Off shell $\kappa$-symmetry of the superparticle and the spinning superparticle

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Abstract

The spinorial local world-line $\kappa$-symmetry of the covariant Brink-Schwarz formulation of the 4-D superparticle is abelian in an off-shell phase-space formulation. The result is shown to generalize to the extended spinorial transformations of the spinning superparticle.

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1 Introduction

The space-time covariant formulation of super $p$-branes is known to have a local fermionic invariance on the world manifold, first discovered by Siegel for the superparticle [1], and subsequently generalized to the case of superstrings in [2]. This invariance helps to balance the number of commuting and anti-commuting degrees of freedom in models with the boson and fermion variables naturally belonging to different representations of the Lorentz group of the target space-time. Indeed, the parameter of this transformation is an anti-commuting space-time spinor $\kappa$, varying in an arbitrary way over the world manifold. In this sense the $\kappa$-invariance is a supersymmetry. However, it is not a classic supersymmetry as provided by the grading of space-time algebras, like the Poincaré or conformal superalgebras. Although it can be related to space-time supersymmetry by gauge-fixing in the light-cone formulation, the algebraic structure underlying the $\kappa$-symmetry has not been fully elucidated. Rather, in most instances it has been found as a local symmetry of specific actions, and not by implementing a given algebraic structure in the context of the general principles of field theory. A related problem is, that in most examples the symmetry has only been realized on-shell, i.e. modulo the equations of motion of the specific model considered. A step forward is therefore provided by examples of $\kappa$-invariance which are realized off-shell, i.e. with an algebraic structure independent of the equations of motion.

In this paper we consider the example of the superparticle and its generalization with additional local world-line supersymmetry, known as the spinning superparticle [3]-[7]. Both models are reparametrization invariant on the worldline, and posses a proper-time dependent $\kappa$-invariance. In addition, the spinning superparticle possesses a local world-line supersymmetry, bringing along a bosonic counterpart of $\kappa$-symmetry with a commuting spinor parameter $\alpha$ [3, 4].

We present results showing that the first order (phase-space) formulations of these models have some remarkable features: they contain a $(D = 1)$ gauge field for local $\kappa$-symmetry [8], the algebra of $\kappa$-transformations and other local world-line symmetries closes off-shell, and the structure of the algebra becomes abelian. This is in strong contrast with the on-shell results [7, 9]. As the $\kappa$-transformations also involve the momenta explicitly, we conclude that the phase-space formulation of these models seems the more natural one, and we make some remarks about prospects for quantisation. How our present formulation is to be implemented for higher-dimensional objects like strings or membranes remains to be investigated.

2 Phase-space description of the superparticle

The standard action for a free massless superparticle described by bosonic and fermionic coordinates $(x^\mu, \theta_\alpha)$ in 4-$D$ space-time is
\[ S_{\text{conf}} = \int d\tau \frac{1}{2e} \left( \dot{x}^\mu - \dot{\theta} \gamma^\mu \dot{\theta} \right)^2. \]  

(1)

Here a dot denotes a proper-time derivative, and \( e \) is the einbein variable making the action reparametrization invariant: under an arbitrary transformation of the affine parameter (proper time) \( \tau \to \tau' \) the action (1) is invariant provided

\[ x'^\mu(\tau') = x^\mu(\tau), \quad \theta'_\alpha(\tau') = \theta_\alpha(\tau), \quad e'(\tau') = \frac{d\tau}{d\tau'} e(\tau). \]  

(2)

For ease of notation, in the following we regularly suppress indices on \( \theta \) and other spinor variables.

