

Evolving the primary constraint gives

$$\{\Phi, H_T\} = \partial^i p_i\,, \tag{3.77}$$

which doesn't vanish on the primary constraint surface and is thus a secondary constraint. Further evolution does not yield a new constraint and by calculating the bracket between the two constraints we find that $\{\Phi_p, \Phi_s\} = 0$ and they are thus first class. The number of degrees of freedom is then calculated to be $N = D - \frac{1}{2}(2 \times 2) = D - 2$, as it should be. Any theory of the form $\mathcal{L}(F_{\mu\nu})$ has a similar constraint structure.

Upon including a mass term the analysis remains largely the same with the exception of the nature of the constraints. Due to the absence of a gauge symmetry in this case the first class constraints become second class, namely $\{\Phi_p, \Phi_s\} \propto \delta(x - y)$, and one finds $N = D - \frac{1}{2}(2) = D - 1$ degrees of freedom. Similar structures emerge for general theories of the form $\mathcal{L}(A_\mu, F_{\mu\nu})$. Further generalising to generalised Proca theories (thus involving $S_{\mu\nu}$ in very specific combinations) one sees that $p_0(x)$ no longer vanishes but equals some function $f(\partial_i A_\mu, A_\mu)$, and thus leads to the primary constraint $\Phi = p_0 - f$. The secondary constraint also takes a more complicated form. For a generic theory involving $S_{\mu\nu}$ no constraints are present.

Chapter 4

Healthy higher derivative theories

In the previous chapters we have already discussed several examples of healthy higher derivative theories, such as the Galileons, Lovelock theories, covariant Galileons, dRGT massive gravity, as well as beyond Proca theories and tensor theories (both in the Stückelberg formulation). So far they all shared the property that their equations of motion are second order, and for a long time the community only considered these in a sense simplest instances of healthy higher derivative theories. As we already discussed in the previous chapter, having second order equations of motion is not necessary in order for a theory to be free of Ostrogradsky ghosts. Only recently this was widely realised and more general theories with higher than second order equations of motion were constructed. In particular in [173] healthy scalar-tensor theories of this type were obtained by starting from Horndeski and performing disformal transformations (although in hindsight these are rather trivial examples since they are related to Horndeski via an invertible redefinition), and in [78, 79] one went beyond Horndeski by letting go of the specific relations between terms and counterterms in the Horndeski Lagrangian. Following these papers several generalisations were soon made, and applied to different field content. This eventually led to the construction of the set of Degenerate Higher-Order Scalar-Tensor (DHOST) theories [14, 15, 43, 44, 120, 121] which are the most general generally covariant scalar-tensor theories in four dimensions that are up to cubic in second derivatives but nevertheless propagate $2 + 1$ degrees of freedom. Analogously, similar constructions for vector-tensor interactions were introduced in [88] and a classification (up to quadratic order) was given in [113].

For all these theories the coupling of a 'healthy' sector to the higher derivative sector is crucial. Although many examples had been constructed, a general analysis of the more formal properties of such theories was so far lacking. This was later adressed in [129] and [II] for mechanical systems, and subsequently further generalised to field theories in [III]. In this chapter we will discuss these general analyses, and what is

to follow is mostly an adaptation of [III]. Specifically, we will analyse bosonic field theories whose Lagrangians depend on M higher derivative fields ϕ_m and A 'healthy' fields q_α:

$$L(\partial_\mu \partial_\nu \phi_m, \partial_\mu \phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)\,. \tag{4.1}$$

We only include dependence up to second derivatives[1]; we will comment on yet higher derivative structures as well as theories involving fermions in the concluding section. Moreover, we make the following assumptions in order to be able to make general statements:

- The higher derivative fields are treated on an equal footing in the sense that we assume all the constraints coming in sets of M. Since we are only interested in the case where the Lagrangians truly depend on the second order time derivatives of the higher derivative fields, we assume that $L_{\ddot{\phi}_m} \neq 0$ for all m. Also, we aim to remove only the Ostrogradsky modes, so we do not consider the case of extra constraints that further reduce the number of degrees of freedom (dof).
- The theories we consider posses no gauge symmetries. In the Lagrangian analysis this means that we do not encounter any gauge identities, i.e. combinations of equations of motion (eom) that vanish identically. In the Hamiltonian analysis this means that no first class constraints are present, i.e. we assume all constraints to be second class. This means that strictly our analysis cannot be applied to for instance scalar-tensor theories, but in many cases the two types of degeneracies, i.e. gauge in the healthy sector versus non-gauge in the higher derivative sector, are effectively decoupled and the conclusions of our analysis can, with minor modifications, be applied.
- We are not interested in possible degeneracies in the healthy sector. We thus assume that the healthy sector itself is non-degenerate, which is precisely the case when the kinetic matrix $L_{\dot{q}_\alpha \dot{q}_\beta}$ is invertible.

No further assumptions are made about the functional dependence of the Lagrangian; f.e. it does not need to be polynomial in the highest derivatives. Also, we do not assume any global symmetry, space-time or internal. This means that we also consider Lorentz violating theories, although we will also specifically address Lorentz invariant ones.

In Section 4.1 we state (the complete analyses can be found in Appendix A.1 and A.2) and interpret our results following from the Lagrangian and Hamiltonian analyses of the theories described above. Specifically, we analyse the conditions to remove Ostrogradsky modes, in particular in relation to the structure of the equations

[1]Note that we do not include dependence on mixed or pure spatial higher order derivatives, e.g. $\partial_i \dot{q}_\alpha$, $\partial_i \ddot{\phi}_m$, $\partial_i \partial_j q_\alpha$, etc. which would be allowed in non-Lorentz invariant field theories. Although including such dependences would in principle modify the analysis and the resulting degeneracy conditions, we believe the general structures will remain unchanged. Therefore, in order to not clutter up the formulae and the discussion, we refrain from this more general analysis.

of motion and the counting of degrees of freedom. We first review the results for mechanical systems and subsequently generalise them to the field theory case. We conclude the section with a discussion of the special properties of Lorentz invariant theories. In Section 4.2 we propose a formal classification for healthy higher derivative theories and analyse their properties in more detail. In particular we discuss how different classes can or cannot be related via different types of redefinitions. Again we give special attention to Lorentz invariant theories. We draw a number of conclusions in Section 4.3. We will discuss several examples in more detail throughout the chapter.

4.1 Degeneracy conditions

In this section we analyse and discuss the degeneracy conditions, and their implication for the field equations, for three different systems: mechanics and general and Lorentz invariant field theories. The full derivations of the degeneracy conditions via both the Lagrangian and the Hamiltonian constraint algorithms can be found in Appendices A.1 and A.2 respectively.

4.1.1 Mechanical systems

We will start with a short recap of the results of [129] and [II]. Starting from a generic Lagrangian

$$L = L(\ddot{\phi}_m, \dot{\phi}_m, \phi_m, \dot{q}_\alpha, q_\alpha)\,, \tag{4.2}$$

that satisfies the assumptions in the introduction, one can put the theory in a first order form using auxiliary fields, and perform a Lagrangian and/or Hamiltonian analysis to determine the number of propagating dof. For a generic theory, i.e. non-degenerate, it follows that no constraints are present and the theory propagates $2M + A$ degrees of freedom, M of which are Ostrogradsky ghosts. As we have extensively discussed, healthy theories are necessarily degenerate (constrained) systems.

A key concept in the discussion of the degeneracy conditions are the vectors

$$v_m^A = (\delta_m^n, V_m^\alpha) \qquad \text{with} \qquad V_m^\alpha \equiv -L_{\ddot{\phi}_m \dot{q}_\beta} L^{-1}_{\dot{q}_\beta \dot{q}_\alpha}\,, \tag{4.3}$$

where the index A spans over the set (n, α). Demanding the existence of M primary Lagrangians constraint equations (or equivalently M primary Hamiltonian constraints) leads one to require that these vectors are null eigenvectors[2] of the Hessian matrix of the Lagrangian with respect to the velocities $\dot{\psi}_A$ of the collection $\psi_A \equiv (\dot{\phi}_m, q_\alpha)$. This is the case precisely when the *primary* conditions:

$$\begin{aligned} 0 = P_{(mn)} &\equiv v_m^A L_{\dot{\psi}_A \dot{\psi}_B} v_n^B \\ &= L_{\ddot{\phi}_m \ddot{\phi}_n} + L_{\ddot{\phi}_m \dot{q}_\alpha} V_n^\alpha\,. \end{aligned} \tag{4.4}$$

[2] Due to the normalization used in (4.3), in the following we will often refer to the components V_m^α as the null eigenvectors themselves.

are satisfied, and as such these are necessary and sufficient conditions to ensure primary degeneracy that removes $\frac{1}{2}M$ degrees of freedom. To fully remove the M ghost degrees of freedom one must additionally satisfy the *secondary* conditions:

$$\begin{aligned} 0 = S_{[mn]} &\equiv 2\, v_m^A L_{\dot{\psi}_{[A}\psi_{B]}} v_n^B \\ &= 2\left(L_{\ddot{\phi}_{[m}\dot{\phi}_{n]}} + V^\alpha_{[m} L_{\dot{q}_\alpha \dot{\phi}_{n]}} + L_{\ddot{\phi}_{[m} q_\beta} V^\beta_{n]} + V^\alpha_m L_{\dot{q}_{[\alpha} q_{\beta]}} V^\beta_n \right) . \end{aligned} \tag{4.5}$$

which in turn in turn guarantee the existence of M secondary Lagrangian constraint equations (or equivalently M secondary Hamiltoninan constraints). Therefore, if one satisfies both conditions a total number of $2M$ constraints are present and we end up with a total of $2M+A-\frac{1}{2}(2M) = M+A$ degrees of freedom and all the Ostrogradsky ghosts associated to the higher derivatives are absent.

The role of the primary and secondary conditions can be made clear at the level of the original equations of motion. First observe that one can always, whether the conditions are satisfied or not, get rid of the third and second order time derivatives of q_α in E_{ϕ_m} by considering the combination:

$$E_{\phi_m} + \frac{d}{dt}(V^\alpha_m E_{q_\alpha}) + U^\alpha_m E_{q_\alpha} = P_{(mn)}\phi^{(4)}_n + \left(\dddot{\phi}, \dot{q}, \dots\right) , \tag{4.6}$$

where U^α_m is defined in (A.21). If the primary and secondary conditions are not satisfied, this is the best one can do. One can in principle solve for ϕ_m if one specifies $4M+2A$ initial conditions, $(\dddot{\phi}_m, \ddot{\phi}_m, \dot{\phi}_m, \phi_m)_0$ and $(\dot{q}_\alpha, q_\alpha)_0$. Since E_{q_α} depends on at most $\dddot{\phi}$ and $\ddot{q}$, one can subsequently solve for $\ddot{q}_\alpha$ without having to specify additional initial conditions. Hence $\frac{1}{2}(4M+2A) = 2M+A$ dof propagate.

On the other hand, if the primary conditions are satisfied, the $\phi^{(4)}_m$ terms and also the terms nonlinear in $\dddot{\phi}_m$ are absent and one finds

$$E_{\phi_m} + \frac{d}{dt}(V^\alpha_m E_{q_\alpha}) + U^\alpha_m E_{q_\alpha} = S_{[mn]}\dddot{\phi}_n + \left(\ddot{\phi}, \dot{q}, \dots\right) . \tag{4.7}$$

If also the secondary conditions hold, the terms linear in $\dddot{\phi}_m$ drop out and one ends up with equations that contain at most $\ddot{\phi}_m$ and $\dot{q}_\alpha$. These particular combinations thus tell us that one can express the initial values $(\dddot{\phi}_m, \ddot{\phi}_m)_0$ in terms of $(\dot{\phi}_m, \phi_m)_0$ and $(\dot{q}_\alpha, q_\alpha)_0$. Therefore, to solve the full set of equations of motion, one only needs to specify $2M+2A$ initial conditions, implying that $M+A$ degrees of freedom propagate and the Ostrogradsky ghosts are absent.

Let us conclude the discussion by observing that $P_{(mn)}$ and $S_{[mn]}$ are generically independent; indeed, there exist theories where the primary conditions are satisfied but the secondary are not. Let us see what this structure implies for the number of degrees of freedom of such theories. First assume that we have an even number of primary constraints. Generically no secondary constraints are present and one finds an integer number of degrees of freedom. Now assume that there are an odd number of primary constraints. In this case there is automatically also 1 secondary constraint:

since $S_{[mn]}$ is antisymmetric and odd-dimensional, it has one null eigenvalue, leading therefore to a secondary constraint. Thus also in the case of an odd number of primary constraints, one generically has an even number of total constraints and so an integer number of degrees of freedom. Note however that these partially degenerate theories are still haunted by Ostrogradsky ghosts unless the secondary constraint is complemented by additional (tertiary, quartic, etc.) ones [130]. Note that the antisymmetry of the secondary conditions implies that if only one higher derivative variable is present, the primary condition actually implies the secondary condition.

4.1.2 Field theories

Now, let us look at the analysis for the field theory case. Starting from

$$L(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)\,, \tag{4.8}$$

again one can put the Lagrangian in a first order form via the introduction of auxiliary fields and perform a Lagrangian and Hamiltonian constraint analysis. We have performed both the analyses whose details are given in Appendix A.1 and A.2.

In particular we find that in order to eliminate the Ostrogradsky modes one must now satisfy three sets of conditions, namely one set of primary conditions and two sets of secondary conditions:

$$\begin{aligned} 0 = P_{(mn)} &\equiv v_m^A L_{\dot\psi_A\dot\psi_B} v_n^B \\ &= L_{\ddot\phi_m\ddot\phi_n} + L_{\ddot\phi_m\dot q_\alpha} V_n^\alpha\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} 0 = (S_i)_{(mn)} &\equiv 2\, v_m^A L_{\dot\psi_{(A}\partial_i\psi_{B)}} v_n^B \\ &= 2L_{\ddot\phi_{(m}\partial_i\dot\phi_{n)}} + 2V^\alpha_{(m}\left(L_{\dot q_\alpha\partial_i\dot\phi_{n)}} + L_{\partial_i q_\alpha\ddot\phi_{n)}}\right) + 2V_m^\alpha L_{\dot q_{(\alpha}\partial_i q_{\beta)}} V_n^\beta\,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} 0 = S_{[mn]} &\equiv 2\, v_m^A L_{\dot\psi_{[A}\psi_{B]}} v_n^B + 2\, v^A_{[m} L_{\dot\psi_A\partial_i\psi_B}\partial_i v^B_{n]} - \partial_i\left(v_m^A L_{\dot\psi_{[A}\partial_i\psi_{B]}} v_n^B\right) \\ &= 2\left(L_{\ddot\phi_{[m}\dot\phi_{n]}} + V^\alpha_{[m} L_{\dot q_\alpha\dot\phi_{n]}} + L_{\ddot\phi_{[m} q_\beta} V^\beta_{n]} + V_m^\alpha L_{\dot q_{[\alpha} q_{\beta]}} V_n^\beta\right) \\ &\quad + \partial_i L_{\partial_i\dot\phi_{[m}\ddot\phi_{n]}} + V^\alpha_{[m}\partial_i L_{\partial_i q_\alpha\ddot\phi_{n]}} + \partial_i L_{\partial_i\dot\phi_{[m}\dot q_\beta} V^\beta_{n]} + V_m^\alpha\partial_i L_{\partial_i q_{[\alpha}\dot q_{\beta]}} V_n^\beta \\ &\quad + \partial_i V^\beta_{[n}\left(L_{\partial_i\dot\phi_{m]}\dot q_\beta} + L_{\ddot\phi_{m]}\partial_i q_\beta} + 2V^\alpha_{m]} L_{\dot q_{(\alpha}\partial_i q_{\beta)}}\right)\,. \end{aligned} \tag{4.11}$$

Similarly to the mechanics case, satisfying the primary conditions enforces the existence of M primary constraints. In order to also have M secondary constraints, one must now satisfy both secondary conditions.

Again the role of the conditions becomes clear when looking at the equations of motion. Regardless of whether one satisfies any of the constraints, one can always get rid of $\dddot q_\alpha$, $\partial_i\ddot q_\alpha$ and $\ddot q_\alpha$ in E_{ϕ_m}, by considering the following combination of equations

$$E_{\phi_m} + \frac{d}{dt}(V_m^\alpha E_{q_\alpha}) + \partial_i(\alpha_m^{i\alpha} E_{q_\alpha}) + U_m^\alpha E_{q_\alpha} = P_{(mn)}\phi_n^{(4)} + \left(\partial_i\dddot\phi, \dddot\phi, \dot q \ldots\right)\,, \tag{4.12}$$

where $\alpha_m^{i\alpha}$ is defined in (A.22). If one satisfies the primary conditions, one can get rid of the $\phi_m^{(4)}$ terms and find

$$E_{\phi_m} + \frac{d}{dt}(V_m^\alpha E_{q_\alpha}) + \partial_i(\alpha_m^{i\alpha} E_{q_\alpha}) + U_m^\alpha E_{q_\alpha} = (S_i)_{(mn)} \partial_i \dddot{\phi}_n + \left(\dddot{\phi}, \dot{q} \ldots\right) . \quad (4.13)$$

Hence, if the symmetric secondary conditions are satisfied, the mixed higher order terms $\partial_i \dddot{\phi}_m$ also drop out leading to

$$E_{\phi_m} + \frac{d}{dt}(V_m^\alpha E_{q_\alpha}) + \partial_i(\alpha_m^{i\alpha} E_{q_\alpha}) + U_m^\alpha E_{q_\alpha} = S_{[mn]} \dddot{\phi}_n + \left(\ddot{\phi}, \dot{q}, \ldots\right) , \quad (4.14)$$

such that if one satisfies the antisymmetric secondary conditions, one can lastly get rid of the $\dddot{\phi}_m$ terms thus yielding equations containing at most $\ddot{\phi}_m$ and $\dot{q}_\alpha$ (and up to second order spatial derivatives thereof). Therefore it is again clear that one does not need to specify the naive amount of $4M + 2A$ initial conditions to solve the equations of motion, but rather only $2M + A$, thus leading to $M + A$ propagating degrees of freedom.

The presence of the additional, independent, symmetric secondary conditions modifies the dof counting (compared to the mechanics case) for partially degenerate theories where only the primary conditions are satisfied. If we have an even number of primary constraints there is no difference: there is an integer number of degrees of freedom. However, if we have an odd number of primary constraints, one generically has a *non-integer* number of degrees of freedom. This is due to the presence of the set of symmetric secondary conditions which, unlike the antisymmetric conditions, is not guaranteed to have a null eigenvalue. Therefore, generically no secondary constraints are present and a “half” degree of freedom propagates. This is known to happen in some Lorentz breaking modifications of GR, such as Horava–Lifschitz [91] and Lorentz breaking massive gravity [39].

4.1.3 Lorentz invariant theories

So far we have made no assumptions concerning possible global symmetries the theories might have. In this section we consider the case of Lorentz invariant theories. We restrict ourselves to the case where all the fields are scalars under Lorentz transformations such that we do not have to worry about additional ghosts that are generically present when dealing with other Lorentz representations (as we have extensively discussed in the previous chapter).

Consider a higher derivative scalar field theory satisfying our assumptions and let us additionally demand it to be Lorentz invariant. Performing a passive Lorentz transformation and noting that the integration measure is invariant we find:

$$\delta\mathcal{L} \equiv \bar{\mathcal{L}} - \mathcal{L} = \partial_\mu \mathcal{M}^\mu , \qquad \mathcal{M} = \mathcal{M}(\phi, \partial_\mu\phi, q) , \quad (4.15)$$

where explicitly

$$\delta\mathcal{L} = \mathcal{L}_{\phi_m}\delta\phi_m + \mathcal{L}_{\partial_\mu\phi_m}\delta(\partial_\mu\phi_m) + \mathcal{L}_{\partial_\mu\partial_\nu\phi_m}\delta(\partial_\mu\partial_\nu\phi_m) + \mathcal{L}_{q_\alpha}\delta q_\alpha + \mathcal{L}_{\partial_\mu q_\alpha}\delta(\partial_\mu q_\alpha)\,, \tag{4.16}$$

and

$$\begin{aligned} \delta\phi_m = \delta q_\alpha = 0, \qquad & \delta(\partial_\mu\phi_m) = \omega_\mu{}^\rho\partial_\rho\phi_m\,, \\ \delta(\partial_\mu q_\alpha) = \omega_\mu{}^\rho\partial_\rho q_\alpha, \qquad & \delta(\partial_\mu\partial_\nu\phi_m) = \omega_\mu{}^\rho\partial_\rho\partial_\nu\phi_m + \omega_\nu{}^\rho\partial_\mu\partial_\rho\phi_m\,. \end{aligned} \tag{4.17}$$

Now, a Lorentz transformation does not change the number of degrees of freedom nor makes one leave our ansatz. As such it does not affect the degeneracy structure, meaning that if $\mathcal{L}$ satisfies the primary condition so does $\bar{\mathcal{L}}$. This means that the variation of the primary condition, i.e.

$$\delta P_{(mn)} \equiv \bar{P}_{(mn)} - P_{(mn)}\,, \tag{4.18}$$

should vanish. One can explicitly calculate this variation to obtain

$$\begin{aligned} \delta P_{(mn)} &= \delta(v_m^A\mathcal{L}_{\dot\psi_A\dot\psi_B}v_n^B) \\ &= (\delta\mathcal{L})_{\ddot\phi_m\ddot\phi_n} + (\delta\mathcal{L})_{\ddot\phi_m\dot q_\alpha}V_n^\alpha + V_m^\beta(\delta\mathcal{L})_{\ddot\phi_n\dot q_\beta} + V_m^\alpha(\delta\mathcal{L})_{\dot q_\alpha\dot q_\beta}V_n^\beta \\ &= v_m^A(\delta\mathcal{L})_{\dot\psi_A\dot\psi_B}v_n^B\,, \end{aligned} \tag{4.19}$$

where we used

$$\delta V_m^\alpha = -\left((\delta L)_{\ddot\phi_m\dot q_\beta} + V_m^\gamma(\delta L)_{\dot q_\gamma\dot q_\beta}\right)L^{-1}_{\dot q_\beta\dot q_\alpha}\,. \tag{4.20}$$

Considering the boost transformation in the i-direction, and denoting the corresponding variation by δ_i, it follows that

$$0 = \delta_i P_{(mn)} = (P_{(mn)})_{\dot\Psi_j}\partial_i\Psi_j + (P_{(mn)})_{\partial_i\Psi_j}\dot\Psi_j + (S_i)_{(mn)}\,, \tag{4.21}$$

where we introduced the notation $\Psi \equiv \{\phi_m, \partial_\mu\phi_m, \partial_\mu\phi_m, q_\alpha\}$. Hence if the primary conditions are satisfied, automatically the symmetric secondary conditions are satisfied as well. Therefore in Lorentz invariant theories only the primary and antisymmetric secondary conditions remain as independent conditions, much resembling the mechanics case.

At the level of the equations of motion, this means that if one can get rid of the fourth order time derivative terms $\phi_m^{(4)}$ in E_{ϕ_m}, then one can automatically also get rid of the mixed terms $\partial_i\dddot\phi_m$. Let us note however that, in general, this cannot be done in a Lorentz covariant manner. This is because the combinations

$$E_{\phi_m} + \frac{d}{dt}(V_m^\alpha E_{q_\alpha}) + \partial_i(\alpha_m^{i\alpha}E_{q_\alpha}) + U_m^\alpha E_{q_\alpha}\,, \tag{4.22}$$

are Lorentz invariant only if $W_m^{\mu\alpha} \equiv (V_m^\alpha, \alpha_m^{i\alpha})$ is a Lorentz vector and U_m^α is a Lorentz scalar which, in general, is not the case. An example of such a theory is given in

the next section (see eq. (4.36)). Therefore, there is generically a tradeoff between manifest Lorentz invariance and manifestly lower order equations of motion: either the equations are manifestly Lorentz invariant and higher order, or the equations are not manifestly Lorentz invariant but lower order. Of course, there are also theories for which it *can* be done in a Lorentz covariant manner. This different behavior divides the set of healthy Lorentz invariant higher derivative theories in two subclasses. We will come back to this point in the next section.

Let us conclude by highlighting an important property of the number of degrees of freedom for partially degenerate Lorentz invariant theories. As noted, the structure of the constraint conditions for Lorentz invariant theories much resembles the one of mechanical systems. Since the symmetric secondary conditions are automatically satisfied if the primary conditions are, the counting of dof goes in the same way as for the mechanics case: one always has an integer number of degrees of freedom. We have thus explicitly shown how Lorentz invariance protects from the propagation of non-integer numbers of degrees of freedom. This is relevant for many theories of interest where there is a single (second class) primary constraint. In these theories, one does not need to check the existence of a companion secondary constraint in order to completely remove the ghost, as its presence is assured as a consequence of Lorentz invariance. We expect that this property still holds for more general cases that go beyond the present analysis of scalar theories; examples of this kind are dRGT massive gravity [49] and degenerate scalar-tensor theories [14].

4.2 Degeneracy classes

Having derived the conditions needed to ensure the absence of Ostrogradsky ghosts in higher derivative theories, we will provide a formal classification according to generic structures one finds within the class of healthy higher derivative theories[3]. In particular we will argue that one should distinguish the following dependences of the nullvectors (4.3):

- **Class I**: $V^\alpha_m = 0$.
- **Class II**: $V^\alpha_m = V^\alpha_m(\phi_n, \partial_\mu\phi_n, q_\beta)$.
- **Class III**: $V^\alpha_m = V^\alpha_m(\phi_n, \partial_\mu\phi_n, q_\beta, \partial_\mu\partial_\nu\phi_n, \partial_\mu q_\beta)$.

Note that we are defining the classes to be disjoint. For each class we will focus on the structure of the constraints and address the question to what extend they truly go

[3]Due to the very complicated nature of the conditions (they constitute a set of highly nonlinear coupled partial differential equations), they cannot be solved in full generality. One could restrict oneself to theories polynomial in $\ddot{\phi}_m$ and $\dot{q}_\alpha$, and do an order by order analysis in the number of fields and the power of the derivative terms. However, this quickly becomes intractable due to the large amount of functional freedom in the general and LI case, again leading to many conditions on these functions given as sets of coupled differential equations that cannot be easily solved. We have therefore refrained from such an analysis.

beyond theories that are first order in time derivatives (but with possibly higher order mixed derivatives). In particular we will examine under what conditions one can put the theories in such a manifestly Ostrogradsky free form, either using suitable total derivatives and/or different types of redefinitions. Again we will consider mechanical systems, generic Lorentz violating field theories and Lorentz invariant field theories.

4.2.1 Class I: trivial constraints

If V_m^α vanishes, there is no coupling between $\ddot{\phi}_m$ and $\dot{q}_\alpha$, and hence the degeneracy is fully contained in the higher derivative sector and not due to the coupling to a healthy sector. In the Hamiltonian picture, the constraints are simply given by the conjugate momenta of the higher order fields. Since the primary conditions reduce to $L_{\ddot{\phi}_m\ddot{\phi}_n} = 0$, these theories are necessarily linear in second order time derivatives. In fact, from the simplified secondary conditions one can see that the equations of motion are automatically free of problematic terms, i.e. they contain at most second order time derivatives of the fields (although they can contain mixed higher order terms like $\partial_i\ddot{\phi}_m$, etc.).

In the case of mechanical systems this class is particularly simple. The primary conditions imply linearity in $\ddot{\phi}_m$,

$$L_I(\ddot{\phi}_m, \dot{\phi}_m, \phi_m, \dot{q}_\alpha, q_\alpha) = \ddot{\phi}_n f^n(\dot{\phi}_m, \phi_m, q_\alpha) + g(\dot{\phi}_m, \phi_m, \dot{q}_\alpha, q_\alpha)\,, \tag{4.23}$$

whereas the secondary conditions, $f^m_{\dot{\phi}_n} = f^n_{\dot{\phi}_m}$, ensure the existence of a function, $F(\dot{\phi}_m, \phi_m, q_\alpha)$, such that $F_{\dot{\phi}_m} = f^m$. As a result the terms linear in $\ddot{\phi}_m$ can be absorbed in a total derivative and one concludes that Class I is actually equal to the class of first order Lagrangians modulo total derivatives:

$$L_I(\ddot{\phi}_m, \dot{\phi}_m, \phi_m, \dot{q}_\alpha, q_\alpha) = L(\dot{\phi}_m, \phi_m, \dot{q}_\alpha, q_\alpha) + \frac{d}{dt}F(\dot{\phi}_m, \phi_m, q_\alpha)\,, \tag{4.24}$$

and as such no truly higher derivatives are present in this class.

Turning to field theories, the primary conditions again imply linearity

$$\begin{aligned} L_I(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha) = \ddot{\phi}_n f^n(\partial_i\partial_\mu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_i q_\alpha, q_\alpha) \\ + g(\partial_i\partial_\mu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)\,, \end{aligned} \tag{4.25}$$

and f^m now has to satisfy the two secondary conditions

$$0 = \frac{\partial f^m}{\partial(\partial_i\dot{\phi}_n)} + \frac{\partial f^n}{\partial(\partial_i\dot{\phi}_m)}\,, \tag{4.26}$$

$$0 = \frac{\partial f^m}{\partial\dot{\phi}_n} - \frac{\partial f^n}{\partial\dot{\phi}_m} - \frac{1}{2}\partial_i\left(\frac{\partial f^m}{\partial(\partial_i\dot{\phi}_n)} - \frac{\partial f^n}{\partial(\partial_i\dot{\phi}_m)}\right)\,. \tag{4.27}$$

It is not clear whether one can always find a total derivative that puts the theory in a form without any second order time derivatives as in the case of mechanical systems. Indeed a suitable total derivative should be of the form

$$\begin{aligned}\frac{d}{dt}F(\partial_i\partial_\mu\phi_m,\partial_\mu\phi_m,\phi_m) &= F_{\dot{\phi}_m}\ddot{\phi}_m + F_{\partial_i\dot{\phi}_m}\partial_i\ddot{\phi}_m + \dots \\ &= \left(F_{\dot{\phi}_m} - \partial_i F_{\partial_i\dot{\phi}_m}\right)\ddot{\phi}_m + \partial_i(F_{\partial_i\dot{\phi}_m}\ddot{\phi}_m) + \dots \end{aligned} \tag{4.28}$$

and hence one must require that $\left(F_{\dot{\phi}_m} - \partial_i F_{\partial_i\dot{\phi}_m}\right) = f^m$. We do not know whether for any f^m satisfying the secondary conditions (4.26) and (4.27), such a function F exists. We do note that if f^m does not depend on $\partial_i\dot{\phi}_n$ (which is always the case when only one higher derivative field is present), condition (4.26) disappears and (4.27) reduces to that of mechanics. As a consequence a total derivative that puts the theory in a manifestly healthy form (i.e. without second order time derivatives but possibly higher order mixed derivatives) can in that case always be found. A necessary condition for a total derivative to exist that puts it in a fully first order form not involving any type of second order derivatives is that the equations of motion are linear in second order derivatives.

Lastly, if the theory at hand is manifestly Lorentz invariant then the equations of motion do not contain any higher order mixed terms and are thus purely second order.[4] All the known Lorentz invariant theories in this class rely on the specific antisymmetric structure we have so often encountered (see also the examples below). This structure in particular implies that f^m never depends on $\partial_i\dot{\phi}_m$ and thus these theories can always be rewritten in a manifestly healthy form via a total derivative. Of course, this total derivative does not need to respect manifest Lorentz invariance and as such generic Lorentz invariant theories in Class I do actually go beyond manifestly healthy theories that are also manifestly Lorentz invariant.

