Construction Formulae for Singular Vectors of the Topological
N=2 Superconformal Algebra

Beatriz Gato-Rivera

Instituto de Matemáticas y Física Fundamental, CSIC,
Serrano 123, Madrid 28006, Spain

NIKHEF-H, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands

ABSTRACT

The Topological N=2 Superconformal algebra has 29 different types of
singular vectors (in complete Verma modules) distinguished by the relative
U(1) charge and the BRST-invariance properties of the vector and of the
primary on which it is built. Whereas one of these types only exists at level
zero, the remaining 28 types exist for general levels and can be constructed
already at level 1. In this paper we write down one-to-one mappings between
16 of these types of topological singular vectors and the singular vectors of the
Antiperiodic NS algebra. As a result one obtains construction formulae for
these 16 types of topological singular vectors using the construction formulae
for the NS singular vectors due to Dörrzapf.

February 1998

*e-mail address: bgato@pinar1.csic.es
1 Introduction and Notation

The N=2 Superconformal algebras provide the symmetries underlying the N=2 strings \[1\][2]. These seem to be related to M-theory since many of the basic objects of M-theory are realized in the heterotic (2,1) N=2 strings [3]. In addition, the topological version of the algebra is realized in the world-sheet of the bosonic string [4], as well as in the world-sheet of the superstrings [4].

Recently it has been shown [5] that the singular vectors of the Topological N=2 algebra can be classified in 29 types, in complete Verma modules, taking into account the relative U(1) charge and the BRST-invariance properties of the vector itself and of the primary on which it is built. In ref. [5] the whole set of singular vectors was explicitly constructed at level 1 (28 types since one type exists only at level 0), whereas the rigorous proofs that these types are the only possible ones will be given in [7].

In this paper we intend to bring to the reader’s attention the fact that one can write down one-to-one mappings between 16 of these types of topological singular vectors and the singular vectors of the NS algebra. As a bonus one obtains construction formulae for the 16 types of topological singular vectors using the construction formulae for the NS singular vectors due to Dörzzapf [8][9]. In section 2 we discuss the basic ingredients to derive the mappings between the topological singular vectors and the NS singular vectors. In section 3 we write down these mappings, which turn into construction formulae for the topological singular vectors once the NS singular vectors are expressed in terms of their construction formulae themselves. Some final remarks are made in section 4.

Notation

*Highest weight (h.w.) states* denote states annihilated by all the positive modes of the generators of the algebra, i.e. \( \mathcal{L}_{n \geq 1} |\chi\rangle = \mathcal{H}_{n \geq 1} |\chi\rangle = \mathcal{G}_{n \geq 1} |\chi\rangle = \mathcal{Q}_{n \geq 1} |\chi\rangle = 0 \).

*Primary states* denote non-singular h.w. states.

*Secondary or descendant states* denote states obtained by acting on the h.w. states with the negative modes of the generators of the algebra and with the fermionic zero modes \( \mathcal{Q}_0 \) and \( \mathcal{G}_0 \). The fermionic zero modes can also interpolate between two h.w. states at the same footing (two primary states or two singular vectors).

*Chiral topological states* \( |\chi\rangle^{G,Q} \) are states annihilated by both \( \mathcal{G}_0 \) and \( \mathcal{Q}_0 \).

*\( \mathcal{G}_0 \)-closed topological states* \( |\chi\rangle^G \) denote non-chiral states annihilated by \( \mathcal{G}_0 \).

*\( \mathcal{Q}_0 \)-closed topological states* \( |\chi\rangle^Q \) denote non-chiral states annihilated by \( \mathcal{Q}_0 \) (they are BRST-invariant since \( \mathcal{Q}_0 \) is the BRST charge).
No-label topological states $|\chi\rangle$ denote states that cannot be expressed as linear combinations of $G_0$-closed and $Q_0$-closed states.

The Verma module associated to a h.w. state consists of the h.w. state plus the set of secondary states built on it. For some Verma modules the h.w. state is degenerate, the fermionic zero modes interpolating between the two h.w. states.

Singular vectors are h.w. zero-norm states.

Secondary singular vectors are singular vectors built on singular vectors. The level-zero secondary singular vectors cannot “come back” to the singular vectors on which they are built by acting with $G_0$ or $Q_0$.

The Topological N=2 superconformal algebra will be denoted as the Topological algebra. The Antiperiodic N=2 superconformal algebra will be denoted as the NS algebra.

2 Basic Concepts

The Topological algebra and the topological twists

The algebra obtained by applying the topological twists on the NS algebra reads

\[
\begin{align*}
[L_m, L_n] & = (m - n)L_{m+n}, & [H_m, H_n] & = \frac{c}{3}m\delta_{m+n,0}, \\
[L_m, G_n] & = (m - n)G_{m+n}, & [H_n, G_n] & = G_{m+n}, \\
[L_m, Q_n] & = -nQ_{m+n}, & [H_m, Q_n] & = -Q_{m+n}, & m, n \in \mathbb{Z}. \quad (2.1)
\end{align*}
\]

where $L_m$ and $H_m$ are the bosonic generators corresponding to the energy momentum tensor (Virasoro generators) and the topological $U(1)$ current respectively, while $Q_m$ and $G_m$ are the fermionic generators corresponding to the BRST current and the spin-2 fermionic current respectively. The eigenvalues of $L_0$ and $H_0$ correspond to the conformal weight $\Delta$ and the $U(1)$ charge $h$ of the states. The “topological” central charge $c$ is the central charge corresponding to the NS algebra. This algebra is topological because the Virasoro generators can be expressed as $L_m = \frac{1}{2}\{G_m, Q_0\}$, where $Q_0$ is the BRST charge. This implies, as is well known, that the correlators of the fields do not depend on the metric.

