Determinant Formula for the Topological N=2 Superconformal Algebra

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ABSTRACT

The Kac determinant for the Topological N=2 superconformal algebra is presented as well as a detailed analysis of the singular vectors detected by the roots of the determinants. In addition we identify the standard Verma modules containing ‘no-label’ singular vectors (which are not detected directly by the roots of the determinants). We show that in standard Verma modules there are (at least) four different types of submodules, regarding size and shape. We also review the chiral determinant formula, for chiral Verma modules, adding new insights. Finally we transfer the results obtained to the Verma modules and singular vectors of the Ramond N=2 algebra, which have been very poorly studied so far. This work clarifies several misconceptions and confusing claims appeared in the literature about the singular vectors, Verma modules and submodules of the Topological N=2 superconformal algebra.

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1 Introduction and Notation

The N=2 superconformal algebras provide the symmetries underlying the N=2 strings [1][2]. These seem to be related to M-theory since many of the basic objects of M-theory are realized in the heterotic (2,1) N=2 strings [3]. In addition, the topological version of the algebra is realized in the world-sheet of the bosonic string [4], as well as in the world-sheet of the superstrings [5].

The Topological N=2 superconformal algebra was written down in 1990 by Dijkgraaf, Verlinde and Verlinde [6], as the symmetry algebra of two-dimensional topological conformal field theory (TCFT). As the authors realized, this algebra can also be obtained by twisting the Neveu-Schwarz N=2 superconformal algebra following the result [7][8] that the modification of the stress-energy tensor of a N=2 superconformal theory, by adding the derivative of the U(1) current, leads to a topological theory, procedure known as topological twist.

A basic tool in the representation theory of infinite dimensional Lie algebras (and superalgebras) is the determinant formula. This is the determinant of the matrix of inner products of a basis for the Verma modules. The zeroes of the determinant formula, which in the case of the N=2 superconformal algebras are given by vanishing surfaces involving the conformal weight \( \Delta \), the U(1) charge \( h \), and the conformal anomaly \( c \), correspond to the Verma modules which contain null vectors. The vanishing surfaces also indicate the borders between regions of positive and negative signature of the metric and therefore the determinant formula is crucial to investigate the unitarity and non-unitarity of the representations [9][29].

In the middle eighties the determinant formulae for the known N=2 superconformal algebras; i.e. the antiperiodic Neveu-Schwarz algebra, the periodic Ramond algebra and the twisted algebra, were computed by several authors [9][10][11] (see also ref. [29]). At that time the Topological N=2 algebra had not been discovered yet and the corresponding determinant formula has remained so far unpublished [1]. This has not prevented, however, from making substantial progress in the study of the topological singular vectors. For example, it has been shown [12][18] that they can be classified in 29 types in complete Verma modules and 4 types in chiral Verma modules, taking into account the relative U(1) charge and the BRST-invariance properties of the vector itself and of the primary on which it is built. In ref. [12] the complete set of topological singular vectors was explicitely constructed at level 1, (28 types in complete Verma modules as one type exists only at level 0), whereas the rigorous proofs that these types are the only possible ones have been given in ref. [18] together with the maximal dimensions of the corresponding singular vector spaces (1, 2, or 3 depending on the type of singular vector). Furthermore 16 types of topological singular vectors can be mapped [12][19] to the standard singular vectors of the Neveu-Schwarz N=2 algebra, for which construction formulae are known [15][16].

Two years ago the chiral N=2 determinant formulae for chiral Verma modules were computed [13] for the Neveu-Schwarz, the Ramond, and the Topological N=2 algebras. As a bonus subsingular

\[ \text{The Neveu-Schwarz, the Ramond and the Topological N=2 superconformal algebras are connected by the spectral flows and/or the topological twists. However their representation theories are different, what has created some confusion in the literature. We thank V. Kac for discussions on this point.} \]
vectors were discovered for these algebras. The reason is that chiral Verma modules are incomplete modules resulting from the quotient of a complete Verma module by the submodule generated by a lowest-level singular vector (level zero for the Topological and Ramond algebras and level $1/2$ for the Neveu-Schwarz algebra). As a consequence, the singular vectors in the chiral Verma modules are either the “surviving” singular vectors which were not descendant of the lowest-level singular vectors that are quotiented out, or they are subsingular vectors in the complete Verma module becoming singular just in the chiral Verma module. Both origins were traced back for the 4 types of singular vectors found in chiral Verma modules [13] [12]. In ref. [18] it was proved that these four types are the only existing types of singular vectors in chiral Verma modules and their corresponding spaces are always one-dimensional.

In this paper we intend to finish the picture presenting the determinant formula for the complete Verma modules of the Topological $N=2$ algebra. In section 2 we review the most basic results regarding the Verma modules and singular vectors of this algebra. In section 3 we present the determinant formulae for the generic (standard) Verma modules and for the ‘no-label’ Verma modules and discuss in very much detail the types of singular vectors detected by the roots of the determinants. In addition we review the chiral determinant formula corresponding to chiral Verma modules adding some new insights. The results obtained are transferred to the Verma modules and singular vectors of the Ramond $N=2$ algebra, which have been very poorly analysed in the literature. Some final remarks are made in section 4. In Appendix A we identify the generic Verma modules with chiral and no-label singular vectors, and we give some examples. The latter are not directly detected by the roots of the determinants. In Appendix B we write down all singular vectors at level 1 in Verma modules with zero conformal weight $\Delta = 0$.

**Notation**

*Highest weight (h.w.) vectors* denote the states with lowest conformal weight. They are necessarily annihilated by all the positive modes of the generators of the algebra, i.e. $L_{n\geq 1}|\chi\rangle = H_{n\geq 1}|\chi\rangle = G_{n\geq 1}|\chi\rangle = Q_{n\geq 1}|\chi\rangle = 0$.

*Primary states* denote h.w. vectors with non-zero norm.

*Secondary or descendant states* denote states obtained by acting on the h.w. vectors with the negative modes of the generators of the algebra and with the fermionic zero modes $Q_0$ and $G_0$. The fermionic zero modes can also interpolate between two h.w. vectors which are on the same footing (two primary states or two singular vectors).

*Chiral states* $|\chi\rangle^{G,Q}$ are states annihilated by both $G_0$ and $Q_0$.

*$G_0$-closed states* $|\chi\rangle^G$ are states annihilated by $G_0$ but not by $Q_0$.

*$Q_0$-closed states* $|\chi\rangle^Q$ are states annihilated by $Q_0$ but not by $G_0$.

*No-label states* $|\chi\rangle$ denote states that cannot be expressed as linear combinations of $G_0$-closed, $Q_0$-closed and chiral states.

The Verma module associated to a h.w. vector consists of the h.w. vector plus the set of secondary
states built on it.

Null vectors are states in the kernel of the inner product which therefore decouple from the whole space of states.

Singular vectors are h.w. null vectors, i.e. the states with lowest conformal weight in the null submodules.

Primitive singular vectors are the singular vectors that cannot be constructed by acting with the generators of the algebra on another singular vector. However, the fermionic zero modes $G_0$ and $Q_0$ can interpolate between two primitive singular vectors at the same level (transforming one into each other).

Secondary singular vectors are singular vectors that can be constructed by acting with the generators of the algebra on another singular vector. The level-zero secondary singular vectors cannot ‘come back’ to the singular vectors on which they are built by acting with $G_0$ or $Q_0$.

Subsingular vectors are non-h.w. null vectors which become singular (i.e. h.w.) in the quotient of the Verma module by a submodule generated by a singular vector.

The singular vectors of the Topological N=2 algebra will be denoted as topological singular vectors $|\chi\rangle$.

The singular vectors of the Ramond N=2 algebra will be denoted as R singular vectors $|\chi_R\rangle$.

2 Verma Modules and Singular Vectors of the Topological N=2 Algebra

2.1 The Topological N=2 algebra

The two possible topological twists of the generators of the Neveu-Schwarz N=2 algebra are:

\[
(T^+_W)^{-1} L_m^{(\pm)} T^+_W = L_m \pm \frac{1}{2} (m+1) H_m, \\
(T^+_W)^{-1} H_m^{(\pm)} T^+_W = \pm H_m, \\
(T^+_W)^{-1} G_m^{(\pm)} T^+_W = G_m^{\pm}, \\
(T^+_W)^{-1} Q_m^{(\pm)} T^+_W = G_m^{\mp},
\]

(2.1)

where $L_m$ and $H_m$ are the bosonic generators corresponding to the stress-energy tensor (Virasoro generators) and the U(1) current, respectively, and $G^{\pm}$ are the spin-3/2 fermionic generators. These twists, which we denote as $T^+_W$, consist of the modification of the stress-energy tensor by adding the derivative of the U(1) current. As a result the conformal spins and modes of the fermionic fields are also modified in such a manner that the spin-3/2 generators $G^{\pm}$, with half-integer modes, are traded by spin-1 and spin-2 generators $Q$ and $G$, respectively, with integer modes, the first
ones having the properties of a BRST current\footnote{Let us stress that the modification of the stress-energy tensor results in the modification of the conformal weights and modes of the fermionic fields. Therefore there are no spectral flows converting the half-integer modes of the Neveu-Schwarz generators into the integer modes of the topological generators, as sometimes confused in the literature (the spectral flows do not modify the conformal weights).}. Observe that the twists are mirrored under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$. The Topological N=2 algebra, obtained by twisting in this way the Neveu-Schwarz N=2 algebra, reads \[\eqref{topoNS} \]

$$
\begin{align*}
\{L_m, L_n\} &= (m - n)L_{m+n}, &\{H_m, H_n\} &= \frac{c}{3} m \delta_{m+n,0}, \\
\{L_m, G_n\} &= (m - n)G_{m+n}, &\{H_m, G_n\} &= G_{m+n}, \\
\{L_m, Q_n\} &= -nQ_{m+n}, &\{H_m, Q_n\} &= -Q_{m+n}, & m, n \in \mathbb{Z}. 
\end{align*}
$$

The eigenvalues of the bosonic zero modes ($L_0$, $H_0$) correspond to the conformal weight and the U(1) charge of the states. These split conveniently as ($\Delta + l$, $h + q$) for secondary states, where $l$ and $q$ are the level and the relative charge of the state and ($\Delta$, $h$) are the conformal weight and the charge of the primary state on which the secondary is built. The ‘topological’ central charge $c$ is the central charge corresponding to the Neveu-Schwarz N=2 algebra. This algebra is topological because the Virasoro generators are BRST-exact, i.e. can be expressed as $L_m = \frac{1}{2}\{G_m, Q_0\}$, where $Q_0$ is the BRST charge. This implies, as is well known, that the correlators of the fields do not depend on the two-dimensional metric \[\eqref{topoNS} \].

An important fact is that the annihilation conditions $G^\pm_{1/2} |\chi_{NS}\rangle = 0$ of the Neveu-Schwarz N=2 algebra read $G_0|\chi\rangle = 0$ after the corresponding twists $T^\pm_W \eqref{topoNS}$. As a result, under $T^+_W$ or $T^-_W$ any state of the Neveu-Schwarz N=2 algebra annihilated by all the positive modes of the NS generators becomes a state of the Topological N=2 algebra annihilated by $G_0$ as well as by all the positive modes of the topological generators, as the reader can easily verify. Conversely, any topological state annihilated by $G_0$ and by all the positive modes of the topological generators transforms into a NS state annihilated by all the positive modes of the NS generators. The zero mode $Q_0$, in turn, corresponds to the negative modes $G^-_{-1/2}$. Observe that a topological state not annihilated by $G_0$ is transformed into a NS state not annihilated by $G^+_{1/2}$ or by $G^-_{1/2}$. Consequently, the Neveu-Schwarz counterpart of the topological primaries and singular vectors not annihilated by $G_0$ are not primary states and singular vectors themselves but rather they are secondary states of no special type, or they are subsingular vectors. To be precise, the $Q_0$-closed topological primaries correspond to NS secondary states obtained by acting with $G^+_{-1/2}$ or $G^-_{-1/2}$ on the NS primaries, the $Q_0$-closed topological singular vectors with non-zero conformal weight correspond to null descendants of NS singular vectors, and the $Q_0$-closed topological singular vectors with zero conformal weight and the no-label topological singular vectors correspond to NS subsingular vectors \[\eqref{topoNS} \]. As to the no-label topological primaries, they do not have NS counterpart.
2.2 Topological Verma modules

Highest weight vectors

In a given representation of the Topological N=2 algebra, the primary states, \textit{i.e.} the states with \textit{lowest} conformal weight denoted traditionally as \textit{highest} weight vectors, require to be annihilated by all the positive modes of the generators (the lowering operators). Therefore the h.w. conditions can be defined unambiguously as the vanishing conditions: \( \mathcal{L}_{n\geq 1}|\chi\rangle = \mathcal{H}_{n\geq 1}|\chi\rangle = \mathcal{G}_{n\geq 1}|\chi\rangle = 0. \)

The zero modes \( \mathcal{G}_0 \) and \( \mathcal{Q}_0 \) provide the BRST-invariance properties of the topological states in the sense that a state annihilated by \( \mathcal{Q}_0 \) is BRST-invariant while a state annihilated by \( \mathcal{G}_0 \) can be regarded as anti-BRST-invariant. The states annihilated by both \( \mathcal{G}_0 \) and \( \mathcal{Q}_0 \) are called chiral, generalizing the fact that the chiral and antichiral primaries\(^4\) of the Neveu-Schwarz N=2 algebra are transformed, under the topological twists (2.1), into topological primaries annihilated by both \( \mathcal{G}_0 \) and \( \mathcal{Q}_0 \): \( (G_{1/2}^\pm, G_{-1/2}^\pm) \rightarrow (\mathcal{G}_0, \mathcal{Q}_0). \) In what follows the states annihilated by \( \mathcal{Q}_0 \) but not by \( \mathcal{G}_0 \) will be called \( \mathcal{Q}_0 \)-closed whereas the states annihilated by \( \mathcal{G}_0 \) but not by \( \mathcal{Q}_0 \) will be called \( \mathcal{G}_0 \)-closed.

From the anticommutator \( \{\mathcal{Q}_0, \mathcal{G}_0\} = 2\mathcal{L}_0 \) one deduces\(^2\) that a topological state (primary or secondary) with non-zero conformal weight \( \Delta \) can be either \( \mathcal{G}_0 \)-closed, or \( \mathcal{Q}_0 \)-closed, or a linear combination of both types:

\[
|\chi\rangle = \frac{1}{2\Delta} \mathcal{Q}_0 \mathcal{G}_0 |\chi\rangle + \frac{1}{2\Delta} \mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle .
\] (2.3)

One deduces also that \( \mathcal{Q}_0 \)-closed (\( \mathcal{G}_0 \)-closed) topological states with non-zero conformal weight are \( \mathcal{Q}_0 \)-exact (\( \mathcal{G}_0 \)-exact) as well. The topological states with zero conformal weight, however, can be \( \mathcal{Q}_0 \)-closed (satisfying \( \mathcal{Q}_0 \mathcal{G}_0 |\chi\rangle^\mathcal{Q} = 0 \)), or \( \mathcal{G}_0 \)-closed (satisfying \( \mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle^\mathcal{G} = 0 \)), or chiral, or no-label (not decomposable into \( \mathcal{G}_0 \)-closed, \( \mathcal{Q}_0 \)-closed and chiral states, satisfying \( \mathcal{Q}_0 \mathcal{G}_0 |\chi\rangle = -\mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle \neq 0 \)).

Hence one can distinguish three different types of topological primaries giving rise to complete Verma modules (provided they do not satisfy additional constraints): \( \mathcal{G}_0 \)-closed primaries \( |\Delta, \hbar|^{\mathcal{G}} \), \( \mathcal{Q}_0 \)-closed primaries \( |\Delta, \hbar|^{\mathcal{Q}} \), and no-label primaries \( |0, \hbar| \). We will not consider primaries \( |\Delta, \hbar| \) which are linear combinations of two or more of these types. Chiral primaries \( |0, \hbar|^{G,Q} \) give rise to incomplete Verma modules because the chirality constraint on the primary state is not required (just allowed) by the algebra.

As to the topological secondary states, in particular singular vectors, they are labelled in addition by the level \( l \) and the relative charge \( q \). Hence the topological secondary states are denoted as \( |\chi\rangle^{(q)}_l \) (\( \mathcal{G}_0 \)-closed), \( |\chi\rangle^{(q)\mathcal{Q}}_l \) (\( \mathcal{Q}_0 \)-closed), \( |\chi\rangle^{(q)G,Q}_l \) (chiral), and \( |\chi\rangle^{(q)}_l \) (no-label). It is convenient also to indicate the conformal weight \( \Delta \), the charge \( \hbar \), and the BRST-invariance properties of the primary state on which the secondary is built. Notice that the conformal weight and the total U(1) charge of the secondary states are given by \( \Delta + l \) and \( \hbar + q \), respectively.

Now we will give a first description of the different kinds of complete Verma modules as well as of the chiral Verma modules.

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\(^4\)Chiral primary states of the N=2 superconformal algebras, which were introduced in ref.\(^{22}\), are of special relevance in physics (see also refs.\(^{11} \) and\(^{14}\)).
Generic Verma modules

The Verma modules built on $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed primary states without additional constraints are called *generic* Verma modules \[12\, 18\]. They are complete in the sense that the constraints on the primaries of being annihilated either by $\mathcal{G}_0$ or by $\mathcal{Q}_0$ are required by the algebra, as we have just discussed. Furthermore for non-zero conformal weight $\Delta \neq 0$ the h.w. vector of any generic Verma module is degenerate, *i.e.* there are two primary states. The reason is that the action of $\mathcal{Q}_0$ on $|\Delta, h\rangle^G$ produces another primary state: $\mathcal{Q}_0|\Delta, h\rangle^G = |\Delta, h-1\rangle^Q$, and the action of $\mathcal{G}_0$ on $|\Delta, h-1\rangle^Q$ brings the state back to $|\Delta, h\rangle^G$ (up to a constant): $\mathcal{G}_0|\Delta, h-1\rangle^Q = 2\Delta|\Delta, h\rangle^G$. For $\Delta = 0$, however, $\mathcal{Q}_0[0, h\rangle^G$ is not a primary state but a level-zero chiral singular vector instead, denoted as $|\chi\rangle_{GQ}^{(-1)GQ}$, and similarly $\mathcal{G}_0[0, h - 1\rangle^Q$ is the chiral singular vector $|\chi\rangle_{0, h-1}^{GQ}$. As a consequence, the h.w. vectors $[0, h\rangle^G$ and $[0, h - 1\rangle^Q$ are located in different Verma modules $V([0, h\rangle^G$ and $V([0, h - 1\rangle^Q$, whereas for $\Delta \neq 0$ the Verma modules built on $|\Delta, h\rangle^G$ and $|\Delta, h - 1\rangle^Q$ coincide: $V(|\Delta, h\rangle^G) = V(|\Delta, h - 1\rangle^Q)$ iff $\Delta \neq 0$.

