Improving Predictions for SUSY Cross Sections

Soft-Gluon Resummation for SUSY-QCD

Irene Niessen
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IMPROVING PREDICTIONS FOR SUSY CROSS SECTIONS
SOFT-GLUON RESUMMATION FOR SUSY-QCD

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Contents

1 Introduction .................................................. 1

2 Supersymmetry ............................................... 5
  2.1 The Minimal Supersymmetric Standard Model ................. 5
  2.2 Why Double the Particle Content? .......................... 7
    2.2.1 Symmetry ............................................. 7
    2.2.2 Open Questions in the Standard Model ................. 9
    2.2.3 Astrophysical Observations ............................ 13
  2.3 Supersymmetry Searches ................................. 16
  2.4 Squarks and Gluinos ...................................... 16

3 Resummation .................................................. 19
  3.1 Infrared Divergences ...................................... 19
    3.1.1 The Eikonal Approximation .......................... 20
    3.1.2 Large Logarithms .................................... 21
  3.2 Factorization .............................................. 23
    3.2.1 Parton Distribution Functions ....................... 23
    3.2.2 Further Factorization of the Partonic Cross Section .... 24
  3.3 Exponentiation ............................................ 28
  3.4 A New Perturbative Series ................................ 30
  3.5 Inverse Mellin Transform and Matching ..................... 32
  3.6 Coulomb Corrections ...................................... 32
  3.7 Summary ................................................. 33

4 Colour and NLL Resummation ............................. 35
  4.1 Colour Bases .............................................. 35
    4.1.1 Product Representations ............................. 36
    4.1.2 Colour-Charge Operators ............................. 37
    4.1.3 Explicit Construction of a Colour Basis .............. 38
  4.2 The Leading Order Cross Section ........................ 39
    4.2.1 Majorana Particles ................................. 39
    4.2.2 Ghost Subtraction .................................. 40
CONTENTS

4.2.3 Colour Decomposition .................................................. 41
4.2.4 Vanishing Cross Sections .............................................. 42
4.2.5 Mellin Transforms ...................................................... 44
4.3 Soft Radiation Factors .................................................... 44
4.4 Numerical Results ......................................................... 46
4.5 Experimental Implications ................................................ 50
4.5.1 NLL-fast ................................................................. 53
4.6 Summary ................................................................. 53

5  NNLL Resummation ........................................................ 55
  5.1 NLO Calculations for SUSY-QCD ....................................... 55
  5.2 Coulomb Contributions .................................................. 57
  5.3 Hard Matching Coefficients ............................................. 58
    5.3.1 Virtual Corrections ................................................. 58
    5.3.2 Real Corrections .................................................... 59
    5.3.3 The Combined Result in Mellin-Moment Space .................. 63
  5.4 Soft Radiation Factors .................................................. 66
  5.5 Numerical Results for $q\bar{q}$ Production ............................. 67
  5.6 Summary ................................................................. 70

6  Conclusion and Outlook ..................................................... 71

A  Product Representations Using Young Tableaux .......................... 73

B  Base Tensors for SUSY-QCD ................................................. 75

C  Colour-Decomposed LO Cross Sections in SUSY QCD .................... 79

D  Mellin Transforms .......................................................... 85
  D.1 Integrals ....................................................................... 85
  D.2 LO Cross Sections and Coulomb Corrections ......................... 87

E  One-Loop Soft Anomalous Dimensions for SUSY-QCD .................. 93

F  Hard Matching Coefficients for SUSY-QCD ................................ 97

Bibliography ................................................................. 117

List of Abbreviations ......................................................... 119

Summary ................................................................. 121

Samenvatting ............................................................... 125

List of Publications ......................................................... 129
CONTENTS

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curriculum Vitae</td>
<td>131</td>
</tr>
<tr>
<td>Dankwoord</td>
<td>133</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

What is the world made of? How does it work? Almost 2500 years ago, the Greek philosopher Empedocles proposed that everything around us consists of four elements: earth, water, fire and air. According to him, these elements were indestructible and would combine in different ways to generate everything around us. Since then, our understanding of nature has evolved, but the basic questions are still the same. What is the world made of and how does it work?

Our current understanding of the fundamental constituents of nature is summarized in the Standard Model (SM) of particle physics. The SM was completed in the 1970’s and consists of six types of quarks and six types of leptons that interact with each other through the exchange of gauge bosons of the electroweak [1–3] and strong interaction [4–6]. Most SM particles obtain a mass through the Higgs mechanism [7–9], sometimes referred to as the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism, which also gives rise to the recently discovered Higgs boson [10, 11].

The SM is an extremely successful theory. Experiments at particle colliders have confirmed its theoretical predictions with astounding accuracy [12]. However, some problems remain [13]. For instance, the SM does not include a description of gravity or dark matter. Also, some parameters seem unnaturally small, and some aspects of the Higgs sector are poorly understood.

Since the SM provides such a good description of nature, it is not unreasonable to assume that there must be some truth to it. However, that does not mean it is the end of the story. The SM can be seen as an effective theory, originating from a more fundamental description of nature. In the same way that the $W$ and $Z$ bosons could only be detected with the energy of the Super Proton Synchrotron (SPS), the effects of this fundamental theory would only become apparent at a higher energy scale.

With gravity missing from the SM, the important question is not if there is a more fundamental theory, but at what energy scale it sets in. If new physics only shows up at the Planck scale, searching for it at present-day colliders is pointless. If, however, it sets in at the TeV scale, we could find it with the new proton-proton collider at CERN, the Large Hadron Collider (LHC).
As we will describe in more detail in Chapter 2, there are good reasons to believe that new physics will occur around the TeV scale. Yet, we do not know what kind of new physics this would be. Several models have been proposed, with very different theoretical foundations. The most common theories that extend the Standard Model introduce extra dimensions [14–19], versions of technicolor [20–22], or supersymmetry [23, 24]. As different as the underlying physical assumptions of these theories might be, they all predict new particles with masses in the TeV range, which could thus be detected at the LHC.

The search for new physics is one of the main goals at the LHC. Searching for such heavy particles is a challenging task, not only because of the experimental conditions, but also on the theory side. In order to identify or exclude specific theories, measurements need to be compared with theoretical predictions. Possibly the most basic prediction for a collider experiment is how many particles will be produced. In other words, the cross section for a given production process. Unfortunately, obtaining an accurate prediction for the cross section of heavy particles is not straightforward.

Cross sections in particle physics are calculated with a perturbative expansion in terms of the coupling of the theory. In most cases that is a good approximation. Unfortunately, it does not work very well for heavy particles. The physical reason for this can already be seen from the behaviour of the leading order (LO) cross section near their production threshold, which is shown schematically for the pair-production of two particles with mass $m$ in Figure 1.1.

![Figure 1.1: Behaviour of a typical cross section $\sigma$ for the production of two particles with mass $m$ as a function of centre-of-mass energy squared $s$ at leading order. At the production threshold $s = 4m^2$, the cross section rises steeply.](image)

The cross section in Figure 1.1 is zero below the production threshold of the particles. At energies just above the production threshold, the cross section rises steeply, so even a small change in the energy has a large effect. Next-to-Leading Order (NLO) effects include radiation of particles, which changes the energy that is available for the process. The probability of a high-energy particle to be radiated is small, but particles with little energy, which change the energy available to the process only by a tiny amount, are radiated easily. For most processes this is a relatively small effect, because the threshold region is only a tiny part of the available phase space. However, particles with masses that are close to the total collider energy are necessarily produced near threshold, because there is simply no more energy available. In the threshold region, where the LO cross section in Figure 1.1 has a steep slope, low-energy radiation can change the cross section considerably, so these higher order effects are important in the production of heavy particles.

The effect is enhanced by the energy distribution inside the protons. Since the proton is not a fundamental particle, its energy is divided over its constituents, the so-called
partons. As this is a stochastic process, it is in principle possible that a parton has a considerable fraction of the total energy of the proton, but it is extremely unlikely. Thus, most processes at the LHC originate from partons that only have a tiny fraction of the proton energy, leaving less energy for the production process. As a result, even particles with masses that are an order of magnitude smaller than the collider energy are usually produced near their production threshold [25]. We can conclude that the threshold region is very important in the search for new physics phenomena at the LHC.

In this thesis, we will study the cross section in the context of supersymmetry (SUSY). Since the LHC is a hadron collider, strong interactions, described by Quantum Chromodynamics (QCD), are expected to play an important role in the discovery of new physics models that contain coloured particles. Therefore we will focus on the cross sections for the coloured SUSY particles. The theoretical predictions for SUSY-QCD pair-production processes are in fairly good shape, since the NLO SUSY-QCD corrections are known [26–28]. However, these corrections are quite large compared to the LO prediction. A significant part of these corrections can be attributed to the threshold region.

As we will explain in more detail in Chapter 3, the NLO corrections near threshold are dominated by soft-gluon emission off the coloured particles in the initial and final state, and by the Coulomb corrections due to the exchange of gluons between the slowly moving massive particles in the final state. The dominant contributions due to soft-gluon emission have the general form

\[ \alpha_s^n \log^m \beta^2, \quad m \leq 2n \quad \text{with} \quad \beta^2 = 1 - \frac{4m_{av}^2}{s}, \]  

(1.1)

where \( \alpha_s \) is the strong coupling, \( s \) is the partonic Centre-of-Mass (CM) energy squared and \( m_{av} \) is the average mass of the final-state particles. Clearly, these corrections become very large if the energy approaches the production threshold, where \( \beta \to 0 \).

The structure of the corrections in Eq. (1.1) suggests that the usual perturbative expansion does not converge very well. The large size of the higher-order corrections is reflected in the theoretical uncertainty of the predictions. At LO, the uncertainty can have the same order of magnitude as the cross section itself. Including the NLO corrections improves the situation, but still leaves a sizeable uncertainty, especially if the masses of the SUSY particles are large. A logical next step might seem to calculate the Next-to-Next-to-Leading Order (NNLO) corrections, but the poor convergence of the perturbative expansion suggests a different strategy.

The soft-gluon corrections from Eq. (1.1) can be taken into account to all orders in perturbation theory by means of threshold resummation techniques. Essentially, this gives rise to a new perturbative expansion, which is not only based on the smallness of \( \alpha_s \), but also on the largeness of the logarithms. In this way, the calculation reflects the importance of the threshold region for these specific processes. Soft-gluon resummation improves the stability of the results and thus the accuracy of the predictions.

In this thesis, we will perform threshold resummation for SUSY-QCD to next-to-leading logarithmic (NLL) accuracy and present the analytical ingredients needed for resummation at next-to-next-to-leading logarithmic (NNLL) accuracy. We will begin by introducing SUSY in Chapter 2. We will then continue with a discussion on the concepts
behind resummation in Chapter 3. In Chapter 4 we present the results for NLL resummation for SUSY-QCD. We will focus on the analytical results and the colour decomposition needed for resummation, although we also discuss the numerical effect of the corrections. The analytical ingredients for NNLL resummation for supersymmetry as well as some numerical results are presented in Chapter 5. The focus of that chapter is on obtaining the matching coefficients needed for NNLL resummation. Finally, we give a summary and outlook in Chapter 6.
Chapter 2

Supersymmetry

Since the Standard Model (SM) of particle physics was formulated, collider experiments have confirmed it to incredible precision [12]. However, the SM is suffering from some theoretical problems [13]. Therefore, theorists have proposed many new theories in their quest to gain a deeper insight into the underlying structure of the SM. Most attempts involve extending the symmetries of the SM in order to make the theory more elegant, usually at the expense of introducing new particles and interactions.

In supersymmetry (SUSY), a new symmetry between bosons and fermions is introduced [23]. As a result, every boson must have a fermionic partner and vice versa. In exact SUSY, the superpartners differ in spin, but all other quantum numbers, including their mass, are the same. A quick inspection of the available particles in the SM shows that exact SUSY is not realized in nature: the required particles, for instance the scalar partner of the electron, have never been observed and thus cannot have the same mass as the SM particles. However, SUSY could be realized at high energies, while it is broken at lower energies. This would result in heavy SUSY partners for the SM particles. The names of these partners are derived from the names of their SM counterparts. The bosonic partners of the fermions get the prefix ‘s’, leading to colourful names such as squarks and sleptons. The fermionic partners of bosons get the suffix ‘ino’, e.g. gaugino.

In this chapter we will first briefly introduce the minimal supersymmetric extension of the SM in Section 2.1. We will then discuss some of the reasons to introduce SUSY in Section 2.2. We will proceed with the status of SUSY searches in Section 2.3 and end with a brief discussion of the relevant properties of SUSY-QCD in Section 2.4.

2.1 The Minimal Supersymmetric Standard Model

In this section, we will briefly introduce the Minimal Supersymmetric Standard Model (MSSM) [13, 24], which is the simplest possible SUSY extension of the SM. Complete introductions can be found in e.g. Refs. [29, 30]. The particle content of the MSSM consists of the SM particles and SUSY partners for each of them, the sparticles. In addition,
we need an extra Higgs doublet with the corresponding Higgsinos to ensure anomaly cancellation [31]. As a result, we have eight real degrees of freedom in the Higgs sector. As in the SM, three of them are absorbed in the longitudinal components of the $Z$ and the $W$ bosons after spontaneous symmetry breaking, leaving three neutral and two charged Higgs bosons. In many specific models, the lightest Higgs boson resembles the SM Higgs boson.

This fixes the number of degrees of freedom in the Higgs sector, but we also have the fermions and their scalar partners. A massless Dirac field has a left-handed and a right-handed component, while a scalar field only has one spin degree of freedom. As a result, every fermion in the SM has two sfermionic partners: one that corresponds to the left-handed degree of freedom and one that corresponds to the right-handed degree of freedom. If sfermions have the same quantum numbers, they can in principle form mixed mass eigenstates.

The gauge bosons have the same number of degrees of freedom as their SUSY partners. However, the partners of the electroweak gauge bosons have the same quantum numbers as the Higgsinos, yielding mixed gaugino mass eigenstates, called neutralinos and charginos for neutral and charged particles respectively. The neutral gauginos, including the gluino, are their own antiparticles, so they are Majorana fermions.

Not only the particle content is supersymmetric, the interactions are invariant under SUSY transformations as well. As a result, SUSY couplings are related to their SM equivalents, and the only additional parameter in the exact MSSM is a Higgs mass term that couples the two Higgs doublets. One could write down other interactions, but in order to suppress proton decay, one usually introduces a conserved quantity called $R$-parity:\footnote{Proton decay can also be suppressed by eliminating only part of the vertices that are forbidden by $R$-parity, but we will not consider such extended models here.}

\[
R = (-1)^{3(B-L)+2S},
\]  

(2.1)

with $B$ the baryon number, $L$ the lepton number and $S$ the spin of the particle [29]. This multiplicative quantum number is $R = +1$ for SM particles and the additional Higgs bosons, and $R = -1$ for their SUSY partners. Conservation of $R$-parity has two important consequences. First, it means that SUSY particles can only be produced in pairs, which has important implications for collider experiments. The second consequence is that the Lightest Supersymmetric Particle (LSP) is stable. This does not only affect collider searches, but we will see in the next section that it also has important consequences for cosmology.

In an exact SUSY theory, the sparticles have the same mass as their SM partners. Since we have not seen selectrons or squarks, we know that SUSY particles must be heavier than their SM counterparts and thus SUSY must be broken. Unfortunately, we do not know how to break SUSY. There are a number of theoretical possibilities, e.g. [32–34], but they all have their drawbacks. Therefore we view broken SUSY as an effective theory and simply add all possible soft SUSY breaking terms to the Lagrangian. In addition to renormalizability and gauge invariance, we require these terms not to introduce any quadratic divergences. This requirement relates to the hierarchy problem, which we will
discuss in the next section. A complete set of Feynman rules for the broken MSSM can be found in Ref. [29].

Unfortunately, we are then left with an effective theory that contains over a hundred new parameters. We can limit the possibilities based on phenomenological considerations [35]. Most of the new parameters would generate unacceptably large CP violation or flavour changing neutral currents and are thus usually taken to be zero. This constrains mixing in the sfermion mass eigenstates to left-right mixing, which is only important for the third generation sfermions, particularly the stops and the staus [36].

Thus, in phenomenologically viable versions of the MSSM, the SUSY parameter dependence of particle mixings and interactions is most prominent in the Higgs and electroweak gaugino sector, and for third generation sfermions. For the other sparticles, the phenomenology is essentially determined by the SM couplings and the sparticle masses.

2.2 Why Double the Particle Content?

One might wonder why anyone would feel the need to look beyond the SM, let alone read a thesis on higher-order corrections for particles that might not even exist. The aim of this section is to convince you that there are plenty of reasons to do so. We will start by discussing general symmetry considerations, then show how SUSY solves some open questions in particle physics, and end with some astrophysical observations that can be explained by SUSY.

2.2.1 Symmetry

Symmetry is one of the guiding principles in the SM. Poincaré symmetry dictates how particles behave in general, while internal symmetries are at the basis of the electroweak and strong interactions. Thus, if we want to go beyond the SM, symmetry is the natural place to start.

Extending the Poincaré Algebra

Let us start by taking a closer look at the Poincaré group. The Coleman-Mandula theorem [37] states that the only possible Lie algebra of this group consists of translations and Lorentz transformations, both of which have a physical meaning and are thus used in nature. There are two ways to extend the symmetry under the Poincaré group. The first is by adding internal symmetries, which commute with the generators of the Poincaré algebra. The gauge groups are an example of this. A second, more interesting option, is to look at the symmetries of the Poincaré group itself. The Coleman-Mandula theorem is restricted to Lie algebras, which correspond to bosonic generators. However, since we have fermions in nature, why would we not have fermionic symmetry generators as well [38]? Working out the simplest example of this possibility leads to fermionic generators \( Q \), which satisfy [29]:

\[
\{Q_a, \bar{Q}_b\} = 2(y^{\mu})_{ab} P_\mu,
\]  

(2.2)
2.2. Why Double the Particle Content?

with \( P_\mu \) the momentum operator and \((\gamma^\mu)_{ab}\) the Dirac matrices with Dirac indices \( a \) and \( b \). One can compare this step to the original steps taken in the development of Quantum Field Theory (QFT). If one sees the Dirac equation as the square root of the massless Klein-Gordon equation, then why would we not take it one step further and take the square root of the Dirac equation as well? From Eq. (2.2) we can see that the product of two SUSY generators is proportional to the momentum operator. In that sense the SUSY generators could be seen as a ‘square root’ of the momentum operator, comparable to the way that the momentum operator follows from taking the ‘square root’ of the massless Klein-Gordon operator [31,39].

Following this line of reasoning, SUSY is a natural extension of what we already know. In particle physics, there seems to be a general consensus that if a symmetry is possible, it should be realized in nature. We should note, however, that we already know of an exception to this rule. Although the SM Lagrangian contains almost all conceivable terms that are allowed by the SM symmetries, the \( \theta \) parameter in \( \mu \) must be either exactly zero or unnaturally small. A Peccei-Quinn symmetry could explain this issue [40], but we have no more evidence for that than we do for SUSY.

**Grand Unification**

Historically, an important argument in favour of SUSY comes from Grand Unified Theories (GUTs). For a long time, combining the SM gauge groups into a single unified group was every theorist’s dream. The first and simplest model that succeeded at this embeds the SM SU(3)\( \times \)SU(2)\( \times \)U(1) group in an SU(5) GUT at some high energy scale [41]. Such an embedding fixes the normalization of the hypercharge, which is absorbed in a rescaling of the U(1) coupling compared to the SM convention.

At the time, experimental data suggested that the couplings of the strong and the weak interaction and the rescaled U(1) coupling had the same value around an energy scale of \( \mathcal{O}(10^{15}) \) GeV. As experimental data on the running couplings became more precise, coupling unification was ruled out for SM Renormalization Group Equations (RGEs), as shown in Figure 2.1.

It was shown in Ref. [42], however, that SUSY still allowed for unification of the gauge couplings. This famous result is shown in Figure 2.2.

![Running of the rescaled U(1) hypercharge coupling \( \alpha_1 \), the SU(2) weak coupling \( \alpha_2 \), and the SU(3) strong coupling \( \alpha_3 \) in the context of an SU(5) GUT and SM RGEs. Reprinted from Ref. [42], Copyright (1991), with permission from Elsevier.](image-url)
Ultimately, symmetry arguments are a matter of taste. Still, this kind of reasoning certainly has its charm and provides a strong motivation to look deeper into the possibility of SUSY.

### 2.2.2 Open Questions in the Standard Model

There are several reasons to look into SUSY that are less philosophical in nature than the ones in the previous section. We will discuss some of them here.

**Gravity**

Arguably the biggest problem in the SM is the lack of any understanding of gravity. Bluntly quantizing general relativity yields a nonrenormalizable theory, so that cannot be the end of the story. A promising candidate for a ‘theory of everything’, or at least one that includes gravity, seems to be string theory. Generally a string theory that includes fermions also predicts SUSY and this is often used as an argument in favour of SUSY. However, even if one ignores the technical difficulties of reproducing the SM within string theory, the type of SUSY predicted is not necessarily useful. A SUSY string theory does not automatically imply low-energy SUSY. If SUSY only manifests itself near the Planck scale, it is not very interesting from a phenomenological point of view.

Nevertheless, low-energy SUSY could be a first step in reconciling general relativity with gauge theories. We have seen in Eq. (2.2) that the product of two SUSY generators corresponds to a translation in spacetime. However, the generators in Eq. (2.2) describe \emph{global} SUSY transformations. We could see SUSY as a gauge theory and allow the SUSY transformations to depend explicitly on spacetime. It turns out that the product of two such \emph{local} SUSY transformations corresponds to a general coordinate transformation [24]. The gauge field of this local SUSY is the spin-$\frac{1}{2}$ gravitino, which is the partner of the graviton. Thus local SUSY automatically implies gravity, which is why it is usually
2.2. Why Double the Particle Content?

called supergravity. Most attempts to reconcile gravity with gauge theories seem to point at supergravity and models that break this symmetry in an elegant way tend to yield a relatively light SUSY particle spectrum [24].

The Hierarchy Problem

The lack of a description of gravity is the first indication that the SM is incomplete. This observation almost inevitably leads to the hierarchy problem, which expresses the unnaturality of the lightness of the Higgs boson given the hierarchy of the energy scales involved. Unlike the other SM particles, the mass of the Higgs boson is not protected by a symmetry. As a result, loop corrections can become very large. As an example, consider a loop contribution to the Higgs self-energy containing a fermion $f$ with mass $m_f$ that couples to the Higgs boson $h$ with momentum $p_h$ through a Yukawa coupling $\lambda_f/\sqrt{2}$:

$$
-2\lambda_f^2 \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{k^2 - m_f^2} + \frac{2m_f^2 - \frac{1}{2}p_h^2}{(k^2 - m_f^2)((k + p_h)^2 - m_f^2)} \right).
$$

The first term in Eq. (2.3) diverges quadratically for large loop momenta $k$. Within the SM that is not a problem, since this divergence can simply be renormalized. However, if the Higgs couples to some kind of new physics at an energy scale $\Lambda$, the SM is only valid up to that energy scale. In that case, $\Lambda$ acts as a cutoff for the integral and the loop corrections to the Higgs boson mass squared are as large as the scale of new physics squared.

Of course, it is possible that the bare mass and the loop corrections conspire to yield a light Higgs boson, but taking the scale of new physics to be the GUT scale, the level of finetuning is somewhat disturbing. If, on the other hand, the only missing ingredient from the SM is gravity, the quadratic divergence in Eq. (2.4) might not be a problem at all. It is argued in Ref. [45] that combining the SM with asymptotic safe gravity could naturally lead to a light Higgs boson. Thus, the SM could be natural, provided that it is valid all the way up to the Planck scale.

We cannot check directly if the SM can be extended up to the Planck scale, but we can use the RGEs in the Higgs sector as a consistency check. It turns out that the Higgs quartic coupling can run into problems when it is evolved to higher energies. If the Higgs mass at the electroweak scale is too large, the theory becomes nonperturbative before the Planck scale, which is known as the triviality bound. If the mass is too small, the coupling reaches the stability bound, where the SM vacuum becomes unstable. These bounds provide an internal consistency check for the SM.

Quite a few assumptions enter this calculation. First, one has to define how large the coupling can be before the theory becomes nonperturbative. In addition, one has to define a stable vacuum. The natural definition might be to require that our universe is at the global minimum of the Higgs potential. However, it is conceivable that we are in fact in a local minimum. In that case, the universe could tunnel to another minimum, but as long as the life expectancy of such a metastable universe is larger than the lifetime of our universe, the model would still be viable. This calculation has been performed using two-loop RGEs in Ref. [46] and the result is shown in Figure 2.3.
Figure 2.3: For a given Higgs mass, one can see at which energy the coupling runs into the triviality (upper) bound or the stability (lower) bound. The two upper bounds in the plot are determined using different definitions for nonperturbativity. The different lower bounds represent different assumptions on the (meta)stability of the universe. The uncertainty bands are based on the experimental uncertainties on the strong coupling and the top quark mass. Reprinted from Ref. [46], Copyright (2009), with permission from Elsevier.

Figure 2.3 shows that there is a small window of Higgs mass values for which the SM is internally consistent all the way up to the Planck scale, even when demanding a stable vacuum. In 2011, it became clear that this window was virtually excluded by the Tevatron [47] and LHC [48, 49] measurements, which when combined with the electroweak precision data [50, 51] confined the Higgs to a mass between 121 GeV and 127 GeV. The recent discovery [10, 11] of a boson that appears to be consistent with the SM Higgs fixes its mass at 125 GeV. One can conclude that either the universe most likely resides in a metastable vacuum [52], or that there is new physics that couples to the Higgs boson before the Planck scale. The latter option brings us back to the hierarchy problem, since then one would expect the mass of the Higgs to be of the order of the scale of the new physics.

This issue could be resolved by SUSY, where in addition to the fermionic loop correction in Eq. (2.3), we also have the corresponding correction from the sfermionic partners of the fermion. They couple to the Higgs with trilinear couplings, as well as a quartic coupling \( \tilde{\lambda}_f \), which is related to the Yukawa coupling of the corresponding fermion as \( \tilde{\lambda}_f = -\lambda_f^2 \). This quartic coupling gives rise to a loop correction involving sfermions \( \tilde{f}_{L,R} \) with mass \( m_{\tilde{f}_{L,R}} \), which is given by:

\[
\tilde{f}_L + \tilde{f}_R = \lambda_f^2 \int \frac{d^4k}{(2\pi)^3} \left[ \frac{1}{k^2 - m_{\tilde{f}_L}^2} + \frac{1}{k^2 - m_{\tilde{f}_R}^2} \right].
\]
2.2. Why Double the Particle Content?

The diagrams in Eq. (2.4) yield a quadratic divergence, which exactly cancels the quadratic divergence in Eq. (2.3). The other self-energy corrections involving sfermions are:

\[
\begin{align*}
\tilde{h} & \tilde{f}_{LR} h + \tilde{h} \tilde{f}_{LR} h.
\end{align*}
\]

These diagrams only yield logarithmic divergences, which do not require excessive fine-tuning. For this to work, however, the scale of the SUSY masses cannot be too high. The logarithmic divergences arising from the diagrams in Eq. (2.5) and the second term of Eq. (2.3) are proportional to the square of the masses in the loop. In unbroken SUSY, the second diagram in Eq. (2.5) vanishes and in addition we have \( m_f = m_{\tilde{f}} \). In that case, the logarithmic divergences cancel [29], expressing the fact that supersymmetry protects the Higgs boson mass. If SUSY is broken, this protection mechanism still works. However, if the difference between the fermion and sfermion masses is too large, the sfermion mass itself introduces a new hierarchy problem, taking the role of the cutoff scale \( \Lambda \). The theory then needs fine-tuning once again, albeit it is reduced considerably provided that the SUSY mass scale is much lower than the GUT scale. Usually finetuning at the percent level is considered acceptable, leading to squark and gluino masses of at most a few TeV, while weakly interacting sparticles should have masses in the sub-TeV range [53].

Spontaneous Symmetry Breaking

SUSY also provides an elegant solution to another puzzle concerning the Higgs in the SM. Although the concept of spontaneous symmetry breaking is well-established theoretically, it remains a mystery why the squared Higgs-mass Lagrangian parameter would become negative.

As we have seen in Section 2.1, in SUSY we need two Higgs doublets rather than one. Due to the RGEs, the running squared mass of one of these Higgs doublets quite naturally becomes negative at low energies, as can be seen in Figure 2.4.

Thus, the MSSM can induce radiative spontaneous symmetry breaking. However, this only happens in broken SUSY. In the unbroken MSSM, the particle spectrum is massless and will remain so, since the RGEs do not generate spontaneous symmetry breaking. In that case the SM particles do not obtain masses either. Thus, SUSY must be broken not only because we have not seen any sparticles yet, but also to explain the masses of the SM particles.

Figure 2.4: From Ref. [30]. RGE running of the SUSY and Higgs running masses starting from a constrained model at the GUT scale. The negative squared mass of the lightest Higgs doublet induces radiative spontaneous symmetry breaking.
particles in the context of SUSY. Of course, this shifts the question from electroweak symmetry breaking to SUSY breaking. Still, it is nice how some pieces of the puzzle seem to fit together in SUSY.

### 2.2.3 Astrophysical Observations

So far we have discussed mostly theoretical problems in particle physics and their relation to SUSY. However, we also have some experimental observations that signal the need for new physics. They were not made by the particle physics community, where all measurements are in good agreement with the SM, but originate from astrophysics.

**Dark Matter**

It is estimated that we can see less than 20% of the matter in the universe. The remaining matter is called dark matter and consists of particles that only interact through the weak and the gravitational force. Evidence for dark matter has been gathering since the 1930’s, when it was first proposed. We will limit ourselves to the most important observations that support the existence of dark matters. A more complete overview can be found e.g. in Ref. [56].

The original motivation for dark matter was the velocity of stars and galaxies. Looking at the velocities of nearby stars, one can determine how much mass is needed to keep our galaxy together. Comparing this to the amount of visible mass, Oort [57] concluded that our galaxy had to contain additional, invisible matter. Zwicky [58] made a similar observation at a much larger scale, based on the velocities of galaxies in the Coma cluster.

A more commonly used argument in favour of dark matter is the shape of the rotation curves of galaxies. Looking at the velocity of stars as a function of the distance to the galactic centre, one would expect that the velocity decreases for large radii. Instead, the curve flattens [59,60]. An example of this behaviour is shown in Figure 2.5, which can be explained by a halo of dark matter that extends beyond the outer reaches of the galaxy. The additional matter ensures gravitational binding of the stars, explaining the shape of the rotation curve.

An alternative explanation for the galaxy rotation curves is Modified Newtonian Dynamics (MOND) [63], or its relativistic version Tensor-Vector-Scalar gravity (TeVeS) [64]. Both theories assume that the gravitational potential has a different behaviour at large distances. However, MOND cannot explain all observations that point towards dark matter. In particular, there is the case of the bullet cluster [65], which is a cluster of galaxies that has collided with another cluster of galaxies. Figure 2.6 shows the visual and X-ray images of the clusters, as well as the mass distribution that was derived from gravitational lensing measurements. The galaxies form a perfect gas and thus pass through each other without friction. The hot intergalactic gas, which contains most of the mass in the clusters, is slowed down in the collision due to friction. Contrary to what one would expect without dark matter, gravitational lensing shows that most of the mass is in fact located

\[ A \text{ possible exception could be the measurement of the anomalous magnetic moment of the muon [54], but even that does not deviate by much more than } 3\sigma \text{ from the SM prediction [55].} \]
2.2. Why Double the Particle Content?

![Graph of NGC 2903 rotation curve](image1)

Figure 2.5: From Ref. [61]. Best-fit rotation curve (solid line) of galaxy NGC 2903. The contributions of the individual components in the fit are stars (dashed), gas (dotted) and the dark matter halo (dash-dot). Without dark matter, one would expect the velocity to decrease with increasing radius. The picture of NGC 2903 on the right-hand side is from [62].

...around the galaxies. This can only be explained with dark matter, which is frictionless as it only interacts weakly.

![Bullet cluster image](image2)

Figure 2.6: Picture of the bullet cluster 1E 0657-56 from [66]. The X-ray image of the hot gas is shown in pink, while the galaxies are the white/orange dots. The mass distribution measured with gravitational lensing is shown in blue and coincides with the location of the galaxies.

Finally, there are the WMAP measurements [67], which chart the temperature fluctuations of the cosmic microwave background radiation. These measurements are fitted to the standard cosmological model ΛCDM, yielding a global dark matter relic density that makes up 23% of the universe. Visible matter only accounts for 4.6% of the energy content of the universe, while the rest is attributed to dark energy, which we will discuss below. The fit only works for cold dark matter, i.e. dark matter with a low velocity. That means that the known neutrino species cannot explain the dark matter relic density, so we
need a new stable particle, which is only subject to weak and gravitational interactions.

As we have seen in Section 2.1, many SUSY models naturally provide such a particle. The conserved $R$-parity quantum number from Eq. 2.1 ensures that the LSP is stable. In many models, the LSP is the lightest neutralino, which is only weakly interacting and thus a perfect dark matter candidate. In fact, if the LSP had electric or colour charge, we would have detected it already. Thus the MSSM almost automatically provides a dark matter candidate.

One of the advantages of the WMAP measurement in this respect is that it gives us the total amount of dark matter in the universe. We can compare this with the prediction for a given SUSY model, which we can calculate from early-universe thermodynamics and the annihilation cross sections involved. For the other measurements that provide evidence for dark matter, this comparison is more difficult, since the local dark matter relic density can vary considerably [68].

There are many attempts to detect dark matter directly, but no consistent signal has been found\(^3\). Within the context of SUSY, the most stringent limits come from the XENON100 experiment [73], but even these limits leave large parts of the SUSY parameter space unaffected, as e.g. we have shown in Ref. [74].

**Dark Energy**

As we have mentioned in our discussion of the WMAP results, most of the universe consists of dark energy. Whereas we have some possible candidates for dark matter, the nature of dark energy is a complete mystery. It corresponds to the cosmological constant in Einstein’s equations and receives contributions from the vacuum energy, which contains the zero-point energy of all fields. Unfortunately, estimating the zero-point energy from dimensional analysis in the SM yields a result that is about 120 orders of magnitude too large [75]. It might be possible that this contribution cancels against a negative bare cosmological constant, but since this is a completely unrelated variable, the amount of finetuning needed makes the hierarchy problem look like a triviality.

SUSY changes this story, at least to a certain extent. In a global SUSY theory, the vacuum energy exactly vanishes [29]. This is not what we want, but one could imagine that having SUSY and breaking it could protect the cosmological constant and keep it small. Indeed, breaking global SUSY yields a vacuum energy that is somewhat better than in the SM, but it is still many orders of magnitude too large.

In supergravity, however, there are some options to solve the issues. In a curved spacetime, gravity naturally gives a negative contribution to the cosmological constant. Therefore it is conceivable that SUSY is broken in such a way that the positive contribution from the SUSY breaking cancels the negative contribution from gravity. This would still need considerable fine-tuning, but at least there is a possibility that the theory will work without a ridiculously large bare cosmological constant.

\(^3\)The DAMA experiment has been observing a possible dark matter signal for years [69] but it is inconsistent with limits obtained by other experiments [70]. The CRESST collaboration [71] and the CDMS collaboration [72] have recently seen hints of dark matter, but no undisputed signal has been found so far.
2.3 Supersymmetry Searches

In the previous section, we have discussed a number of reasons to consider SUSY as a (broken) symmetry of nature. Of course, SUSY has one major problem: no sparticle has ever been observed. The number of ad hoc parameters needed to obtain a viable model is somewhat disturbing. Yet, no matter how appealing or ugly we as physicists think a concept might be, in the end nature decides. If we want to find out if low-energy SUSY exists in nature, we either have to find it, or exclude it.