The two possible topological twists of the NS superconformal generators are:
These twists, which we denote as $T^\pm_W$, are mirrored under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$. Observe that the h.w. conditions $G^\pm_{1/2} |\chi_{NS}\rangle = 0$ of the NS algebra read $G_0 |\chi\rangle = 0$ after the corresponding twists. Therefore, any h.w. state of the NS algebra results in a $G_0$-closed or chiral state of the Topological algebra, which is also h.w. as the reader can easily verify by inspecting the twists (2.2). Conversely, any $G_0$-closed or chiral h.w. topological state (and only these) transforms into a h.w. state of the NS algebra.

**Topological states and topological singular vectors**

From the anticommutator $\{Q_0, G_0\} = 2L_0$ one deduces that a topological state (primary or secondary) with non-zero conformal weight can be either $G_0$-closed, or $Q_0$-closed, or a linear combination of both types. The topological states with zero conformal weight, however, can be $Q_0$-closed, or $G_0$-closed, or chiral, or no-label.

As a first classification of the topological secondary states one considers their level $l$, their relative $U(1)$ charge $q$ and their transformation properties under $Q_0$ and $G_0$ (BRST-invariance properties). The level $l$ and the relative charge $q$ are defined as the difference between the conformal weight and $U(1)$ charge of the secondary state and the conformal weight and $U(1)$ charge of the primary state on which it is built. Hence the topological secondary states will be denoted as $|\chi\rangle^{(q)}_l (G_0\text{-closed})$, $|\chi\rangle^{(q)Q}_l (Q_0\text{-closed})$, $|\chi\rangle^{(q)G,Q}_l$ (chiral), and $|\chi\rangle^{(q)}_l$ (no-label). For convenience we will also indicate the conformal weight $\Delta$, the $U(1)$ charge $h$, and the BRST-invariance properties of the primary state on which the secondary is built. Observe that the conformal weight and the total $U(1)$ charge of the secondary states are given by $\Delta + l$ and $h + q$, respectively.

In complete Verma modules there are 29 types of topological singular vectors, a given type being defined by the relative charge $q$, the BRST-invariance properties of the state itself, and the BRST-invariance properties of the primary on which it is built. The singular vectors with non-zero conformal weight, $\Delta + l \neq 0$, are linear combinations of $G_0$-closed and $Q_0$-closed singular vectors, therefore we can restrict ourselves to these types with well defined BRST-invariance properties. The singular vectors with zero conformal weight, $\Delta + l = 0$, can also be chiral or no-label.

There are three different types of topological primaries giving rise to complete Verma modules: $G_0$-closed primaries $|\Delta, h\rangle^G$, $Q_0$-closed primaries $|\Delta, h\rangle^Q$, and no-label primaries $|0, h\rangle$. We do not consider primaries which are linear combinations of two or more of these types, neither primaries of these types with additional constraints (like chiral primaries)
giving rise to incomplete Verma modules. The topological singular vectors in no-label Verma modules (9 types) cannot be mapped to NS singular vectors. The remaining 20 types are distributed in the following way:

- Ten types built on $G_0$-closed primaries $|\Delta, h\rangle^G$:

<table>
<thead>
<tr>
<th></th>
<th>$q = -2$</th>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}(1)^{G}$</td>
<td>$</td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$</td>
<td>\chi\rangle_{l}(-2)^{Q}$</td>
<td>$</td>
<td>\chi\rangle_{l}(-1)^{Q}$</td>
</tr>
<tr>
<td>chiral</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}(-1)^{G,Q}$</td>
<td>$</td>
</tr>
<tr>
<td>no-label</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}^{(-1)}$</td>
<td>$</td>
</tr>
</tbody>
</table>

(2.3)

- Ten types built on $Q_0$-closed primaries $|\Delta, h\rangle^Q$:

<table>
<thead>
<tr>
<th></th>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}(0)^{G}$</td>
<td>$</td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$</td>
<td>\chi\rangle_{l}(-1)^{Q}$</td>
<td>$</td>
<td>\chi\rangle_{l}^{(0)Q}$</td>
</tr>
<tr>
<td>chiral</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}^{(0)G,Q}$</td>
<td>$</td>
</tr>
<tr>
<td>no-label</td>
<td></td>
<td>$</td>
<td>\chi\rangle_{l}^{(0)}$</td>
<td>$</td>
</tr>
</tbody>
</table>

(2.4)

For $\Delta \neq 0$ the h.w. vector of the Verma module is degenerate: there is one $G_0$-closed primary state as well as one $Q_0$-closed primary state, $Q_0$ and $G_0$ interpolating between them. As a result, for $\Delta \neq 0$ the singular vectors of table (2.3) are equivalent to singular vectors of table (2.4) with a shift on the U(1) charges (for the details see ref. [6]). In particular, the charged (uncharged) chiral singular vectors of table (2.3) are equivalent to the uncharged (charged) chiral singular vectors of table (2.4). With the exception of the no-label singular vectors, all other 16 types of topological singular vectors in tables (2.3) and (2.4) can be mapped to NS singular vectors, as we will see in next section.

