Light Fermion Mass Effects in
\( e^+ e^- \rightarrow 4 \) Fermions

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We investigated the effect of the light fermion masses on cross sections for \( e^+ e^- \rightarrow 4 \) fermions in the Fermion Loop scheme defined in Ref. [1], and approximations to it. The effects are found to be very small, except of course in the collinear region of single \( W \) boson production where the electron mass acts as the cut-off.

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1 Introduction

Properties of the processes $e^+e^- \rightarrow 4$ fermions, which are studied at LEP-2 and higher energy colliders [2], should be computed using a careful treatment of the finite width of the virtual $W$ and $Z$ bosons to obtain correct results [3]. In Refs [4, 1] a scheme was proposed that offered a solution for the fermionic contributions to these finite width effects. In this ‘Fermion Loop scheme’ all fermionic one-loop corrections to the massive boson propagators and three-vector-boson vertices are summed in the massless limit (except for the top quarks). This way strict gauge-invariance is maintained, i.e., the resulting amplitudes satisfy all relevant Ward Identities. As a test-case of this scheme, the process $e^+e^- \rightarrow u\bar{d}e^-\bar{\nu}_e$ (CC20) was studied numerically. It was shown to behave properly in the two critical regions: the collinear region $q_\gamma^2 = (p_e - p'_e)^2 \rightarrow 0$ and the high-energy limit $s \rightarrow \infty$. In the first region even a small violation of U(1) current conservation can give results which are wrong by a factor $\mathcal{O}(m_W\Gamma_W/m_e^2) \approx 10^5$; in the second the well-known SU(2) gauge cancellation fails when SU(2) gauge invariance is broken [4].

In this letter we study the effects of the inclusion of light fermion masses in the fermion loop scheme for the CC20 process mentioned above. The electron mass was taken into account in the partial calculation [4] in order to be able to take the collinear limit, but not in the full results [1]. Throughout, however, we will neglect terms of order $m^2_f/m_W^2$ and $m^2_f/s$, as these are much smaller than the required precision. In principle the inclusion of fermion masses in the Ward Identities is straightforward but tedious. The omission in the expressions used for numerical implementation of some mass terms leads to small violations of these identities; we investigate whether these have non-negligible effects. Finally we study whether the physical effects of the light fermion masses are observable, apart from the obvious case of the electron mass in the forward region, where it is needed as a cut-off.

We thus follow the introduction of light fermion masses through the renormalization scheme, the running couplings and the cross section of the prototypical process $e^+e^- \rightarrow u\bar{d}e^-\bar{\nu}_e$.

2 Renormalization

To include the fermion loops to all orders in a compact way, it is useful to work with the complex renormalization scheme [5], extended to the electroweak sector of the Standard Model. It is straightforward to evaluate it for

\footnote{Note that the converse is not true: the gauge-breaking fixed-width scheme is also properly behaved for $2 \rightarrow 4$ fermion processes.}
non-zero light fermion masses, as all formulae in Ref. [1] are already given for the massive case. The results are listed in Table [1] as expected the deviations are $\mathcal{O}(m^2_f/m_W^2)$ and hence negligible. However, at this stage one cannot discard these yet as it might be possible that cancellations in the amplitude will be spoiled without them.

The next place where the fermion masses can play a role is in the running couplings defined in the fermion loop scheme. Below the light fermion production threshold the slope is changed in accordance with the $\beta$-function, as shown in Fig. [1]. One expects that the deviations from the massless result (except for the top) given in Ref. [1] can lead to a different behaviour of the matrix element in the collinear (small-$q^2$) region.

From the figure we see that even at $\sqrt{p^2}$ as low as 1 GeV the effects of the light fermion masses on the running couplings is less than 1%, and increasing only logarithmically$^2$. As the single-$W$ cross section, which probes $e^2(p^2)$ for lower values, is suppressed relative to the $WW$ cross section by a factor $\mathcal{O}(\Gamma_W/m_W) \log(m_e^2/\text{GeV}) \approx 1/10$, the effect on the total cross section will be small.

$^2$Note that a fit to a collection of straight lines would give rise to effective masses that are a factor 1.5 higher than the physical masses.
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Table 1: Values of the effective top quark mass, pole positions (in GeV) and effective couplings at $m_W^2$ for three different $W$ masses, with massive and massless fermions (except top).
3 Matrix element

A massive tree level matrix element for the reaction $e^+e^- \rightarrow u\bar{d}e^-\bar{\nu}_e$ was generated using MadGraph [3]. This matrix element is formulated in the unitary gauge, so Higgs ghosts do not contribute. However, it neglects the Higgs boson exchanges, which are suppressed by the electron mass squared. This matrix element originally used a gauge-invariance breaking resummation scheme for the vector boson propagators, and hence the collinear and high-energy behaviour were wrong. We modified it to use the massive version of the fermion loop scheme described below to restore the correct collinear and high-energy behaviour. We also added the possibility to compare it against the ad-hoc schemes defined in Ref. [1]: naive running width, fixed width, and high-energy behaviour. We also added the possibility to compare it against the fermion loop scheme described below to restore the correct collinear and high-energy behaviour. We modified it to use the massive version of the fermion loop scheme for the vector boson propagators, and hence the collinear and high-energy behaviour were wrong. We modified it to use the massive version of the fermion loop scheme described below to restore the correct collinear and high-energy behaviour. We also added the possibility to compare it against the ad-hoc schemes defined in Ref. [1]: naive running width, fixed width, and the approximation of the fermion loop scheme valid for $q_r^2 \rightarrow 0$ (which for the propagators just uses the running width).  