As we have seen in the previous section, we would expect the masses of SUSY particles not to exceed a few TeV. That means it should be possible to produce SUSY particles at the LHC. Since the LHC is a hadron collider, one would expect the coloured sparticles to be produced abundantly if their masses are within the energy range of the collider. The squarks and gluinos produced in the hadronic collision decay to other sparticles. The decay cascade ends with the LSP, which escapes the detector undetected. The typical event topology that is searched for at the LHC, is a combination of highly energetic jets and missing transverse energy.

No sign of SUSY has been found yet, but some parts of parameter space are excluded. The Tevatron, with a centre-of-mass energy of \( \sqrt{s} = 1.96 \text{ TeV} \), has set lower limits on squark and gluino masses in the range of 300-400 GeV [76, 77]. The LHC, which has been running with \( \sqrt{s} = 7 \text{ TeV} \) in 2010 and 2011, has extended this limit to around 1 TeV [78, 79]. The LHC is expected to be sensitive to coloured sparticles up to 3 TeV once it reaches its design energy of \( \sqrt{s} = 14 \text{ TeV} \) [80, 81].

It should be noted that these limits are quite model-dependent, since they rely on specific mass patterns and decays of squarks and gluinos. For instance, in order to have an event with highly energetic jets, one needs a considerable mass difference between the coloured sparticles and the LSP. Different mass patterns can easily escape detection, as we have shown for a model with nonuniversal gaugino masses in Ref. [74]. A similar conclusion was reached in the context of the phenomenological MSSM in Ref. [82].

2.4 Squarks and Gluinos

Irrespective of the specific decays of the sparticles, the squark and gluino cross section is an important ingredient in the experimental analyses. Since the SUSY-QCD cross sections heavily depend on the squark and gluino masses, they set the scale for a SUSY discovery as well as the exclusion limits. Although exclusion limits are quite model-dependent, the squark and gluino cross sections are not. With the exception of the stops, which as we have seen in Section 2.1 are subject to left-right mixing, the properties of the coloured sparticles are completely determined by their masses. As we have mentioned in Section 2.1, flavour mixing can in principle occur, but it is highly constrained by measurements in the flavour sector, so we will not consider it here.

Unless otherwise specified, we will use the following notation for the particles in
Chapter 2. Supersymmetry

SUSY-QCD:

\[
\begin{align*}
\text{quark:} & \quad \rightarrow \hspace{1cm} \text{gluon:} \quad \rightarrow \\
\text{squark:} & \quad \rightarrow \hspace{1cm} \text{gluino:} \quad \rightarrow
\end{align*}
\] (2.6)

Neglecting quark masses, and mixing in the quark and squark sectors, the effective SUSY-QCD Lagrangian after SUSY breaking in terms of the gluino fields \( \tilde{g} \) and \( \bar{\tilde{g}} \), the squark fields \( \tilde{q}_j \) and \( \tilde{q}_j^* \), the gluon fields \( g \) and the quark fields \( q_j \) and \( \bar{q}_j \) is given by [29]:

\[
\mathcal{L}_{\text{SUSY-QCD}} = -\frac{1}{4} F^{a \mu \nu} F_{a \mu \nu} + 2 g_s^2 \gamma^\mu (\partial^\mu \delta^{abc} - g_s f^{abc} g_\mu^b) \tilde{g}^c + |D^a \tilde{q}_L|^2 + |D^a \tilde{q}_R|^2 + i \bar{q}_j \gamma^\mu D_\mu q_j \\
- \frac{1}{2} m_{\tilde{g}}^2 \tilde{g}^a - m_{\tilde{q}_L}^2 \tilde{q}_L^a - m_{\tilde{q}_R}^2 \tilde{q}_R^a - \frac{1}{2} g_s^2 (\tilde{q}_L^a T^a \tilde{q}_L^a - \tilde{q}_R^a T^a \tilde{q}_R^a) \\
- \sqrt{2} g_s (\tilde{q}_L T^a \tilde{g}^a \tilde{q}_L + \tilde{q}_L \bar{\tilde{g}}^a T^a \tilde{q}_L - \tilde{q}_R \bar{\tilde{g}}^a T^a \tilde{q}_R - \tilde{q}_R \bar{\tilde{g}}^a T^a \tilde{q}_R) 
\] (2.7)

with \( F^{a \mu \nu} = \partial_\mu g_\nu^a - \partial_\nu g_\mu^a - g_s f^{abc} g_\mu^b g_\nu^c \). Here \( f^{abc} \) are the structure constants of SU(3), \( T^a \) are the generators of the fundamental representation of SU(3) and \( g_s \) is the strength of the strong interaction. The covariant derivative is given by \( D_\mu = \partial_\mu + ig_s T^a g_\mu^a \). Repeated colour indices and the (s)quark flavours \( j \) are summed over. The corresponding Feynman rules for SUSY-QCD are listed in [29].

The most important conclusion we can draw from the Lagrangian (2.7) is that the only parameters that are not yet fixed by the SM are the squark and gluino masses \( m_{\tilde{g}} \) and \( m_{\tilde{g}} \). The SUSY-QCD cross sections strongly depend on the squark and gluino masses and can thus be used to set exclusion limits and, in case SUSY is found, to determine sparticle masses [83]. However, that is only possible if the theoretical predictions for the cross section are sufficiently accurate. Therefore, the aim of the rest of this thesis is to improve the theoretical predictions for squark and gluino cross sections.
Chapter 3

Resummation

If we want to compare experimental SUSY searches to theory, we need good theoretical predictions for sparticle production. As we have seen in the previous chapter, the SUSY-QCD cross sections play a crucial role in SUSY searches. Unfortunately, we do not know how to calculate observables exactly in QFT. Instead, we expand the cross section in terms of the coupling of the theory. The strong coupling $g_s$ is rather large, even at LHC energies. At low energies it becomes so large that the theory becomes non-perturbative and needs an entirely different treatment. Given the precision of present-day colliders, the Leading Order (LO) approximation for QCD cross sections is not good enough, so we have to include at least the next-to-leading order (NLO) corrections.

It turns out that the NLO corrections for SUSY-QCD processes are quite significant [26–28]. The reason for this behaviour is that sparticles are heavy, so they are produced close to their production threshold. In this kinematic regime, the cross section is dominated by logarithms that become large as the velocity of the final-state particles becomes small. The goal of resummation is to sum these logarithms to all orders in perturbation theory. This yields a new perturbative expansion that is better suited for the threshold region.

In this chapter we will first discuss the origin of these large logarithms. Then we will discuss the leading corrections and show that they factorize. Finally we will show how this factorization leads to exponentiation of the logarithms and discuss the perturbative expansion that arises from this procedure.

3.1 Infrared Divergences

The LO matrix elements in $2 \rightarrow 2$ SUSY-QCD processes are quadratic in the coupling, which means they give a $g_s^4$ contribution to the cross section. When it comes to the NLO calculation, we can define two types of Feynman diagrams that give a $g_s^6$ contribution to the cross section. Virtual corrections have two particles in the final state, while real corrections have an additional final-state particle, as shown in Figure 3.1. Both the vir-
3.1. Infrared Divergences

![Virtual and Real Corrections](image)

Figure 3.1: Example of a virtual (a) and a real (b) correction to the $gg \rightarrow \bar{g}g$ process.

virtual and the real corrections have infrared (IR) divergences [84]. In virtual diagrams, they originate from the phase space regime where lines that exchange massless particles become on-shell. Real divergences come from soft and collinear gluon radiation. After phase-space integration, most of these IR divergences cancel. The remaining collinear divergences are absorbed into the parton distribution functions (PDFs).

Divergences have to be defined carefully. Within the context of the SM, one usually uses dimensional regularization [85, 86], where infinities are regularized by setting the number of dimensions to $n = 4 - 2\varepsilon$. After expanding in $\varepsilon$, the divergences appear as $1/\varepsilon$ and $1/\varepsilon^2$ poles at the NLO level. If all contributions are added and are properly renormalized, the IR poles cancel for observables and one can take the $\varepsilon \to 0$ limit. Unfortunately, the divergences leave logarithmic remnants, which are important near threshold. Before we study the origin of these large logarithms, we first introduce the eikonal approximation.

3.1.1 The Eikonal Approximation

Let us take a closer look at gluon radiation. For calculating the IR divergences of soft-gluon radiation, we use the eikonal approximation [87, 88]. This is a valid approximation if the momentum of the gluon is small compared with the momentum of the eikonal line that radiates it. Physically that means that the radiation does not affect the momentum of the eikonal line. Because of its small momentum, the gluon has a very large wavelength and is not sensitive to small length scales. As a result, the structure of the internal process is irrelevant in the eikonal approximation. For instance, consider the diagram in Figure 3.2. If the gluon in the loop has a low energy, it cannot resolve the highly energetic internal quark. Thus the contribution of soft gluons to the loop diagram is suppressed and this diagram does not contribute to the IR divergences. Of course it does contribute to the ultraviolet (UV) divergences originating from high loop momenta, but we are not interested in those. The virtue of the eikonal approximation is that only the external lines are important, so we can ignore the details of the hard, high energy part of the scattering. This simplifies the Feynman rules considerably.

Take a soft gluon with momentum $k$ attached to an eikonal line with momentum $p$. In
the eikonal approximation, we have $k \ll p$, so the propagator that connects the radiated gluon to the rest of the matrix element becomes effectively on-shell. Denoting a generic matrix element by a blob, the generic diagrams and their corresponding Feynman rules are given by [89]:

\[ p \quad \begin{array}{c} a \quad \mu, c \\ \quad \quad \quad k \end{array} b = g_s(T^c_{j})_{ab} \frac{p_\mu}{p \cdot k - i\epsilon} \quad p \quad b \] \quad (3.1)

for an incoming eikonal line and

\[ b \quad \begin{array}{c} a \quad \mu, c \\ \quad \quad \quad k \end{array} p = g_s(T^c_{j})_{ab} \frac{p_\mu}{p \cdot k + i\epsilon} \quad p \quad b \] \quad (3.2)

for an outgoing eikonal line. Here $g_s$ is the strong coupling constant, $\mu$ is the Lorentz index of the gluon and $i\epsilon$ represents the infinitesimal imaginary part of the propagator that connects the matrix element to the radiated gluon. The colour labels of the different particles are denoted by $a, b$ and $c$. The colour-charge operators occurring in Eqs. (3.1) and (3.2) depend on the particle that corresponds to the eikonal line and are given in Table 3.1.

<table>
<thead>
<tr>
<th>Outgoing (s)quark / incoming anti-(s)quark:</th>
<th>$(T^c_{j})<em>{ab} = T^c</em>{ab}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outgoing anti-(s)quark / incoming (s)quark:</td>
<td>$(T^c_{j})<em>{ab} = -T^c</em>{ba} = -(T^c_{ab})^*$</td>
</tr>
<tr>
<td>Gluons / gluinos:</td>
<td>$(T^c_{j})<em>{ab} = F^c</em>{ab} = -if_{abc}$</td>
</tr>
</tbody>
</table>

Table 3.1: Colour-charge operators used in the eikonal Feynman rules.

A notable property of the eikonal Feynman rules is that they do not depend on the energy of the eikonal line. We can see this explicitly by defining a dimensionless vector $v^\mu$ as $p^\mu = Qv^\mu$, where $Q$ is a typical scale of the process of interest. In a $2 \to 2$ process, we typically take $Q^2 = s/2$ with $s$ the CM energy squared. We can then rewrite the eikonal Feynman rules to a scale-invariant form:

\[ p \quad \begin{array}{c} a \quad \mu, c \\ \quad \quad \quad k \end{array} b = g_s(T^c_{j})_{ab} \frac{v_\mu}{v \cdot k - i\epsilon} \quad p \quad b \] \quad (3.3)

\[ b \quad \begin{array}{c} a \quad \mu, c \\ \quad \quad \quad k \end{array} p = g_s(T^c_{j})_{ab} \frac{v_\mu}{v \cdot k + i\epsilon} \quad b \quad p \] \quad (3.4)

3.1.2 Large Logarithms

We are now ready to study the origin of the large threshold logarithms mentioned at the beginning of this section. Due to the finite resolution of detectors, soft and collinear
3.1. Infrared Divergences

gluons cannot be detected, so they have to be integrated out. Now consider a soft gluon that is radiated off an external line $j$. Using the eikonal approximation and splitting the vectors in a temporal and a spatial part, we obtain:

$$ g_s(T_j)_{ab} p^\mu k^0 (1 - |\vec{p}|/\mu_c \cos \theta) + i\epsilon $$

where $\theta$ is the angle between the eikonal line and the radiated gluon. The energy integral has a soft singularity at $k^0 = 0$, which cancels against a similar divergence in the virtual part of the calculation. For a massless eikonal line, we have $p^0 = |\vec{p}|$, and the angular integral gives rise to an additional collinear singularity at $\theta = 0$.

What is the nature of these divergences in a $2 \to 2$ process where an additional gluon with momentum $p_g$ is radiated from one of the external lines? We denote the initial-state momenta by $p_a$ and $p_b$ and the final-state momenta by $p_c$ and $p_d$. Using the eikonal Feynman rules from Section 3.1.1, the square of the matrix element is given by:

$$ \sum_{\text{gluon polarizations}} |M^{ab\rightarrow cdg}|^2 = -g_s^2 \mu^{4-n} \sum_{j,k=[a,b,c,d]} \frac{P_j \cdot P_k}{(p_g \cdot p_j)(p_g \cdot p_k)} (T_j \cdot T_k) \otimes |M^{ab\rightarrow cd}|^2, \quad (3.6) $$

where $\mu$ is the dimensional-regularization scale. The tensor product in colour space $\otimes$ denotes the correct implementation of the colour correlations introduced by radiating a gluon. We will further discuss the colour correlations in Chapter 4. As expected from Eq. (3.5), this expression diverges as $1/E_g^2$ if the gluon becomes soft, i.e. for a gluon energy $E_g \to 0$. For a massless eikonal line, the gluon can become collinear to either $p_j$ or $p_k$, yielding an additional divergence $(1 - \cos \theta)^{-1}$ with $\theta$ the angle between the gluon and the relevant eikonal line$^1$.

To obtain the cross section, we have to combine the matrix element squared with the $n$-dimensional three-particle phase space integration:

$$ \int d\Phi_{cdg} = \int (2\pi)^n \delta^{(n)}(p_g + p_c + p_d - p_a - p_b) \prod_{i=[g,c,d]} \delta_+(p_i^2 - m_i^2) \frac{d^n p_i}{(2\pi)^{n-1}} \quad (3.7) $$

$$ = \int \frac{|\vec{p}_i|^{n-3} E_g^{n-3} \sin^{n-4} \theta_{cg}}{4(2\pi)^{2n-3}} \delta_+(p_i^2 - m_i^2) dE_c d\cos \theta_{cg} d^2 \varphi_{cg} d^2 \Omega_c, $$

where $\delta_+(p_i^2 - m_i^2) = \delta(p_i^2 - m_i^2) \Theta(p_i^0)$. The masses of the final-state particles are denoted by $m_c$ and $m_d$, and in the second line the phase-space integral has been rewritten to an integral over the gluon energy, the angle $\theta_{cg}$ between particle $c$ and the gluon, the corresponding azimuthal angle $\varphi_{cg}$ and the energy $E_c$ and solid angle $\Omega_c$ of the final-state particle $c$, which is defined in $n - 2$ dimensions. In addition to the $E_g^{n-3} \sin^{n-4} \theta_{cg}$ term

$^1$Note that the $(1 - \cos \theta)^{-2}$ contribution vanishes because the gluon cannot become collinear to both $p_j$ and $p_k$ at the same time and the case $j = k$ yields $p_j^2/(p_g \cdot p_j)^2 = 0$ if $p_j$ corresponds to a massless eikonal line.
in the numerator, a non-diverging dependence on \( E_g \) and \( \theta_{cg} \) arises after working out the \( \delta \)-function and performing the \( E_c \)-integral.

The upper limit of the \( E_g \) integral is determined by the distance in energy from the two-particle threshold. It is useful to introduce the threshold parameter \( \beta \), which for a \( 2 \to 2 \) process with partonic CM energy squared \( s \) and two final-state particles with average mass \( m_{av} \) is defined as:

\[
\beta = \sqrt{1 - \frac{4m_{av}^2}{s}}. \tag{3.8}
\]

As we saw in Chapter 2, SUSY-QCD particles are heavy, so they are produced close to their production threshold. Near the two-particle threshold, the maximum energy of the radiated gluon, and thus the available phase space, equals \( E_{\text{max}} = \sqrt{s} - 2m_{av} \approx m_{av}\beta^2 \). Combining Eq. (3.6) and (3.7), the diverging parts of the cross section are given by:

\[
\sigma_{\text{div}} \propto \int_{-1}^{1} \frac{(1 + \cos \theta_{cg})^{-\epsilon} d \cos \theta_{cg}}{(1 - \cos \theta_{cg})^{1+\epsilon}} \int_{0}^{\beta^2 m_{av}/\mu} \frac{d(E_g/\mu)}{(E_g/\mu)^{1+2\epsilon}}. \tag{3.9}
\]

The results of the integrals can be expanded in \( \epsilon \). The energy integral yields a pole in \( \epsilon \), a logarithm of \( \beta^2 \) and a linear term in \( \epsilon \) that contains a double logarithm of \( \beta^2 \). The angular integral gives a \( 1/\epsilon \)-pole and constant terms. The poles cancel against the virtual corrections or are absorbed into the PDFs. What remains are single logarithms for collinear and soft divergences, and a double logarithm for a gluon that is both soft and collinear. Since \( \beta \to 0 \) near threshold, these logarithmic terms are large for heavy particles, which are necessarily produced close to their production threshold.

### 3.2 Factorization

As we saw in the previous section, IR divergences leave logarithmic remnants that become large near threshold. In fixed-order perturbation theory, these logarithms occur with increasing powers as the perturbative order increases, thus spoiling the convergence of the expansion. By resumming these logarithms to all orders in perturbation theory, the behaviour of the perturbative expansion can be improved considerably. The resummation of these logarithms is possible because of the factorization of the different contributions to the cross section.

#### 3.2.1 Parton Distribution Functions

The physical concept underlying factorization is that long-distance behaviour can be separated from short-distance behaviour [90]. This is particularly important in QCD, where due to asymptotic freedom and confinement [91,92] we can use perturbative calculations at high energies, but at low energies we have to deal with non-perturbative quantities.

The most commonly used form of factorization finds its origin in the parton model [93] and states that the PDFs can be decoupled from the hard partonic process. It leads
to an expression for the hadronic cross section $\sigma_{h_1 h_2 \to kl}$ in terms of the hadronic threshold variable $\rho = 4m_{wv}^2 / S$, with $S$ the hadronic CM energy squared:

$$\sigma_{h_1 h_2 \to kl}(\rho) = \sum_{i,j} \int d\hat{x}_1 d\hat{x}_2 d\hat{\rho} \delta \left( \hat{\rho} - \frac{\rho}{x_1 x_2} \right) f_{ij/h_1}(x_1, \mu_F^2) f_{j/h_2}(x_2, \mu_F^2) \sigma_{ij \to kl}(\hat{\rho}, \mu_F^2), \quad (3.10)$$

with $\mu_F$ the factorization scale and $\hat{\rho} = 4m_{wv}^2 / S$ the partonic equivalent of $\rho$. The PDFs $f_{ij/h}(x, \mu_F^2)$ represent the probability of finding a parton $i$ with momentum fraction $x$ of the total hadronic momentum inside hadron $h$. The partonic cross section $\sigma_{ij \to kl}$ can be calculated as a perturbative expansion, while the PDFs are non-perturbative and need to be extracted from data.

Intuitively, Eq. (3.10) can be understood by looking at the different time scales in the process [90]. In the CM frame, the hadrons are Lorentz-contracted with respect to each other and their internal interactions are time-dilated. We are looking at interactions with high energies, so the time it takes for the parton from hadron $h_1$ to traverse hadron $h_2$ is shortened due to the Lorentz contraction, while the internal interactions of hadron $h_2$ take a longer time due to the time dilation. As a result, the hard interaction takes place on much shorter timescales than the interactions within the hadron. Effectively that means that the hadron does not change during the interaction so the partons can indeed be viewed as being specific states with definite momentum.

Nevertheless, separating the PDFs from the hard process is not entirely straightforward. The partonic process is linked to the PDFs through the collinear IR poles that need to be absorbed into the PDFs. We have seen in section 3.1.2 that the soft-collinear divergences give rise to logarithms of $\beta$. Similarly, regularizing a collinear divergence gives rise to a logarithm of the factorization scale. Essentially, the factorization scale defines which gluons are too energetic to be counted as part of the PDF and should be counted as a part of the hard process instead. Therefore, in addition to the poles, we have to move a logarithm of the factorization scale from the partonic cross section to the PDFs.

But now it seems we have arrived at an ambiguity. If we can move around poles and logarithms, how about other terms? In fact, finite parts can always be moved between the hard function and the PDFs [94, 95] as long as they are compensated by a similar change in the partonic cross section. The particular choice of finite terms fixes the renormalization scheme. If you want to compare different cross sections, you have to use the same scheme for all of them. Usually PDFs are defined in the modified Minimal Subtraction (\overline{\text{MS}}) scheme, where in addition to the poles and the logarithms a contribution of $\log(4\pi) - \gamma_E$ is subtracted [84], with $\gamma_E$ Euler’s constant.

### 3.2.2 Further Factorization of the Partonic Cross Section

Although the freedom we have in defining the PDFs might seem like a drawback, it is in fact an asset. If the PDF definition is not suitable for a specific calculation, the higher order corrections can be uncomfortably large. In particular, near threshold we can see from the discussion in Section 3.1.2 that the cross section is dominated by large logarithms
of the form:

$$\alpha_s^n \log^m \beta^2, \quad m \leq 2n, \quad \alpha_s = \frac{g_s^2}{4\pi}, \quad \beta \to 0,$$

which originate from an imperfect cancellation of soft and collinear divergences. It is possible to use the freedom we have in defining the PDFs to further factorize the partonic cross section near threshold [94, 96, 97]. To see how this works, we will look at factorization in a bit more detail.

For convenience, we first turn the convolution of Eq. (3.10) into a completely factorized expression using a Mellin transform, indicated by a tilde:

$$\tilde{\sigma}_{h_1 h_2 \to k l}(N) \equiv \int_0^1 \mathrm{d}\rho \, \rho^{N-1} \sigma_{h_1 h_2 \to k l}(\rho) = \sum_{i,j} \tilde{f}_{i/h_1}(N+1, \mu_F^2) \tilde{f}_{j/h_2}(N+1, \mu_F^2) \tilde{\sigma}_{i j \to k l}(N, \mu_F^2). \quad (3.12)$$

Recently, methods have been developed to perform resummation in $x$-space within the framework of soft-collinear effective theory [98–102]. In this thesis, however, we will use the formalism in Mellin moment space. The logarithmically enhanced terms are then of the form $\alpha_s^n \log^m N, \ m \leq 2n$, with the threshold limit $\beta \to 0$ corresponding to $N \to \infty$.

As we have seen in Section 3.1.2, the NLO IR divergences, and consequently the logarithms, originate from soft and collinear gluons. It can be shown with IR power counting that this is true to all orders [103]. For a $2 \to 2$ process with massless particles, this leads to the schematic picture in Figure 3.3. It shows the leading regions of a hard process $H$ of a given perturbative order with four external jets $\phi_1, \ldots, \phi_4$. The hard part $H$ involves large momentum scales and thus small distances. The jets contain the collinear divergences and consist of on-shell lines with momenta that are proportional to the total momentum of the jet. These lines usually have a large longitudinal momentum, but their transverse momentum is small. Finally, there is a soft subdiagram that contains wide-angle radiation with momenta that are small in all components. The different contributions could in principle exchange momentum through gluons, leading to mixed divergences. This is represented by the connecting gluons in Figure 3.3. Every gluon in Figure 3.3 can stand for multiple gluon insertions with many different possible orderings and any one of them could in principle mix divergences.

Figure 3.3: The leading regions in the amplitude of a $2 \to 2$ process. In principle the soft subdiagram $S$ and the jets $\phi_i$ could be connected to each other and to the hard part $H$ by gluons, leading to a mixing of divergences.
3.2. Factorization

The first step is the decoupling of $H$ and $S$. We have seen before that diagrams such as the one shown in Figure 3.2 do not generate IR divergences at NLO accuracy. This result holds at higher orders: a soft line can only generate an IR divergence if it is connected to an on-shell external line [84]. Thus gluons that connect the soft and the hard part do not contribute to the IR divergences of the given perturbative order of $H$. Within the context of Figure 3.3, where gluons label possible mixing of divergences, that means that the soft and the hard part cannot be connected directly. The physical interpretation of this decoupling is that soft gluons correspond to large length scales, while the hard process takes place at small length scales. Therefore the gluons in the soft function are unable to resolve the internal structure of the hard process.

The second step is decoupling the jets from each other. By definition, all jets move into different directions, so lines in different jets are proportional to different momenta. Since all jets meet at the hard interaction and travel in different directions from there, they cannot rejoin later (or earlier). Thus their collinear divergences do not mix [103] and there is no direct connection between the jets in Figure 3.3.

The leading divergences after these first two steps of factorization are shown in Figure 3.4. As is clear from this picture, there could still be an interplay between the jets and the hard or the soft function. The physical reason for the decoupling of these parts is once again that the wavelengths of the particles in $\phi_j$ and $S$ are not suitable to resolve the substructure of the other parts. The transverse momentum of the gluons in the jets is too small to resolve the small length scales that are relevant in the hard part. However, due to high total momentum of the jets, their substructure cannot be resolved by the soft gluons in $S$.

An actual proof of this decoupling is more subtle than a heuristic argument based on length scales. In the axial gauge, it follows from power counting arguments that the hard part is not connected to the jets by additional gluons [84], but in a general gauge longitudinally polarized gluons spoil this decoupling for individual diagrams. Fortunately, we cannot measure individual diagrams and the sum of all possible diagrams can be shown to vanish at NLO as a result of the Ward identities [105, 106]. This result can be generalized to all orders using Wilson lines, decoupling the jets from the hard part independent of the gauge chosen [107].

A similar procedure can be used to show that the soft function decouples from the jets, but in that case it does not follow immediately from power counting that the connecting gluons can only be longitudinal. In most regions of phase space, the gluon transverse momentum with respect to the jet momentum can easily be neglected compared to its longitudinal momentum, which is typically large for collinear gluons. An exception are the so-called Coulomb or Glauber regions [108], where the gluon momentum becomes small in all components. Fortunately, it turns out that the transverse momentum components
cancel\(^2\) for sufficiently inclusive processes [107]. The remaining longitudinal components of the momentum can be replaced by longitudinally polarized gluons, which cancel after applying the Ward identities [105, 106].

The argumentation so far ensures factorization for sufficiently inclusive processes, but we skipped over an important detail: we are looking at particles that are produced close to their production threshold. Away from threshold, the total cross section is sufficiently inclusive for the transverse components to cancel. Near threshold, however, we have energy restrictions on the gluons. We have already seen in Section 3.1.2 that these energy restrictions lead to large \(\log \beta\) terms in the cross section and in fact they spoil the inclusiveness we needed in the last step. As a result, the soft gluons do not decouple from the rest of the graph and the soft function \(S\) needs to be taken into account in the calculation [94]. Since the soft function involves gluons that connect different external lines, this has considerable implications for the colour flow in the process.

In addition, the usual definition of the PDFs is not particularly suitable for calculations near threshold, since it involves fixing the light-cone momenta of the partons. In light of the energy restrictions on the partons near threshold, it is more natural to fix the partonic energies instead. This redefinition amounts to absorbing additional large logarithms into the PDFs while maintaining factorization [89, 94, 97]. As was mentioned in Section 3.2.1, we always have to use the same PDF definition in order to compare different cross sections to each other. Therefore, in practical calculations, the large logarithms are absorbed into additional jet functions, which contain the difference between the two PDF definitions.

The result is an all-order refactorized partonic cross section in Mellin moment space near threshold [94, 97, 111, 112]:

\[
\bar{\sigma}_{ij \rightarrow kl}(N, \mu^2_F) = J_i(N + 1, \mu^2_F, \mu^2_R) J_j(N + 1, \mu^2_F, \mu^2_R) \times \sum_{JJ} H_{ij \rightarrow kl, JJ}(N, \mu^2_F, \mu^2_R) \bar{S}_{ij \rightarrow kl, JJ}(N, \mu^2_F, \mu^2_R),
\]

where \(\mu_R\) is the renormalization scale. The functions \(J_i\) and \(J_j\) originate from the difference between the PDF definition that we are using and the one that is more suitable near threshold. These functions thus sum the effects of the (soft-)collinear radiation from the incoming partons. They are process-independent and contain the leading logarithmic dependence coming from the soft-collinear gluons, as well as part of the subleading logarithmic behaviour.

The soft function \(\bar{S}_{ij \rightarrow kl, JJ}\) in Eq. (3.13) describes wide-angle soft radiation. It is a matrix in colour-tensor space, with the indices \(IJ\) indicating the colour structure. We need to take the colour content into account explicitly because soft emissions change the colour of an eikonal line. The bar indicates that soft-collinear radiation already included in the \(J_i\) and \(J_j\) factors has been removed to avoid double-counting.

\(^2\)The proof of this cancellation involves light-cone perturbation theory. Since the hard interaction involves large momenta, it has a well-determined location on the light cone. Thus we can identify whether a gluon is radiated before or after the hard interaction. After performing an unweighted sum over all possible light-cone orderings, the transverse components cancel because of a version of the Kinoshita-Lee-Nauenberg theorem [109, 110].
3.3. Exponentiation

The hard function $H_{ij\to kl,II}$ incorporates only higher-order effects of hard, off-shell particles and therefore does not contain any $\log N$ dependence. It is a matrix in colour-tensor space, because the colour content is changed by the soft gluons in $\bar{S}_{ij\to kl,II}$.

Thus factorization leads to a different expression for the partonic cross section in which the large logarithmic corrections are included to all orders. The factorized expression (3.13) is the starting point for threshold resummation.

3.3 Exponentiation

We now have a factorized expression, but we have yet to determine how to calculate its components. A key ingredient in the calculation is that the separate functions in Eq. (3.13) are not invariant under the choice of renormalization scale and gauge, while the cross section is invariant as long as we do not truncate the perturbative series. In addition, because the different functions occur in a product, they are renormalized multiplicatively [97]. Combining this with the conditions of renormalization and gauge invariance, one can show [113] that the logarithms exponentiate in each of the different functions in Eq. (3.13). The expansion of the resulting exponentials has to reproduce the fixed-order calculation, which can thus be used to determine the functions in the exponents. We can, however, calculate the different functions more easily by using their physical interpretation.

As can be concluded from Section 3.2.2, the jet functions $J_i$ and $J_j$ are the ratio of the two PDF definitions. Consequently, they are process-independent and their value is determined by the radiating parton. Since they describe gluon radiation off an external line, the one-loop exponents of the jet functions can be derived from Eq. (3.9). The higher-order exponents follow order by order from the soft singularity in the Altarelli-Parisi splitting function [114] for gluon radiation off the relevant eikonal line.

The soft function $S_{ij\to kl,II}$ is process-dependent, because it describes gluon exchange between different external particles [112, 115]. We could calculate it from the full cross section, but it is easier to use the eikonal approximation and base our calculation on the eikonal cross section instead. In fact, that has several advantages. The eikonal Feynman rules introduced in Section 3.1.1 are not only simpler, they are also scale-invariant. As a result, eikonal cross sections are scale-invariant to all orders. In particular, their divergences do not depend on the scale, so we can obtain the IR divergences directly from the UV divergences, which we can calculate using renormalization-group techniques.

Due to factorization, the different functions in Eq. (3.13) are multiplicatively renormalized [89, 116]. For the matrix-valued soft function, this means:

$$S_{ij\to kl,II}^{(0)} = \left(Z_{S_{ij\to kl}}^{(1)}\right)_{LB}^{LB} S_{ij\to kl,BA}^{LB} \left(Z_{S_{ij\to kl}}^{(1)}\right)_{BA}^{BA},$$

where $S_{ij\to kl}^{(0)}$ is the unrenormalized matrix, which does not depend on the renormalization scale $\mu_R$. The matrix $Z_{S_{ij\to kl}}^{(1)}$ of renormalization constants for the soft function depends on
\[ \mu_R \text{ only through its dependence on the coupling and is of the form} \ \ [117, 118] \]

\[ (Z_{S,i\rightarrow kl})_{LB} = \delta_{LB} + \sum_{n=1}^{\infty} \frac{M_{n, LB}(g_s^n)}{\varepsilon^n}. \]  

(3.15)

The matrices \( M_n \) can be calculated order by order from the eikonal cross section and can be expressed in terms of powers \( g_s^n, i \geq n \). Eq. (3.14) leads to the RGE:

\[ \mu_R \frac{dS_{ij\rightarrow kl, BI}}{d\mu_R} = -\left( \Gamma_{S_{ij\rightarrow kl}}^{\S} \right)_{LB} S_{ij\rightarrow kl, BI} - S_{ij\rightarrow kl, BI} \left( \Gamma_{S_{ij\rightarrow kl}}^{\S} \right)_{AI}. \]  

(3.16)

Here we have introduced the soft anomalous dimension matrix \( \Gamma_S \):

\[ (\Gamma_{S_{ij\rightarrow kl}}^{\S})_{AI} \equiv \mu_R \frac{d(Z_{S_{ij\rightarrow kl}}^{\S})^{AB}}{d\mu_R} (Z_{S_{ij\rightarrow kl}}^{\S^{-1}})_{BI} = \beta(g_s, \varepsilon) \frac{\partial(Z_{S_{ij\rightarrow kl}}^{\S})^{AB}}{\partial g_s} (Z_{S_{ij\rightarrow kl}}^{\S^{-1}})_{BI}. \]  

(3.17)

**Example: one-loop soft anomalous dimension**

At one loop, the QCD beta function follows from the renormalization of the strong coupling in \( 4 - 2\varepsilon \) dimensions [84]:

\[ \beta(g_s, \varepsilon) = \mu_R \frac{dg_s}{d\mu_R} = -\varepsilon g_s + \beta(g_s) \]  

(3.18)

with \( \beta(g_s) \) the one-loop QCD beta function in 4 dimensions. Looking at Eq. (3.17), it seems that the \( \beta(g_s) \) term in Eq. (3.18) yields a \( 1/\varepsilon \) pole on the right-hand side, even though the left-hand side is finite. This apparent contradiction can be resolved by realizing that higher-order corrections to the beta function contain \( 1/\varepsilon \) poles as well. In order for Eq. (3.18) to be valid, these poles will have to cancel the \( \beta(g_s) \) contribution to Eq. (3.17). As a result, only the \( \varepsilon \) part of Eq. (3.18) is left and we can safely take the limit \( \varepsilon \rightarrow 0 \). In that limit, the inverse of the matrix of renormalization constants reduces to \( (Z_{S_{ij\rightarrow kl}}^{\S^{-1}})_{BI} = 1 \), cf. Eq. (3.15) and we obtain:

\[ (\Gamma_{S_{ij\rightarrow kl}}^{\S})_{AB} = -\varepsilon g_s \frac{\partial(Z_{S_{ij\rightarrow kl}}^{\S})^{AB}}{\partial g_s} = -g_s \frac{\partial}{\partial g_s} \text{Res}(Z_{S_{ij\rightarrow kl}}^{\S AB}(g_s, \varepsilon)), \]  

(3.19)

where in the last step we take the residue of the \( 1/\varepsilon \)-pole. This expression is based on the complete one-loop eikonal cross section, so it still contains the soft-collinear radiation that is already contained in the jet functions. This radiation can be removed by subtracting the anomalous dimensions for the eikonal jets [112], thus obtaining an anomalous dimension that describes the wide-angle soft radiation.

