Fermions and world-line supersymmetry*

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Abstract

The world-line path-integral representation of fermion propagators is discussed. Particular attention is paid to the representation of $\gamma_5$, which is connected to the realization of manifest world-line supersymmetry.

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Introduction

The path-integral representation of the evolution operator of dynamical quantum systems provides a direct link to the associated (pseudo-)classical system. This classical theory can suggest symmetries of the quantum theory (see for example [1]) and reformulations taking advantage of various classical descriptions related by co-ordinate transformations (or canonical transformations in the Hamiltonian formalism). As has been emphasized by many authors [2]-[6], Green’s functions in perturbative quantum field theory can also be represented in terms of world-line path integrals, providing additional insight into their properties, like analyticity and causality. This formulation also provides new calculational tools, some of them inspired by considering the point-particle limit of string theory [7].

As was realized long ago [8]-[11] a convenient description of fermions by world-line path integrals requires the introduction of Grassmann variables, and it was found that world-line supersymmetry imposes a structure appropriate to the description of Dirac fermions. Applications of these ideas, as for example the computation of fermion determinants in field theory, require the inclusion of external background fields like scalars, vectors or tensors [11]-[23].

An important issue in the construction of fermion world-line path integrals is the representation of $\gamma_5$, as it appears in mass terms [24], axial couplings [21, 25] and anomalies [26]. In practice there are two approaches to this issue, which have led a more or less independent life in the literature. The first one, motivated by supersymmetry, is to introduce a Grassmann-odd variable $\psi_5$ in addition to the Grassmann-odd vector variables $\psi_\mu$, with its own kinetic terms in the pseudo-classical action [8]-[11]. The other one, motivated by the computation of anomalies using the Witten-index of supersymmetric quantum mechanics [26], is to represent $\gamma_5$ by $(-1)^F$ where $F$ is a suitably defined fermion number in the supersymmetric quantum-mechanical model. In the context of the world-line formulation of spinning particles the latter approach has been advocated in [25]. In this paper I review in some detail the differences between these approaches and their consequences. I show, that the Witten-index approach is equivalent to the bosonic version of $\gamma_5$ presented in [24], and corresponds to an irreducible representation of spinors in four dimensions$^1$, whereas the Grassmann-odd representation of $\gamma_5$ introduces a doubling of the number of degrees of freedom, corresponding to a reducible representation of the Clifford algebra of Dirac matrices used to define spinors.

Dirac equation and supersymmetry

The use of supersymmetry in describing spinning particles results from the formal similarity between the algebra of Dirac operators and the supersymmetry algebra

$^1$Similar results actually hold also in other dimensions
where $Q$ denotes the supercharge and $H$ the hamiltonian operator of a supersymmetric quantum theory. The Dirac operator $\hat{p} = \gamma \cdot p$ is related similarly to the laplacian (the kinetic operator) by

$$\hat{p}^2 = p^2_{\mu}.$$  \hspace{1cm} (2)

In the pseudo-classical theory of a massless fermion the classical quantity corresponding to $\hat{p}$ indeed generates a world-line supersymmetry, just like the classical hamiltonian generates proper-time translations.

For massive fermions the problem is a little more complicated: the Dirac operator $\hat{p} + m$ does not square to the Klein-Gordon operator $p^2_{\mu} + m^2$. However, the situation can be saved by introducing $\gamma_5$ and associate

$$Q \rightarrow (-i\hat{p} + m)\gamma_5, \quad H \rightarrow p^2_{\mu} + m^2.$$ \hspace{1cm} (3)

This defines a representation of the supersymmetry algebra (1) owing to the crucial anti-commutation property of the $\gamma_5$ operator:

$$\hat{p}\gamma_5 + \gamma_5\hat{p} = 0.$$ \hspace{1cm} (4)

Clearly the limit $m \rightarrow 0$ is well-defined and defines an alternative realization of the supersymmetry relation for massless fermions by $Q = -i\hat{p}\gamma_5$.