In the latter scheme we add to the $WW\gamma$ vertex [4] an overall factor $1 + i\Gamma_W/m_W p_2^\mu/(p_2^\mu - p_-^\mu)$, with $p_+$ the momentum of the $s$-channel $W^+$ and $p_-$ of the $t$-channel $W$ boson. This minimal extension restores $U(1)$ current conservation in the limit $q_r^2 \rightarrow 0$, which is sufficiently accurate for LEP2 energies. In case of the full fermion loop scheme we implemented the massive fermion loops as given in Appendix B of Ref. [1], with one simplification: we assumed that the currents to which the triangles are coupled are conserved. In this case we get the following expression for the tree-level and one-loop $W(p_+^\mu)W(p_-^\mu)V(q^\mu)$ vertex

$$V_0^{\kappa\mu\lambda} = 2\delta^{\mu\kappa}q^\lambda - 2\delta^{\mu\lambda}q^\kappa - 2\delta^{\kappa\lambda}p_-^\mu$$

$$V_1^{\kappa\mu\lambda} = N_C \frac{e(q^2)g_w(p_2^\mu)g_w(p_2^\mu)}{32\pi^2} \left\{ F_V^{(1)}[p_-^\mu q^\kappa q^\lambda] 16(-C_{22} + C_{23} + C_{33} - C_{34}) 
\quad + \delta^{\mu\kappa}q^\lambda \{ -B_0(q^2) - B_0(p_+^2) - q^2C_{12} - p_+^2C_{12} - p_-^2(C_0 + C_{12}) 
\quad - m_f^2C_0 - m_f^2C_0 - 8C_{36} \} 
\quad + \delta^{\mu\lambda}q^\kappa \{ B_0(q^2) + B_0(p_+^2) - q^2C_0 - q^2C_{11} + p_+^2C_{11} - p_-^2(C_0 + C_{11}) 
\quad + m_f^2C_0 + m_f^2C_0 - 8C_{24} - 8C_{35} \} 
\quad + \delta^{\kappa\lambda}p_-^\mu \{ B_0(p_+^2) + B_0(p_2^2) + q^2(-C_0 - C_{11} + C_{12}) 
\quad + (p_+^2 + p_-^2 - 2(m_f^2 + m_f^2))(C_{11} - C_{12}) + 8C_{35} - 8C_{36} \} \right\} +$$

$^3$There is one slight difference in definitions: in the fixed-width scheme we take the pole position as $\mu_W = m_W^2 - \Gamma_W^2 - im_W\Gamma_W(1 - \Gamma_W^2/m_W^2)$ to compensate for trivial changes due to the change in definition of the $W$ mass, which is by convention defined in a running-width scheme.


\[ F_V^{(2)}[\delta^{\mu\nu}q^\lambda 2C_{12} + \delta^{\mu\lambda}q^\kappa 2(C_0 + C_{11}) + \delta^{\kappa\lambda}p^\mu 2(C_0 + C_{11} - C_{12})] + \\
F_V^{(3)}[\epsilon^{\alpha\mu\nu\lambda}p_{-\alpha} 8iq^2(-C_{12} - C_{23}) + \epsilon^{\alpha\mu\nu\lambda}q_{\alpha} 2i(-2q^2(C_{12} + C_{23}) - p^2_C12) \\
- (p^2_\perp + m_f^2 - m_W^2)(C_0 + C_{11}) + (m_W^2 - m_f^2)C_{12})] + \\
F_V^{(4)}[\epsilon^{\alpha\mu\nu\lambda}p_{-\alpha} 4i(-C_0 - C_{11} + C_{12}) + \epsilon^{\alpha\mu\nu\lambda}q_{\alpha} 4iC_{12}] \] 

(2)

with the tensor functions \( C_x(p^2_\perp, p^2_\parallel, q^2) \) now in the notation of Ref. [7]. The coefficients are \( F_V^{(i)} \) given by

\[
\begin{align*}
F_\gamma^{(1)} &= -|Q_f| \\
F_\gamma^{(2)} &= 0 \\
F_\gamma^{(3)} &= -Q_f \\
F_\gamma^{(4)} &= 0 \\
F_Z^{(1)} &= |Q_f| \frac{c_w}{s_w} + \frac{1 - 2|Q_f|}{2c_w s_w} \\
F_Z^{(2)} &= |Q_f| \frac{c_w}{s_w} \\
F_Z^{(3)} &= \frac{m_f^2}{c_w s_w} \\
F_Z^{(4)} &= \frac{m_f^2 I_3}{c_w s_w}
\end{align*}
\]

(3)

with \( Q_f \) the charge of fermion \( f \), \( m_f \) its mass, and \( c_w, s_w \) the cosine and sine of the weak mixing angle.

