ANalytical Continuation of the Fermion Determinant with a Finite Cut-Off

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ABSTRACT

We consider the solitonic sector of the SU(2) Nambu-Jona-Lasinio model with vector mesons. We prove that none of the previously proposed procedures to handle the fermion determinant reproduces the perturbative expansion in the vector coupling constant. Hence they are either not consistent with the way the parameters of the model are fixed in the meson sector or they violate Lorentz invariance of the vacuum. We propose a new prescription at the operator level both in the Proper-Time and Pauli-Villars regularization. Our method is in accordance with perturbation theory, explicitly preserves vector gauge invariance and allows for a consistent treatment of the meson and soliton sector. The convergence radius of the perturbative expansion is also studied. We also show how direct calculations in Minkowski space might be performed.

28 May 1995
Submitted to Nuclear Physics A

UG-DFM-2/95

1. Introduction

During the last years, the Nambu-Jona-Lasinio model [1] has been studied extensively as a model of strong interactions. In such model baryons arise naturally as bound states of quarks in a solitonic mesonic background [2]. In particular, the inclusion of vector degrees of freedom and their influence on the NJL soliton has been investigated recently [3, 4, 5, 6]. Contrary to the case without vector mesons, such a program is plagued by difficulties of conceptual and numerical nature. In the presence of mesonic background fields of vector type (especially with $\omega$ meson) the interplay between regularization and Wick rotation of the fermion determinant has been the subject of strong debate and criticism among several groups, with the result that at least three different approaches have been pursued in the numerical search for solitons [7, 8, 9]. This is rather unsatisfactory since a detailed comparison shows that they are mutually incompatible and hence the obvious question arises which is the right one or if any of them is at all correct. In our opinion, the present state of the problem demands further analysis to clarify matters.

In the present paper we show that none of these prescriptions [7, 8, 9] reproduces the perturbative expansion in the vector coupling constant $g_\omega$. This is a clear drawback since in any case perturbation theory is used for the calculation of the meson properties [10, 11] which are in turn needed to fix the parameters for solitonic calculations. In plain terms, the description of both meson and soliton sector is based on different actions, thus contradicting the general idea of the chiral soliton model approach to baryon structure. To overcome these difficulties we propose a new prescription which does reproduce perturbation theory for small but finite values of $g_\omega$. Moreover, we investigate the convergence radius for given meson field configurations with hedgehog symmetry. As we will see it is indeed conceivable that the physically relevant value of the coupling lies beyond the convergence region. In that case, it suffices that one can reach the desired value by following a suitable continuous path in the complex $g_\omega$-plane, but always starting from the origin. In this paper we are not concerned with values of $g_\omega$ which cannot be obtained in such a smooth manner.
Before exposing the problem in more detail it should be emphasized that all the difficulties mentioned above arise from the fact that the NJL Lagrangian is only defined as an effective field theory with a finite cut-off. This is an important point since in a renormalizable field theory, where the infinite cut-off limit can be probed, the problem does not arise. For the same reason the cut-off scheme dependence cannot be removed by renormalization arguments. This is a drawback of the whole NJL approach since it introduces some uncertainty in the model. In general, each cut-off scheme will produce different results although numerical experience with different schemes seems to indicate that the uncertainty is quantitatively small if the parameters are fixed in a way that the pion decay constant and the pion mass are reproduced [10]. On the other hand, this ambiguity allows us to chose freely what objects and how they are to be regularized. In other words, the regularization of all the vertices appearing in the action is not compulsory as one might naively expect. This applies in particular to the imaginary part of the effective action which turns out to be conditionally convergent and to reproduce the anomalous Ward identities of QCD in the infinite cut-off limit (this requires the addition of suitable counterterms as explained in [12]). We see no a priori reason why it should be regularized and choose not to do it. On the contrary, the real part of the action is ultraviolet divergent and must be regularized.

It should also be emphasized that our aim is to bring the soliton description as close as possible to the meson description, i.e. to treat both sectors with one and the same action and at the same level of approximation, say one fermion loop and zero boson loop approximation. We believe that this gives the NJL approach its main appeal and certain predictive power over other hadronic models, i.e. it allows to determine vacuum, meson and baryon properties simultaneously in terms of a unique action which depends on few parameters. This is certainly desirable for consistency reasons. Nevertheless, the main technical difference between meson and soliton sector is the following: whereas in the meson sector one deals with a finite number of external legs, in the soliton sector the number of external legs becomes infinite. Thus, even if the cut-off is kept finite, the study of the meson sector is not an appropriate playground to see the problem of the analytical continuation in the vector fields. This includes in particular heat kernel or gradient expansion approximations to the full momentum dependent mesonic correlation functions. The reason for this lies in the fact that the regularization is generally performed in Euclidean space. The transition from Minkowski to Euclidean space requires the Wick rotation not only of the time coordinate $x_0 \rightarrow iz_4$ but also of the time component of the spin one fields. After the regularization one has to rotate back to the Minkowski space. This last step is well understood in the meson sector since the action is required to second order in the meson fields. In the solitonic sector however we need the meson fields to all orders and hence the transition from Euclidean to Minkowski space is by no means simple. For all the reasons mentioned above, the problem we are concerned about is specific to the appearance of a finite cut-off, the presence of vector couplings and the description of baryons as solitons.