No-label Verma modules

The Verma modules built on no-label primary states are called *no-label* Verma modules $V([0, h\rangle$ \[12\, 18\]. They are complete, obviously, as the no-label primaries are annihilated only by the positive modes of the generators of the algebra. The action of $\mathcal{G}_0$ and $\mathcal{Q}_0$ on $[0, h\rangle$ produce the charged singular vectors $\mathcal{G}_0[0, h\rangle = |\chi\rangle_{0,0}^{G}$ and $\mathcal{Q}_0[0, h\rangle = |\chi\rangle_{0,0}^{(-1)Q}$, which cannot ‘come back’ to $[0, h\rangle$ as the action of $\mathcal{Q}_0$ and $\mathcal{G}_0$, respectively, produces an uncharged chiral singular vector instead: $\mathcal{Q}_0[0, h\rangle = -\mathcal{G}_0\mathcal{Q}_0[0, h\rangle = |\chi\rangle_{0,0}^{(0)GQ}$. (As a matter of fact, one deduces that $\mathcal{G}_0[0, h\rangle$ and $\mathcal{Q}_0[0, h\rangle$ are singular vectors taking into account that $\mathcal{G}_0\mathcal{Q}_0[0, h\rangle$ is a singular vector). The level-zero states in a no-label Verma module consist therefore of the primary state $[0, h\rangle$ plus the two charged singular vectors $\mathcal{G}_0[0, h\rangle$ and $\mathcal{Q}_0[0, h\rangle$, and the uncharged chiral singular vector $\mathcal{Q}_0\mathcal{G}_0[0, h\rangle$.

Chiral Verma modules

The Verma modules built on chiral primary states are called *chiral* Verma modules $V([0, h\rangle^{GQ}$. \[12\, 13\, 18\]. They are not complete because the primary state being annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$ amounts to an additional constraint not required (just allowed) by the algebra. Chiral Verma modules result from the quotient of generic Verma modules, with $\Delta = 0$, by the submodules generated by the level-zero singular vectors. That is, a chiral Verma module can be expressed as the quotient $V([0, h\rangle^{GQ}) = V([0, h\rangle^G)/\mathcal{Q}_0[0, h\rangle^G$ and also can be expressed as the quotient $V([0, h\rangle^{GQ}) = V([0, h\rangle^G)/\mathcal{G}_0[0, h\rangle^Q$, equivalently. Therefore it can be regarded as a complete Verma module with a piece ‘cut off’. The level-zero states in a chiral Verma module consist of only the primary state $[0, h\rangle^{GQ}$, obviously.

2.3 Topological singular vectors

The topological singular vectors can be classified in 29 different types in complete Verma modules and 4 different types in chiral Verma modules, distinguished by the relative charge $q$ and the
BRST-invariance properties of the singular vectors and of the primaries on which they are built. An important question is whether the singular vectors with non-zero conformal weight, $\Delta + l \neq 0$, are linear combinations of $G_0$-closed and $Q_0$-closed singular vectors. From the anticommutator $\{Q_0, G_0\} = 2L_0$ one obtains the decomposition

$$|\chi\rangle_l = \frac{1}{2(\Delta + l)} G_0 Q_0 |\chi\rangle_l + \frac{1}{2(\Delta + l)} Q_0 G_0 |\chi\rangle_l = |\chi\rangle_l^G + |\chi\rangle_l^Q. \quad (2.4)$$

If $|\chi\rangle_l$ is a singular vector, i.e. satisfies the h.w. conditions $L_{n\geq1}|\chi\rangle = H_{n\geq1}|\chi\rangle = G_{n\geq1}|\chi\rangle = Q_{n\geq1}|\chi\rangle = 0$, then $G_0 Q_0 |\chi\rangle_l$ and $Q_0 G_0 |\chi\rangle_l$ satisfy the h.w. conditions too, as one deduces straightforwardly using the algebra (2.2). Therefore, regarding singular vectors with non-zero conformal weight, we can restrict ourselves to $G_0$-closed and to $Q_0$-closed singular vectors. The singular vectors with zero conformal weight, $\Delta + l = 0$, can also be chiral or no-label.

In what follows we will see how the different types of singular vectors are distributed in the generic, no-label and chiral Verma modules. Then we will discuss the action of the fermionic zero modes $G_0$ and $Q_0$ on the singular vectors.

**Generic Verma modules**

The possible existing types of topological singular vectors in generic Verma modules are the following [12][18]:

- Ten types built on $G_0$-closed primaries $|\Delta, h\rangle^G$:

<table>
<thead>
<tr>
<th></th>
<th>$q = -2$</th>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td>$-$</td>
<td>$</td>
<td>\chi\rangle_l^{(-1)G}$</td>
<td>$</td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$</td>
<td>\chi\rangle_l^{(-2)Q}$</td>
<td>$</td>
<td>\chi\rangle_l^{(-1)Q}$</td>
</tr>
<tr>
<td>chiral</td>
<td>$-\quad</td>
<td>\chi\rangle_l^{(-1)G,Q}$</td>
<td>$</td>
<td>\chi\rangle_l^{(0)G,Q}$</td>
</tr>
<tr>
<td>no-label</td>
<td>$-\quad</td>
<td>\chi\rangle_l^{(-1)}$</td>
<td>$</td>
<td>\chi\rangle_l^{(0)}$</td>
</tr>
</tbody>
</table>

(2.5)

- Ten types built on $Q_0$-closed primaries $|\Delta, h\rangle^Q$:

<table>
<thead>
<tr>
<th></th>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td>$-$</td>
<td>$</td>
<td>\chi\rangle_l^{(0)G}$</td>
<td>$</td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$</td>
<td>\chi\rangle_l^{(-1)Q}$</td>
<td>$</td>
<td>\chi\rangle_l^{(0)Q}$</td>
</tr>
<tr>
<td>chiral</td>
<td>$-\quad</td>
<td>\chi\rangle_l^{(0)G,Q}$</td>
<td>$</td>
<td>\chi\rangle_l^{(1)G,Q}$</td>
</tr>
<tr>
<td>no-label</td>
<td>$-\quad</td>
<td>\chi\rangle_l^{(0)}$</td>
<td>$</td>
<td>\chi\rangle_l^{(1)}$</td>
</tr>
</tbody>
</table>

(2.6)

with $l = -\Delta$ in the case of chiral and no-label singular vectors. The maximal dimensions of the corresponding singular vector spaces [18] are two, for the singular vectors of types $|\chi\rangle_l^{(0)G}$.
For $\Delta \neq 0$ the singular vectors of table (2.5) are equivalent to singular vectors of table (2.6) with a shift on the $U(1)$ charges due to the existence of two primary states in the Verma module: one $G_0$-closed primary $|\Delta, h\rangle^G$ and one $Q_0$-closed primary $|\Delta - 1, h\rangle^Q$, as was discussed in subsection 2.2. In particular, the charged (uncharged) chiral singular vectors of table (2.5) are equivalent to uncharged (charged) chiral singular vectors of table (2.6).

An useful observation is that chiral singular vectors $|\chi\rangle_i^{(q)G, Q}$ can be regarded as particular cases of $G_0$-closed singular vectors $|\chi\rangle_i^{(q)G}$ and/or as particular cases of $Q_0$-closed singular vectors $|\chi\rangle_i^{(q)Q}$. That is, some $G_0$-closed and $Q_0$-closed singular vectors ‘become’ chiral when the conformal weight of the singular vector turns out to be zero, i.e., $\Delta + l = 0$. This is always the case for singular vectors of types $|\chi\rangle_i^{(-1)G, |\Delta, h\rangle^G}$, $|\chi\rangle_i^{(0)Q, |\Delta, h\rangle^G}$, $|\chi\rangle_i^{(0)G, |\Delta, h\rangle^Q}$ and $|\chi\rangle_i^{(1)Q, |\Delta, h\rangle^Q}$, as explained in refs. (13) (see below), while this never occurs to singular vectors of types $|\chi\rangle_i^{(1)G, |\Delta, h\rangle^G}$, $|\chi\rangle_i^{(-2)Q, |\Delta, h\rangle^Q}$, $|\chi\rangle_i^{(-1)Q, |\Delta, h\rangle^Q}$ and $|\chi\rangle_i^{(2)G, |\Delta, h\rangle^Q}$, as there are no chiral singular vectors of the corresponding types.

All topological singular vectors in tables (2.5) and (2.6) can be organized into families (2) involving different Verma modules and different levels, every member of a family being mapped to any other member by the topological spectral flows [21][22] and/or the fermionic zero modes $G_0$ and $Q_0$. These families follow infinite many different patterns. Furthermore, with the exception of the no-label singular vectors, all other 16 types of topological singular vectors in tables (2.5) and (2.6) can be easily mapped to the singular vectors of the NS algebra [19]. As a bonus one obtains construction formulae for the 16 types of topological singular vectors using the construction formulae for the NS singular vectors given in refs. [13][13].

**No-label Verma modules**

The possible existing types of topological singular vectors in no-label Verma modules are the following nine types (13)[18] built on no-label primaries $|0, h\rangle$:

\[
\begin{array}{cccccc}
q = -2 & q = -1 & q = 0 & q = 1 & q = 2 \\
G_0\text{-closed} & - & |\chi\rangle_i^{(-1)G} & |\chi\rangle_i^{(0)G} & |\chi\rangle_i^{(1)G} & |\chi\rangle_i^{(2)G} \\
Q_0\text{-closed} & |\chi\rangle_i^{(-2)Q} & |\chi\rangle_i^{(-1)Q} & |\chi\rangle_i^{(0)Q} & |\chi\rangle_i^{(1)Q} & - \\
\text{chiral} & - & - & |\chi\rangle_0^{(0)G, Q} & - & - \\
\end{array}
\tag{2.7}
\]

The maximal dimensions of the corresponding singular vector spaces (18) are three, for the singular vectors of types $|\chi\rangle_i^{(0)G, |0, h\rangle}$, $|\chi\rangle_i^{(1)G, |0, h\rangle}$, $|\chi\rangle_i^{(0)Q, |0, h\rangle}$ and $|\chi\rangle_i^{(-1)Q, |0, h\rangle}$, and one for the remaining types. Observe that the chiral type of singular vector only exists for level zero. It is given by $|\chi\rangle_0^{(0)G, Q} = G_0 Q_0 |0, h\rangle$.

\[4\] The maximal dimension $n$ for a given singular vector space puts a theoretical limit on the number of linearly independent singular vectors of the corresponding type that one can write down at the same level in a given Verma module (13). For the singular vectors given in tables (2.5) and (2.6) the corresponding maximal dimensions have been verified by the low level computations.

\[5\] At levels 1 and 2 there are no 3-dimensional singular spaces (there are 2-dimensional ones though). Further computations are needed to decide whether or not they actually exist at higher levels.
These singular vectors can also be organized into families resulting in two different kinds of families with a unique pattern each: the box diagram consisting of four singular vectors connected by $G_0$, $Q_0$ and the universal odd spectral flow automorphism $\mathcal{A}$.

### Chiral Verma modules

The possible existing types of topological singular vectors in chiral Verma modules are the following four types built on chiral primaries $|0, h\rangle^{G,Q}$:

<table>
<thead>
<tr>
<th></th>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td>$-\chi^{(0)G}_l$</td>
<td>$\chi^{(1)G}_l$</td>
<td></td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$\chi^{(-1)Q}_l$</td>
<td>$-\chi^{(0)Q}_l$</td>
<td></td>
</tr>
</tbody>
</table>

(2.8)

This result was to be expected since a chiral Verma module $V(|0, h\rangle^{G,Q})$ can be viewed as a Verma module $V(|0, h\rangle^G)$, built on a $G_0$-closed primary, with a piece ‘cut off’, but it can also be viewed as a Verma module $V(|0, h\rangle^Q)$, built on a $Q_0$-closed primary, with a piece ‘cut off’. Therefore one might think that the only possible singular vectors in chiral Verma modules should correspond to the types existing in both table (2.3), built on $G_0$-closed primaries, and table (2.6), built on $Q_0$-closed primaries, whereas the non-common types must be projected out belonging to the submodules that are set to zero. Inspecting tables (2.3) and (2.6) one finds that the common types of singular vectors are precisely the ones given in table (2.8) plus chiral and no-label singular vectors which cannot possibly exist in chiral Verma modules (one cannot construct chiral or no-label descendant states on a chiral primary because they would require level zero but $G_0$ and $Q_0$ annihilate the primary). The reasoning we just made is not completely correct in spite of its apparent success, however, because in the generic Verma modules $V(|0, h\rangle^G)$ and $V(|0, h\rangle^Q)$ there exist subsingular vectors which become singular only in the chiral Verma module, i.e. after setting to zero the null submodules generated by the level-zero singular vectors [12]. Nevertheless the resulting types of ‘new’ singular vectors which are only singular in the chiral Verma modules are again of the types shown in table (2.8).

The maximal dimension of the corresponding singular vector spaces is one [18]. These singular vectors can also be organized into families with a unique pattern: the box diagram consisting of four singular vectors, one of each type, connected by $G_0$, $Q_0$ and the spectral flow automorphism $\mathcal{A}$ (see footnote 6).

---

6The topological odd spectral flow for $\theta = 1$, denoted simply as $\mathcal{A}$, acting as $\mathcal{L}_m \to \mathcal{L}_m - m\mathcal{H}_m$, $\mathcal{H}_m \to -\mathcal{H}_m - \frac{\theta}{2}$, $\mathcal{G}_n \to \mathcal{G}_n$, and $\mathcal{Q}_n \to \mathcal{Q}_n$, is an involutive automorphism, i.e. $\mathcal{A}^{-1} = \mathcal{A}$. It is universal in the sense that it transforms every primary state and every singular vector back into another primary state and another singular vector. In particular it is the only spectral flow operator that maps chiral states to chiral states and no-label states to no-label states. Hence any other spectral flow operators (except the identity) ‘destroy’ chiral Verma modules as well as no-label Verma modules.
The fermionic zero modes

Most topological singular vectors come in pairs at the same level in the same Verma module, differing by one unit of relative charge and by the BRST-invariance properties. The reason is that the fermionic zero modes \( G_0 \) and \( Q_0 \) acting on a singular vector produce another singular vector, as can be checked straightforwardly using the algebra \((2.3)\). Therefore, at least in principle, only chiral singular vectors can be ‘alone’, whereas the no-label singular vectors are accompanied by three, rather than one, singular vectors at the same level in the same Verma module. To be precise, inside a given Verma module \( V(\Delta, h) \) and for a given level \( l \) the topological singular vectors with non-zero conformal weight are connected by the action of \( Q_0 \) and \( G_0 \) as:

\[
Q_0 |\chi^{(q)G}_l\rangle \rightarrow |\chi^{(q-1)G}_l\rangle, \quad G_0 |\chi^{(q)Q}_l\rangle \rightarrow |\chi^{(q+1)Q}_l\rangle, \tag{2.9}
\]

where the arrows can be reversed (up to constants), using \( G_0 \) and \( Q_0 \) respectively, since the conformal weight of the singular vector is different from zero, i.e. \( \Delta + l \neq 0 \). Otherwise, on the right-hand side of the arrows one obtains chiral secondary singular vectors which cannot “come back” to the singular vectors on the left-hand side:

\[
Q_0 |\chi^{(q)G}_{l-\Delta}\rangle \rightarrow |\chi^{(q-1)G,Q}_{l-\Delta}\rangle, \quad G_0 |\chi^{(q)Q}_{l-\Delta}\rangle \rightarrow |\chi^{(q+1)G,Q}_{l-\Delta}\rangle. \tag{2.10}
\]

Observe that the non-existence of chiral singular vectors of types \( |\chi^{(-2)G,Q}_{l+\Delta,h}\rangle, |\chi^{(-1)G,Q}_{l+\Delta,h}\rangle, |\chi^{(2)G,Q}_{l+\Delta,h}\rangle, |\chi^{(-1)G}_{l+\Delta,h}\rangle \) and \( |\chi^{(1)G,Q}_{l+\Delta,h}\rangle \), as one can see in tables \((2.3)\) and \((2.4)\), implies in turn the absence of singular vectors of types \( |\chi^{(-1)G,Q}_{l+\Delta,h}\rangle, |\chi^{(0)G,Q}_{l+\Delta,h}\rangle, |\chi^{(1)G,Q}_{l+\Delta,h}\rangle \) and \( |\chi^{(1)Q}_{l+\Delta,h}\rangle \), for \( \Delta + l = 0 \). Therefore for zero conformal weight these types of singular vectors do not exist as such but they always ‘become’ chiral, i.e. of types \( |\chi^{(-1)G,Q}_{l+\Delta,h}\rangle, |\chi^{(0)G,Q}_{l+\Delta,h}\rangle, |\chi^{(1)G,Q}_{l+\Delta,h}\rangle \) and \( |\chi^{(1)Q}_{l+\Delta,h}\rangle \), instead.

Regarding no-label singular vectors \( |\chi^{(q)}_l\rangle \), they always satisfy \( \Delta + l = 0 \). The action of \( G_0 \) and \( Q_0 \) on a no-label singular vector produce three singular vectors:

\[
Q_0 |\chi^{(q)}_{l-\Delta}\rangle \rightarrow |\chi^{(q-1)}_{l-\Delta}\rangle, \quad G_0 |\chi^{(q)}_{l-\Delta}\rangle \rightarrow |\chi^{(q+1)}_{l-\Delta}\rangle, \quad G_0 Q_0 |\chi^{(q)}_{l-\Delta}\rangle \rightarrow |\chi^{(q)G,Q}_{l-\Delta}\rangle. \tag{2.11}
\]

All three are secondary singular vectors which cannot come back to the no-label singular vector \( |\chi^{(q)}_{l-\Delta}\rangle \) by acting with \( G_0 \) and \( Q_0 \), the chiral singular vector being in addition secondary with respect to the \( G_0 \)-closed and the \( Q_0 \)-closed singular vectors.

Summarizing, \( G_0 \) and \( Q_0 \) interpolate between two singular vectors with non-zero conformal weight, in both directions, whereas they produce secondary singular vectors when acting on singular vectors with zero conformal weight.

### 3 Determinant Formulae

In what follows we will write down the determinant formulae for the Topological \( N=2 \) algebra and we will analyse in much detail the interpretation of the roots of the determinants in terms of singular vectors, making use of the results discussed in the last section. We will also investigate the different types of submodules, regarding shape and size, which appear in the Verma modules.
We will start with the generic Verma modules, then we will review the results for the chiral Verma modules \[13\], adding new insights, and finally we will consider the no-label Verma modules. The results obtained will be transferred straightforwardly to the Verma modules of the Ramond N=2 algebra (which have been very poorly studied in the literature) as these turn out to be isomorphic to the Verma modules of the Topological N=2 algebra.