Once the soft anomalous dimension is known, we can in principle use it to calculate the soft function. Unfortunately, the RGE (3.16) is a matrix-valued equation. Its solution is a path-ordered exponential, which is not particularly practical to work with. It turns out that this problem can be solved by working in the so called \( s \)-channel colour basis,
which traces the colour flow through the s-channel. In the s-channel basis, all matrices in Eq. (3.13) are diagonal at threshold [119]. As a result, we do not have to worry about colour correlations and we can simply project all the functions on the irreducible representations that occur in the s-channel colour decomposition. We will discuss this procedure in detail in Chapter 4. Putting everything together, we obtain an expression for the resummed cross section near threshold:

\[
\tilde{\sigma}_{ij \rightarrow kl}^{(n)}(N, \mu_F^2) = \Delta_i(N + 1, \mu_F^2, \mu_R^2) \Delta_j(N + 1, \mu_F^2, \mu_R^2) \times \sum_l \tilde{\sigma}_{ij \rightarrow kl,l}^{(0)}(N, \mu_F^2) C_{ij \rightarrow kl,l}(N, \mu_F^2, \mu_R^2) \Delta_{ij}^{(s)}(N, \mu_F^2) \Delta_{kl}^{(s)}(N, \mu_F^2),
\]

where \( \tilde{\sigma}_{ij \rightarrow kl,l}^{(0)} \) are the colour-decomposed LO cross sections in Mellin-moment space, which depend on \( \mu_R \) through their dependence on the coupling. The sum runs over all irreducible representations \( I \) in the s-channel colour decomposition. The functions \( \Delta_i \) and \( \Delta_j \) sum the effects of the (soft-)collinear radiation from the incoming partons, while the function \( \Delta_{ij}^{(s)}(N, \mu_F^2) \) describes the wide-angle soft radiation. In contrast to the \( J_i \) and \( S \) functions in Eq. (3.13), the \( \Delta \) functions in Eq. (3.20) are defined in such a way that they only contain the exponentials mentioned at the beginning of this section. All factors multiplying these exponentials are absorbed in the LO cross sections \( \tilde{\sigma}_{ij \rightarrow kl,l}^{(0)} \) and the matching coefficients \( C_{ij \rightarrow kl,l} \). The latter are constructed such that they yield the appropriate fixed-order expression when multiplied with \( \tilde{\sigma}_{ij \rightarrow kl,l}^{(0)} \). In the next section, we will discuss this in more detail in the context of the new perturbative series that arises from Eq. (3.20).

### 3.4 A New Perturbative Series

In the previous section, we have seen how factorization leads to resummation. The resulting Eq. (3.20) provides a way of resumming large logarithmic corrections containing \( L \equiv \log(N) \) to all orders. From this equation, a new perturbative expansion arises that combines the usual expansion in powers of \( \alpha_s \) with an expansion for large \( L \). Schematically, the exponentiation of soft-gluon radiation takes the form [94, 111]

\[
\Delta_i \Delta_j \Delta_{ij}^{(s)} \exp \left[ L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots \right].
\]

This exponent captures all dependence on the large logarithm. The \( g_n \) functions are polynomials that start at the first power in \( \alpha_s L \). Keeping only the \( g_1 \) term in Eq. (3.21) constitutes the leading logarithmic (LL) approximation. Including also the \( g_2 \) term is called the next-to-leading logarithmic (NLL) approximation. For the next-to-next-to-leading logarithmic (NNLL) approximation also the \( g_3 \) term needs to be taken into account. Both the \( g_2 \) and the \( g_3 \) term are colour-dependent.

The relation between this new perturbative expansion and fixed-order perturbation theory is shown schematically in Figure 3.5. Comparing the expansion of Eq. (3.21) to Figure 3.5, we can see the role of the matching coefficients \( C_{ij \rightarrow kl,l} \). For the LL and NLL calculations, the matching coefficients equal unity. For NNLL accuracy, we need
to include the NLO contributions to correctly account e.g. for the terms of the form $\alpha_s(\alpha_s L^n)$, $n \geq 0$ in Figure 3.5. Such higher-order corrections are incorporated in the matching coefficients $C_{ij \rightarrow kl,1}$, which contain the Mellin moments of the higher-order contributions without the log($N$) terms. At NNLL accuracy, they can be obtained from the NLO cross section near threshold (cf. Chapter 5).

As we can see from Eq. (3.21), the $g_i$ functions are determined by the expansion of the soft radiation factors $\Delta_i$ and $\Delta^{(s)}$. If the calculation is performed in an $s$-channel colour basis, the function $\Delta_i$ is given by [97, 112, 120]:

$$\log \Delta_i(N, \mu_F^2, \mu_R^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \int_{\mu_F^2}^{4m_{av}^2(1-z)^2} \frac{dq^2}{q^2} A_i(\alpha_s(q^2)),$$  \hspace{1cm} (3.22)

while the function $\Delta^{(s)}$ consists of a colour-independent part $D_i$ for each incoming particle and a colour-dependent part $D_{ij \rightarrow kl,1}$:

$$\log \Delta^{(s)}_{ij \rightarrow kl,1}(N, \mu_R^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[ D_{ij \rightarrow kl,1}(\alpha_s(4m_{av}^2(1-z)^2)) + D_i(\alpha_s(4m_{av}^2(1-z)^2)) \right] \hspace{1cm} (3.23)$$

These functions depend on the renormalization scale $\mu_R$ through their dependence on the coupling $\alpha_s$. Their components are usually given as an expansion:

$$F(\alpha_s(q^2)) = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n F^{(n)}, \hspace{1cm} F = \{A_i, D_i, D_{ij \rightarrow kl,1}\}. \hspace{1cm} (3.24)$$

Comparing Eqs. (3.22) and (3.23) to the expansion in Figure 3.5, we see that at LL accuracy, we only need $A_i$ to first order. At NLL accuracy, we need $A_i$ to second order and both $D$ functions to first order. For the NNLL calculation, we need the third order of $A_i$ and the second order of the $D$ functions.
The resummed cross section of a given order in this new perturbative series is no longer exact and thus depends on the renormalization scale. As in fixed-order calculations, this dependence introduces an additional theoretical uncertainty. However, due to the reorganization of the large logarithms, this uncertainty is typically smaller than in fixed-order calculations.

### 3.5 Inverse Mellin Transform and Matching

The expression in Eq. (3.20) is still in Mellin-moment space. In order to obtain a useful result, we have to go back to $x$-space by performing an inverse Mellin transform. Although this is straightforward in principle, in practice it is a somewhat tricky procedure due to the Landau pole in the running coupling. When performing the $N$-integration, one has to avoid integrating over the Landau pole. Also, subleading corrections have to be treated with care. Strictly speaking these corrections are not part of the logarithmic approximation, but they can give rise to a diverging power series. This introduces spurious divergences that lead to a significant arbitrariness in the result. A solution to both problems is the ‘minimal prescription’ presented in Ref. [121]. The trick is to use an integration contour that crosses the real axis between the Landau pole and the spurious corrections and thus avoids both types of singularities.

In addition, we would like to retain the information contained in the known fixed-order cross sections. This can be done by combining the fixed-order and resummed results through a matching procedure that avoids double counting of the fixed-order terms:

$$
\sigma_{h_1 h_2 \rightarrow k l}^{\text{(matched)}}(\rho, \mu_R^2, \mu_F^2) = \sigma_{h_1 h_2 \rightarrow k l}^{\text{(fixed order)}}(\rho, \mu_R^2, \mu_F^2) + \sum_{i,j} \int_{\text{CT}} \frac{dN}{2\pi i} \rho^{-N} f_{i|h_1}(N + 1, \mu_F^2) f_{j|h_1}(N + 1, \mu_R^2) \times \left[ \tilde{\sigma}_{ij \rightarrow kl}^{\text{(res)}}(N, \mu_F^2, \mu_R^2) - \tilde{\sigma}_{ij \rightarrow kl}^{\text{(res)}}(N, \mu_F^2, \mu_R^2) \bigg|_{\text{fixed order}} \right],
$$

where the integration is over the contour CT described above. From Eq. (3.25) it is clear that we need the PDFs in $N$-space. There are several ways to obtain $N$-space PDFs from the standard $x$-space parametrizations. For the numerical results in later chapters, the method introduced in Ref. [122] has been used.

### 3.6 Coulomb Corrections

The logarithmic terms are not the only important contribution near threshold. Another source of large effects is formed by the Coulomb corrections, which correspond to the exchange of gluons between the slowly moving massive particles in the final state. These corrections are bound-state effects, which are important for non-relativistic particles.

The Coulomb corrections can be summed to all orders by either using a Sommerfeld factor [123–126] or by employing the framework of non-relativistic QCD, where bound-state effects can be included as well [126–131]. In addition, a formalism has been
developed in the framework of effective field theories that allows for the combined resummation of soft and Coulomb gluons in the production of coloured sparticles [119, 132]. In these last papers, it was shown that the NLO correction is the dominant contribution to the Coulomb resummation.

When including Coulomb corrections, however, one has to realize that such bound-state effects can be screened by the finite lifetime of the final-state particles. In SUSY-QCD, the lifetimes of squarks and gluinos vary considerably depending on the sparticle spectrum. As a result, the effect of the Coulomb corrections is model-dependent, making it difficult to draw general conclusions on its magnitude.

In this thesis we will not resum the Coulomb corrections. However, as can be seen from Figure 3.5, the NLO Coulomb contribution enters the NNLL approximation through the matching coefficient $C_{ij \rightarrow kl,j}$ in Eq. (3.20).

3.7 Summary

In this chapter, we have introduced the concepts used in soft-gluon resummation. We have first shown how an imperfect cancellation of IR divergences leads to logarithmic corrections that become large in the threshold region. Then we have introduced the concept of factorization and shown how it leads to a factorization of the cross section in terms of a hard process, and soft and collinear corrections. We then discussed how this form of the cross section naturally leads to exponentiation and resummation.

The result of these steps is the threshold resummation formula in Mellin-moment space (3.20). It leads to a new perturbative expansion, which is particularly suited for the threshold region. This brings us back to the reason we started this investigation. Since SUSY particles are heavy, they are necessarily produced close to their production threshold. Resummation can thus improve the predictions for SUSY cross sections.

Now that we have introduced the basic concepts and equations, we are ready to apply threshold resummation to SUSY-QCD cross sections. This will be the topic of the next chapters.
Chapter 4

Colour and NLL Resummation

The main result of the previous chapter is the threshold resummation formula (3.20). In this chapter, we will discuss its application to NLL resummation for SUSY-QCD. The ingredients we need for this are the colour-decomposed LO cross section in $N$-moment space and the NLL $g_1$ and $g_2$ functions from Eq. (3.21). As mentioned in Section 3.4, the matching coefficients are unity at NLL accuracy.

The $g_1$ function is colour-independent, while the $g_2$ function consists of a colour-independent part that corresponds to the jet functions $\Delta_i$ and $\Delta_j$ in Eq. (3.20), and a colour-dependent part that describes the wide-angle soft radiation contained in $\Delta_{ij}^{(s)}$. At NLL accuracy the colour-dependent part follows from the one-loop soft anomalous dimension matrices.

As discussed in Chapter 3, colour correlations need to be taken into account for processes involving pair-production of coloured particles. To this end, an appropriate colour basis has to be chosen. The most convenient choice is an $s$-channel colour basis, which traces the colour flow through the $s$-channel and has the virtue of rendering the anomalous dimension matrices diagonal at threshold [89, 119]. Both the cross section and the anomalous dimension matrices have to be written in terms of this basis.

The emphasis of this chapter will be on obtaining the colour decomposition. We will first discuss how to construct an $s$-channel colour basis and derive the bases needed for the SUSY-QCD processes. We will then derive the colour-decomposed LO cross sections and soft anomalous dimensions. Finally, we will show some numerical results for NLL resummation and discuss the implications for experiments.

4.1 Colour Bases

In this section we will discuss how to construct an $s$-channel colour basis. This basis is obtained by performing an $s$-channel colour decomposition of the reducible two-particle product representations into irreducible ones. This is a procedure we are familiar with in the context of a system of two spin-$\frac{1}{2}$ particles. A single spin-$\frac{1}{2}$ particle is described by
the fundamental representation of SU(2), which is two-dimensional. Combining two spin-
\(1/2\) particles leads to a four-dimensional product representation that can be decomposed
into two irreducible representations: a singlet corresponding to the spin-0 state and a
triplet corresponding to the spin-1 state.

In the case of SUSY-QCD, the relevant group is SU(3). For sake of generality, the
analytical results will be presented for a general SU\((N_c)\) theory, with \(N_c\) the number of
colours. The quarks and the squarks are in the \(N_c\)-dimensional fundamental representa-
tion, while the gluons and the gluinos are in the adjoint representation, which has dimen-
sion \(N_c^2 - 1\).

In order to study the \(s\)-channel colour decomposition in SU\((N_c)\), we will first study
the relevant product representations. Then we will introduce the notation of colour-charge
operators and derive the colour bases for the SUSY-QCD processes.

### 4.1 Product Representations

For a scattering process, we can define two product representations. The first is based
on the representations of the incoming particles, while the second is based on the out-
goings. Both these product representations can be decomposed into irreducible
representations. Since a physical process cannot change colour content halfway, the ir-
reducible representations of the incoming and outgoing product representations have to
match. For instance, in SU(3) the product representation of two gluons can yield a ten-
dimensional irreducible representation. However, combining a squark and an antisquark
can only yield a singlet and an octet, so the ten-dimensional representation cannot con-
tribute to the \(g\bar{g} \rightarrow \tilde{q}\tilde{q}\) process.

In physics, representations of SU\((N_c)\) are usually denoted by their dimension. The dimen-
sions of the irreducible representations in the product decompositions can be obtained
using Young tableaux. This has been worked out for the relevant product representations
in Appendix A. The colour decompositions for the product representations corresponding
to the squark-gluino production processes are given by:

\[
q\bar{q} \rightarrow \tilde{q}\tilde{q} : \quad 1 \oplus (N_c^2 - 1),
\]

\[
gg \rightarrow \tilde{q}\tilde{q} : \quad 1 \oplus (N_c^2 - 1)_A \oplus (N_c^2 - 1)_S,
\]

\[
q\bar{q} \rightarrow \tilde{g}\tilde{g} : \quad 1 \oplus (N_c^2 - 1)_A \oplus (N_c^2 - 1)_S,
\]

\[
gg \rightarrow \tilde{g}\tilde{g} : \quad 1 \oplus (N_c^2 - 1)_A \oplus (N_c^2 - 1)_S \oplus (N_c^2 - 1)(N_c^2 - 4)/4 \oplus (N_c^2 - 1)(N_c^2 - 4)/4
\]
\[\oplus N_c^2(N_c + 3)(N_c - 1)/4 \oplus N_c^2(N_c - 3)(N_c + 1)/4,\]

\[
qq \rightarrow \tilde{q}\tilde{q} : \quad N_c(N_c - 1)/2 \oplus N_c(N_c + 1)/2,
\]

\[
qg \rightarrow \tilde{g}\tilde{g} : \quad N_c \oplus N_c(N_c + 1)(N_c - 2)/2 \oplus N_c(N_c - 1)(N_c + 2)/2.
\]

In SU(3), the \(N_c(N_c - 1)/2\)-dimensional representation for the \(qq \rightarrow \tilde{q}\tilde{q}\) process coincides
with the antifundamental representation \(\overline{3}\).
The notation for the $gg \rightarrow \tilde{g}\tilde{g}$ colour decomposition (4.4) requires some explanation. First, the labels $\mathbf{A}$ and $\mathbf{S}$ refer to different symmetry properties of the two adjoint representations in the decomposition. Second, most of the representations in the $gg \rightarrow \tilde{g}\tilde{g}$ decomposition are real. The only exceptions are the two representations with dimension $(N_c^2 - 1)(N_c^2 - 4)/4$, which are each others conjugates. By definition, the barred representation is the most antisymmetric of the two. Finally, note that the last representation is zero-dimensional in SU(3) and thus vanishes for the case of SUSY-QCD.

In principle, Eqs. (4.1) - (4.6) define the colour decomposition. However, for practical applications we need the explicit expressions of the base tensors belonging to these irreducible representations. Before we can derive these expressions, we first need to introduce some notation.

### 4.1.2 Colour-Charge Operators

Using the notation of [133], the colour-charge operator $T_j$ follows from the Feynman rule for gluon emission off a particle $j$. Its value for the relevant particles in SUSY-QCD is given in Table 3.1. The operators satisfy the colour-charge algebra:

$$T_i \cdot T_j = T_j \cdot T_i, \quad T_j^2 = C_2(R_j),$$

where $C_2(R_j)$ is the quadratic Casimir invariant belonging to the representation $R_j$ of particle $j$. For the particles in SUSY-QCD, it is given by:

$$C_2(R_q\bar{q}) = C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_2(R_{g,\bar{g}}) = C_A = N_c.$$

(4.8)

Due to colour conservation, we have for a matrix element $M$:

$$\sum_j T_j M = 0.$$  

(4.9)

For a general coloured $2 \rightarrow 2$ process, this colour conservation property can be represented diagrammatically as:

![Diagram](image)

(4.10)

Throughout this chapter, a plain line can denote any particle in SUSY-QCD. Depending on the identity of the particle, the appropriate colour-charge operator can be inserted. In addition to the generators of the fundamental representation $T_{ab}^c$ and the completely antisymmetric structures $F_{ab}^c = -if_{abc}$, we will also need the traceless symmetric octet structures $D_{ab}^c = d_{abc}$ and the singlet colour structures $\delta_{ab}$. The colour labels of the octet structures always have dimension $N_c^2 - 1$, but those of the singlet can belong to particles in either the fundamental or the adjoint representation, so the labels can have dimension $N_c$ or $N_c^2 - 1$.  

37
4.1.3 Explicit Construction of a Colour Basis

We are now ready to construct explicit expressions for the colour basis. Some of the base tensors can be obtained directly from the colour structure of the LO $s$-channel diagrams. An example is the diagram for the $qg \rightarrow \bar{q}g$ process shown in Figure 4.1. Since the quark exchanged in the $s$-channel is in the fundamental representation, we can immediately read off from this diagram that the corresponding $N_c$-dimensional base tensor is $(T^{a_4} T^{a_2})_{a_3 a_1}$.

Unfortunately, most base tensors do not have such a direct physical interpretation, so we need a more general method. The first step is to find a complete basis for a given process. This is easily done by connecting the indices of all external particles in the process with the colour structures from Section 4.1.2 and using them to remove redundancies. For the process $g(a_1)g(a_2) \rightarrow \bar{g}(a_3)\bar{g}(a_4)$, this leads to the independent colour structures:

$$
\begin{align*}
\delta_{a_1 a_2} & \delta_{a_3 a_4} \\
(D^{a_1} D^{a_2})_{a_3 a_4} & (D^{a_1} D^{a_3})_{a_4 a_2} \\
(F^{a_1} D^{a_2})_{a_3 a_4} & (D^{a_1} F^{a_2})_{a_3 a_4}
\end{align*}
$$

Comparing this to Eq. (4.4), it seems we have one structure too many. However, it will turn out that the (complex) structures in the last line of Eq. (4.11) only occur in the combination $(F^{a_1} D^{a_2} + D^{a_1} F^{a_2})_{a_3 a_4}$. In SU(3), there is an additional identity, which relates the symmetric structures to each other, reducing the number of structures to six.

Once we have a complete set of colour structures, we can construct an $s$-channel basis. As the base tensors correspond to irreducible representations, they have to meet three requirements: orthogonality, proper normalization and correct behaviour under the quadratic Casimir operator. Let us discuss these requirements in more detail.

A base tensor corresponds to an irreducible representation, so it has to project all colour structures onto the same invariant subspace. In particular, it should be self-projective and orthogonal to all other base tensors. To implement this requirement, we define an inner product of two colour tensors $c_I$ and $c_J$ based on how one calculates the square of a matrix element. This leads to the condition:

$$
c_I \cdot c_J \equiv c_I(a_1, a_2, a_3, a_4) c_J^*(a_1, a_2, a_3, a_4) = \text{dim}(c_I) \delta_{IJ}. \quad (4.12)
$$

The last equality in Eq. (4.12) fixes the normalization as well as the orthogonality. For our calculations we could choose an arbitrary normalization, but we will normalize the tensors such that the trace yields the dimension of the base tensor.

In addition to being orthogonal and properly normalized, base tensors have to be eigenvectors of the quadratic Casimir operator of the product representation. Denoting
the colour-charge operators of the two initial-state particles as $T_1$ and $T_2$, the quadratic Casimir operator of the product representation is usually written in the compact form $(T_1 + T_2)^2$. Explicitly writing out the indices, we have the condition:

$$
\left( (T_1)_{\alpha_1\beta_1}^{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} + (T_2)_{\alpha_2\beta_2}^{\alpha_2\beta_2} \delta_{\alpha_1\beta_1} \right) \left( (T_1)_{\alpha_3\beta_3}^{\alpha_3\beta_3} \delta_{\alpha_4\beta_4} + (T_2)_{\alpha_4\beta_4}^{\alpha_4\beta_4} \delta_{\alpha_3\beta_3} \right) = C_2(R_I)c_I(a_1, a_2, a_3, a_4).
$$

(4.13)

Diagrammatically, we have in colour-space:

$$
\left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T_1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T_2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \right)^2 \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
c_I
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_2(R_I)
\end{array}
\end{array}
\end{array}
\end{array}
\end{array},
\end{array}
\end{array}

(4.14)

with $C_2(R_I)$ the quadratic Casimir invariant of the representation $R_I$ that corresponds to the base tensor $c_I$. Working out the left-hand side of Eq. (4.14) gives:

$$
(T_1 + T_2)^2 \otimes c_I = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{c}_I
\end{array}
\end{array}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{c}_I
\end{array}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{c}_I
\end{array}
\end{array}
\end{array}
\end{array}.
\end{array}
\end{array}

(4.15)

Inserting this into Eq. (4.14) provides both the value for the quadratic Casimir invariant and an additional constraint for the base tensor.

The combined constraints of orthogonality, normalization and the quadratic Casimir invariant fix the base tensors up to a phase. The resulting base tensors and the corresponding quadratic Casimir invariants for the SUSY-QCD processes can be found in Appendix B. This procedure is equivalent to the methods presented in [119] and [134] for processes that contain particles with the same representations in the initial state and in the final state.

### 4.2 The Leading Order Cross Section

Now that we have complete colour bases for all SUSY-QCD processes, we can decompose the LO cross section in terms of the different irreducible representations. To obtain the colour-decomposed LO cross section, we start from the full LO matrix element. In this section, we will first discuss some technicalities concerning the Majorana nature of gluinos and the ghost contributions to processes involving gluons. We will then continue with the colour decomposition and a discussion on channels that are suppressed near threshold. We will finish with the Mellin transformation of the LO cross sections.

#### 4.2.1 Majorana Particles

Since gluinos are Majorana particles, they can violate fermion number conservation. This is immediately clear from the diagrams for squark-pair production in Eq. (C.13), but in
4.2. The Leading Order Cross Section

that case it does not complicate the calculation. The situation is different for the \( q \bar{q} \rightarrow \tilde{g} \tilde{g} \) process. Consider the \( t \)- and \( u \)-channel diagrams from Eq. (C.18). If we draw the fermion lines in these diagrams, we obtain:

\[
P_1 \quad P_3 \quad P_4
\]

\[
P_2 \quad P_4 \quad P_2 \quad P_3
\]

where the momenta \( p_1 \) and \( p_2 \) are defined as incoming and \( p_3 \) and \( p_4 \) as outgoing. The interference term between these two diagrams yields terms of the form:

\[
|\mathcal{M}|^2_{\text{int}} = \bar{u}(p_3) \Gamma_1 u(p_1) \bar{v}(p_2) \Gamma_2 v(p_4) \bar{v}(p_3) \Gamma_3 v(p_2) \bar{u}(p_1) \Gamma_4 u(p_4),
\]

where the \( \Gamma_i \) stand for some combination of Dirac matrices, in this case either the unit matrix or \( \gamma^5 \). It is clear that this type of interference terms does not automatically have the usual \( \bar{u}u \) and \( \bar{v}v \) combinations one would prefer when calculating a cross section. Fortunately, we can use the charge conjugation matrix, which in the Dirac representation satisfies \( C = i \gamma^0 \gamma^2 = -C^T \), to relate [84]:

\[
u = C \bar{v}^T, \quad \bar{u} = -\bar{v}^T C^{-1}, \quad \bar{v} = -u^T C^{-1},
\]

and rewrite Eq. (4.17) to a more practical form. For example, we have:

\[
\bar{u}(p_1) \Gamma u(p_4) = u^T(p_4) \Gamma^T \bar{u}(p_1) = -\bar{v}(p_4) C \Gamma^T C^{-1} v(p_1) = (-1)^{n_y} \bar{v}(p_4) \Gamma_{\text{rev}} v(p_1),
\]

where \( n_y \) is the number of Dirac matrices occuring in \( \Gamma \), while \( \Gamma_{\text{rev}} \) contains the original combination of the Dirac matrices in reversed order. By working out all the combinations, we can bring the interference term to the desired form.

4.2.2 Ghost Subtraction

When the initial state contains gluons, the sum over the initial-state polarizations yields a rather lengthy expression. However, we know that most terms should cancel in the end result due to gauge invariance. In the simpler case of Quantum Electrodynamics (QED), the sum over the polarization vectors effectively yields

\[
\sum_{\text{polarizations}} \epsilon^\mu(p_1) \epsilon^{\nu*}(p_1) \rightarrow -g^\mu\nu
\]

due to the Ward identities. In QCD, we can use ghost subtraction to simplify the LO matrix element and thus the entire calculation. For instance, look at the process (C.24):

\[
g(p_1, a_1, \mu) g(p_2, a_2, \nu) \rightarrow \tilde{g}(p_3, a_3) \tilde{g}(p_4, a_4),
\]
where the labels indicate the momentum, the colour index, and if applicable the Lorentz index of the particles respectively. We now write the corresponding matrix element as:

\[ \mathcal{M} = \varepsilon_1^\mu(p_1)\varepsilon_2^\nu(p_2)T_{\mu\nu}, \]  

(4.22)

and realize that we can add any linear combination of \( p_1^\mu \) and \( p_2^\nu \)-terms to the matrix element, since these terms vanish upon contraction with the polarization vectors. In particular, we can remove all terms proportional to \( p_1^\mu \) and \( p_2^\nu \) from \( T_{\mu\nu} \) in this way, effectively removing the ghost contributions. This modifies the Ward identity \( p_1^\mu T_{\mu\nu} \sim p_2^\nu \) such that it yields zero, as it does in QED. The LO matrix element for the \( gg \to \tilde{g}\tilde{g} \) process (4.21) then becomes:

\[ \mathcal{M}_{gg\to\tilde{g}\tilde{g}} \to \varepsilon_1^\mu(p_1)\varepsilon_2^\nu(p_2) \left[ \frac{1}{p_2\cdot p_4}(F^{a_1}F^{a_2})_{a_1a_4} + \frac{1}{p_1\cdot p_4}(F^{a_2}F^{a_1})_{a_3a_4} \right] \times ig_2^2\bar{u}(p_3) \left[ \frac{p_2\cdot p_4}{s}(\gamma_\nu\gamma_\mu - 2p_4\gamma_\gamma_\mu) + \frac{p_1\cdot p_4}{s}(\gamma_\mu\gamma_\gamma_\nu - 2p_4\gamma_\gamma_\mu) \right] v(p_4), \]  

(4.23)

where the first line is the colour part and the second line contains the Lorentz structure. The polarization sum simply yields \( -g^{\mu\nu} \) as in Eq. (4.20). Ghost subtraction makes life easier even at LO, but a true appreciation of the procedure only comes when doing an NLO calculation. Since the LO matrix element is an ingredient for the NLO virtual corrections, the calculation is simplified considerably.

### 4.2.3 Colour Decomposition

For soft-gluon resummation, we need colour-decomposed LO cross sections in Mellin-momentum space. We can use the projective properties of our colour bases to obtain the required colour decompositions. The orthogonality of the base tensors ensures that for arbitrary colour structures we can write:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{colour_diagram.png} \\
= \sum_i \frac{1}{\text{dim}(R_i)} \includegraphics[width=0.1\textwidth]{base_tensor.png} \times \includegraphics[width=0.1\textwidth]{base_tensor.png}.
\end{array}
\]  

(4.24)

The colour decomposition of a conjugate matrix element can be found by taking the complex conjugate of this equation (effectively swapping \( c_I \) and \( c_J^\dagger \)). For instance, the colour structure of \( \mathcal{M}_{gg\to\tilde{g}\tilde{g}}^* \), which can be obtained by taking the complex conjugate of Eq. (4.23), can be decomposed in terms of the base tensors (B.12)-(B.18) as:

\[ \mathcal{M}_{gg\to\tilde{g}\tilde{g}}^* \big|_{\text{colour}} = -\frac{1}{2p_2\cdot p_4}(F^{a_2}F^{a_1})_{a_1a_4} - \frac{1}{2p_1\cdot p_4}(F^{a_1}F^{a_2})_{a_3a_4} \]

(4.25)

\[ = -\frac{1}{2p_2\cdot p_4} \left( N_c c_{gg\to\tilde{g}\tilde{g},1} + \frac{N_c}{2} c_{gg\to\tilde{g}\tilde{g},2} + \frac{N_c}{2} c_{gg\to\tilde{g}\tilde{g},3} - c_{gg\to\tilde{g}\tilde{g},6} + c_{gg\to\tilde{g}\tilde{g},7} \right) \]

\[ - \frac{1}{2p_1\cdot p_4} \left( N_c c_{gg\to\tilde{g}\tilde{g},1} + \frac{N_c}{2} c_{gg\to\tilde{g}\tilde{g},2} - \frac{N_c}{2} c_{gg\to\tilde{g}\tilde{g},3} - c_{gg\to\tilde{g}\tilde{g},6} + c_{gg\to\tilde{g}\tilde{g},7} \right). \]
The orthogonality of the base tensors ensures that we only need the colour decomposition of $M^*$, since upon multiplication with $M$ the correct colour structures are projected out automatically. Although this remark is practically of no consequence at LO, it will simplify the calculation of the NLO virtual corrections in Section 5.3.1 significantly. The colour-decomposed LO cross sections for SUSY-QCD can be found in Appendix C.

4.2.4 Vanishing Cross Sections

Near threshold, the $s$-wave contribution to the final state dominates. Higher values of the final-state orbital angular momentum quantum number $L_{\text{fin}}$ are suppressed by higher powers of $\beta$. Thus a cross section can be regarded as being “suppressed” near threshold if the $L_{\text{fin}} = 0$ mode is not accessible due to symmetry properties. In this section, we will first list these general symmetry considerations and then apply them to the different processes.

If the initial or final state consists of two identical particles, the symmetry properties are determined by the eigenvalue $P$ of the permutation operator that interchanges the two particles:

$$P = \mathcal{P}_i S_c (-1)^{L+1} = \begin{cases} +1 & \text{for identical bosons,} \\ -1 & \text{for identical fermions.} \end{cases}$$

(4.26)

Here $\mathcal{P}_i$ is the intrinsic parity quantum number of the particle pair, $S$ the total spin, $S_1$ and $S_2$ the spins of the individual particles and $S_c$ the colour symmetry factor, which is $+1$ for a symmetric colour state and $-1$ for an antisymmetric colour state. In addition, we will use the conservation of parity $\mathcal{P}$ and of $\mathcal{C}\mathcal{P}$ in SUSY-QCD. Finally, because we have $L_{\text{fin}} = 0$ for a nonsuppressed cross section, the conserved angular momentum quantum number equals the final-state spin $J = S_{\text{fin}}$.

$qq \to \bar{q}q$

For the $qq \to \bar{q}q$ process, the contribution from the antisymmetric colour structure is suppressed near threshold if the produced squarks have the same flavour. In that case, we have a system with identical particles in the initial state. We thus know that the initial state should be antisymmetric since quarks are spin-$\frac{1}{2}$ particles. Also, since squarks are scalars, the total spin of the final state is $S_{\text{fin}} = 0$. We need $L_{\text{fin}} = 0$ for a nonsuppressed threshold cross section, so the conserved angular momentum quantum number is $J = 0$, which means $L_{\text{fin}} = S_{\text{fin}}$. Inserting this into Eq. (4.26) for the initial-state quarks, we see that the exponent is always odd. Since the quark pair has positive intrinsic parity, we only have an antisymmetric state if the colour structure is symmetric. Indeed, for equal flavours the antisymmetric colour structure is suppressed near threshold.

$gg \to \bar{q}q$

In a similar way, we can show that for the $gg \to \bar{q}q$ process only the symmetric colour structures contribute near threshold. In this case the gluons need to be in a symmetric
state, since they are spin-1 particles. As with the previous process, we have \( J = 0 \) based on the final state. Using Eq. (4.26) we can see that the exponent is always even, so also in this case only the symmetric colour states can yield a nonsuppressed contribution near threshold.

\[ gg \rightarrow g\bar{g} \]

For the \( gg \rightarrow g\bar{g} \) process, we need the \( P \) invariance of the matrix element as an additional ingredient. Due to their Majorana nature, the gluinos have an intrinsic parity of \( \pm i \) [135]. Thus the gluino pair has a negative intrinsic parity. If we combine this with the fact that we need \( L_{\text{fin}} = 0 \), the parity of the final state is \( P = -1 \). In order to conserve parity, we need \((-1)^{L_{\text{in}}} = -1\), so \( L_{\text{in}} \) must be odd. The final state gives \( J = S_{\text{fin}} \), which can be 0 or 1. In addition, we can use parity conservation to determine the initial-state spin. Denoting the initial-state three-momenta by \( \pm \vec{p} \) and the corresponding spin coordinates by \( \sigma_{1,2} \), we have:

\[
P|g(\vec{p}, \sigma_1), g(-\vec{p}, \sigma_2)\rangle = |g(-\vec{p}, \sigma_1), g(\vec{p}, \sigma_2)\rangle = |g(\vec{p}, \sigma_2), g(-\vec{p}, \sigma_1)\rangle
\]

\[
= -|g(\vec{p}, \sigma_1), g(-\vec{p}, \sigma_2)\rangle,
\]

(4.27)

where the first step is the application of the parity operator, the second step uses the commutation properties of bosons and the last equal sign follows from parity conservation and the fact that the final state has negative parity. It follows from Eq. (4.27) that the initial-state gluons must be in an antisymmetric spin state, so \( S_{\text{in}} = 1 \). According to Eq. (4.26), the initial state should be symmetric under the exchange of the gluons. With an antisymmetric spin and spatial state, this is only possible if the colour structure is symmetric. Indeed, the antisymmetric colour structures are suppressed near threshold.

\[ q\bar{q} \rightarrow g\bar{g} \]

For the \( q\bar{q} \rightarrow g\bar{g} \) process, we use \( CP \)-invariance in addition to parity conservation. First note that gluinos are Majorana particles and therefore unaffected by \( C \). Thus the same argumentation as for the \( gg \rightarrow g\bar{g} \) process shows that we have \( CP = CP = -1 \) based on the final state. Also, we know from the final state that \( J = S_{\text{fin}} \) can be 0 or 1. However, since the \( q\bar{q} \) pair has negative intrinsic parity, in this case \( L_{\text{in}} \) must be even in order to conserve parity. We can also draw conclusions on the initial-state spin based on \( CP \)-conservation:

\[
CP|q(\vec{p}, \sigma_1), \bar{q}(\vec{p}, \sigma_2)\rangle = -|\bar{q}(\vec{p}, \sigma_1), q(\vec{p}, \sigma_2)\rangle = |q(\vec{p}, \sigma_2), \bar{q}(\vec{p}, \sigma_1)\rangle
\]

\[
= -|q(\vec{p}, \sigma_1), \bar{q}(\vec{p}, \sigma_2)\rangle,
\]

(4.28)

where the first step is the application of the \( CP \) operator, which yields a minus sign due to the intrinsic parity of the \( q\bar{q} \) pair. In the second step, the anticommutation properties of the fermions introduces an additional minus sign, while the last equality implements \( CP \)-conservation. According to Eq. (4.28), the quarks must be in an antisymmetric spin state, so \( S_{\text{in}} = 0 \). Combining this with our earlier conclusions that \( J = 0 \) or 1 and \( L_{\text{in}} \) is
4.3 Soft Radiation Factors

even, the only nonsuppressed state has $J = L_{\text{fin}} = 0$. Because of the conservation of angular momentum, this also means $J = S_{\text{fin}} = 0$, so the gluinos are in an antisymmetric spin state. Inserting the final-state quantum numbers into Eq (4.26) shows that the colour state of the gluino pair must be antisymmetric, which is indeed what we observe near threshold.

