The 3-loop QCD calculation of the moments of deep inelastic structure functions

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Abstract

We present the analytic next-to-next-to-leading perturbative QCD corrections in the leading twist approximation for the moments \( N = 2, 4, 6, 8 \) of the flavour singlet deep inelastic structure functions \( F_2 \) and \( F_L \). We calculate the three-loop anomalous dimensions of the corresponding singlet operators and the three-loop coefficient functions of the structure functions \( F_L \) and \( F_2 \). In addition, we obtained the 10\textsuperscript{th} moment for the non-singlet structure functions in the same order of perturbative QCD. We perform an analysis of the obtained results.

1 Introduction

The calculation of the next-to-next-to-leading (NNL) QCD approximation for the structure functions \( F_2 \) and \( F_L \) of deep inelastic electron-nucleon scattering is important for the understanding of perturbative QCD and for an accurate comparison of perturbative QCD with experiment. To obtain the NNL approximation for these structure functions in the operator product expansion (OPE) formalism one needs the 3-loop anomalous dimensions of the operators, the 2-loop Wilson coefficient functions for \( F_2 \) and the 3-loop coefficient functions for \( F_L \). At present, these structure functions are known in the next-to-leading approximation only, since the 3-loop anomalous dimensions and the 3-loop coefficient functions for \( F_L \) were not calculated yet.

The 1-loop anomalous dimensions were calculated in Ref. \cite{1}. The complete 1-loop coefficient functions were obtained in Ref. \cite{2} (see also the references therein). Anomalous dimensions in 2-loop order were obtained in Refs. \cite{3}-\cite{5} and the 2-loop coefficient functions were calculated in Refs. \cite{6}-\cite{11}. In a previous paper \cite{12}, we presented the NNL corrections of the non-singlet type

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in the leading twist approximation for the moments $N = 2, 4, 6, 8$ of the deep inelastic structure functions $F_2$ and $F_L$.

In the present paper we calculate the NNL QCD corrections to the singlet moments $N = 2, 4, 6, 8$ of both structure functions $F_2$ and $F_L$. To this end, we calculate the corresponding 3-loop anomalous dimensions and the 3-loop coefficient functions for the structure function $F_L$. In addition, we present the 3-loop coefficient functions for the structure function $F_2$ for $N = 2, 4, 6, 8$. We also obtain the $N=10$ non-singlet moments of $F_2$ and $F_L$. The calculations are done for the leading twist approximation for zero quark masses.

In the fifth part of this paper we analyze the effects that the calculated 3-loop corrections have on the structure functions.

2 Formalism

We need to calculate the hadronic part of the amplitude for unpolarized deep inelastic electron-nucleon scattering which is given by the hadronic tensor

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi} \int d^4z e^{iqz} \langle p,\text{nucl}|J_\mu(z)J_\nu(0)|\text{nucl},p \rangle$$

$$= e_{\mu\nu} \frac{1}{2x} F_L(x,Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x,Q^2)$$

$$e_{\mu\nu} = (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})$$

$$d_{\mu\nu} = \left( -g_{\mu\nu} - p_\mu p_\nu \frac{4x^2}{q^2} - (p_\mu q_\nu + p_\nu q_\mu) \frac{2x}{q^2} \right),$$

where $J_\mu$ is the electromagnetic quark current, $x = Q^2/(2p \cdot q)$ is the Bjorken scaling variable $(0 < x \leq 1)$, $Q^2 = -q^2$ is the transferred momentum and $|\text{nucl},p \rangle$ is the nucleon state with momentum $p$. Spin averaging is assumed. The longitudinal structure function $F_L$ is related to the structure function $F_1$ by $F_L = F_2 - 2xF_1$.

As one approaches the Bjorken limit, $Q^2 \to \infty$, $x$ fixed, one can show that the integration region in Eq. (1) near the light cone $z^2 \approx 0$ progressively dominates, due to increasingly rapid phase fluctuations of the term $e^{iqz}$ outside the light cone region (and presuming that the integrand $\langle p,\text{nucl}|J_\mu(z)J_\nu(0)|\text{nucl},p \rangle$ varies smoothly outside the light cone). Since we have to deal with this non-local limit $z^2 \approx 0$, a formal operator product expansion in terms of local operators can only be applied together with the dispersion relation technique. These techniques together provide a systematic way to study the leading and non-leading contributions to the hadronic tensor.

The tensor $W_{\mu\nu}$ is, by application of the optical theorem, related to a scattering amplitude $T_{\mu\nu}$ which is a more convenient quantity for practical calculations since it has a time ordered product of currents to which standard perturbation theory applies ($T_{\mu\nu}$ is the amplitude for forward elastic photon–nucleon scattering )

$$W_{\mu\nu}(p,q) = \frac{1}{2\pi} \text{Im} T_{\mu\nu}(p,q), \quad T_{\mu\nu}(p,q) = i \int d^4z e^{iqz} \langle p,\text{nucl}|T (J_\mu(z)J_\nu(0))|\text{nucl},p \rangle.$$

\footnote{For reviews see Refs. [15, 16, 17].}
The operator product expansion in terms of local operators for a time ordered product of the two electromagnetic hadronic currents reads

\[ i \int d^4 z e^{iqz} T(J_{\nu_1}(z)J_{\nu_2}(0)) = \sum_{N,j} \left( \frac{1}{Q^2} \right)^N \left[ \left( g_{\nu_1 \nu_2} - \frac{q_{\nu_1} q_{\nu_2}}{q^2} \right) q_{\mu_1} q_{\mu_2} C_{L,N}^j \left( \frac{Q^2}{\mu^2}, a_s \right) - \left( g_{\nu_1 \mu_1} g_{\nu_2 \mu_2} q^2 - g_{\nu_1 \mu_1} q_{\nu_2} q_{\mu_2} \right) \right] q_{\mu_3} \cdots q_{\mu_N} O^{j;\{\mu_1, \cdots, \mu_N\}}(0) + \text{higher twists}, \]

where everything is assumed to be renormalized (with \( \mu \) being the renormalization scale). The use of the OPE in the short distance regime (\( z \to 0 \)) differs from its use in the light cone region. In the former case the sum over spin-N extends to a finite value for a given approximation, while in the latter (the one we have to deal with) the sum over N extends to infinity. The sum over N runs over the standard set of the spin-N twist-2 irreducible (i.e. symmetrical and traceless in the indices \( \mu_1, \cdots, \mu_N \)) flavour non-singlet quark operators and the singlet quark and gluon operators:

\[
O^{\alpha,\{\mu_1, \cdots, \mu_N\}} = \bar{\psi}\lambda^{\alpha, \gamma} \{\mu_1 D^{\mu_2} \cdots D^{\mu_N}\} \psi, \quad \alpha = 1, 2, \ldots, (n_f^2 - 1) \quad (4)
\]

\[
O^{\psi,\{\mu_1, \cdots, \mu_N\}} = \bar{\psi}\gamma^{\gamma} \{\mu_1 D^{\mu_2} \cdots D^{\mu_N}\} \psi, \quad (5)
\]

\[
O^{G,\{\mu_1, \cdots, \mu_N\}} = G\{\mu_1 D^{\mu_2} \cdots D^{\mu_{N-1}} G^{\mu_N}\}. \quad (6)
\]

Here and in the following we denote the generators of the flavour group \( SU(n_f) \) by \( \lambda^\alpha \), and the covariant derivative by \( D^{\mu_1} \); in addition, it is understood that the symmetrical and traceless part is taken with respect to the indices in curly brackets. The functions \( C_{k,N}^j(Q^2/\mu^2, a_s) \) are the coefficient functions for the above operators. Since the coefficient functions \( C_{k,N}^j \) of non-singlet operators depend trivially on the number \( \alpha \) (see e.g. Ref. [12] or section 5 of the present article) we will use for them the standard notation \( C_{k,N}^{n_s} \). Here and throughout the whole paper we use the notation

\[
a_s = \frac{g^2}{16\pi^2} = \frac{\alpha_s}{4\pi} \quad (7)
\]

for the QCD strong coupling constant. The direct application of the OPE of Eq. (3) to the Green function \( T_{\mu\nu} \) leads to a formal expansion for \( T_{\mu\nu} \) in terms of the variable \( q \cdot p/Q^2 = 1/(2x) \) i.e. an expansion for unphysical \( x \to \infty \),

\[ i \int d^4 z e^{iqz} (p, \text{nucl}|T(J_\mu(z)J_\nu(0))| \text{nucl}, p) = \]

\[ \sum_{N,j} \left( \frac{1}{2x} \right)^N \left[ \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) C_{L,N}^j \left( \frac{Q^2}{\mu^2}, a_s \right) - \left( g_{\mu\nu} + p_\mu p_\nu \frac{4x^2}{q^2} \right) \right] \]
\[(p_{\mu} q_{\nu} + p_{\nu} q_{\mu}) \frac{2x}{q^2} C^j_{2,N} \left( \frac{Q^2}{\mu^2}, a_s \right) \] \( A^{j}_{\text{nucl},N}(m_n^2/\mu^2) + \text{higher twists}, \quad j=n_s, \psi, G \) \hfill (8)

where the spin averaged matrix elements are defined as

\[ \langle p, \text{nucl}|O^j(\mu_1, \ldots, \mu_N)|\text{nucl}, p \rangle = p^{\mu_1} \ldots p^{\mu_N} A^{j}_{\text{nucl},N}(m_n^2/\mu^2) \] \hfill (9)

and \( m_n \) is the nucleon mass.

To perform the proper analytic continuation of the representation Eq. (8) to the physical region \( 0 < x \leq 1 \) one applies a dispersion relation in the complex \( x \) plane to the Green function \( T_{\mu \nu} \). For electron-nucleon scattering where we have hermitian currents \( J_\mu \) one finds that the Mellin moments of the structure functions \( F_k \) are expressed through the parameters of the operator product expansion (3) of

\[ \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s(\mu^2)) \frac{\partial}{\partial a_s(\mu^2)} \right] C^{i}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) = C^{j}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) \gamma^{ji}_{N}(a_s(\mu^2)), \quad i,j=\psi, G \] \hfill (11)

\[ \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s(\mu^2)) \frac{\partial}{\partial a_s(\mu^2)} \right] C^{n_s}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) = C^{n^s}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) \gamma^{n_s}_{N}(a_s(\mu^2)) \] \hfill (12)

Please note that the odd Mellin moments of \( F_k \) are not fixed by this equation. However, all moments in the complex \( N \) plane are fixed by analytic continuation from the even Mellin moments when all the even moments are known. This means that the structure functions in \( x \)-space, \( 0 < x \leq 1 \), can be found by means of the inverse Mellin transformation when the (infinite set of) even moments are known.

The \( Q^2 \)-dependence of the coefficient functions can be studied by the use of the renormalization group equations

\[ \frac{1}{2} \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s(\mu^2)) \frac{\partial}{\partial a_s(\mu^2)} \right] C^{i}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) = C^{j}_{k,N} \left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) \gamma^{ji}_{N}(a_s(\mu^2)) \] \hfill (11)

where Eq. (11) represents the singlet sector where quark and gluon operators mix under renormalization, and Eq. (12) is the non-singlet equation. \( \beta(a) \) is the beta-function that determines the renormalization scale dependence of the renormalized coupling constant. It is known at three loops [18] in the \( \overline{\text{MS}} \) scheme

\[ \frac{\partial a_s}{\partial \ln \mu^2} = \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + O(a_s^5), \]

\[ \beta_0 = \left( \frac{11}{3} C_A - \frac{4}{3} T_F n_f \right) \]
\[
\begin{align*}
\beta_1 &= \left(\frac{34}{3}C_A - 4C_FT_f n_f - \frac{20}{3}C_A T_f n_f\right) \\
\beta_2 &= \left(\frac{2857}{54}C_A^3 + 2C_F^2 T_f n_f - \frac{205}{9}C_F C_A T_f n_f - \frac{1415}{27}C_A^2 T_f n_f + \frac{44}{9}C_F^2 T_f^2 n_f^2 + \frac{158}{27}C_A T_f^2 n_f^2\right)
\end{align*}
\]

(13)

where \(C_F = \frac{4}{3}\) and \(C_A = 3\) are the Casimir operators of the fundamental and adjoint representations of the colour group \(SU(3)\), \(T_F = \frac{1}{2}\) is the trace normalization of the fundamental representation and \(n_f\) is the number of (active) quark flavours. The anomalous dimensions \(\gamma_N(a_s)\) determine the renormalization scale dependence of the operators, that is

\[
\begin{align*}
\frac{d}{d \ln\mu^2} O_R^{j_i\{\mu_1, \ldots, \mu_N\}} &= -\gamma_{ij}^{(a_s)} O_R^{j_i\{\mu_1, \ldots, \mu_N\}}, \quad i,j = \psi, G \\
\frac{d}{d \ln\mu^2} O_R^{ns\{\mu_1, \ldots, \mu_N\}} &= -\gamma_{ns}^{(a_s)} O_R^{ns\{\mu_1, \ldots, \mu_N\}}.
\end{align*}
\]

(14)

(15)

We define renormalized operators in terms of bare operators as \(O_R = Z O_B\) and find

\[
\frac{d}{d \ln\mu^2} O_R = \left(\frac{d}{d \ln\mu^2} Z\right) O_B = \left(\frac{d}{d \ln\mu^2} Z\right) Z^{-1} O_R \quad \Rightarrow \quad \gamma = -\left(\frac{d}{d \ln\mu^2} Z\right) Z^{-1}
\]

(16)

where it is understood that in the singlet case \(Z\) represents a matrix \(Z^{ij}\). The renormalization group equations are solved in the standard form

\[
C_{k,N}^{ns}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = C_{k,N}^{ns}(1, a_s(Q^2)) \times \exp\left(-\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} da_s \frac{\gamma_{ns}(a_s')}{\beta(a_s')}\right)
\]

(17)

The solution for the singlet equations has a similar form but since one gets the exponential of a matrix of anomalous dimensions one has to define the exponential properly in the singlet case (i.e. a \(T\)-ordered exponential, see e.g. Ref. [19]). Here \(a_s(Q^2) \equiv a_s(Q^2/\Lambda_{\text{MS}}^2)\) is the renormalized (i.e. running) coupling constant at the renormalization scale \(Q^2\)

\[
\begin{align*}
a_s\left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right) &= \frac{1}{\beta_0 \ln \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right)} - \frac{\beta_1 \ln \ln \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right)}{\beta_0^2 \ln^2 \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right)} \\
&\quad + \frac{1}{\beta_0^3 \ln^3 \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right)} \left(\beta_1^2 \ln^2 \ln \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right) - \beta_1^2 \ln \ln \left(\frac{Q^2}{\Lambda_{\text{MS}}^2}\right) + \beta_2 \beta_0 - \beta_1^2\right)
\end{align*}
\]

(18)

and \(\Lambda_{\text{MS}}\) is the fundamental scale of QCD in the \(\overline{\text{MS}}\)-scheme. In practice one may use the DGLAP evolution equations [20] for matrix elements of operators at the scale \(\mu^2 = Q^2\) (i.e. \(Q^2\)-dependent parton distributions) instead of the renormalization group equations for the coefficient functions (for perturbative solutions of the DGLAP equations in moment space see e.g. Refs. [3, 21]).
3 The method

In this section we will discuss the method [22] for the calculation of anomalous dimensions and coefficient functions in considerable detail as it applies to the singlet sector. Let us first elaborate on some details specific to the dimensional regularization [23] and the minimal subtraction scheme [24], and its standard modification, the \( \overline{\text{MS}} \)-scheme [2], which form a modern basis for multiloop calculations in QCD. We use the symbol \( a_s \) for the renormalized coupling constant and \( a_b \) for the bare coupling constant. Although renormalization constants \( Z \) contain poles in \( \varepsilon \) in \( D = 4 - 2\varepsilon \) dimensions, anomalous dimensions are finite as \( D \rightarrow 4 \). This fact gives expressions for the higher poles of \( Z \) in terms of the first poles of \( Z \). To see this we write Eq. (16) as

\[
\gamma Z = - \left( \frac{d}{d\ln \mu^2} Z(a_s, \frac{1}{\varepsilon}) \right) = - \left( \frac{\partial}{\partial a_s} Z \right) \frac{da_s}{d\ln \mu^2} Z = - \left( \frac{\partial}{\partial a_s} Z \right) \left[ -\varepsilon a_s + \beta(a_s) \right]
\]

where \( \beta(a_s) \) is the 4-dimensional beta function of Eq. (13) and \( [-\varepsilon a_s + \beta(a_s)] \) is the beta function in \( 4 - 2\varepsilon \) dimensions. This latter function receives no higher order corrections in \( \varepsilon \) due to the form of renormalization factors in the minimal subtraction scheme, viz.:

\[
ap_b = Z_{a_s} a_s, \quad Z_{a_s} = 1 - \frac{\beta_0}{\varepsilon} a_s + \left( \frac{\beta_1}{\varepsilon^2} - \frac{\beta_0}{2\varepsilon} \right) a_s^2 + O(a_s^3)
\]

\[
\frac{d(a_b \mu^{2\varepsilon})}{d\ln \mu^2} = 0 = \varepsilon Z_{a_s} a_s \mu^{2\varepsilon} + \frac{\partial Z_{a_s}}{\partial a_s} \frac{da_s}{d\ln \mu^2} a_s \mu^{2\varepsilon} + Z_{a_s} \frac{da_s}{d\ln \mu^2} \mu^{2\varepsilon}
\]

\[
\Rightarrow \quad \frac{da_s}{d\ln \mu^2} = - \frac{\varepsilon Z_{a_s} a_s}{\frac{\partial Z_{a_s}}{\partial a_s} a_s + Z_{a_s}} = -\varepsilon a_s + \beta(a_s)
\]

where \( a_b \) is a dimensionless object and \( a_b \mu^{2\varepsilon} \) is the bare coupling constant which is invariant under the renormalization group transformations. The factors \( Z^{ij} \) are calculated as series in \( a_s \), and have the well known form

\[
Z^{ij} = Z^{ij(0)} + Z^{ij(1)}(a_s)/\varepsilon + Z^{ij(2)}(a_s)/\varepsilon^2 + \cdots, \quad i, j = \psi, G
\]

with \( Z^{\psi(0)} = Z^{GG(0)} = 1 \), \( Z^{\psi G(0)} = Z^{G\psi(0)} = 0 \). It is helpful to write the matrix equation (13) as 4 separate equations,

\[
\gamma^{\psi\psi} Z^{\psi\psi} + \gamma^{\psi G} Z^{\psi G} = - \left[ -\varepsilon a_s + \beta(a_s) \right] \frac{\partial}{\partial a_s} Z^{\psi\psi}
\]

\[
\gamma^{\psi G} Z^{\psi G} + \gamma^{G G} Z^{G G} = - \left[ -\varepsilon a_s + \beta(a_s) \right] \frac{\partial}{\partial a_s} Z^{\psi G}
\]

\[
\gamma^{\psi G} Z^{G G} = - \left[ -\varepsilon a_s + \beta(a_s) \right] \frac{\partial}{\partial a_s} Z^{G G}
\]

\[
(21)
\]
where we underlined the terms that contribute to the lowest order in $\varepsilon$ (i.e. order $\varepsilon^0$). From these terms one immediately finds that the anomalous dimensions are expressed through the coefficients in front of the first poles of $Z$

$$\gamma^{ij} = a_s \left( \frac{\partial}{\partial a_s} Z^{ij(1)}(a_s) \right) \quad i,j = \psi, G$$

(22)

where $Z^{ij(1)}(a_s)$ was defined as the order 1/\varepsilon part of $Z^{ij}$. The coefficients of higher poles in $Z$ can then be expressed in terms of $\gamma^{ij}$ by substituting the expression for $\gamma^{ij}$ back into equation (21).