Examples. In the case of scalar field theories this class corresponds to the most general set of Lorentz invariant scalar field theories that yield second order equations of motion, and thus equals the generalised Galileons in the single field case and contains multi-Galileons [55,95,144] and their known generalizations [7,145] in the multi-field case. At the present time it is unknown what the most general form of such theories is, however as shown in [158], they are polynomial in second derivatives and have the particular antisymmetric structure. For a vector field on flat space the theory is given by the generalised Proca Lagrangian. In the context of theories of gravity Class I corresponds to the set of Lovelock theories, for scalar-tensor theories it is precisely Horndeski and the corresponding vector-tensor theory is generalised Proca theory properly covariantised by adding counterterms. All the above theories enjoy the same antisymmetric structure, as we have already seen in previous chapters, and indeed they are at most linear in second order time derivatives, no coupling exists between

[4] This implies that not only $V^\alpha_m = 0$ but also $\alpha^{i\alpha}_m = 0$, since if V^α_m vanishes then $E_{q_\alpha} = -\alpha^{i\beta}_m L_{\dot{q}_\beta\dot{q}_\alpha}\partial_i\ddot{\phi}_m + (\dots)$.

those and mixed second order derivatives, the equations of motion are second order and one can always find a total derivative to remove the second order time derivatives from the Lagrangian. Again, generically they do go beyond their manifestly healthy counterparts because of the tradeoff between manifest Lorentz invariance and first order in time derivatives.

4.2.2 Class II: linear constraints

In this class, in contrast to the former one, there is a nontrivial coupling between the healthy and higher derivative sector. This nontrivial coupling is responsible for the appearance of higher order terms in the equations of motion although, as we have seen in the previous section, one can always get rid of these terms by considering appropriate linear combinations of the equations of motion. In the Hamiltonian picture the constraints are now given by linear combinations of the conjugate momenta. Naively one would expect that Class II truly goes beyond Class I, however it turns out that one can always perform a particular derivative dependent field redefinition to put a theory in Class II in a form belonging to Class I: one can always disentangle the higher derivative sector from the healthy one. It is only after demanding additional properties, such as manifest Lorentz invariance, that the two classes are no longer equivalent because the redefinition is not guaranteed to respect the additional properties.

We will now show that the null vector has the Class II form, in other words $V_m^\alpha = V_m^\alpha(q_\beta, \phi_n, \partial_\mu\phi_n)$, if and only if there exists an invertible field redefinition of the form

$$\bar{q}_\alpha = \bar{q}_\alpha(q_\beta, \phi_n, \partial_\mu\phi_n)\,, \tag{4.29}$$

such that

$$L_{II}(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha) = \bar{L}_I(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu\bar{q}_\alpha, \bar{q}_\alpha)\,. \tag{4.30}$$

Necessity is easily established by starting from a theory in Class I, performing such a field redefinition and observing that $V_m^\alpha = -\frac{\partial\bar{q}_\beta}{\partial\dot{\phi}_m}(\frac{\partial\bar{q}_\beta}{\partial q_\alpha})^{-1}$, and therefore $V_m^\alpha = V_m^\alpha(q_\beta, \phi_n, \partial_\mu\phi_n)$. Sufficiency requires a bit more work. Consider the following system of partial differential equations

$$\frac{\partial u}{\partial\dot{\phi}_m} + V_m^\beta(q_\alpha, \phi_n, \partial_\mu\phi_n)\frac{\partial u}{\partial q_\beta} = 0\,. \tag{4.31}$$

Applying Frobenius' theorem one finds that it has A independent solutions, call them $\bar{q}_\alpha$, if and only if the following integrability conditions are satisfied

$$0 = \frac{\partial V_n^\beta}{\partial\dot{\phi}_m} - \frac{\partial V_m^\beta}{\partial\dot{\phi}_n} + V_m^\alpha\frac{\partial V_n^\beta}{\partial q_\alpha} - V_n^\alpha\frac{\partial V_m^\beta}{\partial q_\alpha} \equiv \mathcal{F}_{mn}^\beta\,. \tag{4.32}$$

Explicitly calculating these conditions, using the specific dependence of V_m^α and the fact that L_{II} satisfies the primary conditions, we obtain

$$\mathcal{F}^\beta_{mn} = L^{-1}_{\dot{q}_\beta \dot{q}_\alpha} \frac{\partial}{\partial \dot{q}_\alpha} S_{[mn]} \,. \tag{4.33}$$

Therefore it vanishes by virtue of the antisymmetric secondary conditions. By subsequently using the nondegeneracy of the healthy sector and the fact that $\bar{q}_\alpha$ are independent, one can conclude that $\frac{\partial \bar{q}_\alpha}{\partial q_\beta}$ is invertible. Thus there always exists an invertible field redefinition $\bar{q}_\alpha$ that satisfies (4.31). Now let $\bar{L}(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu\bar{q}_\alpha, \bar{q}_\alpha) \equiv L_{II}(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)$, then their null vectors are related as

$$\bar{V}_m^\alpha = \frac{\partial \bar{q}_\alpha}{\partial \dot{\phi}_m} + V_m^\beta \frac{\partial \bar{q}_\alpha}{\partial q_\beta} \,. \tag{4.34}$$

Thus, since $\bar{q}_\alpha$ satisfies (4.31), we observe that $\bar{V}_m^\alpha = 0$ and the Lagrangian $\bar{L}$ belongs to Class I, concluding our proof.

Turning to manifestly Lorentz invariant theories we note that, although they can be mapped to Class I via the above redefinition, this transformation does not need to be compatible with manifest Lorentz invariance. That is, the transformed Lagrangian might not be manifestly Lorentz invariant. In terms of the equations of motion this means that the particular combinations that are free of problematic terms are not always Lorentz covariant. As we show in Appendix B, a Lorentz invariant field redefinition exists if and only if $W_m^{\mu\alpha} \equiv (V_m^\alpha, \alpha_m^{i\alpha})$ is a Lorentz vector and

$$\frac{\partial W_n^{\mu\beta}}{\partial\partial_\nu\phi_m} - \frac{\partial W_m^{\nu\beta}}{\partial\partial_\mu\phi_n} + W_m^{\nu\alpha}\frac{\partial W_n^{\mu\beta}}{\partial q_\alpha} - W_n^{\mu\alpha}\frac{\partial W_m^{\nu\beta}}{\partial q_\alpha} = 0 \,. \tag{4.35}$$

Therefore, any theory for which this is the case is related to the most general generalised multi-Galileon theory via a Lorentz invariant field redefinition, and thus does not truly go beyond the manifestly healthy ansatz. In the opposite case instead, they really go beyond these theories. To give a simple example merely to illustrate that this set is non-empty, consider for example the following bi-scalar theory

$$L_{II} = (q\Box\phi + 2\partial_\mu q \partial^\mu \phi)^2 \,, \tag{4.36}$$

for which one can easily check that it is healthy, W^μ is not a Lorentz vector and that the corresponding redefinition does not respect manifest Lorentz invariance.

The theories that go beyond second order equations of motion that have been constructed so far all fall within Class II. Examples of these theories have been constructed mainly in the context of scalar-tensor and vector-tensor theories of gravity. The simple bi-scalar theory above tells us about the existence of theories that go beyond the generalised multi-Galileon theories as well, but a thorough analysis has not been performed in this setting.

Example: Degenerate Higher-Order Scalar-Tensor (DHOST) theories. Scalar-tensor analogues of theories in Class II have been constructed in various works: pioneered in [78, 79, 173] and later generalised in [43, 120, 121]. Eventually the most general scalar-tensor theory (in four dimensions) involving terms up to cubic order in second derivatives that nevertheless propagates at most 2 + 1 degrees of freedom was constructed in [14, 15, 44]. The starting point of the analysis is the most general Lagrangian involving terms at most cubic in second derivatives of the scalar field, i.e.

$$\mathcal{L} = \sqrt{-g}\big(f(\phi, X)R + g(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + C_1^{\mu\nu\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma} + C_2^{\mu\nu\rho\sigma\alpha\beta}\phi_{\mu\nu}\phi_{\rho\sigma}\phi_{\alpha\beta}\big)\,, \tag{4.37}$$

where $\phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$ the tensors C_1 and C_2 are the most general tensors one can construct from the metric and the scalar and its first derivative. Via a Hamiltonian analysis the conditions have been derived that the functions f, g and those contained in C_1 and C_2 have to satisfy in order to ensure the absence of the higher derivative scalar degree of freedom. Amongst these theories is of course Horndeski which is in Class I, but the rest of theories all lead to higher order equations of motion and belong to Class II. Their possible relation to Horndeski via general disformal transformations of the form

$$\bar{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\partial_\mu\phi\partial_\nu\phi\,, \tag{4.38}$$

(which are the most general generally covariant redefinitions of the form (4.29)) has been examined. It turns out that not all DHOST theories are related to Horndeski via such transformations and as such these particular subsets truly go beyond Horndeski. However as we have seen, linearity of the constraints does imply that one should be able to perform a non-covariant redefinition to get rid of the second order time derivatives (although our analysis of course strictly does not apply to theories scalar-tensor theories). Let us stress that although the theories do not propagate the scalar Ostrogradsky ghost this does not mean that the theories are well-behaved. Indeed in [52] it was found that most classes of quadratic theories that truly go beyond Horndeski are in fact ill-behaved in the gravitational sector because the tensors are either infinitely strongly coupled or non-dynamical. There is one subclass that does not suffer from this, but it has the property that its vacua spontaneously break Lorentz invariance and many (but not necessarily all) theories in this class are unstable around an FLRW background. A full analysis of the cubic DHOST theories has to our knowledge not been performed. Finally let us note that upon replacing $g_{\mu\nu}$ with $\eta_{\mu\nu}$ one generically finds a scalar field theory that goes beyond the Galileon structure and thus necessarily propagates a ghost: the coupling to the dynamical metric is crucial to eliminate it.

Example: Degenerate vector-tensor theories. One can do a similar analysis in the context of vector-tensor theories and a first step was made in [88] which was subsequently generalised in [113] where a classification up to quadratic order was made of theories that propagate 2+3 degrees of freedom. These include the

covariant generalised Proca theories which belong to Class I, but there are also theories that go beyond second order equations of motion and belong to Class II. Here also the interrelations between covariant generalised Proca and beyond generalised Proca theories under disformal transformations were considered, now being of the form $\bar{g}_{\mu\nu} = C(A_\mu A^\mu)g_{\mu\nu} + D(A_\mu A^\mu)A_\mu A_\nu$. Like for DHOST theories, not all degenerate vector-tensor theories can be mapped to generalised Proca and as such truly go beyond Class I. Finally, the vector-tensor theories can be related to scalar-tensor theories by truncating to the longitudinal mode of the vector, i.e. by substituting $A_\mu = \partial_\mu\phi$. For the covariant generalised Proca theories one ends up in Horndeski, whereas for those in Class II one lands in the correspondingly more general set of DHOST theories. Some work regarding scalar-vector-tensor theories has also been done [87].

Example: Beyond Lovelock? Beyond generalised Proca? Recently it was examined whether one can go beyond Lovelock in the sense of still having two degrees of freedom but with higher order equations of motion [45]. It turns out this is not possible: any theory with higher order equations of motion has additional degrees of freedom. Thus Lovelock stands as the most general healthy covariant tensor theory propagating the two familiar gravitational degrees of freedom. One can ask a similar question in the context of a vector theory on flat space. The generalised Proca theories all give rise to second order equations of motion and the longitudal mode is not cured due to specific coupling to the transverse sector but because it comes in the Galileon combination (in the Stückelberg formulation that is). It would be interesting to see whether one can go beyond this setting, and have non-trivial interactions between the two sectors allowing for three propagating degrees of freedom but higher order equations of motion.

4.2.3 Class III: nonlinear constraints

The dependence of the nullvectors on $\dot{q}$ and $\ddot{\phi}$ implies that the constraints in the Hamiltonian picture are nonlinear, in contrast to the linear ones of Class II. This has several implications regarding the structure of these theories. To examine things further let us focus on mechanical systems, and in particular those systems with only one higher derivative variable but A healthy variables. In this case the primary conditions reduce to the homogeneous Monge-Ampere equation in A dimensions and the secondary conditions are automatically satisfied as explained in Section 4.1.2. A general solution (for which V_α depends on $\ddot{\phi}$ and $\dot{q}$) for the MA equation can be given in parametric form [40, 70] and is given by

$$L = \ddot{\phi}\,\mathcal{F} + \mathcal{G} + \frac{\partial \mathcal{F}}{\partial V_\alpha}\mathcal{H}^\alpha\,. \tag{4.39}$$

Here $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions of the nullvector V_α and also ϕ,

$$\mathcal{H}^\alpha = -\left(\frac{\partial^2 \mathcal{F}}{\partial V_\alpha \partial V_\beta}\right)^{-1} \frac{\partial \mathcal{G}}{\partial V_\beta}\,; \tag{4.40}$$

in turn V_α has to satisfy the following relation

$$\ddot{\phi}\, V_\alpha + \mathcal{H}^\alpha(V_\beta) = \dot{q}_\alpha\,. \tag{4.41}$$

To obtain explicit solutions, one first chooses the functions $\mathcal{F}$ and $\mathcal{G}$ and subsequently solves (4.41) for $V_\alpha(\ddot{\phi}, \dot{q}_\beta, \dot{\phi}, \phi, q_\beta)$. Then plugging it into (4.39), one obtains an explicit Lagrangian in terms of the variables ϕ and q_α.

Given this general solution, we will now examine whether one can put it into manifestly healthy forms via redefinitions. Because it is easy to generate explicit examples we will focus on the $A = 1$ case. Let us first observe that, in contrast to Class II, Class III cannot be rewritten into a simpler class via the redefinitions considered for Class II. This can be seen by noting that the nullvectors of two theories (in any class) related via such transformations, $\bar{q} = \bar{q}(q, \phi, \dot{\phi})$, are related as

$$V = \left(\bar{V} - \frac{\partial \bar{q}}{\partial \dot{\phi}}\right)\left(\frac{\partial \bar{q}}{\partial q}\right)^{-1}\,. \tag{4.42}$$

Hence, starting from a theory in Class I/II, one always ends up in another theory in Class I/II. Therefore, starting from Class III, one always remains in Class III with these redefinitions. Of course one can consider more general redefinitions and as we argue in Appendix C the most general ones relevant to the situation at hand are of the form

$$\bar{t} = at + f(\phi, \dot{\phi}, q), \tag{4.43}$$

$$\bar{\phi} = g(\phi, \dot{\phi}, q), \qquad \bar{\phi}' = G(\phi, \dot{\phi}, q), \qquad \bar{\phi}'' = \tilde{G}(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q}), \tag{4.44}$$

$$\bar{q} = h(\phi, \dot{\phi}, q), \qquad \bar{q}' = H(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q}), \tag{4.45}$$

where f and g must satisfy a set of differential equations given in equation (C.14) and G, $\tilde{G}$ and H follow from f, g and h. These are generically *extended* contact transformations (as introduced in Chapter 2) and include the derivative dependent field redefinitions considered for Class II as special cases. We note that these transformations form a group under composition, which together with fact that any theory in Class II can be mapped to Class I via such a transformation, allows us to directly examine whether one can always map Class III to Class I. Starting from a theory in Class I, $\bar{L}_I$, and performing such a transformation (with $h_q, f_{\dot{\phi}} \neq 0$), one obtains a theory in Class III, L_{III}. In particular one finds

$$L_{III}(\ddot{\phi}, \dot{\phi}, \phi, \dot{q}, q) = \frac{d\bar{t}}{dt} \bar{L}_I(\bar{\phi}'', \bar{\phi}', \bar{\phi}, \bar{q}', \bar{q})\,, \tag{4.46}$$

whose nullvector is given by

$$V = -\frac{\partial \bar{q}'}{\partial \ddot{\phi}} \left(\frac{\partial \bar{q}'}{\partial \dot{q}} \right)^{-1} = \frac{C + \dot{q}}{D + \ddot{\phi}} \,, \tag{4.47}$$

where

$$C = \frac{(f_{\dot{\phi}} h_\phi - h_{\dot{\phi}} f_\phi)\dot{\phi} - h_{\dot{\phi}} a}{f_{\dot{\phi}} h_q - h_{\dot{\phi}} f_q} \,, \quad D = \frac{(h_q f_\phi - f_q h_\phi)\dot{\phi} + h_q a}{f_{\dot{\phi}} h_q - h_{\dot{\phi}} f_q} \,. \tag{4.48}$$

Generic choices for the function $\mathcal{H}$ in (4.41) however, yield nullvectors whose dependence on $\ddot{\phi}$ and $\dot{q}$ is not of this form, and thus not every theory in Class III can be reached from Class I. Interestingly, the simplest option, namely to select $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{H}$ is linear in V, i.e. $\mathcal{H} = B\,V - A$, yields

$$V = \frac{A(\phi, \dot{\phi}, q) + \dot{q}}{B(\phi, \dot{\phi}, q) + \ddot{\phi}} \,. \tag{4.49}$$

However it is not clear to us whether one can, for any A and B, find a redefinition such that $C = A$ and $D = B$. Regardless, one concludes that at most a very small *subset* of Class III can be mapped to Class I via these transformations. We fully expect these conclusions to hold for M higher derivative variables and A healthy variables as well as field theories (Lorentz invariant or not). Thus it seems that most of the theories in Class III are intrinsically higher order due to the non-linear nature of their constraints and that they cannot be brought to a manifestly healthy form via local redefinitions. This is all under the assumption that (4.43) is indeed the most general redefinition to consider (see Appendix C), which excludes the possibility of accidental cancellations that in theory could occur.

To our knowledge no interesting theories in Class III have been constructed so far and it would be interesting to examine whether viable (cosmological) models exist that cannot be mapped to any of the other Classes.

4.3 Conclusions and discussion

In this chapter we have performed a constraints analysis of field theories with two dinstinct sectors: one being higher derivative and the other first order. Restricting to theories without gauge symmetries, we have derived the conditions in order to evade the Ostrogradsky ghosts. They amount to a set of symmetric *primary* conditions and two sets of *secondary* conditions, one symmetric and the other antisymmetric. Remarkably, the symmetric secondary conditions are automatically enforced by Lorentz invariance, explaining how it protects from the propagation of a non-integer number of degrees of freedom. This in principle applies to theories such as those beyond Horndeski, as well as dRGT massive gravity, saving one from a complicated analysis to confirm its existence.

Secondly, we have outlined a number of classes of degenerate theories, depending on the properties of the null vector, and proved a number of equivalence relations between these classes. This classification is illustrated in Figure 4.1 and its most salient features are:

- All Lorentz invariant field theories in Class I can be written in a manifestly healthy, first-order form, modulo a total derivative; however, one generically sacrifices manifest Lorentz invariance in doing so.
- All field theories in Class II can be brought to Class I by means of a derivative dependent field redefinition; again, this does not necessarily preserve manifest Lorentz invariance.
- Only a very small subset of theories in Class III can be brought to Class I by means of extended contact transformations.

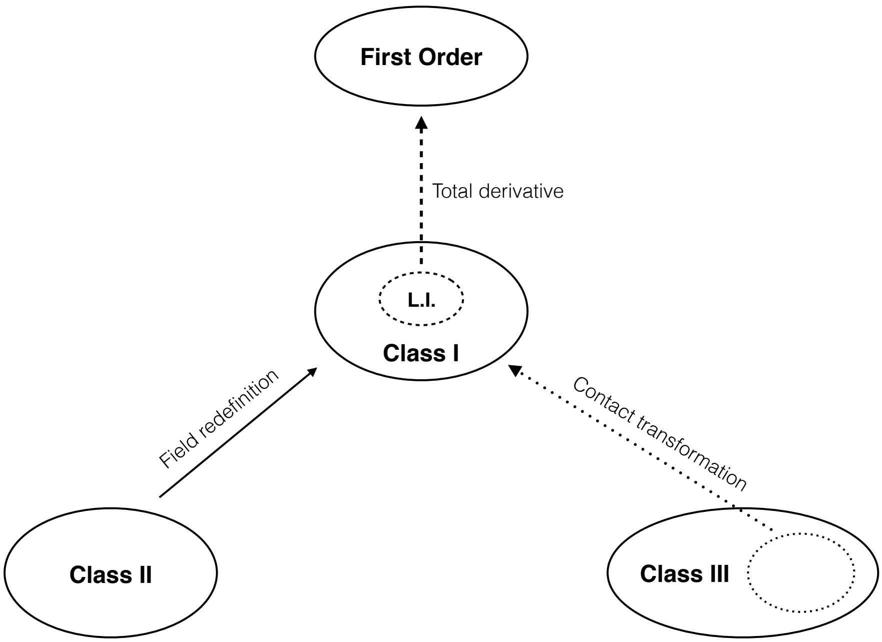

Figure 4.1: *Schematic representation of the three different classes of theories and their connections. Class II theories can always be put in Class I form via derivative dependent field redefinitions. Only a very small subset of theories in Class III can be brought to Class I with extended contact transformations. Finally, Lorentz invariant theories in Class I can be reduced to standard, first order form by adding a total derivative.*

Chapter 5

Nonlinear realisations of space-time symmetries

As previously discussed, the spontaneous breaking of symmetries is of great importance in many areas of physics and is expected to naturally occur when considering the low energy dynamics of a manifestly symmetric UV theory. In this chapter[1] we focus on the formal aspects of the corresponding non-linear realisations and in particular those following from the coset construction. We mainly investigate the subleties involved when dealing with spontaneously broken space-time symmetries. For spontaneously broken spacetime symmetries one does not always need a Goldstone field for every broken generator. Rather there can be some reduced set of essential Goldstones, corresponding to a restricted set of broken generators, which can still non-linearly realise the broken symmetry; the remaining Goldstones are inessential for the realisation and can be eliminated via several methods such as using inverse Higgs constraints or integrating them out. As will become clear, this possibility complicates the universality question for spontaneously broken space-time symmetries.

Before we delve into these subtleties, we first focus on the construction of non-linear realisations and invariant theories using the coset method for both internal and space-time symmetries in section 5.1. We then turn our attention to the possibility of inessential Goldstones and adress the intricate link between the existence of inverse Higgs constraints and the parametrisation of the coset element in section 5.2. Here we also present the conditions on the structure constants which must be satisfied in order to employ the inverse Higgs phenomenon and we will discuss how these conditions differ between coset parametrisations. Here we focus on standard inverse Higgs constraints where the inessential Goldstones are eliminated algebraically by setting a covariant derivative to zero. Later we also discuss the possibility of imposing "generalised" inverse Higgs constraints. These constraints again allow one to eliminate

[1]This chapter is mostly an adaptation of [IV].

inessential Goldstone fields but they do not follow from the usual inverse Higgs phenomenon as outlined in [99]. An example would be an equation of motion either where an inessential Goldstone is an auxilliary field and can be eliminated algebraically in comparison to the standard inverse Higgs constraint or where an inessential Goldstone is integrated out at low energies. In some cases the equations of motion give rise to the standard inverse Higgs constraints [127] but this is not always the case. We will comment on the possible (in)equivalence EFTs obtained via different constraints.

The middle part focusses on how different non-linear realisations of a broken symmetry group are related to each other. We investigate the relations between coset constructions employing different parametrisations of the coset element as well as algebra bases in section 5.3. Prior to imposing inverse Higgs constraints, the relationship between the different non-linear realisations is straightforward: they are related via standard field redefinitions and point transformation for internal and spacetime symmetries respectively. However, imposing inverse Higgs constraints generically complicates the identification of possible relations. If the inverse Higgs constraints are mapped onto each other via the point transformation, there is a naturally induced (extended) contact transformation relating the two theories, but if they do not get mapped it is unclear whether a redefinition relating them exists. This complicates the universality question even within the coset construction itself.

Note that allowing for changes in the algebra basis may seem like a unnecessary complication, but different bases can have different physical motivation. For example, consider the spontaneous breaking of the d-dimensional conformal group $SO(d,2)$ by a n-dimensional Minkowski probe brane embedded in $(d+1)$-dimensional Anti-De Sitter (AdS) space[2]. There are two natural bases for the conformal algebra: the standard conformal basis and the AdS basis. The AdS basis is of interest since the resulting non-linear realisation matches the one derived from the usual probe brane construction using the induced metric and its derivatives. To relate this non-linear realisation to the one derived using the coset construction and the standard conformal basis requires exactly the type of transformations we are considering. Interestingly, for $d = n$ (codimension one) both inverse Higgs constraints are mapped onto each other, thus establishing a contact transformation relating the two non-linear realisations [13, 42] [3]. However, as we will discuss in detail in section 5.4, in higher codimensions with $d > n$, the inverse Higgs constraints of both bases are not mapped onto each other. As a consequence, it is unclear if the equivalence is maintained. In this sense, different algebra bases are a useful way of examining the universality of non-linear realisations of spacetime symmetries.

We end with a conclusion and outlook with particular attention paid to the question of universality for spontaneously broken spacetime symmetries.

[2]In Chapter 6 we will see that an inflationary theory based on this particular coset leads to very interesting predictions regarding the CMB observables.

[3]This transformation reduces to that of the galileon duality [48, 111] after taking the appropriate contractions. The galileon duality transformation can also be extracted more straightforwardly by considering the coset construction for spontaneous breaking of the Galileon group [41, 111].

Notation: Unless otherwise stated, throughout we denote an arbitrary generator of the group G using indices $I, J, \ldots$, a broken generator using $A, B, \ldots$, an unbroken generator using $i, j, \ldots$ and the spacetime coordinates using $\mu, \nu, \ldots$. When we discuss the inverse Higgs phenomenon we will assume that A is reducible under the subgroup H and hence splits into multiple irreps, for which we will use $a, b, \ldots$ for essential and $m, n, \ldots$ for inessential Goldstones.

5.1 Coset construction

In this section we review the coset construction as a tool for constructing non-linear realisations. It allows one to systematically construct realisations of a group G on some set of fields as well as the space-time coordinates that are linear when restricted to a specified subgoup H but non-linear on the remainder G/H. Linearity ensures that any theory invariant under such a realisation is manifestly invariant under H, whereas the G/H is interpreted as being spontaneously broken. We will work in $(d+1)$ space-time dimensions and consider a group G that we assume contains the Poincaré group, i.e. $ISO(1,d)$, as a subgroup. We wish to construct theories that are manifestly invariant under this Poincaré subgroup. One could be tempted to conclude that it should thus be contained in H, but this is not the case from the coset point of view: although the full Poincaré group acts linearly on the fields, the space-time coordinates actually transform non-linearly under translations. Thus, the generators P_μ should be included in G/H rather than H; it is only the Lorentz subgroup that is contained in H. Note that in the extreme cases one can set $H = SO(1,d)$ or $H = G$. We will first construct the non-linear realisations themselves and subsequently discuss how to build invariant theories.

5.1.1 The non-linear realisation

Let us denote the broken generators of G/H (besides P_μ) by T_A and the unbroken ones by T_i. For the coset construction to be applicable, one must assume that P_μ and the set T_A both form (possibly reducible) representations of the subgroup H. This leads to the following commutators

$$\begin{aligned} &[T_i, T_j] = f_{ij}^k T_k, \qquad [T_A, T_i] = f_{Ai}^B T_B, \qquad [T_A, T_B] = f_{AB}^I T_I, \\ &[P_\mu, T_i] = f_{\mu i}^\nu P_\nu, \qquad [P_\mu, T_A] = f_{\mu A}^I T_I. \end{aligned} \tag{5.1}$$

To construct the non-linear realisation, consider an element g from the group G. Locally one can parametrise this group element in terms of the generators of G as

$$g = e^{x^\mu P_\mu} e^{\phi^A T_A} e^{\phi^i T_i}\,, \tag{5.2}$$

which is of course not a unique choice. At the heart of the coset construction lies the coset space G/H which is the set of equivalence classes of G under right multiplication

of H and one can parametrise an element of the coset as

$$\gamma(x,\phi) = e^{x^\mu P_\mu} e^{\phi^A T_A}\,, \tag{5.3}$$

which is again not a unique choice but merely a standard one, see also section (5.3). The space-time coordinates and those corresponding to the other broken generators are collectively referred to as the coset coordinates. One interprets the coordinates ϕ^A as dependent on the space-time coordinates, i.e. they are fields $\phi^A(x)$. Now consider the multiplication

$$g\gamma(x,\phi)h^{-1} \equiv \gamma(x',\phi') = e^{x'^\mu P_\mu} e^{\phi'^A T_A}\,, \qquad g = e^{a^\mu P_\mu} e^{c^A T_A} e^{c^i T_i} e^{\omega^{\mu\nu} M_{\mu\nu}}\,, \tag{5.4}$$

where we used a H transformation, $h = h(x,\phi,g)$, from the right to put the coset representative in the specified form since in general multiplication by any element of G does not preserve this choice. This action on the coset representative defines a non-linear realisation of G on the coordinates x^μ and ϕ^A as

$$g\cdot x^\mu \equiv x'^\mu\,, \qquad g\cdot\phi^A \equiv \phi'^A(x')\,, \tag{5.5}$$

and they take the form of point transformations:

$$x'^\mu = F_g^\mu(x,\phi(x))\,, \qquad \phi'^A(x') = F_g^A(x,\phi(x))\,. \tag{5.6}$$

If $g \in G/H$ then the transformations are non-linear, whereas restricted to H they become linear (which is easy to see by using that the broken generators form representations under H). Explicitly one has the following infinitesimal transformations[4]

$$\delta x^\mu = a^\mu + c^B f_B^\mu(x,\phi(x)) + \omega^\mu{}_\nu x^\nu + c^i f_{i\nu}^\mu x^\nu\,, \tag{5.7}$$

$$\delta\phi^A = c^B f_B^A(x,\phi(x)) + \omega^{\mu\nu} f_{[\mu\nu]B}^A \phi^B(x) + c^i f_{iB}^A \phi^B(x)\,, \tag{5.8}$$

where f_B^A and f_B^μ are non-linear functions of x, $\phi(x)$ and the structure constants. One recognises the standard passive action of the Poincaré group on the space-time coordinates, and the fields ϕ^A are seen to be in Lorentz representations corresponding to their generators T_A.

Next consider some other fields $\psi(x)$ which transform under some linear representation of H but not under the full group G. Using the coset coordinates, the linear action of H can be promoted to a non-linear realisation of G via

$$\psi'^\alpha(x') = g\cdot\psi^\alpha \equiv D_\beta^\alpha(h(x,\phi,g))\psi^\beta\,, \tag{5.9}$$

where the definition of $h(x,\phi,g)$ follows from (6.1). This transformation rule crucially depends on the space-time coordinates and the fields ϕ and together with 5.5 defines a consistent non-linear realisation on x^μ, ϕ^A and ψ^α. When restricted to elements in

[4] Here we have temporarily split the unbroken symmetries as $T_i \to M_{\mu\nu}, T_i$ to make the action of Lorentz transformations explicit.

H it simply reduces to the original linear representation (thus without dependence on x or ϕ).

If one restricts to the case where only internal symmetries are broken and the groups take the simple form $G = ISO(1,d) \times G_i$ and $H = SO(1,d) \times H_i$ where G_i and H_i are internal, such that the broken part is

$$G/H = ISO(1,d)/SO(1,d) \times G_i/H_i \,, \tag{5.10}$$

then the transformations induced by broken generators do not mix the space-time coordinates and the fields, one has $h = h(\phi, g)$, and the fields $\phi^A(x)$ are all scalar fields (because for internal symmetries $[T_A, M_{\mu\nu}] = 0$).

Thus, for space-time symmetries and internal symmetries alike, the coset construction allows us to efficiently construct non-linear realisations where a subgroup is linearly realised (and thus manifest in an invariant theory). Additionally, as we already noted, the real power of this formalism is that *any* non-linear realisation of a compact, semi-simple internal symmetry with a subgroup H that respects the origin in field space, can be put into this form by doing a suitable, locally invertible, field redefinition (also respecting the origin). Thus, any non-linear realisation of a compact semi-simple internal symmetry must involve the fields ϕ^A, one for each broken generator, and as we will see in the next subsection they are always massless and thus indeed correspond to the modes implied by Goldstone's classic theorem. The 'matter' fields ψ^α can acquire a mass since they transform covariantly. Additionally, in constrast to the Goldstones the matter fields are not essential in order to be able to non-linearly realise the symmetry.

Similar universal statements have not been proven for non-compact and/or non-semi-simple internal groups or general space-time groups. It is unclear whether any non-linear realisation can be put in the coset form for these more general groups; f.e. to our knowledge it has not been shown that a group action that respects the origin can be put in a linear form via an appropriate redefinition. For arbitrary internal symmetries one can at least show that all different coset parametrisations lead to equivalent realisations, whereas for space-time symmetries the possible appearance of massive Goldstones that are not essential to the realisation complicates even such a weaker statement. We will come back to this in the following sections.