Although the current attached to the photon propagator is strictly conserved, the ones attached to the massive vector bosons are not. The missing terms are of order \( m_f^2 \), so they are negligible if we can show that they are not scaled by a small factor (like the naive \( O(\Gamma_W) \) gauge violation that becomes \( \Gamma_W m_W/m_e^2 \) in the collinear region). In case of the \( W \) propagators it is clear that this cannot be the case, as no cancellation is associated with these currents, either at small or large \( p_\perp^2 \). However, there is no \textit{a priori} reason why this would hold for the \( Z \) boson in the high-energy limit, due to the SU(2) gauge cancellation. Numerically, we checked \textit{a posteriori} that the cancellation between the three \( W \)-pair production graphs in the high-energy limit is not upset by ignoring these fermion mass terms, or by the omission of the missing Higgs-exchange terms. All disregarded terms are thus \( O(m_f^2/m_W^2) \) and can safely be neglected. The cancellation is not affected by the mass terms in the renormalization either, which means that the renormalization (Table 4) could also be performed massless.

In order to speed up the matrix element we only call the amplitudes in which the electron changes helicity when \( q_\perp^2 < -1000m_e^2 \) and cache the calls to the tensor functions [8] that will reappear for each helicity. With these precautions one can integrate \( 10^5 \) weighted points in 6 hours on a cluster of 7 Sun workstations.
Using the matrix element described in the previous section we study the total cross section for the reaction \(e^+e^- \rightarrow u\bar{d}\bar{e} \bar{\nu}_e\), at a LEP2 energy (including the forward region \(q^2_{\gamma} \rightarrow 0\) that is sensitive to U(1) current conservation) and at very high energies \(s \rightarrow \infty\). In order to keep the results simple we choose not to include initial-state radiation. First it is shown in Fig. 2 that the limit \(q^2_{\gamma} \rightarrow 0\) is well-behaved up to the kinematical limit in all schemes except the naive running width. In fact, the differences between the other schemes are not significant in this plot.

The total cross section at 175 GeV, with an angle cut on the outgoing electron, is shown in Table 2. Except for the total cross-section without cuts, where the electron mass acts as a cut-off, there is no measurable difference between the massive and massless cross sections. Both the fixed-width scheme (with the shifted pole) and the imaginary part of the fermion loops in the limit \(q^2 \rightarrow 0\) (a simple multiplicative factor) are seen to give results which are in good agreement. The slightly higher values of the full fermion scheme result from the inclusion of part of the higher order corrections; the bosonic corrections would likely lower this again [1].

Finally, we show that the inclusion of mass terms does not change the high-energy behaviour. As we are interested in the SU(2) cancellations in \(W\)
Running width & 67524(135) & 1.4264(32) & 0.6332(7) & 0.5918(6) \\
Fixed width & 0.6978(10) & 0.6464(7) & 0.6222(7) & 0.5924(6) \\
Imaginary part for $q^2 \to 0$ & 0.6984(12) & 0.6456(7) & 0.6214(7) & 0.5916(6) \\
Full fermion loops, $m_f = 0$ & ill-defined & 0.6527(14) & 0.6301(8) & 0.5988(7) \\
Full fermion loops, $m_f > 0$ & 0.7038(15) & 0.6514(9) & 0.6298(7) & 0.5992(7) \\

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<th>1°</th>
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Table 2: Total cross sections in pb at $\sqrt{s} = 175$ GeV, without initial state radiation or cuts except between the outgoing electron and either beam.

pair production, rather than single-$W$ production, we impose a cut on the outgoing electron angle of 1° and 10°. The results are shown in Fig. 3, which is entirely comparable to the values in Table 5 in Ref. [1].

5 Conclusion

We have shown how one can incorporate the effects of fermion masses into the fermion loop scheme that consistently resums the $W$ and $Z$-boson propagators, up to terms of order $m_f^2/m_W^2$. Although we loose the rigourous proof of the Ward identities derived in Ref. [1], we show that this does not cause problems. Only in the extreme forward region of phase space $q^2 \approx -m_e^2$ does one notice any effect of the inclusion of fermion mass terms. The effects of the slight shift in the renormalization scheme and the differently running couplings are negligible. We give the total cross section without any cuts of the process $e^+e^- \to u\bar{d}e^-\bar{\nu}_e$ in this scheme.

Acknowledgements We would like to think the rest of the BHF collaboration for their comments.

References


Figure 3: Cross section for the process $e^+e^- \rightarrow u\bar{d}e^-\bar{\nu}_e$ with angular cuts on the outgoing electrons only: $1^\circ$ (upper curves) and $10^\circ$ (lower curves).