The operator product expansion of Eq. (3) is an operator statement and both the coefficient functions $C_{k,N}^i$ and the anomalous dimensions $\gamma^{ij}_N$ of the operators are functions and do therefore not depend on the hadronic states of the Green function to which one wishes to apply the OPE. The information on the hadronic target is contained in the operator matrix elements $A_{N}^{i}$ in Eq. (4) which are generally not calculable perturbatively. It is therefore standard to consider simpler Green functions with quarks and gluons as external particles, instead of the physical nucleon states, in the calculation of coefficient functions and anomalous dimensions. In this case the Green functions can be calculated in perturbation theory as well as the operator matrix elements and the anomalous dimensions and coefficient functions can be extracted as will be shown below in detail.

Let us consider the following 4-point Green functions

$$T_{\mu\nu}^{\gamma\gamma}(p,q) = i \int d^4z e^{iqz} \langle p,\text{quark}\mid T \left( J_{\mu}(z)J_{\nu}(0) \right) \mid \text{quark},p \rangle \quad (23)$$

$$T_{\mu\nu}^{\gamma\gamma}(p,q) = i \int d^4z e^{iqz} \langle p,\text{gluon}\mid T \left( J_{\mu}(z)J_{\nu}(0) \right) \mid \text{gluon},p \rangle \quad (24)$$

where the label $\gamma$ is used to indicate an external photon, $q$ indicates an external quark and $g$ an external gluon. Spin and colour averaging for the quark and gluon states is assumed. Analogously to the decomposition of the hadronic tensor $W_{\mu\nu}$ in terms of $F_2$ and $T_2$ we decompose the Green functions $T_{\mu\nu}$ in terms of $T_L$ and $T_2$. In the leading twist approximation (i.e. dropping non-leading terms in $p^2$) one finds

$$T_L = \frac{-q^2}{(p \cdot q)^2} p^\mu p'^\nu T_{\mu\nu}, \quad T_2 = -\left( \frac{3 - 2\varepsilon}{2 - 2\varepsilon} \frac{q^2}{(p \cdot q)^2} p^\mu p'^\nu + \frac{1}{2 - 2\varepsilon} g^{\mu\nu} \right) T_{\mu\nu}. \quad (25)$$

Applying the OPE to $T_{\mu\nu}^{\gamma\gamma}$ and $T_{\mu\nu}^{\gamma\gamma}$ we find the following equations for the renormalized Green functions

$$T_{k}^{\gamma\gamma}(p,q,a_s,\mu^2,\varepsilon) =$$

$$\sum_{N=2}^{\infty} \left( \frac{1}{2\varepsilon} \right)^N \left[ \left( C_{k,N}^{\gamma}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\gamma}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{G}(a_s,\frac{1}{\varepsilon}) \right) A_{\text{quark},N}(a_s,\frac{p^2}{\mu^2},\varepsilon) \right.$$

$$+ \left( C_{k,N}^{\gamma}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\gamma}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{G}(a_s,\frac{1}{\varepsilon}) \right) A_{\text{quark},N}(a_s,\frac{p^2}{\mu^2},\varepsilon) \right] + O(p^2) \quad (26)$$

$$T_{k}^{\gamma\gamma}(p,q,a_s,\mu^2,\varepsilon) =$$

\[
\sum_{N=2}^{\infty} \left( \frac{1}{2x} \right)^N \left[ \left( C_{k,N}^\psi(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{\psi\psi}(a_s, \frac{1}{\varepsilon}) + C_{k,N}^G(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{GG}(a_s, \frac{1}{\varepsilon}) \right) A_{\text{gluon},N}(a_s, \frac{p^2}{\mu^2}, \varepsilon) 
+ \left( C_{k,N}^\psi(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{\psi\psi}(a_s, \frac{1}{\varepsilon}) + C_{k,N}^G(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{GG}(a_s, \frac{1}{\varepsilon}) \right) A_{\text{gluon},N}(a_s, \frac{p^2}{\mu^2}, \varepsilon) \right] + O(p^2) \quad (27)
\]

where \( k = 2, L \), \( a_s \equiv a_s(\mu^2/\Lambda^2) \) and it is understood that the l.h.s. is renormalized by substituting the bare coupling constant in terms of the renormalized one,

\[
a_b = a_s - \frac{\beta_0}{\varepsilon} a_s^2 + \left( \frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) a_s^3 + O(a_s^4) \quad (28)
\]

The terms \( O(p^2) \) in the r.h.s. of Eqs. (26) and (27) indicate higher twist contributions. The renormalization factors for the external quark and gluon lines are overall factors on both sides of the equations and are omitted. The coefficient functions on the r.h.s are renormalized quantities. The matrix elements \( A_{i,N} \) are the matrix elements of bare operators and are defined as in Eq. (9) with the nucleon states replaced by the appropriate quark or gluon states.

Figure 1. A graphical representation of Eqs. (26) and (27). The symbol \( \otimes \) indicates an appropriate quark or gluon operator (defined in Eqs. (4), (5), (6)). On the l.h.s. of the equations also the crossed diagrams contribute but they are not explicitly shown.
It is known that the gauge invariant operators \( O^\psi \) and \( O^G \) mix under renormalization with unphysical operators (that are BRST variations of some operators or that vanish by the equations of motion) [17, 1, 23]. But physical matrix elements (i.e. on-shell matrix elements with physical polarizations) of such unphysical operators vanish. Since the method that is described below deals with physical matrix elements we omitted the unphysical operators in Eqs. (26) and (27).

Starting from Eqs. (26) and (27), the anomalous dimensions and the coefficient functions are calculated using the method of projections of Ref. [22]. It reduces the calculation of (moments of) coefficient functions and anomalous dimensions to the calculation of diagrams of the propagator type instead of the 4-point diagrams that contribute to \( T_{\mu\nu} \). This method relies heavily on the use of dimensional regularization and the minimal subtraction scheme and implicitly involves a considerable rearrangement of infrared and ultraviolet divergences.

The method consists of applying the following projection operator to both sides of Eqs. (26) and (27).

\[
P_N \equiv \left[ \frac{q^{(\mu_1 \ldots \mu_N)}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \ldots \partial p^{\mu_N}} \right]_{p=0}
\]

Here \( q^{(\mu_1 \ldots \mu_N)} \) is the harmonic (i.e. symmetrical and traceless) part of the tensor \( q^{\mu_1 \ldots \mu_N} \) (see next section). The operator \( P_N \) is applied to the integrands of all Feynman diagrams (nullifying \( p \) before taking the limit \( \varepsilon \to 0 \), to dimensionally regularize the infrared divergences as \( p \to 0 \) for individual diagrams). It is important to realize that this operation does not act on the renormalization constants \( Z_{ij}^O \) and the coefficient functions on the r.h.s. of Eqs. (26), (27). It does however act on the matrix elements \( A_N^i \). The nullification of \( p \) has the effect that of all the diagrams that contribute to the perturbative expansion of \( A_N^i \) only the tree level terms (i.e. with no loops) survive since all diagrams containing loops become massless tadpole diagrams. Massless tadpole diagrams are put to zero in dimensional regularization. Furthermore, the \( N^{th} \) order differentiation in the operator \( P_N \) has the effect that \( P_N \) projects out only the \( N^{th} \) moment since of all the factors \( 1/(2\pi)^N \) only \( 1/(2\pi)^N \) gives a non zero contribution after nullifying \( p \). On the left hand side the effect of \( P_N \) is to effectively reduce the 4-point diagrams that contribute to \( T_{\mu\nu} \) to 2-point diagrams (this follows from the nullification of the momentum \( p \)), which drastically simplifies the calculation. We apply the operator \( P_N \) after the tensor structures \( 2, L \) have been projected out because the operator \( P_N \) would mutilate the tensor structure of \( T^\dagger_{\mu\nu} \). In the projector \( P_N \) we use the harmonic tensor \( q^{(\mu_1 \ldots \mu_N)} \) to remove higher twist contributions (the \( O(p^2) \) terms in Eqs. (26) and (27)) that after differentiation with respect to \( p^\mu \) survive as terms proportional to the metric tensor.

Summarizing, we have after application of the projection operator \( P_N \) to Eqs. (26) and (27)

\[
T_{k,N}^{qq\psi}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) = \left( C_{k,N}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{\psi\psi}(a_s, \frac{1}{\varepsilon}) + C_{k,N}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{G\psi}(a_s, \frac{1}{\varepsilon}) \right) A_{\psi, \text{tree}}^{\text{quark},N}(\varepsilon)
\]

\[
T_{k,N}^{qgG}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) = \left( C_{k,N}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{qG}(a_s, \frac{1}{\varepsilon}) + C_{k,N}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{G\psi}(a_s, \frac{1}{\varepsilon}) \right) A_{\psi, \text{tree}}^{\text{gauge},N}(\varepsilon)
\]
where \( k = 2, L \) and we defined

\[
T_{k,N}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) \equiv \frac{q^{(\mu_1 \ldots q^{\mu_N})}}{N!} \left. \frac{\partial^N}{\partial p^{\mu_1} \ldots \partial p^{\mu_N}} T_k(p,q,a_s,\mu^2,\varepsilon) \right|_{p=0}. \tag{32}
\]

It should be understood that (30) and (31) represent a large coupled system of equations when both sides are expanded in powers of \( a_s \) and \( \varepsilon \) (i.e. \( C \) is expanded in positive powers of \( \varepsilon \) and \( Z \) is expanded in negative powers of \( \varepsilon \)).

After the calculation of \( T_{q,\gamma}^g \) and \( T_{g,\gamma}^q \) in the order \( a_s^2 \) and the determination of the tree level matrix elements \( A_{N,\text{tree}}^q \) one can solve Eqs. (30) and (31) simultaneously to obtain \( C^\psi_k, C^G_k, Z^\psi \) and \( Z^G \) in order \( a_s^2 \) but, unfortunately, \( Z^G_{\psi} \) and \( Z^G \) only in the order \( a_s^2 \). This limitation follows directly from the fact that \( C^\psi_{k} \) starts from the order \( a_s^1 \) but \( C^G_{k} \) starts from order \( a_s \) since the photon couples \emph{directly} only to quarks. In solving the equations it is essential that all poles of the \( Z \) factors are fully expressed in terms of the anomalous dimensions as was discussed in the beginning of this section. Coefficient functions and operator matrix elements are finite as \( \varepsilon \to 0 \) but one must make sure that sufficiently high powers in \( \varepsilon \) are taken into account. For example, one should consider order \( \varepsilon^2 \) contributions for \( C^\psi_{k} \) at order \( a_s \). We stress that by calculating only propagator type diagrams in the l.h.s. of Eqs. (30) and (31) we can get both renormalization constants of operators and coefficient functions.

**Figure 2.** Examples for diagrams contributing to the Green functions \( T_{g,\phi}^\phi \) (a) and \( T_{q,\phi}^q \) (b).

To obtain \( Z^G_{\phi} \) and \( Z^G \) in order \( a_s^2 \) we calculated two more unphysical Green functions \( T_{q,\phi}^q \) and \( T_{g,\phi}^g \) (see fig 2), in which the photon is replaced by an external scalar particle \( \phi \) that couples \emph{directly} only to gluons. The vertices that describe the coupling between the external scalar field \( \phi \) and the gluons follow from adding the simplest gauge invariant interaction term \( \phi G_\mu^a G^\mu_a \) (where \( G_\mu^a \) is the QCD field strength tensor) to the QCD Lagrangian. For the Green functions \( T_{q,\phi}^q \) and \( T_{g,\phi}^g \) an OPE similar to (3) exists with the same operators but with different coefficient functions \( C^G_{\phi} \) and \( C^\psi_{\phi} \), where \( C^\psi_{\phi} \) starts from the order \( a_s \) and \( C^G_{\phi} \) starts from the order \( a_s^0 \).

Repeating the steps that led to Eqs. (30) and (31) one finds for these Green functions the following equations

\[
(Z_G)^2 T_{N,\phi}^g(q\frac{Q^2}{\mu^2}, a_s, \varepsilon) = \left( C^G_{\phi,N}(a_s, q\frac{Q^2}{\mu^2}, \varepsilon) Z^G_{N}(a_s, \frac{1}{\varepsilon}) + C^\psi_{\phi,N}(a_s, q\frac{Q^2}{\mu^2}, \varepsilon) Z^G_{G}(a_s, \frac{1}{\varepsilon}) \right) A_{\text{quon},N}(\varepsilon), \tag{33}
\]
\[(Z_{G2}^2 T_{N}^{\psi \phi}(Q^2/\mu^2, a_s, \varepsilon)) = \left( C_{\phi, N}^G(a, Q^2/\mu^2, \varepsilon) Z_N^{G\psi}(a_s, 1/\varepsilon) + C_{\phi, N}^\psi(a_s, Q^2/\mu^2, \varepsilon) Z_N^{\psi}(a_s, 1/\varepsilon) \right) A_{\text{quark}, N}^{\psi, \text{tree}, N}(\varepsilon) \]  

As is indicated in the l.h.s., the external operators \( G_{\mu\nu}^a G_{\mu\nu}^a \) have to be renormalized

\[
\left( G_{\mu\nu}^a G_{\mu\nu}^a \right)_R = Z_{G2} \left( G_{\mu\nu}^a G_{\mu\nu}^a \right) B + \cdots, \quad Z_{G2} = \frac{1}{1 - \beta(a_s)/(a_s \varepsilon)}
\]

where the dots indicate (unphysical) operators that are omitted since they do not contribute to the on-shell matrix elements with physical spin projections that we consider. (The only physical operator that mixes under renormalization with \( G_{\mu\nu}^a G_{\mu\nu}^a \) is the quark operator \( m_q \psi \bar{\psi} \) that vanishes in our limit of massless quarks.) We emphasize that the 3-loop \( \beta \)-function is required for the present 3-loop calculation.

After the calculation of \( T^{\psi \phi} \) and \( T^{\bar{\psi} \phi \phi} \) in the order \( a_s^3 \) one can solve Eqs. (13) and (14) to obtain \( C_{\phi}^G, C_{\phi}^\psi, Z_{G\psi} \) and \( Z_{GG} \) in the order \( a_s^3 \) (from these equations \( Z_{\psi G} \) and \( Z_{\psi \bar{\psi}} \) can be obtained in the order \( a_s^2 \) only). Please note that the coefficients \( C_{\phi}^G \) and \( C_{\phi}^\psi \) are obtained as a byproduct and are not important for the physical process under consideration. Furthermore, the two sets of equations (13), (14) and (13), (14) determine all the anomalous dimensions of the order \( a_s^2 \) in two independent ways which provides a consistency check on the results.

Solving the system of equations for \( F_L \) is slightly more involved than for \( F_2 \) since the structure function \( F_L \) contains no tree level (i.e. order \( a_s^0 \)) contribution. To solve the sets of equations for \( F_L \) one should add extra information, for example the \( \mathcal{O}(a_s^2) \) contributions to \( Z_{G\psi} \) and \( Z_{GG} \) as determined from the equations for \( F_2 \).

### 4 The calculation

As was discussed in the previous section, we will apply the operator \( \mathcal{P}_N \) for \( N=2,4,6,8 \) to 4 different Green functions, \( T_k^{\psi \phi} \), \( T_k^{\bar{\psi} \phi \phi} \), \( T_k^{\psi \phi} \) and \( T_k^{\bar{\psi} \phi \phi} \) and we sum over the physical spin polarizations of the external quarks and gluons. For the external quarks (in \( T_k^{\psi \phi} \) and \( T_k^{\bar{\psi} \phi \phi} \)) the sum over the polarizations is performed by inserting the projection operator \( \gamma \) between the external quark legs and taking the trace over the strings of gamma matrices. For the external gluons (in \( T_k^{\psi \phi} \) and \( T_k^{\bar{\psi} \phi \phi} \)) the sum over physical spins can be done by contracting the external gluon lines with \(-g^{\alpha\beta} + (p^\alpha q^\beta + p^\beta q^\alpha)/p \cdot q - p^\alpha p^\beta q^2/(p \cdot q)^2\) in which the (on-shell) gluon has momentum \( p \) (with \( p^2 = 0 \)). The presence of the extra powers of \( p \) poses considerable efficiency problems (the operator \( \mathcal{P}_N \) will generate more than 3 times larger intermediate expressions as compared to the case of a simpler \( g^{\alpha\beta} \) projection). Alternatively one may take the sum over physical gluon spins by contracting the external gluon lines with only \(-g^{\alpha\beta}\) and adding external ghost contributions to the Green functions, \( T_k^{\psi \phi} \) to \( T_k^{\bar{\psi} \phi \phi} \), and increase the total number of diagrams that we have to calculate, it still makes the computations more than a factor of 3 faster (since ghost diagrams are of a far simpler nature than
We checked for the lowest moments that the two methods for taking the sum over the gluon spin polarizations gave the same results, but for the higher moments we only applied the ghost method.