**Grassmann representation of spinors**

In view of the above, I represent the Dirac algebra in the following by the pseudo-vector elements $\psi_\mu = -i\gamma_\mu\gamma_5$, which obey the same (Euclidean) anti-commutation relations

$$\{\psi_\mu, \psi_\nu\} = 2\delta_{\mu\nu}.$$ \hspace{1cm} (5)

Notice that one has the relation

$$\psi_5 = \gamma_5 = \frac{1}{4!}\varepsilon^{\mu\nu\kappa\lambda}\psi_{\mu}\psi_{\nu}\psi_{\kappa}\psi_{\lambda},$$ \hspace{1cm} (6)

with the algebraic properties

$$\psi_5^2 = 1, \quad \psi_5\psi_\mu + \psi_\mu\psi_5 = 0.$$ \hspace{1cm} (7)

To obtain a pseudo-classical action for fermions one can introduce another representation of the Dirac algebra in terms of two Grassmann-odd variables $\xi^{1,2}$ as follows:
\[ \psi_1 = \xi^1 + \frac{\partial}{\partial \xi^1}, \quad \psi_2 = -i \left( \xi^1 - \frac{\partial}{\partial \xi^1} \right), \]
\[ \psi_3 = \xi^2 + \frac{\partial}{\partial \xi^2}, \quad \psi_4 = -i \psi_0 = i \left( \xi^2 - \frac{\partial}{\partial \xi^2} \right). \]

(8)

It is straightforward to check, that these Grassmann-odd differential operators satisfy the Clifford algebra (5) and provide an alternative to the Dirac matrices in describing spinors.

This representation of the Dirac algebra in terms of anti-commuting variables can be extended with an element \( \psi_5 \) with the algebraic properties (7) in 2 different ways: first, one can introduce a third Grassmann-odd variable \( \xi^3 \) and define

\[ \psi_5 = \xi^3 + \frac{\partial}{\partial \xi^3}, \quad \psi_6 = -i \left( \xi^3 - \frac{\partial}{\partial \xi^3} \right). \]

(9)

With these definitions one finds

\[ \{ \psi_M, \psi_N \} = 2\delta_{MN}, \quad (M, N) = 1, \ldots, 6. \]

(10)

Therefore both \( \psi_5 \) and \( \psi_6 \) satisfy the properties (7), and one may arbitrarily chose one of them, say \( \psi_5 \). The other one then comes for free, but has no obvious use in the representation of spinors in 4-dimensional space-time at this point (however, see below).

The second way to represent \( \psi_5 \) is by the non-linear Grassmann-even expression

\[ \psi_5 = \frac{1}{4!} \varepsilon^{\mu \nu \kappa \lambda} \psi_\mu \psi_\nu \psi_\kappa \psi_\lambda = \left( 1 - 2\xi^1 \frac{\partial}{\partial \xi^1} \right) \left( 1 - 2\xi^2 \frac{\partial}{\partial \xi^2} \right). \]

(11)

This definition corresponds directly to the \( 4 \times 4 \) matrix representation \( \psi_5 = \gamma_5 \) in (6) and does have the properties (7), in spite of \( \psi_5 \) being Grassmann-even. However, in this case there is no analogue of \( \psi_6 \).

Given these two representations of the Dirac algebra, we consider the properties of spinors, which are simply defined as functions of the Grassmann-variables \( \xi^k \). First, in the minimal representation with non-linear Grassmann-even \( \psi_5 \) a spinor has four components:

\[ \Phi \left( \xi^1, \xi^2 \right) = \phi_2 + \xi^1 \phi_3 - \xi^2 \phi_4 - \xi^1 \xi^2 \phi_1. \]

(12)

The components have been labeled in a somewhat unconventional way, which has the advantage that the differential operators (8, 11) act on the components in the same way as the Dirac matrices on spinors \( \phi_\alpha \) \( (\alpha = 1, \ldots, 4) \) if they are taken in the chiral basis.
\[ \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \]
\[ \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Observe that in this representation \( \gamma_5 \) is diagonal, which is equivalent to the result
\[ \psi_5 \Phi (\xi^1, \xi^2) = \Phi (-\xi^1, -\xi^2). \]

It follows, that in this representation the chirality \( \pm \) of the spinor in the 4-D sense is the same as the Grassmann parity: if \( \Phi = \Phi_+ + \Phi_- \) with
\[ \Phi_+ = \phi_2 - \xi^1 \xi^2 \phi_1, \quad \Phi_- = \xi^1 \phi_3 - \xi^2 \phi_4, \]
and the Grassmann parity operator is \((-1)^F\), then
\[ \psi_5 \Phi = (-1)^F \Phi = \Phi_+ - \Phi_. \]