5.1.2 Invariant theories

Having obtained a non-linear realisation, the aim is to derive the building blocks used to construct actions that are invariant. Since the realisation becomes linear when restricted to H, any invariant Lagrangian must be built out of objects that transform covariantly under H. The trick is to construct all objects which transform covariantly under the full group G, i.e. similar to e.g. ψ^α. Any manifest H invariant Lagrangian one constructs out of these objects will be invariant under the full non-linear transformations. The converse is also true. The relevant objects can be extracted from the

Maurer-Cartan form $\gamma^{-1}d\gamma$ which is part of the Lie algebra of G and can therefore be decomposed with respect to the generators as

$$\gamma^{-1}d\gamma = \omega^\mu P_\mu + \omega^A T_A + \omega^i T_i\,, \tag{5.11}$$

where the Maurer-Cartan components ω^I are thus coset space one-forms whose coefficients are (Taylor expandable) functions of the coset coordinates. Since one interprets the coordinates ϕ as fields one should pull back to space-time to obtain space-time one-forms $\omega^I \equiv (\omega^I)_\mu dx^\mu$ that are in principle linear in first derivatives of the fields $\phi(x)$. From the transformation law for the coset representative one can derive the following transformation rules

$$\begin{aligned} g\cdot(\omega^\mu)_\nu dx^\nu = D(h)^\mu_\nu(\omega^\nu)_\rho dx^\rho, \qquad g\cdot(\omega^A)_\nu dx^\nu = D(h)^A_B(\omega^B)_\rho dx^\rho \\ g\cdot(\omega^i)_\nu dx^\nu = D(h)^i_j(\omega^j)_\rho dx^\rho - D(h)^i_j(h^{-1}\partial_\mu h)^j dx^\mu, \end{aligned} \tag{5.12}$$

i.e. the components $(\omega^I)_\mu$ do not transform covariantly and we must use the ω^I to build invariant Lagrangians since now the coordinates transform. Also, since its the objects $(\omega^\mu)_\nu dx^\nu$ that have nice transformation properties rather than the dx^μ themselves, and to zeroth order in fields $(\omega^\mu)_\nu = \delta^\mu_\nu$, one can interpret the components $(\omega^\mu)_\nu$ as invertible vielbeins

$$e^\mu{}_\nu \equiv (\omega^\mu)_\nu\,, \tag{5.13}$$

enabling one to define a metric and corresponding invariant measure as follows

$$g_{\mu\nu} = e^\rho{}_\mu e^\sigma{}_\nu \eta_{\rho\sigma}, \qquad \sqrt{-g}d^4x = \epsilon_{\mu\nu\rho\sigma}\omega^\mu\wedge\omega^\nu\wedge\omega^\rho\wedge\omega^\sigma. \tag{5.14}$$

We can also define a covariant derivative of the fields, which has the desired covariant transformation properties, by using the Maurer-Cartan components along the directions of the broken generators as

$$\nabla_\mu\phi^A \equiv (e^{-1})^\nu_\mu(\omega^A)_\nu\,, \qquad g\cdot\nabla_\mu\phi^A = D(h)^A_B D(h)^\nu_\mu \nabla_\nu\phi^B, \tag{5.15}$$

and similarly we can define the covariant derivative of the matter fields ψ^α using the components along the directions of the unbroken generators as

$$\nabla_\mu\psi^\alpha \equiv (e^{-1})^\nu_\mu(\partial_\nu\psi^\alpha + (\omega^i)_\nu(T_i)^\alpha_\beta\psi^\beta)\,, \quad g\cdot\nabla_\mu\psi^\alpha = D(h)^\alpha_\beta D(h)^\nu_\mu\nabla_\nu\psi^\beta. \tag{5.16}$$

One can now construct H-invariant combinations out of the objects $\nabla_\mu\phi^A$, ψ^α and ∇_μ and multiply them with the invariant measure $\sqrt{-g}d^dx$ in order to obtain actions that are strongly invariant under G-transformations. Alternatively, one can construct H-invariant d-forms out of the covariantly transforming objects ω^μ, ω^A, ∇_μ, ψ^α to yield an invariant action.

Additionaly, one can also construct actions that are invariant up to a boundary term. In this case the d-form $\mathcal{L}d^dx$ shifts by an exact d-form. As a consequence its exterior derivative is an invariant $(d+1)$-form. Thus by constructing all invariant

exact $(d+1)$-forms (in the full coset space), $\alpha = d\beta$, using the covariant building blocks ω^A of the coset construction, one can find all d-forms β which are invariant either exactly or up to a total derivative. After pulling back to space-time these β's can be used to construct Lagrangians and those that shift by a total derivative are Wess-Zumino terms [169, 170]. A well known example of Wess-Zumino terms for spacetime symmetries is Galileons [81].

Restricting to the case of only broken internal symmetries one finds that the vielbein is trivial, i.e. $e^\mu_\nu = \delta^\mu_\nu$, and as a result $\nabla_\mu \phi^A = (\omega^A)_\mu$ and the invariant measure is simply $d^d x$.

5.1.3 Essential and inessential Goldstones

For internal symmetries the covariant derivatives defined above are of the form

$$\nabla_\mu \phi^A = f^A_B(\phi)\partial_\mu \phi^B \,, \tag{5.17}$$

with f some function depending on the fields and structure constants. From how one constructs invariant Lagrangians it is immediate that pure Goldstone terms always come with derivatives and the fields are thus massless. This remains true when also considering Wess-Zumino terms. Mass terms for the 'matter' fields are allowed by the realisation since ψ^α transforms covariantly. Also, from the definitions it follows that any non-linear realisation following from the coset construction must necessarily contain the Goldstones ϕ^A; they form a minimal set of fields on which to non-linearly realise a given symmetry and must necessarily be included in a low energy EFT. Whether or not one includes additional 'matter' fields depends on the case at hand.

In the case of broken space-time symmetries not every term in a covariant derivative necessarily contains derivatives. To see this consider two broken generators T_a and T_m and assume that they each form irreducible representations under H. Additionally assume that that $[P_\mu, T_m] = f^a_{\mu m} T_a + \dots$ with $f^a_{\mu m}|^n \neq 0$, i.e. there exists a non-zero component of the structure constant $f^a_{\mu m}$ once we project $\mu \times a$ on n. Then, concentrating on the covariant derivative for the ϕ^a field and projecting on n we have

$$\nabla_\mu \phi^a|^n = (e^{-1})^\nu_\mu (\omega^a)_\nu|^n = c\phi^n + \partial_\mu \phi^a|^n + \dots \tag{5.18}$$

where we used the standard parametrisation for the coset element. Thus we see that the Goldstone fields ϕ^n can enter without derivatives and one can construct a mass term. Indeed, to leading order in fields one finds

$$\sqrt{-g}(\nabla_\mu \phi^a|^n)(\nabla^\mu \phi_a|_n) = c^2 \phi^n \phi_n + \dots \tag{5.19}$$

Any Goldstone associated to a broken generator that does not commute with translations into another broken generator remains derivatively coupled and massless. Thus in general the Goldstone fields split into massless Goldstones and massive Goldstones. The massive Goldstones can always be integrated out of the EFT, resulting in a different EFT describing the dynamics of only the massless Goldstones valid up to the

scale set by the mass of the other Goldstones. Thus, the massive Goldstones are *inessential* to describe the effective low energy physics, whereas the massless ones are *essential*. In principle, the resulting non-linear realisation on the essentials might only be well-defined up to some particular order.

In the more restrictive setting where one can find a G-covariant constraint that expresses the inessential Goldstones in terms of the essential ones, a consistent realisation to all orders can be obtained purely in terms of the essentials. To see this consider a covariant constraint $\mathcal{F}^a(\phi^n, \phi^a, \partial_\mu \phi^a, \ldots)$ from which one can algebraically solve for the inessentials in terms of the essentials, i.e.

$$\mathcal{F}^m(\phi^n, \phi^a, \partial_\mu \phi^a, \ldots) = 0 \qquad \Leftrightarrow \qquad \phi^m = f^m(\phi^a, \partial_\mu \phi^a, \ldots)\,. \tag{5.20}$$

The fact that it is covariant means that

$$\mathcal{F}^m(\phi^n, \phi^a, \partial_\mu \phi^a, \ldots) = 0 \qquad \Leftrightarrow \qquad g \cdot \mathcal{F}^m(\phi^n, \phi^a, \partial_\mu \phi^a, \ldots) = 0\,, \tag{5.21}$$

which is equivalent to $(g \cdot \phi^m)|_{\mathcal{F}=0} = (g \cdot f^m)|_{\mathcal{F}=0}$. This implies that one can consistently define a realisation on just the essentials, and of course the space-time coordinates, as follows

$$g|_{\mathcal{F}=0} \cdot x^\mu \equiv (g \cdot x^\mu)|_{\mathcal{F}=0}, \qquad g|_{\mathcal{F}=0} \cdot \phi^a \equiv (g \cdot \phi^a)|_{\mathcal{F}=0}\,. \tag{5.22}$$

It is worthwhile to note that, due to the dependence of the inessentials on derivatives of the essentials, the resulting transformation rule might (but not necessarily does) have derivative dependence as well and can go beyond a standard point transformation. In any case, using this transformation rule one can in principle construct a theory purely in terms of the essential Goldstones that non-linearly realises the symmetry to all orders and energy scales (although in practice this is not needed nor does it make much sense as one is dealing with EFTs). One can also easily construct invariant theories by noting that if $\mathcal{L}(\phi^A, \partial_\mu \phi^A, \ldots)$ is invariant under the original group action, then $\tilde{\mathcal{L}}(\phi^a, \partial_\mu \phi^a, \ldots) \equiv \mathcal{L}(\phi^A, \partial_\mu \phi^A, \ldots)|_{\mathcal{F}=0}$ is invariant under this reduced group action. In the next section we will discuss the possible existence of such covariant constraints and the corresponding elimination of inessential Goldstones in much more detail.

5.2 Eliminating inessential Goldstone modes

In this section we discuss when and how the inessential Goldstones can be algebraically eliminated. For clarity we mostly focus on cases where there are only two Goldstone fields that are thus both irreducible representations with respect to H. One will be essential and one inessential and therefore we will only have to consider a single (inverse Higgs) constraint. We will also comment on more complicated cases that involve multiple inessential and essential Goldstones.

5.2.1 Standard inverse Higgs constraints

The main message we wish to convey in this subsection is that *i*) the existence of standard inverse Higgs constraints is heavily dependent on the parametrisation of the coset element and *ii*) the optimum parametrisation in this regard is not the standard one (5.30) as used in [99, 167] but rather a parametrisation with further splitting of the broken generators (5.33).

Once we have chosen a parametrisation for the coset element we can calculate all objects of interest with regards to the non-linear realisation as explained in section 5.1. In terms of eliminating inessential Goldstone fields the object of most interest is the covariant derivative which in terms of the Maurer-Cartan components is given by

$$\nabla_\mu \phi^A = (e^{-1})_\mu^\nu (\omega^A)_\nu \,. \tag{5.23}$$

From our assumption of two Goldstone fields, it follows that A is reducible under H and splits into two irreps we denote by a and m. Concentrating on the covariant derivative for the ϕ^a field we have

$$\nabla_\mu \phi^a = (e^{-1})_\mu^\nu (\omega^a)_\nu \,, \tag{5.24}$$

which can be expressed in terms of structure constants once we choose a parametrisation for the coset element. The idea of the inverse Higgs phenomenon, as outlined in [99], is to use this covariant derivative to *algebraically* solve for ϕ^m in terms of ϕ^a and $\partial_\mu \phi^a$. Assuming $\mu \times a \supset m$, it is often stated in the literature that if

$$f^a_{\mu m}|^n \neq 0 \,, \tag{5.25}$$

one can solve for ϕ^m in terms of ϕ^a and $\partial_\mu \phi^a$ by setting

$$\nabla_\mu \phi^a|^n = c\phi^n + \partial_\mu \phi^a|^n + \ldots \tag{5.26}$$

to zero. This is because (5.25) ensures that ϕ^m appears linearly. However, in general (5.26) contains $\partial_\mu \phi^m$ terms which *might* restrict one from solving for ϕ^m algebraically. In this sense (5.25) is merely a necessary condition in order to be able to employ the standard inverse Higgs phenomenon and additional conditions on the structure constants must be met. This was touched upon by McArthur in [127] and in the following we give a complimentary discussion with some important differences.

It turns out that (5.25) is a necessary condition for *all* parametrisations of the coset element, however the additional conditions are heavily parametrisation dependent. We illustrate this below with three examples where for clarity we will assume that the covariant derivative forms an irreducible representation of the subgroup H such that the inverse Higgs constraint comes from setting (5.24) to zero, rather than a projection. Now given that the vielbein is non-zero this is equivalent to setting the Maurer-Cartan component $(\omega^a)_\nu$ to zero. In this case we require a commutator of the form

$$[P_\mu, T_m] \supset T_a \,, \tag{5.27}$$

if ϕ^m is to appear linearly in $(\omega^a)_\nu$. Since $(\omega^a)_\nu$ is linear in derivatives, in order to be able to algebraically solve for ϕ^m no $\partial_\mu\phi^m$ terms are allowed to be present.

The first coset parametrisation one might consider is

$$\gamma = e^{x^\mu P_\mu + \phi^A T_A} = e^{x^\mu P_\mu + \phi^a T_a + \phi^m T_m} , \tag{5.28}$$

where all generators appear in a single exponential. This turns out to be a bad choice, not least because the resulting non-linear realisation will have explict space-time coordinate dependence and translations act in a non-standard way on the coset coordinates, but also the condition (5.27) guarantees that $(\omega^a)_\mu$ contains $\partial_\mu\phi^m$ terms. Explicitly we have

$$(\omega^a)_\mu \supset -\frac{1}{2} f^a_{\nu m} x^\nu \partial_\mu \phi^m , \tag{5.29}$$

and therefore one cannot employ the standard inverse Higgs constraint to eliminate the inessential Goldstone field ϕ^m algebraically. This example already clearly demonstrates that ones choice of the coset parametrisation is important with regards to the existence of (standard) inverse Higgs constraints.

The next obvious choice is the following standard parametrisation

$$\gamma = e^{x^\mu P_\mu} e^{\phi^A T_A} = e^{x^\mu P_\mu} e^{\phi^a T_a + \phi^m T_m} , \tag{5.30}$$

as used in the original papers [99,167]. Unlike the previous example this choice ensures that the non-linear realisations have no explict space-time coordinate dependence. By calculating the Maurer-Cartan form for this coset element it follows that

$$\begin{aligned}(\omega^a)_\mu =& \phi^A f^a_{\mu A} + \partial_\mu\phi^a - \tfrac{1}{2!}\phi^A(\phi^B f^I_{\mu A} f^a_{BI} + \partial_\mu\phi^B f^a_{AB}) \\ &+ \tfrac{1}{3!}\phi^A\phi^B(\phi^C f^I_{\mu A} f^J_{BI} f^a_{CJ} + \partial_\mu\phi^C f^I_{BC} f^a_{AI}) \\ &- \tfrac{1}{4!}\phi^A\phi^B\phi^C(\phi^D f^I_{\mu A} f^J_{BI} f^K_{CJ} f^a_{DK} + \partial_\mu\phi^D f^I_{CD} f^J_{AI} f^a_{BJ}) \dots ,\end{aligned} \tag{5.31}$$

and therefore we require those parts of the sequence

$$f^a_{Am}, \quad f^I_{Bm} f^a_{AI}, \quad f^I_{Cm} f^J_{AI} f^a_{BJ}, \quad \dots \tag{5.32}$$

symmetric in $A, B, C, \dots$ to vanish in order for $(\omega^a)_\mu$ to be independent of $\partial_\mu\phi^m$.

Another possibility is to further split the broken generators into three separate exponentials like so

$$\gamma = e^{x^\mu P_\mu} e^{\phi^a T_a} e^{\phi^m T_m} . \tag{5.33}$$

Computing the Maurer-Cartan form for this coset element, it follows that

$$\begin{aligned}(\omega^a)_\mu =& \dots - \tfrac{1}{2!}(\phi^m \partial_\mu \phi^n f^a_{mn} + \dots) + \tfrac{1}{3!}(\phi^m\phi^q\partial_\mu\phi^n f^I_{mn} f^a_{qI} + \dots) + \\ &- \tfrac{1}{4!}(\phi^m\phi^q\phi^r\partial_\mu\phi^n f^I_{mn} f^J_{qI} f^a_{rJ} + \dots) + \dots ,\end{aligned} \tag{5.34}$$

where for brevity we have concentrated only on the $\partial_\mu\phi^n$ dependence. In this case we therefore require those parts of the sequence

$$f^a_{mn}, \quad f^I_{mn}f^a_{qI}, \quad f^I_{mn}f^J_{qI}f^a_{rJ}, \quad \dots \tag{5.35}$$

symmetric in $m, q, r, \dots$ to vanish in order for $(\omega^a)_\mu$ to be independent of $\partial_\mu\phi^m$. It is clear that the two sets of conditions (5.32) and (5.35) are different but interestingly the later conditions are the least stringent. In fact, out of all the possible parametrisations of the coset element, this parametrisation leads to the least stringent conditions on the structure constants and is therefore the best parametrisation to use if one wishes to find a non-linear realisation on a reduced set of fields.

We illustrate these points below with an example which also emphasises the importance of considering the conditions on the structure conditions beyond linear order as we have done here.

Example: Consider the spontaneous breaking of the d-dimensional Poincaré group down to its $(d-1)$-dimensional subgroup i.e. the coset space

$$\mathrm{ISO}(d-1,1)/\mathrm{SO}(d-2,1). \tag{5.36}$$

The d-dimensional Poincaré algebra has the following non-vanishing commutators

$$[M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A\,, \tag{5.37}$$

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC}\,, \tag{5.38}$$

where the indices $A, B, C, \dots$ are d-dimensional spacetime indices and we use the Minkowski metric $\eta_{AB} = (-,+,+,\cdots)$. We initially use the standard parametrisation (5.30) for the coset element such that

$$\gamma = e^{x^\mu P_\mu}e^{\pi P_d + \Omega^\mu M_{\mu d}}\,, \tag{5.39}$$

where $\mu = 0, 1, \dots d-1$ and P_d, $M_{\mu d}$ are respectively the generators of broken translations and rotations. The commutator

$$[P_\mu, M_{\nu d}] = -\eta_{\mu\nu}P_d\,, \tag{5.40}$$

informs us that Ω^μ appears linearly in the Maurer-Cartan component associated with P_d, $(\omega_{P_d})_\mu$. The covariant derivative associated with P_d is indeed irreducible so in principle the inverse Higgs constraint would come from setting $(\omega_{P_d})_\mu = 0$. However the structure constants do not satisfy the series of constraints (5.32) and so this Maurer-Cartan component will contain derivatives of Ω^μ so we cannot set it to zero to solve for Ω^μ as a function of π and $\partial^\mu\pi$. Indeed the would-be inverse Higgs constraint is

$$\frac{\sin\sqrt{\Omega^2}}{\sqrt{\Omega^2}}\partial_\mu\pi - \frac{\sin\sqrt{\Omega^2}}{\sqrt{\Omega^2}}\Omega_\mu + \frac{\pi}{\sqrt{\Omega^2}\Omega^2}(\sqrt{\Omega^2} - \sin\sqrt{\Omega^2})\Omega_\nu\partial_\mu\Omega^\nu = 0. \tag{5.41}$$

As can be seen from expanding $\sin\sqrt{\Omega^2}$, the leading order derivative piece is of the form $\sim \pi\Omega_\nu\partial_\mu\Omega^\nu$ indicating that the leading order condition on the structure constants is satisfied but the next to leading order one i.e. $f^I_{Bm}f^a_{AI} = 0$ is not. So for this particular parametrisation of the coset element it is not possible to non-linearly realise the d-dimensional Poincaré group with a reduced set of fields.

If we instead employ the split parametrisation (5.33) then an inverse Higgs constraint does exist. Now the coset element reads

$$\gamma = e^{x^\mu P_\mu} e^{\pi P_d} e^{\Omega^\mu M_{\mu d}}\,, \tag{5.42}$$

and by setting the Maurer-Cartan component along the broken generator P_d to zero we arrive at the inverse Higgs constraint

$$\cos\sqrt{\Omega^2}\partial_\mu\pi - \frac{\sin\sqrt{\Omega^2}}{\sqrt{\Omega^2}}\Omega_\mu = 0\,, \tag{5.43}$$

which has a linear piece, and is fully algebraic, in Ω^μ so we can use this equation to eliminate all dependence of the Maurer-Cartan form on Ω^μ in favour of the essential Goldstone π. The resulting non-linear realisation corresponds to the DBI galileons [53] in $d-1$ dimensions with the leading order term simply the scalar sector of the $(d-1)$-dimensional DBI action. We refer the reader to [81] for more details.

5.2.2 Generalised inverse Higgs constraints

In some cases it is possible to impose a "generalised" inverse Higgs constraint, i.e. another way of eliminating the inessential Goldstone without spoiling the non-linear realisation. As we mentioned in the introduction, this could be an equation of motion if the inessential Goldstone is an auxilliary field, or it could arise from integrating out the inessential Goldstone at low energies. One can wonder whether for a given symmetry breaking pattern different constraints always give rise to equivalent non-linear realisations on the essential fields and therefore equivalent EFTs, or that one can actually obtain inequivalent realisations in this manner. This possible equivalence has been discussed and/or assumed in the literature before, see for example [21, 59], but to our knowledge no general proof is known nor are any definitive counterexamples.

In many cases however, it is quite easy to see that equivalence is maintained. In particular, if the transformation rules of the essential fields as well as the space-time coordinates do not depend on the inessential, it follows straighforwardly that the resulting realisation on the essential after imposing a covariant constraint as given in (5.22) is independent of the actual constraint used (and incidentally takes the form of a point transformation). Therefore if one constructs the non-linear realisation with only the essential Goldstone from the bottom up, the structure of the effective field theory does not depend on how one eliminates the inessential Goldstone. Of course, starting from a specific theory involving all Goldstones and imposing two different constraints need not lead to the exact same reduced theory: the couplings constants

may attain different values. Below we illustrate this with an informative example[5]. However before we do so we first note that there are, at least in the context of multiple inessentials (see also the next subsection), theories for which the transformation rules of the essential and space-time coordinates do depend on inessential fields (and as such the resulting realisation after imposing a constraint involves derivatives and takes the form of an (extended) contact transformation). An example is the special Galileon theory we already discussed in the first chapter. Whether equivalence is also maintained in such cases is unclear to us.

Example: conformal group in one dimension. Consider the spontaneous breaking of the conformal group in one dimension corresponding to the coset space

$$SO(1,2)/\mathbb{1}. \tag{5.44}$$

The generators are P, D and K and the algebra is

$$[P,D]=P, \qquad [D,K]=K, \qquad [P,K]=-2D. \tag{5.45}$$

Given our discussion in the previous subsection, we take the coset element as

$$\gamma = e^{tP}e^{\phi D}e^{\psi K}\,, \tag{5.46}$$

to maximise our chances of finding a standard inverse Higgs constraint. One can straightforwardly compute the corresponding Maurer-Cartan form which is given by

$$\gamma^{-1}d\gamma = e^{\phi}dtP + (d\phi - 2\psi e^{\phi}dt)D + (d\psi + \psi d\phi - \psi^2 e^{\phi}dt)K\,. \tag{5.47}$$

Now consider the following invariant action

$$S = \int e^{\phi}dt(g_1 - 2g_2\psi + g_3(e^{-\phi}\psi\dot{\phi} - \psi^2))\,, \tag{5.48}$$

where we have taken a linear sum of the Maurer-Cartan components each with a coupling constant g_i and dropped total derivatives.

It is clear that one can set the Maurer-Cartan component associated with the generator D to zero such that we can solve for ψ in terms of ϕ and its derivatives. Doing so yields

$$\psi = \frac{1}{2}e^{-\phi}\dot{\phi}\,. \tag{5.49}$$

Imposing this constraint on our invariant action we arrive at (up to total derivatives)

$$S = \int e^{\phi}dt\left(g_1 + \frac{g_3}{4}e^{-2\phi}\dot{\phi}^2\right). \tag{5.50}$$

[5] We thank Joaquim Gomis for drawing our attention to this example.

However, given that (5.48) is algebraic in the field ψ we can also eliminate it via its equation of motion yielding the new constraint

$$\psi = -\frac{g_2}{g_3} + \frac{1}{2} e^{-\phi}\dot{\phi}\,, \tag{5.51}$$

which differs by a constant from the standard inverse Higgs constraint. Upon imposing this constraint on our invariant action we arrive at (again dropping total derivatives)

$$S = \int e^{\phi} dt \left(g_1 + \frac{g_2^2}{g_3} + \frac{g_3}{4} e^{-2\phi}\dot{\phi}^2 \right) . \tag{5.52}$$

We see that imposing the two different constraints indeed yields the same effective field theory but with different coupling constants.

5.2.3 Multiple inessentials

Everything discussed so far carries over quite naturally to the case of multiple essentials and inessentials without much change. The main difference is that more complicated algebra structures are now possible. In particular this opens up the possibility that certain inessential Goldstones take on a double role. In these cases not all of the inessentials are eliminated using the covariant derivative of the essentials, but rather by using de covariant derivatives from some of the other inessentials. To see this, consider an algebra that includes, P_μ, a set of broken generators denoted by T_0, T_1, ... with corresponding fields $\phi^{m_0}, \phi^{m_1}, \phi^{m_2}, \ldots$ (not necessarily irreps) and additional unbroken generators H. Assume that they have the following schematic commutators with P_μ:

$$[P_\mu, T_i] = T_{i+1} + H, P_\mu \tag{5.53}$$

We then call ϕ^{m_i} an i-th order inessential Goldstone and essential Goldstones of course correspond to $i = 0$.[6] Assuming that indeed all the inessentials can be eliminated (see also below), it is clear from the commutators that one can solve for an i-th order inessential by setting the covariant derivative of the $(i-1)$-th order inessential to zero. By doing so one finds a covariant expression for ϕ^{m_i} in terms of the other fields as well as the derivative of $\phi^{m_{i-1}}$. One can then in principle go through the entire chain of inverse Higgs constraints to finally express all the inessentials in term of the essentials alone. Generically an i-th order inessential will eventually be expressable in terms of the essential and its derivatives up to and including i-th order. Indeed, each additional constraint in the chain generically adds a derivative and thus higher and higher derivatives of the essential Goldstones tend to (but not necesarrily have

[6] In principle one could envision algebras where an unambigious assignment of an order to the fields is not possible, however to our knowledge no such algebras have been constructed. It would be interesting to examine in more detail whether such structures are actually allowed or whether Jacobi identities always forbid them.

to) appear, making it more likely (but not inevitable) for ghost degrees of freedom to emerge in theories.

Like in the single (in)essential case, the above commutator structure involving P_μ gives only a necessary condition in order to be able to solve for all the inessentials. There can be many obstructions to actually being able to solve for all the inessentials and in general additional conditions that are higher order in structure constants need to be met. Again, these depend on the chosen coset parametrisation. Parametrisations giving the least stringent constraints are of the form:

$$\gamma = e^{x^\mu P_\mu} e^{\phi^{a_1} T_{a_1}} \cdots e^{\phi^{a_l} T_{a_l}} e^{\phi^{m_1} T_{m_1}} \cdots e^{\phi^{m_k} T_{m_k}} \,. \tag{5.54}$$

I.e. they are the ones where each irrep is in a seperate exponential and their order of appearance is according to their order as defined above. What the optimal arrangement of multiple fields of the same order is, depends on the case at hand.

Examples of theories we already discussed whose symmetry algebra contains multiple inessentials include the special Galileon (with up to second order inessentials) as well as the free theory (with no bound on the order of inessentials). Let us also note that gauge theories can be cast in a coset form by identifying the global subgroup with Taylor expandable transformation rules. This subgroup will have infinitely many generators and will contain inessentials up to arbitrary order. Carrying out the coset construction for this global subgroup is actually sufficient to reconstruct the theories invariant under the full gauge group. The gauge field will correspond to the essential Goldstone.

5.3 Mapping non-linear realisations

In this section we examine how non-linear realisations obtained from different coset parametrisations are related, both before and after the inessential Goldstones have been eliminated. During our analysis we will encounter various types of transformations relating the different coset constructions that we already discussed in detail in Chapter 2: field redefinitions, point transformations and (extended) contact transformations.

5.3.1 *Prior* to inverse Higgs: point transformations

As already noted, for a given coset space one can parametrise the coset element in many different ways. For some particular basis for the broken generators T_A we can put all the generators in a single exponential, every generator in a separate exponential, or anything inbetween. In addition, the order of the exponentials is freely specifiable. To be more precise, one can consider any partition $A = (a_1, ..., a_k)$ and subsequently parametrise the coset element as

$$\gamma = e^{\phi^{a_1} T_{a_1}} \cdots e^{\phi^{a_k} T_{a_k}} \,, \tag{5.55}$$

where we have temporarily include P_μ in T_A for notational convenience.

A further freedom lies in the choice of algebra basis for the broken generators. That is, one can consider an alternative basis $\bar{T}_A$ invertibly related to the original one by

$$\bar{T}_A = c_A^B T_B + c_A^i T_i, \qquad \det(c_A^B) \neq 0. \tag{5.56}$$

Again in this basis, one can pick any partition $A = (a'_1, ..., a'_l)$ and use the corresponding parametrisation

$$\bar{\gamma} = e^{\bar{\phi}^{a'_1}\bar{T}_{a'_1}} \cdots e^{\bar{\phi}^{a'_l}\bar{T}_{a'_l}}. \tag{5.57}$$

A physically interesting example of such different bases arises in the context of the conformal group and branes in AdS space and will be discussed in section 5.4. Given any two bases related by (5.56) and any two corresponding arbitrary partitions, it follows from the BCH formula that there exists a (locally) invertible redefinition of the coset coordinates relating the corresponding parametrisations. That is, one has

$$\gamma = e^{\phi^{a_1}T_{a_1}} \cdots e^{\phi^{a_k}T_{a_k}} = e^{\bar{\phi}^{a'_1}\bar{T}_{a'_1}} \cdots e^{\bar{\phi}^{a'_l}\bar{T}_{a'_l}} \cdot e^{\bar{\phi}^i T_i} = \bar{\gamma} h, \tag{5.58}$$

where

$$\bar{\phi}^A = \bar{\phi}^A(\phi^B) = (c^{-1})^A_B \phi^B + \text{ terms higher order in coset coordinates,} \tag{5.59}$$
$$\bar{\phi}^i = \bar{\phi}^i(\phi^B) = -c_A^i (c^{-1})^A_B \phi^B + \text{ terms higher order in coset coordinates,} \tag{5.60}$$

and invertibility of (5.59) is guaranteed by the presence of the linear term. The exact form of the resulting mapping can be highly non-trivial on account of the BCH formula. In the case of internal symmetries the coset coordinates only involve fields and the redefinitions are ordinary field redefinitions; if one considers space-time symmetries the space-time coordinates are also included and the redefinitions are thus point transformations.

The relation (5.58) induces an equivalence of the corresponding non-linear realisations. Let us reinstate the space-time coordinates amongst the coset coordinates and let the point transformation relating the two parameterisations be given by $(\bar{x}, \bar{\phi}) = \mathcal{F}(x, \phi)$. By working out the definitions of the transformations rules for both set of coordinates, denoted by $g \cdot (x^\mu, \phi^A) = F_g(x^\mu, \phi^A)$ and $g \cdot (\bar{x}^\mu, \bar{\phi}^A) = \bar{F}_g(\bar{x}^\mu, \bar{\phi}^A)$, one finds that they are compatible with the point transformation:

$$\bar{F}_g(\bar{x}^\mu, \bar{\phi}^A) = (\mathcal{F} \circ F_g \circ \mathcal{F}^{-1})(\bar{x}^\mu, \bar{\phi}^A)\,. \tag{5.61}$$

In other words, the group action on the redefined fields coincides with the one induced by the redefinition. Thus starting from any action $S[x, \phi]$ which is invariant under $g \cdot (x^\mu, \phi^A)$, one can obtain an equivalent barred action $\bar{S}[\bar{x}, \bar{\phi}]$ invariant under $g \cdot (\bar{x}^\mu, \bar{\phi}^A)$ by performing the point transformation (5.59). As a consequence invariant Lagrangians are related to each other as

$$\bar{\mathcal{L}}(\bar{x}, \bar{\phi}(\bar{x}), \ldots) \det\left(\frac{d\bar{x}}{dx}\right) \equiv \mathcal{L}(x, \phi(x), \ldots). \tag{5.62}$$

and the sets of invariant theories using either of the parametrisations are thus equivalent.