An important observation is that chiral singular vectors $|\chi\rangle_{l}^{(q)G,Q}$ can be regarded as particular cases of $G_0$-closed singular vectors $|\chi\rangle_{l}^{(q)G}$ and also as particular cases of $Q_0$-closed singular vectors $|\chi\rangle_{l}^{(q)Q}$. That is, some $G_0$-closed and $Q_0$-closed singular vectors may “become” chiral (although not necessarily, depending on the case) when the conformal weight of the singular vector turns out to be zero, i.e. $\Delta + l = 0$.

The fermionic zero modes

Most topological singular vectors come in pairs at the same level in the same Verma module, differing by one unit of relative charge. The reason is that the fermionic zero
modes $G_0$ and $Q_0$ acting on a singular vector produce another singular vector as can be checked straightforwardly using the Topological algebra (2.1). Therefore only chiral singular vectors can be “alone”, whereas the no-label singular vectors are accompanied by three, rather than one, singular vectors at the same level in the same Verma module. To be precise, inside a given Verma module $V(\Delta, h)$ and for a given level $l$ the topological singular vectors with non-zero conformal weight are connected by the action of $G_0$ and $Q_0$ as:

\[
Q_0 |\chi\rangle_l^{(q)G} \to |\chi\rangle_l^{(q-1)Q}, \quad G_0 |\chi\rangle_l^{(q)Q} \to |\chi\rangle_l^{(q+1)G},
\]

where the arrows can be reversed (up to constants), using $G_0$ and $Q_0$ respectively, since the conformal weight of the singular vectors is different from zero, i.e. $\Delta + l \neq 0$. Otherwise, on the right-hand side of the arrows one obtains chiral secondary singular vectors which cannot “come back” to the singular vectors on the left-hand side:

\[
Q_0 |\chi\rangle_l^{(q)G} \to |\chi\rangle_l^{(q-1)G,Q}, \quad G_0 |\chi\rangle_l^{(q)Q} \to |\chi\rangle_l^{(q+1)G,Q}. \quad (2.6)
\]

Regarding no-label singular vectors $|\chi\rangle_l^{(q)}$, they always satisfy $\Delta + l = 0$. The action of $G_0$ and $Q_0$ on a no-label singular vector produce three singular vectors:

\[
Q_0 |\chi\rangle_l^{(q)} \to |\chi\rangle_l^{(q-1)Q}, \quad G_0 |\chi\rangle_l^{(q)} \to |\chi\rangle_l^{(q+1)G}, \quad G_0 Q_0 |\chi\rangle_l^{(q)} \to |\chi\rangle_l^{(q)G,Q}. \quad (2.7)
\]

All three are secondary singular vectors which cannot come back to the no-label singular vector $|\chi\rangle_l^{(q)}$ by acting with $G_0$ and $Q_0$.

Hence $G_0$ and $Q_0$ interpolate between two singular vectors with non-zero conformal weight, in both directions, whereas they produce secondary singular vectors when acting on singular vectors with zero conformal weight.

**The universal odd spectral flow $A$**

The universal odd spectral flow automorphism $A_1$, denoted simply as $A$, transforms all kinds of primary states and singular vectors back into primary states and singular vectors, mapping chiral states to chiral states. It is given by

\[
A L_m A^{-1} = L_m - mH_m, \\
A H_m A^{-1} = -H_m - \frac{\xi}{3}\delta_{m,0}, \\
A Q_m A^{-1} = G_m, \\
A G_m A^{-1} = Q_m.
\]

with $A^{-1} = A$. It transforms the $(L_0, H_0)$ eigenvalues $(\Delta, h)$ of the states as $(\Delta, -h - \frac{\xi}{3})$, reversing the relative charge of the secondary states and leaving the level invariant, as a consequence. In addition, $A$ also reverses the BRST-invariance properties of the states.
(primary as well as secondary) mapping \( \mathcal{G}_0 \)-closed \( (\mathcal{Q}_0 \)-closed) states into \( \mathcal{Q}_0 \)-closed \( (\mathcal{G}_0 \)-closed) states, and chiral states into chiral states. Hence the action of \( \mathcal{A} \) results in the following mappings between singular vectors in different Verma modules:

\[
\mathcal{A} |\chi\rangle_{l,|\Delta,\hbar\rangle}^{(q)G} \rightarrow |\chi\rangle_{l,|\Delta,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)G}, \quad \mathcal{A} |\chi\rangle_{l,|\Delta,\hbar\rangle}^{(q)Q} \rightarrow |\chi\rangle_{l,|\Delta,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)Q},
\]

