The $\alpha_s^3$ approximation of Quantum Chromodynamics to the Ellis-Jaffe sum rule

S.A. Larin\textsuperscript{a}, T. van Ritbergen\textsuperscript{b}, J.A.M. Vermaseren\textsuperscript{c}

\textsuperscript{a} Theory Division, CERN, CH-1211, Geneva 23, Switzerland
and Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect 7a, Moscow 117312, Russia

\textsuperscript{b} Randall Laboratory of Physics, University of Michigan,
Ann Arbor, MI 48109, USA

\textsuperscript{c} NIKHEF, P.O. Box 41882,
1009 DB, Amsterdam, The Netherlands

Abstract

We present the analytical calculation in perturbative Quantum Chromodynamics of the $\alpha_s^3$ contribution to the Ellis-Jaffe sum rule for the structure function $g_1$ of polarized deep inelastic lepton-nucleon scattering.

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Measurements of the polarized nucleon structure function $g_1$ during the last 20 years have revealed an internal spin structure of the nucleon that is surprisingly different from constituent quark model expectations. The discovery of the disagreement between the combined EMC-SLAC data \cite{1,2} and the constituent quark model expectation \cite{3} \[ \int_0^1 dx g_1^p(x, Q^2) \approx 0.15|g_A| \] attracted a lot of attention and triggered intensive research in the field of polarized deep inelastic scattering. More recently, deuteron scattering data of the SMC \cite{4} and the E143 \cite{5} collaborations, and $^3$He scattering data of the E142 \cite{6} and E154 \cite{7} collaborations also allowed the determination of the neutron sum rule \[ \int_0^1 dx [g_1^n(x, Q^2) - g_1^p(x, Q^2)] \] at present the Bjorken sum rule is confirmed at the 8% level \cite{8} which is an important experimental test of Quantum Chromodynamics.

Higher order perturbative QCD corrections to the sum rules are crucial for an accurate and reliable confrontation of these sum rules with experimental data, see e.g. Ref. \cite{9}. For the Bjorken sum rule, the $\alpha_s$ correction \cite{11}, the $\alpha_s^2$ correction \cite{12}, and the $\alpha_s^3$ correction \cite{13} have been calculated in the leading twist approximation. Higher twist corrections have also been calculated \cite{14}. The Ellis-Jaffe sum rule \[ \int_0^1 dx g_1^{p/n}(x, Q^2) \] for the proton and neutron was calculated to order $\alpha_s$ \cite{15} and to order $\alpha_s^2$ \cite{16} in the leading twist approximation. Power corrections were calculated in \cite{17}.

In this article we obtain the order $\alpha_s^3$ contribution to the Ellis-Jaffe sum rule in the leading twist approximation for massless quarks. We perform the calculations using dimensional regularization \cite{18} in $D = 4 - 2\varepsilon$ space-time dimensions and use the standard modification of the minimal subtraction scheme \cite{19}, the MS-scheme \cite{20}.

Polarized deep inelastic electron-nucleon scattering is described by the hadronic tensor

\[ W_{\mu\nu} = \frac{1}{4\pi} \int d^4z e^{iqz} \langle p, s | J_\mu(z) J_\nu(0) | p, s \rangle \]

\[ = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{1}{p \cdot q} F_2(x, Q^2) + i \varepsilon_{\mu\rho\sigma\nu} q_\rho \left( \frac{s_\sigma}{p \cdot q} g_1(x, Q^2) + \frac{s_\sigma p \cdot q - p_\sigma q \cdot s}{(p \cdot q)^2} g_2(x, Q^2) \right) \] (1)

Here $J_\mu = \sum_{i=1}^{n_f} e_i \psi_i \gamma_\mu \bar{\psi}_i$ is the electromagnetic quark current where $e_i = 2/3, -1/3, -1/3, \cdots$ is the electromagnetic charge of a quark with the corresponding flavour. $x = Q^2/(2p \cdot q)$ is the Bjorken scaling variable and $Q^2 = -q^2$ is the square of the transferred momentum. $|p, s\rangle$ is the nucleon state that is normalized as $\langle p, s | p', s' \rangle = 2p^0(2\pi)^3\delta^{(3)}(p - p')\delta_{ss'}$. The polarization vector of the nucleon is expressed as $s_\sigma = \bar{U}(p, s) \gamma_\sigma \gamma_5 U(p, s)$ where $U(p, s)$ is the nucleon spinor $\bar{U}(p, s) U(p, s) = 2M$.