**Figure 3.** Examples for ghost diagrams contributing to the Green functions $T^{h\gamma h\gamma}$ (a) and $T^{h\phi h\phi}$ (b)

The explicit generation of Feynman diagrams with the corresponding symmetry factors has been done automatically by the use of the program QGRAF\cite{27}. Statistics on the number of diagrams in the different classes $q\gamma q\gamma$, $q\phi q\phi$ etc. is presented in table 1. The generation (and the counting) of the diagrams is non-standard, since a number of tricks have been used (for example, crossed diagrams are not generated explicitly, and the diagrams with 4-gluon vertices are split into a number of parts for which the colour factor necessarily factorizes).

<table>
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<tr>
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<th>Tree</th>
<th>1-loop</th>
<th>2-loops</th>
<th>3-loops</th>
<th>Lorentz projections</th>
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<td>24</td>
<td>697</td>
<td>1</td>
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<tr>
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<td>20</td>
<td>366</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$h\gamma h\gamma$</td>
<td>2</td>
<td>53</td>
<td></td>
<td>2</td>
<td></td>
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<tr>
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<td>241</td>
<td>7219</td>
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<tr>
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<tr>
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<td>3</td>
<td>23</td>
<td>399</td>
<td>10846</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Number of diagrams and Lorentz tensor structures in the classes $q\gamma q\gamma$, $q\phi q\phi$, $g\gamma g\gamma$, $h\gamma h\gamma$, $g\phi g\phi$ and $h\phi h\phi$. Notation: $q$ = quark, $g$ = gluon, $h$ = ghost, $\gamma$ = photon, $\phi$ = scalar particle that couples only to gluons.

It is clear from these statistics that the calculation of the diagrams necessarily has to be automated to a large extent. The calculation is therefore organized as follows:

1. The diagrams are generated automatically with a special version of the diagram generator QGRAF\cite{27}. For every class $q\gamma q\gamma$, $q\phi q\phi$ etc. the full set of diagrams is put into a single file using a dedicated database program MINOS that manages information about thousands of diagrams and can be instructed to call other programs, giving them the proper information from a database file.
2. The representation for a diagram at this point is still a very compact one, and this is explored as follows. We use programs written for the symbolic manipulation program FORM to calculate colour factors for each diagram and bring the diagrams into a representation that explicitly contains information required at later steps in the calculation. For instance, this involves choosing automatically an optimal path (in most cases the shortest path) for the external momentum $p$ to flow through each diagram (we are going to expand in $p$ when the operator $\mathcal{P}_N$ is applied) and determining automatically the diagram’s topology when $p$ is nullified. This information, for all diagrams together, is kept in a single file and is accessible using MINOS.

3. We instruct MINOS to run sequentially, one diagram at a time, a highly optimized FORM program that performs the explicit calculation i.e. it substitutes all the Feynman rules, it performs projections on the Lorentz structures of the Green functions, it Taylor-expands the diagram in the external momentum $p$ (the depth of the expansion increases with the moment index $N$), it takes all the Dirac traces, contracts with the tensors $q^{\mu_1 \cdots \mu_N}$ and finally calls the MINCER integration package to perform the 3-loop scalar integrals of the massless propagator type (using the integration by parts algorithms published in Ref. [30]). The results together with some useful technical information about the calculation (such as the resources used) are again stored into a single file. MINOS will initiate the calculation of a next diagram as soon as the calculation of a previous diagram is completed without any need for human interference.

An important aspect of the calculation is the use of the symmetrical and traceless tensors, $q^{\{\mu_1 \cdots \mu_N\}}$ that are used to extract the leading twist contributions. These tensors are (with the proper normalization) known as ‘harmonic tensors’ $H_n$ satisfying

$$H_{n}^{\mu_1 \cdots \mu_n} \delta_{\mu_1 \mu_2} = 0 \quad \text{(traceless)}$$
$$H_{n}^{\mu_1 \cdots \mu_j \cdots \mu_n} = H_{n}^{\mu_1 \cdots \mu_j \cdots \mu_n} \quad \text{(symmetrical)}$$
$$H_{n}^{\mu_1 \cdots \mu_n} Q^{\mu_n} = H_{n-1}^{\mu_1 \cdots \mu_{n-1}} Q^2 \quad \text{(normalization)}$$

and an explicit construction (in Euclidean space-time) in terms of the Kronecker delta symbols $\delta^{\mu \nu}$ and the momenta $Q^\mu$ reads

$$H_{n}^{\mu_1 \cdots \mu_n} = \sum_{j=0,2,\ldots,n} h_{j}^{n} (Q^2)^{j/2} \sum_{\text{index perm.}} \delta(\mu_1, \cdots, \mu_j)Q^{\mu_{j+1}} \cdots Q^{\mu_n},$$

$$h_{j}^{n} = (-1)^{j/2} 2^{n-j} \frac{\Gamma(2 - 2\varepsilon)\Gamma(1 - \varepsilon + n - j/2)}{\Gamma(2 - 2\varepsilon + n)\Gamma(1 - \varepsilon)}$$

where the second summation is over all the ways to partition the indices into one set containing $j$ indices (put into $\delta$ ) and a second set containing $(n - j)$ indices (put on momenta $Q \cdots Q$). $\Gamma$ is the Euler gamma function (factorial function). The tensor $\delta$ is a completely symmetrical tensor constructed from Kronecker delta symbols only,

$$\delta(\mu_1, \mu_2) = \delta^{\mu_1 \mu_2},$$
$$\delta(\mu_1, \mu_2, \mu_3, \mu_4) = \delta^{\mu_1 \mu_2} \delta^{\mu_3 \mu_4} + \delta^{\mu_1 \mu_3} \delta^{\mu_2 \mu_4} + \delta^{\mu_1 \mu_4} \delta^{\mu_2 \mu_3},$$

etc.
such that the normalization is

$$\delta(\mu_1, \mu_2, \mu_3, \mu_4) = 2^\epsilon \frac{\Gamma(2 - \epsilon + n)}{\Gamma(2 - \epsilon)}$$

For example the tensor with four indices is

$$H_4^{\mu_1, \mu_2, \mu_3, \mu_4} = h_4^1 Q^4 \left[ \delta^{\mu_1 \mu_2} \delta^{\mu_3 \mu_4} + \delta^{\mu_1 \mu_3} \delta^{\mu_2 \mu_4} + \delta^{\mu_1 \mu_4} \delta^{\mu_2 \mu_3} \right] + h_4^0 Q^{\mu_1} Q^{\mu_2} Q^{\mu_3} Q^{\mu_4}$$

$$+ h_2^4 Q^2 \left[ \delta^{\mu_3 \mu_4} Q^{\mu_1} Q^{\mu_2} + \delta^{\mu_2 \mu_4} Q^{\mu_1} Q^{\mu_3} + \delta^{\mu_2 \mu_3} Q^{\mu_1} Q^{\mu_4} \right]$$

$$+ \delta^{\mu_1 \mu_3} Q^{\mu_2} Q^{\mu_4} + \delta^{\mu_1 \mu_4} Q^{\mu_2} Q^{\mu_3} + \delta^{\mu_1 \mu_2} Q^{\mu_3} Q^{\mu_4}].$$

The number of terms in a harmonic tensor increases rapidly with the rank of the tensor and in the present calculation these harmonic tensors are contracted with many other tensors constructed from only a few different (integration) momenta. An efficient implementation of the harmonic tensors that directly takes into account the symmetries of the other tensors can greatly limit the number of terms that are produced in the calculation and is vital for its feasibility.

The results of our calculations are presented in the last section of this article. The results for the second moment N=2, that are presented in this paper, were obtained in an arbitrary covariant gauge for the gluon fields. This means that we keep the gauge parameter $\xi$ that appears in the gluon propagator $i \left( -g^{\mu \nu} + (1 - \xi) q^\mu q^\nu / (q^2 + i\epsilon) \right) / (q^2 + i\epsilon)$ as a free parameter in the calculations. The explicit cancellation of the gauge dependence in the coefficient functions and anomalous dimensions gives an important check of the results.

Another non-trivial check of the calculation is the validity of the singlet relations $\gamma_2^{\psi\psi} + \gamma_2^{G\psi} = 0$, $\gamma_2^{\psi G} + \gamma_2^{G G} = 0$ that follow from the fact that the QCD energy momentum tensor (which contains both the spin-2 quark and gluon operators) is conserved and has therefore zero anomalous dimension.

Please note that at present the only independent check of our 3-loop results is provided by the calculation in Ref. [31] where the leading $n_f$ terms were calculated for the non-singlet anomalous dimensions in all orders in $a_s$ using a large $n_f$ expansion. Our $a_s^3 n_f^2$ terms agree with Ref. [31].

The reader may notice the appearance of the constant $\zeta_4$ in the final 3-loop results for the coefficient functions $C_{2, N}^{\psi N}$ and $C_{2, N}^{G N}$ which seems to be in conflict with the empirical law that in the results for inclusive physical quantities $\zeta_4$ does not appear. However these 3-loop coefficient functions contribute to the next-to-NNL (NNNL) order for $F_2$ and require 4-loop anomalous dimensions to get a complete physical NNNL approximation. Assuming the cancellation of $\zeta_4$ in the complete NNNL approximation, one can derive the coefficients of $\zeta_4$ in the 4-loop anomalous dimensions.

In spite of all optimizations of the integration program MINCER and of many other efficiency-crucial parts of the calculation, the calculation of the singlet moments, $N=2,4,6,8$ as published in this paper, required more than the equivalent of 7500 hours on a 150 Mhz SGI Challenge workstation and required at some instances 2 Gbyte of storage place for the intermediate stages.

\[ \text{it is therefore interesting to mention that an efficient implementation of Eq. (35) exists in FORM-2 and requires only 3 lines of code using the built in combinatorical functions.} \]
in the calculation of one of the diagrams. We noticed that both the required disk storage space and computation time increased with almost a factor 5 when we compared the calculation of each (N+2)th moment to the calculation of the corresponding Nth moment. The calculation of higher moments using the same methods is therefore not feasible at present.

5 Analysis

In this section we will investigate the effects due to the calculated 3-loop coefficient functions and anomalous dimensions (the list of the analytical 3-loop results is given in the next section). We will reconstruct the coefficient functions and splitting functions as distributions in x-space using the calculated moments and incorporating as much as possible the known information about the leading (and in some cases the next-to-leading) singularities in the limits x → 1 and x → 0. Explicit comparison with the known 2-loop coefficient functions and anomalous dimensions shows that our approach to the reconstruction gives very good effective fits of the x-space distributions when only few moments are used and when sufficient information about the two endpoints x = 0, x = 1 is known. This procedure also allows us to estimate the error of the reconstruction.

In order to relate the calculated results for the anomalous dimensions γ_N and the coefficient functions C_N to experiment one must be able to obtain the experimentally measurable structure functions F_k(x, Q^2) from the Mellin moments M_{k,N} of Eq. (10). The rigorous procedure to obtain F_k(x, Q^2) from the moments is to apply the inverse Mellin transform which, however, requires the exact knowledge of M_{k,N} in the complex N-plane (or equivalently the analytic continuation to complex N from all even or odd moments). Because at the 3-loop order we have calculated only a limited number of moments γ_N and C_N, we can obtain only approximate results in x-space. An example of a NNL order analysis in x-space based on a limited number of non-singlet moments can be found in Ref. [32].

As an alternative to studying the Q^2-dependence of F_k(x, Q^2) via the Q^2-dependence of moments M_{k,N} (see Eq. (17)), one may start from the x-space distributions γ(x) and C(x) that are related to γ_N and C_N via the Mellin transform

\[ γ_N = \int_0^1 dx \ x^{N-1} γ(x), \quad C_N = \int_0^1 dx \ x^{N-1} C(x) \] (36)

and do the Q^2-evolution in x-space via the DGLAP equations [20]

\[ Q^2 \frac{d}{dQ^2} \left( \frac{q^+(x, Q^2)}{G(x, Q^2)} \right) = - \left( \begin{array}{ccc} γ^wψ(x, a_s) & γ^wG(x, a_s) \\ γ^Gψ(x, a_s) & γ^Gx^2(x, a_s) \end{array} \right) \otimes \left( \begin{array}{c} q^+(x, Q^2) \\ G(x, Q^2) \end{array} \right) \] (37)

\[ Q^2 \frac{d}{dQ^2} \left( \frac{q^+_f(x, Q^2)}{n_f} \right) = -γ^{ns,+}(x, a_s) \otimes \left( q^+_f(x, Q^2) - \frac{q^+(x, Q^2)}{n_f} \right) \] (38)

\[ Q^2 \frac{d}{dQ^2} \left( q^-_f(x, Q^2) \right) = -γ^{ns,-}(x, a_s) \otimes q^-_f(x, Q^2) \] (39)

4This is called ‘intermediate expression swell’. It is a well known phenomenon in Computer Algebra.
where \( q^+_f = q_f + q_f \) and \( q^+ = \sum_{f=1}^{n_f} q^+_f \) are the quark distributions and \( G \) is the gluon distribution. It is understood that the inverse Mellin transforms of the anomalous dimensions are related to the standard Altarelli-Parisi splitting functions \( P^{ij}(x) \) as \( \gamma^{ij}(x) = -P^{ij}(x) \). The symbol \( \otimes \) indicates the convolution integral

\[
A(x) \otimes B(x) \equiv \int_0^1 dx_1 \int_0^1 dx_2 A(x_1)B(x_2)\delta(x-x_1x_2) = \int_1^1 dx_1 A\left(\frac{x}{x_1}\right)B(x_1). \tag{40}
\]

Since we calculated the moments only for electromagnetic scattering we are restricted to the “+” type anomalous dimensions and coefficient functions for the non-singlet sector, i.e. to the distributions which correspond to analytic continuation from the odd moments and which are relevant for processes that are not symmetric with respect to crossing symmetry such as neutrino scattering via W-boson exchange. The proper relation between the structure functions and parton distributions for electromagnetic scattering is

\[
F_k(x,Q^2) = x \left\{ \sum_{f=1}^{n_f} e_f^2 \left[ C_{k}^{ns,+}(x,a_s) \otimes \left(q_f^+(x,Q^2) - \frac{1}{n_f} q^+(x,Q^2)\right) \right] \\
+ \frac{1}{n_f} \sum_{f=1}^{n_f} e_f^2 \left[ C_{k}^{\psi,+}(x,a_s) \otimes q^+(x,Q^2) + C_{k}^{G,+}(x,a_s) \otimes G(x,Q^2)\right] \right\}, \tag{41}
\]

where \( e_f \) is the electromagnetic charge of a quark of the flavour \( f \).

A special feature of the \( Q^2\)-evolution in \( x\)-space (that is a direct consequence of the convolution integral) is that the structure functions \( F_k(x,Q^2) \) at some value \( x = x_0 \) only depend on the parton distributions, splitting functions and coefficient functions in the region \( x_0 \leq x \leq 1 \). This means that one does not need to know the anomalous dimensions and coefficient functions below the experimentally accessible \( x\)-region (but one always needs them for \( x \approx 1 \)).

One can see that, away from the singular limits \( x \to 0 \) and \( x \to 1 \) that received special attention in the literature, the 1-loop and 2-loop \( \gamma(x) \) and \( C(x) \) \cite{Forte2000,Forte2001} behave smoothly (as a smooth interpolation between the small-\( x \) and large-\( x \) regions). As a consequence the distributions \( \gamma(x) \) and \( C(x) \) can be approximated to a high precision by linear combinations of simple distributions that contain the singular terms of the expansions of the exact \( \gamma(x) \) and \( C(x) \) in the limits \( x \to 0 \) and \( x \to 1 \), i.e. \( \log^i(x)/x \) and \( \log^i(x) \) for small \( x \) and \( [\log^i(1-x)/(1-x)]_+ \), \( \log^i(1-x) \) and \( \delta(1-x) \) for large \( x \) plus finite order polynomials of \( x \) (the order determines the accuracy of the approximation).

(The exact results for \( \gamma(x) \) and \( C(x) \) contain in general complicated polylogarithmic functions but after their expansions in the small-\( x \) and large-\( x \) regions one finds the simple structure of the singularities mentioned above.)

The most singular terms of \( \gamma(x) \) and \( C(x) \) in the limit \( x \to 0 \) (i.e. \( \log^i(x)/x \) ) show up as singularities in moment space for \( N = 1 \). By increasing the moment index \( N \), one increasingly...
probes the region $x \approx 1$ (since the factor $x^{N-1}$ suppresses the contribution from the small-$x$ region). In particular, when the moments grow for large $N$ this is due to terms of the type $[\log^i(1-x)/(1-x)]_+, i \geq 0$ (of which the moments increase as $\log^{i+1}(N)$ for large $N$).

The approximations of $\gamma(x)$ and $C(x)$ in terms of the linear combinations of simple distributions (i.e. $\log^i(x)/x$, $\log^i(1-x)/(1-x)$, $\log^i(1-x)$, $\delta(1-x)$ and low powers of $x$) provides a natural scheme to study the effects of the 3-loop moments $N=2,4,6,8,10$ on the $Q^2$-dependence of structure functions in $x$-space and to quantify the uncertainty due to our lack of knowledge of the higher (i.e. $N > 10$) 3-loop moments. Considering such approximate distributions for the a priori unknown $\gamma(x)$ and $C(x)$ at 3-loops we may impose the correct asymptotic moment behavior for $N \to \infty$ and $N \to 1$ by including in the linear combinations of simple distributions the correct leading (and subleading) terms in the large-$x$ and small-$x$ regions. In some cases information about these leading (and subleading) terms in the large and small $x$ regions is available in the literature.