\[
\mathcal{A} |\chi\rangle_{l,|\Delta,-\hbar\rangle}^{(q)G} \rightarrow |\chi\rangle_{l,|\Delta,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)G}, \quad \mathcal{A} |\chi\rangle_{l,|\Delta,-\hbar\rangle}^{(q)Q} \rightarrow |\chi\rangle_{l,|\Delta,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)Q},
\]

\[
\mathcal{A} |\chi\rangle_{l,|0,\hbar\rangle}^{(q)G} \rightarrow |\chi\rangle_{l,|0,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)G}, \quad \mathcal{A} |\chi\rangle_{l,|0,\hbar\rangle}^{(q)Q} \rightarrow |\chi\rangle_{l,|0,-\hbar,-\frac{\hbar}{2}\rangle}^{(-q)Q},
\]

and their inverses.

\section{Construction Formulae}

About four years ago construction formulae for the singular vectors of the NS algebra were computed by Dörzapf. Using the “fusion method” \([13]\) explicit formulae were obtained \([3]\) for all the charged singular vectors (which only exist for \( q = \pm 1 \)), and for a class of uncharged singular vectors analogous to the BSA singular vectors of the Virasoro algebra \([14]\). Later using the “analytic continuation method” \([15]\) explicit formulae were obtained for all the uncharged singular vectors \([4]\).

In what follows we will show that these construction formulae for the NS singular vectors also provide construction formulae for 16 types of topological singular vectors since one can write down mappings from the NS singular vectors to these 16 types of topological singular vectors; one-to-one mappings in particular. We will proceed in the following way. First we will construct the “box” diagrams obtained by the actions of the fermionic zero modes \( \mathcal{G}_0, \mathcal{Q}_0 \) and the action of the spectral flow automorphism \( \mathcal{A} \), using the results of section 2 (eqns (2.5)-(2.9)). We will be interested only in the box diagrams which contain \( \mathcal{G}_0 \)-closed topological singular vectors built on \( \mathcal{G}_0 \)-closed primaries; \textit{i.e.} of type \( |\chi\rangle_{l,|q\rangle}^{(q)G} \), (see table (2.3)), since only these types have a direct relation with the generic NS singular vectors via the topological (un)twistings. From the box diagrams one can deduce straightforwardly two different mappings from the NS singular vectors to the topological singular vectors, taking into account that in each box diagram the topological singular vector of type \( |\chi\rangle_{l,|q\rangle}^{(q)G} \) can be transformed into two NS singular vectors using the topological (un)twistings \( T^+_W \) \((2.2)\). These two NS singular vectors are mirror-symmetric under the exchange \( H_m \rightarrow -H_m \) and \( G^+_r \leftrightarrow G^-_r \); therefore they have opposite U(1) charges and are located in mirror-symmetric Verma modules:

\[
|\chi\rangle_{l,|\Delta,\hbar\rangle}^{(q)G} = T^+_W |\chi_{NS}\rangle_{l,-q/2,|\Delta-h/2,\hbar\rangle}^{(q)} = T^-_W |\chi_{NS}\rangle_{l,-q/2,|\Delta-h/2,-\hbar\rangle}^{(-q)},
\]

(3.1)
Let us start with the box diagrams which contain an uncharged singular vector of type $|\chi^{(0)G}_{l,|\Delta, h\rangle^G}\rangle$ in the Verma module $V(|\Delta, h\rangle^G)$. For non-zero conformal weight, $\Delta + l \neq 0$, the box diagram, shown in (3.2), consists of singular vectors of the types $|\chi^{(0)G}_{l,|\Delta, h\rangle^G}\rangle$ and $|\chi^{(-1)Q}_{l,|\Delta, h\rangle^Q}\rangle$, at level $l$ in the Verma module $V(|\Delta, h\rangle^Q)$, and singular vectors of the types $|\chi^{(0)Q}_{l,|\Delta, h\rangle^Q}\rangle$ and $|\chi^{(1)G}_{l,|\Delta, h\rangle^G}\rangle$, also at level $l$ in the Verma module $V(|\Delta, -h - c/3\rangle^Q)$.

$$
\begin{align*}
|\chi^{(0)G}_{l,|\Delta, h\rangle^G}\rangle & \xrightarrow{Q_0} |\chi^{(-1)Q}_{l,|\Delta, h\rangle^Q}\rangle \\
\mathcal{A} \uparrow & \uparrow \mathcal{A} \\
|\chi^{(0)Q}_{l,|\Delta, -h - \frac{c}{3}\rangle^Q}\rangle & \xrightarrow{Q_0} |\chi^{(1)G}_{l,|\Delta, -h - \frac{c}{3}\rangle^Q}\rangle
\end{align*}
$$

(3.2)

The arrows $Q_0$ and $G_0$ can be reversed (up to constants) using $Q_0$ and $G_0$ respectively; that is, the fermionic zero modes interpolate between two singular vectors, one charged and one uncharged, at the same level in the same Verma module.