### 3.1 Generic Verma modules

For all the generic Verma modules – either with two h.w. vectors \(|\Delta, h\rangle^G\) and \(|\Delta, h-1\rangle^Q\) (\(\Delta \neq 0\)) or with only one h.w. vector \(|0, h\rangle^G\) or \(|0, h-1\rangle^Q\) – the determinant formula reads

\[
det(\mathcal{M}_T) = \prod_{2 \leq r,s \leq 2l} (f_{r,s})^{2P(l-\frac{r+s}{2})} \prod_{0 \leq k \leq l} (g_k^+)^{2P_k(l-k)} \prod_{0 \leq k \leq l} (g_k^-)^{2P_k(l-k)},
\]

where

\[
f_{r,s}(\Delta, h, t) = -2t\Delta + th - h^2 - \frac{1}{4}t^2 + \frac{1}{4}(s-tr)^2, \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+
\]

and

\[
g_k^\pm(\Delta, h, t) = 2\Delta \mp 2kh - tk(k \mp 1), \quad 0 \leq k \in \mathbb{Z},
\]

defining the parameter \(t = (3 - c)/3\). For \(c \neq 3\) (\(t \neq 0\)) one can factorize \(f_{r,s}\) as

\[
f_{r,s}(\Delta, h, t \neq 0) = -2t(\Delta - \Delta_{r,s}), \quad \Delta_{r,s} = -(h - h_{r,s}^{(0)})(h - \hat{h}_{r,s}),
\]

with

\[
h_{r,s}^{(0)} = \frac{t}{2}(1 + r) - \frac{s}{2}, \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+
\]

\[
\hat{h}_{r,s} = \frac{t}{2}(1 - r) + \frac{s}{2}, \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+.
\]

For \(c = 3\) (\(t = 0\)) one gets however

\[
f_{r,s}(\Delta, h, t = 0) = \frac{1}{4}s^2 - h^2.
\]

For all values of \(c\) one can factorize \(g_k^+\) and \(g_k^-\) as

\[
g_k^\pm(\Delta, h, t) = 2(\Delta - \Delta_k^\pm), \quad \Delta_k^\pm = \pm k(h - h_k^\pm),
\]

with

\[
h_k^\pm = \frac{t}{2}(1 \mp k), \quad k \in \mathbb{Z}^+
\]
The partition functions are defined by
\[
\sum_N P_k(N)x^N = \frac{1}{1+x^k} \sum_n P(n)x^n = \frac{1}{1+x^k} \prod_{0<r\in\mathbb{Z}, 0<m\in\mathbb{Z}} \frac{(1+x^r)^2}{(1-x^m)^2}. \tag{3.10}
\]

It is easy to check, by counting of states, that the partitions \(2P(l - \frac{r}{2}),\) exponents of \(f_{r,s}\) in the determinant formula, correspond to complete Verma submodules whereas \(2P_k(l - k),\) exponents of \(g_k^\pm\) in the determinant formula, correspond to incomplete Verma submodules. Furthermore, as we will see, taking into account the size and the shape one can distinguish four types of submodules. The fact that \(P(0) = P_k(0) = 1\) indicates that the singular vectors come two by two at the same level, in the same Verma module.

The roots of the quadratic vanishing surface \(f_{r,s}(\Delta, h, t) = 0\) and of the vanishing planes \(g_k^\pm(\Delta, h, t) = 0\) are related to the corresponding roots of the determinant formula for the Neveu-Schwarz N=2 algebra \([9][10][11]\) via the topological twists \((2.1)\). These transform the standard h.w. vectors of the Neveu-Schwarz N=2 algebra into h.w. vectors of the null submodule is degenerate. In the case \(\Delta \neq 0\) and \(r,s\) (up to constants), so that they are on the same footing, i.e. the h.w. vector of the null submodule is degenerate. In the case \(\Delta_{r,s} = -l\), however, the singular vectors have zero conformal weight and the action of \(G_0\) or \(Q_0\) on such singular vectors produces chiral secondary singular vectors which cannot ‘come back’ to the primitive singular vector by acting with \(G_0\) and \(Q_0\). As a result, in the case \(\Delta_{r,s} = -l\) the generic situation is that only one of the two singular vectors \(|\chi_l|^{(0)G}\) or \(|\chi_l|^{(-1)Q}\) is the h.w. vector of the null submodule, the other becoming a chiral secondary singular vector (i.e. of type \(|\chi_l|^{(0)G,Q}\) or \(|\chi_l|^{(-1)G,Q}\) instead). However, it can even happen that the two singular vectors \(|\chi_l|^{(0)G}\) and \(|\chi_l|^{(-1)Q}\) become chiral and thus disconnected from each other, both vectors being the h.w. vectors of two different (although overlapping) null submodules. In Appendix A we have deduced the values of \(h\) and \(t\) corresponding to the Verma modules with \(\Delta_{r,s} = -l\) for the three different possibilities: \(|\chi_l|^{(0)G}\) becoming chiral, \(|\chi_l|^{(-1)Q}\) becoming chiral, and both of them becoming chiral. In the first case one finds \(h = h_{r,s}^{(-)}\), eqn. \((1.2)\), for \(t \neq -\frac{s}{n}, n = 1, \ldots, r\). In the second case \(h = h_{r,s}^{(+)}\), eqn.
\((A.1)\), for \(t \neq -\frac{n}{m}, n = 1, \ldots, r\). Finally, in the third case one finds \(h = h^{(+)\, r,s}\) and \(h = h^{(-\, r,s)}\) for \(t = -\frac{n}{m}, n = 1, \ldots, r\) (see also table \((3.11)\)). We also give all these singular vectors at level 1.

It is important to distinguish between the following two possibilities: \(\Delta_{r,s} \neq 0\) and \(\Delta_{r,s} = 0\), as shown in table \((3.11)\), Fig. I and Fig. II. In the first case there is a degeneracy of the ground states, so that the Verma module built on \(|\Delta_{r,s}, h\rangle^G\) has also a \(Q_0\)-closed h.w. vector \(|\Delta_{r,s}, h - 1\rangle^Q, \ Q_0\) and \(Q_0\) interpolating between them. As a result, one can choose to express the descendant states in the Verma module as built on \(|\Delta_{r,s}, h\rangle^G\) or as built on \(|\Delta_{r,s}, h - 1\rangle^Q\) (with a corresponding rearrangement of the relative charges). Consequently, the singular vectors of type \(|\chi\rangle^{(0)\, G}_{l, (\Delta, h - 1)Q}\) are equivalent to singular vectors of type \(|\chi\rangle^{(0)\, G}_{l, (\Delta, h)Q}\), and one can view the pair of singular vectors for \(\Delta_{r,s} \neq 0\) at level \(l = \frac{\mp}{2}\) as two uncharged singular vectors: \(|\chi\rangle^{(0)\, G}_{l, (\Delta, h)Q}\) in the ‘\(G\)-sector’ and \(|\chi\rangle^{(0)\, G}_{l, (\Delta, h - 1)Q}\) in the ‘\(Q\)-sector’ (one of them, or both, becoming chiral for the case \(\Delta_{r,s} = -l\)), in analogy with the (+) and (−) sectors of the Ramond algebra.

\[
\begin{array}{c|c}
\hline
0 \neq \Delta_{r,s} \neq -l & |\chi\rangle^{(0)\, G}_{\Delta, h} = |\chi\rangle^{(+\, \Delta, h - 1)Q}, \ |\chi\rangle^{(0)\, G}_{\Delta, h - 1} = |\chi\rangle^{(1)\, G}_{\Delta, h} \\
0 \neq \Delta_{r,s} = -l, h^{(-\, r,s)}, t \neq -\frac{n}{m}, n = 1, \ldots, r & |\chi\rangle^{(0)^G\, Q}_{\Delta, h} = |\chi\rangle^{(1\, G)}_{\Delta, h - 1}Q, \ |\chi\rangle^{(0)\, G}_{\Delta, h - 1} = |\chi\rangle^{(1)\, G}_{\Delta, h} \\
0 \neq \Delta_{r,s} = -l, h^{(+)\, r,s}, t \neq -\frac{n}{m}, n = 1, \ldots, r & |\chi\rangle^{(0)\, G}_{\Delta, h} = |\chi\rangle^{(1\, G)}_{\Delta, h}Q, \ |\chi\rangle^{(0)\, G}_{\Delta, h - 1} = |\chi\rangle^{(1)\, G}_{\Delta, h} \\
\Delta_{r,s} = 0, h^{(0)}_{r,s} and \hat{h}_{r,s} & |\chi\rangle^{(0)^G\, Q}_{0, h} = |\chi\rangle^{(1\, Q)}_{0, h - 1}Q, \ |\chi\rangle^{(0)\, Q}_{0, h - 1} = |\chi\rangle^{(1)\, Q}_{0, h - 1}Q \\
\hline
\end{array}
\]

(3.11)

For \(\Delta_{r,s} = 0\) there are two sets of solutions for \(h = h_{r,s}\) that annihilate the determinants: \(h_{r,s} = h_{r,s}^{(0)}, \text{eq. (3.3)}\), and \(h_{r,s} = \hat{h}_{r,s}, \text{eq. (3.6)}\). The Verma modules built on the h.w. vectors \(|0, h_{r,s}\rangle^G\) and the Verma modules built on the h.w. vectors \(|0, h_{r,s} - 1\rangle^Q\) are different because \(G_0[0, h_{r,s} - 1]Q\) and \(Q_0[0, h_{r,s}]G\) are level-zero chiral charged singular vectors (corresponding to \(g_0^\pm (\Delta, h, t) = 0\), see below) which cannot be interchanged by the h.w. vectors \(|0, h_{r,s}\rangle^G\) and \(|0, h_{r,s} - 1\rangle^Q\), respectively. Consequently, in these Verma modules the descendant states cannot be expressed in two ways (with the exception of the states inside the null submodules generated by the level-zero singular vectors). Hence in the Verma modules built on the h.w. vectors \(|0, h_{r,s}\rangle^G\), with \(h_{r,s}\) given either by \(h_{r,s}^{(0)}, \text{eq. (3.3)}\), or by \(\hat{h}_{r,s}, \text{eq. (3.6)}\), one finds generically one singular vector of type \(|\chi\rangle^{(0)\, G}_{l, (\Delta, h)}\) and one singular vector of type \(|\chi\rangle^{(0\, Q, l)}\) at level \(l = \frac{\mp}{2}\), \(G_0\) and \(Q_0\) interpolating between them. Moreover, these singular vectors are located outside, or inside, the submodules generated by the level zero singular vectors if \(h_{r,s} = h_{r,s}^{(0)}, \text{or} \ \hat{h}_{r,s}, \text{respectively (see Fig. I)}\). In Verma modules built on the h.w. vectors \(|0, h_{r,s} - 1\rangle^Q\) one finds generically one singular vector of type \(|\chi\rangle^{(0)\, Q}_{l, (\Delta, h)}\) and one singular vector...
of type $|\chi_i^{(1)G}|$ at level $l = \frac{r-s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are located outside, or inside, the submodules generated by the level zero singular vectors if $h_{r,s} = h_{r,s}$ or $h_{r,s} = h_{r,s}^{(0)}$, respectively (see Fig. II). Observe that in the Verma modules with $\Delta = 0$ there are no chiral singular vectors, besides the ones at level zero, because of the condition $\Delta + l = 0$ for chiral singular vectors to exist.

**Case** $f_{r,s}(\Delta, h, t = 0) = 0$

From expression (B.7) one deduces that for $c = 3$ ($t = 0$) the singular vectors of types $|\chi_l^{(0)G}|$ and $|\chi_l^{(-1)Q}|$ at level $l = \frac{r-s}{2}$, built on the $G_0$-closed h.w. vector $|\Delta, h_s^G\rangle$, exist for all values of $\Delta$ and $s$ provided $h_s = \pm \frac{s}{2}$. The analysis of the different possibilities for $\Delta$ yields similar results as in the previous case for $c \neq 3$, as one can see in table (3.12). That is, for $\Delta \neq 0$ the two singular vectors can be viewed as two uncharged singular vectors, one in the $G$-sector and the other in the $Q$-sector, for $\Delta = -l$ one of the two singular vectors becoming chiral (see table (3.12) and Appendix A). For $\Delta = 0$, in the Verma modules built on the h.w. vectors $|0, h_{r,s}\rangle^G$ one finds generically one singular vector of type $|\chi_l^{(0)G}|$ and one singular vector of type $|\chi_l^{(-1)Q}|$. These are located outside, or inside, the submodules generated by the level zero singular vector $Q_0|0, h_{r,s}\rangle^G$ if $h_s = -\frac{s}{2}$, or $h_s = \frac{s}{2}$, respectively. In the Verma modules built on the h.w. vectors $|0, h_s - 1\rangle^Q$ one finds generically one singular vector of type $|\chi_l^{(0)Q}|$ and one singular vector of type $|\chi_l^{(-1)}|$. These are located outside, or inside, the submodules generated by the level zero singular vector $G_0|0, h_s - 1\rangle^Q$ if $h_s = \frac{s}{2}$, or $h_s = -\frac{s}{2}$, respectively.
Fig. 1 For the Verma modules with h.w. vectors \(|0, h_{r,s} - 1\rangle^Q\), for the case \(h_{r,s} = \tilde{h}_{r,s}\) the singular vectors at level \(l\) are located outside the submodule generated by the level zero singular vector \(G_0[0, \tilde{h}_{r,s} - 1]\). Therefore they can only be expressed as type \(|\chi\rangle^{(1)}_l^G\) and type \(|\chi\rangle^{(0)}_l^Q\) descendant states of the h.w. vector \(0, \tilde{h}_{r,s} - 1\rangle^Q\). For the case \(h_{r,s} = h_{r,s}^{(0)}\), however, the singular vectors at level \(l\) are located inside the submodule generated by \(G_0[0, h_{r,s}^{(0)} - 1]\). Therefore they can either be expressed as type \(|\chi\rangle^{(1)}_l^G\) and type \(|\chi\rangle^{(0)}_l^Q\) descendant states of the h.w. vector \(0, h_{r,s}^{(0)} - 1\rangle^Q\), or they can be expressed as secondary singular vectors built on the primitive singular vector \(G_0[0, h_{r,s}^{(0)} - 1]\).

| Singular vectors at level \(rs/2\) for \(f_{r,s}(\Delta, h, t = 0) = 0\) |
|-----------------|-----------------|
| \(0\neq \Delta \neq -l, h_s = \pm \frac{s}{2}\) | \(|\chi\rangle^{(0)}_{l,h}^G = |\chi\rangle^{(1)}_{l,\Delta h 0}^G, |\chi\rangle^{(0)}_{l,\Delta h -1}^Q = |\chi\rangle^{(-1)}_{l,\Delta h}^G\) |
| \(0\neq -l, h_s = -\frac{s}{2}\) | \(|\chi\rangle^{(0)}_{l,h}^G = |\chi\rangle^{(1)}_{l,\Delta h 0}^G, |\chi\rangle^{(0)}_{l,\Delta h -1}^Q = |\chi\rangle^{(-1)}_{l,\Delta h}^G\) |
| \(0\neq \Delta = -l, h_s = \frac{s}{2}\) | \(|\chi\rangle^{(0)}_{l,h}^G = |\chi\rangle^{(1)}_{l,\Delta h 0}^G, |\chi\rangle^{(0)}_{l,\Delta h -1}^Q = |\chi\rangle^{(-1)}_{l,\Delta h}^G\) |
| \(\Delta = 0, h_s = \pm \frac{s}{2}\) | \(|\chi\rangle^{(0)}_{l,h}^G, |\chi\rangle^{(-1)}_{l,h}^G, |\chi\rangle^{(0)}_{l,\Delta h -1}^Q, |\chi\rangle^{(1)}_{l,\Delta h}^G\) |

Case \(g_k^\pm(\Delta, h, t) = 0\)

For \(\Delta = \Delta^+_k\), eq. (3.8), the Verma module built on the \(G_0\)-closed h.w. vector \(|\Delta^+_k, h\rangle^G\) has generically one charged singular vector of type \(|\chi\rangle^{(1)}_l^G\) and one uncharged singular vector of type \(|\chi\rangle^{(0)}_l^Q\) at level \(l = k\). In the case \(\Delta^+_k = -l\) the singular vector of type \(|\chi\rangle^{(0)}_l^Q\) always becomes chiral because there are no chiral singular vectors of type \(|\chi\rangle^{(1)}_l^G\) built on \(G_0\)-closed primaries, as shown in table (3.8), which otherwise would be produced as \(G_0[0, \Delta h]^{(0)}_l^{(0)}\). For \(\Delta^+_k \neq 0\) these singular vectors can be viewed as two \(q = 1\) charged singular vectors: \(|\chi\rangle^{(1)}_l^G\) in the \(G\)-sector and \(|\chi\rangle^{(0)}_l^Q\) in the \(Q\)-sector, as shown in Fig. III, due to the existence of the two h.w. vectors \(|\Delta^+_k, h\rangle^G\) and \(|\Delta^+_k, h - 1\rangle^Q\) in the Verma module, the second type becoming chiral, i.e. of type \(|\chi\rangle^{(1)}_l^G\) in the case \(\Delta^+_k = -l\).

For \(\Delta^+_k = 0\), however, there is only one h.w. vector in the Verma module and one chiral charged singular vector at level zero, corresponding to the solution \(k = 0\), for any value of \(h\): \(Q_0[0, h]^{(0)}_l^G\).
or $G_0|0, h - 1)^Q$ depending on which is the h.w. vector. In addition, in Verma modules built on primaries $|0, h^+_k)^G$, with $h^+_k$ satisfying eq. (3.3), one finds generically one charged singular vector of type $\chi_i^{(1)G}$ and one uncharged singular vector of type $\chi_i^{(0)Q}$ at level $l = k$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are located outside the submodule generated by the level zero singular vector. In Verma modules built on primaries $|0, h^+_k - 1)^Q$ one finds generically two positively charged singular vectors at level $l = k$, one of type $\chi_i^{(1)Q}$ and one of type $\chi_i^{(2)G}$, $G_0$ and $Q_0$ interpolating between them. Moreover, these two singular vectors are located inside the submodule generated by the level zero singular vector (see Fig. III).

![Diagram](image-url)  

**Fig. III**. For $\Delta = \Delta_i^+ \neq 0$ one finds at level $l$ two singular vectors, connected by $Q_0$ and $G_0$, located one unit to the right of the vertical lines over the two h.w. vectors of the Verma module, the singular vector to the right of the vertical over the $G_0$-closed ($Q_0$-closed) h.w. vector being $G_0$-closed ($Q_0$-closed) itself. Therefore the two singular vectors can be appropriately described as two $q = 1$ charged singular vectors, one in the $G$-sector and the other in the $G$-sector. For $\Delta_i^+ = -l$ the $Q_0$-closed singular vector always becomes chiral whereas the $G_0$-closed singular vector never becomes chiral. Consequently the two arrows reduce to the arrow corresponding to $Q_0$. For zero conformal weight $\Delta = \Delta_i^+ = 0$, however, there is only one h.w. vector in the Verma module and there is one singular vector at level zero. For the Verma modules with $G_0$-closed h.w. vectors $|0, h^+_k)^G$, the singular vectors at level $l$ are located outside the submodule generated by the level zero singular vector $Q_0|0, h^+_k)^G$. Therefore they can only be expressed as type $\chi_i^{(1)G}$ and type $\chi_i^{(0)Q}$ descendant states of the h.w. vector $|0, h^+_k)^G$. For the Verma modules with $Q_0$-closed h.w. vectors $|0, h^+_k - 1)^Q$, however, the singular vectors at level $l$ are located inside the submodule generated by the singular vector $G_0|0, h^+_k - 1)^Q$. Therefore they can either be expressed as type $\chi_i^{(2)G}$ and type $\chi_i^{(1)Q}$ descendant states of the h.w. vector $|0, h^+_k - 1)^Q$, or they can be expressed as secondary singular vectors built on the primitive singular vector $G_0|0, h^+_k - 1)^Q$.