The paper is organized as follows. In section 2 we spell out the origin of the problem. In section 3 we present our perturbative continuation to Minkowski space at the operator level both for the Proper-Time and Pauli-Villars regularization. Section 4 contains our numerical results. In section 5 we discuss on a direct formulation in Minkowski space. Section 6 makes some remarks about a self-consistent treatment. In section 7 we compare our new approach with the previous ones used by several groups. Finally, in section 8, we present our conclusions and perspectives for possible future work.

2. Statement of the Problem

Let us briefly review the origin of the problem. We follow here the notation of [4], where further details and physical motivation can also be found. After some simple formal manipulations it has been shown [13] that the NJL Lagrangian is equivalent to the following mesonic non-local action in Euclidean space

\[
S = -Sp\log(i\mathbf{D}) + \frac{g^2}{2G_F} \int d^4x \left( \sigma^2 + \pi^2 \right) + \frac{g^2}{8G_F} \int d^4x \left( \beta^2 + A^2 \right) + \frac{g^2}{2G_F} \int d^4x \omega^2 \]

(2.1)
where the Dirac operator

\[ i \mathcal{D} = -i \partial_t + g_\pi (\sigma + i \gamma_5 \vec{\partial} \vec{\pi}) + g_\rho_2 \left( \vec{\mathbf{\sigma}} \cdot (\vec{\pi} + \vec{\mathbf{A}} \gamma_5) \right) - \omega_\rho \phi + m_0 \]  

(2.2)

and the total functional trace \( \text{Sp} \) (colour, Dirac, flavour and space) have been introduced. As such this trace refers to a complete trace in Hilbert space. The Dirac operator is non-normal, since its hermitian conjugate is

\[ (i \mathcal{D})^\dagger = i \partial_t + g_\pi (\sigma - i \gamma_5 \vec{\partial} \vec{\pi}) - g_\rho_2 \left( \vec{\mathbf{\sigma}} \cdot (\vec{\mathbf{A}} \gamma_5) \right) + \omega_\rho \phi + m_0 \]  

(2.3)

and hence

\[ [\mathcal{D}, \mathcal{D}^\dagger] \neq 0 \]  

(2.4)

what means that, in general, the Dirac determinant is a complex number in Euclidean space and that \( \mathcal{D} \) and \( \mathcal{D}^\dagger \) cannot be diagonalized simultaneously. The fermion determinant is also a divergent quantity and hence some sensible regularization is needed. As discussed above and in previous works [4], we regularize the real part and do not regularize the imaginary part. This has the advantage that in this way vector gauge invariance is automatically preserved and Ward identities can be systematically implemented through addition of counterterms [12]. We use the generalized Proper-Time regularization, leading to the real part

\[ \text{Re} S_f = \frac{1}{2} \int_0^\infty \! \! \, \frac{d\tau}{\tau} \text{Tr} e^{-\tau \mathcal{D}} \Psi(\tau, \Lambda) \]  

(2.5)

with \( \Lambda \) the generalized Proper-Time cut-off. This includes as particular cases the original Proper-Time scheme as well as the Pauli-Villars regularization (see Ref. 4). For finite \( \Lambda \) this expression is meaningful, since the operator appearing in the exponent is positive definite. Thus, for a given set of meson fields the above expression yields a finite number. It should be stressed that eq. (2.5) constitutes the starting point for the calculation of the mesonic properties and hence for the determination of the parameters [4]. This statement holds both for the full momentum dependent Bethe-Salpeter description of mesons [10] as well as the

\[ \frac{1}{\text{Re}} \int_0^\infty \! \! \, \frac{d\tau}{\tau} \text{Tr} e^{-\tau \mathcal{D}} \Phi(\tau, \Lambda) \]  

(2.6)

where \( \text{Re} \) is the imaginary part reads

\[ \text{Im} E = \frac{N_c}{2} \int_{-\infty}^\infty \! \! \, \frac{d\nu}{2\pi} \text{Sp} \left[ \log(i\nu + H) - \log(-i\nu + H^\dagger) \right] \]  

(2.7)

whereas the regularized real part is given by

\[ \text{Re} \mathcal{E}^{PT} = \frac{N_c}{2} \int_{-\infty}^\infty \! \! \, \frac{d\nu}{2\pi} \sum_i \frac{d\tau}{\tau} e^{-\tau \nu} \text{Tr} \left[ \nu^2 + i\nu(\nu^2 - H^\dagger + HH^\dagger) \right] \]  

(2.8)

for the case of Proper-Time (PT) and Pauli-Villars (PV) regularization respectively. The definition of the single particle hamiltonians has been used

\[ H = \hbar - ig_\rho \omega_4 \]  

\[ H^\dagger = \hbar + ig_\rho \omega_4 \]  

(2.9)

where \( \hbar \) denotes the hermitean part.
3. Perturbative Continuation to Minkowski Space