The momenta conjugate to the coordinates \((x^\mu, \theta_\alpha)\) are

\[ p^\mu = \frac{1}{e} \left( \dot{x}^\mu - \dot{\theta} \gamma^\mu \dot{\theta} \right), \quad \pi = -\dot{\theta}. \]  

(3)

The first-class constraint imposed by reparametrization invariance is that the momenta are light-like:

\[ p^\mu p_\mu = 0, \]  

(4)

showing that the particle is massless indeed. The equations of motion imply the conservation of momentum:

\[ \dot{p}_\mu = 0, \]  

(5)

as required by translation invariance in the target space-time. The remaining equations of motion for the fermionic coordinates are

\[ \dot{\pi} = \dot{\theta}, \]  

(6)

which implies

\[ \dot{\theta} = 0. \]  

(7)

As noted by Siegel [1], the first-class constraint (4) and momentum conservation imply that this equation has zero-modes of the form

\[ \theta = \dot{\theta} \kappa, \]  

(8)

where \( \kappa(\tau) \) is an arbitrary spinor-valued function of \( \tau \). The local \( \kappa \)-invariance found by Siegel generates precisely these zero-modes.

We now present a phase-space action for the superparticle in which the coordinates and momenta are independent variables, and which reduces to the above model in terms of solutions of the equations of motion [8]. This model has the advantage that the structure of the \( \kappa \)-symmetry simplifies considerably.
The starting point for this phase-space description of the superparticle is the action
\[
S_{\text{phase}} = \int d\tau \left( -\frac{e}{2} p_\mu^2 + p_\mu \left( \dot{x}^\mu + \bar{\phi} \gamma^\mu \theta \right) + \bar{\pi} \left( \dot{\phi} - \theta \right) \right),
\]
(9)
where \((x^\mu, p_\mu)\) are space-time vectors, and \((\theta, \phi, \pi)\) space-time Majorana spinors. The equations of motion for the new variables guarantee that \((p_\mu, \pi)\) are given by the expressions (3), and furthermore that on shell
\[
\dot{\phi} = \dot{\theta}.
\]
(10)
As a result the action \(S_{\text{phase}}\) indeed reproduces the results of the configuration space formulation based on \(S_{\text{conf}}\). Like the original action, \(S_{\text{phase}}\) is reparametrization invariant, provided \((x^\mu, p_\mu; \theta, \phi, \pi)\) transform as world-line scalars, and \((e, \phi_\alpha)\) like bosonic, respectively fermionic, world-line vectors. In particular the transformation rule for the anti-commuting variables \(\phi_\alpha\) is analogous to that of the einbein \(e\):
\[
\phi_\alpha' (\tau') = \frac{d\tau}{d\tau'} \phi_\alpha (\tau).
\]
(11)
There is a natural explanation for this transformation character of the spinor \(\phi\). This becomes clear from the local \(\kappa\)-symmetry which plays the same role here as previously: to reduce the number of physical fermion degrees of freedom in the solutions for \(\theta\) by generating zero-modes. The infinitesimal transformations with spinor parameter \(\kappa_\alpha (\tau)\), and the infinitesimal reparametrizations with scalar parameter \(\xi (\tau) = \tau - \tau'\), which leave the action \(S_{\text{phase}}\) invariant modulo boundary terms read
\[
\delta e = -2 \dot{\theta} \kappa + 2 \dot{\phi} \kappa + \frac{d (\xi e)}{d\tau}, \quad \delta \phi = \dot{\phi} \kappa + \frac{d (\xi \phi)}{d\tau},
\]
\[
\delta x^\mu = \dot{\kappa} \gamma^\mu \pi + \xi \frac{dx^\mu}{d\tau}, \quad \delta \theta = \dot{\theta} \kappa + \xi \frac{d\theta}{d\tau},
\]
\[
\delta p_\mu = \xi \frac{dp_\mu}{d\tau}, \quad \delta \pi = \xi \frac{d\pi}{d\tau}.
\]
(12)
Clearly, the \(\kappa\)-transformation of \(\theta\) generates the fermion zero-modes, whilst that of \(\phi\) generates the proper-time derivative of these modes. Thus \(\phi\) acts as a \(D = 1\) gauge field for local \(\kappa\)-transformations, and it is not surprising that it transforms as a world-line vector under reparametrizations, rather than as a scalar.