The linear combinations of distributions should also include various functions of a less singular type (compared to the leading singularities), e.g. $\log^i(x)$, $\log^i(1-x)$ and low powers of $x$, which are relevant terms in the intermediate $x$ region. We fit all unknown coefficients in a linear combination of simple distributions such that the moments $N=2,4,6,8,10$ of the linear combinations give the calculated values. Since we know only 4 singlet (and 5 non-singlet) moments explicitly we can only allow up to 4 (or 5) arbitrary coefficients (i.e. types of simple distributions) in the linear combinations. Only in the case of the quark anomalous dimension we can allow more than 5 arbitrary coefficients by considering pure-singlet (ps) and non-singlet (ns) parts separately (since information about the small-$x$ and large-$x$ behaviour is known for these parts separately).

Although at the 3-loop order there are only a finite number of allowed simple distributions of the types $\log^i(x)/x$, $\log^i(1-x)/(1-x)$, $\log^i(1-x)$, $\log^i(x)$, $\log^i(1-x)$ we do not have enough moments to include all of them simultaneously in a fit and we have to choose among the possible simple distributions. Therefore, to obtain an estimate of the uncertainty of the reconstructions due to our lack of knowledge of the higher moments we studied the stability of the $x$-space reconstructions with respect to a change of the simple distributions included in the linear combinations. If one does not find stable results under such a change one should have included a larger number of simple distributions of the types mentioned above. However, as one can see below, in many cases we have sufficient information to obtain stable results.

For a correct $x$-space reconstruction it is crucial that one knows the moments exactly (or with a high precision) since the inversion of moments into $x$-space distributions from a limited set of moments is numerically unstable, i.e. a small change in one of the moments can lead to a large change in the reconstructed result. In relation to this, we note that also the dedicated procedure for reconstructing $\gamma(x)$ and $C(x)$ that we consider here requires very accurate numerical values for the moments. Therefore one can not rely on interpolation between known even moments to increase the number of available moments.

### 5.1 Anomalous Dimensions

We will now consider the 3-loop singlet and non-singlet anomalous dimensions. We will use the following notation for the inverse Mellin transformed anomalous dimensions

$$\gamma^{ns}(x,a_s) = a_s \gamma^{ns,(0)}(x) + a_s^2 \gamma^{ns,(1)}(x) + a_s^3 \gamma^{ns,(2)}(x) + O(a_s^4)$$

(42)
\[ \gamma^{ij}(x, a_s) = a_s \gamma^{ij,(0)}(x) + a_s^2 \gamma^{ij,(1)}(x) + a_s^3 \gamma^{ij,(2)}(x) + O(a_s^4), \quad i, j = \psi, G \]  

where \( a_s = \alpha_s/(4\pi) \). In Ref. [33] it was proven that non-singlet anomalous dimensions are finite as \( N \to 1 \) which means that in \( x \)-space the non-singlet anomalous dimensions are less singular than \( 1/x \) as \( x \to 0 \). The leading singularity has been derived [34] from the leading order small-\( x \) resummation of the non-singlet evolution kernels [33].

\[ \gamma^{ns,(2)}(x \to 0) = -\frac{2}{3} C_F^2 \log^4(x) + O(\log^3(x)). \]  

The singlet anomalous dimensions are in general singular as \( N \to 1 \) (ie \( x \to 0 \)). The following results, which have been derived using small-\( x \) resummations, may be found in the literature [36, 37, 38].

\[ \begin{align*} 
\gamma^{\psi\psi,(2)}(x \to 0) &= \frac{896}{27} n_f C_A C_F \log(x)/x + O(1/x) \\
\gamma^{\psi G,(2)}(x \to 0) &= \frac{896}{27} n_f C_A^2 \log(x)/x + O(1/x) \\
\gamma^{G\psi,(2)}(x \to 0) &= O(\log(x)/x) \\
\gamma^{GG,(2)}(x \to 0) &= O(\log(x)/x). 
\end{align*} \]  

The last two equations simply mean that there are no leading singularities of the type \( \log^2(x)/x \) (since the \( a_s^3 \) term in the BFKL anomalous dimension [36] vanishes).

For the limit \( x \to 1 \) of the anomalous dimensions not much is known except for the conjecture [4] that the diagonal elements \( \gamma^{\psi\psi} \) and \( \gamma^{GG} \) do not rise faster than \( \log(N) \) as \( N \to \infty \), which means that one has in \( x \)-space a leading term \([1/(1-x)]_+ \) (and not \([\log(1-x)]/[(1-x)]_+ \)). This conjecture is based on the explicitly known one and two loop results. But the rise of the 3-loop non-singlet moments observed for \( N = 2, 4, 6, 8, 10 \) indicates that it also holds at the 3-loop level (putting both the terms \([1/(1-x)]_+ \) and \([\log(1-x)]/[(1-x)]_+ \) in the linear combination of simple distributions gives a small numerical coefficient for the second term compared to the first one, although this is clearly not a proof). Furthermore, the one and two loop coefficients of the terms \([1/(1-x)]_+ \) in \( \gamma^{\psi\psi} \) and \( \gamma^{GG} \) have a simple ratio \( C_F/C_A \) and we presume that this ratio is the same at 3-loops.

The off-diagonal singlet anomalous dimensions \( \gamma^{\psi G} \) and \( \gamma^{G\psi} \) go to zero as \( N \to \infty \). The best way to see that they do not contain terms \([\log^i(1-x)/(1-x)]_+ \), \( i \geq 0 \), is to consider the approach of Refs. [4, 5, 8] to calculate the anomalous dimensions, i.e. to study the renormalization of the singlet twist-2 operators where one can see that the terms \([\log^i(1-x)/(1-x)]_+ \) originate from diagrams that only appear for diagonal anomalous dimensions (notice that this method allows the direct calculation of these terms without performing the complete calculation of the anomalous dimensions). Therefore, the leading terms in \( \gamma^{\psi G} \) and \( \gamma^{G\psi} \) as \( x \to 1 \) that one can expect from performing the necessary integrals at 3-loops are \( \log^4(1-x) \).

Finally we presume that the pure singlet (ps) combination \( \gamma^{ps} = \gamma^{\psi\psi} - \gamma^{ns} \) rapidly vanishes as \( N \to \infty \) (i.e. vanishes at least as quick as \( 1/N \) for large \( N \); at 2-loops it is known to vanish as \( 1/N^4 \)). We have observed this tendency already in the low-N moments also at 3-loops. This means that \( \gamma^{\psi\psi} \) and \( \gamma^{ns} \) contain the same terms that are important for \( x \sim 1 \) (since the transform from \( x \)- to N-space for large \( N \) gives \([\log^i(1-x)/(1-x)]_+ \to \log^{i+1}(N), \delta(1-x) \to 1 \)
and \( \log^i(1 - x) \rightarrow \log^i(N)/N \) and that they only differ in terms that are important for small-\( x \). In relation to this, recall that only \( \gamma^{\psi\psi} \) contains a term \( \log(x)/x \) and \( \gamma^{ns} \) does not, see Eqs. (44,45).

**Figure 4.** The exact 2-loop anomalous dimensions and reconstructions (approxs.) based on the lowest 5 moments \( N=2,4,6,8,10 \) \((n_f = 4)\). The approximations are obtained by fitting the sets of distributions of Eq. (46) to these 5 moments.
At 2-loops the analysis based on only a few low-N anomalous dimensions can be compared with the exact results as is shown in figure 4. The indicated approximations in figure 4 are based on matching the linear combinations of simple distributions of Eq. (46) to 5 moments (N=2, 4, 6, 8, 10) of each anomalous dimension $\gamma_{ps}$, $\gamma_{ns}$, $\gamma_{\psi G}$, $\gamma_{G\psi}$ and $\gamma_{GG}$.

$$
\gamma_{ps,(1)} , \gamma_{\psi \psi, (1)} : \left\{ \frac{1}{x}, \log(x), 1, \log(1-x), \log^2(1-x) \right\},
$$

$$
\gamma_{ns,(1)} : \left\{ \log^2(x), \log(x), \log(1-x), \left[ \frac{1}{1-x} \right] + , \delta(1-x) \right\},
$$

$$
\gamma_{GG,(1)} : \left\{ \frac{1}{x}, \log^2(x), 1, \log(1-x), \left[ \frac{1}{1-x} \right] + , \delta(1-x) \right\},
$$

$$
\gamma_{\psi G,(1)} , \gamma_{G\psi, (1)} : \left\{ \frac{1}{x}, \log^2(x), 1, \log(1-x), \left[ \frac{1}{1-x} \right] + , \delta(1-x) \right\},
$$

In these formulae each pair of curly brackets encloses a set of simple distributions that appear in a linear combination with coefficients to be matched. We take 2 different linear combinations for each anomalous dimension to see the stability of the fit.

Please note that all the coefficients in these linear combinations were taken as arbitrary in the fit (i.e. we even did not fix the coefficients of the leading singularities) to have a strong check of the approach. The only additional information that is used in the reconstruction is that the coefficients of the most singular large-$x$ terms $[1/(1-x)]_+$ in $\gamma_{\psi \psi}$ and $\gamma_{GG}$ have a simple ratio $C_F/C_A$.

The considered distributions are singular at $x = 0$ and $x = 1$, and because e.g. $\delta(1-x)$ can not be drawn one should consider the figures only as an indication of the accuracy of the full distributions which are used in convolutions (where all the terms that are singular at $x = 1$ fully contribute). The approximations are stable for $x > .2$ and do dramatically improve in the small-$x$ region when more moments (N=12, 14, …) are used such that more simple distributions can be included in the linear combinations.
The effects of the NNL order (i.e. of the 3-loop anomalous dimensions and of the 3-loop beta-function) on the evolution of the singlet distributions $q(x, Q^2)$ and $G(x, Q^2)$. The parton distributions $q_{NL}^{+}, G_{NL}$ and $q_{NNL}^{+}, G_{NNL}$ are obtained as solutions of the DGLAP evolution equations \cite{37}, in respectively the NL and NNL approximations starting from the parametrization scale $Q_0^2 = 4 \text{ GeV}^2$. The difference between curves for the same $Q^2$ indicates the uncertainty in the NNL order effects due to the lack of knowledge of the higher anomalous dimensions $\gamma_N^{(2)}, N > 8$. The parton distributions at $Q_0^2$ correspond to the MRS(A) set \cite{38}. For simplicity, $n_f = 4$ and $\Lambda_{QCD} = 300 \text{ MeV}$ are taken throughout the evolution.

The effects of the NNL order on the evolution of the singlet distributions $q^+(x, Q^2)$ and $G(x, Q^2)$ is illustrated in figure 5. The evolutions are done using a program based on the Laguerre polynomial technique of Ref. \cite{39}. An estimate of the uncertainty in the NNL effects due to unknown higher moments is made by fitting various linear combinations of simple distributions to the calculated 3-loop moments of the anomalous dimensions. The curves correspond to the sets of simple distributions of Eq. (47) (two sets for each anomalous dimension) which are representative for a larger variation in the choice of simple distributions.

\[
\gamma^{\psi G,(2)} = \left\{ \log(x)/x, \log(x), 1, \log(1 - x), \log^4(1 - x) \right\},
\]

\[
\gamma^{G\psi,(2)} = \left\{ \log(x)/x, \log^2(x), \log^2(1 - x), \log^4(1 - x) \right\},
\]

\[
\gamma^{\psi G,(2)} = \left\{ 1, \log(1 - x), \log^2(1 - x), \log^3(1 - x) \right\},
\]

\[
\gamma^{G\psi,(2)} = \left\{ \log(x)/x, \log^2(x), \log^2(1 - x), \log^4(1 - x) \right\}.
\]
\[ \gamma_{GG,(2)} : \left\{ 1, \log^2(1-x), \log^4(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \right\}, \]
\[ \log(x)/x, 1, \log^2(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \} . \]
\[ \gamma_{ns,(2)} : \left\{ \log^4(x), 1, \log^2(1-x), \log^4(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \right\}, \]
\[ \log^4(x), \log^2(x), 1, \log^2(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \} . \]
\[ \gamma_{ps,(2)} : \{ \log(x)/x, 1/x, \log^3(x), 1, x^2 \} \]
\[ \{ \log(x)/x, \log^4(x), \log^2(x), 1, x^2 \} \]

The coefficients of the leading \( x \to 0 \) terms of \( \gamma_{\psi G,(2)} \), \( \gamma_{ps,(2)} \) and \( \gamma_{ns,(2)} \) are known exactly, see Eqs. (44,45). Please notice that in the variation of simple distributions we changed even the type of the expected leading singularities as given in Eq. (47) (when little information about these leading singularities is known) to obtain a conservative estimate of the accuracy of the reconstructions. One may see that the NNL effects are of the order of a few percent in the region of \( x > 0.1 \).

Approximately 50% of the shown NNL order effect on the evolution of the parton distributions is due to the inclusion of the 3-loop beta-function and the other NNL order terms in the expression for the strong coupling constant, Eq. (18). The other 50% is due to the 3-loop anomalous dimensions. The estimated uncertainty is small in the evolution of \( q^+(x) \) and is somewhat larger in the evolution of \( G(x) \). This is a consequence of the especially good reconstruction of the quark anomalous dimensions (see figure 6), which is itself due to the existence of separate information about the pure-singlet and non-singlet parts.

The curves in figure 6 correspond to matching the following linear combinations of simple distributions to the calculated moments (where the coefficients of the leading \( x \to 0 \) terms are known exactly, see Eqs. (44,45))

\[ \gamma_{ns,(2)} : \{ \log^4(x), \log^2(x), 1, \log^2(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \} , \]
\[ \{ \log^4(x), \log(x), \log(1-x), \log^3(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \} , \]
\[ \{ \log^4(x), 1, \log^2(1-x), \log^4(1-x), \left[ \frac{1}{1-x} \right]_+, \delta(1-x) \} , \]
\[ \gamma_{ps,(2)} : \{ \log(x)/x, 1/x, \log^4(x), \log^2(x), 1 \} \]
\[ \{ \log(x)/x, 1/x, \log^3(x), 1, x^2 \} \]
\[ \{ \log(x)/x, \log^4(x), \log^2(x), 1, x^2 \} \]

(48)
Figure 6. The quality of the reconstruction of the 3-loop quark anomalous dimensions based on the moments $N=2,4,6,8,(10$ for non-singlet only) and the known leading small-$x$ terms for $\gamma^{qs,(2)}$ and $\gamma^{qs,(2)}$. The reconstructions are obtained by fitting the sets of distributions of Eq. (48) to the available moments.

From the foregoing analysis we have concluded that the inclusion of the 3-loop anomalous dimensions in the evolution of parton distributions in the $\overline{\text{MS}}$-scheme changes both the quark and gluon distributions by 1-3% in the region $0.1 < x < 0.8$ for a change in $Q^2$ between $(2 \text{ GeV})^2$ and $(100 \text{ GeV})^2$.

We further remark that it is important to obtain the exact results for the 3-loop anomalous dimensions for all $N$ to do a complete analysis in the NNL order.

5.2 Coefficient Functions

As an example of the effects of the calculated 3-loop coefficient functions on the deep inelastic structure functions we consider the case of $C_2^{ns}$. We use the following notation for the inverse Mellin transformed coefficient function.

$$C_2^{ns}(x, a_s) = C_2^{ns,(0)}(x) + a_s C_2^{ns,(1)}(x) + a_s^2 C_2^{ns,(2)}(x) + a_s^3 C_2^{ns,(3)}(x) + O(a_s^4).$$ (49)

Please note that for the complete NNNL order approximation for $F_2$ one would need, besides the 3-loop $C_2^{ns,(3)}$, the 4-loop anomalous dimensions which are difficult to calculate at present.

Presuming that the anomalous dimensions are not more singular than $\log(N)$ as $N \rightarrow \infty$ one has shown that all the logarithms $\log^i(N)$ that are present in the non-singlet parts of the coefficient functions exponentiate according to the soft gluon resummation formulae [40, 41, 42, 43]. We will
functions we find the following sets of simple distributions to the first 5 moments $N=2, 4, 6, 8, 10$ exact results. In figure 7 we illustrated the quality of the reconstruction based on matching the coefficient functions do contain terms of the type $\log^i N$, and that for the second set only the coefficient of $\log^2(1-x)/(1-x)$ was taken exactly. The singlet coefficient function contains the same non-singlet contribution. The small-$x$ limits of the singlet coefficient functions do contain terms of the type $\log^i(x)/x$, $i \geq 0$. At 3-loops one expects a term $\log^i(x)/x$.

At the two loop order the reconstruction of the coefficient functions can be compared with the exact results. In figure 7 we illustrated the quality of the reconstruction based on matching the following sets of simple distributions to the first 5 moments $N=2, 4, 6, 8, 10$

$$C_{2, N}^{ns, (2)} : \left\{ \begin{array}{c} \log(x), 1, x^2, \delta(1 - x), \left[ \frac{1}{1-x} \right]_+, \left[ \frac{\log^2(1-x)}{1-x} \right]_+, \left[ \frac{\log^3(1-x)}{1-x} \right]_+ \\ \log^2(x), 1, \delta(1 - x), \left[ \frac{1}{1-x} \right]_+, \left[ \frac{\log(1-x)}{1-x} \right]_+, \left[ \frac{\log^2(1-x)}{1-x} \right]_+ \end{array} \right\}$$

(52)

It is understood that for the first set the coefficients of the last two terms are taken from Eq. (51), and that for the second set only the coefficient of $\log^3(1-x)/(1-x)$ was taken exactly. One can see from figure 7 that the reconstruction of the distribution $C_{2, N}^{ns, (2)}$ is quite stable for $x > 0.2$ . We want to emphasize that considering only the two most singular soft gluon terms (i.e. $[\log^3(1-x)/(1-x)]_+$ and $[\log^2(1-x)/(1-x)]_+$) gives a huge overestimate of the true effects. 