The suppressed channels only contribute terms that are threshold-suppressed by higher powers of $\beta$ compared to the other channels. Since the symmetry arguments are quite general, they remain valid at higher orders. As a result, the suppressed channels remain suppressed compared to the other channels when including higher-order corrections. This observation does not affect NLL resummation, but it will be important in the NNLL calculation in Section 5.3.

4.2.5 Mellin Transforms

After obtaining the colour-decomposed LO cross sections in $\beta$-space, we need to perform the Mellin transformations according to Eq. (3.12) to obtain the corresponding expressions in $N$-space. For nonsuppressed cross sections, the leading term of the LO cross section in $N$-space is of the form:

$$4m_{\text{av}}^2 \int_0^1 \frac{1}{\Delta} \frac{z^{N-1} \beta}{s} = \int_0^1 \frac{1}{\Delta} \frac{z^{N} \sqrt{1-z}} = \frac{\sqrt{\pi} \Gamma(N+1)}{2\Gamma(N + \frac{5}{2})},$$

(4.29)

with $z = 4m_{\text{av}}^2/s$. For the threshold limit in Mellin-moment space, i.e. the large-$N$ limit, this expression is proportional to $N^{-3/2}$. An additional power of $\beta$ corresponds to an additional power of $N^{-1/2}$ in the large-$N$ expansion, thus yielding terms that are suppressed with powers of $N^{-1/2}$ compared to the leading terms in $N$-space. The Mellin transforms of the full colour-decomposed LO cross sections are listed in Appendix D.

4.3 Soft Radiation Factors

The final ingredients we need for NLL resummation are the functions $\Delta_{i,j}$ and $\Delta^{(s)}_{ij \rightarrow kl, I}$ in Eq. (3.20). The $\Delta_{i,j}$ functions are defined in Eq. (3.22), while the function $\Delta^{(s)}_{ij \rightarrow kl, I}$ is given in Eq. (3.23). At NLL accuracy we need the $A_i$ function to two-loop order and the $D$ functions to one-loop order. The two-loop expansion (3.24) of $A_i$ is given by [136]:

$$A_i^{(1)} = T_i^2, \quad A_i^{(2)} = \frac{1}{2} T_i^2 \left[ \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} n_f \right],$$

(4.30)

with $n_f = 5$ the number of light flavours and the colour charge operator $T_i$ as defined in Section 4.1.2. The colour-independent $D_i$ function vanishes at one-loop order. The colour factors $D^{(1)}_{ij \rightarrow kl, I}$ follow from the one-loop soft anomalous dimensions, which we will turn to now. As we have seen in Section 3.3, the soft anomalous dimensions are matrices in colour space that are determined by the poles in the eikonal cross section. That means we need to calculate the eikonal diagrams for all possible gluon connections, as shown in Figure 4.2.
The calculation is performed using the eikonal Feynman rules in Section 3.1.1. It consists of two components: the colour algebra and the kinematic part. Let us start with the colour decomposition. We essentially use the same procedure as in Section 4.2.3, except that in this case the end result is a matrix in colour space rather than a vector. First, the diagrams in Figure 4.2 have to be calculated for each of the colour structures \( C_j \) in Eqs. (4.1) - (4.6). The results are then colour-decomposed according to Eq. (4.24), with the blob containing the sum of the diagrams in Figure 4.2 for a particular colour structure. The result of this calculation is a matrix in colour space, with each of the entries corresponding to a combination of two irreducible representations for a given process: \( C_j \) from Figure 4.2 and \( c_f \) from Eq. (4.24).

The explicit form of the \( 1/\varepsilon \)-poles that determine \( Z_{S_{ij\rightarrow{kl}}} \) follows from the kinematic part of the diagrams in Figure 4.2. In particular, we need to integrate over the momentum of the virtual gluon. The calculations are usually performed in the axial gauge with gauge fixing vector \( n^{\mu} \). Then the gluon propagator is given by:

\[
\Pi_s^{\mu\nu}(p, a, b) = \frac{i}{p^2 + i\epsilon} \left( -g^{\mu\nu} + \frac{n^{\mu}p^{\nu} + p^{\mu}n^{\nu}}{n \cdot p} - n^2 \frac{p^{\mu}p^{\nu}}{(n \cdot p)^2} \right) \delta_{ab}, \tag{4.31}
\]

with \( a \) and \( b \) the colour labels and \( p \) the gluon momentum. The eikonal integrals required for the SUSY-QCD processes can be found in Refs. [89, 115, 134]. Inserting the expressions for the UV poles into the colour-decomposed matrix yields the matrix of renormalization constants of the soft function \( Z_{S_{ij\rightarrow{kl}}} \). The soft anomalous dimension matrix can be calculated from \( Z_{S_{ij\rightarrow{kl}}} \) using Eq. (3.19).

Although the combination of the soft and collinear functions in the cross section is gauge invariant, the functions themselves are not automatically separately gauge invariant. Specifically, if we use Eq. (4.31) for the gluon propagator, the soft anomalous dimensions explicitly depend on the gauge fixing vector \( n^{\mu} \). This issue is closely related to the double-counting of soft-collinear radiation mentioned near Eq. (3.13) and so is its solution.

We know that the collinear functions, and in particular their gauge-dependent part, only depend on the colour representations of the incoming partons. Therefore the gauge dependence of the soft function cannot depend on the colour structure of the process either. This implies that we can make the soft and collinear functions separately gauge invariant.
4.4. Numerical Results

We discuss the impact of NLL resummation on squark and gluino production. The purpose of this section is to give a general idea of the importance of the effects. We have presented the full numerical analysis that this section is based on in Refs. [134, 138]. For $\tilde{q}\tilde{q}$ and $\tilde{g}\tilde{g}$ production, the results agree with the analysis first presented in Refs. [137, 139]. Also, agreement was found with the results obtained within the formalism of soft-collinear effective theory in Ref. [140].
We will focus on the LHC with a CM energy of $\sqrt{S} = 7$ TeV. The results for the LHC at higher energies are qualitatively similar provided the final-state sparticle masses are changed accordingly. This is not true for the Tevatron, since it is a proton-antiproton collider. Close to threshold, the valence quarks can be important, sometimes leading to different results.

We will compare the LO, NLO and NLO+NLL matched results, where the full NLO result is combined with the NLL resummed cross section using Eq. (3.25). The LO and NLO cross sections [26–28] are available in the form of the public computer code PROSPINO [141]. As will be explained in detail in Section 5.1, the $\overline{\text{MS}}$-scheme with five active flavours is used to define $\alpha_s$ and the PDFs at NLO. The masses of the squarks and gluinos are renormalized in the on-shell scheme, and the top quark and the SUSY particles are decoupled from the running of $\alpha_s$ and the PDFs.

The production of stops has to be treated separately because of potentially large mixing effects and mass splitting. We have performed NLL resummation for stop production in Ref. [142], but we will not consider it here. For the production of the other squarks, we sum over the five squark flavours and the two chiralities. We also include the charge-conjugated processes. The renormalization and factorization scale are taken to be equal $\mu_R = \mu_F \equiv \mu$ and we limit ourselves to the case of equal squark and gluino masses $m_\tilde{q} = m_\tilde{g}$. The results for unequal masses are in general qualitatively similar.

Both the NLO and NLO+NLL cross sections are obtained with the 2008 NLO MSTW PDFs [143, 144] and the corresponding $\alpha_s(M_Z) = 0.120$. Although this procedure is common practice in top-quark physics, one might be worried about using NLO PDFs for an NLL calculation. In fact, the difference between NLO and NLL PDFs should not be large, since their evolution contains the same divergences [145]. Thus, the only potential problem is fitting the data to either NLO or NLL predictions. The difference between the two calculations is most pronounced in the large $x$ region, where the NLO calculation is not reliable due to large threshold corrections. For this reason, large $x$ data are typically not taken into account in the current PDF fits [146]. This results in large PDF uncertainties near threshold, but also accommodates theoretical consistency when using the NLO PDFs in an NLL calculation.

We first look at the behaviour of the scale dependence of the cross section. Since the choice of scale is arbitrary to some extend, ideally the cross section should not depend on it. Of course, in practice we can only calculate the cross section up to a certain order in perturbation theory, so some scale dependence remains. Including higher-order corrections should reduce the scale dependence and thus stabilize the cross section.

In Figure 4.3 we show the scale dependence in LO, NLO and NLO+NLL for the squark and gluino production processes for a squark and gluino mass of 700 GeV. We observe a significant reduction of the scale dependence when going from LO to NLO. Including the NLL corrections stabilizes the results even further, particularly for the processes involving gluinos. The reason for this behaviour is that gluinos contribute large colour factors in the resummation formula (3.20), so the effect of soft-gluon radiation is more pronounced than for the case of squarks.

Conventionally, the scale uncertainty is quantified by varying the scale by a factor of 2 around its central value, which is usually taken to be the average mass of the produced...
4.4. Numerical Results

![Graphs showing scale dependence of LO, NLO, and NLO+NLL cross sections for different squark and gluino productions.](image)

Figure 4.3: The scale dependence of the LO, NLO and NLO+NLL cross sections for the individual squark and gluino pair-production processes with $m_{\tilde{q}} = m_{\tilde{g}} \equiv m = 700$ GeV.
particles. Although it cannot be interpreted as a confidence interval, the scale uncertainty does provide a reasonable measure for our ignorance due to higher-order corrections. The scale uncertainty as a function of the sparticle mass is shown in Figure 4.4, which shows a comparison of the NLO and the NLO+NLL scale uncertainty for the sum of the SUSY-QCD production processes. Threshold resummation leads to a significant reduction of the scale dependence over the full range of sparticle masses, with an overall scale uncertainty at NLO+NLL of less than 10%. The reduction is more pronounced for higher masses, when the particles are produced closer to their production threshold.

In addition to the scale uncertainty, we need the value of the cross section at the central scale. In order to isolate the effect of the NLL resummation, we define an NLL $K$-factor as:

$$K_{\text{NLL}} = \frac{\sigma_{\text{NLO}+\text{NLL}}}{\sigma_{\text{NLO}}}.$$  

(4.35)

In Fig. 4.5, we show the enhancement of the cross section due to the NLL resummation at the central scale. For comparison, we show both the results for the LHC at 7 TeV CM energy and for the Tevatron.

![Figure 4.4](image)

Figure 4.4: The scale dependence of the NLO and NLO+NLL SUSY-QCD cross sections as a function of the sparticle mass $m \equiv m_{\tilde{q}} = m_{\tilde{g}}$. The upper two curves correspond to a scale of $\mu = m/2$, the lower two curves to $\mu = 2m$.

![Figure 4.5](image)

(a) LHC with 7 TeV CM energy.

(b) Tevatron with 1.96 TeV CM energy.

Figure 4.5: The NLL $K$-factor for inclusive squark and gluino pair-production at the central scale as a function of the sparticle mass $m \equiv m_{\tilde{q}} = m_{\tilde{g}}$.

At the LHC, we see that the $K$-factor increases as the final-state sparticle mass in-
creases and the sparticles are produced closer to threshold. We see a particularly significant effect for the processes involving gluinos, where we find an increase of 10-20% at masses around 1 TeV. The behaviour at the Tevatron is slightly different for several reasons. First, the LHC is essentially a gluon collider due to the high proton energy, while at the Tevatron the valence quarks are more important. Second, the mass range in the Tevatron plot extends to values closer to threshold, putting even more emphasis on the valence quarks for higher masses. In particular, the behaviour of the $K$-factor for the gluino pair-production process at the Tevatron results from the increased importance of the $q\bar{q}$ initial state. In general the Tevatron $K$-factors are larger because the particles are produced closer to threshold.

In a full experimental analysis, a good grasp of the uncertainties involved is as important as knowing the central value. There are actually several sources of theoretical uncertainties in a cross section calculation. We have already discussed the scale uncertainty, which is a measure for the uncertainty due to missing higher-order corrections. In addition, there is the PDF uncertainty, which follows from the multidimensional fit of the PDFs to the data. Finally there is the uncertainty in the value of $\alpha_s$, which affects both the partonic cross section and the PDF fit.

We have added the 68% confidence level PDF and $\alpha_s$ uncertainties in quadrature, and combined this linearly with the scale uncertainty. There are some remarks to be made about this procedure. First, in the MSTW PDF set, $\alpha_s$ is part of the fit, so the $\alpha_s$ and PDF uncertainties are not independent. Thus, strictly speaking they cannot be added in quadrature. We have chosen to do so anyway, since in practice the difference between the official MSTW prescription and adding the uncertainties in quadrature is small. Secondly, as mentioned before, the scale uncertainty cannot be interpreted in terms of a confidence level and should not be treated as such. Therefore we combine it linearly with the other uncertainties. However, it is important to keep in mind that although the total theory uncertainty quantifies the confidence we have in our calculation, it is not in fact a confidence interval.

In Figure 4.6 we plot the full theory uncertainty for the different channels. We find that even though the PDF uncertainty is significant, the inclusion of threshold resummation leads to a sizeable reduction of the overall theory uncertainty. This is particularly true for the case of gluino-pair and squark-gluino production. For gluino-pair production, the total theory uncertainty can be reduced by as much as a factor of two when going from NLO to NLO+NLL. In general, the overall theory uncertainty at NLO+NLL is approximately 20% or smaller.

4.5 Experimental Implications

Cross sections strongly depend on the masses of the produced particles. Thus, if SUSY is found, improved cross section predictions can help determine sparticle masses [83, 147]. As long as we have not seen any sign of SUSY, NLL predictions can be used to improve exclusion limits of squark and gluino masses.
Figure 4.6: The theoretical uncertainties for the individual squark and gluino pair-production processes as a function of the sparticle mass $m_{\tilde{q}} = m_{\tilde{g}} = m$. Included are the scale uncertainty, the uncertainty in the PDF fit and the uncertainty in $\alpha_s$. All graphs have been normalized to the cross section at the central value $\sigma_0$. 
4.5. Experimental Implications

The translation from the numerical results in Section 4.4 to an experimental analysis is no trivial procedure. In particular, we have only presented inclusive cross sections. In a realistic analysis, many cuts are applied to reduce the background from SM processes. This could be a problem if threshold resummation changes the shape of the relevant distributions. We have investigated this issue in the context of stop production in Ref. [142] and found that the $p_T$-distribution becomes somewhat softer, as shown in Figure 4.7. However, we see that the difference between the NLO calculation and the NLO+NLL matched result is small, so rescaling the entire cross section with a $K$-factor is a good approximation.

From Section 4.4, we expect two effects when using NLL predictions instead of the conventional NLO calculation. First, as we have seen in Figure 4.5, NLL resummation increases the prediction of the central value of the cross section. The second effect comes from the reduction in scale uncertainty shown in Figures 4.4 and 4.6.

An estimate of the impact of using the NLL cross section is shown in Figure 4.8. It is based on results from the DØ collaboration [76] and shows the theoretical predictions for the number of observed events depending on the mass. In the same plot, the observed limit is shown. From the increase in the cross section at the central value, we would expect to see more events for a given SUSY mass. Since we have not seen any events so far, that
means we can exclude higher masses with the same experimental data. However, when
determining exclusion limits, one always wants to be conservative. Thus, rather than the
central value, the most important quantity is the lower edge of the uncertainty band in
Figure 4.8. Since NLL resummation reduces the scale uncertainty, the band becomes
smaller. As a result the lower edge moves upwards, yielding an even larger improvement
of the exclusion limit. The combined effect leads to an exclusion limit of the order of
10 GeV better than the one based on the NLO prediction.

We have performed a more extensive analysis comparing the NLO and NLL predic-
tions for data from the CDF collaboration. This analysis also includes a parameter scan
in the squark-gluino mass plane and shows that the exclusion limits are indeed improved
by using the NLL predictions [148].

4.5.1 NLL-fast

For use at the LHC, we have made the numbers available as a grid with an interpolation
routine [149]. The grid covers a wide range of relevant squark and gluino masses. It
includes the central value and the scale uncertainty for the LO, NLO and NLO+NLL
matched predictions. In addition, it contains the NLO 68% confidence level PDF and $\alpha_s$
uncertainties. For a CM energy of 7 TeV, the numbers are obtained using the MSTW PDF
set, while for an 8 TeV CM energy the numbers are provided for both the MSTW-2008
and the CTEQ6.6 [150] PDF sets. The NLL-fast code is currently used in analyses by the
CMS collaboration, see e.g. [151], and the ATLAS collaboration, see e.g. [152].

4.6 Summary

We have discussed NLL resummation for SUSY-QCD pair-production processes. We
have presented explicit analytical results for the anomalous dimension matrices and the
colour-decomposed LO cross sections in $x$- and $N$-space. The emphasis of this chapter
was on the colour decomposition needed for resummation. To this end, we have first
discussed how to construct an s-channel colour basis and then colour-decomposed the
LO cross sections and soft anomalous dimension matrices in terms of this colour basis.
Finally, we have discussed the NLO+NLL matched numerical predictions for all pair-
production processes of coloured sparticles at the LHC with a CM energy of 7 TeV.

The inclusion of NLL corrections leads to a reduction of the scale dependence over
the full mass range that will be probed by the LHC. In addition, they increase the pre-
diction for the cross section at the central scale. The effect of soft-gluon resummation
is most pronounced for squark-gluino production and gluino-pair production, reaching
approximately 12% and 27% respectively for particle masses around 1.2 TeV.

The results presented in this chapter are available in the NLL-fast code, which pro-
vides a grid and an interpolation routine of NLL resummed cross sections for SUSY-QCD
processes. This code is used by both the ATLAS and the CMS experiment and is thus the
current standard in SUSY searches.
Chapter 5

NNLL Resummation

In the previous chapter, we discussed NLL resummation in SUSY-QCD. We saw that including NLL corrections reduces the theoretical uncertainties and increases the size of the cross section at the central scale. The next step in obtaining more accurate predictions is to also include the NNLL contributions. In this chapter, we will present the ingredients needed for this calculation. The emphasis will be on the analytical calculations needed for NNLL resummation, although we will also show some numerical results.

In order to take the next step in resummation, we first go back to the basics. If we look at the threshold resummation formula (3.20), we see that the new ingredients needed for NNLL resummation are the soft radiation factors at NNLL accuracy and the matching coefficients. The matching coefficients contain the Mellin moments of the higher-order contributions without the log(N) terms. To NNLL accuracy, this nonlogarithmic part of the higher-order cross section near threshold factorizes into a part that contains the leading Coulomb correction \( C_{\text{Coul},(1)}^{\text{NNLL}} \) mentioned in Section 3.6, and a part that contains the NLO hard matching coefficients \( C^{(1)} \) [132]:

\[
C_{ij\rightarrow kl,\mu}(N,\mu_F^2,\mu_R^2) = \left( 1 + \frac{\alpha_s}{\pi} C_{ij\rightarrow kl,\mu}^{\text{Coul,(1)}}(N,\mu_R^2) \right) \left( 1 + \frac{\alpha_s}{\pi} C_{ij\rightarrow kl,\mu}^{(1)}(\mu_F^2,\mu_R^2) \right).
\] (5.1)

The matching coefficients follow from the threshold limit of the NLO cross section. Therefore we will start by briefly reviewing NLO calculations in SUSY-QCD. We will then discuss the Coulomb contributions and continue with the calculation of the hard matching coefficients. Finally, we will briefly discuss the soft radiation factors and present some numerical results.

5.1 NLO Calculations for SUSY-QCD

The complete NLO calculations for SUSY-QCD processes were presented in Refs. [26–28]. The results presented in those papers are our starting point for calculating the matching coefficients, so we will briefly discuss the main steps in the NLO calculation.
5.1. NLO Calculations for SUSY-QCD

As mentioned in Section 3.1, the NLO contribution consists of real and virtual corrections. The virtual corrections yield tensor integrals, which can be reduced to a known set of scalar integrals [153] using Passarino-Veltman reduction [154]. These scalar integrals contain the virtual IR divergences. This reduction method yields Gram determinants in the denominator. These determinants vanish in certain parts of phase space and thus need to be treated with care. Since the virtual diagrams correspond to a two-particle final state, the phase space integration is the same as for the LO calculation.

For the real corrections, the phase space integration is an issue. The phase space integrals are generally too complicated to perform analytically, but a numerical implementation is not straightforward due to the IR divergences. The solution to this problem is to isolate the divergences and calculate them analytically. The remaining finite part of the integral can then be computed numerically. There are several ways to accomplish this. The authors of Ref. [28] use phase space slicing, which introduces a cut-off parameter to separate soft and hard gluon radiation. Present-day calculations usually use subtraction methods such as Catani-Seymour dipole subtraction [133, 155] or one of its generalizations. The advantage of such methods is that the diverging contributions are subtracted at the integrand level, so no cut-off needs to be introduced. Subtraction methods provide a systematic way of treating real IR divergences that can be extended to more complicated final states and even to NNLO calculations. We will discuss dipole subtraction in more detail when calculating the hard matching coefficients.

As we have mentioned in Section 3.1, the real and virtual $1/\varepsilon$-poles mostly cancel and the remaining collinear divergences are absorbed into the PDFs. Existing PDF sets are defined in the $\overline{\text{MS}}$ scheme. For each parton $a$ this leads to a collinear counterterm of the form [155, 156]:

\[
\frac{d\sigma^{\text{LO}}_{\alpha}}{d^2 \mathbf{p}_a} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \sum_b \int_0^1 dz \frac{1}{\varepsilon} \left( \frac{4\pi m_{\text{av}}}{z^2 \mu_F^2} \right)^\varepsilon P_{ab}^{\text{alt}}(z) d\sigma^{\text{LO}}_{b}(zp_a) \tag{5.2}
\]

where in the second step we expanded in $\varepsilon$ to obtain the usual $\overline{\text{MS}}$ subtraction term (cf. Section 3.2.1). The functions $P_{ab}^{\text{alt}}(z)$ are the Altarelli-Parisi splitting functions [114], that describe the transition from particle $a$ to particle $b$ when radiating another particle.

After combining the real and virtual corrections and adding the collinear counterterm, all IR divergences cancel. However, the UV divergences associated with the masses and the strong coupling $\alpha_s$ still need to be renormalized. The masses are renormalized in the on-shell scheme when calculating the self-energies. The coupling is renormalized in the $\overline{\text{MS}}$ scheme and heavy particles are decoupled from its running by subtracting logarithms of their masses in addition to the poles. The $\overline{\text{MS}}$ scheme is an obvious choice, because it is used in existing QCD calculations and thus in the measurements of $\alpha_s$. Unfortunately, it introduces a problem in SUSY.

In dimensional regularization, everything is defined in $n$ dimensions. As a result, the gluon has $n - 2$ degrees of freedom, while the gluino still has 2. Thus dimensional regularization explicitly violates SUSY and is not suitable for SUSY calculations.
appropriate scheme is modified Dimensional Reduction (DR), in which fields are defined in four dimensions and momenta in $n$ dimensions. No additional degrees of freedom are introduced when going to $n$ dimensions and SUSY is preserved [157,158]. In fact, we can combine the best of both worlds, so we can use the SM measurements, but also preserve SUSY. It turns out that the SUSY breaking in the $\overline{\text{MS}}$ scheme results in a finite shift in the bare squark-quark-gluino coupling. If we take this shift into account, we can perform the calculation in the $\overline{\text{MS}}$ scheme but still be consistent with SUSY.

The UV divergences were the last ingredient in the NLO calculation for SUSY-QCD. If we expand the resulting cross section in terms of $\beta$, we obtain a constant term and terms that are proportional to positive powers of $\beta$. The constant term in the cross section is the leading Coulomb correction, which enters the Coulomb matching coefficient in Eq. (5.1). The linear term in $\beta$ determines the hard matching coefficient, which we will discuss in Section 5.3. Higher powers of $\beta$ are suppressed near threshold and do not contribute to the matching coefficient.

5.2 Coulomb Contributions

As we have mentioned in Section 3.6, the large logarithms are not the only sizeable contribution near threshold. Another source of large corrections are the Coulomb corrections, which are due to the exchange of gluons between the slowly moving massive sparticles in the final state. The terms in the NLO cross section which give rise to the Coulomb corrections $C_{\text{Coul}}^{(1)}$ in $N$-space do not have the usual phase-space suppression $\propto \beta$, in view of the Coulombic $1/\beta$ enhancement factor. As a result, these terms give a finite contribution at threshold.

In order to calculate the Coulomb corrections $C_{\text{Coul}}^{(1)}$ in $N$-space, we need to know the Coulomb part of the NLO correction in terms of $\beta$. The Coulomb corrections factorize from the rest of the cross section and can be derived from the Coulomb potential [123-126]:

$$\sigma_{ij \rightarrow kl,1}^{\text{Coul},(1)} = -\frac{\alpha_s \pi}{4} \sqrt{\frac{m_{l} m_{l}}{m_{q}^2}} \frac{1}{\beta} \left( C_{2}(R_{l}) - T_{k}^{2} - T_{l}^{2} \right) \sigma_{ij \rightarrow kl,0}^{(0)} . \tag{5.3}$$

The quadratic Casimir invariants $C_{2}(R_{l})$ of all the relevant representations are listed in Appendix B, while the colour operators $T_{k}^{2}$ for SUSY-QCD are given in Eq. (4.8). The Mellin transform $\tilde{\sigma}_{\text{Coul},(1)}$ of Eq. (5.3) is presented in Appendix D. The Coulomb matching coefficient $C_{\text{Coul},(1)}^{C}$ can be obtained by dividing $\tilde{\sigma}_{\text{Coul},(1)}$ by the Mellin transform of the LO cross section, which is listed in Appendix D as well.

We have already mentioned in Section 3.6 that the Coulomb effects can be screened by the width of the sparticles, which is highly model-dependent. The strength of this screening can be estimated from calculations of below-threshold bound-state effects, where the smearing due to finite-width effects can be as large as 50% [129]. For consistency we will stick to the approach adopted in the NLO calculations, where this screening is not taken into account. However, it is important to keep in mind that the size of these corrections can be reduced considerably if the produced sparticles have a large decay width.
5.3 Hard Matching Coefficients

In this section we will discuss the hard matching coefficients \( C^{(1)} \) at one loop. They are determined by the terms in the \( \beta \)-expansion of the NLO cross section that are proportional to \( \beta, \beta \log(\beta) \) and \( \beta \log^2(\beta) \). As we have discussed in Section 4.2.5, terms that contain higher powers of \( \beta \) are suppressed by powers of \( N^{-1/2} \) in Mellin-moment space and do not contribute. In particular, the symmetry arguments discussed in Section 4.2.4 remain valid at higher orders, so the hard matching coefficients of suppressed cross sections only contribute to \( N^{-1/2} \)-suppressed terms. Therefore we take them to be 0.

The NLO hard matching coefficients contain both virtual and real contributions. In this section, we will first discuss the virtual corrections and then the real corrections. Finally, we will combine the results and perform the Mellin transforms.

5.3.1 Virtual Corrections

To obtain the virtual part of the hard matching coefficients, we first need to colour-decompose the NLO virtual correction and then expand it in \( \beta \). For the first step we only need the colour decomposition of the LO matrix element. According to Eq. (4.24), the full matrix element squared is then automatically colour-decomposed due to the orthogonality of the \( s \)-channel colour basis:

\[
|M|_{\text{virt},I}^2 = 2 \text{Re}(M_{\text{NLO}} M_{\text{LO},I}^*).
\]  

We are now left with an expression in terms of masses, Mandelstam variables and scalar integrals. Since we need the cross section to \( O(\beta) \), we have to expand \( |M|^2 \) to zeroth order in \( \beta \).

The most negative power of \( \beta \) in the scalar integrals comes from the Coulomb integrals, which as we saw in Section 5.2 contain a \( 1/\beta \)-divergence. Thus the factors that multiply the scalar integrals need to be expanded to \( O(\beta) \). Due to the Gram determinants, however, these factors contain fake \( 1/\beta \) singularities. For the \( q\bar{q}, g\bar{q} \) and \( g\bar{g} \) production processes, no such singularities are left after expanding the Mandelstam variables to the appropriate order. For the processes involving gluinos, some singularities remain. Fortunately, these terms can be shown to vanish upon phase space integration, so we only need to expand the scalar integrals to zeroth order in \( \beta \).

This simplifies the calculation of the threshold limit for the scalar integrals considerably. To zeroth order in \( \beta \), the two outgoing particles are at rest in the CM frame, so the only nonzero component in their momentum four-vector is the energy component, which is equal to the mass of the particle at threshold. As a result, we have for two final-state particles with momenta \( p_k \) and \( p_l \) and mass \( m_k \) and \( m_l \):

\[
p_k = \frac{m_k}{m_l} p_l,
\]  

which reduces to \( p_k = p_l \) for an equal mass final state. We can us Eq. (5.5) to reduce the number of integrals that need to be expanded. As an example, let us take the integrand of
a four-point function from the $q\bar{q} \rightarrow q\bar{q}$ process with loop momentum $k$. We can rewrite it to three-point functions using:

$$\frac{1}{(k^2 - m_g^2)(k + p_k)^2} = \frac{1}{D_1D_2D_3D_4}$$

(5.6)

$$= \frac{1}{D_4}\left(\frac{A}{D_1D_2} + \frac{B}{D_1D_3} + \frac{C}{D_2D_3}\right).$$

Since this equation must be true for all loop momenta, it leads to three equations, which we can solve for $A$, $B$ and $C$. In this particular case, the solutions are given by:

$$A = \frac{1}{2m_q^2 - 2m_g^2} \equiv -\frac{1}{2m_g^2}, \quad B = \frac{1}{m_\gamma^2}, \quad C = -\frac{1}{2m_\gamma^2},$$

(5.7)

where we have used $p_k^2 = m_q^2$. In this way we can reduce some of the three- and four-point integrals to two- and three-point integrals respectively. This procedure can be used for integrals that contain both outgoing momenta and do not contain Coulomb singularities. The results of the remaining integrals are explicitly expanded to zeroth order in $\beta$.

### 5.3.2 Real Corrections

Now that we have the virtual corrections near threshold, we need to take the threshold limit of the real corrections. In contrast to the case of top-pair production in Ref. [159], there is no full analytic result for the real corrections for SUSY-QCD, so we cannot take the explicit threshold limit. The key observation in our approach is that the real corrections are formally phase-space suppressed near threshold unless the integrand of the phase space integral compensates this suppression. Therefore we can construct the real corrections at threshold from the singular behaviour of the matrix element squared, which can be obtained using dipole subtraction [133, 155]. We will briefly review the procedure of dipole subtraction and specify how only the singular contributions survive in the threshold limit.

Dipole subtraction makes use of the fact that the cross section can be split into three parts: a part with three-particle kinematics $\sigma^{[3]}$, one with two-particle kinematics $\sigma^{[2]}$, and a collinear counterterm $\sigma^C$ that was defined in Eq. (5.2) for each of the initial-state partons. These parts are well-defined in $n = 4 - 2\epsilon$ dimensions, but their constituents diverge for $\epsilon \rightarrow 0$. With the aid of an auxiliary cross section $\sigma^A$, which captures all singular behaviour, all parts are made finite and integrable in four space-time dimensions. This auxiliary cross section is subtracted from the real corrections $\sigma^R$ at the integrand level to obtain $\sigma^{[3]}$ and added to the virtual corrections $\sigma^V$, which defines $\sigma^{[2]}$:

$$\sigma^{NLO} = \int_{[3]} \left[ d\sigma^R - d\sigma^A \right]_{\epsilon=0} + \int_{[2]} \left[ d\sigma^V + d\sigma^C + \int_{[1]} d\sigma^A \right]_{\epsilon=0} \equiv \sigma^{[3]} + \sigma^{[2]}.$$  

(5.8)

Here the first integral is over a three-particle phase space, denoted by [3]. Since $\sigma^A$ captures all singular behaviour, this integral is finite in 4 dimensions and we can safely
5.3. Hard Matching Coefficients

take the $\varepsilon \rightarrow 0$ limit. For the second integral, the three-particle phase space of $\sigma^A$ has been factorized into a two-particle phase space, denoted by [2], and a single-particle phase space, which is labelled [1]. After performing the latter phase space integral, the single-particle integrated auxiliary cross section can be added to the integrand of the virtual cross section, which also contains the renormalization of UV divergences discussed in Section 5.1, and the collinear counterterm $\sigma^C$, yielding a finite two-particle phase space integral.

We will first argue that we can neglect $\sigma^{[3]}$. Compared to the case of two-parton kinematics, the phase space of $\sigma^{[3]}$ is limited by the energy of the third, radiated massless particle. Near the two-particle threshold, the maximum energy of the radiated particle, and thus the available phase space, equals $E_{\text{max}} = \sqrt{s} - 2m_{\text{av}} \propto \beta^2$. Since after subtracting $\sigma^A$ no divergences are left in the integrand of $\sigma^{[3]}$, the leading contribution of $\sigma^{[3]}$ is at most proportional to $\beta^2$ and can thus be neglected. This leaves us with:

$$\sigma^{\text{NLO,thr}} = \sigma^{[2],\text{thr}} = \sigma^{V,\text{thr}} + \sigma^{C,\text{thr}} + \sigma^{A,\text{thr}}, \quad (5.9)$$

so at threshold the real radiation $\sigma^{R,\text{thr}}$ is indeed completely specified by the singular behaviour contained in $\sigma^A$.

In Ref. [155] the general form of $\sigma^A$ is determined by summing over dipoles that correspond to pairs of ordered partons. These dipoles describe the soft and collinear radiation and reproduce the matrix element squared in the soft and collinear limits. To obtain the cross section, the dipole functions need to be integrated over phase space and in particular over the momentum fraction $x$ that is left after radiation. In the threshold limit, the available phase space sets the lower bound of the $x$-integral to $1 - \beta^2$, while the upper bound equals 1. Therefore we cannot get a result of $O(\beta)$ unless the integrand diverges at $x = 1$, which is the case only for soft-gluon radiation. As a result we only need to take into account the dipoles that describe gluon radiation.

Special attention has to be paid to the massive final-state dipole function. In Ref. [155] this dipole function has been modified in order to simplify the integrations. Unfortunately this results in a deformation of the phase space integration which changes exactly the finite terms that we are looking for.