<table>
<thead>
<tr>
<th>$\Delta_i^+$</th>
<th>$h$</th>
<th>Singular vectors at level $l &gt; 0$ for $g_i^+(\Delta, h, t) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \neq \Delta_i^+ = -l$</td>
<td>$\chi_i^{(1)G}$</td>
<td>$\chi_i^{(2)G}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$</td>
</tr>
<tr>
<td>$0 \neq \Delta_i^+ = -l$</td>
<td>$\chi_i^{(1)G}$</td>
<td>$\chi_i^{(2)G}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$</td>
</tr>
<tr>
<td>$\Delta_i^+ = 0$, $h = h^+_1$</td>
<td>$\chi_i^{(1)G}$</td>
<td>$\chi_i^{(2)G}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(1)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$ $\chi_i^{(0)Q}$ $\chi_i^{(0)Q}$ $\chi_i^{(2)G}$</td>
</tr>
</tbody>
</table>
Case \( g_{\overline{\Delta}}(\Delta, h, t) = 0 \)

For \( \Delta = \Delta_{\overline{\Delta}} \), eq. (3.8), the Verma module built on the \( G_0 \)-closed h.w. vector \( |\Delta_{\overline{\Delta}}, h)^G \) has generically two negatively charged singular vectors at level \( l = k \), one of type \( |\chi\rangle_{l}^{(-1)G} \) and one of type \( |\chi\rangle_{l}^{(-2)Q} \). In the case \( \Delta_{\overline{\Delta}} = -l \) the singular vector of type \( |\chi\rangle_{l}^{(-1)G} \) always becomes chiral because there are no chiral singular vectors of type \( |\chi\rangle_{l}^{(-2)G,Q} \), as shown in table (2.8), which otherwise would be produced as \( Q_0|\chi\rangle_{l}^{(-1)G} \). For \( \Delta_{\overline{\Delta}} \neq 0 \) these singular vectors can be viewed as two \( q = -1 \) charged singular vectors: \( |\chi\rangle_{l}^{(-1)G} \) in the \( G \)-sector and \( |\chi\rangle_{l}^{(-1)Q} \) in the \( Q \)-sector, as shown in Fig. IV, due to the existence of the two h.w. vectors \( |\Delta_{\overline{\Delta}}, h)^G \) and \( |\Delta_{\overline{\Delta}} - h - 1)^Q \) in the Verma module, the first type becoming chiral, \( i.e. \) of type \( |\chi\rangle_{l}^{(-1)G,Q} \) in the case \( \Delta_{\overline{\Delta}} = -l \).

For \( \Delta_{\overline{\Delta}} = 0 \), as in the previous case, there is only one h.w. vector in the Verma module and one chiral charged singular vector at level zero for any \( h \). In addition, in Verma modules built on primaries \( |0, h_{\overline{\Delta}})^G \), with \( h_{\overline{\Delta}} \) satisfying eq. (3.4), one finds generically one charged singular vector of type \( |\chi\rangle_{l}^{(-1)G} \) and one charged singular vector of type \( |\chi\rangle_{l}^{(-2)Q} \) at level \( l = k \), \( G_0 \) and \( Q_0 \) interpolating between them. These singular vectors are located inside the submodule generated by the level zero singular vector \( Q_0|0, h_{\overline{\Delta}})^G \). In Verma modules built on primaries \( |0, h_{\overline{\Delta}} - 1)^Q \) one finds generically one charged singular vector of type \( |\chi\rangle_{l}^{(-1)Q} \) and one uncharged singular vector of type \( |\chi\rangle_{l}^{(0)G} \), \( G_0 \) and \( Q_0 \) interpolating between them. These singular vectors are located outside the submodule generated by the level zero singular vector \( G_0|0, h_{\overline{\Delta}} - 1)^Q \) (see Fig. IV).

**Fig. IV.** For \( \Delta = \Delta_{\overline{\Delta}} \neq 0 \) one finds at level \( l \) two singular vectors, connected by \( Q_0 \) and \( G_0 \), located one unit to the left of the vertical lines over the two h.w. vectors of the Verma module, the singular vector to the left of the vertical over the \( G_0 \)-closed (\( Q_0 \)-closed) h.w. vector being \( G_0 \)-closed (\( Q_0 \)-closed) itself. Therefore the two singular vectors can be appropriately described as two \( q = -1 \) charged singular vectors, one in the \( G \)-sector and the other in the \( Q \)-sector. For \( \Delta_{\overline{\Delta}} = -l \) the \( G_0 \)-closed singular vector always becomes chiral whereas the \( Q_0 \)-closed singular vector never becomes chiral. Consequently the two arrows reduce to the arrow corresponding to \( G_0 \). For zero conformal weight \( \Delta = \Delta_{\overline{\Delta}} = 0 \), however, there is only one h.w. vector in the Verma module and there is one singular vector at level zero. For the Verma modules with \( G_0 \)-closed h.w. vectors \( |0, h)^G \) the singular vectors at level \( l \) are located inside the submodule generated by the singular vector \( Q_0|0, h)^G \). Therefore they can either be expressed as type \( |\chi\rangle_{l}^{(-1)G} \) and type \( |\chi\rangle_{l}^{(-2)Q} \) descendant states of the h.w. vector \( |0, h)^G \), or they can be expressed as secondary singular vectors built on the primitive singular vector \( Q_0|0, h)^G \). For the Verma modules with \( Q_0 \)-closed h.w. vectors \( |0, h_{\overline{\Delta}} - 1)^Q \), however, the singular vectors at level \( l \) are located outside the submodule generated by the level zero singular vector \( G_0|0, h_{\overline{\Delta}} - 1)^Q \). Therefore they can only be expressed as type \( |\chi\rangle_{l}^{(0)G} \) and type \( |\chi\rangle_{l}^{(-1)Q} \) descendant states of the h.w. vector \( |0, h_{\overline{\Delta}} - 1)^Q \).
Singular vectors at level $l > 0$ for $g^-_k(\Delta, \mathbf{h}, t) = 0$

\[
\begin{array}{c|c}
\Delta^-_l \neq -l & \vert \chi \rangle_{(\Delta, \mathbf{h}, Q)}^{(1)G} = \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(0)G}, \quad \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(-1)Q} = \vert \chi \rangle_{(\Delta, \mathbf{h})}^{(2)Q} \\
0 \neq \Delta^-_l = -l & \vert \chi \rangle_{(\Delta, \mathbf{h}, Q)}^{(-1)G,Q} = \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(0)G,Q}, \quad \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(-1)Q} = \vert \chi \rangle_{(\Delta, \mathbf{h})}^{(2)Q} \\
\Delta^-_l = 0, \ \mathbf{h} = \mathbf{h}^-_l & \vert \chi \rangle_{(\Delta, \mathbf{h}, Q)}^{(-1)G} , \quad \vert \chi \rangle_{(\Delta, \mathbf{h})}^{(2)Q}, \quad \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(-1)Q} ; \quad \vert \chi \rangle_{(\Delta, \mathbf{h}, -1)Q}^{(0)G} \\
\end{array}
\]

**No-label singular vectors**

Apart from the singular vectors that we have analysed, ‘detected’ by the determinant formula, there may still exist other primitive singular vectors, besides subsingular vectors, hidden by the first ones. Such singular vectors, denoted as isolated, have not been investigated properly for any $N=2$ superconformal algebra. Furthermore, it can even happen that some singular vectors predicted by the determinant formula are secondary with respect to an isolated overlooked singular vector at the same level, the latter being the generator of the detected singular vectors, by acting on it with some zero modes of the algebra.

In the case of the $N=2$ superconformal algebras subsingular vectors have been discovered and partially analysed in refs. [3, 12, 17]. As to the ‘invisible’ isolated singular vectors, in the Topological algebra (as well as in the Ramond algebra) there are certainly such singular vectors, at least the ones which are in addition generators of ‘visible’ secondary singular vectors detected by the determinant formula. Namely, all the no-label singular vectors in generic Verma modules, given in tables (2.5) and (2.6), belong to such category. No-label singular vectors, written down at levels 1 and 2 in refs. [12] and [17], are very scarce as they only exist for specific values of $c$. As deduced in Appendix A, these values are $c = \frac{3r-6}{2}$, corresponding to $t = \frac{2}{r}$. The reason why no-label singular vectors are undetected by the determinant formula is that they only appear in certain Verma modules, for discrete values of $\Delta, \mathbf{h}, t$, where there are intersections, at the same level $l$, of singular vectors corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ with singular vectors corresponding to one of the series $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, with $\frac{2r}{t} = k = l$ and $\Delta = -l$ (see Appendix A for the details).

To be precise, as was explained in section 2, no-label singular vectors generate three secondary singular vectors at the same level, just by the action of $\mathcal{G}_0$ and $Q_0$, which cannot ‘come back’ to the no-label singular vector (see Fig. V). It happens that one of these singular vectors corresponds to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$, another one corresponds to the series $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, and the remaining one corresponds to both series. For example, in the case of an uncharged no-label singular vector $|\chi \rangle_l^{(0)}$ the three secondary singular vectors are of the types: $|\chi \rangle_l^{(1)G} = \mathcal{G}_0|\chi \rangle_l^{(0)}$, $|\chi \rangle_l^{(-1)Q} = Q_0|\chi \rangle_l^{(0)}$ and $|\chi \rangle_l^{(0)G,Q} = Q_0\mathcal{G}_0|\chi \rangle_l^{(0)} = -\mathcal{G}_0Q_0|\chi \rangle_l^{(0)}$ (see Fig. V), whereas in the case of a charged no-label singular vector $|\chi \rangle_l^{(-1)}$ the three secondary singular vectors are of the types: $|\chi \rangle_l^{(0)G} = \mathcal{G}_0|\chi \rangle_l^{(-1)}$, $|\chi \rangle_l^{(-2)Q} = Q_0|\chi \rangle_l^{(-1)}$ and $|\chi \rangle_l^{(-1)G,Q} = Q_0\mathcal{G}_0|\chi \rangle_l^{(-1)}$. In Appendix A we show the no-label singular vector $|\chi \rangle_l^{(0)}$, at level 1, built on the $\mathcal{G}_0$-closed h.w. vector $|0, \mathbf{h} \rangle^G$ together with the three secondary singular vectors at level 1: $|\chi \rangle_l^{(1)G}$, $|\chi \rangle_l^{(-1)Q}$ and $|\chi \rangle_l^{(0)G,Q}$.
Fig. V. The uncharged no-label singular vector $|\chi\rangle_l^{(0)}$ at level $l$, built on the h.w. vector $|-l, h\rangle^G$, is the primitive singular vector generating the three secondary singular vectors at level $l$: $|\chi\rangle_l^{(1)G} = G_0|\chi\rangle_l^{(0)}$, $|\chi\rangle_l^{(-1)Q} = Q_0|\chi\rangle_l^{(0)}$ and $|\chi\rangle_l^{(0)G,Q} = Q_0G_0|\chi\rangle_l^{(0)} = -G_0Q_0|\chi\rangle_l^{(0)}$. These cannot generate the no-label singular vector by acting with the algebra. However, they are the singular vectors detected by the determinant formula, corresponding to the series $f_{r,s}(\Delta, h, t) = 0 \ (|\chi\rangle_l^{(-1)Q}$ and $|\chi\rangle_l^{(0)G,Q})$ and the series $g_r(\Delta, h, t) = 0 \ (|\chi\rangle_l^{(1)G}$ and $|\chi\rangle_l^{(0)G,Q})$.

Types of submodules

Now let us identify the different types of submodules that one finds in the generic Verma modules built on the h.w. vectors $|\Delta, h\rangle^G$ and/or $|\Delta, h - 1\rangle^Q$. We will take into account the size of the submodules as well as their shape at the bottom. The singular vectors corresponding to the series $f_{r,s}(\Delta, h, t) = 0$ belong to two different types of submodules, as shown in Fig. VI, with the same partition functions $P(l - \frac{r^2}{2})$ corresponding to complete Verma submodules, but with drastically different shapes at the bottom of the submodules. In the most general case the bottom of the submodule consists of two singular vectors connected by one or two horizontal arrows corresponding to $Q_0$ and/or $G_0$ (only one arrow if one of the singular vectors is chiral). However, it can also happen that the two singular vectors at the bottom of the submodule are chiral both, and therefore disconnected from each other. An important observation is that a given Verma submodule may not be completely generated by the singular vectors at the bottom. These could generate the submodule only partially, in which case one or more subsingular vectors generate the missing parts.

Fig. VI. The singular vectors corresponding to the series $f_{r,s}(\Delta, h, t) = 0$ belong to two different types of submodules of the same size (complete Verma submodules). In the first type the two singular vectors at the bottom of the submodules are connected by $G_0$ and/or $Q_0$, depending on whether $\Delta \neq -l$ or $\Delta = -l$, $t \neq \frac{\Delta}{n}$, $n = 1, ... , r$ (for which one of the singular vectors is chiral). In the second type, corresponding to $\Delta = -l$, $t = -\frac{\Delta}{n}$, $n = 1, ... , r$, the two singular vectors are chiral and therefore disconnected from each other.
The singular vectors corresponding to the series \( g_k^\pm(\Delta, h, t) = 0 \) belong to only one type of submodule, as shown in Fig. VII, with partition functions \( P_k(l - k) \) corresponding to incomplete Verma submodules. The two singular vectors at the bottom of the submodule are always connected by \( G_0 \) and/or \( Q_0 \) since at most one of these singular vectors can be chiral. Finally, the no-label singular vectors generate their own no-label submodules. As shown in Fig. VII, at the bottom of the no-label submodule one finds four singular vectors: the primitive no-label singular vector and the three secondary singular vectors generated by acting on this one with \( Q_0 \) and \( G_0 \). No-label submodules are therefore much wider than the other types of submodules.

![Diagram showing submodules](image)

Fig. VII. The singular vectors corresponding to the series \( g_k^\pm(\Delta, h, t) = 0 \) belong to only one type of submodules (incomplete Verma submodules). The two singular vectors at the bottom of the submodules are connected by \( G_0 \) and/or \( Q_0 \), depending on whether \( \Delta \neq -l \) or \( \Delta = -l \). The no-label singular vectors generate the widest submodules with four singular vectors at the bottom.

We see therefore that there are four different types of submodules that may appear in generic Verma modules, distinguished by their size and/or the shape at the bottom of the submodule. A more accurate classification of the submodules should take into account also the shape of the whole submodule, including the possible existence of subsingular vectors [28].

### 3.2 Chiral Verma modules

Chiral Verma modules \( V([0, h]^{G,Q}) \) result from the quotient of generic Verma modules of types \( V([0, h]^Q) \) and \( V([0, h]^G) \), by the submodules generated by the level-zero singular vectors \( G_0[0, h]^Q \) and \( Q_0[0, h]^G \), respectively. Therefore there are two possible origins for the singular vectors in chiral Verma modules, as shown in Fig. VIII: they are either the ‘surviving’ singular vectors in \( V([0, h]^G) \) or \( V([0, h]^Q) \) which do not belong to the submodules set to zero, or they appear when subsingular vectors in \( V([0, h]^G) \) or \( V([0, h]^Q) \) are converted into singular vectors in \( V([0, h]^{G,Q}) \) as a consequence of the quotient procedure [28]. In the first case one deduces straightforwardly, from

---

\footnotesize

7In the literature there are claims (without proofs) that there are only two different types of submodules in generic (standard) Verma modules of the Topological \( N=2 \) algebra. In particular, the chiral-chiral submodules on the right of Fig. VI and the no-label submodules on the right of Fig. VII do not fit into that classification.

8We stress the fact that these two are the only possible origins for the singular vectors in chiral Verma modules (by
the results of tables (2.5) and (2.4), that the only possible types of such surviving singular vectors are the four types shown in table (2.8): \(|\chi\rangle^{(0)G}_I, |\chi\rangle^{(1)G}_I, |\chi\rangle^{(0)Q}_I\) and \(|\chi\rangle^{(-1)Q}_I\). In the second case, when the singular vectors in chiral Verma modules originate from subsingular vectors, one cannot deduce their possible types from tables (2.5) and (2.6). However, we know that the only possible types of such singular vectors are again the four types given in table (2.8) since we have proved, in ref. [13], that the space associated to any other type of ‘would-be’ singular vector in chiral Verma modules has maximal dimension zero (i.e. no singular vectors of such types exist). Observe that in chiral Verma modules there are no chiral neither no-label singular vectors. As a consequence the action of either \(G_0\) or \(Q_0\) on a singular vector produces always another singular vector. As a result the singular vectors in chiral Verma modules come always two by two at the same level in the same Verma module, \(G_0\) and \(Q_0\) interpolating between them.

\[
\begin{align*}
V(Q_0|0,h)^G & \equiv \{0\} \\
|\chi\rangle^{(0)G}_I & \equiv 0 \\
V(Q_0|0,h)^G & \equiv \{0\}
\end{align*}
\]

singular vector

\[
\begin{align*}
V(Q_0|0,h)^G & \equiv \{0\} \\
|\chi\rangle^{(1)G}_I & \equiv 0 \\
V(Q_0|0,h)^G & \equiv \{0\}
\end{align*}
\]

subsingular vector

**Fig. VIII.** When the singular vector \(Q_0|0,h)^G\) is set to zero, the generic Verma module \(V([0,h]^G)\) built on the h.w. vector \([0,h]^G\) is divided by the submodule \(V(Q_0|0,h)^G)\). As a consequence, the \(G_0\)-closed h.w. vector \([0,h]^G\) becomes the chiral h.w. vector \([0,h]^{G,Q}\) and the resulting Verma module is an incomplete, chiral Verma module \(V([0,h]^{G,Q})\). The singular vectors in the generic Verma module that are not located inside the submodule \(V(Q_0|0,h)^G)\) remain after the quotient procedure, they are the ‘surviving’ singular vectors. But it can also happen that there are subsingular vectors outside the submodule \(V(Q_0|0,h)^G)\), but descending to it by acting with the generators of the algebra. These subsingular vectors become singular in the resulting chiral Verma module \(V([0,h]^{G,Q})\).

The determinant formulae for chiral Verma modules of the Topological, Neveu-Schwarz and Ramond N=2 superconformal algebras were written down in ref. [13]. They were deduced from the determinant formulae for the complete generic Verma modules by imposing the ansatz that the roots of the determinants are the same in both cases (for the values of the conformal weights \(\Delta\) corresponding to the chiral Verma modules). Although no rigorous proofs were presented for this ansatz, the corresponding expressions were checked until level 4. For the case of the Topological algebra the chiral determinant formula is given as function of the U(1) charges \(h\) as the topological chiral Verma modules have zero conformal weight \(\Delta = 0\). It reads

\[
det(M_T^{T-ch}) = \text{cst.} \prod_{2 \leq r,s \leq 2l} (h - h^{(0)}_{r,s} 2P_r(l - \frac{r}{l}))^2 (h - h^{(1)}_{r,s} 2P_r(l - \frac{r}{l})) \quad r \in \mathbb{Z}^+, \ s \in 2\mathbb{Z}^+, \quad (3.15)
\]

(definition of subsingular vectors). The failure to see this has lead to very confusing statements in the literature where singular vectors in chiral Verma modules are viewed as objects of different nature than the singular and subsingular vectors in the generic Verma modules.
with the roots $h_{r,s}^{(0)}$ and $h_{r,s}^{(1)}$, given by

$$h_{r,s}^{(0)} = \frac{t}{2} (1 + r) - \frac{s}{2}, \quad r \in \mathbb{Z}^+, \ s \in 2 \mathbb{Z}^+, \quad (3.16)$$

$$h_{r,s}^{(1)} = \frac{t}{2} (1 - r) + \frac{s}{2} - 1, \quad r \in \mathbb{Z}^+, \ s \in 2 \mathbb{Z}^+. \quad (3.17)$$

Observe that $h_{r,s}^{(0)}$ is exactly the same as in eq. (3.5) (we write it here again for convenience). These roots satisfy

$$h_{r,s}^{(1)} = -h_{r,s}^{(0)} - c/3,$$

what indicates that the corresponding Verma modules are related by the odd spectral flow automorphism $A$ [13] [22] (see footnote 6).