As noted several times, universality of the coset construction, in the sense that *any* non-linear realisation can be brought back to a specific coset form, has only been proven for internal symmetries described by compact, connected and semi-simple Lie groups and acting as point transformations. For more general internal symmetries as well as spacetime symmetries, there is no proof of universality and therefore it is not clear that *any* non-linear realisation can be brought back to the coset form, even prior to inverse Higgs. There are, however, examples where it is possible. An interesting example relates to supersymmetry, corresponding to the spontaneous breaking of super-Poincaré to the Poincaré group (leading to the Volkov-Akulov theory already discussed in Chapter 3 as the leading order Lagrangian). The corresponding coset element contains a fermion field, the Goldstino, but no inessential Goldstone modes and hence there are no inverse Higgs constraints. However, other methods can be used to arrive at a non-linear realisation of supersymmetry, for example, by imposing a supersymmetric constraint on a linear supermultiplet (see e.g. [29,117,151]). In this case an explicit point transformation relating this non-linear realisation to the one coming from the coset construction of [166] has been constructed [118].

5.3.2 *Post* inverse Higgs: extended contact transformations

We will now examine whether the equivalence between realisations is maintained after eliminating the inessential Goldstones. We again consider parametrisations (5.55) and (5.57) and assume that for both we can consistently employ the standard inverse Higgs mechanism to remove the inessential fields. For clarity we again focus on a single inverse Higgs constraint:

$$\nabla_\mu \phi^a|^m = 0 \qquad \Leftrightarrow \qquad \mathcal{F}^m(\phi^m, \phi^a, \partial_\mu \phi^a) = \phi^m - f^m(\phi^a, \partial_\mu \phi^a) = 0\,, \tag{5.63}$$

$$\bar\nabla_\mu \bar\phi^a|^m = 0 \qquad \Leftrightarrow \qquad \bar{\mathcal{F}}^m(\bar\phi^m, \bar\phi^a, \bar\partial_\mu \bar\phi^a) = \bar\phi^m - \bar f^m(\bar\phi^a, \bar\partial_\mu \bar\phi^a) = 0. \tag{5.64}$$

If the point transformation relating the two sets of coset coordinates *prior* to imposing inverse Higgs constraints is to induce an invertible transformation relating the essential coordinates to each other *post* inverse Higgs, one must demand compatibility in the following sense

$$\bar\phi^A(\phi|_{\mathcal{F}=0}) = \bar\phi^A|_{\bar{\mathcal{F}}=0}\,. \tag{5.65}$$

This is precisely the case when

$$\nabla_\mu \phi^a|^m = 0 \qquad \Leftrightarrow \qquad \bar\nabla_\mu \bar\phi^a|^m = 0, \tag{5.66}$$

i.e. when the two inverse Higgs constraints imply each other based on the point transformation relating the coset elements (see [111, 140] for a discussion related to Galileons and Galileon duality). If this is indeed the case then the induced transformation relating the spacetime coordinates and the essential Goldstones is simply the

point transformation evaluated on the inverse Higgs constraints

$$\bar{x}^\mu = \bar{x}^\mu(x, \phi, f(\phi, \partial\phi)), \qquad \bar{\phi}^a = \bar{\phi}^a(x, \phi, \bar{f}(\phi, \partial\phi)). \tag{5.67}$$

The result is an invertible first order extended contact transformation which reduces to a standard contact transformation when ϕ^a contains a single component. Its invertibility is guaranteed by that of the point transformation. If multiple inessentials, up to some order n, are present in the theory, then generically the resulting redefinition is an n-th order extended contact transformation.

Due to the compatibility of the inverse Higgs conditions and the point transformation, the transformation rules for the essential Goldstones (and spacetime coordinates) are mapped onto each other under the extended contact transformation. Again this ensures physical equivalence of the post inverse Higgs non-linear realisations. In particular, two equivalent Lagrangians prior to inverse Higgs remain equivalent post inverse Higgs. Of course, due to the derivative nature of the extended contact transformations, the order of a Lagrangian is generically not maintained.

On the other hand if (5.66) is not satisfied, it is far from clear if equivalence is maintained post inverse Higgs. What we can say for sure is that if an invertible mapping does exist, it does not directly follow from the point transformation relating the coset elements. This is a somewhat surpising possibility but it is very easy to find situations where it occurs. To see this, consider two Maurer-Cartan forms prior to imposing inverse Higgs constraints where the corresponding coset elements are related by (5.58) but we restrict to the case where $\bar{T}_A = c_A^B T_B + c_A^i T_i + c_A^\mu P_\mu$. Obviously here we do not combine P_μ and T_A. The Maurer-Cartan forms are related by

$$\gamma^{-1} d\gamma = h^{-1}(\bar{\gamma}^{-1} d\bar{\gamma})h + h^{-1}dh\,, \tag{5.68}$$

or in terms of their components we have

$$\begin{aligned}
\omega^\mu &= D(h^{-1})^\mu_\nu \bar{\omega}^\nu + c^\mu_b D(h^{-1})^b_c \bar{\omega}^c + c^\mu_m D(h^{-1})^m_n \bar{\omega}^n, \\
\omega^A &= c^A_b D(h^{-1})^b_c \bar{\omega}^c + c^A_m D(h^{-1})^m_n \bar{\omega}^n, \\
\omega^i &= D(h^{-1})^i_j \bar{\omega}^j + c^i_b D(h^{-1})^b_c \bar{\omega}^c + c^i_m D(h^{-1})^m_n \bar{\omega}^n + (h^{-1}dh)^i.
\end{aligned} \tag{5.69}$$

For simplicitly let us assume that the covariant derivatives which lead to the inverse Higgs constraints are irreducible such that the unbarred inverse Higgs conditions are $\omega^a = 0$ and the barred ones are $\bar{\omega}^a = 0$. It follows from (5.68) that in general we have

$$\bar{\omega}^a = 0\,, \qquad \not\Leftrightarrow \qquad \omega^a = c^a_b D(h^{-1})^b_c \bar{\omega}^c + c^a_m D(h^{-1})^m_n \bar{\omega}^n = 0\,, \tag{5.70}$$

since in general $\bar{\omega}^n \neq 0$ on the inverse Higgs solutions. Here the inverse Higgs constraints are not mapped onto each other under the point transformation and therefore the point transformation does not induce a transformation relating the two non-linear realisations constructed from only the essential Goldstones.

We also note that if one considers two parametrisations with the *same* basis of broken generators, i.e. when $c^A_B = \delta^A_B$ and thus $c^a_m = 0$, one finds

$$\bar{\omega}^a = 0\,, \qquad \Leftrightarrow \qquad \omega^a = D(h^{-1})^a_b \bar{\omega}^b = 0\,, \tag{5.71}$$

such that the inverse Higgs constraints are indeed mapped. It follows that in this case the equivalence between the non-linear realisation is guaranteed to be maintained even after the inessential Goldstones have been eliminated.

Below we show that the non-mapping of the inverse Higgs constraints can indeed occur but does not necessarily imply inequivalence of the two non-linear realisations.

Example: Poincaré in two dimensions. Consider spontaneous breaking of the Poincaré group in two dimensions i.e. the coset space

$$ISO(1,1)/\mathbb{1}. \tag{5.72}$$

We work in two different bases for the algebra, the first with generators P_0, P_1 and M, and the second with $\bar{P}_0 = P_0, \bar{P}_1 = P_1$ and $\bar{M} = M + \alpha P_1$. Since the generators of translations commute with each other the commutators are the same in each basis and are given by

$$[P_0, M] = P_1, \quad [P_0, \bar{M}] = P_1\,, \quad [P_1, M] = P_0\,, \quad [P_1, \bar{M}] = P_0. \tag{5.73}$$

We parametrise the two coset elements as

$$\gamma = e^{tP_0} e^{\pi P_1} e^{\Omega M}, \qquad \bar{\gamma} = e^{\bar{t}P_0} e^{\bar{\pi} P_1} e^{\bar{\Omega}\bar{M}}, \tag{5.74}$$

yielding the two Maurer-Cartan forms

$$\begin{aligned}
\gamma^{-1}d\gamma &= P_0(\cosh\Omega dt + \sinh\Omega d\pi) + P_1(\sinh\Omega dt + \cosh\Omega d\pi) + Md\Omega, \\
\bar{\gamma}^{-1}d\bar{\gamma} &= P_0(\cosh\bar{\Omega} d\bar{t} + \sinh\bar{\Omega} d\bar{\pi}) + P_1(\sinh\bar{\Omega} d\bar{t} + \cosh\bar{\Omega} d\bar{\pi}) + \bar{M}d\bar{\Omega}\,,
\end{aligned} \tag{5.75}$$

which of course have the same structure given that the commutators are the same. The point transformation which relates these two Maurer-Cartan forms is

$$\bar{t} = t + \alpha\cosh\Omega\,, \quad \bar{\pi} = \pi - \alpha\sinh\Omega\,, \quad \bar{\Omega} = \Omega\,, \tag{5.76}$$

which is extracted by equating both expressions in (5.75). The inverse Higgs constraints in both cases come from setting the co-efficient of P_1 in the Maurer-Cartan forms to zero, due to the commutators $[P_0, M] = P_1$ and $[P_0, \bar{M}] = P_1$, yielding the solutions

$$\Omega = \tanh^{-1}(-\dot{\pi})\,, \quad \bar{\Omega} = \tanh^{-1}(-\pi')\,, \tag{5.77}$$

where a prime denotes a derivative with respect to $\bar{t}$. Now these solutions are not mapped onto each other under the point transformations (5.76) therefore the two Maurer-Cartan forms after we impose the inverse Higgs constraints are also not mapped onto each other. This is obvious given that in the unbarred variables the co-efficient of P_1 now vanishes due to the inverse Higgs constraint while it is non-zero in the barred basis after we set $\bar{M} = M + \alpha P_1$.

The resulting building blocks of invariant Lagrangians are

$$\sqrt{1-\dot{\pi}^2}dt, \qquad \frac{\ddot{\pi}}{(1-\dot{\pi}^2)^{3/2}}, \qquad \text{and} \qquad \sqrt{1-\bar{\pi}'^2}d\bar{t}, \qquad \frac{\bar{\pi}''}{(1-\bar{\pi}'^2)^{3/2}}, \tag{5.78}$$

and are therefore mapped onto each other in the trivial manner $\bar{t}=t$ and $\bar{\pi}=\pi$ post inverse Higgs but this has nothing to do with how the coset elements are related. Of course any Wess-Zumino terms will also be mapped.

5.4 Correspondence between AdS and conformal cosets

It turns out that both cases of interest discussed above, i.e. with the inverse Higgs constraints mapped or not, apply to the spontaneous breaking of the d-dimensional conformal group by a codimension $d-n+1$ Minkowski brane embedded in AdS$_{d+1}$, which will be of interest in the next chapter where we discuss inflationary models based on it. The two different bases for the algebra are the standard conformal basis and the AdS basis [13]. The coset space is

$$SO(d,2)/(SO(n-1,1)\times SO(d-n)), \tag{5.79}$$

where the unbroken $SO(d-n)$ transformations correspond to the unbroken Lorentz transformations in the directions transverse to the brane. Whether a mapping between invariant Lagrangians which follows from the point transformation relating the coset elements exists is dependent on the codimension of the brane. It turns out that for codimension one branes there is indeed a well defined mapping of this kind, as discussed in [13,42], but for any other codimension this is not the case as we illustrate below.

5.4.1 Codimension one

Let us begin with the codimension one case corresponding to the coset space

$$SO(d,2)/SO(d-1,1). \tag{5.80}$$

In the standard basis of the conformal algebra the non-vanishing commutators are

$$\begin{aligned}
&[P_A, D] = P_A \\
&[K_A, D] = -K_A \\
&[P_A, K_B] = 2M_{AB} + 2\eta_{AB} D
\end{aligned}
\qquad
\begin{aligned}
&[M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A \\
&[M_{AB}, K_C] = \eta_{AC}K_B - \eta_{BC}K_A
\end{aligned}$$

and $[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC}$. We again use $A, B, C, \ldots$ for d-dimensional spacetime indices. The $d+1$ broken generators correspond to dilatations D and special conformal transformations K_A. Given our discussion in section 5.2 we parametrise the coset element as

$$\gamma = e^{x^A P_A} e^{\phi D} e^{\psi^A K_A}. \tag{5.81}$$

Now the commutator $[P_A, K_B] = 2M_{AB} + 2\eta_{AB}D$ tells us that ψ^A appears linearly in the covariant derivative associated with D and since this covariant derviative is an irrep the standard inverse Higgs constraint would come from setting the Maurer-Cartan component ω_D to zero. Indeed the structure constants satisfy the conditions (5.35) and so we can use this constraint to algebraically eliminate ψ^A in favour of ϕ and $\partial_A\phi$. The resulting non-linear realisation is equivalent to building diffeomorphism invariant scalars out of the effective metric $g_{AB} = e^{2\phi}\eta_{AB}$. In four dimensions the leading terms in a derivative expansion yield the familiar Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4, \tag{5.82}$$

after the field redefinition $\varphi = e^{\phi}$.

The AdS basis is defined by[7] [13]

$$\bar{K}_A = K_A + \frac{1}{2}P_A, \tag{5.83}$$

in which case the non-vanishing commutators are

$$\begin{aligned}
&[P_A, D] = P_A && \\
&[\bar{K}_A, D] = -\bar{K}_A + P_A && [M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A \\
&[P_A, \bar{K}_B] = 2M_{AB} + 2\eta_{AB}D && [M_{AB}, \bar{K}_C] = \eta_{AC}\bar{K}_B - \eta_{BC}\bar{K}_A \\
&[\bar{K}_A, \bar{K}_B] = 2M_{AB} &&
\end{aligned}$$

and $[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC}$. We now parametrise the coset element as

$$\bar{\gamma} = e^{\bar{x}^A P_A} e^{\bar{\phi} D} e^{\bar{\psi}^A \bar{K}_A}, \tag{5.84}$$

and again due to the commutator $[P_A, \bar{K}_B] = 2M_{AB} + 2\eta_{AB}D$, and the fact that the structure constants satisfy the conditions (5.35), we can set $\bar{\omega}_D = 0$ to leave us with a non-linear realisation constructed solely from the dilaton $\bar{\phi}$.

Now the point transformation which maps the two coset elements can be extracted by equating the two corresponding Maurer-Cartan forms. This is the case because whenever the unbroken generator M_{AB} is generated in (6.19) by the BCH formula, the indices are always contracted with copies of $\bar{\psi}^A$ and so it drops out by symmetry. In other words the h of (5.68) is trivial in this case. Importantly, since (5.83) does

[7]To compare with [42] we are working in units where $L = 1/\sqrt{2}$.

not involve the generator D, i.e. we have $c_m^a = 0$ when comparing to (5.56), the two inverse Higgs constraints are mapped by this point transformation, see equation (5.70). A contact transformation relating the non-linear realisations constructed from the dilatons ϕ and $\bar{\phi}$ then follows by evaluating this transformation on the inverse Higgs solutions. This has been done explictly in [13, 42] and we refer the reader there for more details.

5.4.2 Higher codimensions

In higher codimensions the situation is more complicated. Now consider a $d-n+1 > 1$ codimension brane where the broken generators now also include translations and Lorentz transformations. If we let $\mu, \nu, \dots$ label n-dimensional spacetime indices and $i = n+1, \dots, d$, then the broken generators in the conformal basis are

$$P_i, M_{\mu i}, D, K_\mu, K_i, \tag{5.85}$$

and similarly for the AdS basis with $K_A \to \bar{K}_A$. In general there are now $2(d-n)+1$ Goldstone scalars and $d-n+1$ Goldstone vectors. If we parametrise the coset elements as

$$\gamma = e^{x^\mu P_\mu} e^{\pi^i P_i} e^{\phi D} e^{\Omega^{\mu i} M_{\mu i}} e^{\psi^\mu K_\mu} e^{\sigma^i K_i}, \tag{5.86}$$

for the conformal basis and similarly for the AdS basis again with $K_A \to \bar{K}_A$, we can use standard inverse Higgs constraints to remove all inessential Goldstones leaving us with $d-n+1$ essential Goldstone scalars. Of course here there is more than a single inverse Higgs constraint and not all of the relevant covariant derivatives are irreps. Indeed we have to perform traces to eliminate the σ^i fields using the covariant derivatives associated with $\omega_{M_{\mu i}}$. In any case, one of the essential Goldstones is the dilaton and the other $d-n$ correspond to the broken translations and are $SO(d-n)$ invariant.

Let us concentrate on one of these inverse Higgs constraints since this will be enough to draw conclusions about possible mappings. In both bases the commutator $[M_{AB}, P_C] = \eta_{AC} P_B - \eta_{BC} P_A$ tells us that the vectors $\Omega^{\mu i}$ (conformal basis) and $\bar{\Omega}^{\mu i}$ (AdS basis) associated with a broken Lorentz transformation $M_{\mu i}$ appear linearly in the covariant derivatives associated with the broken generator P_i. Since these covariant derivatives are irreps the inverse Higgs constraints can come from setting $\omega_{P_i} = 0$ and $\bar{\omega}_{P_i} = 0$. With (6.24) we can eliminate $\Omega^{\mu i}$ and $\bar{\Omega}^{\mu i}$ algebraically.

However, now given the definition of the AdS basis (5.83), these inverse Higgs constraints *will not* be mapped onto each other under the point transformation which takes us from one coset element to the other unless the Maurer-Cartan component ω_{K_i} vanishes on the inverse Higgs solutions. This is because we now have $c_m^a \neq 0$ in equation (5.56). We have checked explictly for codimension two that $\omega_{K_i} \neq 0$ on the inverse Higgs solutions and one would expect this to hold for higher codimensions too. As we discussed above this leaves us with two possibilities. Either the standard basis and the AdS basis lead to physically different non-linear realisations for the

essential Goldstones when the codimension is higher than one or there is a mapping relating invariant Lagrangians which does not follow from the point transformation which maps the coset elements.

Note that for both bases, our choice for the coset parametrisation (6.24) was inspired by our discussion in section 3. Even though there we primarily concentrated on a single inverse Higgs constraint for clarity, the general principle still applies for multiple inverse Higgs constraints: use the largest number of exponentials which allows one to write the coset element in a H-invariant way, and place the inessential Goldstones to the right. However, of course for multiple inverse Higgs there is also the added subtlety of the order of the inessential Goldstones and this can play an important role. For example, if instead of (6.24) we had chosen

$$\gamma = e^{x^\mu P_\mu} e^{\pi^i P_i} e^{\phi D} e^{\Omega^{\mu i} M_{\mu i}} e^{\sigma^i K_i} e^{\psi^\mu K_\mu}, \tag{5.87}$$

where we have reversed the order of the final two exponentials, then in the AdS basis we would not have been able to remove all inessential Goldstones algebraically since σ^i would appear with derivatives in the Maurer-Cartan form along the broken generator $M_{\mu i}$. This is only problematic in the AdS basis since K_μ and K_i commute in the conformal basis.

5.5 Conclusions and discussion

In this chapter we have extensively discussed the coset construction for both internal as well as space-time symmetries, focusing on the subtleties involved when dealing with the latter. Whereas for large classes of internal symmetries universality of the construction has been proven, this is not the case when dealing with space-time symmetries. The reason is the existence of inessential Goldstones which are not needed in order to non-linearly realise the given symmetry. The fact that one usually has several ways of eliminating them from the realisation greatly complicates the universality question.

For example, given a particular field basis one might be able to impose several different covariant constraints that all alow one to eliminate the inessentials, and it is in general unclear whether using any two of these results in equivalent theories. A similar situation occurs when one uses a different basis for the symmetry algebra. Even though prior to eliminating the inessentials two theories written in terms of different bases are easily seen to be related via a point transformation, it is only when the inverse Higgs constraints are mapped onto eachother via the point transformation that there is a naturally induced (extended contact) transformation relating the two formulations post elimination. If the constraints are not mapped, equivalence is not guaranteed. The possible (in)equivalence remains an interesting open problem.

This crucial distinction concerning the relation of inverse Higgs constraints is beautifully illustrated in our main physical example, focussing on the relation between the

conformal and the AdS basis of the $SO(2,d)$ algebra. We have considered the spontaneous breaking of this algebra as described by a n-dimensional Minkowski probe brane embedded in $(d+1)$-dimensional AdS space. We found that whether the constraints in the conformal and the AdS basis are mapped onto each other depends on the codimension of the brane and hence on the number of essential Goldstone modes. For codimension one, i.e. a single essential Goldstone, the solutions for the inessential Goldstone modes are mapped onto each other, as implicitly used in [13,42]. However, we find that this ceases to be true for higher codimensions which necessarily involve more essential Goldstones. This implies that in the latter case there is no straightforward extended contact transformation relating the two different coset constructions. Clearly this deserves further attention.

Chapter 6

Symmetry breaking patterns for inflation

In this chapter, we are interested in non-linearly realised symmetries in the kinetic sector of inflationary models which are weakly broken by an inflation driving potential. In the absence of such explicit symmetry breaking, the dynamics of the Goldstone modes is strongly constrained by the non-linearly realised symmetries, resulting in specific signatures in observables. Most inflationary models, however, include an explicit symmetry breaking term in the form of the scalar potential. As mentioned before, the simplest example is that of a single canonical field whose kinetic term is invariant under a shift symmetry which in turn is broken by the potential energy $V = \lambda\phi^m$ with integer $m \geqslant 2$ [123]. However, due to the predicted values for r being too large compared to the CMB observations [2, 4] one is lead to consider more elaborate inflationary models whose kinetic sector exhibits more complicated non-linear symmetries. This naturally leads to multi-field models with kinetic sectors which correspond to some non-trivial internal manifold.

Therefore we will consider models of the form

$$S = \int d^4x\sqrt{-g}\left(\tfrac{1}{2}M_{\rm pl}^2 R - \tfrac{1}{2}\Lambda^4 K(\phi^I, \partial_\mu\phi^I) - V(\phi^I)\right), \tag{6.1}$$

where ϕ^I labels n scalar fields, $K(\phi^I, \partial_\mu\phi^I)$ is the dimensionless kinetic sector and contains at least two derivatives, and $V(\phi^I)$ is the symmetry breaking potential. Λ is an arbitrary scale introduced on dimensional grounds. We will be interested in cases where the kinetic sector is fixed by a non-linearly realised symmetry corresponding to a coset space G/H, where G can be an internal symmetry group or a space-time symmetry group. This will be the main ingredient of the scenarios under investigation; our starting point is conventional in that the 4-dimensional Lorentz group is always linearly realised[1] and our scalar sector is minimally coupled to gravity. As we shall see,

[1]Note that here and in what follows we say that the Lorentz group is linearly realised rather

coset spaces are a useful way of characterising kinetic sectors for scalar field theories and we note that this has been considered before in the context of inflation in [26] and coset spaces have been used to classify condensed matter systems in e.g. [134].

In section 2 we will discuss five different forms for the kinetic sector; three which non-linearly realise an internal symmetry and two which non-linearly realise a space-time symmetry. This exhausts all maximally symmetric possibilities (up to group contractions). For internal symmetries, where each generator commutes with those of the 4-dimensional Poincaré group, G corresponds to the symmetries of flat, spherical and hyperbolic geometries. For space-time symmetries, where the group G contains the 4-dimensional Poincaré group, it can correspond to the symmetries of higher-dimensional Minkowski space and anti-de Sitter space[2]. In each case we make use of the coset construction [27, 38, 99, 167] to build invariant kinetic sectors, and will pay most attention to the non-linear realisation of the anti-de Sitter isometries, i.e. the conformal group, since the corresponding coset space has not been well studied in the literature. At this stage let us make it clear that although our scalars are indeed Goldstone bosons, they are not the usual Goldstones appearing in the EFT of inflation [33, 156] since they are not associated with the breaking of time translations.

We add the symmetry breaking potentials in section 3 where we also couple the scalars minimally to gravity in order to drive inflation. We concentrate most on two examples; one with internal symmetries corresponding to a hyperbolic geometry and the other with space-time symmetries which non-linearly realise the conformal group. For clarity we study the $n = 2$ case with two fields, since this captures the main features of the models which are both constructed from a single axion with a shift symmetry and a single dilaton. The former is the α-attractors model [103] while the latter we dub ambient inflation, since the non-linear symmetries are those corresponding to a Minkowski 3-brane fluctuating in an anti-de Sitter ambient space-time.

In both cases we will study the predictions of inflationary trajectories which take place along the dilatonic direction. For a large class of scalar potentials and for order one parameters the predictions of α-attractors are in the sweet spot of the Planck data: $n_s = 0.965 \pm 0.004$ [4]. The spectral index takes values close to 0.960 or 0.967 depending on our choice of 50 or 60 e-folds, while the tensor-to-scalar ratio takes values around $r \sim 0.001$, although it can also be larger. For ambient inflation the spectral index turns out be somewhat bluer than the α-attractors prediction taking values between 0.971 and 0.976 again for e-folds ranging from 50 to 60. There is also a non-trivial difference in the predicted tensor-to-scalar ratio compared to α-attractors

than the full Poincaré group. This because as we already noted in the previous chapter, the full Poincaré group is indeed linearly realised on the fields but translations are non-linearly realised on the space-time coordinates. We thus adopt this terminology to avoid confusion when we we employ the coset construction.

[2]We do not consider the case where G is the de Sitter group since as far as we are aware this group does not have a 4-dimensional Poincaré subgroup so it would be impossible to non-linearly realise the de Sitter isometries with scalar fields in 4-dimensional Minkowski space-time (see e.g. [159] for a discussion of why it is not possible to embed the 4-dimensional Poincaré group into the 5-dimensional de Sitter group).

with ambient inflation naturally predicting $r \sim 0.01$ for order one parameters. Both predictions for the tensor-to-scalar ratio are therefore interesting targets for future ground-based and satellite CMB missions. We present figures in section 3.2 where more accurate values for n_s and r are presented for a range of potentials and parameter values.

We should note the following caveat with regards to large field inflationary models. In comparison to the weak breaking of the shift symmetry in monomial inflation, parameters which break these more exotic symmetries can also be set to a small value in Planck units in a technically natural way. Inflaton loops and graviton loops will not spoil this choice thanks to the weakly broken symmetry. For α-attractors this was investigated in [102] and indeed the hyperbolic geometry of the kinetic sector provides the expected protection. However, a fully fledged theory of quantum gravity is expected to break all continuous global symmetries [107] and this phenomenon will manifest itself via symmetry breaking corrections to the inflationary potential of the form $M_{\rm pl}^{4-n}\phi^n$ with order 1 coefficients. For large field inflationary models this can spoil the slow-roll dynamics. Much work has been done to alleviate this problem in the context of monomial inflation [16, 109, 110, 126, 157] and axion monodromy models [47, 64, 108]. In this chapter our aim is to produce phenomenologically viable theories of inflation which are stable against perturbative quantum gravity effects within EFT. We would therefore require further model building input along the lines above to be sure that the potentially troublesome non-perturbative corrections are under control.

For more details on inflation in general and inflationary models in the context of EFTs we refer to [12].

6.1 Symmetries of the kinetic sector

In this section we construct five interesting choices for the kinetic sector $K(\phi^I, \partial_\mu\phi^I)$ for different coset spaces G/H using the coset construction. We discuss spontaneously broken internal and space-time symmetries separately.

6.1.1 Internal symmetries

Firstly, we assume that the non-linearly realised symmetries of the kinetic sector commute with the 4-dimensional Poincaré group. For maximally symmetric groups, G can be either $ISO(n)$, $SO(n+1)$ or $SO(1,n)$ corresponding to flat, spherical and hyperbolic geometries respectively. Since our aim is to derive kinetic sectors with n scalars we fix $H = SO(n)$ giving us the following three coset spaces

$$\mathbb{R}^n \simeq ISO(n)/SO(n)\,, \qquad \mathbb{S}^n \simeq SO(n+1)/SO(n)\,, \qquad \mathbb{H}^n \simeq SO(1,n)/SO(n)\,. \tag{6.2}$$

In the following we discuss each of these in turn and compute the invariant metrics. For the coset construction we will need the commutators of the Lorentz group in order to compute the Maurer-Cartan form. For $SO(p,q)$ we have

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC} \,, \tag{6.3}$$

where η_{AB} is the metric of a flat space-time with p timelike directions and q spacelike directions. For $ISO(p,q)$ these are augmented with the following nonzero commutators involving the translation generators

$$[M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A \,. \tag{6.4}$$

In all cases one can construct an coset metric of the form $d\sigma^2 = g_{IJ}(\phi)d\phi^I d\phi^J$ which upon pulling back to 4-dimensional space-time gives rise to

$$g_{IJ}(\phi)\partial_\mu\phi^I\partial_\nu\phi^J dx^\mu dx^\nu \,. \tag{6.5}$$

Since here we are considering non-linearly realised internal symmetries, the dx^μ are invariant under G. Therefore the corresponding kinetic sectors are given by

$$K = g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J \,. \tag{6.6}$$

By construction these are invariant under the linearly realised Lorentz group as well as the non-linearly realised isometry group of the coset. A related discussion of these cosets can be found in e.g. [26] (in particular the case $n = 2$).

Flat geometry

For the first coset space the translations of $ISO(n)$ are broken while the rotations are unbroken. If we let i be an $SO(n)$ index then the only H-invariant way of parametrising the coset space is

$$\gamma = e^{\phi^i P_i} \,, \tag{6.7}$$

where ϕ^i are the Goldstone bosons. The Maurer-Cartan form is very simple to calculate in this case since $[P_i, P_j] = 0$ and is given by

$$\gamma^{-1}d\gamma = d\phi^i P_i. \tag{6.8}$$

It follows that the only $SO(n)$ invariant metric one can construct is

$$d\sigma^2 = \delta_{ij}d\phi^i d\phi^j, \tag{6.9}$$

corresponding to a flat scalar manifold. The kinetic sector therefore reads

$$K = \delta_{ij}\partial_\mu\phi^i\partial^\mu\phi^j \,, \tag{6.10}$$

where the Goldstones have the dimension of length such that K is dimensionless. Each Goldstone inherits a shift symmetry $\phi^i \to \phi^i + c^i$ from the spontaneously broken translations. Arbitrary functions of this two-derivative combination (similar to $P(X)$ theories for a single scalar) will also be invariant, but such higher-order corrections will not play a role in our application to slow-roll inflationary models.

Spherical geometry

For a spherical geometry we have $G = SO(1+n)$ and we denote the generators of this group as M_{1i} and M_{ij} where again i, j are $SO(n)$ indices. In this case, the broken generators are M_{1i} and the only H-invariant parametrisation of the coset element is

$$\gamma = e^{\phi^i M_{1i}}. \tag{6.11}$$

The resulting Maurer-Cartan form is not very illuminating but leads to the following unique choice for a G-invariant metric

$$d\sigma^2 = \frac{\sin^2\sqrt{\phi^2}}{\phi^2}\delta_{ij}d\phi^i d\phi^j + \left(1 - \frac{\sin^2\sqrt{\phi^2}}{\phi^2}\right)\frac{\phi_i\phi_j}{\phi^2}d\phi^i d\phi^j, \tag{6.12}$$

where $\phi^2 = \delta_{ij}\phi^i\phi^j$. Note that the metric is manifestly invariant under the linearly realised $SO(n)$, and can be used to construct the corresponding kinetic term. Restricting to $n = 2$ for simplicity, a more familiar form might be

$$d\sigma^2 = L^2(d\theta^2 + \sin^2\theta d\varphi^2) = 4L^4\frac{dZd\bar{Z}}{(L^2 + |Z|^2)^2}, \tag{6.13}$$

where $Z = \phi + i\pi = Le^{i\varphi}\tan\theta/2$ with $0 \leqslant \theta < \pi$ and $0 \leqslant \varphi < 2\pi$. The corresponding kinetic sector is therefore

$$K = \frac{4L^4}{(L^2 + \phi^2 + \pi^2)^2}\left((\partial\phi)^2 + (\partial\pi)^2\right), \tag{6.14}$$

where we have explicitly included the length scale L which sets the radius of the sphere.