\footnote{for a table of the moments of the relevant distributions in the large $N$ limit, see e.g. Ref. [2].}

use this to obtain the large N limit of moments of the 3-loop $C_{2}^{ns}$ coefficient function (i.e. the $x \to 1$ limit of $C_{2}^{ns}$). Exponentiating the $\log^i(N)$ terms from the explicitly known 1 and 2-loop coefficient functions we find

$$C_{2, N}^{ns} \xrightarrow{N \to \infty} \exp \left\{ a_s \left[ 2C_F \log^2(N) + C_F(3 + 4\gamma_E) \log(N) \right] + a_s^2 \left[ \frac{2}{3} C_F(\frac{11}{3} C_A - \frac{2}{3} n_f) \log^3(N) \right. \right.$$ 

$$+ O(\log^2(N)) \left] + O(a_s^3 \log^4(N)) \right\} + O(a_s N^0)$$

(50)

where $\gamma_E$ is the Euler constant. In $x$ space this corresponds to

$$C_{2}^{ns}(x) \xrightarrow{N \to \infty} 1 + a_s \left\{ 4C_F \left[ \frac{\log(1-x)}{1-x} \right]_+ - 3C_F \left[ \frac{1}{1-x} \right]_+ + O(\delta(1 - x)) \right\} + a_s^2 \left\{ 8C_F \left[ \frac{\log^2(1-x)}{1-x} \right]_+ \right.$$ 

$$+ \left( -\frac{22}{3} C_AC_F + \frac{4}{3} C_F n_f - 18C_F \right) \left[ \log^2(1-x)/(1-x) \right]_+ + O \left[ \left( \frac{\log(1-x)}{1-x} \right)_+ \right] \right\}$$

$$+ a_s^3 \left\{ 8C_F \left[ \frac{\log^3(1-x)}{1-x} \right]_+ + ( -\frac{220}{9} C_AC_F^2 + \frac{40}{9} C_F^2 n_f - 30C_F^3 ) \left[ \log^4(1-x)/(1-x) \right]_+ \right.$$ 

$$+ O \left[ \left( \frac{\log^3(1-x)}{1-x} \right)_+ \right] \right\} + O \left[ \left( a_s^4 \left[ \frac{\log^4(1-x)}{1-x} \right]_+ \right) \right].$$

(51)
However, as it was observed in Ref. [10], all the terms of the type \( \log^i(1-x)/(1-x) \), \( i \geq 0 \), together constitute a large fraction of \( C_{ns,2}^{n_s(2)} \otimes \Delta \) in the region \( x > 0.5 \) (more precisely we obtain 40% at \( x = 0.6 \) and 95% at \( x = 0.75 \)).

**Figure 7.** The exact 2-loop coefficient function \( C_{ns,2}^{n_s(2)} \) and reconstructions (approxs.) based on the lowest 5 moments \( N=2,4,6,8,10 \) and the sets of distributions of Eq. (52). The left plot shows the reconstruction for intermediate \( x \). The right plot shows the quality of the reconstructed coefficient function as a distribution in the whole \( x \)-region, i.e. when it is convoluted with a realistic parton distribution \( \Delta = u + \pi - d - \bar{d} \). The quark distributions correspond to the MRS(A) set [38] at its parametrization scale \( Q_0^2 = 4 \text{ GeV}^2, n_f = 4 \).

The present status of the effects of the higher order contributions to \( C_{ns,2}^{n_s} \) is illustrated in figure 8. The 3-loop contribution is obtained by matching the following 4 linear combinations of distributions

\[
C_{2}^{n_s,(3)} : \left\{ 1, x^2, \delta(1-x), \left[ \frac{1}{1-x} \right]_+, \left[ \frac{\log^2(1-x)}{1-x} \right]_+, \left[ \frac{\log^4(1-x)}{1-x} \right]_+ \right\} \\
\left\{ \log(x), 1, x^2, \delta(1-x), \left[ \frac{\log^2(1-x)}{1-x} \right]_+, \left[ \frac{\log^4(1-x)}{1-x} \right]_+ \right\} \\
\left\{ \log(x), 1, x^2, \delta(1-x), \left[ \frac{1}{1-x} \right]_+, \left[ \frac{\log^4(1-x)}{1-x} \right]_+ \right\} \\
\left\{ \log^2(x), 1, x, \delta(1-x), \left[ \frac{1}{1-x} \right]_+, \left[ \frac{\log^4(1-x)}{1-x} \right]_+ \right\}
\]

to the 5 available moments (\( N=2,4,6,8,10 \)), where the numerical coefficients of the terms \( \left[ \log^5(1-x)/(1-x) \right]_+ \) and \( \left[ \log^4(1-x)/(1-x) \right]_+ \) are taken from Eq. (51). Please notice that in the variation of simple distributions in the 4 linear combinations above we changed even the type of the expected small-\( x \) singularities to obtain a conservative estimate of the accuracy of the reconstructions.
Figure 8. Higher order corrections to $F_2$ due to the non-singlet coefficient function $C_{ns}^m$. The different curves show the effects on $F_2$ of higher order corrections to $C_{ns}^m$ via the relation $\frac{1}{2} F_2^{ep}(x, Q^2) - \frac{1}{2} F_2^{en}(x, Q^2) = C_{ns}^m \otimes \frac{1}{3}(u + \bar{u} - d - \bar{d})$, when both the quark distributions and $\alpha_s$ are kept constant. The spread of the different NNNLO curves indicates the uncertainty of the 3-loop contribution (due to variations of the distributions as it is given in Eq. (53)). We used the parton distributions in the MS scheme from the MRS(A) global fit \cite{38} (in the NL order) at its parametrization scale $Q^2 = 4$ GeV$^2$ and $\alpha_s = 0.261, n_f = 4$. (LO = leading order, NLO = next-to-leading order, etc.)
For the non-singlet combination of the nucleon structure functions \( F_2^{ep}(x, Q^2) - F_2^{en}(x, Q^2) \) that is presented in figure 8, \( C_{ns}^2 \) is the only relevant coefficient function. The different curves show the effects on \( F_2 \) of higher order corrections to \( C_{ns}^2 \) when both the quark distributions and \( \alpha_s \) are kept constant. Although the change in \( F_2 \) is indicated when the order of approximation for \( C_{ns}^2 \) changes (LO indicates \( C_{ns}^2(0) \), NLO indicates \( C_{ns}^2(0) + \alpha_s C_{ns}^2(1) \), etc.), it is in fact the quark distributions that may have to be modified to keep \( F_2 \) in agreement with the experiments over a large \((x, Q^2)\) range when higher order corrections are globally included. We noticed that also at 3-loops the two leading soft gluon contributions are to a large extent suppressed by ‘subleading’ contributions in the coefficient functions.

At a relatively low energy scale of \( Q^2 = 4 \text{ GeV}^2 \) of figure 8 we find that the 3-loop contribution \( a_3^2 C_{ns}^2(3) \) gives a sizable correction to the coefficient function \( C_{ns}^2 \) (about 1/3 of the 2-loop contribution \( a_2^2 C_{ns}^2(2) \) in the \( x \) region \( 0.2 \leq x \leq 0.9 \)). Furthermore, from figure 8 one can observe apparent convergence of the QCD perturbation theory up to and including the NNNL order.

### 6 The analytic results of the 3-loop calculation

Before we present the results for the anomalous dimensions and coefficient functions we should explain our conventions for the different flavour factors that appear in the present 3-loop calculation.

\[
FC_2 \quad FC_{11} \quad FC_{02} \quad FC_2^g \quad FC_{11}^g
\]

**Figure 9.** Examples for diagrams in the flavour classes \( FC_2, FC_{11}, FC_{02}, FC_2^g, FC_{11}^g \).

Diagrams in the flavour class \( FC_2 \) (where both photons are attached to the quark line of the external quark legs) have a \( \text{SU}(n_f) \) flavour factor \( \hat{Q}_f^2 \) where the matrix \( \hat{Q}_f \) is the quark charge matrix from the electromagnetic current (\( \hat{Q}_f = \text{diag}(-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \cdots) \) and \( \text{tr}(\hat{Q}_f) = \sum_{f=1}^{n_f} e_f \) where \( e_f \) is the electromagnetic charge of a quark with the corresponding flavour). Diagrams in the flavour class \( FC_{11} \) (where one photon is attached to a closed quark loop and the other one to the quark line of the external quark legs) have a flavour matrix \( \text{tr}(\hat{Q}_f \hat{Q}_f) \) and diagrams in the flavour class \( FC_{02} \) (flavour singlet diagrams, where both photon legs are attached to the same internal quark loop) have a flavour matrix \( \text{tr}(\hat{Q}_f^2) \mathbf{1} \). The diagrams with external gluons are split up into two flavour classes: the class \( FC_2^g \) has a flavour factor \( \text{tr}(\hat{Q}_f^2) \) and the class \( FC_{11}^g \) has a flavour factor \( (\text{tr}(\hat{Q}_f))^2 \).

To project out the non-singlet contributions one should take the flavour trace of the diagrams with the generators \( \lambda^a \) of \( \text{SU}(n_f) \) (due to the diagonal form of \( \hat{Q}_f \) only the diagonal generators are relevant). This means that in the non-singlet projection only the diagrams from the flavour classes \( FC_2 \) and \( FC_{11} \) contribute. In this way the diagrams of the class \( FC_2 \) receive a flavour factor
\[ \text{tr}(\hat{Q}_f^2 \lambda^\alpha), \text{ and the diagrams of the class } FC_{11} \text{ receive a flavour factor } \text{tr}(\hat{Q}_f)\text{tr}(\hat{Q}_f^2 \lambda^\alpha). \text{ Since the ratio of these flavour factors does not depend on the number } \alpha \text{ of a diagonal generator } \lambda^\alpha \]

\[ \frac{\text{tr}(\hat{Q}_f^2 \lambda^\alpha)}{\text{tr}(\hat{Q}_f^2 \lambda^\alpha)} = 3 \]  

(54)

it is possible to factorize the \( \alpha \)-independent coefficient functions \( C_{k,N}^{n,s} \) as follows

\[ \sum_\alpha C_{k,N}^\alpha A_{\text{nucl},N} = \left[ C_{k,N}(FC_2) + \frac{\text{tr}(\hat{Q}_f)\text{tr}(\hat{Q}_f^2 \lambda^\beta)}{\text{tr}(\hat{Q}_f^2 \lambda^\beta)} C_{k,N}(FC_{11}) \right] \times \left[ \left( \sum_\alpha \text{tr}(\hat{Q}_f^2 \lambda^\alpha) \right) A_{\text{nucl},N}^\alpha \right] \]

\[ = C_{k,N}^{n,s} \left( \frac{Q^2}{\mu^2}, a_s \right) \times A_{\text{nucl},N}^{n,s}(p^2/\mu^2), \]

(55)

where the non-singlet \( C_{k,N}^{n,s} \) is independent of \( \alpha \) and the standard normalization is \( C_{k,N}^{n,s} = 1 + O(a_s) \).

To project out the flavour singlet contributions for diagrams with external quark legs one should take the trace with the unit flavour matrix. The explicit values for the flavour factors \( \text{fl}_2, \text{fl}_{11}, \text{fl}_{02}, \text{fl}_2^0, \text{fl}_{11}^0 \) (for the flavour classes \( FC_2, FC_{11}, FC_{02}, FC_2^g \) and \( FC_{11}^g \) respectively) that give the standard normalization of the coefficient functions for the non-singlet and singlet cases are given in table 2. Please notice that singlet flavour factors are chosen such that they reduce to unity if one substitutes for \( \hat{Q}_f \) the unit matrix \( \hat{Q}_f = \text{diag}(1,1,1, \cdots) \). Please notice that the factor 3 in the non-singlet flavour factor \( \text{fl}_{11} \) originates from Eq. (54).

<table>
<thead>
<tr>
<th></th>
<th>( \text{fl}_2 )</th>
<th>( \text{fl}_{11} )</th>
<th>( \text{fl}_{02} )</th>
<th>( \text{fl}_2^0 )</th>
<th>( \text{fl}_{11}^0 )</th>
</tr>
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<td>non-singlet</td>
<td>1</td>
<td>( \frac{1}{n_f} \sum_{f=1}^{n_f} e_f )</td>
<td>0</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>singlet</td>
<td>1</td>
<td>( \frac{1}{n_f} \left( \sum_{f=1}^{n_f} e_f \right)^2 )</td>
<td>( \frac{1}{n_f} \left( \sum_{f=1}^{n_f} e_f \right)^2 )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2. Values for the flavour factors \( \text{fl}_2, \text{fl}_{11}, \text{fl}_{02}, \text{fl}_2^0, \text{fl}_{11}^0 \).

Below we present the results for the 3-loop anomalous dimensions and the 3-loop coefficient functions. Please notice that the results contain both the singlet and non-singlet contributions. To obtain the specific non-singlet or singlet case one should substitute the flavour factors from table 2. In the results we have already put \( \text{fl}_2 = \text{fl}_2^0 = 1 \). The reader may notice an overall factor 2 difference between our results for anomalous dimensions and the results in Ref. [3] which originates from our definition of the renormalization scale derivative \( d/(d \ln(\mu^2)) \) (see Eqs. (14) and (15)) versus the derivative \( d/(d \ln(\mu)) \) in Ref. [3]. In the results for the coefficient functions we have put \( \mu^2 = Q^2 \) (in accordance with the evolution equation Eq.(17)). The Riemann zeta function is written as \( \zeta_n \).
\[ \gamma_2^{\psi} = a_s C_F \frac{8}{3} + a_s^2 \left[ C_F C_A \frac{376}{27} + C_F^2 \left( -\frac{412}{27} \right) + n_f C_F \left( -\frac{64}{27} \right) \right] \\
+ a_s^2 f_{l0} C_F n_f \left( -\frac{40}{27} \right) \\
+ a_s^3 \left[ C_F C_A^2 \left( \frac{2020}{243} + \frac{64}{3} \zeta_3 \right) + C_F^2 C_A \left( -\frac{8528}{243} - 64 \zeta_3 \right) + C_F^3 \left( -\frac{560}{243} + \frac{128}{3} \zeta_3 \right) \\
+ n_f C_F C_A \left( \frac{3128}{243} \right) + n_f C_F^2 \left( -\frac{3412}{243} + \frac{64}{3} \zeta_3 \right) + n_f^2 C_F \left( -\frac{224}{243} \right) \right] \\
+ a_s^3 f_{l0} \left[ C_F C_A n_f \left( \frac{2534}{243} - \frac{64}{3} \zeta_3 \right) + C_F^2 n_f \left( -\frac{3682}{243} + \frac{64}{3} \zeta_3 \right) + C_F n_f^2 \left( -\frac{628}{243} \right) \right] = a_s 3.555555556 \\
+ a_s^2 \left( 48.3292181070 - 3.1604938272 n_f \right) \\
+ a_s^2 f_{l0} \left( -1.9753086420 n_f \right) \\
+ a_s^3 \left( 859.4478371772 - 133.4381617374 n_f - 1.2290809328 n_f^2 \right) \\
+ a_s^3 f_{l0} \left( -42.2118242888 n_f - 3.4458161866 n_f^2 \right) \\
\]

\[ \gamma_4^{\psi} = a_s C_F \frac{157}{30} + a_s^2 \left[ C_F C_A \frac{16157}{675} + C_F^2 \left( -\frac{287303}{54000} \right) + n_f C_F \left( -\frac{13371}{2700} \right) \right] \\
+ a_s^2 f_{l0} n_f C_F \left( -\frac{2117}{2700} \right) \\
+ a_s^3 \left[ C_F C_A^2 \left( \frac{136066373}{972000} + \frac{1439}{75} \zeta_3 \right) + C_F^2 C_A \left( -\frac{267028157}{972000} \right) + \frac{1439}{75} \zeta_3 \right] \\
+ C_F^3 \left( -\frac{714245693}{48600000} + \frac{2878}{75} \zeta_3 \right) + n_f C_F C_A \left( -\frac{8805281}{4860000} \right) - \frac{628}{15} \zeta_3 \right] \\
+ n_f C_F^2 \left( -\frac{1652177563}{48600000} + \frac{628}{15} \zeta_3 \right) + n_f^2 C_F \left( -\frac{219077}{230000} \right) \right] \\
+ a_s^3 f_{l0} \left[ n_f C_F C_A \left( \frac{2485097}{2925000} + \frac{242}{75} \zeta_3 \right) + n_f^2 C_F \left( -\frac{2217031}{2700000} + \frac{242}{75} \zeta_3 \right) + n_f^3 C_F \left( -\frac{618673}{1215000} \right) \right] = a_s 6.9777777778 \\
+ a_s^2 \left( 86.2866502058 - 6.5535802469 n_f \right) \\
+ a_s^2 f_{l0} \left( -0.1060246914 n_f \right) \\
+ a_s^3 \left( 1515.5623634355 - 244.7285919523 n_f - 2.1085157750 n_f^2 \right) \\
+ a_s^4 f_{l0} \left( -5.1701331196 n_f - 0.6789278464 n_f^2 \right) \]
\[\gamma_{\psi} = a_s C_F \frac{709}{105} + a_s^2 \left[C_F C_A \frac{157415}{5292} + C_F^2 \left(- \frac{3173311}{514090} \right) + n_f C_F \left(- \frac{428119}{66150} \right)\right] + a_s^3 [C_F C_A^2 \left(\frac{11523791383}{66679200} + \frac{69862}{3675} \zeta_3\right) + C_F^2 C_A \left(\frac{34855421369}{1166886000} - \frac{69862}{1225} \zeta_3\right) + C_F^3 \left(\frac{854652999073}{5105126500} + \frac{139724}{3675} \zeta_3\right) + n_f C_F C_A \left(\frac{389784373}{686000} - \frac{5672}{105} \zeta_3\right) + n_f C_F^2 \left(\frac{4644018231}{672955000} + \frac{5672}{105} \zeta_3\right) + n_f^2 C_F \left(\frac{80347571}{41645000} \right) + n_s^3 C_F^2 \left(\frac{15602048711}{6336404000} - \frac{968}{735} \zeta_3\right) + n_f C_F^2 \left(\frac{51555763}{45380750} \right)]
= a_s \frac{9.0031746032}{105} + a_s^2 \left(108.0184697117 - 8.6292567397 n_f\right) + a_s^3 [189.15277788314 - 307.4236889402 n_f - 2.5706389919 n_f^2] + a_s^3 \left(- \frac{2.520911925 n_f}{0.288456035 n_f^2}\right)
\]