The partition functions $P_r$ are defined as in eqn. (3.10):

$$\sum_N P_r(N)x^N = \frac{1}{1 + x^r} \prod_{0 < t \in \mathbb{Z}, \ 0 < m \in \mathbb{Z}} \frac{(1 + x^t)^2}{(1 - x^m)^2}. \quad (3.18)$$

Therefore they correspond to incomplete Verma submodules, as we pointed out before. The factors 2 in the exponents of the determinants (3.15) show explicitly again that the singular vectors come two by two at the same level in the same Verma module, as $P_r(0) = 1$. One can also see in the determinant formula (3.15) that the singular vectors corresponding to $h = h_{r,s}^{(0)}$ and the singular vectors corresponding to $h = h_{r,s}^{(1)}$ both belong to the same types of submodules (incomplete Verma submodules as the chiral Verma modules where they are embedded).

The interpretation of the roots of the determinants in terms of singular vectors is much simpler than for generic Verma modules. First, in chiral Verma modules there is only one h.w. vector and therefore only one way to express the singular vectors. Second, in chiral Verma modules there are no chiral singular vectors neither no-label singular vectors. The interpretation of the roots is as follows, as shown in Fig. IX.

**Fig. IX.** In chiral Verma modules built on chiral h.w. vectors $|0, h\rangle^G.Q$, for $h = h_{r,s}^{(0)}$ one finds at level $l = \frac{rs}{2}$ one uncharged singular vector of type $|\chi\rangle^{(0)G}_l$ and one charged singular vector of type $|\chi\rangle^{(-1)G}_l$, connected by $G_0$ and $Q_0$. For $h = h_{r,s}^{(1)}$ one finds at level $l = \frac{rs}{2}$ one uncharged singular vector of type $|\chi\rangle^{(0)Q}_l$ and one charged singular vector of type $|\chi\rangle^{(1)G}_l$, connected by $G_0$ and $Q_0$. 

22
Case $h = h^{(0)}_{r,s}$

For $h = h^{(0)}_{r,s}$ the Verma module built on the chiral h.w. vector $|0, h^{(0)}_{r,s}\rangle^{G,Q}$ has (at least) one uncharged singular vector of type $|\chi\rangle^{(0)G}$ and one charged singular vector of type $|\chi\rangle^{(-1)Q}$ at level $l = \frac{r}{2}$, $G_0$ and $Q_0$ interpolating between them.

Case $h = h^{(1)}_{r,s}$

For $h = h^{(1)}_{r,s}$ the Verma module built on the chiral h.w. vector $|0, h^{(1)}_{r,s}\rangle^{G,Q}$ has (at least) one uncharged singular vector of type $|\chi\rangle^{(0)Q}$ and one charged singular vector of type $|\chi\rangle^{(1)G}$ at level $l = \frac{r}{2}$, $G_0$ and $Q_0$ interpolating between them.

Now let us analyse the origin of the different singular vectors in the chiral Verma modules $V(|0, h^{(0)}_{r,s}\rangle^{G,Q})$ and $V(|0, h^{(1)}_{r,s}\rangle^{G,Q})$ with respect to the singular and subsingular vectors in the generic Verma modules. In other words, let us investigate which of the singular vectors in the generic Verma modules are lost in the quotient, which are not quotiented out, resulting in singular vectors in the chiral Verma modules, and which are ‘new’ singular vectors in the chiral Verma modules that were not singular in the generic Verma modules but subsingular, as a consequence. For this purpose we will consider the two possible quotients which give rise to the same chiral Verma module, i.e. $V(|0, h\rangle^{G,Q}) = V(|0, h\rangle^{G})/V(Q_0|0, h\rangle^{G})$ and $V(|0, h\rangle^{G,Q}) = V(|0, h\rangle^{Q})/V(G_0|0, h\rangle^{Q})$.

Let us start with the realization of the chiral Verma modules as the quotient $V(|0, h\rangle^{G,Q}) = V(|0, h\rangle^{G})/V(Q_0|0, h\rangle^{G})$. As was discussed in the last subsection, setting $\Delta = 0$ in the determinant formula (3.3) one finds the generic Verma modules of type $V(|0, h\rangle^{G})$ which contain singular vectors as follows. From $f_{r,s}(0, h, t \neq 0) = 0$ one gets the solutions $h = h^{(0)}_{r,s}$ and $h = \hat{h}_{r,s}$, eqns. (3.3) and (3.4), whereas from $f_{r,s}(0, h, t = 0) = 0$ one obtains $h = \pm \frac{r}{2}$. For all these solutions the corresponding singular vectors are pairs of types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(-1)Q}$ at level $l = \frac{r}{2}$. From $g^+_k(0, h, t) = 0$ one gets the solution $h = h^+_k$, eqn. (3.9), the corresponding singular vectors being pairs of types $|\chi\rangle^{(1)G}$ and $|\chi\rangle^{(0)Q}$ at level $l = k$. Finally from $g^+_k(0, h, t) = 0$ one obtains $h = h^-_k$, eqn. (3.9), the corresponding singular vectors being pairs of types $|\chi\rangle^{(-1)G}$ and $|\chi\rangle^{(-2)Q}$ at level $l = k$. After the quotient $V(|0, h\rangle^{G})/V(Q_0|0, h\rangle^{G})$ is performed, by setting $Q_0|0, h\rangle^{G} = 0$, one obtains the chiral Verma modules $V(|0, h\rangle^{G,Q})$ which contain singular vectors only for $h = h^{(0)}_{r,s}$ and $h = h^{(1)}_{r,s}$, eqns. (3.3) and (3.17), the corresponding singular vectors being pairs of types $|\chi\rangle^{(0)G}$, $|\chi\rangle^{(-1)Q}$ and $|\chi\rangle^{(1)G}$, $|\chi\rangle^{(0)Q}$, respectively, at level $l = \frac{r}{2}$. A simple analysis of these facts, taking into account that $h^+_l = h^{(1)}_{l,2}$ and $\hat{h}_{r,s} = h^{(1)}_{r,s+2}$, allows us to deduce the following results:

i) In the chiral Verma modules $V(|0, h^{(0)}_{r,s}\rangle^{G,Q})$ all the original pairs of singular vectors in $V(|0, h^{(0)}_{r,s}\rangle^{G})$, of types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(-1)Q}$, with $l = \frac{r}{2}$, remain after the quotient. Therefore they were located outside the submodule generated by the level zero singular vector $Q_0|0, h^{(0)}_{r,s}\rangle^{G}$, as shown in Fig. I.

ii) In the chiral Verma modules $V(|0, h_{r,s}\rangle^{G,Q}) = V(|0, h^{(1)}_{r,s+2}\rangle^{G,Q})$ all the original pairs of singular vectors in $V(|0, h_{r,s}\rangle^{G})$, of types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(-1)Q}$, with $l = \frac{r}{2}$, have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular

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Some of these results were already given in refs. [2] and [3].
vector $Q_0|0, \hat{h}_{r,s}|^G$, as shown in Fig. I. Moreover, in their place new singular vectors appear, of types $|\chi_l^1|^G$ and $|\chi_l^0|^G$, with $l = \frac{r(s+2)}{2}$, which were not present before the quotient. These singular vectors are therefore subsingular in $V(|0, \hat{h}_{r,s}|^G)$.

iii) For $t = 0$ in the chiral Verma modules $V(|0, \h = -\frac{s}{2}|^{G,Q}) = V(|0, \h_{r,s,t+0}|^{G,Q})$ all the original pairs of singular vectors in $V(|0, \h = -\frac{s}{2}|^G)$, of types $|\chi_l^0|^G$ and $|\chi_l^{(-1)}|^G$, with $l = \frac{s}{2}$, have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector $Q_0|0, \h = -\frac{s}{2}|^G$.

iv) For $t = 0$ in the chiral Verma modules $V(|0, \h = \frac{s}{2}|^{G,Q}) = V(|0, \h_{r,s+2,t+0}|^{G,Q})$ all the original pairs of singular vectors in $V(|0, \h = \frac{s}{2}|^G)$, of types $|\chi_l^0|^G$ and $|\chi_l^{(-1)}|^G$, with $l = \frac{s}{2}$, have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector $Q_0|0, \h = \frac{s}{2}|^G$. Moreover, in their place new singular vectors appear, of types $|\chi_l^1|^G$ and $|\chi_l^{(0)}|^G$, with $l = \frac{r(s+2)}{2}$, which were not present before the quotient. These singular vectors are therefore subsingular in $V(|0, \h = \frac{s}{2}|^G)$.

v) In the chiral Verma modules $V(|0, \h^+_r|^G,Q) = V(|0, \h_{r,0}^+|^G,Q)$ all the original pairs of singular vectors in $V(|0, \h^+_r|^G)$, of types $|\chi_l^1|^G$ and $|\chi_l^{(0)}|^G$, remain after the quotient. Therefore they were located outside the submodule generated by the level zero singular vector $Q_0|0, \h^+_r|^G$, as shown in Fig. III.

vi) In the chiral Verma modules $V(|0, \h^-_r|^G,Q) = V(|0, \h_{r,0}^-|^G,Q)$ all the original pairs of singular vectors in $V(|0, \h^-_r|^G)$, of types $|\chi_l^{(-1)}|^G$ and $|\chi_l^{(-2)}|^G$, have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector $Q_0|0, \h^-_r|^G$, as shown in Fig. IV. No new singular vectors appear corresponding to possible subsingular vectors in $V(|0, \h^-_r|^G)$.

Let us now investigate the realization of the chiral Verma modules as the quotient $V(|0, \h|^G,Q) = V(|0, \h|^Q)/V(g_0|0, \h|^Q)$. The generic Verma modules of type $V(|0, \h|^Q)$ which contain singular vectors are the following. From $f_{r,s}(0, \h, t \neq 0) = 0$ one gets the solutions $\h = \h_{r,s}^0 - 1$ and $\h = \h_{r,s} - 1$, where $\h_{r,s}^0$ and $\h_{r,s}$ are given by eqns. (3.5) and (3.6), whereas from $f_{r,s}(0, \h, t = 0) = 0$ one obtains $\h = \pm \frac{s}{2} - 1$. For all these solutions the corresponding singular vectors are pairs of types $|\chi_l^0|^G$ and $|\chi_l^1|^G$ at level $l = \frac{s}{2}$. From $g_k^+(0, \h, t) = 0$ one gets the solution $\h = \h_{r,s}^+ - 1$, with $\h_{r,s}^+$ given by eqn. (3.8). The corresponding singular vectors are pairs of types $|\chi_l^1|^G$ and $|\chi_l^2|^G$ at level $l = k$. Finally from $g_k^-(0, \h, t) = 0$ one obtains $\h = \h_{r,s}^- - 1$, with $\h_{r,s}^-$ given by eqn. (3.9), the corresponding singular vectors being pairs of types $|\chi_l^{(-1)}|^G$ and $|\chi_l^{(0)}|^G$ at level $l = k$. After the quotient $V(|0, \h|^Q)/V(g_0|0, \h|^Q)$ is performed, by setting $g_0|0, \h|^Q \equiv 0$, one obtains as in the previous case the chiral Verma modules $V(|0, \h|^G,Q)$ which contain singular vectors only for $\h = \h_{r,s}^0$ and $\h = \h_{r,s}^1$, eqns. (3.3) and (3.17). Repeating the same analysis as before, taking into account that $\h_{r,s}^0 - 1 = \h_{r,s+2}^0$, $\h_{r,s} - 1 = \h_{r,s+2}^1$ and $\h_{r,s}^- - 1 = \h_{r,s+2}^0$, one finds the following results:

i) In the chiral Verma modules $V(|0, \h_{r,s} - 1|^G,Q) = V(|0, \h_{r,s}^1|^G,Q)$ all the original pairs of singular vectors in $V(|0, \h_{r,s} - 1|^Q)$, of types $|\chi_l^0|^Q$ and $|\chi_l^1|^G$, with $l = \frac{r(s+2)}{2}$, remain after the quotient. Therefore they were located outside the submodule generated by the level zero singular vector $g_0|0, \h_{r,s} - 1|^Q$, as shown in Fig. II.
ii) In the chiral Verma modules \( V(\{0, h_r^0, l\}^{G,Q}) = V(\{0, h_{r,s+l}^{0}G,Q\} \) all the original pairs of singular vectors in \( V(\{0, h_{r,s}^{0} - 1\}^{Q}) \), of types \( |\chi_{l}^{(0)Q}\rangle \) and \( |\chi_{l}^{(1)G}\rangle \), with \( l = \frac{r+s}{2} \), have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector \( G_0(\{0, h_{r,s}^{0} - 1\}^{Q}) \), as shown in Fig. II. Moreover, in their place new singular vectors appear, of types \( |\chi_{l}^{(0)G}\rangle \) and \( |\chi_{l}^{(-1)Q}\rangle \), with \( l = \frac{r+(s+2)}{2} \), which were not present before the quotient. These singular vectors are therefore subsingular in \( V(\{0, h_{r,s}^{0} - 1\}^{Q}) \).

iii) For \( t = 0 \) in the chiral Verma modules \( V(\{0, h = \frac{s}{2} - 1\}^{G,Q}) = V(\{0, h_{r,s,t}^{0,0}G,Q\} \) all the original pairs of singular vectors in \( V(\{0, h = \frac{s}{2} - 1\}^{Q}) \), of types \( |\chi_{l}^{(0)Q}\rangle \) and \( |\chi_{l}^{(1)G}\rangle \), with \( l = \frac{r}{2} \), remain after the quotient. Therefore they were located outside the submodule generated by the level zero singular vector \( G_0(\{0, h = \frac{s}{2} - 1\}^{Q}) \). Moreover, in their place new singular vectors appear, of types \( |\chi_{l}^{(0)G}\rangle \) and \( |\chi_{l}^{(-1)Q}\rangle \), with \( l = \frac{r+(s+2)}{2} \), which were not present before the quotient. These singular vectors are therefore subsingular in \( V(\{0, h = \frac{s}{2} - 1\}^{Q}) \).

iv) For \( t = 0 \) in the chiral Verma modules \( V(\{0, h = -\frac{s}{2} - 1\}^{G,Q}) = V(\{0, h_{r,s+2,t}^{0,0}G,Q\} \) all the original pairs of singular vectors in \( V(\{0, h = -\frac{s}{2} - 1\}^{Q}) \), of types \( |\chi_{l}^{(0)Q}\rangle \) and \( |\chi_{l}^{(1)G}\rangle \), with \( l = \frac{r}{2} \), have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector \( G_0(\{0, h = -\frac{s}{2} - 1\}^{Q}) \). Moreover, in their place new singular vectors appear, of types \( |\chi_{l}^{(0)G}\rangle \) and \( |\chi_{l}^{(-1)Q}\rangle \), with \( l = \frac{r+(s+2)}{2} \), which were not present before the quotient. These singular vectors are therefore subsingular in \( V(\{0, h = -\frac{s}{2} - 1\}^{Q}) \).

v) In the chiral Verma modules \( V(\{0, h^{+} - 1\}^{G,Q}) = V(\{0, h_{r,2}^{0,0}G,Q\} \) all the original pairs of singular vectors in \( V(\{0, h^{+} - 1\}^{Q}) \), of types \( |\chi_{l}^{(-1)Q}\rangle \) and \( |\chi_{l}^{(0)G}\rangle \), remain after the quotient. Therefore they were located outside the submodule generated by the level zero singular vector \( G_0(\{0, h^{+} - 1\}^{Q}) \), as shown in Fig. IV.

vi) In the chiral Verma modules \( V(\{0, h^{+} - 1\}^{G,Q}) \) all the original pairs of singular vectors in \( V(\{0, h^{+} - 1\}^{Q}) \), of types \( |\chi_{l}^{(1)Q}\rangle \) and \( |\chi_{l}^{(2)G}\rangle \), have disappeared after the quotient. Therefore they were located inside the submodule generated by the level zero singular vector \( G_0(\{0, h^{+} - 1\}^{Q}) \), as shown in Fig. III. No new singular vectors appear corresponding to possible subsingular vectors in \( V(\{0, h^{+} - 1\}^{Q}) \).

An interesting observation now [12] is that two series of singular vectors in chiral Verma modules correspond to both singular vectors and subsingular vectors depending on the specific generic Verma modules. Namely, the singular vectors of types \( |\chi_{l}^{(0)G}\rangle \) and \( |\chi_{l}^{(-1)Q}\rangle \) in the chiral Verma modules \( V(\{0, h_{r,s}^{0}G,Q\} \) correspond to singular vectors in \( V(\{0, h_{r,s}^{0}G\} \) but to subsingular vectors in \( V(\{0, h_{r,s}^{0} - 1\}^{Q}) \) for \( s > 2 \). Similarly, the singular vectors of types \( |\chi_{l}^{(0)Q}\rangle \) and \( |\chi_{l}^{(1)G}\rangle \) in the chiral Verma modules \( V(\{0, h_{r,s}^{1}G,Q\} \) correspond to singular vectors in \( V(\{0, h_{r,s}^{1}G\} \) but to subsingular vectors in \( V(\{0, h_{r,s}^{1}G\} \) for \( s > 2 \). These subsingular vectors do not have, a priori, well defined BRST-invariance properties, but become \( G_0 \)-closed or \( Q_0 \)-closed when they become singular. Let us note that this symmetry between singular/subsingular vectors in the \( G_0 \)-closed/\( Q_0 \)-closed Verma modules \( V(\{0, h\}^{G}) \) and \( V(\{0, h - 1\}^{Q}) \) is broken for the Neveu-Schwarz N=2 algebra since the \( Q_0 \)-closed topological h.w. vectors do not correspond to h.w. vectors (but secondary states) of that algebra, as explained in subsection 2.1, and consequently there is no NS counterpart for the \( Q_0 \)-closed Verma modules \( V(\{0, h\}) \). (For \( \Delta \neq 0 \) there is NS counterpart for the Verma modules \( V(\{\Delta, h\}^{G}) \), however, because they coincide with the Verma modules \( V(\{\Delta, h\}^{Q}) \), as explained in subsection 2.2).
3.3 No-label Verma modules

No-label Verma modules $V(|0, \mathbf{h}\rangle)$ built on no-label h.w. vectors appear as submodules in some generic Verma modules for $t = \frac{2}{3}$ ($c = \frac{3\mathcal{r} - 6}{\mathcal{r}}$). They have a counterpart in the Ramond $N=2$ algebra -- no-helicity Verma modules -- as we will see in next subsection. However, they do not have a counterpart in the Neveu-Schwarz $N=2$ algebra as the NS counterpart of no-label h.w. vectors are non-h.w. NS vectors. In fact, the no-label singular vectors in generic Verma modules correspond to NS subsingular vectors, as was proved in ref. [17].