As we have said, the continuation procedure is extremely simple in the perturbative regime. Thus, one might think of an expansion of the action in powers of the fields, compute the trace and rotate each power separately back to Minkowski space, e.g. $\omega_0^3 = (-i \omega_0)^3$. The resulting series can be trivially summed up to the exponential form. This way of proceeding is equivalent to rotate back already at the operator level in eq. (2.6) and eq. (2.7) and serves as our definition of the analytical continuation. Since we have used perturbation theory to justify this procedure, our argument will break down if we approach some singularity in the complex $g_\omega$-plane. Nevertheless, nothing prevents us to use the resulting expression for the total energy beyond the perturbative region provided it is convergent and can be reached continuously along some path in the complex $g_\omega$-plane starting from the origin $g_\omega = 0 + i0$. It should be kept in mind that the coefficients of such a perturbative expansion depend explicitly on the regularization method. Thus, also the detailed singularity structure of the resulting expressions will be regularization scheme dependent.

Rotating the Hamiltonians $H$ and $H^\dagger$ back to Minkowski space yields

$$
H \rightarrow H^+ = h + g_\omega \omega_0 \\
H^\dagger \rightarrow H^- = h - g_\omega \omega_0
$$

and leads to the following exponent in eq. (2.7)

$$
-\tau [\eta^2 + i\omega (H^- - H^+) + H^+ H^-] \equiv -\tau [\eta^2 + \mathcal{H}(\nu)]
$$

(3.2)

Notice that in Minkowski space $H^+$ and $H^-$ are distinct hermitian operators and hence are no longer related to each other by hermitian conjugation. In addition, we also have $(H^+ H^-)^\dagger = H^- H^+ \neq H^+ H^-$. As a consequence, while for the evaluation of the imaginary part two hermitian eigenvalue equations have to be solved

$$
H^+ \Psi_\alpha^+ = \epsilon_\alpha^+ \Psi_\alpha^+ \\
H^- \Psi_\alpha^- = \epsilon_\alpha^- \Psi_\alpha^-
$$

(3.3)

the calculation of the real part requires to solve a non-hermitian eigenvalue problem

$$
\mathcal{H}(\nu) |\lambda_\alpha \rangle = \lambda_\alpha(\nu) |\lambda_\alpha \rangle
$$

(3.4)

which depends non-trivially on the frequency $\nu$. In general, the eigenvalues $\lambda_\alpha(\nu) = \lambda^\nu_\alpha(\nu) + \Omega^\nu_\alpha(\nu)$ are complex valued functions of $\nu$. Of course, the operator $\mathcal{H}(\nu)$ gets hermitian and positive if $\omega$ is constant.

After having done the transition from Euclidean space to Minkowski space it does not make much sense to talk about real and imaginary parts of the energy, since this should be real in Minkowski space, at least for time independent field configurations. Nevertheless it has become customary to talk about such decomposition in Minkowski space meaning the corresponding analytically prolonged part of the Euclidean energy functional. Actually it is more convenient to refer to the even and odd pseudoparity contributions to the energy, defined as the part containing no Levi-Civita pseudotensor $\epsilon_{\mu\nu\alpha\beta}$ and the part containing one respectively. For the sake of clarity we adopt here the latter terminology. We indicate this one to one correspondence by means of the equation

$$
E_R + iE_I \rightarrow E_{\text{even}} + E_{\text{odd}}
$$

(3.5)

The contribution coming from the real Euclidean action (the pseudoparity even in Minkowski) $E_{\text{even}}$ to the energy $E$ reads then

$$
E_{\text{even}}^{\text{PT}} = \frac{N_e}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{-r^2 + \lambda_\alpha(\nu)}
$$

(3.6)

in case of the Proper-Time scheme, and

$$
E_{\text{even}}^{\text{PV}} = \frac{N_e}{2} \sum_{\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_{\alpha} \log[\nu^2 + \lambda_\alpha(\nu) + \lambda^2]
$$

(3.7)

for Pauli-Villars regularization. Similarly, the imaginary part of the Euclidean action (pseudoparity odd in Minkowski) contributes to the energy as

$$
E_{\text{odd}} = -\frac{N_e}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \ln(\nu + \epsilon_\alpha^+) - \ln(-\nu + \epsilon_\alpha^-) \right]
$$

$$
= -\frac{N_e}{4} \sum_{\alpha} |\epsilon_\alpha^+| - |\epsilon_\alpha^-|
$$

(3.8)
while the valence part has the form

$$E_{\text{val}} = N_c \langle \lambda_{\text{val}} \rangle$$  \hspace{1cm} (3.9)$$

The total energy in Minkowski space for a solitonic system is given as a sum of $E_{\text{even}}, E_{\text{odd}}, E_{\text{val}}$ and the mesonic part [4] stemming from the bosonization. By construction, the expressions (3.6) and (3.8) reproduce the perturbative expansion of the effective action in Minkowski space in powers of the vector fields.