\[\gamma_{\psi} = a_s C_F \frac{9883}{1200} + a_s^2 \left[C_F C_A \frac{25870040}{7021845} + C_F^2 \left(- \frac{27040578211}{4009520000}\right) + n_f C_F \left(- \frac{36241043}{400952000}\right)\right] + a_s^3 [C_F C_A^2 \left(\frac{10105995503}{41150992000} + \frac{251047}{132500} \zeta_3\right) + C_F^2 C_A \left(- \frac{3662576909029}{1713201056000} - \frac{251047}{44100} \zeta_3\right) + C_F^3 \left(- \frac{108058710007437939}{6831598695200000} + \frac{251047}{66150} \zeta_3\right) + n_f C_F C_A \left(- \frac{15789415742233}{72013830000} - \frac{19766}{315} \zeta_3\right) + n_f C_F^2 \left(\frac{91675209372043}{168000315840000} + \frac{19766}{315} \zeta_3\right) + n_f^2 C_F \left(- \frac{38920977797}{180053840000}\right) + n_f^3 C_F \left(- \frac{34234829803}{229924712990000} - \frac{1369}{1890} \zeta_3\right) + n_f C_F^2 \left(\frac{3992937384469}{907796553660000} + \frac{1369}{1890} \zeta_3\right) + n_f^2 C_F \left(- \frac{13131081443}{1080921040000}\right) + n_f^3 C_F \left(- \frac{13131081443}{1080921040000}\right)]
= a_s \frac{10.4582010582}{105} + a_s^2 \left(123.7764525165 - 10.1458366227 n_f\right) + a_s^3 \left(- \frac{0.0065864851 n_f}{105}\right) + a_s^3 \left(2164.0918358230 - 352.3116595904 n_f - 2.8824934836 n_f^2\right) + a_s^3 \left(- \frac{1.6821565188 n_f}{0.1620816452 n_f^2}\right)
\]

\[\gamma_{2G} = a_s n_f \left(- \frac{2}{3}\right) + a_s^2 \left[n_f C_F \left(- \frac{74}{27} \right) + n_f C_A \left(\frac{35}{27}\right)\right] + a_s^3 \left[n_f C_F C_A \left(\frac{139}{5} - \frac{104}{105} \zeta_3\right) + n_f C_F^2 \left(- \frac{2155}{243} + \frac{32}{243} \zeta_3\right) + n_f C_A^2 \left(- \frac{3589}{102} + 24 \zeta_3\right) + n_f C_F \left(- \frac{172}{243}\right) + n_f^2 C_F \left(\frac{1058}{243}\right)\right]
= a_s \left(- \frac{0.6666666667 n_f}{105}\right) + a_s^2 \left(- \frac{7.5432098765 n_f}{105}\right) + a_s^3 \left(- \frac{37.6233727456 n_f + 12.1124828532 n_f^2}{105}\right)
\]

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\[ \gamma_4^{\psi G} = a_s n_f \left( -\frac{11}{30} \right) + a_s^2 \left[ n_f C_F \left( -\frac{56317}{18000} \right) + n_f C_A \left( \frac{46387}{9000} \right) \right] + a_s^3 \left[ n_f C_F C_A \left( \frac{278540497}{81000000} - \frac{969}{25} \zeta_3 \right) + n_f C_F^2 \left( -\frac{757117001}{450000000} + \frac{261}{25} \zeta_3 \right) + n_f C_A^2 \left( -\frac{29511093}{12150000} + \frac{708}{25} \zeta_3 \right) + n_f C_F \left( \frac{9613841}{24900000} \right) + n_f C_A \left( \frac{1481529}{243000} \right) \right] = a_s \left( -0.366666667n_f \right) + a_s^2 \left( 1.2907037037n_f \right) + a_s^3 \left( 33.5814927342n_f + 6.0602726200n_f^2 \right) \]

\[ \gamma_6^{\psi G} = a_s n_f \left( -\frac{11}{32} \right) + a_s^2 \left[ n_f C_F \left( -\frac{296083}{92010} \right) + n_f C_A \left( \frac{867311}{29480} \right) \right] + a_s^3 \left[ n_f C_F C_A \left( \frac{1199181909}{3267280000} - \frac{52666}{735} \zeta_3 \right) + n_f C_F^2 \left( -\frac{2933980223981}{163364040000} + \frac{1240}{117} \zeta_3 \right) + n_f C_A^2 \left( -\frac{296083}{261382140} + \frac{582}{245} \zeta_3 \right) + n_f C_F \left( \frac{1539874183}{2722734000} \right) + n_f C_A \left( \frac{86617163}{950500} \right) \right] = a_s \left( -0.2619047619n_f \right) + a_s^2 \left( 2.7611048123n_f \right) + a_s^3 \left( 33.4160213457n_f + 3.5376821018n_f^2 \right) \]

\[ \gamma_8^{\psi G} = a_s n_f \left( -\frac{37}{180} \right) + a_s^2 \left[ n_f C_F \left( -\frac{51090517}{163299000} \right) + n_f C_A \left( \frac{100911011}{408240000} \right) \right] + a_s^3 \left[ n_f C_F C_A \left( \frac{4896295442601}{12962436480000} - \frac{515201}{18900} \zeta_3 \right) + n_f C_F^2 \left( -\frac{4374484944655803}{226842638400000} + \frac{749}{108} \zeta_3 \right) + n_f C_A^2 \left( -\frac{248166560257253}{3150000000} + \frac{6421}{3150} \zeta_3 \right) + n_f C_F \left( \frac{790329784681}{12962436480000} \right) + n_f C_A \left( \frac{10379424541}{22044960000} \right) \right] = a_s \left( -0.2055555556n_f \right) + a_s^2 \left( 3.2439572229n_f \right) + a_s^3 \left( 28.7612614990n_f + 2.2254331118n_f^2 \right) \]
\[
\gamma_{2}^G = a_s C_F \left( -\frac{8}{3} \right) + a_s^{3} \left[ C_F C_A \left( -\frac{326}{27} \right) + C_F \left( \frac{112}{27} \right) + n_f C_F \left( \frac{104}{27} \right) \right]
+ a_s^{3} \left[ C_F C_A^{3} \left( -\frac{20920}{243} \right) - \frac{64}{8} \frac{\zeta_3}{3} \right] + C_F C_A \left( \frac{8528}{243} + 64\zeta_3 \right)
+ C_F^{3} \left( \frac{560}{243} - \frac{128}{3} \frac{\zeta_3}{3} \right) + n_f C_F C_A \left( \frac{97}{2} + \frac{128}{3} \frac{\zeta_3}{3} \right)
+ n_f C_F^{3} \left( \frac{7094}{243} - \frac{128}{3} \frac{\zeta_3}{3} \right) + n_f^{2} C_F \left( \frac{284}{81} \right)
= a_s \left( -3.5555555556 \right)
+ a_s^{3} \left( -48.3292181070 + 5.1358024691 n_f \right)
+ a_s^{3} \left( -859.4478371772 + 175.6499860261 n_f + 4.6748971193 n_f^{2} \right)
\]

\[
\gamma_{4}^G = a_s C_F \left( -\frac{11}{15} \right)
+ a_s^{3} \left[ C_F C_A \left( -\frac{62049}{16380} \right) + C_F^{3} \left( \frac{42109}{16380} \right) + n_f C_F \left( \frac{313}{45} \right) \right]
+ a_s^{3} \left[ C_F C_A^{3} \left( -\frac{325575847}{1215000} - \frac{444}{15} \frac{\zeta_3}{3} \right) + C_F^{2} C_A \left( -\frac{117100723}{243000} + \frac{444}{25} \frac{\zeta_3}{3} \right) \right]
+ C_F^{3} \left( \frac{110687611}{6075000} - \frac{1888}{75} \frac{\zeta_3}{3} \right) + n_f C_F C_A \left( -\frac{799}{12500} + \frac{176}{15} \frac{\zeta_3}{3} \right)
+ n_f C_F^{3} \left( \frac{3651671}{759375} - \frac{176}{15} \frac{\zeta_3}{3} \right) + n_f^{2} C_F \left( \frac{5473}{9000} \right)
= a_s \left( -0.9777777778 \right)
+ a_s^{3} \left( -16.1752427984 + 0.6182716049 n_f \right)
+ a_s^{3} \left( -315.2762549542 + 39.8257102724 n_f + 1.8018436214 n_f^{2} \right)
\]

\[
\gamma_{6}^G = a_s C_F \left( -\frac{44}{105} \right)
+ a_s^{3} \left[ C_F C_A \left( -\frac{237093}{154350} \right) + C_F^{3} \left( \frac{1191271}{1157625} \right) + n_f C_F \left( \frac{2204}{243075} \right) \right]
+ a_s^{3} \left[ C_F C_A^{3} \left( -\frac{200913519537}{163460000} - \frac{37592}{3675} \frac{\zeta_3}{3} \right) + C_F^{2} C_A \left( -\frac{150561431231}{10210252500} + \frac{37592}{1225} \frac{\zeta_3}{3} \right) \right]
+ C_F^{3} \left( \frac{1031558954593}{25105126200} - \frac{75184}{3675} \frac{\zeta_3}{3} \right) + n_f C_F C_A \left( -\frac{253841107}{583413000} + \frac{704}{105} \frac{\zeta_3}{3} \right)
+ n_f C_F^{3} \left( \frac{20157333331}{10521922587} - \frac{704}{105} \frac{\zeta_3}{3} \right) + n_f^{2} C_F \left( \frac{283449}{24367500} \right)
= a_s \left( -0.5587301587 \right)
+ a_s^{3} \left( -9.4963177963 + 0.0888485765 n_f \right)
+ a_s^{3} \left( -188.9088124238 + 19.6794454593 n_f + 1.0878437414 n_f^{2} \right)
\]

\[
\gamma_{8}^G = a_s C_F \left( -\frac{37}{126} \right)
+ a_s^{3} \left[ C_F C_A \left( -\frac{58805263}{2557640000} \right) + C_F^{3} \left( \frac{331619149}{2557640000} \right) + n_f C_F \left( -\frac{12613}{238110} \right) \right]
+ a_s^{3} \left[ C_F C_A^{3} \left( -\frac{810976071727}{129624364800} - \frac{58649}{6615} \frac{\zeta_3}{3} \right) + C_F^{2} C_A \left( -\frac{16504689455671}{90737053100} + \frac{58649}{2205} \frac{\zeta_3}{3} \right) \right]
+ C_F^{3} \left( \frac{1287635200506947}{63515987520000} - \frac{117298}{6615} \frac{\zeta_3}{3} \right) + n_f C_F C_A \left( -\frac{3105820553}{67512690000} + \frac{296}{63} \frac{\zeta_3}{3} \right)
+ n_f C_F^{3} \left( \frac{398139488671}{20343605000} - \frac{296}{63} \frac{\zeta_3}{3} \right) + n_f^{2} C_F \left( \frac{3391846473}{6001129000} \right)
= a_s \left( -0.3915343915 \right)
+ a_s^{3} \left( -6.7576035061 - 0.0706195235 n_f \right)
+ a_s^{3} \left( -134.7055041700 + 12.3754453990 n_f + 0.7536013741 n_f^{2} \right)
\]
\begin{align*}
\gamma_2^{GG} &= a_s n_f \left( \frac{2}{3} \right) \\
&+ a_s^2 \left[ n_f C_F \left( \frac{14}{15} \right) + n_f C_A \left( \frac{35}{24} \right) \right] \\
&+ a_s^3 \left[ n_f C_F C_A \left( -\frac{139}{9} + \frac{104}{3} \zeta_3 \right) + n_f C_F^2 \left( \frac{2155}{243} - \frac{32}{3} \zeta_3 \right) \\
&+ n_f C_A^2 \left( \frac{3589}{162} - 24 \zeta_3 \right) + n_f^2 C_F \left( \frac{173}{243} \right) + n_f^2 C_A \left( -\frac{1058}{243} \right) \right] \\
&= a_s \left( 0.6666666666 n_f \right) \\
&+ a_s^2 \left( 7.5432098765 n_f \right) \\
&+ a_s^3 \left( 37.6233727456 n_f - 12.1124828532 n_f^2 \right)
\end{align*}

\begin{align*}
\gamma_4^{GG} &= a_s \left[ C_A \left( \frac{24}{5} \right) + n_f \left( \frac{3}{4} \right) \right] \\
&+ a_s^2 \left[ C_A^2 \left( \frac{7121}{500} \right) + n_f C_F \left( \frac{40951}{4000} \right) + n_f C_A \left( -\frac{2513}{240} \right) \right] \\
&+ a_s^3 \left[ C_A^3 \left( \frac{933963}{15000} \right) + n_f C_F C_A \left( -\frac{23953517}{400000} + \frac{1632}{25} \zeta_3 \right) \\
&+ n_f C_F^2 \left( \frac{2557151}{357500} \frac{66}{25} \zeta_3 \right) + n_f C_A^2 \left( \frac{5379499}{240000} - \frac{1566}{25} \zeta_3 \right) \\
&+ n_f^2 C_F \left( -\frac{489887}{6975000} + n_f^2 C_A \left( -\frac{75781}{240000} \right) \right] \\
&= a_s \left( 12.6000000000 + 0.6666666667 n_f \right) \\
&+ a_s^2 \left( 128.1780000000 - 13.6494814815 n_f \right) \\
&+ a_s^3 \left( 2066.1927800000 - 401.3127939259 n_f - 10.4315064472 n_f^2 \right)
\end{align*}

\begin{align*}
\gamma_6^{GG} &= a_s \left[ C_A \left( \frac{54}{7} \right) + n_f \frac{3}{4} \right] \\
&+ a_s^2 \left[ C_A^2 \left( \frac{506899}{4088} \right) + n_f C_F \left( \frac{100319}{4695} \right) + n_f C_A \left( -\frac{102997}{1290} \right) \right] \\
&+ a_s^3 \left[ C_A^3 \left( \frac{9639817472}{871274880} \right) + n_f C_F C_A \left( -\frac{690128129}{136691222} + \frac{57256}{75} \zeta_3 \right) \\
&+ n_f C_F^2 \left( -\frac{11024749151}{10641010000} - \frac{176}{114} \zeta_3 \right) + n_f C_A^2 \left( \frac{276346989}{666754400} - \frac{18792}{245} \zeta_3 \right) \\
&+ n_f^2 C_F \left( -\frac{280414331}{2917215000} + n_f^2 C_A \left( -\frac{6507737}{2083725} \right) \right] \\
&= a_s \left( 17.7857142857 + 0.6666666667 n_f \right) \\
&+ a_s^2 \left( 183.0538143829 - 20.4666846633 n_f \right) \\
&+ a_s^3 \left( 2987.0420583375 - 566.6373928210 n_f - 10.7806086102 n_f^2 \right)
\end{align*}

\begin{align*}
\gamma_8^{GG} &= a_s \left[ C_A \left( \frac{319}{11} \right) + n_f \frac{3}{4} \right] \\
&+ a_s^2 \left[ C_A^2 \left( \frac{2232694}{91125} \right) + n_f C_F \left( \frac{658893}{326592} \right) + n_f C_A \left( -\frac{623687}{68040} \right) \right] \\
&+ a_s^3 \left[ C_A^3 \left( \frac{1381390082277}{10333575000} \right) + n_f C_F C_A \left( -\frac{220118238810087}{2592487960000} + \frac{81941}{945} \zeta_3 \right) \\
&+ n_f C_F^2 \left( -\frac{145841795723}{2268126830000} - \frac{37}{34} \zeta_3 \right) + n_f C_A^2 \left( \frac{2808130771161}{70256480000} - \frac{162587}{1890} \zeta_3 \right) \\
&+ n_f^2 C_F \left( -\frac{1747563703}{16878172500} + n_f^2 C_A \left( -\frac{420970849}{1280596000} \right) \right] \\
&= a_s \left( 21.2666666667 + 0.6666666667 n_f \right) \\
&+ a_s^2 \left( 219.6240987654 - 24.6992643216 n_f \right) \\
&+ a_s^3 \left( 3609.3541896322 - 673.9430658122 n_f - 11.2013383657 n_f^2 \right)
\end{align*}
\[ C^{\psi}_{2,2} = 1 + a_{s} C_{F} \left( \frac{g}{3} \right) + a_{s}^{2} f l_{0} n_{f} C_{F} \left( -\frac{133}{81} \right) + a_{s}^{2} n_{f} C_{F} (-4) + a_{s}^{2} C_{F} C_{A} \left( \frac{3077}{1185} - \frac{138}{6} \zeta_{3} \right) + a_{s}^{2} C_{F}^{2} \left( -\frac{4189}{810} + \frac{96}{5} \zeta_{3} \right) + a_{s}^{2} f l_{11} n_{f} C_{F} C_{A} \left( -\frac{480}{5} - \frac{1215}{8} \zeta_{3} - \frac{256}{3} \zeta_{5} \right) + a_{s}^{2} f l_{11} n_{f} C_{F}^{2} \left( \frac{406}{5} - \frac{1215}{8} \zeta_{3} - \frac{256}{3} \zeta_{5} \right) + a_{s}^{2} f l_{02} n_{f} C_{F} C_{A} \left( -\frac{1177679}{21870} - \frac{1376}{3} \zeta_{3} - \frac{32}{3} \zeta_{4} - \frac{64}{3} \zeta_{5} \right) + a_{s}^{2} f l_{02} n_{f} C_{F}^{2} \left( -\frac{28249}{2430} - \frac{17296}{405} \zeta_{3} + \frac{32}{3} \zeta_{4} + \frac{128}{3} \zeta_{5} \right) + a_{s}^{2} f l_{02} n_{f}^{2} C_{F} \left( -\frac{77023}{10935} + \frac{54}{3} \zeta_{3} \right) + a_{s}^{2} n_{f} C_{F} C_{A} \left( -\frac{596153}{10935} + \frac{17432}{405} \zeta_{3} - \frac{32}{3} \zeta_{4} + \frac{80}{3} \zeta_{5} \right) + a_{s}^{2} n_{f} C_{F}^{2} \left( -\frac{441701}{21870} - \frac{1352}{45} \zeta_{3} + \frac{32}{3} \zeta_{4} \right) + a_{s}^{2} n_{f}^{2} C_{F} \left( \frac{7814}{21870} + \frac{64}{81} \zeta_{3} \right) + a_{s}^{2} C_{F}^{2} C_{A} \left( \frac{3067498}{10935} - \frac{4610}{81} \zeta_{3} + \frac{32}{3} \zeta_{4} + 296 \zeta_{5} \right) + a_{s}^{2} C_{F}^{2} C_{A} \left( -\frac{1013578}{10935} + \frac{3076}{45} \zeta_{3} - \frac{32}{3} \zeta_{4} - \frac{1552}{3} \zeta_{5} \right) + a_{s}^{2} C_{F}^{2} \left( -\frac{201577}{7290} - \frac{50864}{405} \zeta_{3} + \frac{64}{3} \zeta_{4} + \frac{416}{3} \zeta_{5} \right) + a_{s}^{2} \left( 0.4444444444 \right) \]