The commutator algebra of these infinitesimal transformations has a very simple structure. Computing the commutator of two transformations with parameters \((\kappa_1, \xi_1)\) and \((\kappa_2, \xi_2)\) results in a similar transformation
\[
[\delta (\kappa_2, \xi_2), \delta (\kappa_1, \xi_1)] = \delta (\kappa_3, \xi_3),
\]
(13)
with the parameters on the right-hand side given by

\[ \kappa_3 = \xi_1 \dot{\kappa}_2 - \xi_2 \dot{\kappa}_1, \quad \xi_3 = \xi_1 \dot{\xi}_2 - \xi_2 \dot{\xi}_1. \]

We observe, that the \( \kappa \)-transformations commute among themselves, and the parameter \( \kappa \) transforms as a scalar under world-line reparametrizations. As these results do not require the use of any equations of motion, this algebra is closed in a model-independent way; indeed, it is shown below that the abelian nature of the \( \kappa \)-transformations survives in a more complicated model including additional physical degrees of freedom, obtained as the world-line supersymmetric extension of the superparticle.

On the other hand, it is presently not clear to us whether such a result would survive in a theory with interactions, which might generate a non-linear extension of the \( \kappa \)-transformations; this is what happens in general relativity, where the general coordinate transformations can be interpreted as a non-linear extension of the abelian gauge transformations of a free massless spin-2 tensor field. It is also true that in different Lagrangians with equivalent Hamiltonian descriptions, various gauge symmetries can be presented in different ways (as between the first order formulation of the scalar relativistic particle with explicit einbein as gauge field, versus the second-order form with Lagrange multiplier to carry the diffeomorphism symmetry).

## 3 The spinning superparticle

The spinning superparticle is a superparticle model with both rigid target spacetime and local world-line supersymmetry [3]-[7]. To construct actions with local world-line supersymmetry we use the conventions of [4], which also describes the second-order action (formulated in configuration space) for the free \( D = 4 \) spinning superparticle in our notation, both in superfield and component form.

In this section we construct a first-order (phase-space) action, equivalent to the standard second-order action after elimination of momenta and lagrange multipliers. The construction requires two sets of super multiplets \((\Sigma^\mu, \Phi_\alpha)\) for the bosonic and fermionic particle coordinates \((x^\mu, \theta_\alpha)\), transforming respectively as a vector and a spinor under the target-space Lorentz group; in addition there are two multiplets \((\Omega_\mu, \Pi_\alpha)\) containing their conjugate momenta \((p_\mu, \pi_\alpha)\). Finally there are gauge-multiplets \((E, Y_\alpha)\) for local world-line supersymmetry and local world-line \( \kappa \)-symmetry.

The component content of these superfields is
\[ \Sigma^\mu = (x^\mu, \psi^\mu), \quad \Phi = (\theta, h), \]
\[ \Omega_\mu = (\omega_\mu, p_\mu), \quad \Pi = (\pi, n), \quad \]
\[ E = (e, \chi), \quad Y = (y, \eta). \]

As before we have suppressed indices of spinor components on \((\Phi, \Pi, Y)\). The superfields \((\Sigma^\mu, Y_\alpha, E)\) have a commuting world-line scalar \((x^\mu, y_\alpha)\) or vector \((e)\) as their first component, whilst the first components of \((\Phi_\alpha, \Omega_\mu, \Pi_\alpha)\) are anti-commuting world-line fermions.