In relation to the total energy one should say [14] that the theory at the level of the effective action fulfills the Current-Field identities for the vector mesons but does not reproduce the QCD anomalous Ward identities as it implicitly involves a vector gauge additive regularization. The addition of appropriate counterterms required to reproduce the QCD anomaly has been considered recently Ref. 12 and the modifications to the Current-Field identities have also been obtained. Nevertheless, the numerical effect on nucleon properties for hedgehog profiles has been found to be rather small ($\approx 1\%$). Hence, these corrections can be ignored in practice at least for the purposes of the present paper.

The part coming from the real Euclidean action deserves some specific comments. Regardless of the particular regularization scheme, the relevant operator $\mathcal{H}(\nu)$ appearing in the expressions (3.4) is not hermitean. However, in the case $g_\omega = 0$ we do have a hermitean operator since then $\mathcal{H} \rightarrow h^2$ and hence $\lambda_{\omega}(\nu) \equiv \zeta_\omega^2$ where $\zeta_\omega$ represents the eigenvalues of the hermitean operator $h$ (2.9). In this case the corresponding eigenfunctions span a complete Hilbert space and can be used to evaluate the total functional trace $\mathcal{S}_\rho$. If we increase the vector coupling constant infinitesimally the operator acquires a small non-hermitean part which endangers the completeness of the corresponding eigenfunctions. If completeness is lost, the functional trace might not be computed using the closure relation for the eigenfunctions. At this point we remind that we are in practice putting the system into a finite box and hence are working in a truncated Hilbert space of finite dimension. In such case the trace can still be evaluated as a sum of eigenvalues even if completeness of the eigenfunctions is lost. In effect, any finite dimensional operator is equivalent to another one in Jordan form, which in general does not possess a complete set of eigenfunctions; nevertheless the operator trace $\mathcal{S}_\rho$, coincides with

the spectral trace $\Sigma_\omega$. We believe that this should also be true in the continuum limit, i.e., in the infinite dimensional case, since the operator under the trace sign is, by construction and due to the regularization, of finite trace. Thus, for finite but perhaps not too large values of the vector coupling constant the functional trace can be evaluated as a sum of eigenvalues. For a certain critical value this will not necessarily be the case. Notice that even if a non-hermitean operator with complex eigenvalues is involved, the final result for the energy in Minkowski space ought to be real and convergent for time independent configurations. This might not happen when approaching a singularity in the $g_\omega$ coupling constant.

With the remarks of the previous paragraph in mind, the odd contribution stemming from the imaginary Euclidean action could also be computed as done and explained in previous works [8, 3, 4, 5]. First, evaluate the trace as sum of eigenvalues in Euclidean space. Second, compute the integral over frequencies. Third, assume that the eigenvalues behave analytically in the vector coupling constant $g_\omega$. This will be certainly true if $g_\omega$ is small enough. This prescription yields the result

$$E_{\text{odd}} = -\frac{N}\zeta \sum_{\omega} \int_{0}^{\infty} \frac{d\nu}{2\pi} \ln \left( \nu \left( \nu + \zeta_\omega^2 \right) \right) \right)$$

\hspace{1cm} (3.10)

\hspace{1cm} \right)$$

which certainly looks quite different as (3.8) and will produce in general different numerical values. Nevertheless, for small enough values of $g_\omega$ (3.8) and (3.10) will yield identical numerical values. If this is not the case there arises the legitimate question which one of the prescriptions is the correct one. Once more, the relevant point is whether the physically relevant $g_\omega$ is sufficiently small.

As we have already said, the singularity structure in the $g_\omega$ coupling depends on the particular regularization. Let us first consider the Proper-Time regularized action. As we see, the operator appearing in the exponent is not necessarily positive. Again, for $g_\omega = 0$ we do have a positive operator. If the vector coupling constant is slightly increased, the resulting complex eigenvalues will still have a positive real part. If for one single particle orbital $\alpha$ and for one $\nu$ of eq. (3.6) the
real part of $\nu^2 + \lambda_0(\nu)$ becomes negative, the $\tau$-integral would be divergent. Since we have started from a positive operator $(g_\omega = 0)$, this also defines a critical vector coupling constant. It is not obvious that such a behaviour finds its correspondence with the Pauli-Villars case.

Thus, the prescription given by eq. (3.6) is well defined for certain values of the coupling constant $g_\omega$ continuously reachable from the origin along the real axis, or equivalently in the region where perturbation theory around the hermitean part of the Hamiltonian $H$ is well defined. If one is at the edge of the perturbative expansion one might go beyond by suitable further continuation, following, if it exists, an appropriate path in the complex $g_\omega$-plane. The hope is that the physically relevant region lies within the region of analyticity mentioned above.

4. Numerical Results

Our prescription embodies all the properties which can be deduced from the use of perturbation theory in the vector coupling constants. Of course, if the perturbation expansion in the vector fields breaks down in the sense that the total energy does not yield a finite real number, then our prescription becomes invalid. This can happen at least in two different ways. Either by the onset of a divergency (corresponding to a pole) or by the appearance of an imaginary part in the expression for the energy in Minkowski space (corresponding to a branch cut). Unfortunately, in the absence of general theorems to our knowledge only numerical investigations can be done. In practice we always work in a truncated finite dimensional Hilbert space. The study of the mathematical continuum limit is not accessible within the present context.