In the present article we will focus on the first Mellin moment of the structure function $g_1$. Moments of deep inelastic structure functions can be expressed \cite{21} (for reviews see Refs. \cite{22,23}) in terms of quantities that appear in the operator product expansion (OPE) of the two currents $J_\mu$. For the first moment of the structure function $g_1$ we need to consider the following expression for the OPE of two electromagnetic currents.
We use the standard notation $C_f$ for the considered sum rule. For the parton model with the parton distributions being the matrix elements of the corresponding operators, there is strictly no well-founded way to get the polarized gluon distribution into the parton model because it is not gauge invariant. Since the formalism of the OPE gives the strict theoretical basis to the parton model, the necessary quantum numbers, but it cannot contribute to the above OPE because it is not gauge invariant. Since the formalism of the OPE gives the strict theoretical basis to the parton model with the parton distributions being the matrix elements of the corresponding operators, there is strictly no well-founded way to get the polarized gluon distribution into the considered sum rule.

The Ellis-Jaffe sum-rule is expressed as

$$
\int_0^1 dx g_1^{p(n)}(x, Q^2) = \mathcal{C}_{\text{ns}}(1, a_s(Q^2))(\pm \frac{1}{12}|g_A| + \frac{1}{36}a_s) + C_{\gamma}(1, a_s(Q^2)) \exp\left(\int_{a_s(\mu^2)}^{a_s(Q^2)} \frac{da_s}{\log(a_s)}\right) \frac{1}{9}a_0(\mu^2)
$$

where the plus (minus) sign before $|g_A|$ corresponds to the proton (neutron) target. The proton matrix elements of the axial currents are defined as

$$
|g_A|s_{\sigma} = 2(p, s|J_{\sigma}^{5,3}|p, s) = (\Delta u - \Delta d)s_\sigma,
$$

$$
a_{ss_{\sigma}} = 2\sqrt{3}(p, s|J_{\sigma}^{5,8}|p, s) = (\Delta u + \Delta d - 2\Delta s)s_\sigma,
$$

$$
a_0(\mu^2)s_{\sigma} = (p, s|J_{\sigma}^0|p, s) = (\Delta u + \Delta d + \Delta s)s_\sigma = \Delta \Sigma(\mu^2)s_\sigma.
$$

Here $|g_A|$ is the absolute value of the constant of the neutron beta-decay, $g_A/g_V = -1.2601 \pm 0.0025$ [26]. $a_s = 0.579 \pm 0.025$ [26, 27] is the constant of hyperon decays. The matrix element

$$
i \int d\sigma^{q\bar{q}} T\{J_{\mu}(z)J_{\nu}(0)\} \overset{Q^2 \to \infty}{=} \epsilon_{\mu\nu\rho\sigma} \frac{q_\rho}{4} \left[ \sum_a C^a \left( \log\left(\frac{\mu^2}{Q^2}\right), a_s(\mu^2)\right) J_{\sigma}^{5,a}(0) + C^s \left( \log\left(\frac{\mu^2}{Q^2}\right), a_s(\mu^2)\right) J_{\sigma}^{5}(0) \right] + \cdots
$$
of the singlet axial current \( a_0(\mu^2) \) will be redefined in a proper invariant way as a constant \( a_0 \) below in Eq. (10).

We use the notation \( \Delta q(\mu^2) s_\sigma = \langle p, s | \gamma_\sigma \gamma_5 q | p, s \rangle \), \( q = u, d, s \), for the polarized quark distributions. We omit the contributions of the nucleon matrix elements for quarks heavier than the \( s \)-quark but it is straightforward to include them. We also avoid to introduce the contribution of the polarized gluon distribution \( \Delta g \) in the expression for the matrix element of the singlet axial current \( a_0(\mu^2) \) for the reason given below Eq. (2).