Hyperbolic geometry

For a hyperbolic geometry we have $G = SO(1, n)$ and we denote the generators of this group as M_{0i} and M_{ij} where again i, j are $SO(n)$ indices. In this case the broken generators are M_{0i} and the only H-invariant parametrisation of the coset element is

$$\gamma = e^{\phi^i M_{0i}}. \tag{6.15}$$

As with the spherical case the Maurer-Cartan form is somewhat complicated, but one can easily show that it leads to the following unique choice for a G-invariant metric

$$d\sigma^2 = \frac{\sinh^2\sqrt{\phi^2}}{\phi^2}\delta_{ij}d\phi^i d\phi^j + \left(1 - \frac{\sinh^2\sqrt{\phi^2}}{\phi^2}\right)\frac{\phi_i\phi_j}{\phi^2}d\phi^i d\phi^j, \tag{6.16}$$

where again $\phi^2 = \delta_{ij}\phi^i\phi^j$. For $n = 2$ we can write this metric as

$$d\sigma^2 = L^2(d\tau^2 + \sinh^2\tau d\theta^2) = 4L^4\frac{dZd\bar{Z}}{(L^2 - |Z|^2)^2}, \tag{6.17}$$

where now $Z = \phi + i\pi = Le^{i\theta} \tanh \tau/2$ with $0 \leqslant \tau < \infty$ and $0 \leqslant \theta < 2\pi$, and the corresponding kinetic sector is

$$K = \frac{4L^4}{(L^2 - \phi^2 - \pi^2)^2} \left((\partial\phi)^2 + (\partial\pi)^2\right), \tag{6.18}$$

where now L sets the curvature radius of the hyperbolic geometry.

6.1.2 Space-time symmetries

Now we consider non-linearly realised symmetries which do not commute with the 4-dimensional Poincaré group. If we again assume maximal symmetries then in principle we would have three possibilities for G corresponding to the isometries of higher-dimensional Minkowski space, de Sitter space and anti-de Sitter space. However, since there is no way to embed the 4-dimensional Poincaré group into the de Sitter group we only have the two remaining possibilities. In each case we take $H = SO(1,3) \times SO(p)$, leading to the following two coset spaces

$$\begin{aligned} &\mathrm{Mink}_{4+n} : ISO(1, 3+n)/(SO(1,3) \times SO(n)), \\ &\mathrm{AdS}_{4+n} : SO(2, 3+n)/(SO(1,3) \times SO(n-1)). \end{aligned} \tag{6.19}$$

All of these yield non-linear realisations constructed from n scalars, where one has to impose inverse Higgs constraints to remove the additional so-called inessential Goldstone modes.

Since the non-linearly realised symmetries no longer commute with the 4-dimensional Poincaré group, the dx^μ are not invariant and one cannot construct invariant kinetic sectors in the same way as for the internal case. Rather, the invariant that is lowest order in derivatives is a Poincaré invariant combination of four copies of the Maurer-Cartan components $e^\mu{}_\nu dx^\nu$ (corresponding to the translation generators P_μ) in the following way

$$\epsilon_{\mu\nu\rho\sigma}(e^\mu{}_\alpha dx^\alpha) \wedge (e^\nu{}_\beta dx^\beta) \wedge (e^\rho{}_\gamma dx^\gamma) \wedge (e^\sigma{}_\delta dx^\delta). \tag{6.20}$$

As we will see, the above term is not always strictly a kinetic term with at least two derivatives. In some cases there is also a potential term necessary to ensure invariance.

We will now briefly review the flat case, and then discuss the AdS_{4+n} coset space in more detail, since to our knowledge the resulting invariants have not been constructed before for general n.

Minkowski space

For the Minkowski coset space, the broken generators are translations P_i and Lorentz transformations $M_{\mu i}$ where i is an $SO(n)$ index and μ is a 4-dimensional space-time

index. Given the intricate link between the coset parametrisation and one's ability to impose inverse Higgs constraints, we parametrise the coset element as

$$\gamma = e^{x^\mu P_\mu} e^{\phi^i P_i} e^{\Omega^{\mu i} M_{\mu i}} \,, \tag{6.21}$$

from which we can compute the Maurer-Cartan form using the commutators of the $(4+n)$-dimensional Poincaré group. Again the full form of the Maurer-Cartan form is not particularly useful but after we impose inverse Higgs constraints to remove the vectors $\Omega^{\mu i}$ the kinetic sector is seen to be [81]

$$K = \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi_i)} \,. \tag{6.22}$$

One can also derive this term by calculating the induced metric corresponding to a Minkowski 3-brane embedded in higher-dimensional Minkowski space [95]. The kinetic sector then simply corresponds to the measure of this metric. Similarly, invariants like the Einstein-Hilbert term give rise to higher-order invariants of this symmetry[3].

Note that for space-time symmetries we need to include a whole tower of operators to non-linearly realise the broken symmetry group in the kinetic sector, whereas for the internal symmetries discussed above this could be achieved order by order in derivatives. This crucial difference follows from the transformation properties of the metrics derived from the coset construction: they are invariant for the internal cases while they transform covariantly for the space-time symmetry cases.

Anti-de Sitter space

The final coset we consider corresponds to the spontaneous breaking of the anti-de Sitter isometries, corresponding to the coset space (6.19). To derive our kinetic sector we will make use of the AdS basis for the conformal algebra defined by the following non-vanishing commutators[4] [13]

$$\begin{aligned}
&[P_A, D] = P_A \\
&[\hat{K}_A, D] = -\hat{K}_A + P_A && [M_{AB}, P_C] = \eta_{AC} P_B - \eta_{BC} P_A \\
&[P_A, \hat{K}_B] = 2M_{AB} + 2\eta_{AB} D && [M_{AB}, \hat{K}_C] = \eta_{AC} \hat{K}_B - \eta_{BC} \hat{K}_A \\
&[\hat{K}_A, \hat{K}_B] = 2M_{AB}
\end{aligned}$$

and $[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} - \eta_{BC} M_{AD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC}$. Here $A = 0, 1, \ldots 2+n$. The relation to the standard basis is given by

$$\hat{K}_A = K_A + \frac{1}{2} P_A. \tag{6.23}$$

[3] In the single field case, these are so-called DBI galileons [53] which have interesting behaviour in their soft amplitudes due to the non-linearly realised symmetry [35, 146], as mentioned before.

[4] Note that we have set the AdS radius $L = 1/\sqrt{2}$ but will reintroduce it later on.

Here, in addition to the Poincaré generators, we have D and K_A which are the generators of dilatations and special conformal transformations respectively. The AdS basis is useful since the resulting kinetic sector matches the one from embedding Minkowski 3-branes in $(4+n)$-dimensional anti-de Sitter space.

Again given the discussion on inverse Higgsability, we parametrise the coset element as

$$\gamma = e^{x^\mu P_\mu} e^{\pi^i P_i} e^{\varphi D} e^{\Omega^{\mu i} M_{\mu i}} e^{\psi^\mu \hat{K}_\mu} e^{\sigma^i \hat{K}_i}\,, \tag{6.24}$$

where we have assumed $n > 1$ and $\mu = 0, \ldots 3$ and $i = 4 \ldots 2+n$. Non-linear realisations of this symmetry breaking can then be constructed from the Maurer-Cartan form which can be written as

$$\gamma^{-1} d\gamma = \omega^\mu P_\mu + \omega^A T_A + \omega^i T_i\,, \tag{6.25}$$

where T_A are the broken generators of the conformal group and T_i are the unbroken ones i.e. $M_{\mu\nu}$ and M_{ij}. To match our previous notation we have $\omega^\mu = e^\mu{}_\nu dx^\nu$ once we pull back to 4-dimensional space-time. The commutators

$$[P_\mu, M_{\nu i}] \supset \eta_{\mu\nu} P_i\,, \qquad [P_\mu, \hat{K}_\nu] \supset \eta_{\mu\nu} D\,, \qquad [P_\mu, \hat{K}_i] \supset M_{\mu i}\,, \tag{6.26}$$

ensure that the fields $\Omega^{\mu i}$, ψ^μ and σ^i appear linearly in the Maurer-Cartan components along the broken generators P_i, D and $M_{\mu i}$ respectively and given our choice for the coset parametrisation they only appear algebraically. We can therefore impose inverse Higgs constraints to eliminate all of these fields in favour of φ, π^i and their derivatives leaving us with a non-linear realisation constructed from the dilaton and $n-1$ axions.

The axion fields π^i are guaranteed to be shift symmetric since they are the Goldstones of broken translations and they will inherit a linearly realised $SO(n-1)$ symmetry due to the unbroken rotations M_{ij}. Without loss of generality we can therefore simply consider the case where $n = 2$ i.e. where there is a single axion $\pi^4 = \pi$ then augment the resulting non-linear realisation by adding the other axions in an $SO(n-1)$ invariant manner. We therefore consider the coset space and coset element[5]

$$SO(2,5)/SO(3,1)\,, \qquad \gamma = e^{x^\mu P_\mu} e^{\pi P_4} e^{\varphi D} e^{\Omega^\mu M_{\mu 4}} e^{\psi^\mu \hat{K}_\mu} e^{\sigma \hat{K}_4}\,, \tag{6.27}$$

for concreteness.

As with all of the previous cases our aim is to compute a metric from which we can derive G-invariant theories. From the coset construction the metric is fixed in terms of ω^μ so we only need to compute this contribution to the Maurer-Cartan form and the necessary inverse Higgs constraints. A somewhat lengthy calculation yields the following kinetic sector[6]

$$K = \sqrt{-\det\left(e^{2\varphi/L}(\eta_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi) + \partial_\mu\varphi\partial_\nu\varphi\right)}, \tag{6.28}$$

[5]Here we have $\Omega^{\mu 4} = \Omega^\mu$ and $\sigma^4 = \sigma$.

[6]Here we have made the rescaling $\varphi \to \sqrt{2}\varphi$ and reintroduced the AdS radius L.

and by adding the remaining axions we arrive at

$$K = \sqrt{-\det\left(e^{2\varphi/L}(\eta_{\mu\nu} + \partial_\mu\pi^i\partial_\nu\pi_i) + \partial_\mu\varphi\partial_\nu\varphi\right)}. \tag{6.29}$$

Again a full tower of operators is required to non-linearly realise the conformal symmetries.

As we mentioned above this is precisely what one gets from computing the world-volume of a Minkowski 3-brane embedded in $(4+n)$-dimensional AdS space. This is easy to see with the AdS_{4+n} metric written in the Poincaré patch

$$d\sigma^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dy_i dy^i + dz^2)\,, \tag{6.30}$$

where L is the AdS radius, x^μ are the 4-dimensional brane directions, y^i are the axionic directions and z is the dilatonic direction. In fact this metric motivates us to make the field redefinition $\phi = Le^{-\varphi/L}$ such that the kinetic sector becomes[7]

$$K = \sqrt{-\det\left(\frac{L^2}{\phi^2}\left(\eta_{\mu\nu} + \partial_\mu\pi^i\partial_\nu\pi_i + \partial_\mu\phi\partial_\nu\phi\right)\right)}. \tag{6.31}$$

We remind the reader that K is dimensionless and the Goldstones ϕ and π^i have dimension of length.

In what follows we will use this kinetic structure, which is protected by the non-linearly realised conformal symmetry, to realise slow-roll inflation. When we expand the square root we see that there is already a potential of the form $(L/\phi)^4$, which prevents us from interpreting this as a purely kinetic term. In the case where $n=1$, corresponding to $\pi^i = 0$, we know that there is a Wess-Zumino term which one can add to the action to remove this potential without breaking the symmetries; see [53] for a discussion of this term in the context of embedded branes and [81] for a derivation using the corresponding Maurer-Cartan form. In the following we shall assume that there is a Wess-Zumino term for arbitrary n which we can add to the action which reduces to the $n=1$ case when we send $\pi^i = 0$. We simply denote this Wess-Zumino as W.Z such that our symmetric kinetic sector is

$$K = \frac{L^4}{\phi^4}\sqrt{-\det(\eta_{\mu\nu} + \partial_\mu\pi^i\partial_\nu\pi_i + \partial_\mu\phi\partial_\nu\phi)} + \mathrm{W.Z}\,, \tag{6.32}$$

which has a tower of higher-derivative operators in order to realise the non-linear AdS_{4+n} symmetry. Some of the non-linear AdS_{4+n} symmetries are realised order by order in derivatives, namely, the shifts in π^i associated to the translation generators P_i, and the dilations D. In contrast, the transformations associated with the broken Lorentz generators $M_{\mu i}$ and the special conformal transformations K_μ and K_i require the full tower. The precise from of these final symmetries is quite lengthy so we don't

[7] In the single field case this is the kinetic sector of DBI inflation [6] up to field redefinitions. Note that only in the single field case can we canonically normalise.

present them here but one can easily extract them along the lines discussed in [95] by using the isometries of the AdS_{4+n} space and the appropriate embedding functions of the Minkowski 3-brane. This way of computing the symmetries is more straight forward than via the coset construction.

6.1.3 Geometrical considerations

In this chapter we are interested in slow-roll inflationary applications of the above kinetic sectors. Under this approximation, we can neglect higher-order terms in derivatives. This yields a two-derivative kinetic sector which is amenable to a geometric interpretation. For simplicity we will discuss the two-field case in this section (while the generalisation to more scalars is rather straightforward). We therefore have different geometries on the complex plane, with three interesting geometric possibilities, in addition to the flat case.

There are two possibilities with negative curvature which are interesting to compare against each other. In particular, the hyperbolic kinetic sector is based on the geometry (6.17) in terms of disc coordinates. An alternative parametrisation is in terms of half-plane coordinates T, which are related via the Cayley transformation

$$\frac{T}{L} = \frac{L - Z}{L + Z}\,, \tag{6.33}$$

where we keep all coordinates as lengths and in the following drop all order one factors since they can always be absorbed into a redefintion of L. This brings the hyperbolic geometry to the form

$$d\sigma^2 = \frac{L^2}{(T + \bar{T})^2} dT d\bar{T}\,. \tag{6.34}$$

In contrast, the truncation of the square root structure in the AdS kinetic sector (6.32) at two derivative order naturally leads us to consider the geometry[8]

$$d\sigma^2 = \frac{L^4}{(T + \bar{T})^4} dT d\bar{T}\,. \tag{6.35}$$

Both of these lead to an axion-dilaton system, with the dilaton being the real part of T and the shift symmetric axion being the imaginary part i.e. $T = \phi + i\pi$. Importantly, the couplings between the two scalars is different in both cases, dictated either by an internal or space-time symmetry. Again, this crucial difference follows from the invariance or covariance of the associated coset metric.

[8] A more detailed knowledge of the structure of the Wess-Zumino term is required to be sure that at the two derivative level the AdS kinetic sector gives rise to the geometry (6.35). In this paper our ultimate interest is in inflationary dynamics along the dilatonic direction where this subtlety plays no role, but here we will point out the geometric properties at the two derivative level assuming that the Wess-Zumino allows for a truncation to (6.35).

The two relevant geometries are therefore special cases of the more general metric

$$d\sigma^2 = \frac{L^p}{(T+\bar{T})^p} dT d\bar{T}\,, \tag{6.36}$$

for an arbitrary power p, which we will refer to as the order of the pole. The above discussion singles out three values, $p = 0, 2$ and 4, as being special from a symmetry perspective. The former two have an enhanced isometry group at the two-derivative level while the latter realises the full symmetries of the AdS_6 group once the higher order corrections are taken into account.

Of the maximally symmetric possibilities, $p = 0$ is flat and has an $ISO(2)$ isometry group. The case $p = 2$ is the hyperbolic half-plane and has the Mobius transformations as isometry group, isomorphic to $SO(2,1)$. This ensures that the curvature of the manifold is constant and negative with scaling $R \sim L^{-2}$. Moreover, the Mobius transformations include the inversion $T \to 1/T$. As a consequence, there is a pole of order two at $T = 0$ as well as at $T = \infty$. Indeed, one can see that the proper distance to both points is infinite. For a geometry of the form (6.36) $p = 0$ and $p = 2$ are the only ones with maximal symmetries and no singularities.

For any other value of p there is only a single isometry: the shift in the axionic direction. However, as we have discussed, $p = 4$ is special for other reasons since when we include the higher order operators we can realise the full AdS_6 symmetries. At the two derivative level, of these symmetries the linearly realised Lorentz group of course survives the truncation to the two-derivative level but so does the shift in π, $\pi \to \pi + c$, corresponding to the broken translation P_4 and the scale symmetry. For the field basis used in (6.32) this symmetry reads

$$\phi \to \phi + \lambda(\phi - x^\mu \partial_\mu \phi) \qquad \pi \to \pi + \lambda(\pi - x^\mu \partial_\mu \pi), \tag{6.37}$$

where λ is the infinitesimal parameter of dilatations and here we do not transform the space-time coordinates. The other symmetries corresponding to the broken Lorentz transformations and special conformal transformations require higher order operators for invariance of the action. In the following we will investigate the effects of these symmetries on inflationary dynamics but we note that spontaneous breaking of scale invariance has been studied in the context of inflation before, e.g. [46], but in a different set-up to ours.

For $p > 2$ the proper distance to the pole at $T = 0$ is infinite but this does not hold for the point $T = \infty$. Given that in these cases the curvature scales as $R \sim (T+\bar{T})^{p-2}$, the geometry has a singularity at this point. Note that p and $4-p$ yield identical results along the real line but this is not true for the entire complex plane. We can use the Cayley transformation (6.33) to go to disc-like coordinates, which highlights both special points at $T = 0$ and $T = \infty$ and moves them to finite coordinate values. This leads to the space-time interval

$$d\sigma^2 = \frac{L^4 (L+Z)^{p-2} (L+\bar{Z})^{p-2}}{(L^2 - Z\bar{Z})^p} dZ d\bar{Z}\,. \tag{6.38}$$

For $p = 2$, this choice of coordinates highlights the $SO(2)$ isometry and corresponds to the Poincaré disc parametrisation of the hyperbolic geometry. Along the real axis, the above metric reduces to the interval

$$d\sigma^2 = \frac{L^4 (L+Z)^{p-4}}{(L-Z)^p} dZ^2 \,. \tag{6.39}$$

For $p = 2$, there is equivalence between the two special points at the real line, $Z = \pm 1$. For values $p > 2$ we have mapped the pole to $Z = 1$ and the singularity to $Z = -1$, while these two are interchanged for $p < 2$. Again $p = 0$ and $p = 4$ are special in that there is no singularity at one side. We refer the reader to [28] for a more detailed discussion for $p = 2$ and to [112] for a further discussion on the role of geometry in scale invariant models of inflation.

6.2 Symmetry breaking potentials

6.2.1 Universality classes of inflation

The symmetric kinetic sectors of the previous section provide an attractive starting point for inflationary scenarios. To this end, one has to introduce a scalar potential in order to introduce the required energy for the accelerated expansion, as well as evolution towards the end of inflation. At the same time, the weakly broken symmetry in the kinetic sector protects the model against large quantum corrections within EFT. This ties in with the smallness of the observed level of quantum fluctuations: the inflationary energy scale is orders of magnitude below the Planck scale and small symmetry breaking parameters are technically natural.

As alluded to in the introduction, the simplest of such examples consists of a single scalar field with a canonical kinetic term. The symmetries of this model include a constant shift which will be broken by the introduction of a generic potential. We will assume a Minkowski minimum somewhere in field space, which can be taken at $\phi = 0$ without loss of generality. Different manners of breaking this shift symmetry then correspond to e.g. a quadratic or a quartic scalar potential around this point, or a combination of such monomials.

The inflationary predictions of such models are particularly simple under the assumption that a single monomial dominates the inflaton dynamics at the observable window of $N = 50$ to 60 e-folds. Taking $V = \lambda \phi^m$ as the simplest example of this class, the resulting predictions are

$$n_s = 1 - \frac{2+m}{2} \frac{1}{N} \,, \qquad r = \frac{4m}{N} \,. \tag{6.40}$$

The leading order $1/N$ scaling for the tensor-to-scalar ratio means that the simplest of these models with e.g. $m = 2$ or $m = 4$ are virtually ruled out [2, 4].

The underlying assumption in the above is that $N = 50$ to 60 e-folds is a generic window on the primordial quantum fluctuations, i.e. there is nothing special about the moment we probe inflation, leading to an expansion in the small parameter $1/N$. Different models (e.g. of polynomial character) will lead to the same predictions at leading order in $1/N$ (under the assumption that $N = 50$ to 60 is dominated by a single monomial) and only lead to different, model-dependent subleading terms at higher order in $1/N$.

However, the above predictions (6.40) are one out of two possible perturbative expansions[9] in $1/N$. The alternative has predictions that can be written as (with $p > 1$) [153]

$$n_s = 1 - \frac{p}{p-1}\frac{1}{N}\,, \qquad r = \frac{r_0}{N^{p/(p-1)}}\,, \tag{6.41}$$

at leading order in the $1/N$ expansion. Instead of a monomial expansion of the scalar potential, the second class of inflationary predictions can be conveniently parametrised in terms of the kinetic sector and can result in a suppression of tensor modes. Rather than having a canonical kinetic term, one can allow for a pole in the kinetic sector of the theory. The latter is a natural possibility in multi-field inflation, as suggested by UV theories, where in general one cannot canonically normalise the fields. Along the single-field trajectory, the kinetic sector has the general Laurent expansion [76], see also [24, 165],

$$K = \left(\frac{a_p}{\phi^p} + \frac{a_{p-1}}{\phi^{p-1}} + \ldots\right)(\partial\phi)^2\,, \tag{6.42}$$

where p is the order of the pole. The assumption in this scenario is that V is regular around $\phi = 0$ i.e. we have

$$V = V_0(1 + c_1\phi + c_2\phi^2 + \ldots). \tag{6.43}$$

As inflation takes place close to the pole, it is only the leading term in the scalar potential which determines the inflationary predictions and pole inflation can therefore be seen as a very convenient parametrisation of these inflationary models: all relevant information about the prediction is stored in the leading term of the kinetic sector. Note that for $p = 2$ the coefficient c_1 drops out of all observables due to the scaling symmetry in the kinetic sector, while for other values it can always be set equal to unity by a rescaling of the field and redefinitions of a_p, a_{p-1} etc. Many models fall in the same universality class, with the same model-independent leading predictions and different model-dependent sub-leading terms at higher order in $1/N$.

Our previous discusssion has singled out two types of poles that have an enhanced symmetry in the kinetic sector. The first is the well-known inflationary model based on the hyperbolic geometry (6.34) with $p = 2$ which is commonly referred to as α-attractors [72, 103]. In this case, under the assumption of a regular potential at the

[9] A third possibility has a non-pertubative expansion in $1/N$ instead [77], which includes natural inflation [75]. These can be associated to the kinetic sector with the positively curved internal manifold of section 2.

pole, the inflationary predictions yield

$$n_s = 1 - \frac{2}{N}, \qquad r = \frac{8a_2}{N^2}, \tag{6.44}$$

to leading order in $1/N$ where we have set $M_{\rm pl} = \Lambda = 1$ and the higher order corrections are sufficiently suppressed [153]. As discussed in the literature, the leading terms in the $1/N$ expansion are model-independent for α-attractors, with robust predictions. The fixed $2/N$ deviation from scale invariance is in very good agreement with Planck constraints [4], leading to e.g. $n_s = 0.960$ to 0.967 for a range of $N = 50$ to 60 (as will be the case for all following quotes). Moreover, the $1/N^2$ scaling implies that r generically takes values of a few permille, assuming order one values for $a_2 = 3\alpha/2$. The benchmark model $a_2 = 3/2$, which corresponds to Starobinsky, has $r = 0.005$ to 0.003, while other constructions can boost this to percent level values, see e.g. [71, 106].

The other inflationary model based on the AdS kinetic sector, which has not been discussed previously in the literature, corresponds to pole inflation with $p = 4$ as shown in (6.35), plus higher order corrections in the kinetic sector which will be suppressed during inflation. Inflation proceeds along one of the isometries of the AdS space: as inflation proceeds, the 3-brane moves from through the ambient space. It is therefore natural to refer to this set-up as ambient inflation. It follows from the above discussion that the inflationary predictions to leading order in $1/N$ are

$$n_s = 1 - \frac{4}{3N}, \qquad r = \frac{8a_4^{1/3}}{3^{4/3}} \frac{1}{N^{4/3}}, \tag{6.45}$$

where again we have set $M_{\rm pl} = \Lambda = 1$. For order one values of a_4 this leads to a spectral index with a range $n_s = 0.973$ to 0.978 which is compatible with observational constraints if the number of e-folds would be on the low side of the range from 50 to 60 [4]. However, it turns out that the next to leading order correction to n_s scales as $1/N^{4/3}$ and given the sensitivity of CMB experiments this can produce important corrections. Fortunately this next correction comes in with a minus sign so it can decrease the value of n_s thereby moving it towards the sweet spot of the Planck data. We will discuss this further in subsection 3.2. The $1/N^{4/3}$ scaling for the tensor-to-scalar ratio naturally takes values at the percent level, for instance $r = 0.010$ to 0.008 for $a_4 = 1$. The tensors therefore come out an order of magnitude higher than the generic α-attractor prediction. Both models therefore provide interesting observational targets for upcoming ground-based (e.g. CMB-S4) and satellite (e.g. Litebird and Core) CMB polarisation experiments.

For $p > 1$ the number of inflationary e-folds is given by

$$N = \int \frac{a_p}{\phi^p} d\phi \sim \frac{a_p \phi^{1-p}}{(p-1)}, \tag{6.46}$$

from which we can extract the field range during inflation. For α-attractors ($p = 2$) we have $N \sim \phi^{-1}$. After we canonically normalise the kinetic sector we see that the field

range scales as $\sim \log(N)$ in Planck units. On the other hand, for ambient inflation ($p = 4$) the field range of the canonically normalised field scales as $\sim N^{1/3}$ in Planck units. Both of these field ranges are smaller than for chaotic inflation where the field range scales as $\sim N^{1/2}$. However, note that both pole models are essentially large-field models with super-Planckian excursions. They could therefore be susceptible to UV considerations such as the weak gravity conjecture, see e.g. [9, 23, 115], which questions the validity of the EFT but this question is far from settled and remains an area of active research.

Even in the context of EFT, in the above we have assumed that inflation takes place along the dilatonic direction, while setting the axion constant. For α-attractors the resulting predictions turn out to be a very good approximation to those where the axion is not stabilised thanks to the hyperbolic geometry of the scalar manifold [1]. In the absence of some mechanism to stabilise the axion in ambient inflation we would require a similar analysis to be sure that our predictions are stable against turning on axionic fluctuations. Again in ambient inflation the kinetic sector is fixed by symmetry so there is reason to believe that this will indeed be the case but the reader should bear in mind that a full multi-field analysis would require a more detailed knowledge of the form of the Wess-Zumino term in the AdS kinetic sector (6.32).

6.2.2 Adding curvature to reduce tensors

A natural question concerns the relation between the different cases above: is there a limit in which the curved cases, with poles of order 2 and 4, reduce to the flat case with $p = 0$ and the monomial expansion? We will now show that this is indeed the case. The simplest way to this connection is in terms of the Z coordinates introduced in subsection 6.1.3. Along the real line, this yields kinetic sectors

$$K = \frac{1}{(1-\phi^2/L^2)^2}(\partial\phi)^2\,, \qquad K = \frac{1}{(1-\phi/L)^4}(\partial\phi)^2\,, \tag{6.47}$$

for $p = 2$ and $p = 4$, respectively. In this section we will again work with $M_{\rm pl} = \Lambda = 1$. In this parametrisation, the kinetic structure is regular around the minimum at $\phi = 0$ while the pole is located at $\phi = L$. One can thus go from the flat case, with L infinite, to the curved case by bringing the pole in from infinity to a finite distance from the minimum of the potential. This allows for a continuous interpolation between the different flat and curved inflationary scenarios of the previous subsection.

We have illustrated the behaviour of the different inflationary predictions in the presence of a symmetry breaking potential $V = \lambda\phi^m$ in the plots below. Included are the predictions based on the flat geometry with a weakly broken shift symmetry, as well as the two geometries with negative curvatures.

- The inflationary predictions based on the flat geometry include those of monomial inflationary models, with a $1/N$ scaling for both the spectral index as well as the tensor-to-scalar ratio, indicated by the dotted line.

Turning on the curvature scale L of the two geometries, the predictions converge to a similarly precise relation (at least when L is order one):

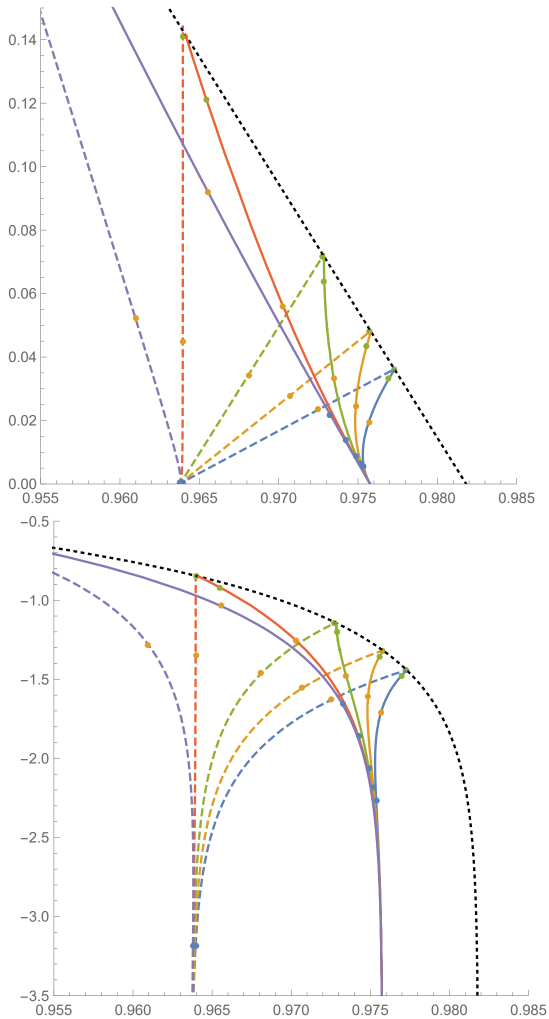

Figure 6.1: *The interpolation in* (n_s, r) *(in linear and log plots) from the flat predictions* (6.40) *to the* α*-attractor predictions* (6.48) *based on an internal hyperbolic symmetry (dashed lines) or to the ambient inflation predictions* (6.49) *based on a spacetime AdS symmetry (solid lines). The different lines indicate models with a monomial potential* ϕ^m *with* $m = (4, 2, 1, 2/3, 1/2)$ *from left to right, while the curvature scale* L *in the kinetic sectors* (6.47) *varies along each line, with blue, orange and green dots at* $L = 1, 10, 100$ *respectively. The black dotted line indicates the flat limit* $L \to \infty$ *with canonical kinetic term and monomial potentials. We have taken* $N = 55$ *throughout.*

- For the internal symmetry based on the hyperbolic geometry, one is led to

$$n_s = 1 - \frac{2}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) , \quad r = \frac{2L^2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) . \tag{6.48}$$

  The spectral index has a model-independent leading term, while subleading terms are sufficiently suppressed. Moreover, the $1/N^2$ scaling leads to permille values of the tensors, indicated by the dashed lines. Note that these only depend on L. As a consequence, the $L = 1$ points (denoted by blue dots) for all monomials coincide.