For $\Delta + l = 0$, the conformal weight of the singular vectors is zero, so that the corresponding arrows $Q_0$, $G_0$ cannot be reversed, producing secondary chiral singular vectors $|\chi^{(-1)G, Q}_{l,|-l, h\rangle^G}\rangle$ and $|\chi^{(1)G, Q}_{l,|-l, -h - c/3\rangle^Q}\rangle$ on the right-hand side, at level zero with respect to the singular vectors on the left-hand side. The corresponding box diagram is therefore:

$$
\begin{align*}
|\chi^{(0)G}_{l,|-l, h\rangle^G}\rangle & \xrightarrow{Q_0} |\chi^{(-1)G, Q}_{l,|-l, h\rangle^G}\rangle \\
\mathcal{A} \uparrow & \uparrow \mathcal{A} \\
|\chi^{(0)Q}_{l,|-l, -h - \frac{c}{3}\rangle^Q}\rangle & \xrightarrow{Q_0} |\chi^{(1)G, Q}_{l,|-l, -h - \frac{c}{3}\rangle^Q}\rangle
\end{align*}
$$

(3.3)

The untwisting of the uncharged singular vector $|\chi^{(0)G}_{l,|\Delta, h\rangle^G}\rangle$, using $T^+_W$ (2.2), produces two uncharged mirror-symmetric NS singular vectors located in mirror-symmetric Verma modules. Conversely, the twisting of two uncharged mirror-symmetric NS singular vectors, using $T^+_W$ and $T^-_W$ respectively, produces the same uncharged topological singular vector of type $|\chi^{(0)G}_{l,|\Delta, h\rangle^G}\rangle$. That is, one has the mappings:

$$
|\chi^{(0)G}_{l,|\Delta + h/2, h\rangle^G}\rangle = T^+_W |\chi_{NS}^{(0)G}_{l,|\Delta, h\rangle} = T^-_W |\chi_{NS}^{(0)G}_{l,|\Delta, -h\rangle},
$$

(3.4)

where we have redefined $\Delta$ as the conformal weight of the NS primaries. The mappings from the NS singular vectors to the remaining topological singular vectors in diagrams (3.2) and (3.3) can be derived now resulting in the following expressions:
\[
\begin{align*}
|\chi\rangle_{l,|\Delta-h/2,-h-\frac{c}{3}Q}) & = \mathcal{A} T^+_W \chi_{NS}|\Delta,h\rangle^{(0)Q} \\
|\chi\rangle_{l,|\Delta+h/2,-h-\frac{c}{3}Q}) & = \mathcal{Q}_0 T^+_W \chi_{NS}|\Delta,h\rangle^{(0)Q} \\
|\chi\rangle_{l,|\Delta-h/2,-h-\frac{c}{3}Q}) & = \mathcal{A} \mathcal{Q}_0 T^+_W \chi_{NS}|\Delta,h\rangle^{(0)Q} \\
|\chi\rangle_{l,|\Delta+h,-\frac{c}{3}Q}) & = \mathcal{Q}_0 T^+_W \chi_{NS}|\Delta,h\rangle^{(0)Q} \\
|\chi\rangle_{l,|\Delta-h,-\frac{c}{3}Q}) & = \mathcal{A} \mathcal{Q}_0 T^+_W \chi_{NS}|\Delta,h\rangle^{(0)Q}
\end{align*}
\]

and similar expressions using \(T^-_W \chi_{NS}|\Delta,-h\rangle^{(0)}\). Observe that the two last mappings, to chiral singular vectors, are not invertible since the arrows \(\mathcal{Q}_0, \mathcal{G}_0\) in diagram (3.3) cannot be reversed.

One finds similar box diagrams associated to the charge \(q = 1\) singular vector \(|\chi\rangle^{(1)G}_{l,|0\rangle} \) in the Verma module \(V(|\Delta,h\rangle^G)\), as shown in (3.6) and (3.7). For \(\Delta + l \neq 0\) the box diagram consists of singular vectors of the types \(|\chi\rangle^{(1)G}_{l,|0\rangle} \) and \(|\chi\rangle^{(0)Q}_{l,|0\rangle} \) at the same level \(l\) in the Verma module \(V(|\Delta,h\rangle^G)\), and singular vectors of the types \(|\chi\rangle^{(-1)Q}_{l,|0\rangle} \) and \(|\chi\rangle^{(0)Q}_{l,|0\rangle} \) also at level \(l\) in the Verma module \(V(|\Delta,-h-c/3\rangle^Q)\).

\[
\begin{align*}
|\chi\rangle^{(1)G}_{l,|\Delta,h\rangle^G} & \xrightarrow{\mathcal{Q}_0} |\chi\rangle^{(0)Q}_{l,|\Delta,h\rangle^G} \\
\mathcal{A} \uparrow & \uparrow \mathcal{A} \\
|\chi\rangle^{(-1)Q}_{l,|\Delta,-h-\frac{c}{3}Q}) & \xrightarrow{\mathcal{G}_0} |\chi\rangle^{(0)G}_{l,|\Delta,-h-\frac{c}{3}Q})
\end{align*}
\]