The supersymmetry transformation rules and the construction of invariant actions are discussed in [4]. For the supersymmetry gauge variables \((e, \chi)\) the infinitesimal supertransformations are

\[ \delta_c e = -2i \varepsilon \chi, \quad \delta_c \chi = \dot{\varepsilon}. \quad (16) \]

For scalar multiplets like \(Y = (y, \eta)\) they take the form

\[ \delta_\varepsilon y = -i \varepsilon \eta, \quad \delta_\varepsilon \eta = \varepsilon \frac{1}{e} \mathcal{D}_\tau y = \varepsilon \frac{1}{e} (\dot{y} + i \chi \eta). \quad (17) \]

Finally, for the fermionic multiplets such as \(\Phi = (\theta, h)\) the infinitesimal component transformations are

\[ \delta_\varepsilon \theta = \varepsilon h, \quad \delta_\varepsilon h = -i \varepsilon \frac{1}{e} \mathcal{D}_\tau \theta = -i \varepsilon \frac{1}{e} (\dot{\theta} - \chi h). \quad (18) \]

The component action is constructed from the following fermionic superfield expression

\[ \Lambda \equiv (\lambda, \ell) = \left( -\frac{1}{2} \mathcal{D} \Omega^\mu + \mathcal{D}^2 \Sigma^\mu \right) \times \Omega_\mu + i \mathcal{D} \Omega_\mu \times \left( Y^\mu \Phi \right) - i \left( \mathcal{D} \Phi - Y \right) \times \Pi, \quad (19) \]

where \(\mathcal{D}\) denotes the super derivative [4]. The explicit result for the locally supersymmetric component action is

\[ S = \int d\tau (e \ell - i \chi \lambda) \]

\[ = \int d\tau \left( -\frac{e}{2} \mathcal{P}_\mu^2 + i \left( \psi^\mu + i \bar{\eta} \gamma^\mu \theta - \frac{1}{2} \omega_\mu \right) \dot{\omega}^\mu - i e \bar{\eta} (y - h) \right) \]

\[ + p_\mu \left( \mathcal{D}_\tau x^\mu + e \bar{\eta} \gamma^\mu \theta + i e \bar{\eta} \gamma^\mu h \right) - \bar{\pi} \left( \dot{\theta} - e \eta - \chi y \right). \quad (20) \]

In addition to local world-line supersymmetry, the action is invariant under local \(\kappa\)-transformations which are a direct extension of those for the ordinary superparticle:
\[ \delta_\kappa e = -2\bar{\theta}\kappa + 2e\bar{\eta}\kappa + 2\chi\bar{\eta}\kappa, \quad \delta_\kappa \chi = 0, \]

\[ \delta_\kappa x^\mu = \bar{\kappa}\gamma^\mu \pi, \quad \delta_\kappa \psi^\mu = -i\bar{\eta}\gamma^\mu \rho\kappa + i\frac{\delta_\mu e}{2e} \bar{\eta}\gamma^\mu \theta, \]

\[ \delta_\kappa \theta = \bar{\kappa} \rho, \quad \delta_\kappa h = -\frac{\delta_\kappa e}{2e} h, \]

\[ \delta_\kappa \omega_\mu = 0, \quad \delta_\kappa p_\mu = 0, \]

\[ \delta_\kappa n = -\frac{\delta_\kappa e}{2e} n, \]

\[ \delta_\kappa y = -\frac{\delta e}{2e} y, \quad \delta_\kappa (e\eta) = \bar{\rho}\kappa + \frac{\delta_\kappa e}{2e} \chi y. \]

Working in \( D = 4 \) space-time dimensions, the parameter \( \kappa(\tau) \) is an anti-commuting Majorana spinor. Comparing the action and \( \kappa \)-transformations to those of the standard superparticle, it is convenient to introduce new variables

\[ \phi \equiv e\eta + \chi y, \quad \lambda^\mu \equiv \psi^\mu + i\bar{\eta}\gamma^\mu \theta, \quad \zeta^\mu \equiv \omega^\mu - \lambda^\mu. \]