\( \beta(a_s) \) is the beta function that determines the renormalization scale dependence of the renormalized coupling constant. It is presently known at four loops \( \beta(a_s) = -\frac{\beta_0}{\alpha_s} - \frac{\beta_1}{\alpha_s^2} - \frac{\beta_2}{\alpha_s^3} - \frac{\beta_3}{\alpha_s^4} + O(\alpha_s^5) \)

with the SU(3) values

\[
\begin{align*}
\beta_0 &= 11 - \frac{2}{3} n_f \\
\beta_1 &= 102 - \frac{38}{3} n_f \\
\beta_2 &= \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{3} n_f^2 \\
\beta_3 &= \left( -\frac{149753}{6} + 3564\zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f \\
&\quad + \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3
\end{align*}
\]

in which \( \zeta \) is the Riemann zeta-function \((\zeta_3 \approx 1.202056903\ldots)\).

The singlet anomalous dimension \( \gamma_s \) that appears in Eq. (4) determines the renormalization scale dependence of the axial singlet current \( J_5^R \),

\[
\frac{d}{d \ln \mu^2} [J_5^R] = \gamma_s [J_5^R]_R
\]

where subscript \( R \) means that a current is renormalized.

The axial singlet current is not conserved – the axial anomaly \( \gamma_s \) – and this causes the singlet anomalous dimension to be non-zero starting from the order \( a_s^2 \)

\[
\gamma_s(a_s) = \gamma_1 a_s^2 + \gamma_2 a_s^3 + \gamma_3 a_s^4 + O(a_s^5).
\]

That is why \( a_0(\mu^2) \) in Eq. (4) depends on the renormalization point \( \mu^2 \) and it is therefore not a physical quantity.

The non-singlet axial current is conserved in the limit of massless quarks and the anomalous dimension for the non-singlet axial current therefore vanishes. This is why the non-singlet contribution to the Ellis-Jaffe sum rule Eq. (4) does not involve an exponential factor and that is why the non-singlet matrix elements \( g_A \) and \( a_8 \) are renormalization group invariant \((\mu^2 \text{ independent})\).

The non-singlet contribution to the Ellis-Jaffe sum rule is known in the order \( a_s^3 \) from [13] where the polarized Bjorken sum rule \( \int_0^1 dx (g_1^p - g_1^n) \) was calculated in this order. In this article we obtain the order \( a_s^3 \) contribution to the singlet part of the Ellis Jaffe sum rule:
\[
C^s(1, a_s(Q^2)) \exp \left[ \int_{a_s(\mu^2)}^{a_s(Q^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right] \frac{1}{9} a_0(\mu^2)
\]

\[
= C^s(1, a_s(Q^2)) \left[ 1 + a_s(Q^2) - \frac{\gamma_1}{\beta_0} + \left( a_s(Q^2) \right)^2 - \frac{\gamma_2}{\beta_0} + \frac{\gamma_1 \beta_1 + \gamma_3^2}{2 \beta_0^2} \right]
\]

\[
+ \left( a_s(Q^2) \right)^3 - \frac{2 \gamma_3 \beta_0^2 + 2 \gamma_1 \beta_2 \beta_0 + 2 \beta_1 \gamma_2 \beta_0 - 2 \gamma_1 \beta_1^2 + 3 \gamma_1 \gamma_2 \beta_0 - 3 \gamma_1^2 \beta_1 - \gamma_3^3}{6 \beta_0^4} \right] \frac{1}{9} \hat{a}_0
\]

where we introduce the notation

\[
\hat{a}_0 = \exp \left( - \int_{a_s(\mu^2)}^{a_s(Q^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right) a_0(\mu^2)
\]

for the renormalization group invariant (i.e. \( \mu^2 \) independent) nucleon matrix element of the singlet axial current. Since \( \hat{a}_0 \) is the renormalization group invariant it should be considered as a physical constant on the same ground as the constants \( g_A \) and \( a_s \). That is why from now on we will only use the notation \( \hat{a}_0 \) for the matrix element of the singlet axial current in the expression for the sum rule.