**How Velocity Factors Can Deform the Phase Space Integration**

As we have seen in Section 3.1.2, the soft singularities of the matrix element follow from the eikonal approximation. Consider the process $ij \rightarrow kl$ with an additional radiated gluon $g$. We can rewrite the eikonal matrix element squared from Eq. (3.6) to split the collinear singularities for massless particles:

$$\sum_{\text{gluon polarizations}} |M^{i \rightarrow k|g|l}|^2 = -2g_s^2 \alpha_s^{4-n} \sum_n \frac{1}{p_g \cdot p_n} \sum_{n'} \frac{p_n \cdot p_{n'}}{p_g \cdot (p_n + p_{n'})}(T_{n'} T_{n'}) \otimes |M^{i \rightarrow k|l}|^2, \quad (5.10)$$
where the sums over $n$ and $n'$ run over $\{i, j, k, l\}$. Using colour conservation (4.9), we can rewrite the terms for which $n = n'$:
\[
\sum_n \frac{p_n^2}{2(p_g \cdot p_n)^2} T_n^2 \otimes |\mathcal{M}^{ij-kl}|^2 = -\sum_n \frac{m_n^2}{2(p_g \cdot p_n)^2} \sum_{n' \neq n} (T_{n'} T_{n'}^*) \otimes |\mathcal{M}^{ij-kl}|^2,
\]
(5.11)
yielding an eikonal matrix element squared of the form:
\[
\sum_{\text{gluon polarizations}} |\mathcal{M}^{ij-kl}|^2 = -2g_s^2 \mu^{4-n} \sum_n \frac{1}{p_g \cdot p_n} \times \sum_{n' \neq n} \left( \frac{p_n \cdot p_{n'}}{p_g \cdot p_n + p_g \cdot p_{n'}} - \frac{1}{2} \frac{m_n^2}{p_g \cdot p_n} \right) (T_n \cdot T_{n'}) \otimes |\mathcal{M}^{ij-kl}|^2.
\]
(5.12)
Because this expression is based on the eikonal approximation, it correctly reproduces the soft limit. It is the basis for the dipole functions used in dipole subtraction. However, the final-state contributions of Eq. (5.12) vanish at threshold, so the final-state dipoles do not contribute in the threshold limit. The final-state dipole function defined in Ref. [155] does yield a finite result due to additional velocity factors.

We will explicitly show the effect of this modification by considering the example of a gluon that is radiated from a heavy quark with mass $m_Q$ in a $Q\bar{Q}$ production process. The modification affects the second term of Eq. (5.12), which is denoted as $\Gamma_{\text{coll}}^{\text{unchanged}}$ in Eq. (5.23) of Ref. [155]. In Ref. [155] finite pieces of the integrand are taken into account as well, but since we just argued that the only contribution at threshold comes from the singular part of the integrand, we will omit these terms. The singular term of the integrand yields a $1/\varepsilon$-pole and a finite piece. The pole cancels the pole of the first term of the dipole function (5.12), while the finite piece contributes to the hard matching coefficient. In its unmodified form, the finite piece is given by:
\[
\Gamma_{\text{coll, unchanged}}^{\text{unchanged}} \bigg|_{\text{fin}} = 2 \int_{0}^{\gamma^*} dy \left[ \frac{1}{y} - \frac{\mu^2 Q^2 \sqrt{[2\mu^2 Q^2 + (1 - 2\mu^2 Q)(1 - y)]^2 - 4\mu^2 Q^2}}{y(\mu^2 Q^2 + y(1 - 2\mu^2 Q)) \sqrt{1 - 4\mu^2 Q^2}} \right]
\approx 2 \int_{0}^{2(1-2\mu_Q)} dy \frac{\sqrt{1 - 4\mu^2 Q^2 - (1 - y/2)^2 - 4\mu^2 Q}}{y \sqrt{1 - 4\mu^2 Q}}
\]
(5.13)
with
\[
y = \frac{p_g \cdot p_Q}{p_g \cdot p_Q + p_g \cdot p_Q + p_Q \cdot p_Q}, \quad \gamma^* = \frac{1 - 2\mu_Q}{1 - 2\mu_Q Q^2} \quad \text{and} \quad \mu_Q = \frac{m_Q}{\sqrt{3}}.
\]
(5.14)
The approximation in the second line of Eq. (5.13) is suitable near threshold, where we have $\mu_Q \approx 1/2$. Exactly at threshold, the finite part equals $4 - 4\log(2)$ and exactly cancels the contribution from the first term of Eq. (5.12).
5.3. Hard Matching Coefficients

In Ref. [155], the integrand in $F_{Q,Q}^{\text{coll}}$ has been multiplied by velocity factors in order to simplify the integration:

$$\frac{\tilde{v}_{QQ}}{v_{QQ}} = \frac{(1 - y) \sqrt{1 - 4\mu_Q^2}}{\sqrt{[2\mu_Q^2 + (1 - 2\mu_Q^2)(1 - y)]^2 - 4\mu_Q^2}} \approx \frac{\sqrt{1 - 4\mu_Q^2}}{\sqrt{(1 - y/2)^2 - 4\mu_Q^2}}, \quad (5.15)$$

where the approximation in the second step holds near threshold. The velocity factor effectively replaces $(1 - 4\mu_Q^2)^{1/2}$ in the denominator of Eq. (5.13) by $((1 - y/2)^2 - 4\mu_Q^2)^{1/2}$, which amounts to a shift comparable in size to the value of the numerator. In the strict soft limit $y$ vanishes and the velocity factors have no effect. However, we are integrating over gluons that are not soft compared to the energy above threshold $\sqrt{s} - 2m_Q$, so we also need the correct behaviour away from the strict soft limit. In fact, if the velocity factors are included the integral $F_{Q,Q}^{\text{coll}}$ vanishes at threshold, so it does no longer cancel the contribution from the first term of Eq. (5.12).

Usually the velocity factors do not pose a problem in calculations using dipole subtraction, since the terms are subtracted from the real part and added to the virtual part. Therefore it does not matter if a dipole function is deformed, as long as the pole is reproduced. Finite contributions can always be moved between therefor it does not matter if a dipole function is deformed as long as the pole is reproduced since the terms are subtracted from the real part and added to the virtual part. For the other dipole functions and the collinear counterterm we can use the equations in [155] and take the threshold limit. The result in $n = 4 - 2\epsilon$ dimensions is given by:

$$\sigma_{ij}^{R,\text{thr}} + \sigma_{ij}^{C,\text{thr}} = 16\pi\alpha_s S_n \sigma_{ij}^{1,\text{LO,thr}} \left( C_2(R_{\ell}) \left( \frac{1}{2\epsilon} - \log(8\beta^2) + 3 \right) \right) + \sum_{n=|i,j|} T_n^2 \left[ \frac{1}{2\epsilon^2} - \frac{1}{\epsilon} \left( \log(2) - \frac{\gamma_n}{2T_n} \right) \right] + \log^2(8\beta^2) - 4 \log(8\beta^2) + 8 - \frac{11\pi^2}{24} - \log \left( \frac{\mu_F^2}{m_{\text{spin}}^2} \right) \left( \log(8\beta^2) - 2 + \frac{\gamma_n}{2T_n} - \log(2) \right) \right]. \quad (5.16)$$

Thus the real threshold cross section $\sigma_{ij}^{R,\text{thr}}$ corresponding to a representation $I$ is proportional to the colour-decomposed LO cross section at threshold $\sigma_{I}^{1,\text{LO,thr}}$. The final-state contributions are weighted by the quadratic Casimir invariant $C_2(R_{\ell})$ of the representation $R_{\ell}$, which is listed in Appendix B. The sum in the last two lines of Eq. (5.16) runs over the initial-state partons. The colour operators $T_n^2$ are given in Eq. (4.8), while the values of the flavour coefficients $\gamma_n$ are determined by the partons in the initial state:

$$\gamma_q = \frac{3}{2} C_F, \quad \gamma_g = \frac{11}{6} C_A - \frac{1}{3} n_f. \quad (5.17)$$
where \( n_f = 5 \) is the number of light flavours. Finally, the factor \( S_n \) is given by:

\[
S_n = \frac{1}{16\pi^2} e^{-\gamma_E - \log(4\pi)} \left( \frac{\mu^2}{m^2_{\text{av}}} \right)^\E
\]

with \( \gamma_E \) being Euler’s constant and \( \mu_R \) the renormalization scale. Eq. (5.16) is valid for all pair-production processes in SUSY-QCD that have a linear threshold behaviour in \( \beta \) at LO, including the unequal mass case of squark-gluino production.

### 5.3.3 The Combined Result in Mellin-Moment Space

The hard matching coefficients are defined in Mellin-moment space. As we have seen in Section 4.2.5, the Mellin transforms of terms that are linear in \( \beta \) are proportional to \( N^{-3/2} \) in the large-\( N \) limit. The Coulomb contribution is constant, so its Mellin transform is proportional to \( 1/N \). To obtain the Mellin transforms of the logarithmic terms in Eq. (5.16), we define \( z = 4m^2_{\text{av}}/s \). In the large-\( N \) limit, the relevant Mellin transforms are then given by:

\[
\int_0^1 \! dz z^{N-1} \beta \log^2(8\beta^2) \xrightarrow{N \to \infty} N^{-3/2} \left[ \gamma_E^2 + \frac{1}{2} \pi^2 + \log^2(2) - \gamma_E(4 + 2 \log(2)) + 4 \log(2) - (4 - 2\gamma_E + 2 \log(2)) \log(N) + \log^2(N) \right],
\]

\[
\int_0^1 \! dz z^{N-1} \beta \log(8\beta^2) \xrightarrow{N \to \infty} N^{-3/2} \left[ 2 - \gamma_E + \log(2) - \log(N) \right].
\]

To obtain the hard matching coefficients, we add the Mellin transforms of the real and virtual threshold cross sections and the collinear counterterm. We then omit the \( \log(N) \) and Coulomb terms and divide the result by the colour-decomposed LO threshold cross section in \( N \)-space \( \sigma_{\text{LO,thr}} \). This procedure results in:

\[
C_{ij \to kl}^{(1)} = \frac{1}{\hat{\sigma}_{ij \to kl,I}^{\text{LO,thr}}} \left\{ 16\pi\alpha_s S_n \hat{\sigma}_{ij \to kl,I}^{\text{LO,thr}} \left[ C_2(R_I) \left( \frac{1}{2\E} + \gamma_E - \log(2) + 1 \right) \right. \right.
\]

\[
+ \sum_{n=[i,j]} T_n^2 \left[ \frac{1}{2\E^2} - \frac{\gamma_n}{2T_n^2} \right] \left[ \log(2) - \frac{\gamma_n}{2T_n^2} \right] + \gamma_E^2 + \log^2(2) - 2\gamma_E \log(2) + \frac{\pi^2}{24}
\]

\[
\left. - \left( \frac{\gamma_n}{2T_n^2} - \gamma_E \right) \log\left( \frac{\mu^2}{m^2_{\text{av}}} \right) \right\} \left. + \left( \hat{\sigma}_{ij \to kl}^{\text{V,thr}} - \hat{\sigma}_{ij \to kl}^{\text{Coul,thr}} \right) \right\}.
\]

Here \( \hat{\sigma}_{ij \to kl,I}^{\text{LO,thr}} \) is the term proportional to \( N^{-3/2} \) in the threshold expansion of the LO cross section in Mellin-moment space. The complete expressions for the hard matching coefficients of the SUSY-QCD processes can be found in Appendix F.

These hard matching coefficients have been checked numerically using PROSPINO [141] and agree within the numerical accuracy of PROSPINO. The virtual part of the corrections of the \( gg \to \tilde{g}\tilde{g} \) process agrees with the analytical results for vanishing top-quark mass.
5.3. Hard Matching Coefficients

presented in [131], provided that one translates their DR result to our \( \overline{\text{MS}} \) result and manually decouples the heavy particles from the running of \( \alpha_s \). In addition, we have compared the matching coefficients for \( \tilde{q}\tilde{g} \) production with the \( a_1 \) coefficients presented in Ref. [160]. These \( a_1 \) coefficients were obtained with a numerical fit to PROSPINO that was not tailored to the threshold region and agree with our results to a few percent. The behaviour for a varying squark-gluino mass ratio \( r = m_{\tilde{g}}/m_\tilde{q} \) is shown in Figure 5.1.

As mentioned at the beginning of this section, the hard matching coefficients of suppressed cross sections have been set to 0. We see that the other hard matching coefficients can be quite large. Particularly the ones belonging to the higher-dimensional representations are large in view of the high colour charge that features in these representations. A contribution of 50% or more of the LO cross section is not unusual. Thus we can expect the hard matching coefficients to have a significant effect on the magnitude of the resummed cross section.

There are a few remarks to be made about the graphs in Figure 5.1. First, the processes that involve gluinos have a singularity if the gluino mass equals the sum of the squark and the top mass. This singularity could be removed by taking the finite gluino width into account.

Second, for many of the hard matching coefficients there is a branch cut in the complex plane starting or ending at \( r = 1 \), when the squark and gluino mass are equal. For most processes, we only see a small feature, but the effect is quite drastic in the \( gg \rightarrow \tilde{q}\tilde{g} \) process.

Finally, the nonvanishing hard matching coefficient of the \( q\bar{q} \rightarrow \tilde{g}\tilde{g} \) process becomes extremely large for small squark-gluino mass differences. Even worse, the hard matching coefficient is ill-defined at \( r = 1 \). Since we want the hard matching coefficient to be well-defined for all squark and gluino masses, this issue requires further investigation.

The \( q\bar{q} \rightarrow \tilde{g}\tilde{g} \) Hard Matching Coefficient

The problematic behaviour of \( C_{q\bar{q} \rightarrow \tilde{g}\tilde{g}}^{(1)} \) near \( r = 1 \) originates from the threshold behaviour of the LO cross section. The hard matching coefficient is defined in Eq. (5.21) as being the \( O(N^{-3/2}) \) term in the NLO cross section in \( N \)-space divided by the \( O(N^{-3/2}) \) term in the LO expression. As long as the LO cross section is nonzero, this expression is well-defined.

As we have seen in Section 4.2.4, some of the LO cross sections are \( \beta \)-suppressed near threshold. In most cases, this suppression is due to symmetry arguments, which remain valid at higher orders. Thus the corresponding matching coefficients only contribute with terms that are suppressed by powers of \( N^{-1/2} \) compared to the leading contributions and have therefore been set to 0. For the \( q\bar{q} \rightarrow \tilde{g}\tilde{g} \) process, the leading term in the LO cross section is proportional to \( (m_\tilde{g}^2 - m_\tilde{q}^2)^2 \) and thus vanishes ‘accidentally’ for \( m_\tilde{q} = m_\tilde{g} \) due to destructive interference between the LO diagrams [28]. There is no symmetry that causes this behaviour, so it is not surprising that it does not hold at higher orders. In fact, the \( O(N^{-3/2}) \) term of the NLO cross section contains a term that is proportional to \( m_\tilde{g}^2 - m_\tilde{q}^2 \), which results in a \( (m_\tilde{g}^2 - m_\tilde{q}^2)^{-1} \) divergence in the hard matching coefficient.

To solve this issue, we define a modified matching coefficient, which includes higher
Figure 5.1: Mass dependence of the colour-decomposed NLO hard matching coefficients for the SUSY-QCD processes. The common renormalization and factorization scale have been set equal to the average mass of the produced particles \(m_{av} = 1.2\) TeV, while the mass ratio \(r = m_{\tilde{g}}/m_q\) has been varied. The top quark mass is taken to be \(m_t = 172.9\) GeV.

order terms in \(N\) in the LO cross section to regularize the divergence. Instead of using the \(O(N^{-3/2})\) term, we expand the threshold cross section in the denominator of Eq. (5.21)
5.4. Soft Radiation Factors

to $N^{-7/2}$, which is the first nonvanishing term in the expansion for equal masses. After working out the Mellin transforms, the modified hard matching coefficient takes the form:

$$C_{q\bar{q} \to \tilde{g} \tilde{g}}^{(1, \text{mod})} = \left[ 1 - \frac{4 + B - B^2}{N + 3/2} + \frac{1 + 2B + 39B^2 + 34B^3 - 12B^4 - 4B^5 + 9B^6}{4B^2(N + 3/2)(N + 5/2)} \right]^{-1} C_{q\bar{q} \to \tilde{g} \tilde{g}}^{(1)}, \quad (5.22)$$

with $B = \frac{\alpha^2}{2\pi + 1}$. This deeper expansion of the LO cross section ensures a well-behaved hard matching coefficient, which vanishes at $r = 1$. The behaviour of this modified hard matching coefficient for different values of $N$ is shown in Figure 5.2.

![Figure 5.2](image)

Figure 5.2: Modified hard matching coefficient for the $q\bar{q} \to \tilde{g} \tilde{g}$ process for different values of $N$. As in Figure 5.1, the common renormalization and factorization scale have been set equal to the average mass of the produced particles $m_{av} = 1.2$ TeV, while the mass ratio $r = m_\tilde{g}/m_t$ has been varied. The top quark mass is taken to be $m_t = 172.9$ GeV.

For large values of $N$, the modified hard matching coefficient matches the behaviour of the unmodified matching coefficient. Even for relatively low values of $N$ the modified matching coefficient resembles the $N \to \infty$ case quite well. This was to be expected, since the modification only amounts to an $N^{-2}$ effect near $r = 1$. When combining the hard matching coefficient with the rest of the calculation, we integrate over $N$ according to Eq. (3.25), so the weight of the higher $N$ values depends on how close we are to threshold and on the PDFs.Either way, since the $q\bar{q} \to \tilde{g} \tilde{g}$ cross section is suppressed near $r = 1$, the gluino-pair cross section will usually be dominated by the $gg \to \tilde{g} \tilde{g}$ channel, leaving a relatively small numerical effect of the $q\bar{q} \to \tilde{g} \tilde{g}$ channel.

### 5.4 Soft Radiation Factors

For NNLL accuracy, we need the $A_i$ and $D$ functions in the functions $\Delta_i$ and $\Delta_{ij}^{(3)}$ from Eqs. (3.22) and (3.23) to third and second order respectively. The expansion of $A_i$ up to two loops was given in Eq. (4.30), while the three-loop coefficient $A_i^{(3)}$ is given by [136]:

$$A_i^{(3)} = \frac{1}{4} T_i^2 \left[ \left( \frac{245}{24} - \frac{67}{9} \zeta(2) + \frac{11}{6} \zeta(3) + \frac{11}{5} \zeta(2) \right) C_A^2 + \left( - \frac{55}{24} + 2 \zeta(3) \right) C_F n_l \right. \right.$$
$$\left. + \left( - \frac{209}{108} + \frac{10}{9} \zeta(2) - \frac{7}{5} \zeta(3) \right) C_A m_l - \frac{1}{27} n_l^2 \right], \quad (5.23)$$
with \( \zeta \) the Riemann zeta function, \( C_A = N_c \), \( C_F = \frac{N_c^2 - 1}{2N_c} \) and \( n_l = 5 \) the number of light flavours. The colour charge operator \( T_i \) is defined in Section 4.1.2. The two-loop coefficient \( D_i^{(2)} \) is given by [161, 162]:

\[
D_i^{(2)} = \frac{1}{2} T_i^2 \left[ C_A \left( -\frac{101}{27} + \frac{11}{3} \zeta(2) + \frac{7}{2} \zeta(3) \right) + \left( \frac{14}{27} - \frac{2}{3} \zeta(2) \right) n_l \right], \tag{5.24}
\]

while the one-loop contribution vanishes. Finally, for the colour-dependent part, we need the two-loop coefficient \( D_{ijkl,1}^{(2)} \) in addition to the one-loop result from Eq. (4.34). It follows from the two-loop soft anomalous dimension matrices and has been calculated in coordinate space [163, 164] and in the framework of soft collinear effective theory [119]. For total cross sections, the two-loop soft anomalous dimension matrices become diagonal and the diagonal components are given by [119]:

\[
D_{ijkl,1}^{(2)} = C_2(R_f) \left\{ - C_A \left( \frac{115}{36} - \frac{1}{2} \zeta(2) + \frac{1}{2} \zeta(3) \right) + \frac{11}{18} n_l \right\}. \tag{5.25}
\]

The quadratic Casimir invariants for the SUSY-QCD processes are listed in Appendix B. Explicit expressions for the \( g_3 \) term in Eq. (3.21) and its ingredients for the singlet and octet production channels are given in Refs. [136, 165, 166], although one has to correct for an extra minus sign in front of all \( D_{Q\bar{Q}} \) terms in Eq. (A9) of [165]. We have listed the specific \( g_3 \) functions needed for \( \tilde{q} \tilde{\bar{q}} \) production in [167].

### 5.5 Numerical Results for \( \tilde{q} \tilde{\bar{q}} \) Production

In this section we present numerical results for the NNLL-resummed squark-antisquark cross sections with and without the Coulomb contributions. We show the results for squark-antisquark pair-production at the LHC for a CM energy of 7 TeV. As we have shown in Ref. [167], results for 14 TeV are qualitatively similar. In order to evaluate hadronic cross sections we use the 2008 NLO MSTW PDFs [143, 144] with the corresponding \( \alpha_s(M_Z^2) = 0.120 \). We have used a top quark mass of \( m_t = 172.9 \) GeV [12]. The numerical results have been obtained with two independent computer codes.

As we have mentioned in Section 5.2, the Coulomb effects can be screened by the width of the final-state sparticles. For consistency we will stick to the approach adopted in the NLO calculations, where this screening is not taken into account. In order to study the effects from the hard matching coefficients and the Coulomb corrections separately, we will compare several cross sections with the NLO result and discuss their contribution:

- The NNLL matched cross section without Coulomb contributions to the resummation \( \sigma^{\text{NLO+NNLL w/o Coulomb}} \) contains the soft-gluon resummation to NNLL accuracy matched to the full NLO result. The matching is performed according to Eq. (3.25). The Coulomb correction to the resummation is not included, so \( C_{\text{Coul},(1)} \) in Eq. (5.1) is set to zero.

- The NNLL matched cross section \( \sigma^{\text{NLO+NNLL}} \) does include the Coulomb contribution \( C_{\text{Coul},(1)} \) from equation (5.1). Also in this case Eq. (3.25) has been used to match the cross section to the complete NLO result.
5.5. Numerical Results for $\tilde{q}\tilde{q}$ Production

We also show the NLL matched cross section $\sigma^{NLO+NLL}$ as introduced in Chapter 4. The NLO cross sections are calculated using the publicly available PROSPINO code [141], which is based on the calculations that were presented in Ref. [28] and that we briefly discussed in Section 5.1. No top-squark final states are considered. We sum over squarks with both chiralities ($\tilde{q}_L$ and $\tilde{q}_R$), which are taken as mass degenerate. The renormalization and factorization scales are taken to be equal, i.e. $\mu_F = \mu_R \equiv \mu$.

We first discuss the scale dependence of the cross sections. Figure 5.3a shows the squark-antiquark cross section for $m_\tilde{q} = m_\tilde{g} = 1.2$ TeV as a function of the renormalization and factorization scale $\mu$. The value of $\mu$ is varied around the central scale $\mu_0 = m_\tilde{q}$ from $\mu = \mu_0/5$ up to $\mu = 5\mu_0$.

![Scale dependence](image)

(a) Scale dependence. The renormalization and factorization scale have been set equal.

Figure 5.3: The scale dependence of the LO, NLO, NLO+NLL and NLO+NNLL (both with the Coulomb part $C_{\text{Coul},(1)}$ and without it) squark-antiquark cross sections for the LHC at 7 TeV. The squark and gluino masses have been set to $m_\tilde{q} = m_\tilde{g} = 1.2$ TeV.

We see the usual scale reduction going from LO to NLO. Including the NLL correction and the NLL contribution without the Coulomb part $C_{\text{Coul},(1)}$ improves the behaviour for moderate values of $\mu/\mu_0$, but a fairly strong scale dependence for small values of $\mu/\mu_0$ remains. Upon inclusion of the Coulomb corrections $C_{\text{Coul},(1)}$ the scale dependence stabilises over the whole range. Similar results have been found in the context of top quark physics [168–171].

Figure 5.3b shows the mass dependence of the scale uncertainty for the different cross sections. The squark and gluino mass have been taken equal and the scale has been varied in the range $m_\tilde{q}/2 \leq \mu \leq 2m_\tilde{q}$. As was to be expected from Figure 5.3a, the scale uncertainty reduces as the accuracy of the predictions increases. In the range of squark masses considered here, the NNLL resummation without the Coulomb corrections $C_{\text{Coul},(1)}$ already reduces the scale uncertainty to at most 10%. The inclusion of the Coulomb term
in the resummed NNLL prediction results in a scale uncertainty of only a few percent. The improvement is particularly striking in the higher mass range, suggesting that resummation to NNLL accuracy describes the threshold behaviour extremely well.

Finally we study the $K$-factors with respect to the NLO cross section:

$$K_x = \frac{\sigma^x}{\sigma^{\text{NLO}}},$$

(5.26)

where $x$ can be NLO+NNL, NLO+NNLL w/o Coulomb or NLO+NNLL. We show the three $K$-factors for equal squark and gluino masses in Figure 5.4a. At the central scale $\mu = m_\tilde{q}$ the $K$-factor, and thus the theoretical prediction of the cross section, increases as more corrections are included. Also, the effect becomes more pronounced for higher squark masses. This was to be expected, since in that case the particles are produced closer to threshold. As can be seen in Figure 5.4a, the NNLL resummation without the Coulomb corrections $C^{\text{Coul}(1)}$ already results in a 25% increase of the cross section with respect to the NLO cross section for squarks of 2 TeV. The contribution from the Coulomb term to the resummed NLO+NNLL cross section is larger than the contributions provided by the $g_3$ term in the exponential and the hard matching coefficient $C^{(1)}$, yielding a total $K$-factor of 1.45. Although the effect from the Coulomb corrections could be somewhat smaller in reality due to the finite lifetime of the squarks, Figure 5.4a suggests that the NNLL contribution will remain large.

![Figure 5.4](image)

Figure 5.4: The $K$-factor with respect to the NLO cross section of the NLO+NLL and NLO+NNLL squark-antisquark cross sections with and without the Coulomb contributions $C^{\text{Coul}(1)}$ for the LHC at 7 TeV. The common renormalization and factorization scale has been set equal to the squark mass.

We also show the NLO+NNLL w/o Coulomb and NLO+NNLL $K$-factors for different ratios of the squark and gluino mass in Figure 5.4b. As can be seen in Figure 5.4b, the
effect of the gluino mass is small. In addition we find that the dependence of the hard matching coefficients on the squark mass is smaller than the dependence on \( r \), so one would expect that nondegenerate squark masses mainly affect the LO cross section and thus can be captured by a simple reweighting. Consequently the NNLL-resummed results are relatively independent of the relation between squark and gluino masses.

The scale dependence of the cross section shows the best stability after including both the hard matching coefficients \( C^{(1)} \) and the Coulomb contributions \( C^{\text{Coul},(1)} \). This indicates that all these contributions should be taken into account to achieve the observed cancellation. However, the observed reduction in the scale dependence might be modified somewhat by the inclusion of the width of the particles or by matching to the full NNLO result, which is not available. In this context we note that, as a consequence of the NNLL accuracy of resummation, our matched cross section receives additional non-logarithmic NNLO contributions, which would have been consistently treated if matching to NNLO had been possible. A very conservative estimate of the scale uncertainty is provided by the NLO+NNLL w/o Coulomb results, which do not include the Coulomb corrections in the resummed expression.

### 5.6 Summary

In this chapter, we have discussed the NNLL resummation of threshold corrections for SUSY-QCD. In particular, the previously unknown hard matching coefficients \( C^{(1)} \), needed at this level of accuracy, has been calculated analytically. To this end, we have discussed the calculation of the NLO corrections and taken the threshold limit. We have presented the full analytical result for all SUSY-QCD processes. In addition, we have discussed the treatment of the Coulomb corrections and the soft radiation factors.

We have also numerically evaluated the NNLL resummed cross section for squark-antisquark production, matched to the NLO fixed-order expression, for the LHC with a CM energy of 7 TeV, and found that the total cross section increases at the central scale. For a squark mass of 2 TeV, the NLO+NNLL squark-antisquark cross section is larger than the corresponding NLO cross section by as much as 45%. The correction is reduced to 25% if the contributions due to Coulombic interactions are not taken into account in the resummation. In that case, the scale uncertainty is reduced to approximately 50% of the NLO scale uncertainty. After inclusion of the Coulomb corrections in the resummation, the scale uncertainty is only a few percent.

The improved cross sections can be used to improve current limits on SUSY masses. In the case that SUSY is found, they can help to more accurately determine the masses of the sparticles.
Chapter 6

Conclusion and Outlook

In the past few years, the LHC has been pushing the limits for SUSY particle masses up to the TeV scale. Therefore, we know that if low-energy SUSY exists, the sparticles will be produced close to their production threshold at the LHC. As a result, conventional perturbation theory is not particularly suitable for calculating SUSY-QCD cross sections, since each order will be dominated by large logarithmic corrections.

In this thesis, we have taken a different approach to calculating the SUSY-QCD cross section near threshold, which is based on the resummation of soft-gluon effects. After a discussion on SUSY in Chapter 2 and an introduction to the main concepts in resummation in Chapter 3, we have presented the calculation for resummation at NLL accuracy in Chapter 4. We have presented colour bases for all SUSY-QCD processes, as well as the colour decomposition of the LO cross section and the one-loop soft anomalous dimension matrices. We have seen that NLL resummation stabilizes the theoretical predictions and increases the cross section at the central scale compared to the NLO result. The effects are most pronounced for processes involving gluinos, with corrections of 27% for pair production of 1.2 TeV gluinos at the LHC with a CM energy of 7 TeV. The results of the NLL calculation are the current state-of-the art predictions that include all SUSY-QCD processes. They are publicly available in the NLL-fast code, which is currently used by the major experiments.

We have then taken the next step in Chapter 5, and presented the ingredients needed for NNLL resummation for the SUSY-QCD processes. In particular, we have calculated the matching coefficients, which contain the contributions from NLO corrections to the cross section near threshold. We have given analytical expressions for the Mellin transformations needed for the Coulomb contributions as well as the full analytical result for the hard matching coefficients. We have presented results for the NNLL resummed squark-antisquark pair production processes and found a striking stabilization of the cross section. At the central scale, the effect of the hard matching coefficients and the soft radiation factors are as large as 25% of the NLO cross section for a squark mass of 2 TeV at the LHC with a CM energy of 7 TeV. Including the Coulomb corrections in the resummed cross section as well gives a 45% correction compared to the NLO result.
The next steps in this research depend on what nature has in store for us. If SUSY does not exist, we want to exclude it. In that case, the next step would be to complete the numerical implementation of the NNLL calculation. Based on the results for the NLL calculation, one might expect that the NNLL corrections for processes involving gluinos are considerable. Thus, a full NNLL calculation could significantly improve the theoretical predictions for squark and gluino production, particularly in view of the small scale uncertainty of the squark-antisquark NNLL predictions. The final goal would be to make the results available to the experimental community, so they can use it to improve exclusion limits for squark and gluino masses.

To put the NNLL results on stronger footing, it might be useful to further investigate the observed stabilization of the cross section prediction in more detail, for instance by varying the renormalization and factorization scale separately. Also, since no NNLO results are available, the NNLL results are currently matched to NLO predictions. A full treatment of the NNLO corrections would enable us to use NNLO PDFs in the calculation. This would reduce theoretical uncertainties in the calculation and thus allow for a less conservative use of the results.

If low-energy SUSY does exist in nature, the LHC should be able to find it and the NNLL predictions can be used to improve the determination of sparticle masses. In that case, it is worth investigating the more model-dependent parts of the calculation. First, one could extend the calculation to models with unequal squark masses. The main effects of different squark masses enter through the LO cross section and can thus be captured by a simple rescaling. However, higher-order corrections do cause some small effects. If the differences between the squark masses are large, including these corrections would improve the result even further.

More importantly, however, the finite lifetime of the SUSY particles can affect the results of resummation. The Coulomb corrections in particular are sensitive to the width of the particles, since they are essentially bound-state effects. If SUSY is found, the width of the sparticles can be determined. Depending on the lifetime of the sparticles, it could be worth to include these effects in the calculation as well.

Perhaps SUSY exists in nature, perhaps it does not. In the former case, resummed results can help determine particle masses, while in the latter case, they can be used to improve exclusion limits. Either way, resummation improves the quality of the conclusions that can be drawn from the experimental results. Since the technique can be extended to other new physics models that contain heavy coloured particles, resummation will remain an important theoretical ingredient for the new physics searches at the LHC.

It is a small piece of the puzzle, but it could help us get a little bit further with answering the questions Empedocles already asked 2500 years ago: what is the world made of and how does it work?
Appendix A

Product Representations Using Young Tableaux

An intuitive way to study representations is provided by Young tableaux, which graphically depict representations by using their symmetry properties. For the interested reader, we will work out the product representations relevant to SUSY-QCD processes and their colour decomposition into irreducible representations using Young tableaux.

Every box in a Young tableau stands for an $N_c$-dimensional index. Boxes in the same row depict symmetric combinations of indices, while boxes in the same column stand for antisymmetric combinations. The fundamental representation is depicted by a single box, while the adjoint representation has a mixed symmetry. The dimension of a given irreducible representation can be determined from the corresponding Young tableau using the hook length formula. A pedagogical introduction to Young tableaux, including the multiplication rules and the hook length formula, can be found in e.g. [172, 173].

Using Young tableaux, we can work out product representations and their decomposition into irreducible representations. The simplest example is the decomposition of the quark-quark (or squark-squark) product representation:

$$\begin{array}{c}
1 \otimes [a] = \begin{array}{c}
1 \\
\vdots \\
\bullet
\end{array} + \begin{array}{c}
1 \\
\vdots \\
\bullet
\end{array} = N_c(N_c - 1)/2 \oplus N_c(N_c + 1)/2.
\end{array} \quad (A.1)$$

For this simple example the labels in the boxes are superfluous, but in more complicated cases they help keeping track of all possible combinations of the boxes. The decomposition for a antiquark-quark (or antisquark-squark) product representation is given by:

$$\begin{array}{c}
N_c - 1 \begin{array}{c}
1 \\
\vdots \\
\bullet
\end{array} \otimes [a] = 1 \oplus N_c - 1 \begin{array}{c}
1 \\
\vdots \\
\bullet
\end{array} = 1 \oplus (N_c^2 - 1),
\end{array} \quad (A.2)$$
while for a gluon-quark (or gluino-squark) combination, the product representation is given by:

\[
N_c - 1 \begin{pmatrix} 1 & 1 \\ 2 & : \\ \vdots & \ddots \end{pmatrix} \otimes a = 1 \oplus N_c - 1 \begin{pmatrix} 1 & 2 & a \\ \vdots & : & \ddots \end{pmatrix} \oplus N_c - 1 \begin{pmatrix} 1 & 1 & a \\ \vdots & : & \ddots \end{pmatrix}
\]

\[
= N_c \oplus N_c(N_c + 1)(N_c - 2)/2 \oplus N_c(N_c - 1)(N_c + 2)/2.
\]

The quantities next to the curly brackets denote the length of the corresponding column.

The most involved example is the decomposition of the gluon-gluon (or gluino-gluino) product representation:

\[
N_c - 1 \begin{pmatrix} 1 & 1 \\ 2 & : \\ \vdots & \ddots \end{pmatrix} \otimes N_c - 1 \begin{pmatrix} a & a \\ b & : \\ \vdots & \ddots \end{pmatrix} = 1 \oplus N_c - 1 \begin{pmatrix} 1 & a \\ \vdots & \ddots \end{pmatrix} \oplus N_c - 1 \begin{pmatrix} 1 & a \\ \vdots & \ddots \end{pmatrix} \oplus N_c - 2 \begin{pmatrix} 1 & a & a \\ \vdots & : & \ddots \end{pmatrix}
\]

\[
= 1 \oplus (N_c^2 - 1)_A \oplus (N_c^2 - 1)_S \oplus (N_c^2 - 1)(N_c^2 - 4)/4 \oplus (N_c^2 - 1)(N_c^2 - 4)/4 \oplus N_c^2(N_c - 1)/4 \oplus N_c^2(N_c - 3)(N_c + 1)/4.
\]
Appendix B

Base Tensors for SUSY-QCD

In this appendix, all the base tensors needed for the $2 \to 2$ SUSY-QCD processes as well as their dimensions and quadratic Casimir invariants are listed. They are obtained with the method described in Section 4.1.3 and are given for a general SU($N_c$) theory in terms of Kronecker deltas in colour space $\delta_{ab}$, the generators of the fundamental representation $T^c_{ab}$, the structure constants $f_{abc}$ and their symmetric counterparts $d_{abc}$. We use colour labels $a_1$ and $a_2$ for the initial-state particles and labels $a_3$ and $a_4$ for the final-state particles. Summation over repeated indices is implied.