- In the case of the spacetime symmetry based on the AdS geometry, one finds

$$n_s = 1 - \frac{4}{3N} - \frac{(mL)^{2/3}}{(3N)^{4/3}} + \mathcal{O}\left(\frac{1}{N^{5/3}}\right) , \quad r = \frac{8(mL)^{2/3}}{(3N)^{4/3}} + \mathcal{O}\left(\frac{1}{N^2}\right) . \tag{6.49}$$

  While the leading term for the spectral index is again model-independent, there is a subleading term which is only $1/N^{1/3}$ suppressed and hence can make a (possibly observable) model-dependent contribution. Similarly, the $1/N^{4/3}$ leads to percent values for the tensors indicated by the solid lines. Their value in this case depends on mL. Therefore the $L = 1$ points differ slightly in their (n_s, r) values.

These can be seen as prototypes for chaotic inflation, α-attractors and ambient inflation, respectively. The robustness of the latter two is beautifully illustrated by the funnel-like behaviour in the logarithmic (n_s, r) plane.

For intermediate values of L one finds an interpolation between the different cases. Note that these do not conform to either of the $1/N$ scalings introduced in the previous subsection. The reason for this is the interplay between the large values of N and L in these cases: while certain terms might be higher-order in $1/N$, their L-prefactor might offset this and make them equally important. We are not aware of any simple expansion or organising principle in this intermediate regime. Moreover, in this regime the model-independence is lost; for e.g. α-attractors there are different interpolations between the regimes of L infinite and of order one.

Note that there are specific flat limits which have the same spectral index as either of the curved models as we decrease L. The corresponding models can be extracted by equating the spectral indices in (6.40) and (6.41), leading to the relation $m = 2/(p - 1)$ between the pole p of the curved model and the power m of the corresponding monomial. For α-attractors this is the quadratic potential. In a sense, this model can therefore be seen as the simplest flat counterpart to the α-attractors: as one varies L, the spectral index is invariant while the tensor-to-scalar ratio interpolates from a $1/N^2$ to a $1/N$ scaling. In contrast, for ambient inflation the special flat limit yields a $\phi^{2/3}$ behaviour. Again, as one varies L, the spectral index is roughly invariant while the $1/N$ scaling of r varies from $4/3$ to 1. It is tantalising to notice that exactly this fractional behaviour surfaced in the first realisation of axion monodromy

inflation [157]. One might hope that there is an extension of this scenario where one includes the curvature scale L and obtains a microscopic realisation of the ambient inflation scenario.

6.3 Conclusions and discussion

While the present discussion has highlighted the symmetries of the bosonic scalar sectors, it is natural to embed these in more formal constructions in order to connect to high-energy theories. In this context, α-attractors arise very naturally in scenarios with minimal supersymmetry or superconformal symmetry [72,103] as well as no-scale supergravity [67, 68]. Such set-ups often have a Kähler potential with specific symmetries which are weakly broken by the superpotential (alternatively, the symmetry breaking can also be realized in terms of the Kähler potential, indicating an interesting link with anti-D3 brane geometry [105, 128]). Moreover, there are proposals to realise the same inflationary scenarios in maximally supersymmetric theories as well as string theory [25,71,104,106]. For the new bosonic construction, based on the Anti-de Sitter ambient space, it would be very interesting to investigate similar set-ups. In particular, this would involve the spontaneous breaking of the 6-dimensional Anti-de Sitter superalgebra with eight supercharges to 4-dimensional minimal super-Poincaré, which can be seen as the curved version of DBI-Volkov-Akulov [152].

On a final note, while in this chapter our non-linear symmetries were weakly broken by the inflationary potential, it would also be interesting to consider the effects of unbroken non-linearly realised symmetries on cosmological correlators both in the single field and multi-field case. This is the cosmological version of studying the effects of non-linear symmetries in the soft limits of scattering amplitudes [34,35,146]. For a single field with a shift symmetry this was very recently explored in [22, 73] and novel features can arise. Based on what we know about scattering amplitudes, one would expect new novel features to arise if one considers space-time extensions of this shift symmetry. This is an interesting avenue for future work.

Chapter 7

Conclusions and outlook

In this thesis we have discussed several formal aspects of cosmological models. In particular we focused on two topics. Firstly, we investigated higher derivatives in physical models, how they generically lead to ghost degrees of freedom, and importantly how these ghosts can be avoided. Secondly, we examined the intricacies of theories with spontaneously broken space-time symmetries, the accompanying non-linear realisations, and the role of essential and inessential Goldstones.

The first two chapters were introductory and covered the basics needed to thoroughly discuss these topics in depth: the Lagrangian and Hamiltonian formalisms, degenerate theories and degrees of freedom, symmetries and their realisations, and redefinitions of space-time coordinates and fields. Special attention was given to the possible appearance of (higher) derivatives in these contexts, since standard treatments mostly focus on first order theories and symmetry transformations and redefinitions not involving derivatives. This led us to discuss point, contact and the more general Lie-Bäcklund symmetries and redefinitions that turn out to occur naturally when discussing the structures of healthy higher derivative theories, as well as in theories with sponanteously broken space-time symmetries.

Healthy higher derivative theories. In Chapter 4 we have performed a constraint analysis of field theories with two dinstinct sectors: one with up to second order derivatives and the other being first order in derivatives. Restricting to theories without gauge symmetries, we have derived the conditions in order to evade the Ostrogradsky ghosts. They amount to a set of symmetric *primary* conditions and two sets of *secondary* conditions, one symmetric and the other antisymmetric. In Lorentz invariant theories the symmetric secondary conditions were found to be automatically enforced, implying the impossibility of a non-integer number of propagating degrees of freedom in these cases.

We also introduced a classification of healthy theories based on the properties of null vector of the relevant degeneracy matrix. Theories whose relevant part of the

null vector actually vanishes belong to Class I. For such theories the degeneracy is not due to coupling to a healthy sector and they are at most linear in second time derivatives. Theories in the remainig classes, Class II and III, do have nontrivial coupling between the higher derivative sector and the healthy one and this coupling is essential to their degenerate structure and allows for non-linear dependence on second derivatives. This is reflected in their non-vanishing null vectors, which for Class II (Class III) are independent of (dependent on) the highest derivatives of the fields, i.e. $\ddot{\phi}$ and $\dot{q}$.

Lastly, we examined whether healthy higher derivative theories can or cannot be cast in manifestly healthy form. For theories in Class I one can usually get rid of the second order time derivatives by adding a suitable total derivative, which puts them in a manifestly healthy form. It turns out that all theories in Class II can be mapped to Class I by suitable invertible derivative dependent field redefinitions. A subclass of Class III can be mapped to Class I via more complicated extended contact transformations, whereas for the remainder it seems impossible to find any sort of redefinition that achieves this. We note that the total derivatives and/or redefinitions used to rewrite theories do not always respect manifest symmetries that might be present. Whenever this is the case, these theories go beyond the healthy first order ansatz with manifest symmetries, and are thus interesting to consider.

Amongst the topics we have not touched upon here is the inclusion of degeneracies in the healthy sector, e.g. arising from gauge symmetries or the absence of specific kinetic terms. This option would be necessary in order to go beyond scalar fields and discuss other Lorentz representations. We expect the implications of Lorentz invariance regarding the structure of the constraints to be similar in such cases. A more detailed examination of how different types of gauge symmetries might be able to avoid Ostrogradsky ghosts in higer derivative theories, f.e. by rendering them pure gauge, would be interesting for future research.

Another generalisation of our analysis would be to also include higher than second order time derivatives. It is clear however that it will be significantly more involved than the one presently performed since one must remove more would be Ostrogradsky ghosts. A first analysis of beyond second order mechanical systems has very recently been put forward in [131, 132]. However the analysis is not yet fully general and is mostly restricted to obtaining degeneracy conditions (most of which are not explicitly given in terms of the Lagrangian). An in depth analysis of degeneracy classes and possible relations between them via different type of redefinitions is still missing and also the generalisation to field theories is yet to be made.

Finally, one could consider theories whose unhealthy sector consists of first order fermions, rather than higher derivative bosons. A preliminary analysis has been performed in [114], where degeneracy conditions for mechanical systems involving first order fermions and a healthy first order bosonic sector have been obtained. However, like in the case of higher than second order derivatives, a thorough analysis of the structures amongst such theories in terms of degeneracy classes and possible redefinitions as well as the generalisation to field theories are again still to be done. Also,

an analysis of theories with higher than first order fermions would be interesting.

Spontaneously broken space-time symmetries. Chapters 5 and 6 mostly dealt with theories with broken space-time symmetries, where we focused on the formal aspects of such theories in the former chapter, and applications of such theories to cosmology in the latter one. We have seen that coset constructions are a powerful tool for constructing theories with non-linearly realised symmetries for both internal as well as space-time symmetries. For spacetime symmetries, however, they generically involve a number of inessential Goldstone modes that are dispensable for the non-linear realisation. This makes it hard to see whether all non-linear realisations or even coset constructions are equivalent, something which has been proven for large classes of internal symmetries. Motivated by this, we have addressed two crucial aspects with regards to the inessential Goldstones.

First of all, we have investigated different ways of eliminating the inessential Goldstones. In the literature, this often proceeds via imposing inverse Higgs constraints. In contrast to existing claims, we have demonstrated that the existence of such constraints actually requires the structure constants to satisfy a sequence of conditions as also discussed in [127]. Moreover, the severity of these conditions depends on the form of the coset element, with the standard parametrisation being a suboptimal choice. Instead, the least stringent conditions arise for a coset element that consists of the largest number of exponential factors. We have also discussed the possible (in)equivalence of theories obtained using other methods of eliminating the inessential Goldstones, algebraically or otherwise.

The second issue concerns the relation between coset constructions employing different parametrisations and/or basis choices. Again the inessential Goldstones play a crucial role. Prior to the process of elimination, all coset constructions are related to each other by means of a point transformation, involving the set of essential and inessential Goldstones as well as spacetime coordinates. This naturally generalises the field redefinitions relating all coset constructions for internal symmetries. However, such a point transformation does not necessarily relate the inverse Higgs constraints for the inessential Goldstone modes. In the case where they are related, one inherits an extended contact transformation, involving the essential Goldstones, their derivatives and the spacetime coordinates, that maps the different non-linear realisations onto each other. More generally, if we have n inverse Higgs constraints then the extended contact transformation could in principle be n-th order.

However, we have seen that in the cases where the inverse Higgs constraints are not related by the point transformation, there is no such inherited extended contact transformation. A natural expectation would be that the resulting theories for the essential Goldstones are inequivalent. However, we have shown that this is not necessarily the case in a simple example where the inverse Higgs constraints are unrelated but the non-linear realisations result in equivalent physics. As of now it is unclear whether the same holds for all such theories or whether this is a consequence of the simplicity of our example. Thus, whether or not a given spacetime symmetry breaking

pattern does indeed produce universal dynamics for the essential Goldstones remains an interesting open question. Either way the inessential Goldstones will certainly play an important role.

The above subtleties aside, the coset construction has been proven very useful also in settings involving non-linearly realised space-time symmetries, especially when combined with a formal Lie algebraic classification. This has been recently illustrated in [18, 19], where a systematic classification and construction of scalar field models with non-linear symmetries was given, thereby reproducing results obtained earlier via the recent investigation of enhanced soft limits of scattering amplitudes [34,35,146] in a complementary and arguably simpler manner. Lastly, following such an approach we have very recently shown in [VI] that under the assumption of universality, no healthy theories of a single gauge boson exist that have non-linear symmetries beyond those included in the gauge symmetry. It seems fruitful to also use this Lie algebraic approach in other settings, for example in the context of SUSY or condensed matter systems.

Inflation. Lastly, we investigated potentially interesting inflationary models making systematic use of the coset construction. The recent CMB data indicates that the most simple single field inflationary models do not seem to describe the physics of the very early universe, since they predict a tensor-to-scalar ratio above the current upper bound [2, 4]. This requires model builders to construct slightly more complicated inflationary models but one should aim to maintain the very nice field theory properties of these original models, namely, radiative stability within EFT. This is crucial to ensure that slow-roll inflation can take place while perturbative quantum corrections are under control. For simple single field models this is the case thanks to a weakly broken shift symmetry in the kinetic sector.

Multi-field inflation is a popular alternative to single field models given that high-energy frameworks such as string theory and supergravity often involve a large number of moduli. In Chapter 6 we have constructed kinetic sectors for multi-field inflation which have a non-linearly realised symmetry which can be weakly broken by a potential to drive inflation, similar to what happens in simple chaotic inflation. Indeed, throughout our motivation has been to build observationally consistent inflationary models which maintain radiative stability within EFT. We have not examined the delicate problem of (in)stability of slow-roll conditions under full fledged non-perturbative quantum gravity effects, which is something that would provide interesting future research directions.

We have classified possible kinetic structures for n scalar fields which are fixed by a non-linearly realised symmetry corresponding to a coset space G/H similar to what was done in [26]. Concentrating on cases where G is maximally symmetric, five different kinetic sectors are possible and each come with their own interesting structures. Three of these arise when G is an internal group and the other two arise when it is a space-time extension of the 4-dimensional Poincaré group. In all cases we made use of the coset construction to extract the invariant kinetic sectors and for the

space-time symmetric cases we imposed various inverse Higgs constraints to arrive at a theory of interacting scalar fields. Out of these five possibilites our main focus has been on two examples which when one adds the symmetry breaking potential correspond to the well known α-attractors model and a new inflationary model we called ambient inflation. In the two field limit these models describe different interactions between an axion and a dilaton.

Although α-attractors and ambient inflation have a comparable origin with their kinetic sectors dictated by symmetry, their inflationary predictions differ. The spectral index for α-attractors naturally lies close to the point $n_s = 0.965$ whereas for ambient inflation one finds predictions closer to $n_s = 0.975$. A similar difference can be seen in the tensor-to-scalar ratios which naturally differ by an order of magnitude due their $1/N^2$ and $1/N^{4/3}$ scaling for order one parameters. These lead to a range of permille and percent level values for the tensors, well within the reach of future CMB missions.

List of publications

I R. Klein, M. Ozkan and D. Roest, *Galileons as the Scalar Analogue of General Relativity*, Phys. Rev. D **93** (2016) no.4, 044053, [arXiv:1510.08864 [hep-th]].

II R. Klein and D. Roest, *Exorcising the Ostrogradsky ghost in coupled systems*, JHEP **1607** (2016) 130, [arXiv:1604.01719 [hep-th]].

III M. Crisostomi, R. Klein and D. Roest, *Higher Derivative Field Theories: Degeneracy Conditions and Classes*, JHEP **1706** (2017) 124, [arXiv:1703.01623 [hep-th]].

IV R. Klein, D. Roest and D. Stefanyszyn, *Spontaneously Broken Spacetime Symmetries and the Role of Inessential Goldstones*, JHEP **1710** (2017) 051, [arXiv:1709.03525 [hep-th]].

V R. Klein, D. Roest and D. Stefanyszyn, *Symmetry Breaking Patterns for Inflation*, JHEP **1806** (2018) 006, [arXiv:1712.05760 [hep-th]].

VI R. Klein, E. Malek, D. Roest and D. Stefanyszyn, *No-go Theorem for a Gauge Vector as a Spacetime Goldstone mode*, Phys. Rev. D **98** (2018) no.6, 065001, [arXiv:1806.06862 [hep-th]].

Nederlandse samenvatting

Het onderzoek waaruit dit proefschrift is samengesteld is voornamelijk gedaan met het oog op toepassingen in de fysica van het universum als geheel, ook wel bekend als de kosmologie. Om een goed beeld te geven van de redenen achter het onderzoek en de beoogde resultaten geven we eerst een inleiding in en een korte geschiedenis van de moderne kosmologie en zullen we gaandeweg de openstaande vraagstukken en problemen introduceren.

Inleiding in de kosmologie

Heeft het universum altijd al bestaan? Zo niet, hoe is het dan ontstaan? Hoe zal het er in de toekomst uitzien? Waaruit is het universum opgebouwd? En wat is de aard van de ruimte, en die van de tijd? Dit is een greep uit de vragen die men tracht te beantwoorden binnen de kosmologie.

Voor de 20e eeuw beperkte de kosmologie zich, afgezien van filosofische vraagstukken, voornamelijk tot het proberen te verklaren van de bewegingen van de (op dat moment bekende) hemellichamen. De eerste modellen waren met name geocentrische modellen, waarbij de aarde als middelpunt van het universum een speciale plek innam. In de 16e eeuw veranderde dit beeld toen achtereenvolgens Kepler, Copernicus en Galilei heliocentrische modellen beschouwden waar de zon als middelpunt van het universum fungeerde. Hoewel deze modellen een vooruitgang betekenden was er nog geen principe dat alle waargenomen bewegingen kon verklaren. Pas toen in 1687 Newton zijn zwaartekrachtstheorie opstelde kwam er een elegante en universeel toepasbare theorie en werd tot op zekere hoogte het probleem van de beweging van de hemellichamen opgelost.

Newton's zwaartekrachtstheorie gaat uit van een onveranderlijke ruimte waarbinnen de verschillende hemellichamen zich bewegen. Daarnaast beschouwt men het lopen van de tijd als absoluut, dat wil zeggen onveranderlijk en hetzelfde in elk punt in de ruimte. Met de komst van Einstein en zijn twee relativiteitstheorieën, de speciale en de algemene, veranderde dit beeld radicaal. Allereerst kwam met de speciale relativiteitstheorie de realisatie dat tijd niet absoluut is en dat tijd en ruimte samen een samenhangend geheel vormen: de ruimte-tijd. Het is deze ruimte-tijd waarbinnen

zich alles afspeelt. Deze werd overigens nog wel als een onveranderlijke achtergrond beschouwd. Echter toen Einstein de zwaartekracht wilde verenigen met de principes van de speciale relativiteitstheorie kwam hij tot de conclusie dat de ruimte-tijd niet onveranderlijk is: materie binnen de ruimte-tijd zorgt voor kromming van diezelfde ruimte-tijd, en omgekeerd zorgt de kromming van ruimte-tijd weer voor beweging van de materie. Er is dus een essentiële wisselwerking tussen ruimte-tijd en materie.

Einstein voerde zijn idee van veranderlijke ruimte-tijd in eerste instantie niet in extremis door: hij geloofde dat op de grootste afstandsschalen het universum nagenoeg onveranderlijk zou zijn. Anderen namen een dergelijke veranderlijkheid wel serieus, en in 1922 postuleerde Friedmann dat het universum een steeds groter wordende, ook wel uitdijende, ruimte-tijd met daarin bewegende materie behelst. Het uiteindelijke antwoord werd gegeven toen Hubble in 1929 observeerde dat alle sterrenstelsels zich van ons af bewegen. En dat niet alleen, het bleek dat hoe verder de sterrenstelsels van ons af staan des te sneller ze zich van ons af bewegen. De manier om dit te verklaren is dat het niet zozeer de sterrenstelsels zelf zijn die van ons af bewegen, maar dat het de uitdijing van de ruimte-tijd zelf is dat dit effect teweeg brengt.

Door het gegeven van een uitdijend heelal kan men concluderen dat als men terug gaat in de tijd het universum steeds kleiner wordt. Als men lang genoeg terug gaat in de tijd zal dus op een gegeven moment het gehele universum samen moeten komen in één punt. Dit noemt men ook wel de oerknal, iets wat door de priester Lemaitre in 1927 al was verwoord. Op dit idee werd sindsdien voortgeborduurd en in 1948 werd de zogeheten hete oerknaltheorie geformuleerd door Gamov en Alpher. Hierin werd een nauwkeurige uiteenzetting van de evolutie van het universum gegeven, beginnende bij een zeer hete kleine brij van elementaire deeltjes die vervolgens uitdijde en daarbij afkoelde. Door die afkoeling kunnen die deeltjes vervolgens samen iets grotere structuren vormen, te beginnen bij protonen en neutronen en vervolgens lichte atoomkernen zoals die van waterstof en helium. Verdere afkoeling zorgt voor het binden van elektronen aan die atoomkernen om daadwerkelijke atomen te vormen. Dit moment van recombinatie is tevens het moment dat het universum voor het eerst doorzichtig werd. Dat wil zeggen: vanaf dit moment krijgt licht vrij spel en kan het door de ruimte bewegen zonder continu te worden geabsorbeerd door materie. Het licht dat op dat moment werd uitgezonden zou tegenwoordig nog steeds waarneembaar moeten zijn en wordt de kosmische achtergrondstraling genoemd. Toen in 1960 deze achtergrondstraling bij toeval werd waargenomen door Wilson en Penzias werd de hete oerknal algemeen aangenomen als juiste hypothese.

Gaandeweg kwamen er steeds verfijndere versies van de hete oerknaltheorie maar uiteindelijk kwam men erachter dat naast alle zichtbare materie het universum twee exotische componenten bevat. Zo concludeerde men in de jaren tachtig, door de steeds betere waarnemingen van de structuur van het universum op de grootste schaal als ook die van de kosmische achtergrondstraling, dat er een hele boel ongeziene massa, genaamd donkere materie, moet bestaan in het universum. Dit omdat de zwaartekracht van al het zichtbare materie op zichzelf niet voldoende is om dergelijke structuurvorming te verklaren. De aard van deze donkere materie is tot op heden onbekend en hier wordt zeer actief onderzoek naar gedaan.

In 1998 werd vastgesteld dat het universum op dit moment versneld uitdijt. Dit was zeer verassend omdat zowel zichtbare als donkere materie door hun zwaartekracht uitdijing tegenwerken en men op basis daarvan een vertraagde uitdijing zou verwachten. Een versnelde expansie kan verklaard worden door de aanwezigheid van wat men donkere energie noemt, die de bijzondere eigenschap heeft uitdijing in de hand te spelen. Ook van deze donkere energie is de preciese aard tot op heden onbekend. Deze inzichten hebben uiteindelijk geleid tot het succesvolle standaardmodel der kosmologie, het ΛCDM-model, door deze componenten aan de oerknaltheorie toe te voegen.

Alhoewel dit model erg succesvol is, kan het nog enkele dingen moeilijk te verklaren. Zoals genoemd poneert het het bestaan van donkere materie en donkere energie zonder daarbij inzicht te geven in hun ware aard. Daarnaast geeft het ook geen goede uitleg voor de specifieke eigenschappen van de kosmische achtergrondstraling. Deze is over het geheel genomen namelijk bijzonder gelijkmatig, maar heeft daarop zeer kleine doch specifieke variaties. Een populair idee dat dit laatste kan verklaren is dat het hele jonge universum een periode van exponentieel versnelde uitdijing heeft doorgemaakt, genaamd inflatie. Door deze enorm snelle uitdijing werden al bestaande macroscopische verschillen in het jonge universum als het ware gladgestreken wat de bijzondere mate van gelijkmatigheid van de achtergrondstraling verklaart. Tevens werden minuscule quantumfluctuaties opgeblazen tot macroscopische proporties die vervolgens uitgroeiden tot de kleine waargenomen variaties.

Een interessant idee als inflatie dient natuurlijk geconcretiseerd te worden door fysische modellen te construeren die kwantitatieve voorspellingen doen die vervolgens getest kunnen worden. De meest gangbare manier om inflatie te bewerkstelligen is door een zogenaamd scalair veld te introduceren die op een gunstige wijze wisselwerkt met het zwaartekrachtsveld. De simpelste optie is door een standaard minimale koppeling tussen scalair en zwaartekracht en een juist gekozen potentiaalfunctie voor het scalair veld aan te nemen. Men kan op dergelijk wijze al zeer succesvolle modellen verkrijgen die consistent zijn met onze waarnemingen tot nu toe. Echter, het is de moeite waard om algemenere theorieën te beschouwen waarbij de wisselwerking ingewikkelder is door ook niet-minimale koppelingen toe te staan. Door dit toe te staan modificeert men de werking van zwaartekracht met als bijkomend effect dat in potentie de effecten van donkere materie en donkere energie op deze wijze deels zouden kunnen verklaard.

Wanneer men standaardtheorieën aan wenst te passen dient men met een aantal zaken rekening te houden; willekeurige aanpassingen zullen in de regel verre van bruikbare modellen opleveren. Er zijn verscheidene eisen die men op diverse gronden zou willen stellen aan een potentieel interessante theorie, die niet zondermeer vooraf gegarandeerd zijn. Eén van die eisen is een zeker stabiliteitscriterium. Het wil namelijk zo zijn dat zeer grote klassen van modellen lijden aan instabiliteiten door de aanwezigheid van hogere afgeleiden. Deze hogere afgeleiden leiden generiek tot zogenaamde spookdeeltjes in de theorie, genaamd Ostrogradski spoken. Waar normale deeltjes een positieve energie hebben, hebben deze spookdeeltjes een negatieve energie die tot desastreuze gevolgen leiden voor de theorie. In zo'n theorie zal door de

wisselwerking tussen deeltjes van positieve en negatieve energieën uit het niets allerlei deeltjes gecreeërd worden en zal het universum vrijwel instantaan vervallen naar een hete brij van deeltjes. Dit is iets wat, gelukkig, niet daadwerkelijk gebeurt en daarmee kunnen dergelijke modellen onmogelijk een beschrijving geven van het universum en zijn ze in dat opzicht niet relevant.

Langere tijd werden alleen hogere orde theorieën beschouwd die leiden tot tweede orde bewegingsvergelijkingen, wat over het algemeen een voldoende eis is om vrij te zijn van spookdeeltjes. De algemene relativiteitstheorie is hier een voorbeeld van, alsook de standaard minimale koppeling daarvan met een scalair veld. De meest algemene theorie met een scalair die tot tweede orde bewegingsvergelijkingen leidt, was al in 1974 geconstrueerd door Horndeski, en herondekt in 2009 met het oog op toepassingen in de kosmologie. Gedurende vijf jaar werd deze als meest algemene spookvrije theorie beschouwd, tot men realiseerde dat je nog algemenere theorieën kunt bekijken, waarbij de bewegingsvergelijkingen hoger zijn van orde die toch leiden tot gezonde fysica. Wat hierbij een cruciale rol speelt is degenerativiteit: als een hogere orde theorie niet degeneratief is is hij onvermijdelijk onstabiel, maar anders is er de mogelijkheid dat deze wel stabiel is. Om de klasse van potentieel interessante theorieën te verkennen dient men dus naar degeneratieve theorieën te kijken. Echter dergelijke modellen zijn doorgaans lastig te analyseren en alhoewel vele voorbeelden zijn geconstrueerd die verder gaan dan de theorie van Horndeski en het duidelijk is dat een koppeling tussen een gezonde en een ongezonde sector cruciaal is, ontbreekt een solide begrip van de onderliggende structuren en is tot op heden onbekend wat de meest algemene theorie is.

Het onderzoek

De eerste helft van ons onderzoek is toegespitst op het identificeren van de onderliggende structuren van potentieel interessante hogere orde theorieën. We hebben daartoe een uitgebreide analyse gedaan van de specifieke vorm van degenerativiteit die nodig is om de theorie vrij te waren van de Ostrogradski spookdeeltjes, aannemende dat er geen ijksymmetrieën aanwezig zijn. We hebben hierbij gebruik gemaakt van twee algoritmes zowel vanuit het oogpunt van de Lagrangiaan als die van de Hamiltoniaan. We hebben algemene condities afgeleid waaraan een tweede orde Lagrangiaan moet voldoen wil deze vrij zijn van spookdeeltjes. Aan de hand hiervan hebben we een klassificatie weten te geven van gezonde theorieen met tweede orde afgeleiden. We hebben drie verschillende klassen geïdentificeerd (genaamd I, II en III), die oplopend zijn in complexiteit aangaande de degenerativiteit. Lagrangianen in klasse I zijn linear in tweede orde tijdsafgeleiden en is er geen wezenlijke wisselwerking tussen gezonde en ongezonde sectoren. Daarnaast leiden ze tot bewegingsvergelijkingen die tweede orde zijn in de tijd. In klasse II en III komen de afgeleiden niet-linear voor, is de wisselwerking tussen gezonde en ongezonde sectoren cruciaal en is die in zekere zin linear en niet-linear respectievelijk. Deze klassificatie kan ons helpen bij het construeren en identificeren van interessante nog niet eerder beschouwde theorieën.

Vervolgens hebben we gekeken in hoeverre gezonde tweede-orde theorieën daadwerkelijk essentieel anders zijn dan die van de standaard eerste-orde. Men dient namelijk rekening te houden met de mogelijkheid dat twee theorieën die op het eerste gezicht niets met elkaar te maken lijken te hebben, in werkelijkheid dezelfde fysica zouden kunnen beschrijven. Het is dus van belang om te karakteriseren welke klassen van theorieën werkelijk verder gaan dan de al eerder beschouwde modellen, en welke eigenlijk niets anders zijn dan een herformulatie. Afgezien van de praktische kant is dit ook met name theoretisch interessant om een goed begrip te krijgen van de abstractere eigenschappen van gezonde hogere order theorieen. Wat is gebleken is dat alle theorieen in klasse II middels herdefinities terug te brengen zijn tot klasse I. Daarnaast is een subklasse van klasse III ook om te schrijven naar klasse I, ditmaal middels zogenaamde contacttransformaties en veralgemeniseringen daarvan. Men dient hierbij wel rekening te houden met het feit dat de transformaties een gegeven symmetrie niet noodzakelijkerwijs respecteren. Dat wil zeggen: het feit dat een theorie uit klass II een bepaalde manifeste symmetrie heeft, wil niet zeggen dat die symmetrie ook manifest is in de omgeschreven theorie uit klasse I. In dat opzicht gaat zo'n theorie uit klasse II of III wel degelijk verder dan de standaard eerste-orde ansatz en is deze potentieel interessant.

Er zijn twee duidelijke lijnen van verder onderzoek. Allereerst is er de mogelijkheid om ijksymmetrie toe te laten wat in veel fysische modellen van essentieel belang is. In dergelijke gevallen kunnen ander typen van degenerativiteit ook leiden tot vrijwaring van spookdeeltjes en het is interessant te onderzoeken op welke wijze dit precies kan. Daarnaast ligt het voor de hand om ook hoger-dan-tweede-orde afgeleiden toe te staan in de theorie. De verwachting structuren naar voren komen die zeer vergelijkbaar zijn met het tweede-orde scenario. Losstaand van de oorspronkelijke eerdergenoemde motivatie om dergelijk theorieën te onderzoeken, is het vraagstuk ook opzichzelfstaand interessant. Dat wil zeggen: de vraag hoe ver kan men gaan met het toevoegen van hogere afgeleiden zonder daarmee de ongewilde spookdeeltjes te introduceren, is ook zonder directe toepassing van belang. Inderdaad, een (gedeeltelijk) antwoord op deze vraag zou ook in andere gebieden van de natuurkunde zijn nut kunnen bewijzen.

Het tweede gedeelte van het onderzoek heeft zich voornamelijk gericht op het onderzoeken van een ander aspect van veel interessante modellen. Deze hebben vaak zogenoemde niet-linear gerealiseerde symmetrieën die samenhangen met symmetriebreking, een belangrijk concept in de natuurkunde. Voor puur interne symmetrieën is er een (vrijwel) universele methode om dergelijke modellen te construeren, de zogenaamde coset constructie. Deze constructie kan ook worden toegepast op klassen van symmetrieën waarbij ook de ruimte-tijd een rol speelt. In dat scenario is de universaliteit nooit aangetoond en is het onduidelijk of dat zo is. Wij hebben verschillende fundamentele aspecten van dit soort theorieën met niet-linear gerealiseerde ruimte-tijd symmetrieën onderzocht. We hebben met name gekeken naar het zogenoemde inverse Higgs-effect en diens belangrijke rol met betrekking tot het universaliteitsvraagstuk. We hebben gezien dat in bepaalde situaties hierdoor de mogelijkheid bestaat van het bestaan inequivalente manieren om theorieën te construeren. Echter, uitsluitsel kan vooralsnog niet gegeven worden en het blijft een interessant vraagstuk om aan te

werken.

Tenslotte hebben we geïnspireerd door onze bevindingen en berekeningen met betrekking tot de niet-lineare realisaties, een klasse van inflatiemodellen geconstrueerd. Deze modellen hebben een interessante hoger-dimensionele interpretatie en niet-lineaire symmetrieën die in grote mate hun eigenschappen vastleggen. De klasse geeft zeer robuuste voorspellingen die ruim binnen de nu bestaande waargenomen waarden vallen. Daarnaast voorspellen ze generiek tensorfluctuaties van een dusdanige grootte dat de modellen in de nabije toekomst getest kunnen worden middels de nieuwe generatie satelietexperimenten die gepland staan.