As the mass-term in the Dirac operator (3) contains \( \gamma_5 \), the Dirac equation becomes
\[ [p^\mu \psi_\mu + m\psi_5] \Phi(\xi^1, \xi^2) = 0. \]

It is therefore clear that the mass-term mixes spinor components of opposite Grassmann parity. We also observe, that the spinors \( \Phi(\xi^1, \xi^2) \) can be of either Dirac or Majorana type. With the charge conjugation operator represented by
\[ C = \left( \xi^1 - \frac{\partial}{\partial \xi^1} \right) \left( \xi^2 - \frac{\partial}{\partial \xi^2} \right), \]
a Majorana spinor has the same expansion (12) as a Dirac spinor, but with the additional restriction that \( \phi_3 = -\phi_2^*, \phi_4 = \phi_1^* \); in terms of the chiral components:
\[ \Phi_- = -i\sigma_2 \Phi_. \]

Next we consider the representation if the Dirac algebra in terms of three anti-commuting variables \( \xi^k, k = 1, 2, 3 \). In this case a general spinor is a function
\[ \Phi(\xi^1, \xi^2, \xi^3) = \Phi_1(\xi^1, \xi^2) + \xi^3 \Phi_2(\xi^1, \xi^2), \]
where each of the functions \( \Phi_{1,2}(\xi^1, \xi^2) \) is a 4-component spinor of the type discussed above. The Dirac equation:
\[ [p^\mu \psi_\mu + m\psi_5] \Phi(\xi^1, \xi^2, \xi^3) = 0, \]
is now equivalent to an $8 \times 8$ matrix equation
\[
(\Gamma_\mu p^\mu + m\Gamma_5) \Phi = \begin{pmatrix} -i\hat{p}\gamma_5 \\ m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0.
\] (22)

Clearly $(\Gamma_\mu, \Gamma_5)$ define a reducible 8-dimensional representation of the 4-D Dirac algebra, as is obtained by dimensional reduction from six space-time dimensions. This also explains the appearance of $\psi_6$ in eq.(10). The four-component spinors $\Phi_{1,2}$ then describe a degenerate pair of fermions in four space-time dimensions.

The conclusion from this analysis is, that the minimal (Dirac or Majorana) representation of spinors by Grassmann variables is the four-component one with a Grassmann-even non-linear representation of $\psi_5$, whilst the anti-commuting variable representation with Grassmann-odd $\psi_5$ describes a reducible representation of the Dirac algebra with a doubling of the number of physical degrees of freedom.

Ordered symbols

A general linear operator $A$ on a two-component vector $v = (v_1, v_2)$ with matrix
\[
A \equiv \begin{pmatrix} a_0 & a_2 \\ a_1 & a_3 \end{pmatrix}
\] (23)
can be represented as an ordered differential operator acting on a function $v(\xi) = v_1 + \xi v_2$ of an anti-commuting variable $\xi$, with ordered expansion
\[
A = a_0 + a_1 \xi + a_2 \frac{\partial}{\partial \xi} + (a_3 - a_0)\xi \frac{\partial}{\partial \xi}.
\] (24)

The action of such an ordered differential operator is equivalent to an integral operator
\[
[Av] (\xi) = \int d\xi_1 d\bar{\xi}_1 e^{\bar{\xi}_1(\xi_1 - \xi)} \bar{A} (\xi_1, \bar{\xi}_1) v(\xi_1),
\] (25)

where $\bar{A}(\xi, \bar{\xi})$ is the ordered symbol \cite{27, 28}, defined as
\[
\bar{A} (\xi, \bar{\xi}) = a_0 + a_1 \xi + a_2 \bar{\xi} + (a_3 - a_0)\xi \bar{\xi},
\] (26)

the ordered operator with every derivative $\partial/\partial \xi$ replaced by the Grassmann variable $\bar{\xi}$. Clearly there is a one-to-one correspondence between linear operators and ordered symbols. Note that the unit operator has components $a_0 = a_3 = 1$, $a_1 = a_2 = 0$, and its symbol is the number 1.