For the process $q(a_1)\bar{q}(a_2) \to \bar{q}(a_3)\bar{q}(a_4)$ we have:

$$c_{q\bar{q} \to \bar{q}\bar{q},1} = \frac{1}{N_c} \delta_{a_1 a_3} \delta_{a_2 a_4}, \quad \text{dim}(R_1) = 1, \quad C_2(R_1) = 0, \quad (B.1)$$

$$c_{q\bar{q} \to \bar{q}\bar{q},2} = 2 T^c_{a_1 a_3} T^c_{a_2 a_4}, \quad \text{dim}(R_2) = N^2_c - 1, \quad C_2(R_2) = N_c. \quad (B.2)$$

For the process $g(a_1)g(a_2) \to \bar{q}(a_3)\bar{q}(a_4)$ there are three structures:

$$c_{gg \to \bar{q}\bar{q},1} = \frac{1}{\sqrt{N_c(N_c^2 - 1)}} \delta_{a_1 a_2} \delta_{a_3 a_4}, \quad \text{dim}(R_1) = 1, \quad C_2(R_1) = 0, \quad (B.3)$$

$$c_{gg \to \bar{q}\bar{q},2} = \frac{i \sqrt{2}}{\sqrt{N_c}} f_{a_1 a_2 c} T^c_{a_3 a_4}, \quad \text{dim}(R_2) = N^2_c - 1, \quad C_2(R_2) = N_c, \quad (B.4)$$

$$c_{gg \to \bar{q}\bar{q},3} = \frac{\sqrt{2N_c}}{\sqrt{N_c^2 - 4}} d_{a_1 a_2 c} T^c_{a_3 a_4}, \quad \text{dim}(R_3) = N^2_c - 1, \quad C_2(R_3) = N_c. \quad (B.5)$$

For squark-pair production $q(a_1)q(a_2) \to \bar{q}(a_3)\bar{q}(a_4)$ the base tensors are:

$$c_{qq \to \bar{q}\bar{q},1} = \frac{1}{2} (\delta_{a_1 a_4} \delta_{a_2 a_3} - \delta_{a_1 a_3} \delta_{a_2 a_4}), \quad c_{qq \to \bar{q}\bar{q},2} = \frac{1}{2} (\delta_{a_1 a_4} \delta_{a_2 a_3} + \delta_{a_1 a_3} \delta_{a_2 a_4}), \quad (B.6)$$
and their dimension and quadratic Casimir invariants are given by:

\[
\begin{align*}
\dim(R_1) &= \frac{1}{2} N_c (N_c - 1), \\
C_2(R_1) &= \frac{(N_c + 1)(N_c - 2)}{N_c}, \\
\dim(R_2) &= \frac{1}{2} N_c (N_c + 1), \\
C_2(R_2) &= \frac{(N_c - 1)(N_c + 2)}{N_c}.
\end{align*}
\]

The colour structure of the process \( q(a_1)\bar{q}(a_2) \to \bar{g}(a_3)g(a_4) \) is similar to Eqs. (B.3-B.5):

\[
\begin{align*}
c_{q\bar{q} \to \bar{g}g,1} &= \frac{1}{\sqrt{N_c(N_c^2 - 1)}} \delta_{a_1a_2} \delta_{a_3a_4}, \\
\dim(R_1) &= 1, \\
C_2(R_1) &= 0, \\
c_{q\bar{q} \to \bar{g}g,2} &= \frac{i \sqrt{2}}{\sqrt{N_c}} f_{a_1a_2c} T^c_{a_2a_1}, \\
\dim(R_2) &= N_c^2 - 1, \\
C_2(R_2) &= N_c, \\
c_{q\bar{q} \to \bar{g}g,3} &= \frac{\sqrt{2N_c}}{\sqrt{N_c^2 - 4}} d_{a_2a_4c} T^c_{a_2a_1}, \\
\dim(R_3) &= N_c^2 - 1, \\
C_2(R_3) &= N_c.
\end{align*}
\]

For the \( g(a_1)\bar{g}(a_2) \to \bar{g}(a_3)g(a_4) \) process, the base tensors are given by:

\[
\begin{align*}
c_{gg \to \bar{g}g,1} &= \frac{1}{N_c^2 - 1} \delta_{a_1a_2} \delta_{a_3a_4}, \\
c_{gg \to \bar{g}g,2} &= \frac{1}{N_c^2 - 4} d_{a_1a_2} d_{c a_3a_4}, \\
c_{gg \to \bar{g}g,3} &= \frac{1}{N_c} f_{a_1a_2c} f_{ca_3a_4}, \\
c_{gg \to \bar{g}g,4} &= \frac{1}{4} \left( \delta_{a_1a_2} \delta_{a_3a_4} - \delta_{a_1a_4} \delta_{a_2a_3} \right) - \frac{f_{a_1a_2c} f_{ca_3a_4}}{2N_c} + \frac{i}{4} \left( d_{a_1a_2c} f_{ca_3a_4} + f_{a_1a_2c} d_{ca_3a_4} \right), \\
c_{gg \to \bar{g}g,5} &= \frac{1}{4} \left( \delta_{a_1a_2} \delta_{a_3a_4} - \delta_{a_1a_4} \delta_{a_2a_3} \right) - \frac{f_{a_1a_2c} f_{ca_3a_4}}{2N_c} - \frac{i}{4} \left( d_{a_1a_2c} f_{ca_3a_4} + f_{a_1a_2c} d_{ca_3a_4} \right), \\
c_{gg \to \bar{g}g,6} &= -\frac{N_c + 2}{2N_c(N_c + 1)} \delta_{a_1a_2} \delta_{a_3a_4} + \frac{N_c + 2}{4N_c} \left( \delta_{a_1a_2} \delta_{a_3a_4} + \delta_{a_1a_4} \delta_{a_2a_3} \right) \\
&\quad - \frac{N_c + 4}{4(N_c + 2)} d_{a_1a_2c} d_{a_3a_4c} + \frac{1}{4} \left( d_{a_1a_2c} d_{a_2a_4c} + d_{a_2a_3c} d_{a_1a_4c} \right), \\
c_{gg \to \bar{g}g,7} &= \frac{N_c - 2}{2N_c(N_c - 1)} \delta_{a_1a_2} \delta_{a_3a_4} + \frac{N_c - 2}{4N_c} \left( \delta_{a_1a_2} \delta_{a_3a_4} + \delta_{a_1a_4} \delta_{a_2a_3} \right) \\
&\quad + \frac{N_c - 4}{4(N_c - 2)} d_{a_1a_2c} d_{a_3a_4c} - \frac{1}{4} \left( d_{a_1a_2c} d_{a_2a_4c} + d_{a_2a_3c} d_{a_1a_4c} \right).
\end{align*}
\]
while the corresponding dimensions and quadratic Casimir invariants are:

\[
\begin{align*}
\text{dim}(R_1) &= 1, & C_2(R_1) &= 0, \quad (B.19) \\
\text{dim}(R_2) &= N_c^2 - 1, & C_2(R_2) &= N_c, \quad (B.20) \\
\text{dim}(R_3) &= N_c^2 - 1, & C_2(R_3) &= N_c, \quad (B.21) \\
\text{dim}(R_4) &= (N_c^2 - 4)(N_c^2 - 1)/4, & C_2(R_4) &= 2N_c, \quad (B.22) \\
\text{dim}(R_5) &= (N_c^2 - 4)(N_c^2 - 1)/4, & C_2(R_5) &= 2N_c, \quad (B.23) \\
\text{dim}(R_6) &= N_c^2(N_c + 3)(N_c - 1)/4, & C_2(R_6) &= 2(N_c + 1), \quad (B.24) \\
\text{dim}(R_7) &= N_c^2(N_c - 3)(N_c + 1)/4, & C_2(R_7) &= 2(N_c - 1). \quad (B.25)
\end{align*}
\]

Note that since the dimension of \(c_{gq \rightarrow \tilde{g} \tilde{g}}\) vanishes for \(N_c = 3\), this representation does not contribute in (SUSY)-QCD. Finally, the base tensors for squark-gluino production \(q(a_1)g(a_2) \rightarrow \tilde{q}(a_3)\tilde{g}(a_4)\) are given by:

\[
\begin{align*}
c_{gq \rightarrow \tilde{g} \tilde{g},1} &= \frac{2N_c}{N_c^2 - 1} (T^{a_4}T^{a_2})_{a_3a_1}, \\
c_{gq \rightarrow \tilde{g} \tilde{g},2} &= \frac{N_c - 2}{2N_c} \delta_{a_3a_4} \delta_{a_1a_3} - d_{a_4a_3}T^{c}_{a_3a_1} + \frac{N_c - 2}{N_c - 1} (T^{a_4}T^{a_2})_{a_3a_1}, \\
c_{gq \rightarrow \tilde{g} \tilde{g},2} &= \frac{N_c + 2}{2N_c} \delta_{a_3a_4} \delta_{a_1a_3} + d_{a_4a_3}T^{c}_{a_3a_1} - \frac{N_c + 2}{N_c + 1} (T^{a_4}T^{a_2})_{a_3a_1}.
\end{align*}
\]

The dimensions and quadratic Casimir invariants for the corresponding representations are:

\[
\begin{align*}
\text{dim}(R_1) &= N_c, & C_2(R_1) &= \frac{N_c^2 - 1}{2N_c}, \quad (B.29) \\
\text{dim}(R_2) &= \frac{1}{2} N_c(N_c + 1)(N_c - 2), & C_2(R_2) &= \frac{(N_c - 1)(3N_c + 1)}{2N_c}, \quad (B.30) \\
\text{dim}(R_3) &= \frac{1}{2} N_c(N_c - 1)(N_c + 2), & C_2(R_3) &= \frac{(N_c + 1)(3N_c - 1)}{2N_c}. \quad (B.31)
\end{align*}
\]
Appendix C

Colour-Decomposed LO Cross Sections in SUSY QCD

In this appendix we present the colour-decomposed LO cross sections for $2 \rightarrow 2$ processes in SUSY-QCD and their threshold limits. We sum over final state spin, chirality and flavour and average over initial-state spin and colour. We exclude top squarks from the final state in view of potentially large mixing effects and mass splitting in the stop sector. The number of light squark flavours is denoted by $n_l$. The squark and gluino mass are denoted as $m_{\tilde{q}}$ and $m_{\tilde{g}}$ respectively. We first define the following shorthand notation:

$$L_1 = \log \left( \frac{s + 2m^2_{\tilde{q}} - s\beta}{s + 2m^2_{\tilde{q}} + s\beta} \right), \quad m^2_{\tilde{g}} = m^2_{\tilde{g}} - m^2_{\tilde{q}},$$

$$L_2 = \log \left( \frac{s - 2m^2_{\tilde{q}} - s\beta}{s - 2m^2_{\tilde{q}} + s\beta} \right), \quad m^2_{\tilde{q}} = m^2_{\tilde{g}} + m^2_{\tilde{q}},$$

$$L_3 = \log \left( \frac{s + m^2_{\tilde{q}} - \kappa s\beta}{s + m^2_{\tilde{q}} + \kappa s\beta} \right), \quad \beta = \sqrt{1 - \frac{4m^2_{\text{av}}}{s}},$$

$$L_4 = \log \left( \frac{s - m^2_{\tilde{q}} - \kappa s\beta}{s - m^2_{\tilde{q}} + \kappa s\beta} \right), \quad \kappa = \sqrt{1 - \frac{(m_{\tilde{q}} - m_{\tilde{g}})^2}{s}},$$

with $s$ the CM energy squared and $m_{\text{av}}$ the average mass of the produced particles. Furthermore we define:

$$\alpha_s = \frac{g^2_s}{4\pi} \quad \text{and} \quad \hat{\alpha}_s = \frac{\hat{g}^2_s}{4\pi},$$

where $g_s$ is the QCD gauge coupling, while $\hat{g}_s$ is the corresponding quark-squark-gluino coupling in the MS scheme, as explained in Section 5.1. We give the LO diagrams and colour-decomposed cross sections for SU$(N_c)$ in $n = 4-2\epsilon$ dimensions. The cross sections are labelled such that they correspond to the colour structures in Appendix B.
\( q\bar{q} \to q\bar{q} \)

For the \( q\bar{q} \to q\bar{q} \) process, we have the following diagrams:

\[
\mathcal{M}^{(0)}_{q\bar{q} \to q\bar{q}} = \quad + \quad ,
\]

and the colour-decomposed LO cross section is given by:

\[
\sigma^{(0)}_{q\bar{q} \to q\bar{q},1} = \frac{\pi \hat{s}^2 (N_c^2 - 1)^2}{s} \left( - \left( \frac{m^4}{m_-^2 + s m_k^2} + 1 \right) \beta - \left( \frac{2m_-^2}{s} + 1 \right) L_1 \right),
\]

\[
\sigma^{(0)}_{q\bar{q} \to q\bar{q},2} = \frac{\sigma^{(0)}_{q\bar{q} \to q\bar{q},1}}{N_c^2 - 1} \left( \frac{(N_c^2 - 1)\pi}{2N_c^3 s} \left( \alpha_s \hat{s} \left( \frac{2m_-^2 + s}{s} \beta + \frac{2m_-^2}{s^2} L_1 \right) + \frac{\alpha_s^2}{3} n_1 N_c \beta^2 \right) \right) \delta_{f_1 f_2},
\]

where the part proportional to \( \delta_{f_1 f_2} \) only contributes if the initial-state partons have the same flavour. Near threshold this reduces to:

\[
\sigma^{(0), \text{thr}}_{q\bar{q} \to q\bar{q},1} = \frac{\hat{s}^2 \pi (N_c^2 - 1)^2 m_k^2}{2N_c^4 m_-^4} \beta,
\]

\[
\sigma^{(0), \text{thr}}_{q\bar{q} \to q\bar{q},2} = \frac{\hat{s}^2 \pi (N_c^2 - 1) m_k^2}{2N_c^4 m_-^4} \beta.
\]

\( gg \to q\bar{q} \)

For the \( gg \to q\bar{q} \) process, we have four diagrams:

\[
\mathcal{M}^{(0)}_{gg \to q\bar{q}} = \quad + \quad + \quad + \quad ,
\]

and the colour-decomposed LO cross sections are:

\[
\sigma^{(0)}_{gg \to q\bar{q},1} = \frac{\pi \alpha_s^2 n_1}{N_c(0^2 - 1)(0 - \varepsilon)^2 s} \left( \frac{4m_0^2}{s} - \frac{8m_0^4}{s^2} \right) \log \left( \frac{1 - \beta}{1 + \beta} \right) + \left( 1 - \varepsilon + \frac{4m_0^2}{s} \right) \beta, \quad \text{(C.7)}
\]

\[
\sigma^{(0)}_{gg \to q\bar{q},2} = \frac{\alpha_s^2 n_1 N_c}{2} \sigma^{(0)}_{gg \to q\bar{q},1}
\]

\[
+ \frac{\pi \alpha_s^2 n_1 N_c}{N_c^2 - 1}(0 - \varepsilon)^2 s \left( \frac{8m_0^4}{s^2} \log \left( \frac{1 - \beta}{1 + \beta} \right) + \frac{2(5 + \varepsilon) m_0^2}{3s} - \frac{1 - \varepsilon}{3} \right) \beta, \quad \text{(C.8)}
\]

\[
\sigma^{(0)}_{gg \to q\bar{q},3} = \frac{N_c^2 - 4}{2} \sigma^{(0)}_{gg \to q\bar{q},1}. \quad \text{(C.9)}
\]
Near threshold, the colour decomposition becomes:

\[
\sigma^{(0),\text{thr}}_{gg\to \tilde{q}\tilde{q},1} = \frac{\alpha_s^2 \pi n_l \beta}{4N_c(N_c^2 - 1)(1 - \varepsilon)m_q^2}, \\
\sigma^{(0),\text{thr}}_{gg\to \tilde{q}\tilde{q},2} = 0 , \\
\sigma^{(0),\text{thr}}_{gg\to \tilde{q}\tilde{q},3} = \frac{\alpha_s^2 \pi(N_c^2 - 4)n_l \beta}{8N_c(N_c^2 - 1)(1 - \varepsilon)m_q^2}.
\]

**qq → \tilde{q}\tilde{q}**

In squark-pair production, fermion number conservation is violated. This is possible due to the Majorana nature of the gluino, cf. Section 4.2.1. For the production of equal-flavoured squarks, we then have two diagrams:

\[
\mathcal{M}^{(0)}_{qq\to \tilde{q}\tilde{q}} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1} \\
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array}.
\]

If the squarks have different flavours, only one diagram remains. If the squarks have the same flavour and chirality, one has to include a statistical factor of \(\frac{v}{w}\) in the calculation of the cross section. The contributions of the different \(s\)-channel colour structures to the \(qq \to \tilde{q}\tilde{q}\) cross section are:

\[
\sigma^{(0)}_{qq\to \tilde{q}\tilde{q},1} = \frac{\pi \hat{a}_s^2(N_c^2 - 1)(N_c + 1)}{4N_c^3 s} \left\{ \left( \frac{2m^2_{\tilde{q}} \delta_{f_1 f_2}}{2m^2_s + s} - \frac{2m^2_s + s}{s} \right) L_1 - \left( 1 + \frac{m^4_s}{m^4 + s m^2_{\tilde{q}}} \right) \beta \right\} ,
\]

\[
\sigma^{(0)}_{qq\to \tilde{q}\tilde{q},2} = \frac{N_c - 1}{N_c + 1} \sigma^{(0)}_{qq\to \tilde{q}\tilde{q},1} - \frac{\pi \hat{a}_s^2(N_c^2 - 1)(N_c - 1)}{N_c^3 s} \left\{ \delta_{f_1 f_2} \frac{m^2_{\tilde{q}}}{2m^2_s + s} - L_1 \right\} .
\]

Near threshold they are given by:

\[
\sigma^{(0),\text{thr}}_{qq\to \tilde{q}\tilde{q},1} = \frac{\hat{a}_s^2 \pi(N_c^2 - 1)(N_c + 1)m^2_{\tilde{q}}}{4N_c^3 m^4_s} (1 - \delta_{f_1 f_2}) \beta ,
\]

\[
\sigma^{(0),\text{thr}}_{qq\to \tilde{q}\tilde{q},2} = \frac{\hat{a}_s^2 \pi(N_c^2 - 1)(N_c - 1)m^2_{\tilde{q}}}{4N_c^3 m^4_s} (1 + \delta_{f_1 f_2}) \beta .
\]

Note that the contribution of the first colour structure is suppressed near threshold if the produced squarks have the same flavour.
\( \bar{q}q \to \tilde{g}\tilde{g} \)

For the \( \bar{q}q \to \tilde{g}\tilde{g} \) process, we have three diagrams:

\[
\mathcal{M}^{(0)}_{\bar{q}q \to \tilde{g}\tilde{g}} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array}
\end{array}. \tag{C.18}
\]

The colour decomposition is given by:

\[
\sigma^{(0)}_{\bar{q}q \to \tilde{g}\tilde{g},1} = \frac{\pi \hat{\alpha}_s^2(N_c^2 - 1)}{2N_c s} \left\{ \left( \frac{2m_{\tilde{g}}^2}{s - 2m_{\tilde{g}}^2} - \frac{2m_{\tilde{g}}^2}{s} \right) L_2 + \left( 1 + \frac{m_4^4}{m_4^4 + sm_{\tilde{g}}^2} \right) \beta \right\}, \tag{C.19}
\]

\[
\sigma^{(0)}_{\bar{q}q \to \tilde{g}\tilde{g},2} = \frac{\pi (N_c^2 - 1)}{2N_c s} \left\{ \alpha_s^2 \left[ \frac{4m_{\tilde{g}}^2}{3s} + \frac{2}{3} - \varepsilon \right] \beta \right. \tag{C.20}
\]

\[
+ \alpha_s \hat{\alpha}_s \left[ \left( - \frac{2m_{\tilde{g}}^2}{s} - 1 + 2\varepsilon \right) \beta + \left( \frac{2m_4^4}{s} + \frac{2m_{\tilde{g}}^2}{s} - \frac{2m_4^2\varepsilon}{s} \right) L_2 \right]
\]

\[
+ \hat{\alpha}_s^2 \left[ \frac{2m_4^4 + m_{\tilde{g}}^2s}{2(m_4^4 + m_{\tilde{g}}^2s)} \beta + \frac{(2m_{\tilde{g}}^2 - m_4^2s - 2m_4^4)}{(2m_4^2 - s)s} L_2 \right],
\]

\[
\sigma^{(0)}_{\bar{q}q \to \tilde{g}\tilde{g},3} = \frac{N_c^2 - 4}{2} \sigma^{(0)}_{\bar{q}q \to \tilde{g}\tilde{g},1}. \tag{C.21}
\]

Near threshold this reduces to:

\[
\sigma^{(0),\text{thr}}_{\bar{q}q \to \tilde{g}\tilde{g},1} = \sigma^{(0),\text{thr}}_{\bar{q}q \to \tilde{g}\tilde{g},3} = 0, \tag{C.22}
\]

\[
\sigma^{(0),\text{thr}}_{\bar{q}q \to \tilde{g}\tilde{g},2} = \frac{\pi (N_c^2 - 1)}{2N_c} \left( \hat{\alpha}_s^2 \frac{m_{\tilde{g}}^2}{m_4^4} - \frac{\alpha_s \hat{\alpha}_s (1 - \varepsilon)}{m_4^2 m_{\tilde{g}}^2} + \frac{\alpha_s^2 (1 - \varepsilon)}{4m_{\tilde{g}}^2} \right) \beta. \tag{C.23}
\]
The colour decomposition is given by:

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}} = \frac{2\alpha_s^2 \pi N_c^2}{(N_c^2 - 1)^2(1 - \varepsilon)^2 s} \left( \varepsilon - 1 - \frac{4m_{\tilde{g}}^2}{s} \right) \beta \]

\[ + \left( \frac{8m_{\tilde{g}}^4}{s^2} - \frac{4m_{\tilde{g}}^2}{s} - (1 - \varepsilon)^2 \right) \log \left( \frac{1 - \beta}{1 + \beta} \right), \]

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 1} = \frac{1}{4} (N_c^2 - 1) \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 1}, \]

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 3} = \frac{1}{4} (N_c^2 - 1) \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 1} - \frac{\alpha_s^2 \pi N_c^2}{(N_c^2 - 1)(1 - \varepsilon)^2 s} \left( \frac{8m_{\tilde{g}}^4}{s^2} \right) \log \left( \frac{1 - \beta}{1 + \beta} \right) \]

\[ + \left( \frac{4m_{\tilde{g}}^2}{s} - \frac{(s + 2m_{\tilde{g}}^2)(1 - \varepsilon)}{3s} + (1 - \varepsilon)^2 \right) \beta, \]

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 4} = \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 5} = 0, \]

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 6} = \frac{1}{4} (N_c + 3)(N_c - 1) \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 1}, \]

\[ \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 7} = \frac{1}{4} (N_c - 3)(N_c + 1) \sigma^{(0)}_{gg \rightarrow \tilde{g}\tilde{g}, 1}. \]

Near threshold this becomes:

\[ \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 1} = \frac{\alpha_s^2 \pi N_c^2 (1 - 2\varepsilon)}{2(N_c^2 - 1)^2(1 - \varepsilon)m_{\tilde{g}}^2}, \]

\[ \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 2} = \frac{\alpha_s^2 \pi N_c^2 (1 - 2\varepsilon)}{8(N_c^2 - 1)(1 - \varepsilon)m_{\tilde{g}}^2}, \]

\[ \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 3} = \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 4} = \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 5} = 0, \]

\[ \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 6} = \frac{\alpha_s^2 \pi N_c^2 (N_c + 3)(1 - 2\varepsilon)}{8(N_c^2 - 1)(N_c + 1)(1 - \varepsilon)m_{\tilde{g}}^2}, \]

\[ \sigma^{(0), \text{thr}}_{gg \rightarrow \tilde{g}\tilde{g}, 7} = \frac{\alpha_s^2 \pi N_c^2 (N_c - 3)(1 - 2\varepsilon)}{8(N_c^2 - 1)(N_c - 1)(1 - \varepsilon)m_{\tilde{g}}^2}. \]
\( qg \rightarrow \bar{q}\bar{g} \)

For the \( qg \rightarrow \bar{q}\bar{g} \) process, we have the following diagrams:

\[
M^{(0)}_{qg \rightarrow \bar{q}\bar{g}} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array},
\]

while the colour decomposition is given by:

\[
\sigma^{(0)}_{qg \rightarrow \bar{q}\bar{g}, 1} = \frac{\alpha_s \hat{\alpha} \pi}{(N_c^2 - 1)(1 - \varepsilon)s} \left( \frac{2m_g^2m_{\bar{g}}^2}{s^2} - \frac{(2m_g^2 + s)^2 + (1 - 2\varepsilon)s^2}{4s^2} \right) L_3 + \left( \frac{(3N_c^2 + 1)(N_c^2 - 1)(\varepsilon - 1)}{4N_c^2 s} - \frac{(N_c^2 - 1)^2(7 + \varepsilon)m_g^2}{4N_c^2 s} - \frac{2m_g^2}{s} \right) \kappa \beta \left( \frac{m_g^2 - s}{N_c^2 s} + \frac{2m_g^2}{s} \right) L_4 \right),
\]

\[
\sigma^{(0)}_{qg \rightarrow \bar{q}\bar{g}, 2} = \frac{\alpha_s \hat{\alpha} \pi (N_c - 2)}{(N_c - 1)(1 - \varepsilon)s} \left( \frac{m_g^2(m_g^2 - s)}{2s^2} L_4 - \frac{m_g^2}{s} \kappa \beta \left( \frac{m_g^2 - s}{N_c^2 s} + \frac{2m_g^2}{s} \right) L_3 \right),
\]

\[
\sigma^{(0)}_{qg \rightarrow \bar{q}\bar{g}, 3} = \frac{(N_c + 2)(N_c - 1)}{(N_c - 2)(N_c + 1)} \sigma^{(0)}_{qg \rightarrow \bar{q}\bar{g}, 2}.
\]

In the threshold limit, these cross sections become:

\[
\sigma^{(0), \text{thr}}_{qg \rightarrow \bar{q}\bar{g}, 1} = \frac{\alpha_s \hat{\alpha} \pi}{(m_{\bar{g}} + m_g)^2} \sqrt{m_{\bar{g}}} \frac{(N_c^2 m_{\bar{g}} + m_g)^2}{N_c^2(N_c^2 - 1)(m_{\bar{g}} + m_g)^2},
\]

\[
\sigma^{(0), \text{thr}}_{qg \rightarrow \bar{q}\bar{g}, 2} = \frac{\alpha_s \hat{\alpha} \pi}{(m_{\bar{g}} + m_g)^2} \sqrt{m_{\bar{g}}} \frac{N_c - 2}{2(N_c - 1)},
\]

\[
\sigma^{(0), \text{thr}}_{qg \rightarrow \bar{q}\bar{g}, 3} = \frac{\alpha_s \hat{\alpha} \pi}{(m_{\bar{g}} + m_g)^2} \sqrt{m_{\bar{g}}} \frac{N_c + 2}{2(N_c + 1)}.
\]
Appendix D

Mellin Transforms

In this appendix, we will present the Mellin transforms of the LO cross sections listed in Appendix C. In addition, we give the corresponding integrals for the Coulomb corrections, which receive an additional factor 1/β compared to the LO cross section. We will use the notation introduced in Appendix C, as well as the following definitions:

\[ z = \frac{4m_{\text{av}}^2}{s} = 1 - \beta^2, \quad r = \frac{m^2}{m_q}, \quad A = \frac{r - 1}{r + 1}, \quad B = \frac{r^2 - 1}{r^2 + 1}. \]

We will first list the integrals needed for the calculation and then give the full results for all processes.

D.1 Integrals

The solutions to the integrals are expressed in terms of the Γ-function:

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \]

and the generalized hypergeometric function:

\[ \pFq{a_1, \cdots, a_p; b_1, \cdots, b_q}{x} = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n) \cdots \Gamma(a_p+n)}{\Gamma(a_1) \cdots \Gamma(a_p)} \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(b_1+n) \cdots \Gamma(b_q+n)} \frac{x^n}{n!}. \]

First we have the integrals that correspond to the linear terms in β in the LO cross section. When including the 1/β factor from the Coulomb correction, these terms become constants. Such integrals occur in most processes and are given by:

\[ K(N) = \int_0^1 dz z^N \sqrt{1 - z} = \frac{\sqrt{\pi} \Gamma(N + 1)}{2 \Gamma(N + \frac{3}{2})}, \quad (D.1) \]

\[ \int_0^1 dz z^N = \frac{1}{N + 1}, \quad (D.2) \]

85
\textbf{D.1. Integrals}

For the quark-initiated processes we also need:

\begin{equation}
K_1(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} = \frac{1}{(r^2 + 1)^2} \frac{\sqrt{\pi} \Gamma(N + 1)}{2 \Gamma(N + \frac{5}{2})} 2F_1(1, \frac{3}{2}; N + \frac{5}{2}; B^2), \tag{D.3}
\end{equation}

\begin{equation}
M_1(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} = \frac{1}{(r^2 + 1)^2(N + 1)} 2F_1(1, 1; N + 2; B^2). \tag{D.4}
\end{equation}

For the $qg \rightarrow \bar{q}g$ case we need integrals for the $\kappa \beta$ terms:

\begin{equation}
K_2(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} = \frac{\sqrt{\pi} \Gamma(N + 1)}{2 \Gamma(N + \frac{5}{2})} 2F_1(-\frac{1}{2}, N + 1; N + \frac{5}{2}, A^2), \tag{D.5}
\end{equation}

\begin{equation}
M_2(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} = \frac{1}{N + 1} 2F_1(-\frac{1}{2}, N + 1; N + 2; A^2). \tag{D.6}
\end{equation}

For the gluon-initiated processes, we need the additional integrals:

\begin{equation}
K_3(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} \log \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right) = -\frac{\sqrt{\pi} \Gamma(N + 1)}{(N + 1) \Gamma(N + \frac{3}{2})}, \tag{D.7}
\end{equation}

\begin{equation}
M_3(N) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} \log \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right) = -2 \frac{\sqrt{\pi}}{N + 1} \frac{\Gamma(N + 1)}{\Gamma(N + \frac{3}{2})} 2F_2(\frac{1}{2}, 1, 1; \frac{3}{2}, N + 2), \tag{D.8}
\end{equation}

which can be calculated using the identity [174]:

\[
\log \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right) = -2 \sqrt{z} \frac{\sqrt{\pi}}{N + 1} \frac{\Gamma(N + 1)}{\Gamma(N + \frac{3}{2})} 2F_2(\frac{1}{2}, 1, 1; \frac{3}{2}, N + 2). \]

For the $q\bar{q} \rightarrow \bar{q}q$ process, we also need two integrals containing $L_1$. These can be obtained by rewriting the logarithm:

\[
\log \left( \frac{1 + \frac{1}{2}(r^2 - 1)z - \sqrt{1 - z}}{1 + \frac{1}{2}(r^2 - 1)z + \sqrt{1 - z}} \right) = \log \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right) + \log \left( \frac{1 + B \sqrt{1 - z}}{1 - B \sqrt{1 - z}} \right),
\]

which leads to the solution:

\begin{equation}
K_4(N, r) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} \log \left( \frac{1 + \frac{1}{2}(r^2 - 1)z - \sqrt{1 - z}}{1 + \frac{1}{2}(r^2 - 1)z + \sqrt{1 - z}} \right) \tag{D.9}
\end{equation}

\[
= K_3(N) + B \frac{\sqrt{\pi} \Gamma(N + 1)}{(N + 1) \Gamma(N + \frac{3}{2})} 2F_1(\frac{1}{2}, 1; N + \frac{5}{2}; B^2),
\]

\begin{equation}
M_4(N, r) = \int_0^1 \frac{dz}{z(r^2 - 1)^2 + 4r^2} \log \left( \frac{1 + \frac{1}{2}(r^2 - 1)z - \sqrt{1 - z}}{1 + \frac{1}{2}(r^2 - 1)z + \sqrt{1 - z}} \right) \tag{D.10}
\end{equation}

\[
= M_3(N) + \frac{2B}{N + 1} \frac{\sqrt{\pi} \Gamma(N + 1)}{(N + 1) \Gamma(N + \frac{3}{2})} 3F_2(\frac{1}{2}, 1, 1; \frac{3}{2}, N + 2; B^2).
\]
The corresponding integrals for $L_2$, which are needed for the $q\bar{q} \rightarrow g\bar{g}$ process, can be obtained by substituting $r \rightarrow 1/r$, which corresponds to $B \rightarrow -B$. For the $qq \rightarrow q\bar{q}$ process, we also need:

\[ K_5(N, r) = \int_0^1 \frac{dz}{z r + 1} \frac{z^N}{2 + (r^2 - 1)z} \log \left( \frac{1 + \frac{1}{2}(r^2 - 1)z - \sqrt{1 - z}}{1 + \frac{1}{2}(r^2 - 1)z + \sqrt{1 - z}} \right) \]  \hspace{1cm} (D.11)

\[ = -\frac{\Gamma(N + 1)}{r^2 + 1} \sum_{n,k=0}^\infty \frac{B^k(1 - B^{2n+1})}{n + \frac{1}{2}} \frac{\Gamma(n + k + \frac{3}{2})}{\Gamma(n + k + 2)} \frac{2F_1(1, n + \frac{3}{2}; N + n + \frac{5}{2}; B)}{\Gamma(n + n + \frac{5}{2})} . \]

\[ M_5(N, r) = \int_0^1 \frac{dz}{z r + 1} \frac{z^N}{(2 + (r^2 - 1)z)\sqrt{1 - z}} \log \left( \frac{1 + \frac{1}{2}(r^2 - 1)z - \sqrt{1 - z}}{1 + \frac{1}{2}(r^2 - 1)z + \sqrt{1 - z}} \right) \]  \hspace{1cm} (D.12)

\[ = -\frac{\Gamma(N + 1)}{r^2 + 1} \sum_{n,k=0}^\infty \frac{B^k(1 - B^{2n+1})}{n + \frac{1}{2}} \frac{\Gamma(n + k + 1)}{\Gamma(n + k + n + N + 2)} \frac{2F_1(1, n + 1; n + N + 2; B)}{\Gamma(n + N + 2)} . \]

Also in this case, the corresponding integrals for $L_2$, which are needed for the $q\bar{q} \rightarrow g\bar{g}$ process, can be obtained by substituting $r \rightarrow 1/r$.