Acknowledgements

First of all, I would like to stress that the research for this PhD thesis was made possible by a generous private donation to the Dutch funding agency the Foundation for Fundamental Research on Matter (FOM), part of the Netherlands Organisation for Scientific Research (NWO), and in this light I would like to express my sincere gratitude.

Secondly, I would like to thank my first and daily supervisor, Diederik Roest, and my second supervisor, Eric Bergshoeff. **Diederik**, we have known each other for the larger part of a decade now, and you have been my supervisor for almost as long; first during my days as a Bachelor and Master student and more recently during my period as a PhD student. We have quite different styles and personalities, but I'd say our track record goes to show that we have both enjoyed working together. In fact, I think your unwavering enthousiasm and optimism and strong intuition coupled to my slighty more temperate personality and affinity for mathematical rigor, has turned out to be quite fruitful. Anyway, thank you for your supervision and accompanying advice and constructive critisisms throughout the years; I've learned a lot! **Eric**, we haven't had much contact in a student/supervisor setting, but nevertheless thank you for taking on the role of second supervisor (with its corresponding duties).

Next, let me thank the reading committee, consisting of **Holger Waalkens**, **Jan-Willem van Holten** and **David Langlois**, for carefully going through the manuscript and deeming it worthy of a defense.

My thanks also go to my collaborators for their contributions to the work portrayed in this thesis: **Mehmet Ozkan**, **Marco Crisostomi**, **David Stefanyszyn** and **Emanuel Malek**. In particular David; I think we had a good run the last year and a half of my PhD period.

I also want to thank my paranymphs **Anne Tuijp** and **Jan-Willem Bryan**. Much appreciated!

Of course I would also like to thank all the members of the Van Swinderen Institute, being the senior scientific staff, the secretaries and of course all the (PhD) students and postdocs who have been or are still working at the institute, for all their help, company and shared hardships. I won't list all your names here, but you know who you are. Cheers!

Let me conclude by thanking all my friends and family for their interest and support!

Appendix A

Constraint analysis of higher derivative theories

A.1 Lagrangian analysis

In this Appendix we perform the Lagrangian constraint analysis [154, 162] for the general Lagrangian (4.1):

$$L(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)\,, \tag{A.1}$$

and derive the conditions that, within our assumptions, are necessary and sufficient for the existence of the right amount of primary and secondary Lagrangian constraints to ensure that the Ostrogradsky degrees of freedom are eliminated.

A.1.1 Non-degenerate Lagrangians

First we put the theory in a first order form. This can be done in several equivalent ways, and we opt for the following:

$$\begin{aligned} L(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha) \approx L(&\dot{A}_m, \partial_i A_m, A_m, \partial_i\partial_j\phi_m, \partial_i\phi_m, \phi_m, \dot{q}_\alpha, \partial_i q_\alpha) \\ &+ \lambda^m(\dot{\phi}_m - A_m)\,. \end{aligned} \tag{A.2}$$

Now we can proceed with the constraint algorithm, starting off with determining the equations of motion

$$E_{A_m} \equiv L_{\dot{A}_m\dot{A}_n}\ddot{A}_n + L_{\dot{A}_m\dot{q}_\beta}\ddot{q}_\beta + L_{\dot{A}_m\psi}\dot{\psi} + L_{\partial_i A_m\chi}\partial_i\chi - L_{A_m} + \lambda^m\,, \tag{A.3}$$

$$E_{q_\alpha} \equiv L_{\dot{q}_\alpha\dot{A}_n}\ddot{A}_n + L_{\dot{q}_\alpha\dot{q}_\beta}\ddot{q}_\beta + L_{\dot{q}_\alpha\psi}\dot{\psi} + L_{\partial_i q_\alpha\chi}\partial_i\chi - L_{q_\alpha}\,, \tag{A.4}$$

$$E_{\phi_m} \equiv -\partial_i\partial_j L_{\partial_i\partial_j\phi_m} + \partial_i L_{\partial_i\phi_m} - L_{\phi_m} + \dot{\lambda}^m\,, \tag{A.5}$$

$$E_{\lambda^m} \equiv -(\dot{\phi}_m - A_m)\,. \tag{A.6}$$

Here we introduced the short hand notation:

$$\chi \equiv \{\dot{A}_m, \partial_i A_m, A_m, \partial_i\partial_j\phi_m, \partial_i\phi_m, \phi_m\, \dot{q}_\alpha, \partial_i q_\alpha\}\,,$$

and $\psi \equiv \chi\backslash\{\dot{A}_m, \dot{q}_\alpha\}$. If the Lagrangian is non-degenerate the only constraint equations are

$$C_{\phi_m} \equiv E_{\phi_m}\,, \tag{A.7}$$

$$C_{\lambda^m} \equiv E_{\lambda^m}\,. \tag{A.8}$$

Time evolving them yields

$$\frac{d}{dt}C_{\phi_m} = \ddot{\lambda}^m + (C_{\phi_m})_{\dot{A}_m}\ddot{A}_m + (C_{\phi_m})_{\dot{q}_\alpha}\ddot{q}_\alpha + \ldots\,, \tag{A.9}$$

$$\frac{d}{dt}C_{\lambda^m} = -\ddot{\phi}_m + \ldots\,. \tag{A.10}$$

Here we only included the terms that contain purely second order time derivatives, because it is already clear from these (specifically the $\ddot{\lambda}^m$ term) that no secondary constraint equations can be formed. Therefore the algorithm terminates and one concludes that in total $2M$ constraints are present, which are purely due to the redundant first order description. The theory thus propagates $3M + A - \frac{1}{2}(2M) = 2M + A$ degrees of freedom (of which M are ghosts) as a non-degenerate higher derivative theory should.

A.1.2 Degenerate Lagrangians

Turning to the degenerate case, we see that in order to have M additional primary constraints we must demand that

$$L_{\dot{A}_m\dot{A}_n} - L_{\dot{A}_m\dot{q}_\alpha}L^{-1}_{\dot{q}_\alpha\dot{q}_\beta}L_{\dot{q}_\beta\dot{A}_n} = 0\,, \tag{A.11}$$

which is equivalent to the existence of M null vectors, $v^A_m = (\delta^n_m, V^\alpha_m)$, of the Hessian of L w.r.t. $\dot{A}_m$ and $\dot{q}_\alpha$. Specifically we have

$$V^\alpha_m = -L_{\dot{A}_m\dot{q}_\beta}L^{-1}_{\dot{q}_\beta\dot{q}_\alpha}\,. \tag{A.12}$$

In terms of the original variables only, i.e. using the identification $A_m = \dot{\phi}_m$, (A.11) reduces to the primary conditions (4.9). The M additional primary constraints are then given by:

$$\begin{aligned} C_m &\equiv E_{A_m} + V^\alpha_m E_{q_\alpha} \\ &= (L_{\dot{A}_m\psi} + V^\alpha_m L_{\dot{q}_\alpha\psi})\dot{\psi} + (\partial_i L_{\partial_i A_m} + V^\alpha_m\partial_i L_{\partial_i q_\alpha}) - (L_{A_m} + V^\alpha_m L_{q_\alpha})\,. \end{aligned} \tag{A.13}$$

Time evolving them yields

$$\begin{aligned} \frac{dC_m}{dt} &= (C_m)_{\dot{A}_n}\ddot{A}_n + (C_m)_{\partial_i\dot{A}_n}\partial_i\ddot{A}_n + (C_m)_{\dot{q}_\beta}\ddot{q}_\beta + (C_m)_{\partial_i\dot{q}_\beta}\partial_i\ddot{q}_\beta \\ &\quad + (C_m)_{\dot{\phi}_n}\ddot{\phi}_n + (C_m)_{\partial_i\dot{\phi}_n}\partial_i\ddot{\phi}_n + (C_m)_{\partial_i\partial_j\dot{\phi}_n}\partial_i\partial_j\ddot{\phi}_n + \ldots\,. \end{aligned} \tag{A.14}$$

Next we must demand that M secondary constraints exist in order to fully remove the ghost degrees of freedom. The most general such constraints will have the following form:

$$\begin{aligned} D_m = &\frac{d}{dt}C_m + U^\alpha_m E_{q_\alpha} + \alpha^{i\alpha}_m \partial_i E_{q_\alpha} \\ &+ (C_m)_{\dot\phi_n}\frac{d}{dt}C_{\lambda^n} + (C_m)_{\partial_i\dot\phi_n}\partial_i\frac{d}{dt}C_{\lambda^n} + (C_m)_{\partial_i\partial_j\dot\phi_n}\partial_i\partial_j\frac{d}{dt}C_{\lambda^n}\,. \end{aligned} \tag{A.15}$$

One can see this by first noting that no terms involving E_{A_m} or its spatial derivatives are present since, by virtue of the primary conditions, their relevant higher order derivative terms are not independent of those of E_{q_α} and its spatial derivatives. In addition, no higher order spatial derivatives of the equations of motion are present, as these will actually introduce even higher order problematic terms.

Now, depicting the relevant higher order terms in these combinations yields:

$$\begin{aligned} D_m = &\{(C_m)_{\dot A_n} + U^\alpha_m L_{\dot q_\alpha \dot A_n} + \alpha^{i\alpha}_m \partial_i L_{\dot q_\alpha \dot A_n}\}\ddot A_n \\ &+ \{(C_m)_{\dot q_\beta} + U^\alpha_m L_{\dot q_\alpha \dot q_\beta} + \alpha^{i\alpha}_m \partial_i L_{\dot q_\alpha \dot q_\beta}\}\ddot q_\beta \\ &+ \{(C_m)_{\partial_i \dot A_n} + \alpha^{i\alpha}_m L_{\dot q_\alpha \dot A_n}\}\partial_i \ddot A_m + \{(C_m)_{\partial_i \dot q_\beta} + \alpha^{i\alpha}_m L_{\dot q_\alpha \dot q_\beta}\}\partial_i \ddot q_\alpha + \ldots\,. \end{aligned} \tag{A.16}$$

From this one can see that U^α_m and $\alpha^{i\alpha}_m$ exist such that all these terms vanish, if and only if the following conditions are met

$$(C_m)_{\dot A_n} + (C_m)_{\dot q_\alpha}V^\alpha_n - (C_m)_{\partial_i \dot q_\beta} L^{-1}_{\dot q_\beta \dot q_\alpha}(\partial_i L_{\dot q_\alpha \dot A_n} + \partial_i L_{\dot q_\alpha \dot q_\rho}V^\rho_n) = 0\,, \tag{A.17}$$

$$(C_m)_{\partial_i \dot A_n} + (C_m)_{\partial_i \dot q_\beta}V^\beta_n = 0\,. \tag{A.18}$$

Using explicit expressions we obtain

$$\begin{aligned} 0 = &(\partial_i L_{\partial_i A_m \dot A_n} + V^\alpha_m \partial_i L_{\partial_i q_\alpha \dot A_n} + \partial_i L_{\partial_i A_m \dot q_\beta}V^\beta_n + V^\alpha_m \partial_i L_{\partial_i q_\alpha \dot q_\beta}V^\beta_n) \\ &+ \partial_i V^\beta_n (L_{\partial_i A_m \dot q_\beta} + L_{\dot A_m \partial_i q_\beta} + 2V^\alpha_m L_{\dot q_{(\alpha}\partial_i q_{\beta)}}) \\ &+ (L_{\dot A_m A_n} - L_{A_m \dot A_n}) + V^\alpha_m (L_{\dot q_\alpha A_n} - L_{q_\alpha \dot A_n}) \\ &+ (L_{\dot A_m q_\beta} - L_{A_m \dot q_\beta})V^\beta_n + V^\alpha_m (L_{\dot q_\alpha q_\beta} - L_{q_\alpha \dot q_\beta})V^\beta_n\,, \end{aligned} \tag{A.19}$$

$$0 = 2L_{\dot A_{(m}\partial_i A_{n)}} + 2V^\alpha_{(m}(L_{\dot q_\alpha \partial_i A_{n)}} + L_{\partial_i q_\alpha \dot A_{n)}}) + 2V^\alpha_m L_{\dot q_{(\alpha}\partial_i q_{\beta)}}V^\beta_n\,, \tag{A.20}$$

and

$$U^\alpha_m = ((C_m)_{\dot q_\beta} - \alpha^{i\rho}_m \partial_i L_{\dot q_\rho \dot q_\beta})L^{-1}_{\dot q_\beta \dot q_\alpha}\,, \tag{A.21}$$

$$\alpha^{i\alpha}_m = -(L_{\partial_i A_m \dot q_\beta} + L_{\dot A_m \partial_i q_\beta} + 2V^\rho_m L_{\dot q_{(\rho}\partial_i q_{\beta)}})L^{-1}_{\dot q_\beta \dot q_\alpha}\,. \tag{A.22}$$

Therefore we conclude that if and only if the primary conditions (A.11) hold, M additional ($3M$ in total) primary constraint equations are present. Moreover, if and only if in addition the secondary conditions (A.19) and (A.20) are satisfied, M secondary constraint equations exist. Assuming that no further conditions are imposed, no tertiary constraint equations will be present and the theory then propagates

$3M + A - \frac{1}{2}(3M + M) = M + A$ degrees of freedom and the M Ostrogradsky ghosts are not present.

Note that the symmetric part of (A.19) is in fact the spatial derivative of (A.20). Hence one ends up with one symmetric and one antisymmetric set of conditions, which, when written in terms of the original variables, precisely yield the symmetric (4.10) and antisymmetric (4.11) secondary conditions.

A.2 Hamiltonian analysis of higher derivative theories

A.2.1 Non-degenerate Lagrangians

In this Appendix we perform the canonical analysis, using the Dirac method for constrained systems [63], of the general Lagrangian (4.1)

$$L(\phi_m, \partial_\mu \phi_m, \partial_\mu \partial_\nu \phi_m, q_\alpha, \partial_\mu q_\alpha) \approx L(\phi_m, A^m_\mu, \partial_\nu A^m_\mu, q_\alpha, \partial_\mu q_\alpha) + \lambda^\mu_m (\partial_\mu \phi_m - A^m_\mu)\,. \tag{A.23}$$

Using the relations imposed by the Lagrangian multipliers λ^μ_m, we have that $\partial_\mu A^m_\nu = \partial_\nu A^m_\mu$ and we can replace $\dot{A}^m_i = \partial_i A^m_0$. To be precise, these relations hold only on-shell, i.e. on the phase space of constraints, however since they are second class constraints, they can be consistently imposed during the analysis.

Separating the space and time components, the Lagrangian (A.23) becomes

$$L = L(\phi_m, A^m_0, A^m_i, \dot{A}^m_0, \partial_i A^m_0, \partial_i A^m_j, q_\alpha, \dot{q}_\alpha, \partial_i q_\alpha) + \lambda^0_m(\dot{\phi}_m - A^m_0) + \lambda^i_m(\partial_i \phi_m - A^m_i)\,. \tag{A.24}$$

The momenta conjugated to the fields and the primary constraints associated to the Lagrangian (A.24) are

- $\pi_m \equiv \frac{\partial L}{\partial \dot{\phi}_m} = \lambda^0_m \quad \Rightarrow \quad (\pi_m - \lambda^0_m) \approx 0 \quad$ M primary constraints
- $\Lambda^0_m \equiv \frac{\partial L}{\partial \dot{\lambda}^0_m} = 0 \quad \Rightarrow \quad \Lambda^0_m \approx 0 \quad$ M primary constraints
- $\Lambda^i_m \equiv \frac{\partial L}{\partial \dot{\lambda}^i_m} = 0 \quad \Rightarrow \quad \Lambda^i_m \approx 0 \quad$ $M \cdot i$ primary constraints
- $P^m_i \equiv \frac{\partial L}{\partial \dot{A}^m_i} = 0 \quad \Rightarrow \quad P^m_i \approx 0 \quad$ $M \cdot i$ primary constraints
- $P^m_0 \equiv \frac{\partial L}{\partial \dot{A}^m_0} \quad \Rightarrow \quad \dot{A}^m_0 = f^m(P^n_0, \phi_n, A^n_0, A^n_i, \partial_i A^n_0, \partial_i A^n_j, q_\alpha, \partial_i q_\alpha, p_\alpha)$
- $p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha} \quad \Rightarrow \quad \dot{q}_\alpha = g_\alpha(P^n_0, \phi_n, A^n_0, A^n_i, \partial_i A^n_0, \partial_i A^n_j, q_\beta, \partial_i q_\beta, p_\beta)$

where i refers to the number of spatial dimensions. In the last two lines we have not assumed any extra degeneracy for the moment. The total Hamiltonian is the sum of the canonical Hamiltonian plus the primary constraints enforced through multipliers

$$H_T = H_C + \int d^3x \left[a_m(\pi_m - \lambda^0_m) + b^m_0 \Lambda^0_m + b^m_i \Lambda^i_m + c^m_i P^m_i \right] , \tag{A.25}$$

where $H_C = \int d^3x \, \mathcal{H}_C$ and

$$\begin{aligned}\mathcal{H}_C = {} & P^m_0 f^m + p_\alpha g_\alpha - L(\phi_n, A^n_0, A^n_i, \partial_i A^n_0, \partial_i A^n_j, q_\beta, \partial_i q_\beta, f^n, g_\beta) \\ & + \lambda^0_m A^m_0 - \lambda^i_m(\partial_i \phi_m - A^m_i) \, .\end{aligned} \tag{A.26}$$

Here, a_m, b^m_0, b^m_i, c^m_i are the multipliers used to enforce the primary constraints.

Evolving the primary constraints we get

- $\left\{\Lambda^i_m, H_T\right\} = \partial_i \phi_m - A^m_i \approx 0$ $\qquad M \cdot i$ secondary constraints
- $\left\{P^i_m, H_T\right\} = \frac{\partial L}{\partial A^m_i} - P^n_0 \frac{\partial f^n}{\partial A^m_i} - \lambda^i_m \approx 0$ $\qquad M \cdot i$ secondary constraints
- $\left\{\Lambda^0_m, H_T\right\} = a_m - A^m_0 \approx 0 \qquad \Rightarrow \qquad a_m = A^m_0$
- $\left\{\pi_m - \lambda^0_m, H_T\right\} \approx 0 \qquad \Rightarrow \qquad b^m_0 = \frac{\partial L}{\partial \phi_m} - P^n_0 \frac{\partial f^n}{\partial \phi_m} - \partial_i \lambda^i_m$

The evolution of Λ^i_m and P^i_m gives $2\,M \cdot i$ secondary constraints, instead from the evolution of Λ^0_m and $(\pi_m - \lambda^0_m)$ we can solve for two (out of four) set of multipliers, namely a_m and b^m_0.

Finally we need to evolve the secondary constraints

- $\{\partial_i \phi_m - A^m_i, H_T\} \approx 0 \qquad \Rightarrow \qquad c^m_i = \{\partial_i \phi_m, H_T\}$
- $\left\{\frac{\partial L}{\partial A^m_i} - P^n_0 \frac{\partial f^n}{\partial A^m_i} - \lambda^i_m, H_T\right\} \approx 0 \qquad \Rightarrow \qquad b^m_i = \left\{\frac{\partial L}{\partial A^m_i} - P^n_0 \frac{\partial f^n}{\partial A^m_i}, H_T\right\}$

All the multipliers are now completely determined and the procedure stops. It is easy to verify that all these constraints are second class, indeed they are simply associated with the redundancy of description we have used to reduce the order of the Lagrangian. We started with $2(3M + 2M \cdot i + A)$ canonical variables and we found $2(M + M \cdot i)$ constraints, therefore we are left with $2(2M + A)$ canonical dof, or $2M + A$ physical dof. As it is well known, M of these dof are due to the higher derivative terms in the Lagrangian (A.23) and usually are associated with instabilities.

The safest of the solutions is to require that none of them actually propagate, demanding the existence of M extra primary constraints in the (A^m_0, P^m_0) sector. Since we are not considering here gauge invariant theories, we will also need to demand that these primary constraints generate M secondary ones.

A.2.2 Degenerate Lagrangians

As we have seen, the fields A_i^m and λ_m^i don't play any significant rule so can be ignored in the rest of the analysis. Also, to simplify the notation, from now on we drop the suffix "zero" from A_0 and P_0.

Requiring the existence of extra M primary constraints means that the system of momenta $P^m = \partial L/\partial \dot{A}^m$ cannot be inverted anymore and solved in terms of the velocities $\dot{A}^m$. The constraints therefore take the form

$$\chi^m \equiv P^m - F^m(A^n, \partial_i A^n, q_\alpha, \partial_i q_\alpha, p_\alpha) \approx 0\,, \tag{A.27}$$

and need to be added to the total Hamiltonian as

$$H_T = H_C + \int d^3x\, \xi_m \chi^m\,, \tag{A.28}$$

where ξ_m are the usual multipliers and we have omitted the other primary constraints already analysed in the former section as they do not interact with the new ones.

It can be shown [129] that the existence of the constraints (A.27) is in one-to-one correspondence with the degeneracy of the Hessian matrix of the Lagrangian with respect to the velocities $\dot{A}^m$ and $\dot{q}_\alpha$, i.e. conditions (A.11). Therefore, in order to have the primary constraints (A.27), our Lagrangian has to satisfy the conditions (A.11).

The evolution of the constraints (A.27) gives

$$\{\chi^m(x), H_T\} = \{\chi^m(x), H_C\} + \left\{\chi^m(x), \int d^3y\, \xi_n(y)\chi^n(y)\right\}\,, \tag{A.29}$$

and the last term is composed by the following parts

$$\begin{aligned}\left\{P^m(x), \int d^3y\, \xi_n(y) F^n(y)\right\} &= \left(-\frac{\partial F^n}{\partial A^m} + \partial_i \frac{\partial F^n}{\partial(\partial_i A^m)}\right)\xi_n \\ &+ \frac{\partial F^n}{\partial(\partial_i A^m)}\partial_i \xi_n\,,\end{aligned} \tag{A.30}$$

$$\left\{F^m(x), \int d^3y\, \xi_n(y) P^n(y)\right\} = \frac{\partial F^m}{\partial A^n}\xi_n + \frac{\partial F^m}{\partial(\partial_i A^n)}\partial_i \xi_n\,, \tag{A.31}$$

$$\begin{aligned}\left\{F^m(x), \int d^3y\, \xi_n(y) F^n(y)\right\} &= \left(\frac{\partial F^m}{\partial q_\alpha}\frac{\partial F^n}{\partial p_\alpha} - \frac{\partial F^m}{\partial p_\alpha}\frac{\partial F^n}{\partial q_\alpha}\right. \\ &+ \left.\frac{\partial F^m}{\partial(\partial_i q_\alpha)}\partial_i \frac{\partial F^n}{\partial p_\alpha} + \frac{\partial F^m}{\partial p_\alpha}\partial_i \frac{\partial F^n}{\partial(\partial_i q_\alpha)}\right)\xi_n \\ &+ \left(\frac{\partial F^m}{\partial(\partial_i q_\alpha)}\frac{\partial F^n}{\partial p_\alpha} + \frac{\partial F^m}{\partial p_\alpha}\frac{\partial F^n}{\partial(\partial_i q_\alpha)}\right)\partial_i \xi_n\end{aligned} \tag{A.32}$$

The Poisson brackets (A.29) have therefore the form

$$\{\chi^m(x), H_T\} = \{\chi^m(x), H_C\} + S^{mn}\xi_n + (S_i)^{mn}\partial_i\xi_n \,, \tag{A.33}$$

and in order to give secondary constraints we need to remove their dependency from ξ_n. This gives the new conditions $S^{mn} = (S_i)^{mn} = 0$, whose specific form is easily obtainable from equations (A.30) – (A.32).

Using the primary constraints (A.27), it is possible to relate the derivatives of F^m to those of the Lagrangian, namely

$$\begin{aligned}
\frac{\partial F^m}{\partial p_\alpha} &= -V^m_\alpha \,, \qquad \frac{\partial F^m}{\partial q_\alpha} = L_{\dot{A}_m q_\alpha} + L_{\dot{q}_\beta q_\alpha} V^m_\beta \,, \\
\frac{\partial F^m}{\partial(\partial_i q_\alpha)} &= L_{\dot{A}_m \partial_i q_\alpha} + L_{\dot{q}_\beta \partial_i q_\alpha} V^m_\beta \,, \quad \frac{\partial F^m}{\partial A^n} = L_{\dot{A}_m A_n} + L_{\dot{q}_\alpha A_n} V^m_\alpha \,, \\
\frac{\partial F^m}{\partial(\partial_i A^n)} &= L_{\dot{A}_m \partial_i A_n} + L_{\dot{q}_\alpha \partial_i A_n} V^m_\alpha \,.
\end{aligned} \tag{A.34}$$

Finally, substituting these relations in the above conditions, we get exactly equations (A.19) and (A.20).

Appendix B

Lorentz invariant field redefinitions

In this Appendix we prove the following statement: a manifestly Lorentz invariant theory $L_{II}(\partial_\mu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu q_\alpha, q_\alpha)$, belonging to Class II, can be put in a manifestly Lorentz invariant form $\bar{L}_I(\partial_\nu\partial_\nu\phi_m, \partial_\mu\phi_m, \phi_m, \partial_\mu\bar{q}_\alpha, \bar{q}_\alpha)$ (with $\bar{q}_\alpha = \bar{q}_\alpha(q, \phi, \partial\phi)$ being Lorentz scalars), if and only if $W_m^{\mu\alpha} \equiv (V_m^\alpha, \alpha_m^{i\alpha})$ is a Lorentz vector and

$$\frac{\partial W_n^{\mu\beta}}{\partial\partial_\nu\phi_m} - \frac{\partial W_m^{\nu\beta}}{\partial\partial_\mu\phi_n} + W_m^{\nu\alpha}\frac{\partial W_n^{\mu\beta}}{\partial q_\alpha} - W_n^{\mu\alpha}\frac{\partial W_m^{\nu\beta}}{\partial q_\alpha} = 0\,. \tag{B.1}$$

Let us start with necessity. Assume that both L_{II} and $\bar{L}_I$ are manifestly Lorentz invariant and related via a field redefinition of the specified form. Since $\bar{L}_I$ is Lorentz invariant, not only $\bar{V}_m^\alpha = 0$ but also $\bar{\alpha}_m^{i\alpha} = 0$ (as noted in Section 4.2.1). Then, by calculating $W_m^{\mu\alpha}$ one finds

$$\frac{\partial\bar{q}_\alpha}{\partial\partial_\mu\phi_m} + W_m^{\mu\beta}\frac{\partial\bar{q}_\alpha}{\partial q_\beta} = 0\,. \tag{B.2}$$

Therefore, since $\bar{q}_\alpha$ is Lorentz invariant, we conclude that $W_m^{\mu\alpha} = -\left(\frac{\partial\bar{q}_\beta}{\partial\partial_\mu\phi_m}\right)\left(\frac{\partial\bar{q}_\beta}{\partial q_\alpha}\right)^{-1}$ is a Lorentz vector. Lastly, one notes that the consistency conditions corresponding to (B.2) are precisely (B.1), which are thus automatically satisfied.

Now, for sufficiency we first note that since $W_m^{\mu\beta}$ is a Lorentz vector and $V_m^\beta = V_m^\beta(q_\alpha, \phi_n, \partial_\mu\phi_n)$, the most general form is given by

$$W_m^{\mu\beta}(q_\alpha, \phi_p, \partial_\mu\phi_p) = A_m^{n\beta}\partial^\mu\phi_n, \qquad A_m^{n\beta} = A_m^{n\beta}(q_\alpha, \phi_p, X_{p,q}), \qquad X_{p,q} \equiv \frac{1}{2}\partial_\mu\phi_p\partial^\mu\phi_q\,. \tag{B.3}$$

Plugging this specific expression into (B.1) it follows that

$$\left(A_n^{m\beta} - A_m^{n\beta}\right)\eta^{\mu\nu} + \left(\frac{\partial A_n^{p\beta}}{\partial X_{mq}} - \frac{\partial A_m^{q\beta}}{\partial X_{np}} + A_m^{p\alpha}\frac{\partial A_n^{q\beta}}{\partial q_\alpha} - A_n^{q\alpha}\frac{\partial A_m^{p\beta}}{\partial q_\alpha}\right)\partial^\mu\phi_p\partial^\nu\phi_q = 0\,. \tag{B.4}$$

Since both terms in parenthesis are Lorentz invariant, one sees that $A_n^{m\beta} = A_m^{n\beta}$. Next we observe that because by assumption the consistency conditions (B.1) are satisfied, one can always find independent $\bar{q}_\alpha$ that satisfy (B.2). Picking precisely such a redefinition and calculating its variation under Lorentz transformations yields

$$\begin{aligned}
\delta\bar{q}_\alpha &= \frac{\partial\bar{q}_\alpha}{\partial\partial_\mu\phi_m}\delta\partial_\mu\phi_m \\
&= -W_m^{\mu\beta}\left(\delta\partial_\mu\phi_m\right)\frac{\partial\bar{q}_\beta}{\partial q_\alpha} \\
&= \left(A_m^{n\beta}\partial^\mu\phi_n\omega_{\mu\nu}\partial^\nu\phi_m\right)\frac{\partial\bar{q}_\beta}{\partial q_\alpha} \\
&= 0\,,
\end{aligned} \tag{B.5}$$

where we used the symmetry of $A_n^{m\beta}$. Thus, we conclude that $\bar{q}_\alpha$ is a Lorentz scalar and hence describes a manifestly Lorentz invariant field redefinition (and so is its inverse). Starting from a manifestly Lorentz invariant theory and performing this redefinition one obtains a Lagrangian belonging to Class I (since (B.2) implies that $\bar{W}_m^{\mu\alpha} = 0$) that is also manifestly Lorentz invariant.