The product of two operators $A$ and $B$ is represented by the symbol
\[
[AB] (\xi, \bar{\xi}) = \int d\xi_1 d\bar{\xi}_1 e^{\bar{\xi}_1(\xi_1 - \xi)} \bar{A} (\xi_1, \bar{\xi}_1) B (\xi_1, \bar{\xi}).
\] (27)
It is straightforward to write down the ordered symbols and their multiplication rules for operators on functions of multiple Grassmann variables. In particular, the symbols for the operators $\psi_\mu$ in eq. (8) take the simple form

$$\bar{\psi}_1 = \xi^1 + \bar{\xi}^1, \quad \bar{\psi}_2 = -i (\xi^1 - \bar{\xi}^1),$$

(28)

$$\bar{\psi}_3 = \xi^2 + \bar{\xi}^2, \quad \bar{\psi}_4 = -i \bar{\psi}_0 = i (\xi^2 - \bar{\xi}^2).$$

For the symbol of $\psi_5$ we again have two choices: one is to introduce Grassmann variables $(\xi^3, \bar{\xi}^3)$ and define

$$\bar{\psi}_5 = \xi^3 + \bar{\xi}^3, \quad \bar{\psi}_6 = -i (\xi^3 - \bar{\xi}^3),$$

(29)

for the reducible spinor representation. The other one is to define

$$\bar{\psi}_5 = \left(1 - 2\xi^1\bar{\xi}^1\right)\left(1 - 2\xi^2\bar{\xi}^2\right) = e^{-2\xi\cdot\bar{\xi}}. \quad (30)$$

This is the correct form for the irreducible bosonic representation of $\psi_5$ corresponding to the Grassmann parity operator $(-1)^F$ [26].

**From fields to point particles**

The quantum theory of (free) fields has an interpretation in terms of the propagation and exchange of point particles. As emphasized by Schwinger [29], external sources $J(x)$ for the fields can excite the vacuum, and the amplitude for exchanging quanta of energy-momentum, spin etc. are proportional to

$$Z[J] \sim e^{\frac{i}{\hbar} \int J G_F \cdot J},$$

(31)

where $G_F(x - y)$ is the Feynman propagator. This propagator has a direct interpretation in terms of classical point particles through the path-integral formalism.² The connection is made as follows: for the particular case of free fermions, the Feynman propagator in momentum space is

$$G_F(p) = \frac{1}{ip + m - i\varepsilon}.$$

(32)

Here the $i\varepsilon$-prescription guarantees the causal behaviour of the theory. In view of our previous discussion it is actually preferable to work with the equivalent quantity

$$G_F(p)\gamma_5 = \frac{1}{-ip\gamma_5 + (m - i\varepsilon)\gamma_5}.$$  

(33)

This operator can be represented by its symbol in terms of Grassmann variables $(\xi^k, \bar{\xi}^k)$ by

²For a recent discussion see [24] and references therein.
\[ \langle G_{F_{\gamma 5}} \rangle (p; \xi, \bar{\xi}) = \frac{p \cdot \bar{\psi} + m \bar{\psi}_5}{p^2 + m^2}. \]  

(34)

This expression can be rewritten using a generalized Schwinger representation with a pair of bosonic and fermionic super proper-time parameters \((T, \sigma)\) [30]:

\[ \langle G_{F_{\gamma 5}} \rangle (p; \xi, \bar{\xi}) = -\frac{i}{2m} \int_0^\infty dT \int d\sigma e^{-\frac{iT}{2m}(p^2 + m^2) - \sigma(p \cdot \bar{\psi} + m \bar{\psi}_5)} \]  

(35)

To make the whole exponent proportional to the proper time \(T\), so as to obtain additivity in this parameter, it is customary to define a new anti-commuting parameter \(\chi = m\sigma/T\) and write

\[ \langle G_{F_{\gamma 5}} \rangle (p; \xi, \bar{\xi}) = -\frac{i}{2} \int_0^\infty dT \int d\chi \bar{K}(p; \xi, \bar{\xi}; T, \chi), \]  

(36)

where the integrand is

\[ \bar{K}(p; \xi, \bar{\xi}; T, \chi) = e^{-\frac{iT}{2m} \left[ (p^2 + m^2) - i\epsilon - \frac{i}{2} \sigma(p \cdot \bar{\psi} + m \bar{\psi}_5) \right]} \]  

(37)