For the $qg \rightarrow q\bar{g}$ process, we need the Mellin transforms of $L_3$ and $L_4$:

\[ K_6^\pm(N, r) = \int_0^1 \frac{dz}{z r + 1} \frac{z^N}{2 + (r^2 - 1)z} \log \left( \frac{1 \pm Az - \sqrt{(1 - z)(1 - A^2z)}}{1 \pm Az + \sqrt{(1 - z)(1 - A^2z)}} \right) \]  \hspace{1cm} (D.13)

\[ = -\frac{\sqrt{\pi} \Gamma(N + 1)}{\Gamma(N + \frac{3}{2}) \Gamma(N + N + 1)} \frac{2F_1(1, \frac{3}{2}; N + 1; N + \frac{3}{2}; A^2)}{2F_1(\frac{1}{2}, N + 2; N + \frac{5}{2}; A^2)} + A \frac{\sqrt{\pi} \Gamma(N + 1)}{\Gamma(N + \frac{5}{2})} \frac{2F_1(1, \frac{3}{2}; N + 2; N + \frac{5}{2}; A^2)}{2F_1(\frac{1}{2}, N + 2; N + \frac{5}{2}; A^2)} , \]

\[ M_6^\pm(N, r) = \int_0^1 \frac{dz}{z r + 1} \frac{z^N}{\sqrt{1 - z}} \log \left( \frac{1 \pm Az - \sqrt{(1 - z)(1 - A^2z)}}{1 \pm Az + \sqrt{(1 - z)(1 - A^2z)}} \right) \]  \hspace{1cm} (D.14)

\[ = -\frac{\Gamma(n + 1) \Gamma(N + 1)}{\Gamma(n + N + 2)} \frac{2F_1(n + \frac{1}{2}, N + 1; n + N + 2; A^2)}{2F_1(n + \frac{1}{2}, N + 2; n + N + 2; A^2)} + \frac{2A}{\sqrt{1 - A^2}} \frac{1}{N + 1} \frac{3F_2(\frac{1}{2}, \frac{1}{2}, 1; \frac{3}{2}, N + 2; \frac{A^2}{A^2-1})}{\sqrt{1 - A^2}} . \]

### D.2 LO Cross Sections and Coulomb Corrections

We will now list the Mellin transforms of the LO cross sections in Appendix C and the Coulomb corrections as derived using Eq. (5.3).
\[ q\bar{q} \to q\bar{q} \]

For the \( q\bar{q} \to q\bar{q} \) process, the Mellin transforms of the LO cross sections are:

\[
\hat{\sigma}^{(0)}_{q\bar{q} \to q\bar{q},1} = -\frac{\alpha_s^2 \pi (N_c^2 - 1)^2}{8m_q^2 N_c^4} \left[ 2K(N) - 4r^2 K_1(N) + K_3(N, r) + \frac{1}{2} r^2 K_4(N+1, r) \right],
\]
\( \text{D.15} \)

\[
\hat{\sigma}^{(0)}_{q\bar{q} \to q\bar{q},2} = \frac{1}{N_c^2 - 1} \hat{\sigma}^{(0)}_{q\bar{q} \to q\bar{q},1} + \delta_{f_1 f_2} \frac{\alpha_s^2 \pi (N_c^2 - 1)}{24m_q^2 N_c^2} \left( K(N) - K(N+1) \right)
\]
\[+ \delta_{f_1 f_2} \frac{\alpha_s^2 \pi (N_c^2 - 1)}{8m_q^2 N_c^4} \left[ K(N) + \frac{1}{2} r^2 (K(N+1) + \frac{1}{2} r^2 K_4(N+1, r) + \frac{1}{8} r^2 K_4(N+2, r) \right],
\]
\( \text{D.16} \)

while the Mellin transforms of the Coulomb corrections are given by:

\[
\hat{\sigma}^{\text{Coul.(1)}}_{q\bar{q} \to q\bar{q},1} = \frac{\alpha_s^2 \pi (N_c^2 - 1)^3}{32m_q^2 N_c^5} \left[ -2 \frac{N+1}{N+1} + 4r^2 M_1(N) - M_4(N, r) - \frac{1}{2} (r^2 - 1) M_4(N+1, r) \right],
\]
\( \text{D.17} \)

\[
\hat{\sigma}^{\text{Coul.(1)}}_{q\bar{q} \to q\bar{q},2} = -\frac{1}{(N_c^2 - 1)^2} \hat{\sigma}^{\text{Coul.(1)}}_{q\bar{q} \to q\bar{q},1} - \delta_{f_1 f_2} \frac{\alpha_s^2 \pi (N_c^2 - 1)}{96m_q^2 N_c^2} \left[ \frac{1}{(N+1)(N+2)} \right.
\]
\[\left. - \delta_{f_1 f_2} \frac{\alpha_s^2 \pi (N_c^2 - 1)}{32m_q^2 N_c^4} \left[ \frac{1}{N+1} + \frac{r^2 - 1}{2(N+2)} + \frac{1}{4} r^2 M_4(N+1, r) + \frac{1}{8} (r^2 - 1)^2 M_4(N+2, r) \right] \right].
\]
\( \text{D.18} \)

\[ gg \to q\bar{q} \]

For the \( gg \to q\bar{q} \) process, the Mellin transforms of the LO cross sections are given by:

\[
\hat{\sigma}^{(0)}_{gg \to q\bar{q},1} = \frac{\alpha_s^2 \pi n_t}{4m_q^2 N_c (N_c^2 - 1)} \left[ K(N) + K(N+1) + K_3(N+1) - \frac{1}{2} K_3(N+2) \right],
\]
\( \text{D.19} \)

\[
\hat{\sigma}^{(0)}_{gg \to q\bar{q},2} = \frac{\alpha_s^2 \pi n_t N_c}{4m_q^2 (N_c^2 - 1)} \left[ \frac{1}{2} K(N) + \frac{3}{4} K(N+1) + \frac{1}{2} K_3(N+1) + \frac{1}{4} K_3(N+2) \right],
\]
\( \text{D.20} \)

\[
\hat{\sigma}^{(0)}_{gg \to q\bar{q},3} = \left( \frac{1}{2} (N_c^2 - 4) - 2 \right) \hat{\sigma}^{(0)}_{gg \to q\bar{q},1},
\]
\( \text{D.21} \)

and the Coulomb corrections in \( N \)-space are:

\[
\hat{\sigma}^{\text{Coul.(1)}}_{gg \to q\bar{q},1} = \frac{\alpha_s^2 \pi n_t}{16m_q^2 N_c^2} \left[ \frac{1}{N+1} + \frac{1}{N+2} + M_3(N+1) - \frac{1}{2} M_3(N+2) \right],
\]
\( \text{D.22} \)

\[
\hat{\sigma}^{\text{Coul.(1)}}_{gg \to q\bar{q},2} = -\frac{\alpha_s^2 \pi n_t}{16m_q^2 (N_c^2 - 1)} \left[ \frac{1}{6(N+1)} + \frac{4}{3(N+2)} + \frac{1}{2} M_3(N+1) + \frac{1}{4} M_3(N+2) \right],
\]
\( \text{D.23} \)

\[
\hat{\sigma}^{\text{Coul.(1)}}_{gg \to q\bar{q},3} = -\frac{1}{2} \frac{N_c^2 - 4}{N_c^2 - 1} \hat{\sigma}^{\text{Coul.(1)}}_{gg \to q\bar{q},1},
\]
\( \text{D.24} \)

88
Appendix D. Mellin Transforms

$qq \to \bar{q}\bar{q}$

The structure of the $qq \to \bar{q}\bar{q}$ process has much in common with the singlet channel of the $q\bar{q} \to \bar{q}q$ process:

\[
\begin{align*}
\bar{\sigma}_{qq\to\bar{q}\bar{q},1}^{(0)} &= \frac{\alpha_s^2 \pi^2 (N_c^2 - 1)(N_c + 1)}{16m_Q^2 N_c^3} \left[ -2K(N) + 4r^2 K_1(N) - K_4(N, r) \right] (D.25) \\
&\quad - \frac{1}{2} (r^2 - 1) K_4(N + 1, r) + r^2 \delta_{f_1 f_2} K_5(N + 1, r), \\
\bar{\sigma}_{qq\to\bar{q}\bar{q},2}^{(0)} &= \frac{\alpha_s^2 \pi^2 (N_c^2 - 1)(N_c + 1)}{16m_Q^2 N_c^3} \left[ -2K(N) + 4r^2 K_1(N) - K_4(N, r) \right] (D.26) \\
&\quad - \frac{1}{2} (r^2 - 1) K_4(N + 1, r) - r^2 \delta_{f_1 f_2} K_5(N + 1, r),
\end{align*}
\]

and the Mellin transforms of the Coulomb corrections are:

\[
\begin{align*}
\bar{\sigma}_{qq\to\bar{q}\bar{q},1}^{\text{Coul}(1)} &= \frac{\alpha_s^3 \pi^2 (N_c^2 - 1)(N_c + 1)^2}{64m_Q^2 N_c^4} \left[ \frac{-2}{N + 1} + 4r^2 M_1(N) - M_4(N, r) \right] (D.27) \\
&\quad - \frac{1}{2} (r^2 - 1) M_4(N + 1, r) + r^2 \delta_{f_1 f_2} M_5(N + 1, r), \\
\bar{\sigma}_{qq\to\bar{q}\bar{q},2}^{\text{Coul}(1)} &= \frac{\alpha_s^3 \pi^2 (N_c^2 - 1)(N_c - 1)^2}{64m_Q^2 N_c^4} \left[ \frac{2}{N + 1} - 4r^2 M_1(N) + M_4(N, r) \right] (D.28) \\
&\quad + \frac{1}{2} (r^2 - 1) M_4(N + 1, r) + r^2 \delta_{f_1 f_2} M_5(N + 1, r).
\end{align*}
\]

$q\bar{q} \to g\bar{g}$

The Mellin transforms of the LO cross sections of the $q\bar{q} \to g\bar{g}$ process are given by:

\[
\begin{align*}
\bar{\sigma}_{q\bar{q}\to g\bar{g},1}^{(0)} &= \frac{\alpha_s^2 \pi (N_c^2 - 1)}{8m_Q^2 N_c^3} \left[ 2K(N) - 4r^2 K_1(N) - \frac{r^2 - 1}{2r^2} K_4(N + 1, \frac{1}{r}) \right] (D.29) \\
&\quad + K_5(N + 1, \frac{1}{r}), \\
\bar{\sigma}_{q\bar{q}\to g\bar{g},2}^{(0)} &= \frac{\alpha_s^2 \pi (N_c^2 - 1)}{16m_Q^2 N_c} \left[ \frac{3 - r^2}{3r^2} K(N + 1) - 4r^2 K_1(N) + \frac{4}{3} K(N) \right] (D.30) \\
&\quad + \frac{r^2 + 1}{2r^2} K_4(N + 1, \frac{1}{r}) + \frac{(r^2 - 1)^2}{4r^4} K_4(N + 2, \frac{1}{r}) - K_5(N + 1, \frac{1}{r}), \\
\bar{\sigma}_{q\bar{q}\to g\bar{g},3}^{(0)} &= \frac{1}{2}(N_c^2 - 4) \bar{\sigma}_{q\bar{q}\to g\bar{g},1}^{(0)}, (D.31)
\end{align*}
\]
D.2. LO Cross Sections and Coulomb Corrections

and the Mellin-transformed Coulomb corrections are:

\[
\hat{\sigma}_{q\overline{q}ightharpoonup g\overline{g},1}^{\text{Coul}} = \frac{\alpha_s^2 \pi^2 (N_c^2 - 1)}{16 m_g^2 N_c^2} \left[ \frac{2}{N+1} - \frac{r^2 - 1}{2r^2} M_1(N) - \frac{M_1(N+1, 1/r)}{2r^2} \right] + M_5(N+1, 1/r),
\]

\[
\hat{\sigma}_{q\overline{q}ightharpoonup g\overline{g},2}^{\text{Coul}} = \frac{\alpha_s^2 \pi^2 (N_c^2 - 1)}{64 m_g^2} \left[ \frac{3 - r^2}{3r^2 \cdot N+2} - \frac{4r^2}{3(N+1)} \right] + \frac{r^2 + 1}{2r^2} M_4(N+1, 1/r) + \frac{(r^2 - 1)^2}{4r^4} M_4(N+2, 1/r) - M_5(N+1, 1/r),
\]

\[
\hat{\sigma}_{q\overline{q}ightharpoonup g\overline{g},3}^{\text{Coul}} = \frac{1}{4}(N_c^2 - 4)\hat{\sigma}_{q\overline{q}ightharpoonup g\overline{g},1}^{\text{Coul}}.
\]

\[gg \rightarrow \tilde{g}\tilde{g}\]

For the \(gg \rightarrow \tilde{g}\tilde{g}\) process, the Mellin transforms of the LO cross sections are given by:

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{(0)} = \frac{-\alpha_s^2 \pi N_c^2}{2m_g^2 (N_c^2 - 1)} \left[ K(N) + K(N+1) + K_3(N) + K_3(N+1) - \frac{1}{2}K_3(N+2) \right],
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},2}^{(0)} = \frac{-\alpha_s^2 \pi N_c^2}{8m_g^2 (N_c^2 - 1)} \left[ \frac{7}{3} K(N) + \frac{8}{3} K(N+1) + K_3(N) + K_3(N+1) + \frac{1}{2}K_3(N+2) \right],
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},3}^{(0)} = \frac{1}{4}(N_c^2 - 1)\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{(0)},
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},4}^{(0)} = \hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},5}^{(0)} = 0,
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},6}^{(0)} = \frac{1}{4}(N_c+3)(N_c-1)\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{(0)},
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},7}^{(0)} = \frac{1}{4}(N_c-3)(N_c+1)\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{(0)},
\]

and the Mellin transforms of the Coulomb corrections are:

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{\text{Coul}} = \frac{-\alpha_s^2 \pi^2 N_c^3}{4m_g^2 (N_c^2 - 1)^2} \left[ \frac{1}{N+1} + \frac{1}{N+2} + M_3(N) + M_3(N+1) - \frac{1}{2}M_3(N+2) \right],
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},2}^{\text{Coul}} = \frac{-\alpha_s^2 \pi^2 N_c^3}{32m_g^2 (N_c^2 - 1)} \left[ \frac{7}{3(N+1)} + \frac{8}{3(N+2)} + M_3(N) + M_3(N+1) \right]
\]

\[
+ \frac{1}{2}M_3(N+2),
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},3}^{\text{Coul}} = \frac{1}{8}(N_c^2 - 1)\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{\text{Coul}},
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},4}^{\text{Coul}} = \hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},5}^{\text{Coul}} = 0,
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},6}^{\text{Coul}} = -\frac{(N_c+3)(N_c-1)}{4N_c} \hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{\text{Coul}},
\]

\[
\hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},7}^{\text{Coul}} = \frac{(N_c-3)(N_c+1)}{4N_c} \hat{\sigma}_{gg\rightharpoonup \tilde{g}\tilde{g},1}^{\text{Coul}}.
\]
Appendix D. Mellin Transforms

$qg \rightarrow \bar{q}g$

The Mellin transforms of the LO cross sections of the $qg \rightarrow \bar{q}g$ process are given by:

\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,1}^{(0)} = \frac{\alpha_s^2 \pi}{4m_g^2(N_c^2-1)} \left[ \left( \frac{1}{2} + \frac{1}{4N_c^2} - \frac{3N_c^2}{4} \right) K_2(N) - \frac{N_c^2}{2} K_6^+(N) \right] \]  
\[ + \left( \frac{3}{2} - \frac{7}{4N_c^2} - \frac{7N_c^2}{4} \right) A K_2(N+1) + \frac{A(1+A)^2}{2} K_6^+(N+2) - A N_c^2 K_6^+(N+1) \]
\[ - A^2 N_c^2 K_6^+(N+2) + \frac{A(1-A)^2}{2} K_6^+(N+1) - \frac{A}{2} K_6^+(N+1) + \frac{A}{4} (1+A^2) K_6^+(N+2) \right], \]  
\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,2}^{(0)} = \frac{\alpha_s^2 \pi (N_c-2)}{4m_g^2(N_c-1)} \left[ \frac{A(1+A)^2}{4} K_6^-(N+2) - A K_2(N+1) \right] \]
\[ - \frac{A}{2} K_6^-(N+1) - \frac{1}{4} K_6^+(N) + \frac{A}{4} (1-A) K_6^+(N+1) + \frac{A}{4} (1+A^2) K_6^+(N+2) \right], \]
\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,3}^{(0)} = \frac{N_c+2}{N_c+1} \frac{N_c-1}{N_c-2} \tilde{\sigma}_{qg \rightarrow \bar{q}g,2}^{(0)}, \]  

and the Coulomb corrections in $N$-space are:

\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,1}^{\text{Coul}(1)} = \frac{\alpha_s^2 \pi^2 N_c}{16m_g^2(N_c^2-1)} \sqrt{m_g m_q} \left[ \left( \frac{1}{2} + \frac{1}{4N_c^2} - \frac{3N_c^2}{4} \right) M_2(N) - \frac{N_c^2}{2} M_6^+(N) \right] \]  
\[ + \left( \frac{3}{2} - \frac{7}{4N_c^2} - \frac{7N_c^2}{4} \right) A M_2(N+1) + \frac{A(1+A)^2}{2} M_6^+(N+2) - A N_c^2 M_6^+(N+1) \]
\[ - A^2 N_c^2 M_6^+(N+2) + \frac{A(1-A)^2}{2} M_6^+(N+1) - \frac{A}{2} M_6^+(N+1) + \frac{A}{4} (1+A^2) M_6^+(N+2) \right], \]  
\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,2}^{\text{Coul}(1)} = \frac{\alpha_s^2 \pi^2 (N_c-2)}{16m_g^2(N_c-1)} \sqrt{m_g m_q} \left[ \frac{A(1+A)^2}{4} M_6^-(N+2) - A M_2(N+1) \right] \]
\[ - \frac{A}{2} M_6^-(N+1) - \frac{1}{4} M_6^+(N) + \frac{A}{4} (1-A) M_6^+(N+1) + \frac{A}{4} (1+A^2) M_6^+(N+2) \right], \]
\[ \tilde{\sigma}_{qg \rightarrow \bar{q}g,3}^{\text{Coul}(1)} = \frac{N_c+2}{N_c+1} \frac{N_c-1}{N_c-2} \tilde{\sigma}_{qg \rightarrow \bar{q}g,2}^{\text{Coul}(1)}, \]  

91
Appendix E

One-Loop Soft Anomalous Dimensions for SUSY-QCD

In this appendix we will list the soft anomalous dimensions for SUSY QCD. The one-loop soft anomalous-dimension matrices for the squark-antisquark and gluino-pair production processes agree with the calculation in Ref. [137]. We will present the matrices in terms of the colour bases in Appendix B. Labelling the initial-state partons by 1 and 2 and the final-state sparticles by 3 and 4, we introduce the definitions:

\[
L_{34} = \frac{\kappa^2 + \beta^2}{2\kappa\beta} \left[ \log\left(\frac{\kappa - \beta}{\kappa + \beta}\right) + i\pi \right], \quad \beta = \sqrt{1 - \frac{(m_3 + m_4)^2}{s}},
\]

\[
\Lambda_{34} = \frac{1}{2} \left[ T(m_3) + T(m_4) + U(m_3) + U(m_4) \right], \quad \kappa = \sqrt{1 - \frac{(m_3 - m_4)^2}{s}},
\]

\[
\Omega_{34} = \frac{1}{2} \left[ T(m_3) + T(m_4) - U(m_3) - U(m_4) \right],
\]

with \( s \) the CM energy squared. The quantities \( \Lambda \) and \( \Omega \) are defined in terms of the \( t \)- and \( u \)-channel logarithms:

\[
T(m_3) = \log\left( \frac{2p_1 \cdot p_3}{m_3\sqrt{s}} \right) - \frac{1 - i\pi}{2}, \quad U(m_3) = \log\left( \frac{2p_2 \cdot p_3}{m_3\sqrt{s}} \right) - \frac{1 - i\pi}{2},
\]

\[
T(m_4) = \log\left( \frac{2p_2 \cdot p_4}{m_4\sqrt{s}} \right) - \frac{1 - i\pi}{2}, \quad U(m_4) = \log\left( \frac{2p_1 \cdot p_4}{m_4\sqrt{s}} \right) - \frac{1 - i\pi}{2}.
\]

Note that for processes where the final-state particles have equal masses the quantities \( L_{34}, \Lambda_{34} \) and \( \Omega_{34} \) reduce to the more familiar forms:

\[
L_{34} \xrightarrow{m_3=m_4} \frac{1 + \beta^2}{2\beta} \left[ \log\left(\frac{1 - \beta}{1 + \beta}\right) + i\pi \right],
\]

\[
\Lambda_{34} \xrightarrow{m_3=m_4} T(m) + U(m), \quad \Omega_{34} \xrightarrow{m_3=m_4} T(m) - U(m).
\]
\( q\bar{q} \to \tilde{q}\bar{\tilde{q}} \)

The one-loop soft anomalous dimension matrix for the \( q\bar{q} \to \tilde{q}\bar{\tilde{q}} \) process is given by:

\[
\tilde{\Gamma}^{(1)}_{q\bar{q} \to \tilde{q}\bar{\tilde{q}}} = \frac{\alpha_s}{2\pi} \left( \begin{array}{ccc}
\tilde{\Gamma}_{11} & 2\sqrt{2(N_c^2 - 1) \Omega_{q\bar{q}}} & 0 \\
2\sqrt{\frac{2}{N_c^2 - 1} \Omega_{q\bar{q}}} & \tilde{\Gamma}_{22} & \sqrt{N_c^2 - 4 \Omega_{q\bar{q}}} \\
0 & \frac{\sqrt{N_c^2 - 4 \Omega_{q\bar{q}}}}{N_c} & \tilde{\Gamma}_{33}
\end{array} \right),
\]

(E.1)

The quadratic Casimir invariants for the \( q\bar{q} \to \tilde{q}\bar{\tilde{q}} \) process are listed in Eqs. (B.1) and (B.2).

\( gg \to \tilde{q}\bar{\tilde{q}} \)

The one-loop soft anomalous dimension matrix for the \( gg \to \tilde{q}\bar{\tilde{q}} \) process is given by:

\[
\tilde{\Gamma}^{(1)}_{gg \to \tilde{q}\bar{\tilde{q}}} = \frac{\alpha_s}{2\pi} \left( \begin{array}{ccc}
\tilde{\Gamma}_{11} & 2\sqrt{2(N_c^2 - 1) \Omega_{q\bar{q}}} & 0 \\
2\sqrt{\frac{2}{N_c^2 - 1} \Omega_{q\bar{q}}} & \tilde{\Gamma}_{22} & \sqrt{N_c^2 - 4 \Omega_{q\bar{q}}} \\
0 & \frac{\sqrt{N_c^2 - 4 \Omega_{q\bar{q}}}}{N_c} & \tilde{\Gamma}_{33}
\end{array} \right),
\]

(E.2)

with:

\[
\begin{align*}
\tilde{\Gamma}_{11} &= -\frac{N_c^2 - 1}{N_c} (L_{q\bar{q}} + 1), \\
\tilde{\Gamma}_{22} &= C_2(R_2) \Lambda_{q\bar{q}} + \frac{1}{N_c} (L_{q\bar{q}} + 1), \\
\tilde{\Gamma}_{33} &= C_2(R_3) \Lambda_{q\bar{q}} + \frac{1}{N_c} (L_{q\bar{q}} + 1).
\end{align*}
\]

The quadratic Casimir invariants for the \( gg \to \tilde{q}\bar{\tilde{q}} \) process can be found in Eqs. (B.3-B.5).

\( qq \to \tilde{q}\bar{\tilde{q}} \)

The one-loop soft anomalous dimension matrix for the \( qq \to \tilde{q}\bar{\tilde{q}} \) process is given by:

\[
\tilde{\Gamma}^{(1)}_{qq \to \tilde{q}\bar{\tilde{q}}} = \frac{\alpha_s}{2\pi} \left( \begin{array}{cc}
\tilde{\Gamma}_{11} & -(N_c + 1) \Omega_{q\bar{q}} \\
-(N_c - 1) \Omega_{q\bar{q}} & \tilde{\Gamma}_{22}
\end{array} \right),
\]

(E.3)

with the diagonal elements given by:

\[
\begin{align*}
\tilde{\Gamma}_{11} &= C_2(R_1) \Lambda_{q\bar{q}} - \frac{N_c + 1}{N_c} (L_{q\bar{q}} + 1), \\
\tilde{\Gamma}_{22} &= C_2(R_2) \Lambda_{q\bar{q}} + \frac{N_c - 1}{N_c} (L_{q\bar{q}} + 1).
\end{align*}
\]

The quadratic Casimir invariants for the squark-pair production process are listed in Eqs. (B.7) and (B.8).
\( q\bar{q} \to \tilde{g}\tilde{g} \)

The one-loop soft anomalous dimension matrix for the \( q\bar{q} \to \tilde{g}\tilde{g} \) process is given by:

\[
\tilde{\Gamma}^{(1)}_{q\bar{q} \to \tilde{g}\tilde{g}} = \frac{\alpha_s}{2\pi} \begin{pmatrix}
\tilde{\Gamma}_{11} & -2\sqrt{2(N_c^2 - 1)} \Omega_{\tilde{g}\tilde{g}} & 0 \\
-2\sqrt{\frac{2}{N_c^2-1}} \Omega_{\tilde{g}\tilde{g}} & \tilde{\Gamma}_{22} & -\sqrt{N_c^3 - 4} \Omega_{\tilde{g}\tilde{g}} \\
0 & -\sqrt{N_c^3 - 4} \Omega_{\tilde{g}\tilde{g}} & \tilde{\Gamma}_{33}
\end{pmatrix},
\]

(E.4)

where the diagonal elements are given by:

\[
\begin{align*}
\tilde{\Gamma}_{11} &= -2N_c(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{22} &= C_2(R_2)\Lambda_{\tilde{g}\tilde{g}} - N_c(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{33} &= C_2(R_3)\Lambda_{\tilde{g}\tilde{g}} - N_c(L_{\tilde{g}\tilde{g}} + 1).
\end{align*}
\]

The quadratic Casimir invariants for the \( q\bar{q} \to \tilde{g}\tilde{g} \) process can be found in Eqs. (B.9-B.11).

\( gg \to \tilde{g}\tilde{g} \)

The one-loop soft anomalous dimension matrix for the \( gg \to \tilde{g}\tilde{g} \) process is given by:

\[
\tilde{\Gamma}^{(1)}_{gg \to \tilde{g}\tilde{g}} = \frac{\alpha_s}{2\pi} \begin{pmatrix}
\tilde{\Gamma}_{11} & 0 & 4N_c \Omega_{\tilde{g}\tilde{g}} & 0 & 0 & 0 & 0 \\
0 & \tilde{\Gamma}_{22} & N_c \Omega_{\tilde{g}\tilde{g}} & N_c \Omega_{\tilde{g}\tilde{g}} & N_c \Omega_{\tilde{g}\tilde{g}} & 0 & 0 \\
0 & N_c \Omega_{\tilde{g}\tilde{g}} & \tilde{\Gamma}_{33} & 0 & 0 & \tilde{\Gamma}_{36} & \tilde{\Gamma}_{37} \\
0 & 0 & \tilde{\Gamma}_{42} & 0 & \tilde{\Gamma}_{44} & 0 & \tilde{\Gamma}_{46} & \tilde{\Gamma}_{47} \\
0 & 0 & 0 & \tilde{\Gamma}_{52} & 0 & 0 & \tilde{\Gamma}_{55} & \tilde{\Gamma}_{56} & \tilde{\Gamma}_{57} \\
0 & 0 & 4N_c \Omega_{\tilde{g}\tilde{g}} & \tilde{\Gamma}_{64} & \tilde{\Gamma}_{65} & \tilde{\Gamma}_{66} & 0 \\
0 & 0 & 4N_c \Omega_{\tilde{g}\tilde{g}} & \tilde{\Gamma}_{74} & \tilde{\Gamma}_{75} & 0 & \tilde{\Gamma}_{77}
\end{pmatrix},
\]

(E.5)

with:

\[
\begin{align*}
\tilde{\Gamma}_{11} &= -2N_c(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{22} &= C_2(R_2)\Lambda_{\tilde{g}\tilde{g}} - N_c(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{33} &= C_2(R_3)\Lambda_{\tilde{g}\tilde{g}} - N_c(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{44} &= C_2(R_4)\Lambda_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{55} &= C_2(R_5)\Lambda_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{66} &= C_2(R_6)\Lambda_{\tilde{g}\tilde{g}} + 2(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{77} &= C_2(R_7)\Lambda_{\tilde{g}\tilde{g}} - 2(L_{\tilde{g}\tilde{g}} + 1), \\
\tilde{\Gamma}_{42} &= \tilde{\Gamma}_{56} = \frac{N_c(N_c-2)}{N_c+1} \Omega_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{36} &= \tilde{\Gamma}_{57} = \frac{N_c(N_c-3)}{N_c+2} \Omega_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{37} &= \frac{N_c(N_c-3)}{N_c+1} \Omega_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{47} &= \frac{N_c(N_c-3)}{N_c+2} \Omega_{\tilde{g}\tilde{g}}, \\
\tilde{\Gamma}_{42} &= \frac{4N_c}{N_c^2-4} \Omega_{\tilde{g}\tilde{g}}.
\end{align*}
\]

The quadratic Casimir invariants for the \( gg \to \tilde{g}\tilde{g} \) process are listed in Eqs (B.19-B.25).
\[ qg \rightarrow \bar{q}\bar{g} \]

The one-loop soft anomalous dimension matrix for the \( qg \rightarrow \bar{q}\bar{g} \) process is given by:

\[
\hat{\Gamma}^{(1)}_{qg\rightarrow \bar{q}\bar{g}} = \frac{\alpha_s}{2\pi} \begin{pmatrix}
\hat{\Gamma}_{11} & \frac{N_c(N_c-2)}{N_c-1} \Omega_{\bar{q}\bar{g}} & \frac{N_c(N_c+2)}{N_c+1} \Omega_{\bar{q}\bar{g}} \\
\frac{2N_c}{N_c^2-1} \Omega_{\bar{q}\bar{g}} & \hat{\Gamma}_{22} & \frac{N_c(N_c-2)}{N_c-1} \Omega_{\bar{q}\bar{g}} \\
\frac{2N_c}{N_c^2-1} \Omega_{\bar{q}\bar{g}} & \frac{N_c(N_c-2)}{N_c-1} \Omega_{\bar{q}\bar{g}} & \hat{\Gamma}_{33}
\end{pmatrix},
\]

(E.6)

with:

\[
\hat{\Gamma}_{11} = C_2(R_1) \Lambda_{\bar{q}\bar{g}} + [C_F + \frac{1}{2N_c}] \Omega_{\bar{q}\bar{g}} - \frac{N_c^2+1}{2N_c} [T(m_\bar{q}) - T(m_\bar{g})] - N_c (L_{\bar{q}\bar{g}} + 1),
\]

\[
\hat{\Gamma}_{22} = C_2(R_2) \Lambda_{\bar{q}\bar{g}} + [C_F - \frac{1}{N_c-1}] \Omega_{\bar{q}\bar{g}} - \frac{N_c^2+1}{2N_c} [T(m_\bar{q}) - T(m_\bar{g})] - (L_{\bar{q}\bar{g}} + 1),
\]

\[
\hat{\Gamma}_{33} = C_2(R_3) \Lambda_{\bar{q}\bar{g}} + [C_F - \frac{1}{N_c+1}] \Omega_{\bar{q}\bar{g}} - \frac{N_c^2+1}{2N_c} [T(m_\bar{q}) - T(m_\bar{g})] + (L_{\bar{q}\bar{g}} + 1),
\]

where \( C_F = \frac{N_c^2-1}{2N_c} \) and the quadratic Casimir invariants for the \( qg \rightarrow \bar{q}\bar{g} \) process can be found in Eqs. (B.29-B.31).
Appendix F

Hard Matching Coefficients for SUSY-QCD

Here we present the exact expressions for the hard matching coefficients $C^{(1)}$ for the SUSY-QCD production processes. We sum over squarks with both chiralities ($\tilde{q}_L$ and $\tilde{q}_R$). No top-squark final states are considered and all squarks are considered to be mass-degenerate with mass $m_{\tilde{q}}$. Top squarks are taken into account in the loops, where they are taken to be mass-degenerate with the other squarks. The calculation is outlined in section 5.3 and was done with FORM [175]. We first define:

$$\beta_{12}(q^2) = \sqrt{1 - \frac{4m_1m_2}{q^2 - (m_1 - m_2)^2}}, \quad x_{12}(q^2) = \frac{\beta_{12}(q^2) - 1}{\beta_{12}(q^2) + 1}$$

and

$$m_\pm^2 = m_\tilde{g}^2 - m_{\tilde{q}}^2, \quad m_{\tilde{q}}^2 = m_\tilde{g}^2 + m_{\tilde{q}}^2,$$

where $m_\tilde{g}$ is the gluino mass and $m_{\tilde{q}}$ the squark mass. Furthermore, $m_{\text{avg}}$ is the average mass of the produced particles. Denoting the number of light flavours by $n_l = 5$, the total number of flavours by $n_f = 6$ and the number of colours by $N_c$, we also define:

$$\gamma_q = \frac{3}{2} C_F, \quad C_F = \frac{N_c^2 - 1}{2N_c},$$

$$\gamma_g = \frac{11}{6} C_A - \frac{1}{3} m_l, \quad C_A = N_c.$$

We denote the factorization scale by $\mu_F$, the renormalization scale by $\mu_R$ and Euler’s constant by $\gamma_E$. The dilogarithm is defined as

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1 - t)}{t} \, dt.$$
\( q\bar{q} \rightarrow q\bar{q} \) and \( qq \rightarrow q\bar{q} \)

We split the hard matching coefficients into a representation-independent part and a part that depends on the irreducible representation in the colour decomposition of Appendix B. Then the hard matching coefficients for the \( q\bar{q} \rightarrow q\bar{q} \) and the \( qq \rightarrow q\bar{q} \) process are the same provided that the appropriate representations are used:

\[
C_{q\bar{q} \rightarrow q\bar{q},I}^{(1)} = C_{qq \rightarrow q\bar{q},I}^{(1)} = \text{Re} \left\{ \frac{2C_F}{3} \pi^2 + \gamma_g \log \left( \frac{\mu_q^2}{m_q^2} \right) - \gamma_q \log \left( \frac{\mu_q^2}{m_q^2} \right) + \frac{19N_c}{24} + \frac{23}{8N_c} \right\} C_F \\
- \frac{2}{N_c} \log(2) + \left( \frac{7N_c}{6} + \frac{2m_q^2}{m_+^2} C_F \right) \log \left( \frac{m_q^2}{m_q^2} \right) - \frac{m_q^2}{m_q^2} \log \left( \frac{m_q^2}{m_q^2} \right) + 1 \right\} C_F \\
+ F_0(m_q, m_{\bar{q}}, m_t) - \frac{1}{2N_c} \left( \frac{m_q^2}{m_{\bar{q}}^2} - 3 \right) F_1(m_q, m_{\bar{q}}) + \left( \frac{m_q^2}{2m_q^2} C_F + \frac{1}{N_c} \right) F_2(m_q, m_{\bar{q}}) \\
+ 2C_F \left[ \gamma_q^2 + \gamma_E \log \left( \frac{\mu_q^2}{4m_q^2} \right) \right] + \frac{m_q^2}{2m_+^2} \log \left( \frac{m_q^2}{m_q^2} \right) - 1 \right\} C_F + \frac{3N_q^2}{N_c} \log \left( \frac{m_q^2}{m_q^2} \right) \\
+ \left\{ - \frac{\pi^2}{4} + \log \left( \frac{m_q^2}{m_q^2} \right) - \log(2) - \frac{m_q^2}{m_q^2} \log \left( \frac{m_q^2}{m_q^2} \right) + 2 + \gamma_E \\
- \frac{1}{8} \left( \frac{m_q^2}{m_q^2} \right)^3 \right\} \left[ F_1(m_q, m_{\bar{q}}) + F_2(m_q, m_{\bar{q}}) \right] \right\} C_2(R_I) \right\}.
\]

In this equation the last two lines are proportional to the quadratic Casimir invariants of the representations, which are given in Eqs. (B.1) and (B.2) for the \( q\bar{q} \rightarrow q\bar{q} \) process and in Eqs. (B.7) and (B.8) for the \( qq \rightarrow q\bar{q} \) process. Furthermore we have defined the functions:

\[
F_0(m_q, m_{\bar{q}}, m_t) = \frac{m_q^2}{2m_{\bar{q}}^2} - \left( 1 + \frac{m_q^2}{m_{\bar{q}}^2} \right) n_f + \left[ \frac{m_q^2}{2m_+^2} \log \left( \frac{m_q^2}{m_q^2} \right) \right] + \frac{4m_q^2}{m_q^2} \log(2) n_f \\
+ \left[ \frac{m_q^4}{2m_+^2m_q^2} - \frac{(m_q^2 - m_t^2)^2}{4m_q^4} + \frac{m_q^2 - m_t^2}{m_q^2} - \frac{1}{12} \right] \log \left( \frac{m_q^2}{m_q^2} \right) \\
- \frac{m_q^2(m_q^2 - m_t^2)(m_q^2 - m_{\bar{q}}^2 + m_t^2)}{2m_q^4} \beta_q(m_q^2) \log \left( x_{q\bar{q}}(m_q^2) \right) \\
+ \frac{m_q^4 - 2m_q^2m_t^2 + 4m_q^4m_t - 4m_q^4}{m_q^2m_{\bar{q}}^2} \beta_{q\bar{q}}(-m_q^2) \log \left( x_{q\bar{q}}(-m_q^2) \right) ,
\]

\[
F_1(m_q, m_{\bar{q}}) = \text{Li}_2 \left( \frac{m_q^2}{2m_q^2} \right) + \text{Li}_2 \left( 1 - \frac{m_q^2}{2m_q^2} \right) + \log \left( \frac{m_q^2}{2m_q^2} \right) \log \left( \frac{m_q^2}{2m_q^2} \right) + \frac{1}{2} \log^2 \left( \frac{m_q^2}{m_q^2} \right) + \frac{\pi^2}{12} ,
\]

\[
F_2(m_q, m_{\bar{q}}) = \text{Li}_2 \left( \frac{m_q^2}{m_q^2} \right) - \text{Li}_2 \left( - \frac{m_q^2}{m_q^2} \right) + \log \left( \frac{m_q^2}{m_q^2} \right) \log \left( \frac{m_q^2}{m_q^2} \right) .
\]
For the $gg \rightarrow q\bar{q}$ process the antisymmetric representation in Eq. (B.4) does not contribute because it yields a $p$-wave contribution, which vanishes at threshold. The hard matching coefficients for the representations of Eqs. (B.3) and (B.5) do contribute:

$$C^{(1)}_{gg \rightarrow q\bar{q},2} = 0,$$

$$C^{(1)}_{gg \rightarrow q\bar{q},1} = \text{Re}\left\{(5N_c - \frac{C_F}{4})\pi^2 + \gamma_E \log\left(\frac{\mu_R^2}{\mu_F^2}\right) - \frac{m_g^2 N_c}{2m_q^2} \log^2\left(x_{g\bar{g}}(4m_q^2)\right)\right. \right.$$

$$+ C_F \left[\frac{m^2 m_q^2}{2m_q^4} \log\left(\frac{m^2}{m_q^2}\right) - \frac{m^2 N_c}{m_q^2} - 3\right] + \frac{m^2 N_c}{2m_q^2} \left[\text{Li}_2\left(-\frac{m^2}{m_q^2}\right) - \text{Li}_2\left(\frac{m^2}{m_q^2}\right)\right]$$

$$\left. + 2C_A \gamma_E + \gamma_E \log\left(\frac{\mu_R^2}{4m_q^2}\right)\right\} \left(\frac{\pi^2}{8} - \frac{1}{2} \text{Li}_2\left(-\frac{m^2}{m_q^2}\right) + \frac{1}{2} \text{Li}_2\left(\frac{m^2}{m_q^2}\right) + \frac{m^2}{4m_q^2} \log^2\left(x_{g\bar{g}}(4m_q^2)\right) + 2 + \gamma_E\right)C_2(R_i),$$

where in the second equation the representation $I$ can be either of the symmetric representations, with colour tensors given by Eq. (B.3) or (B.5), and the last line is proportional to the corresponding quadratic Casimir invariant.