The variable \( \phi \) acts as the gauge field of \( \kappa \)-transformations on the world line, whilst the anti-commuting Lorentz vectors \( \lambda^\mu \) and \( \zeta^\mu \) are \( \kappa \)-invariant:

\[ \delta_\kappa \phi = \delta(e\eta + \chi y) = \rho\kappa, \quad \delta_\kappa e = -2\bar{\theta}\kappa + 2\bar{\rho}\kappa, \]

\[ \delta_\kappa \lambda^\mu = 0, \quad \delta_\kappa \zeta^\mu = 0. \]

In addition, the supersymmetric structure of the model allows yet another local world-line invariance with a \textit{commuting} Majorana spinor parameter \( \alpha \) [3, 4], which is implemented here by

\[ \delta_\alpha e = -2ie\bar{\alpha}(y + h), \quad \delta_\alpha \chi = 0, \]

\[ \delta_\alpha y = \rho\alpha - \frac{\delta_\alpha e}{2e} y, \quad \delta_\alpha \phi = \delta_\alpha (e\eta + \chi y) = 0, \]

\[ \delta_\alpha x^\mu = 0, \quad \delta_\alpha \lambda^\mu = \delta_\alpha (\psi^\mu + i\bar{\eta}\gamma^\mu \theta) = 0, \]

\[ \delta_\alpha \theta = 0, \quad \delta_\alpha h = \rho\alpha - \frac{\delta_\alpha e}{2e} h, \]

\[ \delta_\alpha n = -\frac{\delta_\alpha e}{2e} n, \quad \delta_\alpha y = -\frac{\delta e}{2e} y. \]
\[ \delta_\alpha \zeta_\mu = 0, \quad \delta_\alpha p_\mu = 0, \quad (24, \text{cont’d}) \]
\[ \delta_\alpha \pi = 0, \quad \delta_\alpha n = -\frac{\delta_\alpha e}{2e} n. \]

From this point of view the simplest form of the \(\alpha\)- and \(\kappa\)-transformation rules is obtained by defining

\[ u = \frac{\sqrt{e}}{2} (y - h), \quad v = \frac{\sqrt{e}}{2} (y + h), \quad w = 2\sqrt{e} n, \quad (25) \]

whilst at the same time redefining the spinor parameter by

\[ \alpha' = \sqrt{e} \alpha. \quad (26) \]

This is allowed, because \(\alpha(\tau)\) is an arbitrary function of proper time. Then, dropping the prime on the parameter: \(\alpha' \to \alpha\), the complete spinor-symmetry transformation rules become

\[ \delta' e = 2 (\tilde{\phi} \kappa - \tilde{\theta} \bar{\kappa}) - 4i \bar{\alpha} v, \quad \delta' v = \bar{\phi} \alpha, \]
\[ \delta' \phi = \bar{\phi} \kappa, \quad (27) \]
\[ \delta' \theta = \bar{\kappa}, \quad \delta' x^\mu = \bar{\kappa} \gamma^\mu n, \]

with all other variables invariant: \(\delta'(\chi, \lambda_\mu, \zeta^\mu, p_\mu, \pi, u, w) = 0\). Comparison of the two representations of \(\kappa\)-symmetry in (12) for the scalar superparticle and (27) for the spinning superparticle are identical.

In terms of the new variables the action becomes

\[ S = \int d\tau \left( -\frac{e}{2} p_\mu^2 + p_\mu \left( \dot{x}^\mu + i \chi \lambda^\mu + \tilde{\phi} \gamma^\mu \theta + i \bar{\psi} \gamma^\mu \bar{\psi} - i \bar{\psi} x^\mu u \right) \right. \]
\[ \left. - \frac{i}{2} \tilde{\zeta}_\mu \hat{\zeta}^\mu + \frac{i}{2} \lambda_\mu \hat{\lambda}^\mu - i \bar{\omega} u + \bar{\pi} (\phi - \bar{\theta}) \right). \quad (28) \]

Of course, the above variable redefinitions do not respect the multiplet structure of world-line supersymmetry, and checking invariance of the new action under world-line supersymmetry and reparametrizations is more complicated.