For the case of zero conformal weight \(\Delta + l = 0\) the uncharged singular vectors on the right-hand side become secondary chiral singular vectors:

\[
\begin{align*}
|\chi\rangle^{(1)G}_{l,|l\rangle} & \xrightarrow{\mathcal{Q}_0} |\chi\rangle^{(0)G,Q}_{l,|l\rangle} \\
\mathcal{A} \uparrow & \uparrow \mathcal{A} \\
|\chi\rangle^{(-1)Q}_{l,|l\rangle} & \xrightarrow{\mathcal{G}_0} |\chi\rangle^{(0)G,Q}_{l,|l\rangle}
\end{align*}
\]

The untwisting of the charged singular vector \(|\chi\rangle^{(1)G}_{l,|\Delta,h\rangle^G} \) , using \(T^+_W (2.2)\), produces two charged mirror-symmetric NS singular vectors located in mirror-symmetric Verma modules. Conversely, the twisting of two charged mirror-symmetric NS singular vectors, using \(T^+_W \) and \(T^-_W \) respectively, produces the same charged (\(|q| = 1\)) topological singular vector. For charge \(q = 1\) one finds the mappings:

\[
|\chi\rangle^{(1)G}_{l,|\Delta+h/2,|\Delta,-h\rangle} = T^+_W \chi_{NS}|\Delta+h/2,|\Delta,-h\rangle = T^-_W \chi_{NS}|\Delta-h\rangle^Q.
\]
where we have redefined $\Delta$ again, for convenience. The mappings from the NS singular vectors to the remaining topological singular vectors in diagrams (3.4) and (3.7) are given by:

$$
\begin{align*}
|\chi\rangle_{l,|\Delta+h/2,-h-\frac{\xi}{2}Q}^{(-1)Q} &= A \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} \\
|\chi\rangle_{l,|\Delta+h/2,0}^{(0)Q} &= Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} \\
|\chi\rangle_{l,|\Delta+h/2,-h-\frac{\xi}{2}Q}^{G} &= A \ Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} \\
|\chi\rangle_{l,|\Delta+h/2,0}^{G} &= Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} \\
|\chi\rangle_{l,-l,|h-\frac{\xi}{2}Q}^{(2)G} &= A \ Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} \\
|\chi\rangle_{l,-l,|h-\frac{\xi}{2}Q}^{G} &= A \ Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)}
\end{align*}
$$

and similar expressions using $T_W^- \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}$. As before, the two last mappings, to chiral singular vectors, are not invertible since the arrows $Q_0$, $G_0$ in diagram (3.7) cannot be reversed.

Finally let us take a charge $q = -1$ singular vector of type $|\chi\rangle_{l,|\Delta,h}^{(-1)G}$ in the Verma module $V(|\Delta, h)^G)$. For $\Delta + l \neq 0$ the box diagram, shown in (3.10), consists of singular vectors of the types $|\chi\rangle_{l,|\Delta,h}^{(-1)G}$ and $|\chi\rangle_{l,|\Delta,h}^{(-2)G}$, at the same level $l$ in the Verma module $V(|\Delta, h)^G)$, and singular vectors of the types $|\chi\rangle_{l,|\Delta,h}^{(1)G}$ and $|\chi\rangle_{l,|\Delta,h}^{(2)G}$ also at level $l$ in the Verma module $V(|\Delta, -h - c/3)^G)$.

$$
\begin{align*}
|\chi\rangle_{l,|\Delta,h}^{(-1)G} &\xrightarrow{Q_0} |\chi\rangle_{l,|\Delta,h}^{(2)G} \\
A &\uparrow \quad \uparrow A \\
|\chi\rangle_{l,|\Delta,-h-\frac{\xi}{2}}^{(1)G} &\xrightarrow{G_0} |\chi\rangle_{l,|\Delta,-h-\frac{\xi}{2}}^{(2)G}
\end{align*}
$$

As was pointed out before, the twisting of two charged mirror-symmetric NS singular vectors, using $T_W^+$ and $T_W^-$ respectively, produces the same charged $(|q| = 1)$ topological singular vector. For charge $q = -1$ one finds the mappings:

$$
|\chi\rangle_{l,|\Delta,h}^{(-1)G} = T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)} = T_W^- \ |\chi_{NS}\rangle_{l+1/2,|\Delta,-h}^{(1)}.
$$

The mappings from the NS singular vectors to the remaining topological singular vectors in diagram (3.10) result as follows:

$$
\begin{align*}
|\chi\rangle_{l,|\Delta+h/2,-h-\frac{\xi}{2}}^{(1)Q} &= A \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,h}^{(-1)} \\
|\chi\rangle_{l,|\Delta+h/2,0}^{(-2)Q} &= Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,h}^{(-1)} \\
|\chi\rangle_{l,|\Delta+h/2,0}^{(2)G} &= A \ Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,h}^{(-1)} \\
|\chi\rangle_{l,|\Delta+h/2,-h-\frac{\xi}{2}}^{G} &= A \ Q_0 \ T_W^+ \ |\chi_{NS}\rangle_{l+1/2,|\Delta,h}^{(-1)}
\end{align*}
$$
and similar expressions using $T^-_W |\chi_{NS}\rangle_{l+1/2,|\Delta,-h\rangle}$.