$q\bar{q} \rightarrow g\bar{g}$

In the case of the $q\bar{q} \rightarrow g\bar{g}$ process, only the antisymmetric representation in Eq. (B.10) yields a nonzero matching coefficient, the cross sections for the other representations are suppressed near threshold:

$$C^{(1)}_{q\bar{q} \rightarrow g\bar{g},1} = C^{(1)}_{q\bar{q} \rightarrow g\bar{g},3} = 0,$$

$$C^{(1)}_{q\bar{q} \rightarrow g\bar{g},2} = \text{Re}\left\{\frac{m^2 C_F}{4m_q^2 m_+^2} + \frac{7}{4N_c} + \frac{3m_q^2}{8m_q^2 N_c} + \frac{5(2m_q^2 + m_N^2) N_c}{24m_q^2} + \gamma_E \log\left(\frac{\mu_R^2}{m_q^2}\right) - \gamma_q \log\left(\frac{\mu_R^2}{m_q^2}\right)\right. \right.$$

$$\left. + \left[\frac{N_c^2 - 4}{12N_c}\pi^2 + 2C_F\gamma_E + \gamma_E \log\left(\frac{\mu_R^2}{4m_q^2}\right)\right] + \left[\frac{m_q^2 m_N^2}{4m_q^2} + \frac{3}{8N_c} + \frac{m_q^2}{m_q^2 N_c} + \frac{m_q^2 C_F}{2m_q^2 m_+^2}\right]\log\left(\frac{m_q^2}{m_+^2}\right) + \frac{1}{2} N_c \left[\frac{m_q^2}{m_+^2 N_c} + \frac{m_q^2}{m_q^2 N_c}\right]\log\left(\frac{m_q^2}{m_+^2}\right) + F_5(m_q, m_g, m_r) + \gamma_E N_c\right.$$

$$\left. + \left[\frac{2m_q^2 m_+^2 C_F}{m_q^2 m_+^2} - \frac{m_q^2}{4m_q^2 N_c} - \frac{m_q^2}{m_q^2 N_c}\right]\log\left(\frac{m_q^2}{m_+^2}\right) + \frac{5}{2} N_c \log(2) - \frac{m_q^2 (5m_q^4 - 3m_q^2 m_+^2 + m_+^4)}{32m_q^4 m_q^2 N_c} F_4(m_q, m_q)\right\},$$

99
\(- \frac{(3m_8^2 - m_q^2)m_+^4}{8m_8^3m_q^3N_c} \log (x_{q\bar{q}}(4m_8^2)) \beta_{q\bar{q}}(4m_8^2) \)
\(+ \frac{1}{N_c} \left[ \frac{m_8^2(m_\bar{q}^2 - 3m_8^2)}{2m_8^4} - \frac{m_8^2}{8m_8^2} - \frac{1}{4} \right] F_1(m_8, m_\bar{q}) \)
\(+ \frac{N_c}{8} \left[ \frac{m_\bar{q}^2 - m_8^2}{2m_8^4} - \frac{3(3m_8^2 + m_\bar{q}^2)}{m_8^2} \right] F_2(m_8, m_\bar{q}) \),

where we have defined the additional functions:

\(F_4(m_8, m_\bar{q}) = \text{Li}_2 \left( 1 - x_{q\bar{q}}(4m_8^2) \frac{m_8^2}{m_\bar{q}^2} \right) + \text{Li}_2 \left( 1 - \frac{1}{x_{q\bar{q}}(4m_8^2)} \frac{m_\bar{q}^2}{m_8^2} \right) - 2\text{Li}_2 \left( 1 - \frac{m_8^2}{m_\bar{q}^2} \right) + \log^2 (x_{q\bar{q}}(4m_8^2)) \),

\(F_5(m_\bar{q}, m_8, m_t) = \frac{(m_8^2 - m_\bar{q}^2)m_f(m_\bar{q}^2 - m_8^2)}{m_8^2m_\bar{q}^2} + \left[ \frac{2m_\bar{q}(3m_8^2 + m_\bar{q}^2)m_8^2(m_t - m_\bar{q})}{3m_8^2 m_\bar{q}^2 (m_t^2 - m_\bar{q}^2)} - \frac{2m_\bar{q}^2m_\bar{q}m_t}{3m_8^4 m_\bar{q}^2} \right]
+ \frac{(5m_8^2 - 4m_\bar{q}^2)(2m_\bar{q} - m_t)m_\bar{q}^3}{3m_8^2 m_\bar{q}^2} + \frac{2m_\bar{q}^2(m_t - m_\bar{q})m_t}{3m_8^2 (m_\bar{q}^2 + m_t^2)} + \frac{m_\bar{q}^2 m_t^2}{m_\bar{q}^2 m_t^2}
- \frac{2m_\bar{q}^2 (2m_\bar{q} - 5m_t)}{3m_\bar{q}^2} + \frac{4m_\bar{q}m_t (m_\bar{q}m_t - m_\bar{q}^2)}{m_\bar{q}^2 (m_\bar{q}^2 - (m_\bar{q} + m_t)^2)} \right] \log \left( x_{q\bar{q}}(m_8^2) \right) \beta_{q\bar{q}}(m_8^2)
+ \left[ \frac{m_\bar{q}^2 m_t}{m_\bar{q}^2 + 2m_\bar{q}^2 (m_\bar{q}^2 - m_\bar{q}^2)} - \frac{8m_\bar{q}^4 - 13m_\bar{q}^6 + 5m_\bar{q}^4}{6m_\bar{q}^2 m_t^2} \right] \log \left( \frac{m_\bar{q}^2}{m_\bar{q}^2} \right)
+ \frac{m_\bar{q}^2}{m_\bar{q}^2} \left[ \frac{m_\bar{q}^2}{3m_\bar{q}^2 + m_t^2} - \frac{m_\bar{q}^2 m_f}{6m_\bar{q}^2} \right] \log \left( x_{q\bar{q}}(4m_8^2) \right) \beta_{q\bar{q}}(4m_8^2)
+ \frac{m_\bar{q}^2}{3m_\bar{q}^2} \left[ \frac{2 + m_\bar{q}^2}{2m_8^2} - \frac{3m_\bar{q}^2 + m_t^2}{m_\bar{q}^2 - m_t^2} \right] \log \left( x_{qt}(4m_8^2) \right) \beta_{qt}(4m_8^2)
+ \frac{4m_\bar{q}^2 m_\bar{q}^2 m_t}{3m_\bar{q}^2} \log \left( \frac{m_\bar{q}^2}{m_\bar{q}^2} \right) + \frac{2n_t}{3} \log(2) \).

\(gg \rightarrow \tilde{g}\tilde{g}\)

For the hard matching coefficients of the \(gg \rightarrow \tilde{g}\tilde{g}\) process, the representation-dependent part does not scale with the quadratic Casimir invariants from Eq. (B.11) due to contributions from box diagrams. Therefore we introduce the additional colour factors \(C'(R_i)\) for convenience:

\(C'(R_1) = \frac{C_F}{2N_c}, \quad C'(R_2) = \frac{N_c^2 - 4}{4N_c^2}, \quad C'(R_5) = C'(R_7) = 0 \).
Appendix F. Hard Matching Coefficients for SUSY-QCD

We also introduce the function:

\[
F_3(q_1^2, q_2^2, m_q, m_t) = \log^2 \left( x_{q\bar{q}}(q_2^2) \frac{m_q}{m_t} \right) - \log^2 \left( x_{q\bar{q}}(q_1^2) \frac{m_q}{m_t} \right) - 2 \text{Li}_2 \left( 1 - x_{q\bar{q}}(q_2^2) \frac{m_q}{m_t} \right) + 2 \text{Li}_2 \left( 1 - x_{q\bar{q}}(q_2^2) \frac{m_q}{m_t} \right) + 2 \text{Li}_2 \left( 1 - x_{q\bar{q}}(q_1^2) \frac{m_q}{m_t} \right) - 2 \text{Li}_2 \left( 1 - x_{q\bar{q}}(q_1^2) \frac{m_q}{m_t} \right).
\]

Then the hard matching coefficients are given by:

\[
C^{(1)}_{gg \to \bar{q}q, 3} = C^{(1)}_{gg \to \bar{q}q, 4} = C^{(1)}_{gg \to \bar{q}q, 5} = 0,
\]

\[
C^{(1)}_{gg \to \bar{q}q, l} = \text{Re} \left\{ \frac{2N_c \pi^2}{3} + \frac{m_t^2 + m_q^2 n_f}{2m_g^2} + 2 \left[ \gamma_E + \gamma_q \log \left( \frac{\mu_F^2}{4m_g^2} \right) - 2 \right] N_c + \gamma_q \log \left( \frac{\mu_R^2}{\mu_F^2} \right) \right. \\
+ \frac{m_q^2 m_t^2}{2m_g^2} \log \left( \frac{m_q^2}{m_t^2} \right) n_f - (m_q^2 - m_t^2)(m_q^2 + m_t^2) - 2 m_g^2 (m_q^2 + m_t^2) - (m_q^2 + m_t^2)(m_q^2 - m_t^2) \\
+ \frac{m_q^2}{2m_g^2} \beta_{q\bar{q}}(m_q^2) \log \left( x_{q\bar{q}}(m_q^2) \right) \\
+ \frac{m_q^2 + m_t^2}{4(m_q^2 - m_t^2)} \left[ \frac{m_q^2}{m_q^2} F_3(-m_q^2, m_q^2, m_t, m_q) + \frac{m_t^2}{m_t^2} F_3(m_q^2, -m_q^2, m_q, m_t) \right] \\
+ \frac{2 + \gamma_q - \pi^2}{8} C_2(R_I) \\
- \left\{ \frac{m_q^2}{m_q^2 - m_t^2} F_3(m_q^2, -m_q^2, m_q, m_t) + \frac{m_t^2(m_q^2 + m_t^2)}{(m_q^2 - m_t^2)(m_q^2 - m_t^2)} F_3(-m_q^2, m_q^2, m_t, m_q) \\
+ \frac{m_t^2}{m_t^2 - m_t^2} \log^2 \left( x_{q\bar{q}}(4m_q^2) \right) + 2 n_f \left[ \text{Li}_2 \left( \frac{m_q^2}{m_q^2} \right) - \text{Li}_2 \left( \frac{m_q^2}{m_q^2} \right) \right] \right\} C'(R_I) \right\}.
\]

As mentioned in Section 4.1.1, the representation \( c_{gg \to \bar{q}q, 7} \) is zero-dimensional in SU(3) and does thus not contribute in the case of SUSY-QCD.

\( gg \to q\bar{q} \)

The hard matching coefficients for the \( gg \to q\bar{q} \) process also have a representation-dependent part that is not described by a simple quadratic Casimir invariant. In fact, the first representation, in Eq. (B.26), has an additional part compared to the representations in Eqs. (B.27) and (B.28), presumably due to the \( s \)-channel contribution that plays a role in the former. The hard matching coefficients are given by:

\[
C_{gg \to q\bar{q}, l} = \text{Re} \left\{ \left[ \frac{m_q^2 - m_q^2}{2m_{av}} \right] N_c + \frac{m_q^2 - 6m_q^2 m_q + 7m_q^2}{48m_{av}^2} C_F - \frac{m_q^2 m_q^2 N_c}{8m_{av}^2} \right] \pi^2 + \gamma_q \log \left( \frac{\mu_R^2}{4m_{av}^2} \right) \\
- \gamma_q + \gamma_q \log \left( \frac{\mu_F^2}{4m_{av}^2} \right) + (C_F + C_A) \left[ \gamma_E + \gamma_q \log \left( \frac{\mu_F^2}{4m_{av}^2} \right) + \frac{3(m_t^2 - m_q^2 n_f)}{4m_g^2} \right].
\]

101
\[
+ \left[ -\frac{m_q^2 n_f}{4m_q m_{av}} + \frac{(3m_q - m_\bar{q}) C_F}{2m_{av}} + \frac{(m_\bar{q} - 2m_q)^2}{4N_c m_q m_{av}} + \frac{1}{2N_c} \right] \text{Li}_2 \left( -\frac{m_\bar{q}}{m_q} \right) \\
- \left[ \frac{m_\bar{q}^2}{4m_{av}} - \frac{m_\bar{q}(m_\bar{q}^2 - m_q^2 N_c^2)}{8N_c m_q m_{av}^2} \right] \log \left( \frac{m_\bar{q}^2}{m_q^2} \right) \log \left( 1 - \frac{m_\bar{q}}{m_q} \right) \\
+ \frac{m_\bar{q}^2}{16m_q m_{av}} \left[ \frac{m_\bar{q}^2 (m_\bar{q}^2 - m_q^2 (N_c^2 - 2) - 2)}{64N_c m_q^2 m_{av}^2} - \frac{2m_q^2 m_{av}^2}{N_c m_q^4} + \frac{m_\bar{q}^2 (m_\bar{q}^2 + 3m_q)^2}{8N_c m_q^4} \right] \\
- \frac{m_\bar{q}^2}{8m_q m_{av}} F_3(m_\bar{q}^2, -m_\bar{q} m_q, m_q, m_q) + \frac{m_\bar{q}^2}{8m_q m_{av}} F_3(m_\bar{q}^2, -m_\bar{q} m_q, m_q, m_q) \\
(m_\bar{q} - 2m_q)\frac{m_\bar{q}^2}{4N_c m_q^2} + \frac{(N_c^2 + 2)m_q^2}{4N_c m_q^2} F_6(m_\bar{q}, m_q) \\
- \frac{m_\bar{q}^2}{2N_c m_q m_{av}^4} \left( (m_\bar{q} - m_q)^2 + m_\bar{q} m_q (m_\bar{q}^2 - m_q^2) \right) F_6(m_\bar{q}, m_q) \\
- \frac{m_\bar{q}^2}{8m_q m_{av}} \left[ \frac{(m_\bar{q} - m_q)^2 + m_\bar{q} m_q (m_\bar{q}^2 - m_q^2)}{2m_\bar{q} m_q m_{av}^2} \log (x_{\bar{q}}(-m_\bar{q} m_q)) \beta_{\bar{q}}(-m_\bar{q} m_q) \right] \\
- \left. \frac{m_\bar{q}^2 (m_\bar{q} - 2m_q)}{2m_q^2 m_{av}} \left[ m_\bar{q} F_6(m_\bar{q}, m_q) - m_q F_6(m_\bar{q}, m_q) \right] - \frac{m_\bar{q} m_q}{16m_q m_{av}} \log^2 \left( \frac{m_\bar{q}^2}{m_q^2} \right) + 2 \right) \\
- \frac{m_\bar{q} - m_q}{2m_{av}} \left[ \frac{m_\bar{q}}{m_q} \right] \text{Li}_2 \left( \frac{m_\bar{q}}{m_q} \right) + \frac{1}{2} \log \left( \frac{m_\bar{q}^2}{m_q^2} \right) + \frac{\pi^2}{12} + \gamma_E - \frac{m_\bar{q} m_q}{8m_q^2 m_{av}} \pi^2 C_2(R_{f_1}) \right],
\]

where \( \delta_{f,1} = 1 \) for the first representation and vanishes for the other representations. Also, we have defined the function:

\[
F_6(m_q, m_\bar{q}) = \text{Li}_2 \left( 2 - \frac{m_q}{m_\bar{q}} \right) - \text{Li}_2 \left( 1 - \frac{m_q}{m_\bar{q}} + \frac{m_\bar{q}^2}{m_q^2} \right).
\]
Bibliography


Bibliography


Bibliography


Bibliography


Bibliography


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Bibliography


## List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR</td>
<td>Modified Dimensional Reduction</td>
<td>57</td>
</tr>
<tr>
<td>MS</td>
<td>Modified Minimal Subtraction</td>
<td>24</td>
</tr>
<tr>
<td>CM</td>
<td>Centre-of-Mass</td>
<td>3</td>
</tr>
<tr>
<td>GUT</td>
<td>Grand Unified Theory</td>
<td>8</td>
</tr>
<tr>
<td>IR</td>
<td>Infrared</td>
<td>20</td>
</tr>
<tr>
<td>LHC</td>
<td>Large Hadron Collider</td>
<td>2</td>
</tr>
<tr>
<td>LL</td>
<td>Leading Logarithmic</td>
<td>30</td>
</tr>
<tr>
<td>LO</td>
<td>Leading Order</td>
<td>2</td>
</tr>
<tr>
<td>LSP</td>
<td>Lightest Supersymmetric Particle</td>
<td>6</td>
</tr>
<tr>
<td>MSSM</td>
<td>Minimal Supersymmetric Standard Model</td>
<td>5</td>
</tr>
<tr>
<td>NLL</td>
<td>Next-to-Leading Logarithmic</td>
<td>4</td>
</tr>
<tr>
<td>NLO</td>
<td>Next-to-Leading Order</td>
<td>2</td>
</tr>
<tr>
<td>NNLL</td>
<td>Next-to-Next-to-Leading Logarithmic</td>
<td>4</td>
</tr>
<tr>
<td>NNLO</td>
<td>Next-to-Next-to-Leading Order</td>
<td>3</td>
</tr>
<tr>
<td>PDF</td>
<td>Parton Distribution Function</td>
<td>20</td>
</tr>
<tr>
<td>QCD</td>
<td>Quantum Chromodynamics</td>
<td>3</td>
</tr>
<tr>
<td>QED</td>
<td>Quantum Electrodynamics</td>
<td>40</td>
</tr>
<tr>
<td>QFT</td>
<td>Quantum Field Theory</td>
<td>8</td>
</tr>
<tr>
<td>RGE</td>
<td>Renormalization Group Equation</td>
<td>8</td>
</tr>
<tr>
<td>SM</td>
<td>Standard Model</td>
<td>1</td>
</tr>
<tr>
<td>SUSY</td>
<td>Supersymmetry</td>
<td>3</td>
</tr>
<tr>
<td>UV</td>
<td>Ultraviolet</td>
<td>20</td>
</tr>
</tbody>
</table>
Summary

What is the world made of? How does it work? I have been wondering about these questions for as long as I can remember. If I put a hair under a microscope, I see that it is made of cells. If I zoom in further, it turns out that these cells are made of molecules, that in turn are made of atoms. But it does not stop there. Atoms are composed of even smaller pieces. So where does it stop? What are the smallest building blocks?

As far as we know, the building blocks of the universe are described by a theory called the Standard Model. It describes all the particles we have observed: quarks, leptons, force particles, and the recently discovered Higgs boson. We have also observed the corresponding antiparticles, and the theory passes all the tests it has been put to. Except for one.

Astrophysicists have observed something called ‘dark matter’. It is matter that cannot be seen, so ‘invisible matter’ would have been a better name. Still, we know it exists, because its gravity changes the way the stars move. This is a problem, since according to the Standard Model there is no invisible particle that could possibly explain this dark matter.

So it seems that the Standard Model is not complete and we have to look for a better theory. One possibility is supersymmetry, which high energy physicists lovingly call SUSY. This theory predicts that every particle we know has a partner particle. These supersymmetric partners are quite similar to the corresponding Standard Model particles, except that they have a different spin and they are heavier. Their high mass is particularly important to this thesis.

If supersymmetric particles exist, we want evidence. This can best be achieved by making them ourselves. So how could we do that? Einstein taught us that \( E = mc^2 \), meaning you can turn mass into energy and vice versa. The more mass you want, the more energy you need. Since supersymmetric particles are heavier than the particles we already know of, we need a great deal of energy. We can get this by colliding particles as hard as possible in a particle accelerator. The most powerful particle accelerator today is the Large Hadron Collider (LHC) at CERN. It is built in a 27 kilometre tunnel, a hundred metres below the surface, and accelerates particles to nearly the speed of light. When these particles collide, they can produce new particles.

To understand the measurements at the LHC, we have to compare them to theoretical predictions. In particular, we need to know how many supersymmetric particles to expect. To find out, we need to look at particle collisions in more detail. So let us take a look at
Summary

what happens if we collide a quark and an antiquark and produce their supersymmetric partners, a squark and an antisquark. We can schematically draw what happens in what are known as Feynman diagrams:

In the first diagram, the quark and the antiquark temporarily combine into a gluon, which then breaks up into a squark and an antisquark. In the second diagram, the quark splits into a squark and the supersymmetric partner of the gluon, the gluino. This gluino combines with the antiquark, creating an antisquark. Both diagrams depict ways to make a squark and an antisquark.

If these were the only possibilities, the answer would be fairly straightforward. Unfortunately, there are more options. Nothing prevents nature from temporarily making more particles, as we can see in the diagrams below:

We can go on, adding more and more particles. We can also produce extra particles, in addition to the squark and the antisquark, so the possibilities are endless.

But now we have a problem. As a rule of thumb, a more complex diagram gives a more difficult calculation. We can handle the calculation for the diagrams shown here, but we cannot do much more. So how can we best calculate how many squarks to expect at the LHC?

The good news is that diagrams with many particles turn out to be less important than diagrams with few particles. Therefore, up until now we only had to calculate diagrams with at most one extra particle. Unfortunately, the results are not accurate enough to compare with the measurements at the LHC, so we have to come up with a smarter approach.

A sensible place to start is the diagrams with one extra particle. We can figure out which of these diagrams contribute the most to the final result. It now becomes very significant that supersymmetric particles are heavy. For heavy particles, the most important contribution comes from a specific type of diagram with an additional gluon. These diagrams are relatively simple and have a unique property that is very useful. It turns out that we can combine parts of these very important diagrams in our calculation. As a result, we can calculate the most important contributions of diagrams with one, two, three, up to infinitely many extra gluons, all at once. This way of collecting these contributions is called resummation.

Resummation is a different way of organizing the calculation and improves the result. This thesis is about resummation for the production of squarks and gluinos. The improved
calculation shows that the LHC should produce more supersymmetric particles than we expected. This new result is used at the LHC to compare the measurements to the theory.

It is not yet clear whether supersymmetry exists. If we find supersymmetry, we can use this research to better determine its properties. But so far there is no trace of supersymmetry. Many people find this disappointing. However, even if we demonstrate that supersymmetry does not exist, that would still tell us something about nature.

At this time, the results in this thesis are used to exclude certain supersymmetric particles. Thus, they are helping us answer the question if supersymmetry exists or not. In that way, this piece of research is a tiny step towards answering the big questions I have been wondering about for so long: what is the world made of and how does it work?
Samenvatting

Waar is de wereld van gemaakt? Hoe werkt hij? Het zijn vragen die me al bezighouden zolang ik me kan herinneren. Als ik een haar onder een microscoop leg, zie ik dat die is opgebouwd uit cellen. Als we het beeld nog verder vergroten, blijken die cellen gemaakt te zijn van moleculen, die weer gemaakt zijn van atomens – dan zijn we er nog niets. Aanvankelijk kunnen we opbreken in nog kleinere stukjes. Wanneer houdt dit op? Wat zijn de kleinste bouwstenen van het universum?

Voorzover we nu weten, worden de bouwstenen van het universum beschreven door een theorie die het Standaardmodel heet. Deze theorie beschrijft alle elementaire deeltjes die we ooit waargenomen hebben: quarks, leptonen, krachtdeeltjes, en het recent ontdekte Higgsdeeltje. Van alle geladen deeltjes zijn ook de bijbehorende antideeltjes waargenomen, en de theorie doorstaat de testen waar zij aan onderworpen is met glans.


Het lijkt er dus op dat het Standaardmodel niet compleet is en dat we op zoek moeten naar een betere theorie. Een mogelijkheid hiervoor is supersymmetrie, meestal liefdevol SUSY genoemd. Dit is een theorie die voorspelt dat ieder deeltje dat we kennen een partnerdeeltje heeft. Deze supersymmetrische partners lijken heel veel op de bijbehorende Standaardmodeldeeltjes, maar ze hebben een andere spin en zijn zwaarder. Vooral het laatste is belangrijk in dit proefschrift.

Als supersymmetrische deeltjes bestaan, willen we ze natuurlijk vinden. Een manier om dat te doen is door ze in een wetenschappelijk experiment te maken. Dankzij Einstein weten we dat \( E = mc^2 \), wat betekent dat je van massa energie kunt maken en andersom. Voor meer massa heb je meer energie nodig. Aangezien supersymmetrische deeltjes zwaarder zijn dan de deeltjes die we al kennen, hebben we dus heel veel energie nodig om ze te maken. Die enorme hoeveelheid energie krijgen we door deeltjes zo hard mogelijk op elkaar te laten botsen in een deeltjesversneller. De meest krachtige deeltjesversneller op dit moment is de Large Hadron Collider (LHC) bij CERN. Hier botsen, honderd meter onder de grond, in een tunnel van ruim 27 kilometer lang, deeltjes met...
bijna de lichtsnelheid tegen elkaar om nieuwe deeltjes te produceren.

Om de metingen die bij de LHC gedaan worden te begrijpen, moeten we ze vergelijken met theoretische voorspellingen. Een van de belangrijkste voorspellingen is hoeveel deeltjes we verwachten dat er geproduceerd worden. Daarvoor moeten we deeltjesbotsingen wat preciezer bestuderen. Laten we dus eens kijken wat er gebeurt als er een quark en een antiquark op elkaar botsen en hun supersymmetrische partners, een squark en een antisquark, produceren. Dat proces geven we schematisch weer in zogenaamde Feynmandiagrammen:

![Diagram](image)

In het eerste diagram combineren het quark en het antiquark tijdelijk tot een gluon, dat vervolgens weer uit elkaar valt in een squark en een antisquark. In het tweede diagram splitst het quark in een squark en de supersymmetrische partner van het gluon, het gluino. Dat gluino combineert weer met het antiquark, waardoor een antisquark ontstaat. Ook dit is een manier om een squark en een antisquark te maken.

Als dit de enige mogelijkheden waren, was het redelijk overzichtelijk. Helaas is dat niet het geval. Er kunnen namelijk nog meer tijdelijke deeltjes gemaakt worden, zoals bijvoorbeeld in onderstaande diagrammen:

![Diagram](image)

Hier kunnen we natuurlijk mee doorgaan door steeds meer deeltjes toe te voegen. Daarnaast is het mogelijk dat behalve het squark en het antisquark er ook nog andere deeltjes geproduceerd worden, dus de mogelijkheden zijn eindeloos.

Maar nu hebben we een probleem. Een vuistregel is dat een ingewikkelder diagram een moeilijkere berekening oplevert. Bovenstaande diagrammen zijn nog te doen, maar veel moeilijker moet het niet worden. Dus hoe kunnen we dan uitrekenen hoeveel squarks we verwachten bij de LHC?

Het goede nieuws is, dat diagrammen met veel deeltjes minder belangrijk blijken te zijn dan diagrammen met weinig deeltjes. We hoeven dus niet eindeloos door te gaan met extra deeltjes toevoegen in diagrammen. Tot nog toe werden daarom alleen de diagrammen met maximaal één extra deeltje meegenomen in de berekening. Helaas is dat niet goed genoeg om te kunnen vergelijken met metingen bij de LHC. We moeten het dus slimmer aanpakken.

De truc is nu om te kijken welke diagrammen met extra deeltjes de grootste bijdrage geven aan het eindresultaat. Daarbij is het belangrijk dat supersymmetrische deeltjes veel zwaarder zijn dan de deeltjes die we al kennen. Voor zware deeltjes komt de grootste bijdrage namelijk van een specifiek soort diagrammen waarin een extra gluon gemaakt
wordt. Deze diagrammen zijn relatief simpel en hebben een heel mooie eigenschap. Het blijkt namelijk dat we delen van deze belangrijke diagrammen samen kunnen nemen. Op deze manier kunnen we de belangrijkste bijdragen van diagrammen met één extra gluon, twee extra gluonen, drie extra gluonen, enzovoorts, allemaal tegelijk uitrekenen. Deze manier van bijdragen samennemen heet hersommatie.

Met hersommatie organiseren we de berekening op een andere manier, waardoor het resultaat beter wordt. Dit proefschrift gaat over hersommatie voor de productie van squarks en gluinos. Uit de verbeterde berekening blijkt dat er meer supersymmetrische deeltjes geproduceerd zouden moeten worden bij de LHC dan we hiervoor dachten. Deze resultaten worden gebruikt bij de LHC om metingen te vergelijken met de theorie.

Het is nog niet duidelijk of supersymmetrie bestaat of niet. Als supersymmetrie ooit gevonden wordt, kunnen we met dit onderzoek de eigenschappen ervan beter bepalen. Maar tot nog toe is er geen spoor van supersymmetrie te bekennen. Voor veel mensen is dat een teleurstelling, maar ook als we erachter komen dat supersymmetrie niet bestaat, vertelt dat ons iets over hoe de wereld in elkaar zit.

Op dit moment worden de resultaten uit dit proefschrift gebruikt om bepaalde supersymmetrische deeltjes uit te sluiten. Ze helpen zo mee om de vraag te beantwoorden of supersymmetrie bestaat of niet. En daarmee is dit onderzoek een heel klein puzzelstukje in het antwoord op die grote vragen die me al zo lang bezighouden: waar is de wereld van gemaakt en hoe werkt hij?
List of Publications


Conference Proceedings


Curriculum Vitae

Irene Niessen was born in Amsterdam on 26 March 1982. She went to elementary school at Vrije School Mareland and attended high school at Stedelijk Gymnasium Leiden, where she graduated cum laude in a broad variety of courses. She won awards in the junior biology olympiad and the mathematics-olympiad and organized events for the pupil’s association.

In 2001, she started her physics study at Radboud University Nijmegen. During her studies, she spent three months at CERN as a Summer Student, working on the high voltage system of the ATLAS electromagnetic calorimeter. She also was part of the organization of the BBB-CarrièreBeurs for two years and won several awards at Model United Nations conferences as a member of the United Netherlands delegation. She obtained her Master of Science cum laude in 2008 with the thesis “Supersymmetric phenomenology in the mSUGRA parameter space”, for which she was awarded a Shell stipend. She also received the poster prize at the Trends in Theory symposium in 2009 for this research.

In 2008, she started the PhD research described in this thesis. At Radboud University Nijmegen, she continued working on supersymmetry, studying threshold resummation for SUSY-QCD. She also did a side project on the implications of the observed Higgs mass on models with non-universal gaugino masses. In addition, she worked on two projects using top polarization to disentangle properties of new physics, for which she spent two weeks with the group of Prof. R.M. Godbole at the Indian Institute of Science in Bangalore.

During her PhD, she taught quantum mechanics and quantum field theory tutorials and supervised four bachelor students. She was also part of the PhD council of the Dutch Research School of Theoretical Physics, and as such took part in organizing the DRSTP PhD Day twice and was part of the organizing committee of the Trends in Theory symposium in 2011.
Dankwoord

Relax, take it easy, for there is nothing that we can do.
Mika

De afgelopen vier jaar waren een prachtige ervaring, vooral dankzij de mensen om me heen. Nu mijn proefschrift vrijwel af is, rest me de bijna onmogelijke taak om al die mensen in een kort stukje tekst te bedanken.

Ik wil beginnen met mijn begeleider Wim Beenakker, die met zijn eindeloze geduld en aanstekelijk optimisme de meest geweldige begeleider en collega was die ik me kon wensen. En mijn promotor Ronald Kleiss, die ik in de afgelopen vier jaar met de dag meer ben gaan waarderen.

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Dankwoord

Het was niet alleen maar werk de afgelopen vier jaar, hoewel sommige mensen dat misschien anders zullen zien. Ik ben heel blij dat ons oude vriendinnenclubje Lise, Agnes en Karen nog steeds contact heeft. Hoe vaak we elkaar zien mag dan wisselen, we spreken altijd weer een keer af. Gerben, eigenlijk geldt voor jou hetzelfde, maar al met al kennen we elkaar inmiddels bijna ons halve leven. José, ik hoop dat je er ooit nog van overtuigd raakt dat dit boekje toch ook onderzoek is, maar we raken sowieso nooit uitgepraat.

Tot slot wil ik mijn familie bedanken. Vooral mijn ouders, omdat jullie er altijd voor me zijn. Esther, je blijft altijd een beetje mijn kleine zusje, maar de afgelopen jaren zijn jij en Bob vooral ook goede vrienden geworden. En ik wil Emmie, Sieger en Roel bedanken dat jullie me zo hartelijk in het gezin verwelkomd hebben.

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