Even if one knows that formula (3.6) is reliable from a theoretical point of view, it is far from clear if it can be evaluated with a moderate numerical effort. One has to diagonalize the non-hermitean operator $\mathcal{H}(\nu)$ for each value of $\nu$ and then perform the $\nu$ and, in the Proper-Time case, the $\tau$ integration. For the numerical evaluation of the real part the calculation is organized as follows. 1) We diagonalize $\mathcal{H}(\nu)$ for a certain value of $\nu$; 2) We sum over all eigenvalues $\lambda_\nu(\nu)$;

3) We integrate over $\tau$, in the Proper-Time case and 4) We integrate over $\nu$. In what follows we will discuss certain qualitative features which can be inferred numerically from our formula (3.6). In the present work it is not our aim to make an exhaustive numerical investigation, but rather to show that our prescription allows for a systematic analysis. We will discuss later how this can be extended for self-consistent calculations.

We first note that the exponential factor in (3.6) is in fact a complex number, nevertheless we always observe that for small non-critical $g_\omega$ the imaginary part of eq. (3.6) does in fact vanish to a high degree of accuracy (less than 1 MeV at soliton mass of 1200 MeV) after numerically evaluating the integration over the frequency $\nu$. Therefore our formula produces, as it should be, real energies in Minkowski space for not too large $g_\omega$-couplings. We look now for the critical behaviour of formula (3.6) with respect to the vector coupling constant $g_\omega$. For this purpose we use as input profiles the self-consistent solutions of Ref. 4 and the corresponding values for the parameters based on the calculation of the one loop on-shell masses. The explicit form of the fields is not a very essential point at the present stage and hence we do not worry about the fact that the profiles of [4] are not self-consistent profiles of the present procedure; we only want to feed formula (3.6) with mesonic fields with reasonable strength, shape and size. It is clear that definitive statements can be only made by performing a self consistent calculation in the present procedure. We will come back below to this point. In figure 1 we consider the spectrum of the operator for fixed $g_\omega$ and increasing $\nu$. As it can be seen, most eigenvalues show a very weak dependence on the frequency $\nu$, with the exception of the lowest eigenvalue (the would-be squared valence eigenvalue in case $g_\omega = 0$, $v_{\text{val}}^2$), which increases with increasing $\nu$. We have checked that this is a rather general feature of the spectrum.

4.1 Proper-Time scheme

In the Proper-Time case the convergence condition for the $\tau$ integral reads

$$\nu^2 + \lambda_0(\nu) > 0$$

meaning that the sign of the real part of the lowest eigenvalue $\lambda_0(\nu)$ at $\nu = 0$
is crucial for (3.6) to be justified. The value for \( g_\omega \) where \( \lambda_0(0) \) becomes negative defines a critical coupling constant and hence the convergence radius for a perturbative expansion starting from the Proper-Time regularized action. For illustration we show in figure 2 the real part of the eigenvalue \( \lambda_0(0) \) as a function of the constituent quark mass in the simpler case with \( g_\rho = 0 \). In this case the \( \omega \) coupling constant depends directly on the constituent quark mass by requiring the \( \omega \) meson propagator to have a pole at the physical value \( m_\omega = 783 \text{ MeV} \). (See Ref. 10, 4). We see that there indeed exists a critical value for the constituent quark mass and hence for \( g_\omega \), for which \( \lambda_0(0) \) becomes negative. The influence of the \( \rho \) and \( A \) meson can be inferred from figure 3. The situation is qualitatively different as compared to figure 2. Indeed, we see a whole range of values between \( M = 345 \text{ MeV} \) and \( M = 410 \text{ MeV} \), for which the lowest eigenvalue is clearly negative. In this case, our prescription is not applicable. However, one should have in mind that in a self-consistent treatment based on the present approach the detailed behaviour might be different.

Finally, after the steps 1, 2 and 3 mentioned above have been carried out the sea part of the energy stemming from the real part has the form

\[
E_R = \frac{N_c}{2} \int \frac{d\nu}{2\pi} f(\nu)
\]

where \( f(\nu) \) is the function of the \( \omega \) meson mass. The result is presented in figure 4. As we see, both functions are rather smooth, with a maximum at the origin. This suggests the use of the Gauss-Laguerre integration method. Generally speaking, we have found that high accuracy can be obtained with only few Gauss points (five to ten). This simple feature makes systematic numerical investigations with formula eq. (3.6) possible.

4.2 Pauli-Villars scheme

Many of the features appearing in the Proper-Time method find its correspondence in the Pauli-Villars scheme. The only exception is related to the divergency of the \( \gamma \)-integration in case of a negative eigenvalue. We do not expect such a behaviour in the Pauli-Villars scheme since the only singularities which might appear within a logarithm which has to be integrated over the frequency. One might rather expect the occurrence of an imaginary part for the energy in Minkowski space.