Appendix C

Redefinitions in the $(\phi(t), q(t))$ case

Here we determine the most general relevant transformation in the case of mechanical systems with a single higher derivative variable and a single healthy variable. Let us consider a Lagrangian, $\bar{L}(\bar{\phi}, \bar{\phi}', \bar{\phi}'', \bar{q}, \bar{q}')$, belonging to any of the three degeneracy classes as discussed in Section 4.2. Upon performing a general local and invertible redefinition

$$\begin{aligned}
\bar{t} &= \bar{t}(t, \phi, \dot{\phi}, ..., \phi^{(n)}, q, \dot{q}, ..., q^{(m)})\,, \\
\bar{\phi}(\bar{t}) &= \bar{\phi}(t, \phi, \dot{\phi}, ..., \phi^{(p)}, q, \dot{q}, ..., q^{(q)})\,, \\
\bar{q}(\bar{t}) &= \bar{q}(t, \phi, \dot{\phi}, ..., \phi^{(r)}, q, \dot{q}, ..., q^{(s)})\,,
\end{aligned} \tag{C.1}$$

the Lagrangian transforms as

$$L = \frac{d\bar{t}}{dt}\bar{L}\,. \tag{C.2}$$

To stay within our ansatz one should only consider redefinitions for which

$$L = L(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})\,. \tag{C.3}$$

In order for this to be the case in general, i.e. modulo non-generic structures, we must demand that the same holds for $\frac{d\bar{t}}{dt}$, $\bar{\phi}$, $\bar{\phi}'$, $\bar{\phi}''$, $\bar{q}$ and $\bar{q}'$. Thus, first requiring that $\frac{d\bar{t}}{dt} = \frac{d\bar{t}}{dt}(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})$, yields

$$\bar{t} = at + f(\phi, \dot{\phi}, q)\,, \tag{C.4}$$

where f is arbitrary, and $a \neq 0$ is a constant. Next starting from

$$\bar{\phi} = \bar{\phi}(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})\,, \tag{C.5}$$

$$\bar{q} = \bar{q}(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})\,, \tag{C.6}$$

and demanding the same dependence for their first derivatives

$$\bar{\phi}' = \frac{d\bar{\phi}}{d\bar{t}} = (\frac{d\bar{t}}{dt})^{-1}(\bar{\phi}_{\ddot{\phi}}\dddot{\phi} + \bar{\phi}_{\dot{q}}\ddot{q} + ...)\,, \tag{C.7}$$

$$\bar{q}' = \frac{d\bar{q}}{d\bar{t}} = (\frac{d\bar{t}}{dt})^{-1}(\bar{q}_{\ddot{\phi}}\dddot{\phi} + \bar{q}_{\dot{q}}\ddot{q} + ...)\,, \tag{C.8}$$

yields $\bar{\phi}_{\ddot{\phi}} = \bar{\phi}_{\dot{q}} = \bar{q}_{\ddot{\phi}} = \bar{q}_{\dot{q}} = 0$. Thus in fact we find

$$\bar{\phi} = \bar{\phi}(\phi, \dot{\phi}, q)\,, \tag{C.9}$$

$$\bar{q} = \bar{q}(\phi, \dot{\phi}, q)\,. \tag{C.10}$$

Subsequently calculating the second derivative of $\bar{\phi}$ yields

$$\bar{\phi}'' = \frac{d^2\bar{\phi}}{d\bar{t}^2} = (\frac{d\bar{t}}{dt})^{-1}(\bar{\phi}'_{\ddot{\phi}}\dddot{\phi} + \bar{\phi}'_{\dot{q}}\ddot{q} + ...)\,, \tag{C.11}$$

from which we conclude that

$$0 = \bar{\phi}'_{\ddot{\phi}} \qquad \Rightarrow \qquad 0 = (a + \bar{t}_\phi\dot{\phi} + \bar{t}_{\dot{\phi}}\ddot{\phi} + \bar{t}_q\dot{q})\bar{\phi}_{\dot{\phi}} - (\bar{\phi}_\phi\dot{\phi} + \bar{\phi}_{\dot{\phi}}\ddot{\phi} + \bar{\phi}_q\dot{q})\bar{t}_{\dot{\phi}}\,, \tag{C.12}$$

$$0 = \bar{\phi}'_{\dot{q}} \qquad \Rightarrow \qquad 0 = (a + \bar{t}_\phi\dot{\phi} + \bar{t}_{\dot{\phi}}\ddot{\phi} + \bar{t}_q\dot{q})\bar{\phi}_q - (\bar{\phi}_\phi\dot{\phi} + \bar{\phi}_{\dot{\phi}}\ddot{\phi} + \bar{\phi}_q\dot{q})\bar{t}_q\,, \tag{C.13}$$

which can be rewritten as:

$$\begin{aligned} 0 &= \bar{t}_q\bar{\phi}_{\dot{\phi}} - \bar{\phi}_q\bar{t}_{\dot{\phi}}\,, \\ 0 &= (a + \bar{t}_\phi\dot{\phi})\bar{\phi}_{\dot{\phi}} - \bar{\phi}_\phi\dot{\phi}\bar{t}_{\dot{\phi}}\,, \\ 0 &= (a + \bar{t}_\phi\dot{\phi})\bar{\phi}_q - \bar{\phi}_\phi\dot{\phi}\bar{t}_q\,. \end{aligned} \tag{C.14}$$

Thus we conclude that the only redefinitions that satisfy our demands are of the form

$$\begin{aligned} &\bar{t} = at + f(\phi, \dot{\phi}, q)\,, \\ &\bar{\phi} = g(\phi, \dot{\phi}, q), \qquad \bar{\phi}' = G(\phi, \dot{\phi}, q), \qquad \bar{\phi}'' = \tilde{G}(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})\,, \\ &\bar{q} = h(\phi, \dot{\phi}, q), \qquad \bar{q}' = H(\phi, \dot{\phi}, \ddot{\phi}, q, \dot{q})\,, \end{aligned} \tag{C.15}$$

where f and g have to satisfy the differential equations (C.14) and G, $\tilde{G}$ and H follow from f, g and h. Of course one must also require invertibility of the transformation, which is precisely the case if one can solve $\bar{\phi}$, $\bar{\phi}'$ and $\bar{q}$ for ϕ, $\dot{\phi}$ and q.

Bibliography

[1] Ana Achúcarro, Renata Kallosh, Andrei Linde, Dong-Gang Wang, and Yvette Welling. Universality of multi-field α-attractors. *JCAP*, 1804(04):028, 2018.

[2] P. A. R. Ade et al. Improved Constraints on Cosmology and Foregrounds from BICEP2 and Keck Array Cosmic Microwave Background Data with Inclusion of 95 GHz Band. *Phys. Rev. Lett.*, 116:031302, 2016.

[3] P. A. R. Ade et al. Planck 2015 results. XX. Constraints on inflation. *Astron. Astrophys.*, 594:A20, 2016.

[4] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. 2018.

[5] Andreas Albrecht and Paul J. Steinhardt. Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking. *Phys. Rev. Lett.*, 48:1220–1223, 1982. [Adv. Ser. Astrophys. Cosmol.3,158(1987)].

[6] Mohsen Alishahiha, Eva Silverstein, and David Tong. DBI in the sky. *Phys. Rev.*, D70:123505, 2004.

[7] Erwan Allys. New terms for scalar multi-Galileon models and application to SO(N) and SU(N) group representations. *Phys. Rev.*, D95(6):064051, 2017.

[8] R.L. Anderson and N.H. Ibragimov. *Lie-Bäcklund transformations in applications.* Society for Industrial and Applied Mathematics, Philadelphia, 1979.

[9] Nima Arkani-Hamed, Lubos Motl, Alberto Nicolis, and Cumrun Vafa. The String landscape, black holes and gravity as the weakest force. *JHEP*, 06:060, 2007.

[10] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. The Dynamics of general relativity. *Gen. Rel. Grav.*, 40:1997–2027, 2008.

[11] Valentin Assassi, Daniel Baumann, and Daniel Green. On Soft Limits of Inflationary Correlation Functions. *JCAP*, 1211:047, 2012.

[12] Daniel Baumann and Liam McAllister. *Inflation and String Theory.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2015.

[13] S. Bellucci, E. Ivanov, and S. Krivonos. AdS / CFT equivalence transformation. *Phys. Rev.*, D66:086001, 2002. [Erratum: Phys. Rev.D67,049901(2003)].

[14] Jibril Ben Achour, Marco Crisostomi, Kazuya Koyama, David Langlois, Karim Noui, and Gianmassimo Tasinato. Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order. *JHEP*, 12:100, 2016.

[15] Jibril Ben Achour, David Langlois, and Karim Noui. Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations. *Phys. Rev.*, D93(12):124005, 2016.

[16] Marcus Berg, Enrico Pajer, and Stefan Sjors. Dante's Inferno. *Phys. Rev.*, D81:103535, 2010.

[17] Dario Bettoni and Stefano Liberati. Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action. *Phys. Rev.*, D88:084020, 2013.

[18] Mark P. Bogers and Tomáš Brauner. Lie-algebraic classification of effective theories with enhanced soft limits. *JHEP*, 05:076, 2018.

[19] Mark P. Bogers and Tomáš Brauner. The Geometry of Multi-Flavor Galileon-Like Theories. 2018.

[20] D. G. Boulware and Stanley Deser. Can gravitation have a finite range? *Phys. Rev.*, D6:3368–3382, 1972.

[21] Tomáš Brauner and Haruki Watanabe. Spontaneous breaking of spacetime symmetries and the inverse Higgs effect. *Phys. Rev.*, D89(8):085004, 2014.

[22] Rafael Bravo, Sander Mooij, Gonzalo A. Palma, and Bastián Pradenas. A generalized non-Gaussian consistency relation for single field inflation. *JCAP*, 1805(05):024, 2018.

[23] Jon Brown, William Cottrell, Gary Shiu, and Pablo Soler. Fencing in the Swampland: Quantum Gravity Constraints on Large Field Inflation. *JHEP*, 10:023, 2015.

[24] Benedict J. Broy, Mario Galante, Diederik Roest, and Alexander Westphal. Pole inflation — Shift symmetry and universal corrections. *JHEP*, 12:149, 2015.

[25] C. P. Burgess, M. Cicoli, S. de Alwis, and F. Quevedo. Robust Inflation from Fibrous Strings. *JCAP*, 1605(05):032, 2016.

[26] C. P. Burgess, M. Cicoli, F. Quevedo, and M. Williams. Inflating with Large Effective Fields. *JCAP*, 1411:045, 2014.

[27] Curtis G. Callan, Jr., Sidney R. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. 2. *Phys. Rev.*, 177:2247–2250, 1969.

[28] John Joseph M. Carrasco, Renata Kallosh, Andrei Linde, and Diederik Roest. Hyperbolic geometry of cosmological attractors. *Phys. Rev.*, D92(4):041301, 2015.

[29] R. Casalbuoni, S. De Curtis, D. Dominici, F. Feruglio, and Raoul Gatto. Nonlinear Realization of Supersymmetry Algebra From Supersymmetric Constraint. *Phys. Lett.*, B220:569–575, 1989.

[30] Javier Chagoya and Gianmassimo Tasinato. A geometrical approach to degenerate scalar-tensor theories. *JHEP*, 02:113, 2017.

[31] Ali H. Chamseddine and Viatcheslav Mukhanov. Mimetic Dark Matter. *JHEP*, 11:135, 2013.

[32] Athanasios Chatzistavrakidis, Fech Scen Khoo, Diederik Roest, and Peter Schupp. Tensor Galileons and Gravity. *JHEP*, 03:070, 2017.

[33] Clifford Cheung, Paolo Creminelli, A. Liam Fitzpatrick, Jared Kaplan, and Leonardo Senatore. The Effective Field Theory of Inflation. *JHEP*, 03:014, 2008.

[34] Clifford Cheung, Karol Kampf, Jiri Novotny, Chia-Hsien Shen, and Jaroslav Trnka. A Periodic Table of Effective Field Theories. *JHEP*, 02:020, 2017.

[35] Clifford Cheung, Karol Kampf, Jiri Novotny, and Jaroslav Trnka. Effective Field Theories from Soft Limits of Scattering Amplitudes. *Phys. Rev. Lett.*, 114(22):221602, 2015.

[36] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified Gravity and Cosmology. *Phys. Rept.*, 513:1–189, 2012.

[37] Sidney R. Coleman and J. Mandula. All Possible Symmetries of the S Matrix. *Phys. Rev.*, 159:1251–1256, 1967.

[38] Sidney R. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. 1. *Phys. Rev.*, 177:2239–2247, 1969.

[39] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo. Degrees of Freedom in Massive Gravity. *Phys. Rev.*, D86:101502, 2012.

[40] Denis Comelli, Fabrizio Nesti, and Luigi Pilo. Massive gravity: a General Analysis. *JHEP*, 07:161, 2013.

[41] Paolo Creminelli, Marco Serone, Gabriele Trevisan, and Enrico Trincherini. Inequivalence of Coset Constructions for Spacetime Symmetries. *JHEP*, 02:037, 2015.

[42] Paolo Creminelli, Marco Serone, and Enrico Trincherini. Non-linear Representations of the Conformal Group and Mapping of Galileons. *JHEP*, 10:040, 2013.

[43] Marco Crisostomi, Matthew Hull, Kazuya Koyama, and Gianmassimo Tasinato. Horndeski: beyond, or not beyond? *JCAP*, 1603(03):038, 2016.

[44] Marco Crisostomi, Kazuya Koyama, and Gianmassimo Tasinato. Extended Scalar-Tensor Theories of Gravity. *JCAP*, 1604(04):044, 2016.

[45] Marco Crisostomi, Karim Noui, Christos Charmousis, and David Langlois. Beyond Lovelock gravity: Higher derivative metric theories. *Phys. Rev.*, D97(4):044034, 2018.

[46] Csaba Csaki, Nemanja Kaloper, Javi Serra, and John Terning. Inflation from Broken Scale Invariance. *Phys. Rev. Lett.*, 113:161302, 2014.

[47] Guido D'Amico, Nemanja Kaloper, and Albion Lawrence. Monodromy inflation at strong coupling: 4π in the sky. 2017.

[48] Claudia de Rham, Matteo Fasiello, and Andrew J. Tolley. Galileon Duality. *Phys. Lett.*, B733:46–51, 2014.

[49] Claudia de Rham, Gregory Gabadadze, and Andrew J. Tolley. Resummation of Massive Gravity. *Phys. Rev. Lett.*, 106:231101, 2011.

[50] Claudia de Rham, Gregory Gabadadze, and Andrew J. Tolley. Ghost free Massive Gravity in the Stúckelberg language. *Phys. Lett.*, B711:190–195, 2012.

[51] Claudia De Rham, Luke Keltner, and Andrew J. Tolley. Generalized galileon duality. *Phys. Rev.*, D90(2):024050, 2014.

[52] Claudia de Rham and Andrew Matas. Ostrogradsky in Theories with Multiple Fields. *JCAP*, 1606(06):041, 2016.

[53] Claudia de Rham and Andrew J. Tolley. DBI and the Galileon reunited. *JCAP*, 1005:015, 2010.

[54] C. Deffayet, S. Deser, and G. Esposito-Farese. Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors. *Phys. Rev.*, D80:064015, 2009.

[55] C. Deffayet, S. Deser, and G. Esposito-Farese. Arbitrary p-form Galileons. *Phys. Rev.*, D82:061501, 2010.

[56] C. Deffayet, Gilles Esposito-Farese, and A. Vikman. Covariant Galileon. *Phys. Rev.*, D79:084003, 2009.

[57] C. Deffayet, Xian Gao, D. A. Steer, and G. Zahariade. From k-essence to generalised Galileons. *Phys. Rev.*, D84:064039, 2011.

[58] Cedric Deffayet, Gilles Esposito-Farese, and Daniele A. Steer. Counting the degrees of freedom of generalized Galileons. *Phys. Rev.*, D92:084013, 2015.

[59] Luca V. Delacrétaz, Solomon Endlich, Alexander Monin, Riccardo Penco, and Francesco Riva. (Re-)Inventing the Relativistic Wheel: Gravity, Cosets, and Spinning Objects. *JHEP*, 11:008, 2014.

[60] Luca V. Delacrétaz, Alberto Nicolis, Riccardo Penco, and Rachel A. Rosen. Wess-Zumino Terms for Relativistic Fluids, Superfluids, Solids, and Supersolids. *Phys. Rev. Lett.*, 114(9):091601, 2015.

[61] Nathalie Deruelle, Misao Sasaki, Yuuiti Sendouda, and Daisuke Yamauchi. Hamiltonian formulation of f(Riemann) theories of gravity. *Prog. Theor. Phys.*, 123:169–185, 2010.

[62] Stanley Deser. Selfinteraction and gauge invariance. *Gen. Rel. Grav.*, 1:9–18, 1970.

[63] P. Dirac. Lectures on Quantum Mechanics. *Dover Books on Physics. Dover Publications*, 2001.

[64] Xi Dong, Bart Horn, Eva Silverstein, and Alexander Westphal. Simple exercises to flatten your potential. *Phys. Rev.*, D84:026011, 2011.

[65] Bogar Díaz, Daniel Higuita, and Merced Montesinos. Lagrangian approach to the physical degree of freedom count. *J. Math. Phys.*, 55:122901, 2014.

[66] G. Efstathiou, J. R. Bond, and S. D. M. White. Cobe background radiation anisotropies and large-scale structure in the universe. *Monthly Notices of the Royal Astronomical Society*, 258(1):1P–6P, 1992.

[67] John Ellis, Dimitri V. Nanopoulos, and Keith A. Olive. No-Scale Supergravity Realization of the Starobinsky Model of Inflation. *Phys. Rev. Lett.*, 111:111301, 2013. [Erratum: Phys. Rev. Lett.111,no.12,129902(2013)].

[68] John Ellis, Dimitri V. Nanopoulos, and Keith A. Olive. Starobinsky-like Inflationary Models as Avatars of No-Scale Supergravity. *JCAP*, 1310:009, 2013.

[69] Jose María Ezquiaga, Juan García-Bellido, and Miguel Zumalacárregui. Field redefinitions in theories beyond Einstein gravity using the language of differential forms. *Phys. Rev.*, D95(8):084039, 2017.

[70] D. B. Fairlie and A. N. Leznov. General solutions of the Monge-Ampere equation in n-dimensional space. *J. Geom. Phys.*, 16:385–390, 1995.

[71] Sergio Ferrara and Renata Kallosh. Seven-disk manifold, α-attractors, and B modes. *Phys. Rev.*, D94(12):126015, 2016.

[72] Sergio Ferrara, Renata Kallosh, Andrei Linde, and Massimo Porrati. Minimal Supergravity Models of Inflation. *Phys. Rev.*, D88(8):085038, 2013.

[73] Bernardo Finelli, Garrett Goon, Enrico Pajer, and Luca Santoni. Soft Theorems For Shift-Symmetric Cosmologies. *Phys. Rev.*, D97(6):063531, 2018.

[74] D. J. Fixsen. The Temperature of the Cosmic Microwave Background. *Astrophys. J.*, 707:916–920, 2009.

[75] Katherine Freese, Joshua A. Frieman, and Angela V. Olinto. Natural inflation with pseudo - Nambu-Goldstone bosons. *Phys. Rev. Lett.*, 65:3233–3236, 1990.

[76] Mario Galante, Renata Kallosh, Andrei Linde, and Diederik Roest. Unity of Cosmological Inflation Attractors. *Phys. Rev. Lett.*, 114(14):141302, 2015.

[77] Juan Garcia-Bellido and Diederik Roest. Large-N running of the spectral index of inflation. *Phys. Rev.*, D89(10):103527, 2014.

[78] Jérôme Gleyzes, David Langlois, Federico Piazza, and Filippo Vernizzi. Exploring gravitational theories beyond Horndeski. *JCAP*, 1502:018, 2015.

[79] Jérôme Gleyzes, David Langlois, Federico Piazza, and Filippo Vernizzi. Healthy theories beyond Horndeski. *Phys. Rev. Lett.*, 114(21):211101, 2015.

[80] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. Broken Symmetries. *Phys. Rev.*, 127:965–970, 1962.

[81] Garrett Goon, Kurt Hinterbichler, Austin Joyce, and Mark Trodden. Galileons as Wess-Zumino Terms. *JHEP*, 06:004, 2012.

[82] Garrett Goon, Kurt Hinterbichler, Austin Joyce, and Mark Trodden. Einstein Gravity, Massive Gravity, Multi-Gravity and Nonlinear Realizations. *JHEP*, 07:101, 2015.

[83] X. Gracia and J. M. Pons. Gauge Generators, Dirac's Conjecture and Degrees of Freedom for Constrained Systems. *Annals Phys.*, 187:355, 1988.

[84] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev.*, D23:347–356, 1981. [Adv. Ser. Astrophys. Cosmol.3,139(1987)].

[85] S. F. Hassan and Rachel A. Rosen. Bimetric Gravity from Ghost-free Massive Gravity. *JHEP*, 02:126, 2012.

[86] Lavinia Heisenberg. Generalization of the Proca Action. *JCAP*, 1405:015, 2014.

[87] Lavinia Heisenberg. Scalar-Vector-Tensor Gravity Theories. 2018.

[88] Lavinia Heisenberg, Ryotaro Kase, and Shinji Tsujikawa. Beyond generalized Proca theories. *Phys. Lett.*, B760:617–626, 2016.

[89] M. Henneaux and C. Teitelboim. *Quantization of gauge systems.* 1992.

[90] M. Henneaux, C. Teitelboim, and J. Zanelli. Gauge Invariance and Degree of Freedom Count. *Nucl. Phys.*, B332:169–188, 1990.

[91] Marc Henneaux, Axel Kleinschmidt, and Gustavo Lucena Gómez. Remarks on Gauge Invariance and First-Class Constraints. *Proc. Steklov Inst. Math.*, 272(1):141, 2011.

[92] Kurt Hinterbichler. Theoretical Aspects of Massive Gravity. *Rev. Mod. Phys.*, 84:671–710, 2012.

[93] Kurt Hinterbichler, Lam Hui, and Justin Khoury. Conformal Symmetries of Adiabatic Modes in Cosmology. *JCAP*, 1208:017, 2012.

[94] Kurt Hinterbichler and Austin Joyce. Goldstones with Extended Shift Symmetries. *Int. J. Mod. Phys.*, D23(13):1443001, 2014.

[95] Kurt Hinterbichler, Mark Trodden, and Daniel Wesley. Multi-field galileons and higher co-dimension branes. *Phys. Rev.*, D82:124018, 2010.

[96] Gregory Walter Horndeski. Second-order scalar-tensor field equations in a four-dimensional space. *Int. J. Theor. Phys.*, 10:363–384, 1974.

[97] Matthew Hull, Kazuya Koyama, and Gianmassimo Tasinato. Covariantized vector Galileons. *Phys. Rev.*, D93(6):064012, 2016.

[98] C. J. Isham, Abdus Salam, and J. A. Strathdee. Spontaneous breakdown of conformal symmetry. *Phys. Lett.*, 31B:300–302, 1970.

[99] E. A. Ivanov and V. I. Ogievetsky. The Inverse Higgs Phenomenon in Nonlinear Realizations. *Teor. Mat. Fiz.*, 25:164–177, 1975.

[100] Renata Kallosh. Volkov-Akulov theory and D-branes. 1997. [Lect. Notes Phys.509,49(1998)].

[101] Renata Kallosh, Anna Karlsson, and Divyanshu Murli. Origin of Soft Limits from Nonlinear Supersymmetry in Volkov-Akulov Theory. *JHEP*, 03:081, 2017.

[102] Renata Kallosh and Andrei Linde. Cosmological Attractors and Asymptotic Freedom of the Inflaton Field. *JCAP*, 1606(06):047, 2016.

[103] Renata Kallosh, Andrei Linde, and Diederik Roest. Superconformal Inflationary α-Attractors. *JHEP*, 11:198, 2013.

[104] Renata Kallosh, Andrei Linde, Diederik Roest, Alexander Westphal, and Yusuke Yamada. Fibre Inflation and α-attractors. *JHEP*, 02:117, 2018.

[105] Renata Kallosh, Andrei Linde, Diederik Roest, and Yusuke Yamada. $\overline{D3}$ induced geometric inflation. *JHEP*, 07:057, 2017.

[106] Renata Kallosh, Andrei Linde, Timm Wrase, and Yusuke Yamada. Maximal Supersymmetry and B-Mode Targets. *JHEP*, 04:144, 2017.

[107] Renata Kallosh, Andrei D. Linde, Dmitri A. Linde, and Leonard Susskind. Gravity and global symmetries. *Phys. Rev.*, D52:912–935, 1995.

[108] Nemanja Kaloper and Albion Lawrence. London equation for monodromy inflation. *Phys. Rev.*, D95(6):063526, 2017.

[109] Nemanja Kaloper, Albion Lawrence, and Lorenzo Sorbo. An Ignoble Approach to Large Field Inflation. *JCAP*, 1103:023, 2011.

[110] Nemanja Kaloper and Lorenzo Sorbo. A Natural Framework for Chaotic Inflation. *Phys. Rev. Lett.*, 102:121301, 2009.

[111] Karol Kampf and Jiří Novotny. Unification of Galileon Dualities. *JHEP*, 10:006, 2014.

[112] Georgios K. Karananas and Javier Rubio. On the geometrical interpretation of scale-invariant models of inflation. *Phys. Lett.*, B761:223–228, 2016.

[113] Rampei Kimura, Atsushi Naruko, and Daisuke Yoshida. Extended vector-tensor theories. *JCAP*, 1701(01):002, 2017.

[114] Rampei Kimura, Yuki Sakakihara, and Masahide Yamaguchi. Ghost free systems with coexisting bosons and fermions. *Phys. Rev.*, D96(4):044015, 2017.

[115] Daniel Klaewer and Eran Palti. Super-Planckian Spatial Field Variations and Quantum Gravity. *JHEP*, 01:088, 2017.

[116] Tsutomu Kobayashi, Masahide Yamaguchi, and Jun'ichi Yokoyama. Generalized G-inflation: Inflation with the most general second-order field equations. *Prog. Theor. Phys.*, 126:511–529, 2011.

[117] Zohar Komargodski and Nathan Seiberg. From Linear SUSY to Constrained Superfields. *JHEP*, 09:066, 2009.

[118] Sergei M. Kuzenko and Simon J. Tyler. Relating the Komargodski-Seiberg and Akulov-Volkov actions: Exact nonlinear field redefinition. *Phys. Lett.*, B698:319–322, 2011.

[119] David Langlois, Michele Mancarella, Karim Noui, and Filippo Vernizzi. Mimetic gravity as DHOST theories. 2018.

[120] David Langlois and Karim Noui. Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability. *JCAP*, 1602(02):034, 2016.

[121] David Langlois and Karim Noui. Hamiltonian analysis of higher derivative scalar-tensor theories. *JCAP*, 1607(07):016, 2016.

[122] Andrei D. Linde. A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. *Phys. Lett.*, 108B:389–393, 1982. [Adv. Ser. Astrophys. Cosmol.3,149(1987)].

[123] Andrei D. Linde. Chaotic Inflation. *Phys. Lett.*, 129B:177–181, 1983.

[124] Andrei D. Linde. Chaotic Inflation With Constrained Fields. *Phys. Lett.*, B202:194–197, 1988.

[125] D. Lovelock. The Einstein tensor and its generalizations. *J. Math. Phys.*, 12:498–501, 1971.

[126] Liam McAllister, Eva Silverstein, and Alexander Westphal. Gravity Waves and Linear Inflation from Axion Monodromy. *Phys. Rev.*, D82:046003, 2010.

[127] I. N. McArthur. Nonlinear realizations of symmetries and unphysical Goldstone bosons. *JHEP*, 11:140, 2010.

[128] Evan McDonough and Marco Scalisi. Inflation from Nilpotent Kähler Corrections. *JCAP*, 1611(11):028, 2016.

[129] Hayato Motohashi, Karim Noui, Teruaki Suyama, Masahide Yamaguchi, and David Langlois. Healthy degenerate theories with higher derivatives. *JCAP*, 1607(07):033, 2016.

[130] Hayato Motohashi and Teruaki Suyama. Third order equations of motion and the Ostrogradsky instability. *Phys. Rev.*, D91(8):085009, 2015.

[131] Hayato Motohashi, Teruaki Suyama, and Masahide Yamaguchi. Ghost-free theory with third-order time derivatives. 2017.

[132] Hayato Motohashi, Teruaki Suyama, and Masahide Yamaguchi. Ghost-free theories with arbitrary higher-order time derivatives. 2018.

[133] Viatcheslav F. Mukhanov, H. A. Feldman, and Robert H. Brandenberger. Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions. *Phys. Rept.*, 215:203–333, 1992.

[134] Alberto Nicolis, Riccardo Penco, Federico Piazza, and Riccardo Rattazzi. Zoology of condensed matter: Framids, ordinary stuff, extra-ordinary stuff. *JHEP*, 06:155, 2015.

[135] Alberto Nicolis, Riccardo Penco, and Rachel A. Rosen. Relativistic Fluids, Superfluids, Solids and Supersolids from a Coset Construction. *Phys. Rev.*, D89(4):045002, 2014.

[136] Alberto Nicolis, Riccardo Rattazzi, and Enrico Trincherini. The Galileon as a local modification of gravity. *Phys. Rev.*, D79:064036, 2009.

[137] Emmy Noether. Invariant Variation Problems. *Gott. Nachr.*, 1918:235–257, 1918. [Transp. Theory Statist. Phys.1,186(1971)].

[138] Johannes Noller and James H. C. Scargill. The decoupling limit of Multi-Gravity: Multi-Galileons, Dualities and More. *JHEP*, 05:034, 2015.

[139] Johannes Noller, Vishagan Sivanesan, and Mikael von Strauss. On extended symmetries for the Galileon. *Phys. Rev.*, D92(6):064009, 2015.

[140] Jiri Novotny. Geometry of special Galileons. *Phys. Rev.*, D95(6):065019, 2017.

[141] Peter J. Olver. *Equivalence, Invariants and Symmetry.* Cambridge University Press, 1995.

[142] P.J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986.

[143] M. Ostrogradski. *Mem. Ac. St. Petersbourg*, VI 4:385, 1850.

[144] Antonio Padilla, Paul M. Saffin, and Shuang-Yong Zhou. Bi-galileon theory I: Motivation and formulation. *JHEP*, 12:031, 2010.

[145] Antonio Padilla and Vishagan Sivanesan. Covariant multi-galileons and their generalisation. *JHEP*, 04:032, 2013.

[146] Antonio Padilla, David Stefanyszyn, and Toby Wilson. Probing Scalar Effective Field Theories with the Soft Limits of Scattering Amplitudes. *JHEP*, 04:015, 2017.

[147] S. Perlmutter et al. Measurements of Omega and Lambda from 42 high redshift supernovae. *Astrophys. J.*, 517:565–586, 1999.

[148] J. M. Pons. New Relations Between Hamiltonian and Lagrangian Constraints. *J. Phys.*, A21:2705, 1988.

[149] Bharat Ratra and P. J. E. Peebles. Cosmological Consequences of a Rolling Homogeneous Scalar Field. *Phys. Rev.*, D37:3406, 1988.

[150] Adam G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.*, 116:1009–1038, 1998.

[151] M. Rocek. Linearizing the Volkov-Akulov Model. *Phys. Rev. Lett.*, 41:451–453, 1978.

[152] M. Rocek and Arkady A. Tseytlin. Partial breaking of global D = 4 supersymmetry, constrained superfields, and three-brane actions. *Phys. Rev.*, D59:106001, 1999.

[153] Diederik Roest. Universality classes of inflation. *JCAP*, 1401:007, 2014.

[154] H.J. Rothe and K.D. Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems.* World Scientific Publishing, Singapore, 2010.

[155] Angnis Schmidt-May and Mikael von Strauss. Recent developments in bimetric theory. *J. Phys.*, A49(18):183001, 2016.

[156] Leonardo Senatore and Matias Zaldarriaga. The Effective Field Theory of Multifield Inflation. *JHEP*, 04:024, 2012.

[157] Eva Silverstein and Alexander Westphal. Monodromy in the CMB: Gravity Waves and String Inflation. *Phys. Rev.*, D78:106003, 2008.

[158] Vishagan Sivanesan. Generalized multiple-scalar field theory in Minkowski space-time free of Ostrogradski ghosts. *Phys. Rev.*, D90(10):104006, 2014.

[159] Julian Sonner and Paul K. Townsend. Dilaton domain walls and dynamical systems. *Class. Quant. Grav.*, 23:441–464, 2006.

[160] Thomas P. Sotiriou and Valerio Faraoni. f(R) Theories Of Gravity. *Rev. Mod. Phys.*, 82:451–497, 2010.

[161] E. C. G. Stueckelberg. Theory of the radiation of photons of small arbitrary mass. *Helv. Phys. Acta*, 30:209–215, 1957.

[162] E.C.G. Sudarshan and N. Mukunda. *Classical Dynamics: A Modern Perspective.* John Wiley and Son, New York, 1974.

[163] Raman Sundrum. Effective field theory for a three-brane universe. *Phys. Rev.*, D59:085009, 1999.

[164] Gianmassimo Tasinato. Cosmic Acceleration from Abelian Symmetry Breaking. *JHEP*, 04:067, 2014.

[165] Takahiro Terada. Generalized Pole Inflation: Hilltop, Natural, and Chaotic Inflationary Attractors. *Phys. Lett.*, B760:674–680, 2016.

[166] D. V. Volkov and V. P. Akulov. Is the Neutrino a Goldstone Particle? *Phys. Lett.*, 46B:109–110, 1973.

[167] Dmitri V. Volkov. Phenomenological Lagrangians. *Fiz. Elem. Chast. Atom. Yadra*, 4:3–41, 1973.

[168] Steven Weinberg. The Cosmological Constant Problem. *Rev. Mod. Phys.*, 61:1–23, 1989. [,569(1988)].

[169] J. Wess and B. Zumino. Consequences of anomalous Ward identities. *Phys. Lett.*, 37B:95–97, 1971.

[170] Edward Witten. Global Aspects of Current Algebra. *Nucl. Phys.*, B223:422–432, 1983.

[171] Richard P. Woodard. Avoiding dark energy with 1/r modifications of gravity. *Lect. Notes Phys.*, 720:403–433, 2007.

[172] Richard P. Woodard. Ostrogradsky's theorem on Hamiltonian instability. *Scholarpedia*, 10(8):32243, 2015.

[173] Miguel Zumalacárregui and Juan García-Bellido. Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian. *Phys. Rev.*, D89:064046, 2014.