Let us come back to diagram (3.10). For $\Delta + l = 0$ the “would be” secondary chiral singular vectors with $|q| = 2$ simply do not exist, as follows from the results in tables (2.3) and (2.4). As a consequence, the singular vectors of types $|\chi\rangle^{(-1)G}_{l_{|q|=0}}$ and $|\chi\rangle^{(1)Q}_{l_{|q|=0}}$ “become” actually chiral for zero conformal weight, i.e. of types $|\chi\rangle^{(-1)GQ}_{l_{|q|=0}}$ and $|\chi\rangle^{(1)GQ}_{l_{|q|=0}}$ instead, and the box diagram reduces to two chiral singular vectors, connected by $A$. Thus one has the mappings:

$$
|\chi\rangle^{(-1)GQ}_{l_{|q|=0}} = T^+_W |\chi_{NS}\rangle^{(-1)}_{l+1/2,-l-h/2,h} = T^-_W |\chi_{NS}\rangle^{(1)}_{l+1/2,-l-h/2,-h} \\
|\chi\rangle^{(1)GQ}_{l_{|q|=0}} = A T^+_W |\chi_{NS}\rangle^{(-1)}_{l+1/2,-l-h/2,h} = A T^-_W |\chi_{NS}\rangle^{(1)}_{l+1/2,-l-h/2,-h} .
$$

(3.13)

An important observation here is the following. The fact that $|\chi\rangle^{(-1)G}_{l_{|q|=0}}$ becomes chiral for $\Delta = -l$ implies necessarily that the NS singular vector $|\chi_{NS}\rangle^{(-1)}_{l+1/2,|l-h/2,h\rangle}$ is antichiral (annihilated by $G^{-1/2}_{-1}$), whereas the NS singular vector $|\chi_{NS}\rangle^{(1)}_{l+1/2,|l-h/2,-h\rangle}$ is chiral (annihilated by $G^{+1/2}_{-1}$). The reason is that $Q_0 = T^+_W G^{-1/2}_{-1} = T^-_W G^{+1/2}_{-1}$, so that the condition of being annihilated by $Q_0$ is transformed into the conditions of being annihilated by $G^{-1/2}_{-1}$ and $G^{+1/2}_{-1}$, respectively, under the (un)twistings $T^+_W$ and $T^-_W$. Thus we have found that charged NS singular vectors $|\chi_{NS}\rangle^{(\pm 1)}_{l_{|q|=0},|\Delta',h\rangle}$ with $\Delta' + l' = \pm \frac{(h+1)}{2}$ are chiral (upper signs) and antichiral (lower signs).

The uncharged chiral singular vectors are equivalent to charged chiral singular vectors, as was pointed out in section 2. Namely

$$
|\chi\rangle^{(0)GQ}_{l_{|q|=0}} = |\chi\rangle^{(-1)GQ}_{l_{|q|=0}}, \quad |\chi\rangle^{(0)GQ}_{l_{|q|=0}} = |\chi\rangle^{(1)GQ}_{l_{|q|=0}},
$$

(3.14)

by exchanging the primary states of the Verma module: $|l, h\rangle^G = G_0 |l, h - 1\rangle^Q$ and $|l, -h - h_3\rangle^Q = Q_0 |l, -h - h_3 + 1\rangle^G$.

Hence we have found invertible mappings from the NS singular vectors to the chiral topological singular vectors. These are, in addition, simpler than the mappings (3.3) and (3.9) deduced from diagrams (3.3) and (3.7).

The mappings given by eqns. (3.3), (3.4), (3.8), (3.9), (3.11), (3.12), and (3.13) turn into construction formulae for the topological singular vectors just by expressing the NS singular vectors $|\chi_{NS}\rangle^{(0)}$, $|\chi_{NS}\rangle^{(1)}$ and $|\chi_{NS}\rangle^{(-1)}$ in terms of their corresponding construction formulae, given in refs. [8] and [9]. The explicit expressions for the charged singular vectors at level $k$ read [8]

$$
|\chi_{NS/k}^{(+\pm)} = W^{\pm} E^{\pm}(k-1/2) T^{\pm}(k-1) \xi^{\pm}(k-3/2) T^{\pm}(k-2) \cdots \xi^{\pm}(1) T^{\pm}(1/2) \Psi^{\pm} ,
$$

(3.15)
where $\mathcal{E}^\pm(k)$ and $\mathcal{T}^\pm(k)$ are even and odd recursion step matrices, respectively, and $W^\pm$ and $\Psi^+_0$ are vectors, the latter depending on the initial low level singular vectors. The spectrum of $\Delta$ and $\hbar$ for which the NS Verma modules $V_{NS}(\Delta, \hbar)$ contain charged singular vectors $|\chi_{NS}^{(\pm 1)}\rangle$ is given (at least) by the zeroes of the NS determinant formula which are solutions to the vanishing planes $g_{\pm k}(\Delta, \hbar) = 0$ [14].