Turning to the algebra of infinitesimal spinorial transformations (27), it is readily observed that they all commute:

\[ [\delta'(\kappa_2, \alpha_2), \delta'(\kappa_1, \alpha_1)] = 0. \quad (29) \]

For the \(\kappa\)-transformations this follows directly from the commutator algebra (13); for the \(\alpha\)-transformations it follows because commuting Majorana spinors \(\alpha_i\) satisfy the transposition rule \(\bar{\alpha}_1 \gamma^\mu \alpha_2 = \bar{\alpha}_2 \gamma^\mu \alpha_1\). Thus the extended algebra of
off-shell $\kappa$- and $\alpha$-transformations is again abelian for free spinning superparticles.

Although the commutator algebra (29) has been established for the transformations with parameter $\alpha$ redefined as in (26), the abelian character holds for the original version (24) as well. This is because in the original version the commutator of two $\alpha$-transformations is only modified by terms containing the commutator on the einbein:

$$[\delta'_{\alpha_2}, \delta'_{\alpha_1}] \propto \frac{1}{\sqrt{e}} [\delta_{\alpha_2}, \delta_{\alpha_1}] \sqrt{e},$$

which vanishes. This we have checked by an explicit calculation. However, the commutator of $\alpha$ and $\kappa$ transformations is modified in a non-trivial way, which can only be redressed by redefining the anti-commuting spinor parameters $\kappa$ as well.

In summary, we have found a phase-space version of spinorial $\kappa$ transformations for the superparticle, and a bosonically extended version for the spinning superparticle, with an off-shell abelian commutator algebra. The basic multiplets of local $\kappa$- and $\alpha$-transformations are $(e, \phi, \theta, p, \pi)$ and $(x^\mu, \pi)$ for $\kappa$ transformations, and $(e, v)$ for the $\alpha$-transformations. The transformation rules are given in (27), the corresponding action in (28).

Finally, we make some remarks on prospects for quantisation from the perspective of the present work. Although systems with reducible gauge example [10] for a recent discussion), our present off-shell phase space formulation may allow new approaches to covariant gauge fixing. For example for the simple (scalar) superparticle introduce anti-commuting scalar ghosts $(b, c)$ and lagrange multiplier $k$ for reparametrizations, and commuting spinor ghosts $(\beta, \gamma)$ with lagrange multiplier $\rho$ for $\kappa$-symmetry. The anti-commuting nilpotent variational BRST derivative $\delta$ then is, for the physical variables, formally given by (12) above with ghosts $c, \gamma$ in place of the parameters $\xi, \kappa$ of the infinitesimal diffeomorphisms and Siegel transformations respectively. For the ghost variables the non-zero BRST variations read:

$$\delta c = ec, \quad \delta \gamma = c \gamma, \quad \delta b = k, \quad \delta \beta = \rho,$$

for which nilpotence may be easily verified. If the action of diffeomorphisms and $\kappa$-supersymmetries on phase space $(x^\mu, p_\mu, \theta, \pi)$ is free for generic points$^1$, a suitable gauge fixing in the total space $(x^\mu, p_\mu, \theta, \pi, e, \phi)$ is induced [11] from that on the space of connections $(e, \phi)$. In these circumstances gauge classes such as $\dot{\phi} = \dot{e} = \text{const}$ may be admissible. Such conditions are natural concomitants of the full BRST-BFV formalism, in which connections with the representation theory of extended spacetime superalgebras have recently been reported [12]. Further work along these lines is in progress.

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$^1$On shell with $\kappa = \phi c'$, we have however $\delta_\kappa \theta = \delta_\kappa \pi = \delta_\kappa p = 0$ but $\delta_\kappa x^\mu \propto p'^\nu \kappa^\nu \pi$
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References