The integral kernel may be interpreted as the Fourier transform of the real-space expression

\[ K(x - y; \xi, \bar{\xi}; T, \chi) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x - y)} \bar{K}(p; \xi, \bar{\xi}; T, \chi) \]  

\[ = -i \left( \frac{m}{2\pi T} \right)^2 e^{\frac{i\pi}{2} \left( T \chi \bar{\psi}_5 - \frac{m^2}{4} - \chi \bar{\psi} \cdot (x-y) \right)}. \]  

(38)

The integral kernel (38) has two useful properties:

A. in the limit \(T \to 0\) it reduces to the unit distribution

\[ \lim_{T \to 0} K(x - y; \xi, \bar{\xi}; T, \chi) = \delta^4(x - y); \]  

(39)

B. it satisfies the composition rule

\[ \int d^4z \prod_k \left[ d\xi'^k d\bar{\xi}^\kappa \right] e^{(\xi' - \bar{\xi}) \cdot (\xi' - \bar{\xi})} K(x - z; \xi, \bar{\xi}; T', \chi) K(z - y; \xi', \bar{\xi}; T'', \chi) = K(x - y; \xi, \bar{\xi}; T' + T'', \chi). \]  

(40)

Note that all results so far are true for both representations of \(\psi_5\). The properties A and B permit a path-integral representation of the propagator, by iterating the composition rule \(N\) times for periods \(\Delta T = T/(N + 1)\) and taking the limit \(N \to \infty\):
Dirac operator is obtained with anti-periodic boundary conditions. However, note that this is in contrast to the formulation of [26], where the trace of the propagator in correspondence with the quantum field theory of fermions described by the pseudo-classical theory defines the action. This effective action takes the form

\[
S_{\text{eff}}[\psi_5] = \int \mathcal{D}^4 x \mathcal{D}^n \bar{\psi} \mathcal{D}^n \chi e^{i S_T[x(\tau),\xi(\tau),\bar{\xi}(\tau)]}.
\]

Here I have labeled \(x_0 = y, x_{N+1} = x, \bar{\xi}_0 = \bar{\xi} \) and \(\xi_{N+1} = \xi\), and \(n = 2\) or \(3\) depending on the representation of \(\psi_5\). In the continuum limit the classical action in the exponent takes the form

\[
S_T[x(\tau),\xi(\tau),\bar{\xi}(\tau)] = \int_0^T d\tau \left[ \frac{m}{2} (\dot{x}_\mu^2 - 1) - \frac{i}{2} (\dot{\bar{\xi}} - \bar{\xi} : \dot{\xi} : \bar{\xi}) + i\chi \bar{\psi} \cdot \dot{x} + i\chi \bar{\psi}_5 \right],
\]

modulo boundary terms. This effective action defines the pseudo-classical theory in correspondence with the quantum field theory of fermions described by the propagator \(G_F\). It is well-known [26] that for closed propagators (loops) with anti-periodic fermionic boundary conditions: \(\xi_k = -\xi_{N+k}, \bar{\xi}_k = -\bar{\xi}_{N+k}\), the boundary terms disappear. The resulting path-integral expression then corresponds to the trace of \(\gamma_5\) times the Dirac propagator of the field-theory:

\[
\text{Tr} \left[ \gamma_5 (\hat{\mathcal{D}} + m)^{-1} \right] = -\frac{i}{2} \int_0^\infty \frac{dT}{T} \int d\chi \int_{PBC} \mathcal{D}^4 x(\tau) \int_{ABC} \mathcal{D}^n \bar{\xi}(\tau) \mathcal{D}^n \xi(\tau) e^{i S_T[x(\tau),\xi(\tau),\bar{\xi}(\tau)]},
\]

where \(PBC\) and \(ABC\) denote periodic and anti-periodic boundary conditions respectively.