To obtain this singlet contribution, we need to calculate the 3-loop contribution to the singlet coefficient function \( C^s \) and the 4-loop contribution to the anomalous dimension \( \gamma^s \). For the treatment of the \( \gamma_5 \) matrix in dimensional regularization we use a technique described in [18] which is based on the original definition of \( \gamma_5 \) in [18].

In the \( \overline{\text{MS}} \) scheme the proper normalization of the axial singlet current requires the introduction of a finite renormalization constant \( Z_{\text{MS}}^5 \) in addition to the standard ultraviolet renormalization constant\(^1 \) \( Z_{\text{MS}}^5 \) that contains only poles in the regularization parameter \( \varepsilon \)

\[
[J^5_\mu]_R = Z_{\text{MS}}^5 Z_{\text{MS}}^8 [J^5_\mu]_B = Z_{\text{MS}}^5 Z_{\text{MS}}^8 \frac{i}{5} \epsilon_{\mu \rho \sigma \tau} \overline{B} \gamma_\rho \gamma_\sigma \gamma_5 \psi_B
\]

where \( B \) denotes the bare, unrenormalized quantity.

This extra renormalization constant is introduced to keep the exact 1-loop Adler-Bardeen form [31] for the operator anomaly equation \( \partial_\mu j_\mu^5 = a_s \frac{n_L^2}{2} \epsilon_{\mu \rho \lambda \nu} G^a_{\mu \rho} G^a_{\lambda \nu} \) within dimensional regularization. Here \( G^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig f^{abc} A^b_\mu A^c_\nu \) is the QCD field strength tensor.

This finite renormalization constant \( Z_{\text{MS}}^5 \) affects the results for the coefficient function and anomalous dimension but the sum rule, i.e. the combination in the r.h.s. of Eq. (4) is independent of the choice of \( Z_{\text{MS}}^5 \). This fact is evident since the sum rule as a physical object can not depend of the choice of the normalization of a non-physical object such as a flavour singlet axial current. More precisely, \( Z_{\text{MS}}^5 \) enters in the coefficient function and anomalous dimension as

\[
C^s = \frac{C^s}{Z_{\text{MS}}^5},
\]

\[
\gamma^s = \gamma^s + \beta(a_s) \frac{d}{da_s} \log(Z_{\text{MS}}^5)
\]

\(^1\) Please note that ultraviolet renormalization constants coincide in the MS and \( \overline{\text{MS}} \) schemes and we can therefore use the notation \( Z_{\text{MS}} \) instead of \( Z_{\text{MS}}^5 \).
where $\overline{C}^s$ and $\overline{\gamma}^s$ are calculated in the $\overline{\text{MS}}$ scheme without a factor $Z_5^s$ (i.e., with $Z_5^s = 1$). Please notice that $Z_5^s$ enters the anomalous dimension only at the $a_s^2$ order, since $\beta(a_s)$ starts from $a_s^2$. Presently the singlet constant $Z_5^s$ is unknown in the order $a_s^3$. We will therefore obtain $\overline{C}^s$ and $\overline{\gamma}^s$ which is sufficient to obtain the sum rule.

To obtain the 3-loop coefficient function $\overline{C}^s$ we apply the method of projectors \cite{32} which gives us the following formula

$$\overline{C}^s = \frac{1}{24} \frac{1}{Z_{s}^{\overline{\text{MS}}}} R_{\overline{\text{MS}}} \int dze^{iqz} \langle 0|T\{\bar{\psi}(p)\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\psi(q)p_{\mu}\psi(0)J_\mu(z)\}J_\nu(0)|0\rangle|_{p=0}$$

where $\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\psi = \gamma_{\mu}\gamma_{\nu}q^\rho - \delta_{\mu\nu} q^\rho + q^\mu \gamma_{\nu} - q^\nu \gamma_{\mu}$. The $R_{\overline{\text{MS}}}$ operation performs the renormalization of the ultraviolet divergences. The infrared divergences that are produced by putting $p = 0$ are removed by the ultraviolet renormalization factor of the axial singlet current $Z_{s}^{\overline{\text{MS}}}$. In this way we need to evaluate 3-loop massless propagator diagrams which is done with the package MINCER written for the symbolic manipulation program FORM \cite{33}. The set of 3-loop flavour singlet diagrams that contribute to this coefficient function is identical to the set "$q_7 q_7 q_7$" that previously appeared in Ref. \cite{34}. In this way we obtained the following result for $\overline{C}^s$:

$$\overline{C}^s = 1 + a_s C_F(-7) + a_s^2 \left[ C_A C_F \left( -\frac{314}{9} \right) + C_F n_f T_F \left( \frac{109}{9} \right) + 16 \zeta_3 + C_F^2 \left( \frac{89}{2} \right) \right] + a_s^3 \left[ C_A C_F n_f T_F \left( \frac{4124}{81} \right) + 1124 \frac{9}{9} \zeta_3 + C_A C_F^2 \left( \frac{13004}{27} \right) - \frac{656}{3} \zeta_3 \right] + C_A^2 C_F \left( -\frac{13021}{27} \right) + 56 \zeta_3 + C_F n_f T_F^2 \left( -\frac{460}{27} \right) - \frac{320 \zeta_3}{9} + C_F^3 \left( -\frac{1397}{6} \right) + 96 \zeta_3 \right] \tag{14}$$

where $C_F = 4/3$ and $C_A = 3$ are the quadratic Casimir operators of the fundamental and adjoint representation of the colour group $SU(3)$, $T_F = 1/2$ is the trace normalization of the fundamental representation and $n_f$ is the number of (active) quark flavours. The Riemann zeta function is written as $\zeta_n$. It is interesting to mention that diagrams with one external photon in a closed quark loop have colour factors proportional to the cubic Casimir operator $d_{abc} d_{abc}$. Individually these diagrams are non-zero but this higher group invariant cancels in the sum.

To obtain the 4-loop contribution to the anomalous dimension $\overline{\gamma}^s$ we need to calculate the $\overline{\text{MS}}$ renormalization factor $Z_{s,n}^{\overline{\text{MS}}}$ of Eq. \cite{11} (and of Eq. \cite{13} where the lower order $Z_{s,n}^{\overline{\text{MS}}}$ appeared) which contains apart from a leading constant 1 only poles in the regularization parameter $\varepsilon$,

$$Z_{s,n}^{\overline{\text{MS}}}(a_s, \varepsilon) = 1 + Z_{s,n}^{\overline{\text{MS}}(1)}(a_s)/\varepsilon + \ldots$$

The anomalous dimension $\overline{\gamma}^s$ is then expressed through the coefficient in front of the first pole in $Z_{s,n}^{\overline{\text{MS}}}$

$$\overline{\gamma}^s = -a_s \left( \frac{\partial}{\partial a_s} Z_{s,n}^{\overline{\text{MS}}(1)}(a_s) \right) \tag{16}$$
To obtain the renormalization factor \( Z_{\text{MS}} \) we calculated the overall ultraviolet divergence of the Green function

\[
G_{\bar{\psi}[J_\mu]\psi} = \int dxdye^{iqx+i\eta y} \langle 0 | T \{ \bar{\psi}(x)J_\mu(y)\psi(0) \} | 0 \rangle
\]

(17)

This Green function is renormalized multiplicatively and the standard \( \overline{\text{MS}} \) renormalization factor (containing only poles in \( \varepsilon \)) is equal to \( Z_{\text{MS}}^a / Z_2 \), where \( Z_2 \) is the renormalization factor of the inverted quark propagator.

To obtain the required anomalous dimension \( \overline{\gamma} \) in the \( a_s^4 \) order we calculated the overall divergences of both the quark propagator and the Green function \( G_{\bar{\psi}[J_\mu]\psi} \) in the 4-loop order. This can be conveniently done using the technique that is described in Ref. [28]. This general technique is based on the direct calculation of 4-loop massive vacuum (bubble) integrals and provides a procedure that is well suited for the automatic evaluation of huge numbers of Feynman diagrams. For the present calculation we needed to evaluate of the order of 10000 4-loop diagrams. These diagrams were generated with the program QGRAF [35].