In table 1 we present the results for the total sea even and sea odd contributions in dependence on the constituent quark mass both for the Proper-Time case and the Pauli-Villars case. In the first case, one can see that the sea energy coming from the real Euclidean action becomes infinite at the same point where the valence eigenvalue \( \epsilon^0_{\text{val}} \) has changed sign. Nevertheless, the energy becomes finite again at about \( M = 450 \text{ MeV} \). Clearly, our prescription cannot be applied in the intermediate region, i.e. between \( M = 340 \text{ MeV} \) and \( M = 410 \text{ MeV} \). In contrast, in the Pauli-Villars method the sea energy coming from the real part of the Euclidean action is always a well defined real and finite number.

5. Formulation in Minkowski Space

One may wonder, if it is at all possible to make the whole calculation in Minkowski space without recoursing to Euclidean space as an intermediate step. In fact, the main motivation to go to Euclidean space is to ensure a well defined behaviour of the Feynman path integral and an easy implementation of the regularization procedure. This can be also achieved by introducing an \( \imath \epsilon \)-mass term for the corresponding fields, namely \( m \to m - \imath \epsilon \) for the fermions and \( \frac{\partial^2}{\partial \nu^2} \to \frac{\partial^2}{\partial \nu^2} - \imath \epsilon \) for the spin zero and spin one mesons. The only problem is that of introducing a finite cut-off regularization in Minkowski space which at the same time preserves Ward identities. For instance the usual Proper-Time regularization cannot be applied directly since no obvious positive definite operator, as \( D^\dagger D \) in Euclidean space, is available. We choose to use the Pauli-Villars scheme (which incidentally can be formally cast as an imaginary Proper-Time regularization). For details of this scheme in the present context we refer to previous work [4]. In the case of external space-like momenta the action in Minkowski space is a real quantity. This is the case for static solitonic configurations. For time dependent fields this is no longer the case and pair production can occur for excitation energies beyond the quark-antiquark threshold, resulting in an imaginary part for
the action in Minkowski space [15]. In the present context we do not know if an imaginary Minkowski energy for static solitonic configurations admits sensible physical interpretation or rather reflects a deficiency of the theory.

As already mentioned in section 3 it is more appropriate to distinguish between normal parity and abnormal parity parts of the action instead of real and imaginary parts as it would be done in the Euclidean case. In this section we will show that a direct formulation in Minkowski space is indeed possible. In fact, besides some small but important details, our treatment mimics to a large extent the Euclidean calculation.

5.1 Regularization in Minkowski Space

Our starting point is the fermionic contribution to the effective action in Minkowski space. In the notation of Ref. 12 it reads

$$S = -iS_p \log(i\mathbf{D} + ic)$$

where the Dirac operator is given by

$$i\mathbf{D} = i\partial - g_\sigma (\sigma + i\gamma_5 T^\sigma) - g_\gamma_5 \frac{\gamma}{2} (\gamma + \vec{\gamma} \gamma_5) - g_w \phi + m_0$$

Notice that we have included the $ic$ prescription explicitly to specify the proper Feynman boundary condition for quarks and antiquarks. We now introduce the operator $\mathbf{D}_5$ defined as

$$-i\mathbf{D}_5 = -i\partial - g_\sigma (\sigma - i\gamma_5 T^\sigma) + g_\gamma_5 \frac{\gamma}{2} (\gamma + \vec{\gamma} \gamma_5) + g_w \phi + m_0$$

In general, the $\mathbf{D}_5$ operator cannot be obtained from the operator $\mathbf{D}$ by any linear operation in Minkowski space. Nevertheless in $SU(2)$ one has the antilinear operator $Q = \gamma_5 G$ with $G$ the conventional $G$-parity symmetry (charge conjugation times 180-degrees rotation in isospace). Moreover, if we were in Euclidean space this transition would correspond to hermitian conjugation. We rewrite the action now as the sum of a $\gamma_5$-even part and a $\gamma_5$-odd part as follows

$$S_{\text{even}} = -iS_p \log(i\mathbf{D}_5 + ic)$$
$$S_{\text{odd}} = -iS_p \left[ \log(i\mathbf{D} + ic) - \log(i\mathbf{D}_5 + ic) \right]$$

The $\gamma_5$-even part contains the normal parity vertices (no Levi-Civita tensor) and it is ultraviolet divergent. We will use the Pauli-Villars method, which preserves gauge invariance and can be directly applied in Minkowski space. The regularized normal parity action reads

$$S_{\text{even}} = \frac{i}{2} \sum_i c_i S_p \log(\mathbf{D}_5 + \lambda_i^2 + ic)$$

with $\lambda_0 = 0$ and $c_0 = 1$ and $\sum_i c_i \lambda_i^2 = 0$.