In principle, we could define no-label Verma modules also for $\Delta \neq 0$. In this case, however, the no-label h.w. vector $|\Delta, \mathbf{h}\rangle$ can be decomposed into a $\mathcal{G}_0$-closed h.w. vector and a $\mathcal{Q}_0$-closed h.w. vector so that $V(|\Delta, \mathbf{h}\rangle)$ is simply a direct sum of the generic Verma module built on $\mathcal{G}_0|\Delta, \mathbf{h}\rangle = |\Delta, \mathbf{h} + 1\rangle^G$ and the generic Verma module built on $\mathcal{Q}_0|\Delta, \mathbf{h}\rangle = |\Delta, \mathbf{h} - 1\rangle^Q$. Therefore, the corresponding determinant formula for $\Delta \neq 0$ is simply the product of the two generic determinant formulae and its study does not lead to anything more than what we have found for the generic Verma modules.

For the only interesting no-label case, i.e. for $\Delta = 0$, the no-label Verma module is no longer a direct sum of two generic Verma modules, however. In fact, the level-zero vectors $\mathcal{G}_0|0, \mathbf{h}\rangle$, $\mathcal{Q}_0|0, \mathbf{h}\rangle$, and $\mathcal{G}_0\mathcal{Q}_0|0, \mathbf{h}\rangle$ become singular for $\Delta = 0$ and therefore the no-label determinant formula for $\Delta = 0$ vanishes for all levels. Nevertheless, we can still use the roots of the determinants of the no-label Verma modules in the neighbourhood of $\Delta = 0$ in order to find the singular vectors for no-label Verma modules with $\Delta = 0$. For this purpose we take the product of the determinant formulae for the generic Verma modules $V(|\Delta, \mathbf{h} + 1\rangle^G)$ and $V(|\Delta, \mathbf{h} - 1\rangle^Q)$ and factorise their roots in the limit $\Delta \to 0$. The terms that vanish in this limit we combine in a factor $\alpha (\Delta = 0) = 0$ which is responsible for the fact that the no-label determinant vanishes at each level. Taking into account the terms that do not vanish in this limit one finally finds the following expression

$$
det(\mathcal{M}_t^{T-nl}) = \alpha (\Delta = 0) \prod_{2 \leq r,s \leq 2l} (\mathbf{h} - \mathbf{h}_{r,s}^{(0)})^{2P(l - \frac{r}{\mathcal{r}})} (\mathbf{h} - \mathbf{h}_{r,s}^{(1)})^{2P(l - \frac{r}{\mathcal{r}})} (\mathbf{h} - \mathbf{h}_{r,s}^{(0)} + 1)^{2P(l - \frac{r}{\mathcal{r}})}$$

$$
(h - \hat{h}_{r,s})^{2P(l - \frac{r}{\mathcal{r}})} \prod_{0 \leq k \leq l} (\mathbf{h} - \mathbf{h}_k^+)^{2P_k(l - k)} (\mathbf{h} - \mathbf{h}_k^-)^{2P_k(l - k)},$$

with $\mathbf{h}_{r,s}^{(0)}$, $\mathbf{h}_{r,s}^{(1)}$, $\hat{h}_{r,s}$, $\mathbf{h}_k^+$ and $\mathbf{h}_k^-$ given by eqns. (3.5), (3.11), (3.6) and (3.8). The partition functions $P$ and $P_k$ are defined in eqn. (3.10). As was explained in subsection 3.1, the partitions $2P(l - \frac{r}{\mathcal{r}})$ correspond to complete Verma submodules of generic type, whereas the partitions $2P_k(l - k)$ correspond to incomplete Verma submodules. Once again we see that the singular vectors come two by two at the same level, in the same Verma module, as $P(0) = P_k(0) = 1$.

The factor $\alpha (\Delta = 0) = 0$ indicates that the determinant always vanishes for any value of $\mathbf{h}$, as happens for the generic Verma modules with $\Delta = 0$ (with one singular vector at level zero). Similarly as in that case, the other roots of the determinants identify the no-label Verma modules with singular vectors other than the ones at level zero (which is an important information in order to study embedding patterns). Like for chiral Verma modules, in no-label Verma modules there are
no chiral singular vectors nor no-label singular vectors. The interpretation of the roots in terms of
the singular vectors that one finds in the no-label Verma modules is as follows.

Case $h = h_{r,s}^{(0)}$

For $h = h_{r,s}^{(0)}$ the Verma module built on the no-label h.w. vector $|0, h_{r,s}^{(0)}\rangle$ has (at least) one
uncharged singular vector of type $|\chi_l^{(0)}G\rangle$ and one charged singular vector of type $|\chi_l^{(-1)}Q\rangle$ at
level $l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the chiral
submodule generated by the level zero chiral singular vector $\mathcal{G}_0 \mathcal{Q}_0 |0, h_{r,s}^{(0)}\rangle$.

Case $h = h_{r,s}^{(1)}$

For $h = h_{r,s}^{(1)}$ the Verma module built on the no-label h.w. vector $|0, h_{r,s}^{(1)}\rangle$ has (at least) one
uncharged singular vector of type $|\chi_l^{(0)}Q\rangle$ and one charged singular vector of type $|\chi_l^{(1)}G\rangle$ at
level $l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the chiral
submodule generated by the level zero chiral singular vector $\mathcal{G}_0 \mathcal{Q}_0 |0, h_{r,s}^{(1)}\rangle$.

Case $h = h_{r,s}^{(0)} - 1$

For $h = h_{r,s}^{(0)} - 1$ the Verma module built on the no-label h.w. vector $|0, h_{r,s}^{(0)} - 1\rangle$ has (at least) one
uncharged singular vector of type $|\chi_l^{(0)}Q\rangle$ and one charged singular vector of type $|\chi_l^{(1)}G\rangle$ at
level $l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the submodule
generated by the level zero singular vector $\mathcal{Q}_0 |0, h_{r,s}^{(0)} - 1\rangle$ (outside the chiral submodule).

Case $h = h_{r,s}$

For $h = h_{r,s}$ the Verma module built on the no-label h.w. vector $|0, h_{r,s}\rangle$ has (at least) one
uncharged singular vector of type $|\chi_l^{(0)}G\rangle$ and one charged singular vector of type $|\chi_l^{(-1)}Q\rangle$ at
level $l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the submodule
generated by the level zero singular vector $\mathcal{Q}_0 |0, h_{r,s}\rangle$ (outside the chiral submodule).

Case $h = h_l^-$

For $h = h_l^-$ the Verma module built on the no-label h.w. vector $|0, h_l^-\rangle$ has (at least) one
charged singular vector of type $|\chi_l^{(-1)}G\rangle$ and one charged singular vector of type $|\chi_l^{(-2)}Q\rangle$ at
level $l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the submodule
generated by the level zero singular vector $\mathcal{Q}_0 |0, h_l^-\rangle$ (outside the chiral submodule).

Case $h = h_l^+ - 1$

For $h = h_l^+ - 1$ the Verma module built on the no-label h.w. vector $|0, h_l^+ - 1\rangle$ has (at least) one
charged singular vector of type $|\chi_l^{(1)}Q\rangle$ and one charged singular vector of type $|\chi_l^{(2)}G\rangle$ at level
$l = \frac{r_s}{2}$, $G_0$ and $Q_0$ interpolating between them. These singular vectors are inside the submodule
generated by the level zero singular vector $\mathcal{G}_0 |0, h_l^+ - 1\rangle$ (outside the chiral submodule).
Cases $h = h_i^+$ and $h = h_i^- - 1$

These roots do not give new singular vectors as $h_i^+ = h_{i,2}^{(1)}$ and $h_i^- - 1 = h_{i,2}^{(0)}$.

FIG. X The no-label Verma module built on the no-label h.w. vector $|0, h\rangle$ has three singular vectors at level zero: $|0, h + 1\rangle^G = G_0|0, h\rangle$, $|0, h - 1\rangle^Q = Q_0|0, h\rangle$ and $|0, h\rangle^{G,Q} = G_0Q_0|0, h\rangle$. The submodules generated by $|0, h + 1\rangle^G$ and $|0, h - 1\rangle^Q$ are generic Verma modules intersecting in the chiral Verma module built on $|0, h\rangle^{G,Q}$.

One important remark is that all the singular vectors ‘detected’ by the determinant formula (3.19), that belong to the series that we have just described, are located inside the submodules generated by the level zero singular vectors $G_0|0, h\rangle$, $Q_0|0, h\rangle$ and $G_0Q_0|0, h\rangle$. There must exist, however, singular or subsingular vectors outside these submodules. The reason is that after dividing the no-label Verma module by these submodules a chiral Verma module is left with the corresponding singular vectors for $h = h_{r,s}^{(0)}$ and $h = h_{r,s}^{(1)}$, as discussed in subsection 3.2. Our conjecture is that outside the submodules built on the level zero singular vectors there are generically only subsingular vectors, which are singular just for some discrete values of $t$ (i.e. of $c$). At level 1 this is the case for $t = -2$ ($c = 9$), as one can see in Appendix B, and at level 2 this is the case for $t = -1$ ($c = 6$), $t = -2$ ($c = 9$) and $t = 2$ ($c = -3$).

3.4 Verma modules and singular vectors of the Ramond N=2 algebra

The determinant formula for the standard Verma modules of the Ramond N=2 superconformal algebra was computed in the middle eighties [9][11]. In spite of this, the Verma modules and singular vectors of the Ramond N=2 algebra have not been given much attention in the literature so far (see however refs. [26], [17] and [18]). More recently the determinant formula for the ‘chiral’ Verma modules (built on the Ramond ground states) has been computed in ref. [13]. As a bonus, subsingular vectors were discovered for this algebra. Our purpose now is to show that the Verma modules of the Ramond N=2 algebra are isomorphic to the Verma modules of the Topological N=2 algebra. In particular one can construct a one-to-one mapping between every Ramond singular vector and every topological singular vector at the same levels and with the same relative charges. As a consequence all the results we have obtained in this section for the Topological N=2 algebra, summarized in the four tables (3.11) – (3.14), and the ten figures Fig. I – Fig. X, can be transferred...
straightforwardly to the Ramond N=2 algebra.

To start let us say a few words about the Ramond N=2 superconformal algebra given by \[ L_m, L_n \] \[ L_m, G^+_r \] \[ L_m, H_n \] \[ \{ G^-_r, G^+_s \} \]

\[
\begin{align*}
[L_m, L_n] & = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G^+_r] & = (\frac{m}{2} - r)G^\pm_{m+r}, \\
[L_m, H_n] & = -nH_{m+n}, \\
\{ G^-_r, G^+_s \} & = 2L_{r+s} - (r - s)H_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0},
\end{align*}
\]

where all the generators, bosonic \(L_n, H_n\), and fermionic \(G^+_r\), are integer-moded. The fermionic zero modes characterize the states as being \(G^+_0\)-closed or \(G^-_0\)-closed, as the anticommutator \(\{ G^+_0, G^-_0 \} = 2L_0 - \frac{c}{12}\) shows. We will denote these states as ‘helicity’(+)- and ‘helicity’(-)- states. However, for the conformal weight \(\Delta = \frac{c}{24}\) the ground states are annihilated by both \(G^+_0\) and \(G^-_0\) (we will call them chiral), and there also exist indecomposable ‘no-helicity’ states which cannot be expressed as linear combinations of helicity-(+), helicity-(−) and chiral states. These no-helicity states, singular vectors in particular, have been reported very recently for the first time in ref. [17], and they were completely overlooked in the early literature.

We already see that the case \(\Delta = \frac{c}{24}\) for the Ramond N=2 algebra is the equivalent to the case \(\Delta = 0\) for the Topological N=2 algebra. Furthermore, the helicity-(+) and helicity-(−) R states are analogous to the \(G_0\)-closed (\(G\)) and \(Q_0\)-closed (\(Q\)) topological states, and the chiral (+−) and no-helicity R states are analogous to the chiral (\(G, Q\)) and no-label topological states, respectively.

In order to simplify the analysis that follows we will define the U(1) charge for the states of the Ramond N=2 algebra in the same way as for the states of the Topological and Neveu-Schwarz N=2 algebras. Namely, the U(1) charge of the states will be denoted by \(h\) (instead of \(h \pm \frac{1}{2}\), as was the standard notation in the past [6]), and the relative charge \(q\) of a secondary state will be defined as the difference between the \(H_0\)-eigenvalue of the state and the \(H_0\)-eigenvalue of the primary on which it is built. Therefore, the relative charges of the R states are defined to be integer, like for the other N=2 algebras. We will denote the R singular vectors as \(|\chi_{Rl}^{(q)}\rangle\), \(|\chi_{Rl}^{(q-)}\rangle\), \(|\chi_{Rl}^{(q+)}\rangle\) or \(|\chi_{Rl}^{(q)}\rangle\), where, in addition to the level \(l\) and the relative charge \(q\), the helicities indicate that the vector is annihilated by \(G^+_0\) or \(G^-_0\), or both, or none, respectively.

Now we will construct a one-to-one mapping between the R singular vectors and the topological singular vectors that preserves the level \(l\) and the relative charge \(q\). For this let us compose the topological twists [21], which transform the Topological N=2 algebra into the Neveu-Schwarz N=2 algebra, with the spectral flows, which transform the latter into the Ramond N=2 algebra. Let us consider the odd spectral flow [27] [22], which is the only fundamental spectral flow, as explained in ref. [22]. It is given by the one-parameter family of transformations

\[
\begin{align*}
A_\theta L_m A_\theta & = L_m + \theta H_m + \frac{c}{6}\theta^2 \delta_{m,0}, \\
A_\theta H_m A_\theta & = -H_m - \frac{c}{3}\theta \delta_{m,0}, \\
A_\theta G^+_r A_\theta & = G^-_{r-\theta}, \\
A_\theta G^-_r A_\theta & = G^+_{r+\theta},
\end{align*}
\]

satisfying \(A_\theta^{-1} = A_\theta\) and giving rise to isomorphic N=2 algebras. (It is therefore an involution).
If we denote by \((\Delta, h)\) the \((L_0, H_0)\) eigenvalues of any given state, then the eigenvalues of the transformed state \(A_\theta |\chi\rangle\) are \((\Delta + \theta h + \frac{\theta^2}{2}, -h - \frac{\theta}{2})\). If the state \(|\chi\rangle\) is a level-\(l\) secondary state with relative charge \(q\), then one deduces straightforwardly that the level of the transformed state \(A_\theta |\chi\rangle\) changes to \(l + \theta q\) while the relative charge \(q\) reverses its sign. For half-integer values of \(\theta\) the spectral flows interpolate between the Neveu-Schwarz N=2 algebra and the Ramond N=2 algebra.

By analysing the composition of the topological twists (2.1) and the spectral flows (3.21), combining all possibilities, one obtains that the only mapping that transforms the topological states into R states, preserving the level and the relative charge, is the composition \(A_{-1/2} (T_W^\dagger)^{-1}\):

\[
|\chi_R\rangle_l^{(q)} = A_{-1/2} (T_W^\dagger)^{-1} |\chi\rangle_l^{(q)} .
\] (3.22)

Any other combination of the topological twists (2.1) and the spectral flows (3.21) changes either the level or the relative charge of the states, or both, as the reader can easily verify\(^\dagger\). Furthermore this mapping is one-to-one because it transforms every topological state into a R state, and the other way around: \(|\chi\rangle_l^{(q)} = (T_W) A_{-1/2} |\chi_R\rangle_l^{(q)} .

Now let us show that if \(|\chi\rangle_l^{(q)}\) is singular, i.e. satisfies the topological h.w. conditions \(L_{n\geq 1} |\chi\rangle_l^{(q)} = H_{n\geq 1} |\chi\rangle_l^{(q)} = Q_{n\geq 1} |\chi\rangle_l^{(q)} = 0\), then \(|\chi_R\rangle_l^{(q)}\) is also singular, satisfying in turn the R h.w. conditions \(L_{n\geq 1} |\chi_R\rangle_l^{(q)} = H_{n\geq 1} |\chi_R\rangle_l^{(q)} = 0\). To see this we have to study first the transformation, under \(A_{-1/2} (T_W^\dagger)^{-1}\), of the h.w. vectors of the topological Verma modules. Thus we have four cases to analyse, corresponding to the topological h.w. vectors being \(G_0\)-closed \(|\Delta, h\rangle^G\), \(Q_0\)-closed \(|\Delta, h\rangle^Q\), chiral \(|0, h\rangle^G.Q\), and no-label \(|0, h\rangle\). By carefully keeping track of the transformation of the positive and zero modes of the topological generators one obtains the following results:

i) The \(G_0\)-closed topological h.w. vectors \(|\Delta, h\rangle^G\) are mapped by \(A_{-1/2} (T_W^\dagger)^{-1}\) to helicity-(+) R h.w. vectors \(|\Delta_R, h_R\rangle_R^+ = |\Delta + \frac{\theta}{2}, h + \frac{\theta}{2}\rangle_R^+\).

ii) The \(Q_0\)-closed topological h.w. vectors \(|\Delta, h\rangle^Q\) are mapped by \(A_{-1/2} (T_W^\dagger)^{-1}\) to helicity-(−) R h.w. vectors \(|\Delta_R, h_R\rangle_R^- = |\Delta + \frac{\theta}{2}, h + \frac{\theta}{2}\rangle_R^-\).

iii) The chiral topological h.w. vectors \(|0, h\rangle^G.Q\) are mapped by \(A_{-1/2} (T_W^\dagger)^{-1}\) to chiral R h.w. vectors \(|\Delta_R, h_R\rangle_R^\dagger^+ = |\frac{\theta}{2}, h + \frac{\theta}{2}\rangle_R^\dagger^+\), usually denoted as Ramond ground states.

iv) The no-label topological h.w. vectors \(|0, h\rangle\) are mapped by \(A_{-1/2} (T_W^\dagger)^{-1}\) to no-helicity R h.w. vectors \(|\Delta_R, h_R\rangle_R = |\frac{\theta}{2}, h + \frac{\theta}{2}\rangle_R\).

Now by taking into account that singular vectors are just particular cases of h.w. vectors (secondary states that satisfy the h.w. conditions, to be precise) one deduces that the topological singular vectors are transformed under \(A_{-1/2} (T_W^\dagger)^{-1}\) into R singular vectors, and with the helicities determined by the exchange \(G \rightarrow +, Q \rightarrow −\). That is, the \(G_0\)-closed topological singular vectors are mapped to helicity-(+) R singular vectors, the \(Q_0\)-closed topological singular vectors are mapped to helicity-(−) R singular vectors, the chiral topological singular vectors are mapped to chiral R singular vectors and the no-label topological singular vectors are mapped to no-helicity R singular vectors.\(^\dagger\)

\(^\dagger\)See also a related discussion in ref. [17] (subsection 3.3).
vectors. In addition, as we pointed out, these R singular vectors have the same level $l$ and relative charge $q$ as the topological singular vectors. Therefore all the results we have obtained for the topological singular vectors and Verma modules, summarized in tables (3.11) – (3.14) and figures Fig. 1 – Fig. X are also valid for the singular vectors and Verma modules of the Ramond N=2 algebra, simply by taking into account that $\Delta_R = \Delta + \frac{c}{24}$, $h_R = h + \frac{c}{6}$, and exchanging the labels $G \rightarrow +$ and $Q \rightarrow -$.