\[
\overline{\gamma} = a_s^2 \left[ C_A C_F \left( -\frac{44}{3} \right) + C_F n_f T_F \left( -\frac{20}{3} \right) \right] + a_s^3 \left[ C_A C_F^2 \left( \frac{308}{3} \right) + C_A^2 C_F \left( -\frac{3578}{27} \right) + C_A C_F n_f T_F \left( -\frac{298}{27} \right) + C_F^2 n_f T_F \left( \frac{44}{3} \right) + C_F n_f^2 T_F^2 \left( -\frac{104}{27} \right) \right] + a_s^4 \left[ C_A C_F^3 \left( -\frac{1870}{3} + 1056 \zeta_3 \right) + C_A^2 C_F^2 \left( \frac{58618}{27} - 1760 \zeta_3 \right) + C_A C_F \left( -\frac{36607}{27} + 616 \zeta_3 \right) + C_A C_F n_f T_F \left( -\frac{3794}{27} + \frac{368}{3} \zeta_3 \right) + C_F n_f T_F \left( -\frac{15593}{81} + \frac{1748}{3} \zeta_3 \right) + C_F n_f^2 T_F^2 \left( -\frac{58}{3} - 384 \zeta_3 \right) + C_F^2 n_f T_F \left( -\frac{6808}{27} - \frac{896}{3} \zeta_3 \right) + C_A C_F n_f^2 T_F^2 \left( \frac{496}{81} + \frac{848}{3} \zeta_3 \right) + C_F n_f^3 T_F^3 \left( \frac{560}{81} \right) \right]
\]

(18)

The results of Eqs. (14,18) are obtained in an arbitrary covariant gauge for the gluon field. This means that we keep the gauge parameter \( \xi \) that appears in the gluon propagator \( i [ -g^{\mu\nu} + (1 - \xi) q^\mu q^\nu / (q^2 + i\epsilon) ] / (q^2 + i\epsilon) \) as a free parameter in the calculations. The explicit cancellation of the gauge dependence in the coefficient function and the anomalous dimension gives an important check of the results. The results for individual diagrams that contribute to \( \overline{C_\psi} \) and \( \overline{\gamma} \) also contain (apart from the constant \( \zeta_3 \)) the constants \( \zeta_4, \zeta_5 \). The cancellation of these constants at various stages in the calculation provides additional checks of the results. We should also note that the various higher order colour factors [see Ref. [28]] that appear in the separate results for \( Z_2 \) and \( (Z_{\text{MS}}^a / Z_2) \) all canceled in Eq. (18).

At this point it is interesting to compare the obtained result for \( \overline{\gamma} \) with the analogously defined non-singlet anomalous dimension \( \overline{\gamma}^{\text{ns}} \) since this can be constructed at 4-loops from
known 3-loop results. Since the non-singlet anomalous dimension \( \gamma_{ns} \) vanishes, we have [see, e.g. Eq. (12)] \( \gamma_{ns} = -\beta(a_s) \frac{d}{da_s} \log(Z_{n}^{ns}) \) where \( Z_{n}^{ns} \) is the finite renormalization constant for the non-singlet axial current which is known in the order \( a_s^3 \). Since the difference between the singlet and non-singlet sector is in terms of at least one power of \( n_f \) we must have that all the \( n_f \)-independent terms in \( \gamma_{ns} \) and \( \gamma \) coincide, and indeed they do. This gives another strong check to the calculations.

Substitution of the obtained anomalous dimension and coefficient function in Eq. (3) gives the following result for the Ellis-Jaffe sum rule.

\[
\int_0^1 dx g_1^{p(n)}(x, Q^2) = \left[ 1 + \left( \frac{\alpha_s}{\pi} \right) d_1^{ns} + \left( \frac{\alpha_s}{\pi} \right)^2 d_2^{ns} + \left( \frac{\alpha_s}{\pi} \right)^3 d_3^{ns} \right] \left( \pm \frac{1}{12} |g_A| + \frac{1}{36} a_8 \right)
\]

\[
+ \left[ 1 + \left( \frac{\alpha_s}{\pi} \right) d_1^{s} + \left( \frac{\alpha_s}{\pi} \right)^2 d_2^{s} + \left( \frac{\alpha_s}{\pi} \right)^3 d_3^{s} \right] \frac{1}{9} \hat{a}_0
\]