The explicit expressions for the uncharged singular vectors at level $l = \frac{t^2}{2}$ read [4]

$$|\chi_{NS}^{(0)}\rangle_{r,s} = \epsilon^+_{r,s}(t, \hbar) \Delta_{r,s}(1, 0) + \epsilon^-_{r,s}(t, \hbar) \Delta_{r,s}(0, 1), \quad (3.16)$$

where $\Delta_{r,s}(1, 0)$ and $\Delta_{r,s}(0, 1)$ are two basis vectors taken from the analytically continued Verma module, $t = \frac{3-\xi}{3}$ parametrizes the central charge, and

$$\epsilon^\pm_{r,s}(t, \hbar) = \prod_{m=1}^{r} \left( \pm \frac{s - rt}{2t} + \frac{\hbar}{t} \pm \frac{1}{2} \pm m \right), \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+. \quad (3.17)$$

The spectrum of $\Delta$ and $\hbar$ for which the NS Verma modules $V_{NS}(\Delta, \hbar)$ contain uncharged singular vectors $|\chi_{NS}^{(0)}\rangle_{r,s}$ is given (at least) by the zeroes of the NS determinant formula which are solutions to the quadratic vanishing surface $f_{r,s}(\Delta, \hbar) = 0$ [14].

The simultaneous vanishing of the two curves, $\epsilon^+_{r,s}(t, \hbar) = 0$ and $\epsilon^-_{r,s}(t, \hbar) = 0$, leads to the appearance of two linearly independent uncharged NS singular vectors at the same level, in the same Verma module [4]. The topological twists $T^\pm_W(2.2)$ let these conditions invariant, extending the existence of the two-dimensional space of singular vectors to the topological singular vectors of types $|\chi\rangle^{(0)G}_{L|\phi\rangle}\chi, |\chi\rangle^{(-1)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(1)G}_{L|\phi\rangle}\chi$ and $|\chi\rangle^{(0)Q}_{L|\phi\rangle}\chi$, as the generic uncharged NS singular vectors are transformed necessarily into these four types of topological singular vectors via the mappings (3.4) and (3.5). The chiral types of singular vectors, which are related to particular, non-generic uncharged NS singular vectors via the mappings (3.3), do not admit two-dimensional spaces however [14]. Nevertheless, they may appear as partners of $G_0$-closed or $Q_0$-closed singular vectors in D"{o}rrzapf pairs since they are just particular cases of $G_0$-closed and $Q_0$-closed singular vectors (see Appendix C in ref. [3]).

For the NS charged singular vectors there are no two-dimensional spaces either [9]. This also implies the absence of two-dimensional spaces for all types of topological singular vectors connected to them generically via the mappings (3.8)-(3.12); that is, for the eight types: $|\chi\rangle^{(-2)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(-1)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(0)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(1)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(-1)G}_{L|\phi\rangle}\chi, |\chi\rangle^{(0)G}_{L|\phi\rangle}\chi, |\chi\rangle^{(1)G}_{L|\phi\rangle}\chi$ and $|\chi\rangle^{(2)G}_{L|\phi\rangle}\chi$.

With the same reasoning, and taking into account that there are no NS singular vectors $|\chi_{NS}^{(q)}\rangle_l$ with $|q| \geq 2$ (as was proved in ref. [4]), one can prove the non-existence of topological singular vectors of types $|\chi\rangle^{(q)G}_{L|\phi\rangle}\chi$ with $|q| \geq 2$ (because of result (3.1)), and the non-existence of all types of topological singular vectors in the “would-be” box-diagrams associated to $|\chi\rangle^{(q)G}_{L|\phi\rangle}\chi, |q| \geq 2$; that is, singular vectors of the types $|\chi\rangle^{(q-1)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(-q)Q}_{L|\phi\rangle}\chi, |\chi\rangle^{(1-q)G}_{L|\phi\rangle}\chi, |\chi\rangle^{(1-q)G,Q}_{L|\phi\rangle}\chi$ and $|\chi\rangle^{(q-1)G,Q}_{L|\phi\rangle}\chi$, with $|q| \geq 2$. 

11
4 Final Remarks

We have written down one-to-one mappings between the NS singular vectors and 16 types of topological singular vectors: the ones given in tables (2.3) and (2.4) with the exception of the no-label types. One can write many other (chains of) transformations among the topological singular vectors [1]. apart from the ones given by the box diagrams, used in this work. However, for the purpose of mapping the NS singular vectors to the topological singular vectors, the additional transformations are of little interest. In particular there are no mappings from the NS singular vectors to the no-label topological singular vectors nor to the topological singular vectors in no-label Verma modules. It seems that these types of topological singular vectors can only be mapped either to NS subsingular vectors [17] or to null non-highest weight descendants of NS singular vectors.

Two important consequences of the analysis performed in this paper are: i) the non-existence of topological singular vectors of types $|\chi^{(q)}_{l,|\phi\rangle,\alpha}\rangle$ with $|q| \geq 2$, and of all the “would-be” topological singular vectors in their box-diagrams, and ii) the fact that the charged NS singular vectors $|\chi_{NS}^{(\pm 1)}_{l,|\Delta,\phi\rangle}\rangle$ with $\Delta + l = \pm \frac{(h+1)}{2}$ are chiral (upper signs) and antichiral (lower signs).

Acknowledgements

I am grateful to M. Dörrzapf for reading the manuscript and for several important suggestions.

References

N. Marcus, talk at the Rome String Theory Workshop (1992), hep-th/9211059