For the case of the bosonic realization \(\psi_5 = (-1)^F\) (indicated by +) the trace of \((\hat{\mathcal{D}} + m)^{-1}\) itself is then obtained from the same path-integral with periodic boundary conditions:

\[
\text{Tr} G_F^{(+)} = \frac{\partial}{\partial m} \log \det (\hat{\mathcal{D}} + m)
\]

\[
= -\frac{i}{2} \int_0^\infty \frac{dT}{T} \int d\chi \int_{PBC} \mathcal{D}^4 x(\tau) \int_{PBC} \mathcal{D}^2 \bar{\xi}(\tau) \mathcal{D}^2 \xi(\tau) e^{i S_T^{(+)}[x(\tau),\xi(\tau),\bar{\xi}(\tau)]},
\]

Note that this is in contrast to the formulation of [26], where the trace of the Dirac operator is obtained with anti-periodic boundary conditions. However,
integration over $\chi$ brings down another $\bar{\psi}_5$ which changes the boundary conditions back to anti-periodic:

$$\text{Tr} \ G_F^{(+)} = \frac{i}{2} \int_0^\infty dT \oint_{\text{PBC}} \mathcal{D}^4 x(\tau) \oint_{\text{ABC}} \mathcal{D}^2 \bar{\xi}(\tau) \mathcal{D}^2 \xi(\tau) \ e^{i S_F^{(+)}[x(\tau),\xi(\tau),\bar{\xi}(\tau) \mid x = 0}.$$  \hspace{1cm} (45)

For the case of the linear fermionic realization of $\psi_5$ (indicated by $-$) a direct calculation shows that after integration over $(\xi^3, \bar{\xi}^3)$ and $\chi$ the trace of the propagator can be written as

$$\text{Tr} \ G_F^{(-)} = i \int_0^\infty dT \oint_{\text{PBC}} \mathcal{D}^4 x(\tau) \oint_{\text{ABC}} \mathcal{D}^2 \bar{\xi}(\tau) \mathcal{D}^2 \xi(\tau) \ e^{i S_F^{(-)}[x(\tau),\xi(\tau),\bar{\xi}(\tau) \mid x = \xi^3 = \bar{\xi}^3 = 0},$$  \hspace{1cm} (46)

with only an integration over the paths for the remaining two components of $(\xi(\tau), \bar{\xi}(\tau))$, satisfying anti-periodic boundary conditions, left. The results (45) and (46) are the same for both prescriptions for dealing with $\psi_5$, up to a factor of 2. This is precisely the factor one expects from doubling the dimension of spinor space, as I have argued to be associated with the Grassmann-odd representation of $\psi_5$.

**Supersymmetry**

In terms of the covariant fermionic co-ordinates $\bar{\psi}_\mu$, eq.(28), the classical action (42) for $\chi = 0$ (and in the appropriate case $\xi^3 = \bar{\xi}^3 = 0$) becomes

$$S_T \left[ x(\tau), \bar{\psi}(\tau) \right]_{\chi = 0} = \int_0^T d\tau \left[ \frac{m}{2} \left( \dot{x}_\mu^2 - 1 \right) + \frac{i}{4} \bar{\psi} \cdot \dot{\bar{\psi}} \right].$$  \hspace{1cm} (47)

Both path-integrals (45) and (46) therefore admit a rigid supersymmetry transforming $x^\mu$ into $\bar{\psi}_\mu$

$$\delta x^\mu = \bar{\psi}_\mu \varepsilon, \hspace{1cm} \delta \bar{\psi}_\mu = 2m \dot{x}^\mu \varepsilon.$$  \hspace{1cm} (48)

This is of course to be expected from the algebraic starting point (3). However, in the full action (42) with $\chi$ reinstated only the action with fermionic $\bar{\psi}_5$ can be interpreted as the gauge-fixed version of a model with local world-line supersymmetry, where the einbein field is fixed to the constant value $T$ and the gravitino fixed to the Grassmann constant $\chi$ [23]. In this formulation there is apparently no matching between the four bosonic and the six fermionic degrees of freedom, but this is because $\bar{\psi}_{5,6}$ are members of a fermionic supermultiplet of which the bosonic component is an auxiliary field (a recent review of $D = 1$ supergravity can be found in [23]). The additional fermion variable $\psi_6$ enters the action in the path integral through the kinetic terms; however, in the free-particle theory it is completely decoupled from the rest of the physical degrees of freedom and actually describes a simple topological quantum theory [24].
In contrast, in the case of a Grassmann-even $\psi_5$ local supersymmetry of the action $S_T$ seems violated by the Grassmann-odd term $i\chi\bar{\psi}_5$, which is rather unusual in a classical Lagrangian formalism. However, the presence of such a term is well-understood as the fermion mass-term flips chirality, which in the present formalism is equivalent to a change in Grassmann parity of the state.

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References