\[ = 1 + a_{s} \left( -2.1893004115 n_{f} f l_{02} + 17.6937658911 - 5.3333333333 n_{f} \right) + a_{s} \left( -24.0920133486 n_{f} f l_{11} - 79.044861424 f l_{02} n_{f} + 3.3255044776 f l_{02} n_{f}^{2} + 442.7409692714 - 165.1971095394 n_{f} + 6.0302724150 n_{f}^{2} \right) \]

\[ C^{\psi}_{2,4} = 1 + a_{s} C_{F} \left( \frac{91}{27} \right) + a_{s}^{2} f l_{02} n_{f} C_{F} \left( \frac{3035523}{10800000} \right) + a_{s}^{2} n_{f} C_{F} \left( -\frac{1376051}{10935} \right) + a_{s}^{2} C_{F} C_{A} \left( \frac{469639}{50000} - \frac{269}{5} \zeta_{3} \right) + a_{s}^{2} C_{F}^{2} \left( -\frac{12624159}{6480000} + \frac{231}{3} \zeta_{3} \right) + a_{s}^{2} f l_{11} n_{f} C_{F} C_{A} \left( -\frac{454}{15} + \frac{76}{5} \zeta_{3} + 16 \zeta_{5} \right) + a_{s}^{2} f l_{11} n_{f} C_{F}^{2} \left( \frac{1232}{45} - \frac{408}{5} \zeta_{3} - \frac{138}{3} \zeta_{5} \right) + a_{s}^{2} f l_{02} n_{f} C_{F} C_{A} \left( 20953881467 - \frac{10438}{1125} \zeta_{3} - \frac{121}{75} \zeta_{4} \right) + a_{s}^{2} f l_{02} n_{f} C_{F}^{2} \left( 10781550739 - \frac{161336}{108576} \zeta_{3} + \frac{124}{75} \zeta_{4} \right) + a_{s}^{2} f l_{02} n_{f}^{2} C_{F} \left( -\frac{16205516}{6834375} + \frac{1204}{2025} \zeta_{3} \right) + a_{s}^{2} n_{f} C_{F} C_{A} \left( -\frac{36959762439}{122172000} + \frac{365878}{2895} \zeta_{3} - \frac{314}{15} \zeta_{4} \right) \]
\[ C_{2,6}^\psi = 1 + a_s \left( 6 \cdot 0.666666666667 \right) \]
\[ + a_4^2 \left( 0.4858308642 n_f f l_{02} + 142.3434719201 - 16.9879135802 n_f \right) \]
\[ + a_3^2 \left( -18.2188461805 n_f f l_{11} + 16.6483484853 n_f f l_{02} - 2.2086306890 n_f^2 f l_{02} \right) \]
\[ + 4169.2678883902 - 901.2351625706 n_f + 23.3550392440 n_f^2 \]
\[ C_{2,8}^\psi = 1 + \alpha_s C_F \left( \frac{58703}{5040} + \frac{83382717493}{3885414400} \right) + a_s^2 n_f C_F \left( \frac{24004512009}{24004512009} \right) + a_s^2 n_F C_F \left( \frac{324822023689}{324822023689} \right) - \frac{17071}{210} \zeta_3 + \frac{2396}{35} \zeta_3 \] 
\[ + a_s^2 n_f C_F A \left( \frac{203714735078509}{203714735078509} \right) + a_s^2 C_F^2 \left( \frac{58703}{5040} + \frac{83382717493}{3885414400} \right) + \frac{36443}{36443} \zeta_3 + \frac{200}{3} \zeta_5 \] 
\[ + a_s^3 n_f C_F A \left( \frac{1907199980}{1907199980} \right) + \frac{13577}{13577} \zeta_3 - \frac{1600}{9} \zeta_5 \] 
\[ + a_s^3 n_f C_F A \left( \frac{918296234140453}{918296234140453} \right) - \frac{24344003}{24344003} \zeta_4 - \frac{1369}{1369} \zeta_4 \] 
\[ + a_s^3 n_f C_F A \left( \frac{10181259543168231}{10181259543168231} \right) + \frac{117875930}{117875930} \zeta_3 + \frac{1360}{3780} \zeta_4 \] 
\[ + a_s^3 n_f C_F A \left( \frac{1695985520839}{1695985520839} \right) + \frac{21517}{21517} \zeta_3 \] 
\[ + a_s^3 n_f C_F A \left(- \frac{588845637600000}{588845637600000} \right) + \frac{56295559}{56295559} \zeta_3 - \frac{9883}{315} \zeta_4 \] 
\[ + a_s^3 n_f C_F A \left( -\frac{119581362341463641}{119581362341463641} \right) - \frac{4139607}{4139607} \zeta_3 + \frac{9883}{315} \zeta_4 \] 
\[ + a_s^3 n_f C_F A \left( -\frac{3144850682200000}{3144850682200000} \right) - \frac{523908}{523908} \zeta_3 \] 
\[ + a_s^3 n_f C_F A \left( -\frac{212111666800000}{212111666800000} \right) + \frac{19766}{19766} \zeta_3 \] 
\[ + a_s^3 n_f C_F A \left( \frac{32717639914938763}{32717639914938763} \right) + \frac{1614116849}{1614116849} \zeta_3 + \frac{2510407}{2510407} \zeta_4 + \frac{47044}{63} \zeta_5 \] 
\[ + a_s^3 n_f C_F A \left( \frac{517186169425292529}{517186169425292529} \right) + \frac{916839000}{916839000} \zeta_3 - \frac{2510407}{2510407} \zeta_4 - \frac{1982}{1982} \zeta_5 \] 
\[ + a_s^3 n_f C_F A \left( \frac{14319953431756525891}{14319953431756525891} \right) + \frac{3131600990227}{3131600990227} \zeta_3 + \frac{2510407}{2510407} \zeta_4 + \frac{13290}{664} \zeta_5 \] 
\[ + 1 + a_s \left( \frac{15.52998941799}{15.52998941799} \right) \right) + 470.8074190065 - 37.924827990 n_f \right) \] 
\[ + a_s^3\left(-15.0920382729 n_f f_{11} + 22.3320193843 n_f f_{11} - 1.0360391217 n_f^2 f_{11} - 17162.3724471532 - 2787.2976921073 n_f + 61.911779688 n_f^2 \right) \] 
\[ C_{2,2}^\psi = a_s n_f \left( -\frac{1}{2} \right) + a_s^2 n_f C_F \left( \frac{4709}{810} + \frac{16}{3} \zeta_3 \right) + a_s^2 n_f C_F A \left( \frac{115}{324} - 2 \zeta_3 \right) + a_s^2 f l_{11}^2 n_f^2 C_F \left( -4 + \frac{272}{15} \zeta_3 - \frac{44}{5} \zeta_5 \right) + a_s^2 f l_{11}^2 n_f^2 C_F A \left( \frac{2}{3} - \frac{34}{3} \zeta_3 + 8 \zeta_5 \right) + a_s^2 n_f C_F A \left( -\frac{16128}{6934} + \frac{15}{15} \zeta_3 - \frac{52}{3} \zeta_4 + 40 \zeta_5 \right) + a_s^2 n_f C_F A \left( \frac{28403}{4860} + \frac{4148}{4860} \zeta_3 + \frac{16}{3} \zeta_4 \right) + a_s^2 n_f C_F A \left(- \frac{1444493}{844860} + \frac{1828}{4860} \zeta_3 + 12 \zeta_4 + 12 \zeta_5 \right) + a_s^2 n_f C_F A \left( \frac{23291}{12840} - \frac{104}{105} \zeta_3 \right) + a_s^2 n_f C_F A \left( \frac{139219}{21879} - \frac{122}{105} \zeta_3 \right) + \frac{1}{5} \right) \]
\[ C_{2,4} = \begin{align*}
&= a_3 n_f(-\frac{133}{180}) \\
&+ a_2 n_f C_F \left( \frac{6410867}{72000} + \frac{18}{5} \zeta_3 \right) \\
&+ a_0 n_f C_A \left( \frac{1812467}{32400} - \frac{9}{5} \zeta_3 \right) \\
&+ a_3 f_{l_{11}} n_f^2 C_F \left( -\frac{410}{9} - \frac{10592}{35} \zeta_3 + \frac{928}{3} \zeta_5 \right) \\
&+ a_2 f_{l_{11}} n_f^2 C_A \left( \frac{205}{12} + \frac{1324}{15} \zeta_3 - 116 \zeta_5 \right) \\
&+ a_0 n_f C_F C_A \left( -\frac{1571010034879}{61236000000} - \frac{25747}{630} \zeta_3 - \frac{269}{30} \zeta_4 + 82 \zeta_5 \right) \\
&+ a_0 n_f C_F^2 \left( \frac{1560936087251}{122412000000} + \frac{2868060}{35} \zeta_3 + \frac{261}{10} \zeta_4 - \frac{292}{3} \zeta_5 \right) \\
&+ a_0 n_f C_A^2 \left( \frac{77953723983}{7200000000} + \frac{1312997}{20250} \zeta_3 + \frac{354}{25} \zeta_4 - \frac{28}{3} \zeta_5 \right) \\
&+ a_0 n_f^2 C_F \left( \frac{707842890021}{61236000000} - \frac{595883}{141750} \zeta_3 \right) \\
&+ a_0 n_f^2 C_A \left( \frac{109146757}{9112500} + \frac{59}{2025} \zeta_3 \right) \\
&= a_3 \left( -0.73888888889 n_f \right) \\
&+ a_2 \left( -14.2715869197 n_f \right) \\
&+ a_0 \left( -1.6118165124 n_f^2 f_{l_{11}}^0 - 346.4612756184 n_f + 46.5201756441 n_f^2 \right)
\end{align*} \]

\[ C_{2,6} = a_3 n_f(-\frac{1777}{2920}) \\
+ a_2 n_f C_F \left( \frac{12660217}{1295440} + \frac{20}{7} \zeta_3 \right) \\
+ a_0 n_f C_A \left( \frac{16794543}{1779240} - \frac{10}{7} \zeta_3 \right) \\
+ a_3 f_{l_{11}} n_f^2 C_F \left( -\frac{1330183}{9072} - \frac{1673}{3} \zeta_3 + \frac{16400}{21} \zeta_5 \right) \\
+ a_2 f_{l_{11}} n_f^2 C_A \left( \frac{13489183}{24192} + \frac{1673}{8} \zeta_3 - \frac{2050}{15} \zeta_5 \right) \\
+ a_0 n_f C_F C_A \left( -\frac{137225393600000}{39583264061093} - \frac{4535294}{23125} \zeta_4 - \frac{11833}{33} \zeta_4 + \frac{1040}{21} \zeta_5 \right) \\
+ a_0 n_f C_F^2 \left( \frac{286797600800253}{137225393600000} + \frac{65999719}{138910} \zeta_3 - \frac{620}{42} \zeta_4 - \frac{144}{7} \zeta_5 \right) \\
+ a_0 n_f C_A^2 \left( \frac{22053438063807}{1630090032200} + \frac{43721551}{694575} \zeta_3 + \frac{2911}{245} \zeta_4 - \frac{80}{21} \zeta_5 \right) \\
+ a_0 n_f^2 C_F \left( \frac{888375865061}{57174110000} - \frac{485917}{158915} \zeta_3 \right) \\
+ a_0 n_f^2 C_A \left( \frac{4519686643989}{294055272000} + \frac{143}{7638} \zeta_3 \right) \\
= a_3 \left( -0.7051587302 n_f \right) \\
+ a_2 \left( -20.0684982842 n_f \right) \\
+ a_0 \left( -1.4960369382 n_f^2 f_{l_{11}}^0 - 715.0372438398 n_f + 61.2854509604 n_f^2 \right)\]
\[ C'_{2,8} = a_s n_f \left( -\frac{16231}{25200} \right) + a_s^2 n_f C_F \left( -\frac{2399462871}{2351432000} + \frac{7}{3} \zeta_3 \right) + a_s^2 n_f C_A \left( \frac{29149688167}{17503240000} - \frac{7}{6} \zeta_3 \right) + a_s^2 f l_{11}^9 n_f C_F \left( -\frac{692580563}{1913625} - \frac{2357104}{2025} \zeta_3 + \frac{15232}{9} \xi_5 \right) + a_s^2 f l_{11}^9 n_f C_A \left( \frac{294638}{695} \zeta_3 - \frac{1904}{9} \xi_5 \right) + a_s^3 n_f C_F C_A \left( -\frac{31}{5980873744} \right) + a_s^3 n_f C_F^2 \left( -\frac{25534}{1295560} + \frac{19443654720000}{3593175651200153} \zeta_3 + \frac{5056}{45} \zeta_3 + \frac{358}{3} \zeta_5 \right) + a_s^3 n_f C_A^2 \left( -\frac{40536}{1215} \right) + a_s^3 n_f C_F \left( \frac{243285660000000}{243285660000000} \right) + a_s^3 n_f C_A \left( \frac{243285660000000}{243285660000000} \right) + a_s^3 n_f^2 C_F \left( -\frac{40536}{1215} \right) + a_s^3 n_f^2 C_A \left( \frac{40536}{1215} \right) + a_s^3 \left( -0.6440873016 n_f \right) + a_s^3 \left( -23.1787352382 n_f \right) + a_s^3 \left( -1.2864009150 n_f^2 f l_{11}^9 - 996.5038709496 n_f + 68.6646730444 n_f^2 \right) \]

\[ C'_{L,2} = a_s \left( \frac{1}{27} \right) + a_s^2 f l_{02} n_f C_F \left( -\frac{80}{27} \right) + a_s^2 n_f C_F \left( -\frac{80}{27} \right) + a_s^2 C_F C_A \left( \frac{2678}{135} - \frac{32}{9} \zeta_3 \right) + a_s^2 C_F^2 \left( -\frac{1906}{135} + \frac{54}{9} \zeta_3 \right) + a_s^2 f l_{11} n_f C_F C_A \left( -\frac{164}{9} + \frac{1008}{9} \zeta_3 - 192 \zeta_5 \right) + a_s^2 f l_{11} n_f C_F^2 \left( \frac{1412}{135} - \frac{2088}{9} \zeta_3 + 512 \zeta_5 \right) + a_s^2 f l_{02} n_f C_F C_A \left( -\frac{40568}{1215} + \frac{304}{9} \zeta_3 - \frac{256}{3} \zeta_5 \right) + a_s^2 f l_{02} n_f C_F^2 \left( -\frac{46998}{1215} - \frac{5056}{45} \zeta_3 + \frac{312}{3} \zeta_5 \right) + a_s^2 f l_{02} n_f^2 C_F \left( \frac{3364}{405} + \frac{64}{9} \zeta_3 \right) + a_s^2 n_f C_F C_A \left( -\frac{204548}{1215} - \frac{1508}{45} \zeta_3 + \frac{420}{3} \zeta_5 \right) + a_s^2 n_f C_F^2 \left( -\frac{25534}{405} - \frac{2848}{45} \zeta_3 \right) + a_s^2 n_f^2 C_F \left( \frac{2168}{243} \right) + a_s^2 C_F C_A^2 \left( \frac{54868}{1215} - \frac{3680}{9} \zeta_3 + 224 \zeta_5 \right) + a_s^2 C_F^2 C_A \left( -\frac{41536}{405} + \frac{27504}{45} \zeta_3 + \frac{528}{3} \zeta_5 \right) + a_s^2 C_F^3 \left( -\frac{222278}{1215} - \frac{39432}{45} \zeta_3 + \frac{358}{3} \zeta_5 \right) \]