5.2 Static Energy

In the case of static fields we have

$$\mathbf{D} = \gamma_0 (i\partial_t - H_+); \quad -i\mathbf{D}_5 = (-i\partial_t + H_-)\gamma_0$$

where $H^+$ and $H^-$ have been defined in section 3. Although both are hermitian operators the product $H^+H^-$ is not and hence there is no guarantee that the corresponding eigenvalues are real. Nevertheless, for $g_w = 0$ we do have a hermitian positive operator $h^2$ and thus real eigenvalues. If $g_w$ is slightly increased the eigenvalues will remain real until a critical value of the coupling constant is reached. For couplings below this critical value it makes sense to take the branch of the logarithm as given by the formula

$$\log(x + ic) = \log|x| - ic \Theta(x)$$

Assuming that $g_w$ is small enough, the natural parity part of the energy reads then

$$E_{\text{even}} = -\frac{\pi}{4} \sum_i c_i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \log \left( \omega^2 + H_+ H_- + \omega(H_+ - H_-) + \lambda_i^2 \right)$$

where we have used that $\text{sign}(x) = 2\Theta(x) - 1$ and the property $\sum_i c_i = 0$. In the particular case where no vector fields are present $g_w = 0$ the relation $H_- = H_+ = h$ holds, and hence the $\omega$ integral has the simpler form

$$E_{\text{even}} = -\frac{\pi}{4} \sum_i \sum_a c_i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{sign} \left[ -\omega^2 + e_a^2 + \lambda_i^2 \right]$$

which yields after evaluating the $\omega$ integral the known result for the Pauli-Villars regularized total energy [16] in the absence of vector mesons. Thus, expression
(5.8) allows us to compute the the normal parity energy in Minkowski space even if vector mesons are included.

For the abnormal parity part we make use of the identity

\[ \frac{i}{T} \log(iD + i\epsilon) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \Theta(H) \log(\omega - H + i\epsilon) + \Theta(-H) \log(\omega - H - i\epsilon) \right] \]  

(5.10)

where we have used \( i\gamma^t = i\text{sign}(H) \), valid for not too strong fields and \( H \) stands for both \( H^+ \) and \( H^- \). Integration by parts yields \( \text{up to a infinite constant independent of } H \)

\[ \frac{i}{T} \log(iD + i\epsilon) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \frac{\Theta(H)\omega}{\omega - H + i\epsilon} + \frac{\Theta(-H)\omega}{\omega - H - i\epsilon} \right] \]  

(5.11)

Direct calculation yields

\[ -\frac{i}{T} S_{odd} = -\frac{1}{4} \text{Tr} \text{sign}(H_+)H_+ - \text{sign}(H_-)H_- \]  

(5.12)

which coincides with our analytically continued imaginary Euclidean action. It is important to notice that the choice for the branch cut of the logarithm is not unique since one could add and arbitrary imaginary constant so that \( \text{e.g. } \text{Im } \log(x + i\epsilon) = \pi\theta(-x) \). For the Pauli-Villars regularized energy the particular choice is completely irrelevant since the difference cancels due to the property \( \sum_i c_i = 0 \) of the Pauli-Villars regulators.

5.3. Relation with the Analytical Continuation

Formula (5.8) appears not to be directly equivalent with the our expressions for the analytical continued expression from Euclidean to Minkowski space. To see the connection let us use the logarithm representation of the total energy for static configurations

\[ E_{even} = \frac{1}{2} \sum_i c_i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \log(\omega - H_+)(\omega - H_+) + \Lambda_i^2 + i\epsilon) \]  

(5.13)

Due to the presence of the Pauli-Villars regulator the \( \omega \) integral is convergent, so that we can write it as a contour integration including the upper half of the complex plane provided the argument of the logarithm does not vanish in this region. If so we can rotate the integration contour as follows

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) = +i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(i\omega) \]  

(5.14)

if we further assume that the real part of the eigenvalues are positive definite then a Proper-Time representation can be used for each of the logarithm. eq. (5.8) is reproduced if the Proper-Time regularization function is substituted. It appears, however, that the use of a Proper-Time representation is more restrictive as compared with Pauli-Villars. Finally, one should say that we consider formulas (3.6) and (3.7) more suitable for numerical computations.

6. Remarks on a self-consistent treatment

The results of section 4 cannot be considered definitive. In effect, the applicability of our prescription eq. (3.6) depends quantitatively on the particular profiles used to evaluate the total energy. Hence it is of course desirable to obtain self-consistent solutions. The way of proceeding is clear, one looks for the stationary solutions extremizing the total energy, and then solves them iteratively until convergence is achieved. The resulting expressions can be obtained in a straightforward manner and are rather cumbersome, so we will not reproduce them here. The only new feature is the appearance of double sums over occupied levels as compared to the usual single sums. We have attempted to solve this complicated set of equations by the customary feed-back method. This turns out to dramatically increase the required computational power for reliable numerical calculations. In fact, the needed CPU time is augmented by a factor 30 for a single iteration, with respect to previous calculations with vector mesons. It should be clear that some other method would be needed to solve these equations in practice. One possible way would be using a direct minimization of the total energy with respect to the value of the fields on a given grid as proposed for the purely scalar Nambu–Jona-Lasinio model [17]. Another possibility would be expanding in powers of the omega field, as suggested very recently [18]. Although this is certainly very appealing, since it simplifies the calculational effort tremendously, it is not
clear whether such a perturbative expansion is convergent for non-linearly coupled fields. In addition, the latter method would not allow to make definitive conclusions on the convergence radius.