These results imply that the standard classification of the Ramond singular vectors in two sectors, the + sector and the – sector, where the singular vectors and the h.w. vectors on which they are built have the same helicities, is not complete. The reason is that there also exist: i) singular vectors with both helicities, like the level-zero singular vectors $|\chi_R(1)+\rangle_0, |\chi_R(-1)+\rangle_0$ for $\Delta = \frac{c}{24}$, ii) indecomposable singular vectors with no helicity\footnote{\textsuperscript{11}Some examples of no-helicity R singular vectors were given in [17].}, and iii) singular vectors with the opposite helicity than the h.w. vector of the Verma module. The first and the second possibilities can only occur if $\Delta_R + l = \frac{c}{24}$. In addition, no-helicity singular vectors only exist for $\Delta = \frac{c}{24}$ (c = 3r−6), like no-label topological singular vectors, as the spectral flows and topological twists do not modify the value of c. The third possibility occurs for $\Delta = \frac{c}{24}$ due to the fact that the h.w. vectors $|\frac{c}{24}, h\rangle^+_R$ and $|\frac{c}{24}, h - 1\rangle^-_R$ are in different Verma modules, i.e. there is only one h.w. vector in the Verma modules (plus one level-zero chiral singular vector), so that one cannot chose the helicity of the h.w. vector in order to build the singular vector of a given helicity. In addition in these Verma modules the number of singular vectors with helicity (+) is the same as the number of singular vectors with helicity (−).

As to the no-helicity h.w. vectors and Verma modules, they always have level-zero singular vectors. After dividing the Verma modules by these one is left with a chiral Verma module built on a Ramond ground state annihilated by both $G^+_0$ and $G^-_0$.

To finish, an interesting observation is that one gets exactly the same results by constructing the mapping using the even spectral flow $\mathcal{U}_\theta$ instead of the odd spectral flow $A_\theta$. In that case one finds that the only mapping that preserves the level and the relative charge of the states is $\mathcal{U}_{-1/2} (T^+_W)^{-1}$. But the transformation of the topological h.w. vectors is again $|\Delta, h\rangle^G \rightarrow |\Delta + \frac{c}{24}, h + \frac{c}{6}\rangle^+_R$, $|\Delta, h\rangle^Q \rightarrow |\Delta + \frac{c}{24}, h + \frac{c}{6}\rangle^-_R$. Hence this mapping is exactly equivalent to the mapping produced by $A_{-1/2} (T^-_W)^{-1}$. Notice that the composition of the topological twists and the spectral flows does not give any transformation exchanging the labels $G \rightarrow -$, $Q \rightarrow +$ while preserving the level and the relative charge of the states.

4 Conclusions and Final Remarks

In this paper we have presented the determinant formulae for the Verma modules of the Topological N=2 superconformal algebra: the generic (standard) Verma modules, the chiral Verma modules and the no-label Verma modules. These were the last N=2 determinant formulae which remained unpublished. (The absence of these formulae has created some confusion in the recent literature, in fact). Then we have analysed in very much detail the interpretation of the roots of the
determinants in terms of singular vectors. As a bonus we have obtained a first classification of the different types of submodules, regarding size and shape, finding four different types (at least), in contradiction with some claims in the literature that there are only two. A complete classification of the different types of submodules requires a deeper understanding of the shape of these, which in turn requires a classification of the subsingular vectors that is still lacking [28].

We have also identified the Verma modules which contain chiral and no-label singular vectors. The later are indecomposable primitive singular vectors which are not directly detected by the determinant formulae (but their level-zero secondaries are) and only exist in generic Verma modules. As to the no-label Verma modules, built on no-label h.w. vectors, they have two-dimensional singular spaces already at level 1, and they reduce to chiral Verma modules once the quotient by the level-zero singular vectors is performed. We have found the interesting result that the singular vectors of these chiral Verma modules are mostly subsingular vectors in the original no-label Verma module.

Finally we have transferred our analysis and results to the Verma modules and singular vectors of the Ramond N=2 algebra, which have been very insufficiently studied so far. In order to do this we have found a one-to-one mapping between the topological singular vectors and the R singular vectors which preserves the grading (level and relative charge). Under this mapping, which is a composition of the topological twists and the spectral flows, the topological singular vectors are transformed into R singular vectors in the following way: $|\chi\rangle^{(q)G}_l \rightarrow |\chi^{(q)+}_R \rangle_l$, $|\chi\rangle^{(q)Q}_l \rightarrow |\chi^{(q)-}_R \rangle_l$, $|\chi\rangle^{(q)G,Q}_l \rightarrow |\chi^{(q)+-}_R \rangle_l$, and $|\chi\rangle^{(q)}_l \rightarrow |\chi^{(q)}_R \rangle_l$. Our results imply that the standard classification of the Ramond singular vectors in two sectors, the $+$ sector and the $-$ sector, where the singular vectors and the h.w. vectors on which they are built have the same helicities, is not complete. The reason is that there also exist: singular vectors with both helicities (chiral), indecomposable singular vectors with no helicity, and singular vectors with the opposite helicity than the h.w. vector of the Verma module. In particular, we have identified the standard Verma modules with chiral and indecomposable no-helicity singular vectors, which have been overlooked until very recently in the literature. The no-helicity Verma modules and submodules, built on no-helicity h.w. vectors, have also been overlooked consequently.

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Appendix A

In this appendix we will identify the generic Verma modules $V(|\Delta, h)^G)$ which contain chiral singular vectors of types $|\chi\rangle^{(0)G,Q}_l$ or $|\chi\rangle^{(-1)G,Q}_l$ (or both), and those which contain no-label singular
vectors $|\chi\rangle^{(0)}_l$ or $|\chi\rangle^{(-1)}_l$. Zero conformal weight, $\Delta + l = 0$, is necessary for chiral and no-label singular vectors to exist. Therefore a first requirement for the generic Verma modules to contain chiral or no-label singular vectors at level $l$ is $\Delta = -l$.

**Chiral singular vectors**

As explained in section 3, the determinant formula (3.1) implies the existence of singular vectors of the types $|\chi\rangle^{(0)G}_{l,|\Delta, h\rangle}$ and $|\chi\rangle^{(-1)Q}_{l,|\Delta, h\rangle}$ at level $l = \frac{rs}{2}$ built on $G_0$-closed h.w. vectors $|\Delta, h\rangle^G$ with conformal weight $\Delta = \Delta_{r,s}$, eq. (3.4), for $c \neq 3$, or with any conformal weight $\Delta$ for $h_\pm = \pm \frac{s}{7}$, for $c = 3$. For simplicity in what follows we will denote these singular vectors as $|\chi\rangle^{(0)G}_{r,s}$ and $|\chi\rangle^{(-1)Q}_{r,s}$. Our analysis showed that $G_0$ and $Q_0$ interpolate between $|\chi\rangle^{(0)G}_{r,s}$ and $|\chi\rangle^{(-1)Q}_{r,s}$, except in the cases when these vectors have zero conformal weight, that is $\Delta_{r,s} + \frac{rs}{2} t = 0$, for which this interpolation breaks down. Namely, at least one of the singular vectors $|\chi\rangle^{(0)G}_{r,s}$ or $|\chi\rangle^{(-1)Q}_{r,s}$ necessarily becomes chiral and therefore is annihilated by both $G_0$ and $Q_0$. A priori it is not possible to tell which one of the two singular vectors becomes chiral, or if both become chiral, but by analysing some of their coefficients we will in the following give an answer to this question.

The condition of zero conformal weight $\Delta + \frac{rs}{2} t = 0$ yields two curves of solutions for the charge $h$ as functions of $r$, $s$ and $t = \frac{3-s}{s}$:

$$C^+ : \quad h_{r,s}^{(+)}(t) = \frac{s(t+1)}{2} t, \quad \text{(A.1)}$$

$$C^- : \quad h_{r,s}^{(-)}(t) = -\frac{s(t-1)}{2} t. \quad \text{(A.2)}$$

In ref. [18] we showed that singular vectors can be identified by their coefficients with respect to the ordering kernel. If two singular vectors agree in the coefficients of the ordering kernel then they are identical. In particular, if the coefficients with respect to the ordering kernel all vanish then the vector has to be trivial. The advantage of this procedure is that the ordering kernel has at most two elements in our case, which means that we can identify singular vectors and decide if they are trivial just by comparing only two coefficients. In addition we can also deduce if the vectors are $G_0$-closed, $Q_0$-closed or chiral, as we will see. The ordering kernel for the uncharged $G_0$-closed singular vectors $|\chi\rangle^{(0)G}_{r,s}$ at level $l$ is given, for the general case $c \neq 3$ ($t \neq 0$), by [18] $\{\mathcal{L}_{-1}^l, \mathcal{L}_{-1}^l G_{-1} Q_0\}$. The coefficients of these vectors can easily be obtained from the coefficients of the uncharged singular vectors of the Neveu-Schwarz N=2 algebra[12] given in eq. (3.11) of ref. [18]. For

$$|\chi\rangle^{(0)G}_{r,s} = \left\{ \alpha \mathcal{L}_{-1}^{\frac{rs}{2}} + \beta \mathcal{L}_{-1}^{\frac{rs}{2}-1} G_{-1} Q_0 + \gamma \mathcal{H}_{-1} \mathcal{L}_{-1}^{\frac{rs}{2}-1} + \delta \mathcal{L}_{-1}^{\frac{rs}{2}-2} G_{-1} Q_{-1} + \ldots \right\} |\Delta_{r,s}, h\rangle^G, \quad \text{(A.3)}$$

we use the notation $|\chi\rangle^{(0)G}_{r,s} = (\alpha, \beta, \gamma, \delta)^{0,G}$ (for convenience we show two other coefficients besides the coefficients with respect to the ordering kernel) finding

$$|\chi\rangle^{(0)G}_{r,s} = (1, \frac{1}{2}(P_{r,s} - 1), \frac{rs}{2} h + \frac{1}{l}, \frac{rs}{2} h + \frac{1}{l} - t - \frac{1}{2}(P_{r,s} - 1)|\Delta_{r,s}^{0,G}, \quad \text{(A.4)}$$

[12]The uncharged $G_0$-closed singular vectors built on $G_0$-closed h.w. vectors correspond to the uncharged singular vectors of the Neveu-Schwarz N=2 algebra by performing the topological twists.
As before we analyse whether these conditions are satisfied for the curves that both conditions hold for the curve \( C \). Again, there are two conditions which have to be satisfied, for the general case points instead, on the curve \( C \). One finds that these vectors to become chiral: the singular vectors \( Q_1 \) also becomes chiral. Finally, in the limit \( t \to 0 \) along curve \( C^- \) the singular vectors \( |\chi\rangle_{r,s}^{(0)G} \) are well defined, and chiral also as a consequence.

In the same way we analyse the singular vectors \( |\chi\rangle^{(-1)Q}_{r,s} \), which are obtained generically from the singular vectors \( |\chi\rangle^{(0)G}_{r,s} \) under the action of \( Q_0 \). Dividing out overall factors one then obtains the leading coefficients for \( |\chi\rangle^{(-1)Q}_{r,s} \). Using the notation \( |\chi\rangle^{(0)G}_{r,s} = (\alpha, \beta, \gamma, \delta)^{\pm 1}G \) for

\[
|\chi\rangle^{(-1)Q}_{r,s} = \left\{ \alpha \mathcal{L}^{r/2}_1 Q_0 + \beta \mathcal{L}^{-r/2}_1 Q_1 + \gamma \mathcal{H}_1 \mathcal{L}^{-r/2}_2 Q_1 + \gamma \mathcal{H}_1 \mathcal{L}^{-r/2}_2 Q_1 + \delta \mathcal{L}^{-r/2}_1 \mathcal{G}_1 Q_1 + \ldots \right\}|\Delta_{r,s}, h\rangle^G,
\]

one finds

\[
|\chi\rangle^{(-1)Q}_{r,s} = \left( \Pi_{r-1,s}, \frac{1}{2t} \left( (h + H(0))(h - H(1))\Pi_{r-1,s} - (h - H(r + 1))(h - H(r)) \right) \right) |\Delta_{r,s}, h\rangle^G,
\]

\[
= \frac{r\alpha(h-1)}{2t} \Pi_{r-1,s}, -\frac{1}{4t}(h - H(r + 1))(h - H(r)) - \frac{1}{2t} \Pi_{r-1,s} \{ \Delta_{r,s} + \frac{r\alpha(h-1)}{2t} \}^{-1} Q_0.
\]

Again, there are two conditions which have to be satisfied, for the general case \( c \neq 3 \) (\( t \neq 0 \), for these vectors to become chiral:

\[
- \alpha \frac{r}{2t} + \beta = 0, \quad \gamma + 2\delta - \alpha \frac{r}{2t} = 0.
\]

As before we analyse whether these conditions are satisfied for the curves \( C^+ \) or \( C^- \). One finds that both conditions hold for the curve \( C^+ \). Hence, \( |\chi\rangle^{(-1)Q}_{r,s} \) becomes chiral, i.e. of type \( |\chi\rangle^{(-1)G,Q}_{r,s} \) instead, on the curve \( C^+ \). However, for \( C^- \) these conditions are generically not satisfied and therefore \( |\chi\rangle^{(-1)Q}_{r,s} \) does not become chiral. The only exceptions to the generic case are again the discrete points \( t_{n,s} = -\frac{s}{n} \) for \( n = 1, \ldots, r \) on \( C^- \) where \( |\chi\rangle^{(-1)Q}_{r,s} \) also becomes chiral. Finally, in the limit \( t \to 0 \) along curve \( C^+ \) the singular vectors \( |\chi\rangle^{(-1)Q}_{r,s} \) are well defined, and chiral also consequently.

We can summarise these results as follows: for the values \( h = h^{(1)}_{r,s} \), given by curve \( C^+ \), the singular vectors \( |\chi\rangle^{(-1)Q}_{r,s} \) become chiral for all values of \( t \) whilst \( |\chi\rangle^{(0)G}_{r,s} \) stay generically \( G_0 \)-closed. Only for the values \( t = -\frac{s}{n}, n = 1, \ldots, r \), on \( C^+ \) (corresponding to \( c = \frac{3(n+s)}{n} \)) both types of
vectors become chiral. Similarly, for the values \( \mathbf{h} = \mathbf{h}_{r,s}^{(-)} \), given by curve \( C^- \), the singular vectors \( |\chi\rangle^{(0)G}_{r,s} \) become chiral for all values of \( t \) whilst \( |\chi\rangle^{(-1)Q}_{r,s} \) only become chiral for the values \( t = -\frac{s}{n}, n = 1, \ldots, r. \)

As an example, the chiral singular vectors \( |\chi\rangle^{(0)G,Q}_{r,s} \) and \( |\chi\rangle^{(-1)G,Q}_{r,s} \) at level 1 are given by:

\[
|\chi\rangle^{(0)G,Q}_{1,|−1,−1}G} = (−2\mathcal{L}_{−1} + \mathcal{G}_{−1}Q_{0})|−1,−1\rangle^G,
\]
\[
|\chi\rangle^{(−1)G,Q}_{1,|−1,−1}G} = (\mathcal{L}_{−1}Q_{0} + \mathcal{H}_{−1}Q_{0} + \mathcal{Q}_{−1})|−1,−\frac{6−c}{3}\rangle^G,
\]

Observe that for \( c = 9 (t = −2) \) both chiral singular vectors are together in the same Verma module \( V(|−1,−1\rangle^G) \), in agreement with our analysis.

**No-label singular vectors**

Now we will investigate the appearance of no-label singular vectors at level \( l \) in generic Verma modules \( V(|−l,\mathbf{h}\rangle^G) \). Let us start with the uncharged no-label singular vectors \( |\chi\rangle^{(0)}_{l,|−l,\mathbf{h}}\rangle^G \) which will be denoted simply as \( |\chi\rangle^{(0)}_l \). An uncharged no-label singular vector \( |\chi\rangle^{(0)}_l \) is necessarily accompanied at the same level \( l \) in the same Verma module by three secondary singular vectors which cannot ‘come back’ to this one by acting with the algebra: one charged \( \mathcal{G}_{0} \)-closed singular vector \( |\chi\rangle^{(1)G}_{l} = \mathcal{G}_{0}|\chi\rangle^{(0)}_l \), one charged \( \mathcal{Q}_{0} \)-closed singular vector \( |\chi\rangle^{(−1)Q}_{l} = \mathcal{Q}_{0}|\chi\rangle^{(0)}_l \), and one uncharged chiral singular vector \( |\chi\rangle^{(0)G,Q}_{l} = \mathcal{G}_{0}\mathcal{Q}_{0}|\chi\rangle^{(0)}_l = −\mathcal{Q}_{0}\mathcal{G}_{0}|\chi\rangle^{(0)}_l \). Observe that the singular vector \( |\chi\rangle^{(−1)Q}_l \) must be non-chiral necessarily since otherwise \( \mathcal{G}_{0}\mathcal{Q}_{0}|\chi\rangle^{(0)}_l = 0 \) while \( \mathcal{Q}_{0}\mathcal{G}_{0}|\chi\rangle^{(0)}_l \neq 0 \), as the singular vectors \( |\chi\rangle^{(1)G}_l \) (built on \( |\Delta,\mathbf{h}\rangle^G \)) never become chiral. The fact that two charged singular vectors of types \( |\chi\rangle^{(−1)Q}_{r,s} \) and \( |\chi\rangle^{(1)G}_{r,s} \) exist at the same level in the same generic Verma module \( V(|\Delta,\mathbf{h}\rangle^G) \) requires \( f_{r,s}(\Delta,\mathbf{h},t) = q^+_t(\Delta,\mathbf{h},t) = 0 \), given by eqns. (3.2) and (3.3), where \( l = \frac{ra}{2} \). For the general case \( c \neq 3 (t \neq 0) \) the corresponding equation \( \Delta_{r,s} = \Delta^+_l \) is satisfied by two curves of solutions for \( \mathbf{h}_{r,s}(t) \):

\[
C^1: \quad \mathbf{h}_{r,s}(t) = \frac{1}{2}(s−t(r+rs−1)),
\]
\[
C^2: \quad \mathbf{h}_{r,s}(t) = −\frac{1}{2}(s−t(r−rs+1)),
\]

whereas for \( c = 3 (t = 0) \) one simply has \( \Delta^+_l = \pm \frac{ra^2}{4} \). The charged singular vector \( |\chi\rangle^{(1)G}_l \) is accompanied by the uncharged singular vector \( |\chi\rangle^{(0)Q}_l = \mathcal{Q}_{0}|\chi\rangle^{(1)G}_l \) with the following leading coefficients for \( c \neq 3 (t \neq 0) \):

\[
|\chi\rangle^{(0)Q}_l = (2,−1,−2\frac{(t+1)(h+1)}{1−t},\frac{(t+1)(h+1)}{1−t})^0Q,\]

where we choose the same notation as for \( |\chi\rangle^{(0)G}_{r,s} \), eqn. (A.3). (Observe that \( |\chi\rangle^{(0)Q}_l \) is \( \mathcal{Q}_{0} \)-closed while the h.w. vector \( |−l,\mathbf{h}\rangle^G \) is \( \mathcal{G}_{0} \)-closed). The charged singular vector \( |\chi\rangle^{(−1)Q}_l \), in turn, is accompanied by the uncharged singular vector \( |\chi\rangle^{(−1)G}_{r,s} = \mathcal{G}_{0}|\chi\rangle^{(−1)Q}_{r,s} \), the latter one with coefficients given in eqn. (A.4).
Thus, following curve $C^1$ or curve $C^2$ by varying the parameter $t$ we can assume that we have at level $\frac{r s}{T}$ the two uncharged singular vectors $|\chi\rangle_{r,s}^{(0)Q}$ and $|\chi\rangle_{r,s}^{(0)G}$, which are generically different (i.e. not proportional) except in the case when both of them become chiral: $|\chi\rangle_{r,s}^{(0)G,Q} = |\chi\rangle_{r,s}^{(0)G,Q}$.