\[ = a_s \left( 1.7777777778 \right) + a_s^2 \left( -3.9506172840 n_f f l_{02} + 56.7553015166 - 4.5432098765 n_f \right) + a_s^2 \left( -7.7366982885 n_f f l_{11} - 213.9253075658 n_f f l_{02} + 17.9132652795 n_f^2 f l_{02} + 2544.5980873744 - 421.6908884762 n_f + 11.8957475995 n_f^2 \right) \]
\[ C_{L,A}^\psi = a_s C_F \left( \frac{4}{7} \right) + a_s^2 f l o_2 n_f C_F \left( \frac{586}{1125} \right) + a_s^2 n_f C_F \left( \frac{64}{25} \right) + a_s^2 C_F C_A \left( \frac{4763}{225}, - \frac{48}{5} \zeta_3 \right) + a_s^2 C_F^2 \left( - \frac{19967}{1125}, + \frac{96}{5} \zeta_3 \right) + a_s^3 f l o_1 n_f C_F C_A \left( \frac{12359}{90}, + \frac{856}{15} \zeta_3 - 192 \zeta_5 \right) + a_s^3 f l o_1 n_f C_F^2 \left( \frac{94136}{139}, - \frac{6848}{45} \zeta_3 + 512 \zeta_5 \right) + a_s^3 f l o_2 n_f C_F C_A \left( - \frac{5074537}{405900}, - \frac{824}{675} \zeta_3 \right) + a_s^3 f l o_2 n_f C_F^2 \left( \frac{7715012}{1069425}, + \frac{29488}{4125} \zeta_3 \right) + a_s^3 f l o_2 n_f^2 C_F \left( \frac{106981}{60750} \right) + a_s^3 n_f C_F C_A \left( - \frac{14259893}{945000}, + \frac{55904}{945} \zeta_3 \right) + a_s^3 n_f C_F^2 \left( \frac{258828431}{2835000}, - \frac{33344}{315} \zeta_3 \right) - \frac{82688}{10125} + a_s^3 n_f^2 C_F \left( \frac{582157141}{3780000}, + \frac{57356}{945} \zeta_3 - 192 \zeta_5 \right) = a_s \left( 1.06666666667 \right) + a_s^2 \left( -0.6945181585 n_f f l o_2 + 47.9938983120 - 3.4133333333 n_f \right) + a_s^3 \left( -5.0588695123 n_f f l o_1 - 55.5530455971 n_f f l o_2 + 2.3480045470 n_f^2 f l o_2 + 2523.7390200791 - 383.0520013416 n_f + 10.8889547325 n_f^2 \right) \]

\[ C_{L,6}^\psi = a_s C_F \left( \frac{4}{7} \right) + a_s^2 f l o_2 n_f C_F \left( \frac{1464}{1175} \right) + a_s^2 n_f C_F \left( \frac{1480}{1480} \right) + a_s^2 C_F C_A \left( \frac{172106}{11025}, - \frac{48}{7} \zeta_3 \right) + a_s^2 C_F^2 \left( - \frac{257318}{11025}, + \frac{96}{7} \zeta_3 \right) + a_s^3 f l o_1 n_f C_F C_A \left( \frac{5869993}{63000}, + \frac{944}{21} \zeta_3 - \frac{960}{7} \zeta_5 \right) + a_s^3 f l o_1 n_f C_F^2 \left( - \frac{9899993}{29024}, - \frac{752}{210} \zeta_3 + \frac{256}{7} \zeta_5 \right) + a_s^3 f l o_2 n_f C_F C_A \left( - \frac{41368883483}{816820000}, - \frac{7384}{11025} \zeta_3 \right) + a_s^3 f l o_2 n_f C_F^2 \left( - \frac{193479693407}{6126161000}, + \frac{416}{15} \zeta_3 \right) + a_s^3 f l o_2 n_f^2 C_F \left( \frac{279005687}{18602050}, 23602050 \right) + a_s^3 n_f C_F C_A \left( - \frac{29754107}{6043950}, + \frac{88916}{195} \zeta_3 \right) + a_s^3 n_f C_F^2 \left( \frac{66489992539}{875164500}, - \frac{31312}{135} \zeta_3 \right) + a_s^3 n_f^2 C_F \left( \frac{908672}{135915}, 135915 \right) \]
\[\begin{align*}
+ a_3^2 C_F C_A^2 & \left( \frac{318722330}{6174000} - \frac{4977821}{11025} \zeta_3 + \frac{1040}{7} \zeta_5 \right) \\
+ a_3^2 C_F C_A & \left( \frac{433877446411}{8751645000} + \frac{2071508}{2205} \zeta_3 - 480 \zeta_5 \right) \\
+ a_3^2 C_F^2 & \left( \frac{90265366481}{2268944000} - \frac{53658}{2205} \zeta_3 + \frac{2560}{9} \zeta_5 \right) \\
= \quad a_s \left( 0.7619047619 \right) \\
+ a_2^2 & \left( -0.2524824533 n_f f_{l_02} + 40.9961975991 - 2.6956916100 n_f \right) \\
+ a_3^3 & \left( -3.7056125257 n_f f_{l_{11}} - 24.0132253893 n_f f_{l_{02}} + 0.7652692585 n_f^2 f_{l_{02}} \\
+ 2368.1937754336 - 340.0691069253 n_f + 9.4721904282 n_f^2 \right) \\
C_{L,8}^{\phi} = \quad a_s C_F \left( \frac{9}{4} \right) + a_3^2 f_{l_02} n_f C_F \left( \frac{6523}{1144} \right) \\
+ a_2^2 n_f C_F \left( - \frac{14234}{5905} \right) + a_2^2 C_F C_A \left( \frac{14741729}{1190700} - \frac{16}{3} \zeta_3 \right) \\
+ a_2^2 C_F^2 \left( - \frac{21694349}{3572100} + \frac{32}{9} \zeta_3 \right) + a_3^3 n_f C_F C_A \left( \frac{35555774437}{80094000} + \frac{170756}{4725} \zeta_3 - \frac{320}{3} \zeta_5 \right) \\
+ a_3^3 f_{l_11} n_f C_F C_A \left( \frac{35555774437}{18713509} - \frac{136694}{4725} \zeta_3 + \frac{2560}{9} \zeta_5 \right) \\
+ a_3^3 f_{l_02} C_F C_A \left( \frac{266509460173}{426450761073} - \frac{12892}{9} \zeta_3 \right) \\
+ a_3^3 f_{l_02} n_f C_F C_A \left( \frac{55505717822203}{31190682780000} + \frac{1327342}{1091473} \zeta_3 \right) \\
+ a_3^3 f_{l_02} n_f^2 C_F \left( \frac{13857324703}{135025380000} \right) + a_3^3 n_f C_F C_A \left( \frac{22153083634152}{198037244000} + \frac{18459136}{363826} \zeta_3 \right) \\
+ a_3^3 n_f C_F^2 \left( \frac{11844644404299}{1988972241000} - \frac{914992}{16382} \zeta_3 \right) + a_3^3 C_F C_A \left( \frac{1435876}{229635} \right) \\
+ a_3^3 C_F C_A \left( \frac{7653142912007}{16065340000} - \frac{9503518}{19845} \zeta_3 + \frac{2240}{9} \zeta_5 \right) \\
+ a_3^3 C_F C_A \left( \frac{7667007621800089}{27725211360000} + \frac{2476882549}{2182950} \zeta_3 - \frac{2720}{3} \zeta_5 \right) \\
+ a_3^3 C_F \left( - \frac{861671664918457}{49905380448000} - \frac{111066693}{2182950} \zeta_3 + \frac{7360}{9} \zeta_5 \right) \\
= \quad a_s \left( 0.5925925926 \right) + a_2^2 \left( -0.1217307796 n_f f_{l_{02}} + 35.8766440564 - 2.2314716833 n_f \right) \\
+ a_3^3 \left( -2.9137025628 n_f f_{l_{11}} - 12.9718526710 n_f f_{l_{02}} + 0.3443623910 n_f^2 f_{l_{02}} \\
+ 2215.2108750618 - 305.4730328944 n_f + 8.3371495344 n_f^2 \right) \\
\end{align*}\]
\[ C^F_{L,2} = a_n f \left( \frac{6}{5} \right) + a^2_n f C_F \left( \frac{116}{135} - \frac{16}{8} \zeta_3 \right) + a^2_n f C_A \left( \frac{173}{24} \right) + a^2_n f t^{g}_{11} n^2 f C_F \left( \frac{8}{3} + \frac{3808}{15} \zeta_3 - \frac{896}{3} \zeta_5 \right) + a^2_n f t^{g}_{11} n^2 f C_A \left( -1 - \frac{476}{3} \zeta_3 + 112 \zeta_5 \right) + a^3_n f C_F C_A \left( -\frac{71657}{1215} - \frac{248}{5} \zeta_3 + \frac{80}{3} \zeta_5 \right) + a^3_n f C_F^2 \left( \frac{51283}{1215} + \frac{928}{5} \zeta_3 - \frac{160}{3} \zeta_5 \right) + a^3_n f C_A^2 \left( \frac{235283}{2430} - \frac{148}{5} \zeta_3 + \frac{64}{3} \zeta_5 \right) + a^3_n f C_F \left( \frac{9031}{1215} + \frac{296}{5} \zeta_3 \right) + a^3_n f C_A \left( -\frac{5431}{405} + \frac{4}{15} \zeta_3 \right) = a_n f \left( 0.6666666667 n_f \right) + a^2_n f \left( 12.9477670897 n_f \right) + a^3_n f \left( -0.3889396640 n^2 f t^{g}_{11} + 407.2806319596 n_f - 20.239597492 n^2 f \right) \]

\[ C^F_{L,4} = a_n f \left( \frac{4}{5} \right) + a^2_n f C_F \left( -\frac{2761}{1125} \right) + a^2_n f C_A \left( \frac{19226}{3375} \right) + a^3_n f t^{g}_{11} n^2 f C_F \left( -\frac{5528}{135} + \frac{4384}{135} \zeta_3 \right) + a^3_n f t^{g}_{11} n^2 f C_A \left( \frac{691}{45} - \frac{548}{45} \zeta_3 \right) + a^3_n f C_F C_A \left( -\frac{201555163}{1890000} - \frac{272858}{4725} \zeta_3 + 128 \zeta_5 \right) + a^3_n f C_F^2 \left( \frac{912559079}{11300000} + \frac{225908}{4725} \zeta_3 - 128 \zeta_5 \right) + a^3_n f C_A^2 \left( \frac{172348913}{1215000} + \frac{5036}{675} \zeta_3 - 32 \zeta_5 \right) + a^3_n f C_F \left( \frac{4921307}{680400} - \frac{32}{105} \zeta_3 \right) + a^3_n f C_A \left( -\frac{455666}{3087} - \frac{8}{15} \zeta_3 \right) = a_n f \left( 0.2666666667 n_f \right) + a^2_n f \left( 13.8165925926 n_f \right) + a^3_n f \left( -0.3984298404 n^2 f t^{g}_{11} + 767.7125420910 n_f - 36.7841923177 n^2 f \right) \]

\[ C^F_{L,6} = a_n f \left( \frac{2}{3} \right) + a^2_n f C_F \left( -\frac{137761}{92610} \right) + a^2_n f C_A \left( \frac{509779}{123487} \right) + a^3_n f t^{g}_{11} n^2 f C_F \left( \frac{2994213}{340200} + \frac{61778}{315} \zeta_3 - \frac{1760}{9} \zeta_5 \right) + a^3_n f t^{g}_{11} n^2 f C_A \left( \frac{994213}{960720} - \frac{30889}{420} \zeta_3 + \frac{660}{7} \zeta_5 \right) + a^3_n f C_F C_A \left( -\frac{1620888888209}{1638404000} - \frac{51181}{2205} \zeta_3 + \frac{480}{7} \zeta_5 \right) + a^3_n f C_F^2 \left( \frac{101548438631}{2043005600} + \frac{43574}{2205} \zeta_3 - \frac{480}{7} \zeta_5 \right) \]
\[ + a_s^3 C_A^2 \left( \frac{380361802767}{3267828800} + \frac{10817}{4410} \zeta_3 - \frac{120}{7} \zeta_5 \right) \\
+ a_s^3 n_f C_F^2 \left( \frac{60190474091}{12252383000} - \frac{32}{63} \zeta_3 \right) \\
+ a_s^3 n_f^2 C_A \left( \frac{847812559}{7792400} - \frac{7}{8} \zeta_3 \right) \\
= a_s \left( 0.1428571429 n_f \right) \\
+ a_s^2 \left( 10.2609536407 n_f \right) \\
+ a_s^3 \left( -0.3055276256 n_f^2 \left( f l_{11}^0 + 694.5092121136 n_f - 27.9895081033 n_f^2 \right) \right) \\
C_{L,8}^G = a_s n_f (\frac{4}{75}) \\
+ a_s^2 n_f C_F \left( \frac{51097}{51080} \right) \\
+ a_s^2 n_f C_A \left( \frac{7712669}{2501500} \right) \\
+ a_s^3 f l_{11}^0 n_f^2 C_F \left( \frac{366571401}{23671000} + \frac{617672}{1575} \zeta_3 - \frac{1664}{3} \zeta_5 \right) \\
+ a_s^3 f l_{11}^0 n_f^2 C_A \left( \frac{366571401}{9556000} - \frac{72299}{525} \zeta_3 + 208 \zeta_5 \right) \\
+ a_s^3 n_f C_F C_A \left( \frac{508552379600003}{129341064000} - \frac{6119609}{1947900} \zeta_3 + \frac{128}{3} \zeta_5 \right) \\
+ a_s^3 n_f^2 C_F^2 \left( \frac{238440812495187}{7129340000000} + \frac{7723411}{77205} \zeta_3 - \frac{128}{3} \zeta_5 \right) \\
+ a_s^3 n_f^2 C_A \left( \frac{11324757281}{1371810600} - \frac{8}{33} \zeta_3 \right) \\
+ a_s^3 n_f^2 C_A \left( \frac{3564070032000}{1371810600} - \frac{608}{2727} \zeta_3 \right) \\
= a_s \left( 0.08888888889 n_f \right) \\
+ a_s^2 \left( 7.7335451042 n_f \right) \\
+ a_s^3 \left( -0.232211886 n_f^2 \left( f l_{11}^0 + 592.3307972098 n_f - 21.3033368117 n_f^2 \right) \right) \\
\]
\[
C_{2,10}^{ns} = 1 + a_s C_F \left( \frac{2006299}{138600} \right) + a_s^2 n_f C_F \left( -\frac{561457267420757}{19999006276000} \right) + a_s^2 C_F C_A \left( \frac{6124093918924187}{290454598290000} - \frac{104674}{1155} \zeta_3 \right) + a_s^2 C_F^2 \left( \frac{558708799987320133}{111509202898800000} + \frac{88708}{1155} \zeta_3 \right) + a_s^3 n_f l_1 n_f C_F C_A \left( -\frac{3753913187503}{8676090000} - \frac{162770}{1302208} \zeta_3 + \frac{896}{11} \zeta_5 \right) + a_s^3 n_f l_1 n_f C_F^2 \left( \frac{3753913187503}{192491136000} + \frac{1302208}{11} \zeta_3 - \frac{7168}{11} \zeta_5 \right) + a_s^3 n_f C_F C_A \left( -\frac{216644962939357214987}{2355034934311200000} + \frac{24567525}{693} \zeta_3 - \frac{24110}{693} \zeta_4 \right) + a_s^3 n_f C_F \left( -\frac{1521387668689946101939}{3376008045899569600000} - \frac{399754476}{693} \zeta_3 + \frac{24110}{693} \zeta_4 \right) + a_s^3 C_F C_A \left( \frac{25708422807485151111}{18711} \zeta_3 \right) + a_s^3 C_F C_A \left( \frac{1092211008547909695237}{163509009092900358968829} - \frac{14713925739913}{12432357900000} - \frac{14157962999}{1500215930681} \zeta_3 + \frac{157190858}{261} \zeta_4 - \frac{299}{261} \zeta_5 \right) + a_s^3 C_F C_A \left( \frac{32477953237909263776010131}{92175812816020753168000000} + \frac{218220882524282}{1022461215375} \zeta_3 + \frac{157190858}{8004150} \zeta_4 - \frac{75212}{99} \zeta_5 \right) + a_s \left( \frac{19.3006156806}{146.86131384159 n_f} \right) + a_s^2 \left( 639.2106629599 - 46.86131384159 n_f \right) + a_s^3 \left( -14.4587445075 n_f l_1 + 24953.1349702005 - 3770.10121201303 n_f + 80.5209797251 n_f^2 \right)
\]

\[
C_{E,10}^{ns} = a_s C_F \left( \frac{163679}{114345} \right) + a_s^2 n_f C_F \left( -\frac{89670761}{873598} \right) + a_s^2 C_F C_A \left( \frac{52827396}{19999019607} + \frac{96}{11} \zeta_3 \right) + a_s^3 l_1 n_f C_F C_A \left( \frac{50705934324963}{8801644000} + \frac{3641546}{121275} \zeta_3 - \frac{960}{11} \zeta_5 \right) + a_s^3 C_F C_A \left( -\frac{50705934324963}{33989234000} - \frac{3641546}{121275} \zeta_3 + \frac{3504}{11} \zeta_5 \right) + a_s^3 C_F C_A \left( -\frac{176183756988227333}{1099159381920000} + \frac{55485434}{1260215} \zeta_3 \right) + a_s^3 n_f C_F C_A \left( \frac{1008874339397637}{10396079628590000} \right) + a_s^3 n_f C_F \left( -\frac{53272639}{11320155} \right) + a_s^3 n_f C_F \left( 236905392418105137 \right) + a_s^3 C_F C_A \left( -\frac{3213189048004267269}{3347505475000000} + \frac{299041911}{17325} \zeta_3 - \frac{14240}{11} \zeta_5 \right) + a_s^3 C_F C_A \left( -\frac{8875238660849567383}{41062189926298000000} - \frac{457031607234}{46822775} \zeta_3 + \frac{13440}{11} \zeta_5 \right) + a_s \left( 0.48484848484 \right) + a_s^2 \left( 32.0176594698 \right) - 1.9085982480 n_f \right) + a_s^3 \left( -2.3976416945 n_f l_1 + 2081.2132221274 - 278.0172176870 n_f + 7.4525056120 n_f^2 \right)
\]
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