\[
d_1^{ns} = -1
\]
\[
d_2^{ns} = -\frac{55}{12} + \frac{1}{3} n_f
\]
\[
d_3^{ns} = \left( -\frac{1344}{216} - \frac{44}{9} \zeta_3 + \frac{55}{2} \zeta_5 \right) + n_f \left( \frac{10339}{1296} + \frac{61}{36} \zeta_3 - \frac{5}{3} \zeta_5 \right) + n_f^2 \left( \frac{115}{648} \right)
\]
\[
d_1^{s} = \left( 1/\beta_0 \right) \left[ -11 + n_f (\hat{S}) \right]
\]
\[
d_2^{s} = \left( 1/\beta_0 \right)^2 \left[ -\frac{6655}{12} + n_f \left( \frac{235}{2} + \frac{142}{3} \zeta_3 \right) + n_f^2 \left( -\frac{55}{18} - \frac{88}{9} \zeta_3 \right) + n_f^3 \left( \frac{16}{81} + \frac{8}{27} \zeta_3 \right) \right]
\]
\[
d_3^{s} = \left( 1/\beta_0 \right)^3 \left[ \left( -\frac{1842371}{216} - \frac{58564}{9} \zeta_3 + \frac{73205}{2} \zeta_5 \right) + n_f \left( \frac{4631373}{1296} + \frac{312785}{54} \zeta_3 - \frac{113135}{9} \zeta_5 \right) + n_f^2 \left( \frac{2353243}{1440} - \frac{20976}{27} \zeta_3 + \frac{13310}{9} \zeta_5 \right) + n_f^3 \left( \frac{4647815}{14064} + \frac{22594}{243} \zeta_3 - \frac{220}{3} \zeta_5 \right) \right]
\]

where \( \alpha_s = \frac{\alpha_s(Q^2)}{3} = 4\pi a_s(Q^2), \beta_0 = 11 - 2/3n_f \) is the 1-loop coefficient of the beta function and \( \hat{a}_0 \) is the invariant matrix element of the singlet axial current defined in Eq. (1).

In particular, for \( n_f = 3 \) we find

\[
\int_0^1 dx g_1^{p(n)}(x, Q^2) = \left[ 1 - 3.5833 \left( \frac{\alpha_s}{\pi} \right)^2 - 20.2153 \left( \frac{\alpha_s}{\pi} \right)^3 \right] \left( \pm \frac{1}{12} |g_A| + \frac{1}{36} a_8 \right)
\]

\[
+ \left[ 1 - 0.33333 \left( \frac{\alpha_s}{\pi} \right) - 0.54959 \left( \frac{\alpha_s}{\pi} \right)^2 - 4.44725 \left( \frac{\alpha_s}{\pi} \right)^3 \right] \frac{1}{9} \hat{a}_0
\]

It is interesting to compare our result Eq. (19) with a recent estimate [34] of the singlet coefficient \( d_3^s \) for \( n_f = 3 \). The estimate gives the value -2 which is slightly less than half the value obtained in the present article.

In table 1 we have listed the numerical values of the second and third-order coefficients for the Ellis-Jaffe sum rule for \( n_f = 3, 4, 5, 6 \). One can observe the sign-constant character of perturbative QCD series both for non-singlet and singlet contributions. The series tends to preserve its sign-constant character even when perturbative coefficients of the singlet contribution change their signs around the value \( n_f = 4 \).
Table 1. Second and third-order coefficients for the Ellis-Jaffe sum rule.

Another deep inelastic sum rule, the Bjorken sum rule for neutrino-nucleon scattering, which is also known in the $\alpha_3$-order [3], also exhibits the sign-constant behaviour of the perturbative QCD series. One can see that the obtained perturbative coefficients of the Ellis-Jaffe sum rule grow rather moderately. If we assume that the error of the truncated asymptotic series is determined by the last calculated term, then the obtained $\alpha_3$ approximation for this sum rule provides a good theoretical framework for extraction of the fundamental constant $\hat{a}_0$ from experiment.

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References