The reason is that two chiral singular vectors at the same level with the same charge never span a two dimensional singular vector space, as we proved in ref. [18]. When the singular vectors have zero conformal weight, i.e. in the case $\Delta_{r,s} = \Delta_{r,s}^+ = -\frac{r s}{T}$ the singular vectors $|\chi\rangle_{r,s}^{(0)Q}$ always become chiral, as explained in section 3, whereas the singular vectors $|\chi\rangle_{r,s}^{(0)G}$ become chiral only under certain conditions that we have deduced a few paragraphs above. Observe that the existence of no-label singular vectors requires that these two uncharged singular vectors become chiral, and therefore proportional, whereas the charged singular vector of type $|\chi\rangle_{r,s}^{(-1)Q}$ should remain non-chiral. These are necessary, although not sufficient, conditions to guarantee the existence of the no-label singular vectors.

Let us therefore analyse the different possibilities when $|\chi\rangle_{r,s}^{(0)G}$ becomes chiral and $|\chi\rangle_{r,s}^{(-1)Q}$ stays non-chiral. Let us start with the general case $c \neq 3$ ($t \neq 0$). First of all, the condition $\Delta_{r,s} = -\frac{r s}{T}$, necessary for chiral singular vectors $|\chi\rangle_{r,s}^{(0)G,Q}$ to exist, leads to the solutions given by curves $C^+$ and $C^-$, eqns. (A.1) and (A.2). The solutions along curve $C^+$, however, involve chiral singular vectors of type $|\chi\rangle_{r,s}^{(1)G,Q}$. As a consequence we only need to investigate the solutions to the conditions $\Delta_{r,s} = \Delta_{r,s}^+ = -\frac{r s}{T}$ given by the intersections of curve $C^-$ (for which $|\chi\rangle_{r,s}^{(0)G}$ always becomes chiral) with curves $C^1$ and $C^2$. In the first case the only intersection points correspond to $t = \frac{2}{T}$ whereas in the second case there are solutions for all $t$ provided $s = 2$. A closer look at these solutions reveals that no-label singular vectors only exist for $t = \frac{2}{T}$, in agreement with the low level computations of these vectors given in refs. [12] and [7]. The argument goes as follows.

Along curve $C^1$, varying $t$, there are two uncharged singular vectors at level $l = \frac{r s}{2T}$: $|\chi\rangle_{r,s}^{(0)Q}$ and $|\chi\rangle_{r,s}^{(0)G}$. Since these correspond to vectors in the kernel of the inner product matrix for the Verma module, one finds that for the case of both becoming chiral, and thus proportional, the rank of the inner product matrix would rise for these particular points on the curve $C^1$. But the inner product matrix of the Verma module contains only rational functions of $t$, $\Delta$, and $\mathbf{h}$ as entries and therefore its rank is upper semi-continuous. As a consequence the rank cannot rise for particular values of $t$ and therefore for $t = \frac{2}{r}$ at least one additional uncharged null vector $|\chi\rangle_{rs}^{(0)Q}$ needs to exist at level $\frac{r s}{T}$. For the solution $s = 2$ for all $t$, however, one finds that curves $C^2$ and $C^-$ are identical. Therefore, the vectors $|\chi\rangle_{r,s}^{(0)Q}$ and $|\chi\rangle_{r,s}^{(0)G}$ are both chiral and proportional all along the curve $C^2$. As a result the corresponding space of uncharged singular vectors is just one-dimensional and consequently an additional null vector is not required.

The additional null vectors found for $t = \frac{2}{r}$ cannot be subsingular vectors as there are no singular vectors at lower levels than $|\chi\rangle_{rs}^{(0)Q}$ themselves. Neither they can be $Q_0$-closed as this would result in two-dimensional spaces of uncharged singular vectors annihilated by $Q_0$, which do not exist, as we proved in ref. [18]. Finally, in ref. [10] conditions were given for Verma modules containing two-dimensional spaces of uncharged singular vectors annihilated by $G_0$. Comparing these conditions

\footnote{See ref. [28] where this mechanism is explained in more detail.}
with the solutions $t = \frac{2}{r}$ on $\mathcal{C}^1$ shows that $|\chi\rangle_{\mathcal{C}|2}$ are neither $\mathcal{G}_0$-closed. The only possibility left is therefore that $|\chi\rangle_{\mathcal{C}|2}$ are uncharged no-label singular vectors.

Finally, for the case $c = 3$ ($t = 0$) the two conditions $\Delta_{r,s}^+ = -\frac{rs}{2}$ and $\Delta_{r,s}^- = \pm\frac{rs^2}{4}$ lead to the unique solution $s = 2$, $\Delta_{r,s}^+ = -r$. For this solution, however, the corresponding space of uncharged singular vectors is just one-dimensional so that no-label singular vectors do not exist. (This solution can be viewed in fact as the case described above where $\mathcal{C}^-$ and $\mathcal{C}^2$ are identical, in the limit $t \to 0$).

These results can easily be transferred to the other types of no-label singular vectors using the mappings analysed in ref. [12]. Since these mappings do not modify the value of $t$, one deduces that no-label singular vectors only exist for generic Verma modules with $t = \frac{2}{r}$. In particular, no-label singular vectors of types $|\chi\rangle_l^{(0)}$ and $|\chi\rangle_l^{(-1)}$, built on $\mathcal{G}_0$-closed h.w. vectors $|\Delta_{r,s},h\rangle^G$, appear at level $l = \frac{rs}{2}$ for $t = \frac{2}{r}$ (corresponding to $c = \frac{3r-6}{r}$), $\Delta_{r,s} = -\frac{rs}{2}$, and $h = 1 - \frac{s}{2} + \frac{1}{r}$ and $h = 1 + \frac{s}{2} + \frac{1}{r}$, respectively.

As an example, let us write the no-label singular vector $|\chi\rangle_l^{(0)}$ built on a $\mathcal{G}_0$-closed h.w. vector $|\Delta_{r,s},h\rangle^G$, at level 1, together with the three secondary singular vectors that it generates at level 1 by the action of $\mathcal{G}_0$ and $\mathcal{Q}_0$:

$$|\chi\rangle_{1,\mathcal{C}|2}^{(0)} = (\mathcal{L}-1 - \mathcal{H}-1)|-1, -1, t = 2\rangle^G,$$  \hspace{1cm} (A.13)

$$|\chi\rangle_{1,\mathcal{C}|2}^{(1)} = \mathcal{G}_0|\chi\rangle_{1,\mathcal{C}|2}^{(0)} = 2\mathcal{G}_1|1, -1, t = 2\rangle^G,$$ \hspace{1cm} (A.14)

$$|\chi\rangle_{1,\mathcal{C}|2}^{(-1)} = \mathcal{Q}_0|\chi\rangle_{1,\mathcal{C}|2}^{(0)} = (\mathcal{L}_1 \mathcal{Q}_0 - \mathcal{H}_1 \mathcal{Q}_0 - \mathcal{Q}_1)|-1, -1, t = 2\rangle^G,$$ \hspace{1cm} (A.15)

$$|\chi\rangle_{1,\mathcal{C}|2}^{(0,G)} = \mathcal{G}_0 \mathcal{Q}_0|\chi\rangle_{1,\mathcal{C}|2}^{(0)} = 2(-2\mathcal{L}_1 + \mathcal{G}_1 \mathcal{Q}_0)|1, -1, t = 2\rangle^G.$$ \hspace{1cm} (A.16)

The no-label singular vector only exists for $t = 2$ ($c = -3$) whereas the three secondary singular vectors are just the particular cases, for $t = 2$, of the one-parameter families of singular vectors of the corresponding types, which exist for all values of $t$. (The singular vectors of the Topological $\mathcal{N}=2$ algebra at level 1 were given in ref. [12]). Moreover, the singular vector $|\chi\rangle_{1,\mathcal{C}|2}^{(1)}$ is always primitive for any value of $t$, except for $t = 2$, the singular vector $|\chi\rangle_{1,\mathcal{C}|2}^{(-1)}$ is also primitive, except for $t = 2$, even when it becomes chiral (for $t = -2$), and the singular vector $|\chi\rangle_{1,\mathcal{C}|2}^{(0,G)}$ is always secondary, except for the value $t = -2$ ($c = 9$), for which it becomes primitive together with the other chiral singular vector $|\chi\rangle_{1,\mathcal{C}|2}^{(-1,G)}$. 

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Appendix B

In what follows we present the complete set of singular vectors at level 1 in Verma modules of the Topological N=2 algebra with zero conformal weight $\Delta = 0$. That is, the singular vectors at level 1 in: generic Verma modules $V([0, h]^G)$ and $V([0, h]^Q)$, chiral Verma modules $V([0, h]^G_Q)$, and no-label Verma modules $V([0, h])$. Several of these singular vectors were written down in ref. \[12\] (for any value of $\Delta$ in the case of generic Verma modules).

**Generic Verma modules**

In generic Verma modules $V([0, h]^G)$ and $V([0, h - 1]^Q)$ one finds singular vectors at level 1 for the following values of $h$, in agreement with eqns. \[3.5\], \[3.4\] and \[3.3\]: $h_{1,2}^{(0)} = t - 1 = -\frac{c}{3}$, $h_{1,2}^+ = 1$, $h_{1,2}^- = t = \frac{2c}{3}$. The corresponding singular vectors in $V([0, h]^G)$ are

\[
|\chi\rangle^{(0)G}_{1,0,-\frac{c}{3}} = (\frac{c + 3}{3} L_{-1} + \frac{c + 3}{3} H_{-1} - \mathcal{G}_{-1} Q_0) |0, -\frac{c}{3}\rangle^G,
\]

\[
|\chi\rangle^{(-1)Q}_{1,0,-\frac{c}{3}} = ((c - 3) L_{-1} Q_0 + (c + 3) H_{-1} Q_0 + (c - 3) Q_{-1}) |0, -\frac{c}{3}\rangle^G,
\]

\[
|\chi\rangle^{(0)G}_{1,0,0} = \mathcal{G}_{-1} Q_0 |0, 1\rangle^G, \quad |\chi\rangle^{(-1)Q}_{1,0,0} = L_{-1} Q_0 |0, 1\rangle^G,
\]

\[
|\chi\rangle^{(1)G}_{1,0,0} = \mathcal{G}_{-1} |0, 0\rangle^G, \quad |\chi\rangle^{(0)Q}_{1,0,0} = Q_0 \mathcal{G}_{-1} |0, 0\rangle^G,
\]

\[
|\chi\rangle^{(-1)G}_{1,0,\frac{3-c}{3}} = (L_{-1} Q_0 + H_{-1} Q_0) |0, \frac{3-c}{3}\rangle^G, \quad |\chi\rangle^{(-2)Q}_{1,0,\frac{3-c}{3}} = Q_{-1} Q_0 |0, \frac{3-c}{3}\rangle^G.
\]

The singular vectors in $V([0, h - 1]^Q)$ are

\[
|\chi\rangle^{(0)Q}_{1,0,-\frac{c+3}{3}} = \mathcal{Q}_{-1} G_0 |0, -\frac{c + 3}{3}\rangle^Q, \quad |\chi\rangle^{(1)G}_{1,0,-\frac{c+3}{3}} = (L_{-1} G_0 + H_{-1} G_0) |0, -\frac{c + 3}{3}\rangle^Q,
\]

\[
|\chi\rangle^{(0)Q}_{1,0,0} = (\frac{c + 3}{3} L_{-1} - Q_{-1} G_0) |0, 0\rangle^Q,
\]

\[
|\chi\rangle^{(1)G}_{1,0,0} = (\frac{c + 3}{6} L_{-1} G_0 - H_{-1} G_0 + \frac{c + 3}{6} H_{-1} G_0) |0, 0\rangle^Q,
\]

\[
|\chi\rangle^{(1)Q}_{1,0,-1} = L_{-1} G_0 |0, -1\rangle^Q, \quad |\chi\rangle^{(2)Q}_{1,0,-1} = G_{-1} G_0 |0, -1\rangle^Q.
\]
\[ |\chi\rangle^{(-1)Q}_{1,\{0, -\frac{c}{3}\}} = \mathcal{Q}_{-1} |0, -\frac{c}{3}\rangle^Q, \quad |\chi\rangle^{(0)G}_{1,\{0, -\frac{c}{3}\}} = \mathcal{G}_0 \mathcal{Q}_{-1} |0, -\frac{c}{3}\rangle^Q. \] (B.10)

All these singular vectors also apply for \( c = 3 \) (\( t = 0 \)) and no additional singular vectors appear for this value. Chiral singular vectors and no-label singular vectors at level 1 require \( \Delta = -1 \), therefore they are absent for \( \Delta = 0 \).

**Chiral Verma modules**

In chiral Verma modules \( V(\{0, h\}^{G,Q}) \) one finds singular vectors at level 1 for the following values of \( h \), in agreement with eqns. (3.16) and (3.17): \( h_{1,2}^{(0)} = t - 1 = -\frac{c}{3} \) and \( h_{1,2}^{(1)} = 0 \). The corresponding singular vectors, for all values of \( c \), are

\[ |\chi\rangle^{(0)G}_{1} = (\mathcal{L}_{-1} + \mathcal{H}_{-1}) |0, -\frac{c}{3}\rangle^G, \quad |\chi\rangle^{(-1)Q}_{1} = \mathcal{Q}_{-1} |0, -\frac{c}{3}\rangle^Q, \] (B.11)

\[ |\chi\rangle^{(1)G}_{1} = \mathcal{G}_{-1} |0, 0\rangle^G, \quad |\chi\rangle^{(0)Q}_{1} = \mathcal{L}_{-1} |0, 0\rangle^Q. \] (B.12)

**No-label Verma modules**

In no-label Verma modules \( V(\{0, h\}) \) one finds singular vectors at level 1 for the following values of \( h \), in agreement with our analysis in subsection 3.3: \( h_{1,2}^{(0)} = h_1^1 - 1 = t - 1 = -\frac{c}{3} \), \( h_{1,2} = 1 \), \( h_{1,2}^{(0)} - 1 = t - 2 = -\frac{c+3}{3} \), \( h_{1,2}^{(1)} = h_1^+ = 0 \), \( h_1^+ - 1 = -1 \) and \( h_1^- = t = \frac{3-c}{3} \). For \( c = 9 \) (\( t = -2 \)) there are even two-dimensional singular spaces. The corresponding singular vectors, for all values of \( c \), are

\[ |\chi\rangle^{(0)G}_{1,\{0, -\frac{c}{3}\}} = (\mathcal{L}_{-1} \mathcal{G}_0 \mathcal{Q}_0 + \mathcal{H}_{-1} \mathcal{G}_0 \mathcal{Q}_0) |0, -\frac{c}{3}\rangle^G, \quad |\chi\rangle^{(-1)Q}_{1,\{0, -\frac{c}{3}\}} = \mathcal{Q}_{-1} \mathcal{G}_0 \mathcal{Q}_0 |0, -\frac{c}{3}\rangle^Q, \] (B.13)

\[ |\chi\rangle^{(0)G}_{1,\{0, 0\}} = \left(\frac{3-c}{6} \mathcal{L}_{-1} \mathcal{G}_0 \mathcal{Q}_0 + \mathcal{H}_{-1} \mathcal{G}_0 \mathcal{Q}_0 - \frac{c+3}{6} \mathcal{G}_{-1} \mathcal{Q}_0 \right) |0, 1\rangle, \] (B.14)

\[ |\chi\rangle^{(-1)Q}_{1,\{0, 0\}} = \left(\frac{c+3}{3} \mathcal{L}_{-1} \mathcal{Q}_0 + \mathcal{Q}_{-1} \mathcal{G}_0 \right) |0, 1\rangle, \] (B.15)

\[ |\chi\rangle^{(0)Q}_{1,\{0, -\frac{c+3}{3}\}} = ((c-3) \mathcal{L}_{-1} \mathcal{Q}_0 + (c+3) \mathcal{H}_{-1} \mathcal{Q}_0 + (c+3) \mathcal{Q}_{-1} \mathcal{Q}_0) |0, -\frac{c+3}{3}\rangle, \] (B.16)

\[ |\chi\rangle^{(1)G}_{1,\{0, -\frac{c+3}{3}\}} = \left(\frac{c+3}{3} \mathcal{L}_{-1} \mathcal{G}_0 + \frac{c+3}{3} \mathcal{H}_{-1} \mathcal{G}_0 + \mathcal{G}_{-1} \mathcal{Q}_0 \mathcal{Q}_0 \right) |0, -\frac{c+3}{3}\rangle^Q, \] (B.17)
In addition, for $c = 9$ ($t = -2$) one finds the two-dimensional singular spaces generated by the singular vectors

$$|\chi\rangle_{1,0}^{(0)G} = (\mathcal{L}_1 + \mathcal{H}_1)G0Q0|0,0\rangle, \quad |\chi\rangle_{1,0}^{(0)Q} = (\mathcal{L}_1 - \mathcal{H}_1)G0Q0|0,0\rangle,$$

(B.18)

$$|\chi\rangle_{1,0}^{(1)Q} = Q0G1Q0|0,-1\rangle, \quad |\chi\rangle_{1,0}^{(2)G} = G1Q0|0,-1\rangle,$$

(B.19)

$$|\chi\rangle_{1,0}^{(-1)G} = G0Q1Q0|0,\frac{3-c}{3}\rangle, \quad |\chi\rangle_{1,0}^{(-2)Q} = Q1Q0|0,\frac{3-c}{3}\rangle.$$

(B.20)

In addition, for $c = 9 (t = -2)$ one finds the two-dimensional singular spaces generated by the singular vectors

$$|\chi\rangle_{1,0}^{(0)G} = (\mathcal{L}_1 + \mathcal{H}_1)G0Q0|0,0\rangle, \quad |\chi\rangle_{1,0}^{(0)Q} = (\mathcal{L}_1 - \mathcal{H}_1)G0Q0|0,0\rangle,$$

(B.21)

$$|\chi\rangle_{1,0}^{(-1)Q} = Q1Q0|0,-1\rangle, \quad |\chi\rangle_{1,0}^{(-1)Q} = (\mathcal{L}_1 - \mathcal{H}_1)Q0\mathcal{Q}0\mathcal{Q}0|0,-1\rangle,$$

(B.22)

$$|\chi\rangle_{1,0}^{(1)G} = G1Q0|0,0\rangle, \quad |\chi\rangle_{1,0}^{(1)G} = (\mathcal{L}_1 - \mathcal{H}_1)G0Q0|0,0\rangle,$$

(B.23)

$$|\chi\rangle_{1,0}^{(0)Q} = L1G0Q0|0,0\rangle, \quad |\chi\rangle_{1,0}^{(0)Q} = (\mathcal{L}_1 - \mathcal{H}_1)G0Q0|0,0\rangle.$$

(B.24)

For each of these pairs the singular vector on the left is the particular case, for $c = 9$, of the one-parameter family of singular vectors that exists for all values of $c$, given in eqns. (B.13) or (B.18). The singular vectors on the right, however, are the particular cases of a one-parameter family of subsingular vectors that turn out to be singular just for $c = 9$.

References


[3] E. Martinec, M-theory and N=2 Strings, [hep-th/9710123] (1997), and references there