An interesting aspect in the search of self-consistent solutions is given by the interplay between stationarity and the analytical continuation. The structure of the complex Euclidean action prevents from having non-trivial real solutions for the \( \omega \) field. This might seem a contradiction. Actually, it does not have to be like that, at least within a Pauli-Villars regularization. If one considers the extension of the action to complex fields according to the even or odd pseudoparity nature as discussed above, it can be shown that an analytical continuation to Minkowski space does in fact represent a stationary point of such an action. This is the case provided the vector field is not too strong as to endanger the analytical continuation. Obviously, this does not prevent in principle the existence of other stationary field configurations with no well defined hermiticity but a lower and real energy.

7. Comparison with previous approaches

In this section we compare our new prescription eq. (3.6) with three previous approaches, namely the Tokyo-group [7], the Tübingen-group [8, 9] and the Bochum-group [8, 3, 4, 5]. In all cases we will examine the perturbative regime showing that none of the methods reproduces the results of perturbation theory. For our purposes it is enough to proceed up to second order in the vector fields. We will not discuss the motivations for any of the approaches as they can be found in the corresponding references [7, 8, 9]. As a general feature all these prescriptions reproduce the standard Proper-Time regularized energy in the case in which no \( \omega \) meson is present. The differences appear in the lowest non-vanishing order in perturbation theory.

The second order term in the vector fields as obtained directly from eq. (3.6) can be expressed in the form

\[
E^{(2)} = N \frac{1}{2 \sqrt{\pi}} \int_0^\infty \frac{d\tau}{\tau^{3/2}} \frac{\beta(\tau, \Lambda)}{\sum_{\alpha, \beta}} \langle \omega | g_{\omega} \omega \beta | \beta \rangle > \langle \omega | g_{\omega} \omega | \beta \rangle^2 f(\varepsilon_{\alpha}, \varepsilon_{\beta}, \tau) \tag{7.1}
\]

where the correct result for the function \( f \) is given by

\[
f(\varepsilon_{\alpha}, \varepsilon_{\beta}, \tau) = e^{-\tau \varepsilon_{\beta}^2 - \tau \varepsilon_{\beta}^2} - \tau \frac{\varepsilon_{\alpha} e^{-\tau \varepsilon_{\beta}^2} + \varepsilon_{\beta} e^{-\tau \varepsilon_{\beta}^2}}{\varepsilon_{\beta}^2 - \varepsilon_{\beta}^2} \tag{7.2}
\]

which fulfills \( f(\varepsilon_{\alpha}, \varepsilon_{\alpha}, \tau) = 0 \). This expression coincides with the one obtained for the moment of inertia of the soliton [19] if one makes the formal replacement \( g_{\omega} \omega \to \tilde{\omega} \cdot \tilde{f}/2 \) and can be obtained by the well-known Schwinger operator expansion. In the limit in which the eigenvalues are evaluated in the plane wave basis, this expression becomes identical to the second order term in the \( \omega \) field of [10]. It has been obtained by going from Minkowski to Euclidean space using \( t \to -i\tau \) and \( \omega \to -i \omega \). After that \( D^1 D \) has been evaluated and the proper time regularization applied to the real part of the action. The expansion in powers of \( \omega \) has then been performed to second order. The first order term vanishes and the second order term second order term has been rotated back to the Minkowski space by \( \omega \to i \omega \). This second order term is very important since the \( \omega \)-meson propagator, which is used to adjust the \( g_{\omega} \) coupling constant in the model, is extracted from it. In the following we shall see, that none of the previous approaches [7, 9, 6, 8, 3, 4, 5] reproduces eq. (7.1) and eq. (7.2).

We shall discuss the methods [7, 9, 6, 8, 3, 4, 5] also under the aspect how far the infinite cut-off limit \( \Lambda \to \infty \) is properly reproduced. In this limit there is no regularization and one can formally work in Minkowski space by explicitly evaluating \( S_0 \log(1D) \). It is of course necessary for any regularization scheme to be formally correct in the limit \( \Lambda \to \infty \).

Another interesting aspect to judge the validity of the proposed methods is given by the gauge and Lorentz invariance of the action for a finite value of the cut-off \( \Lambda \). This means in particular that the final expression for the energy coming from the real part of the Euclidean action has to be invariant under the transformations \( \omega_{\mu} \to \omega_{\mu} + \partial_{\mu} a(x) \). This implies that if the space components of the \( \omega \) field vanish
and the time component is a constant, the final expression must be independent of this constant (the energy should not depend on the origin of potential). A consequence of this is that the second order energy must vanish. Indeed, in that case $\langle \alpha|\omega|\beta \rangle = \delta_{\alpha\beta}\omega_0$ and since $f(\epsilon_\alpha, \epsilon_\beta, \tau) = 0$ one has $E^{(2)} = 0$.

The last criterion is how far in the limit $g_\omega \to 0$ or vanishing $\omega$ field the well known expressions involving scalar and pseudoscalar mesons are reproduced. In this limit one is on safe grounds since the Wick rotation does not affect the spin zero fields.

Since all the previous methods produce formulas for the energy in terms of the eigenvalues of $H_-$ [7], $H_+$ and $H_-$ [8, 3, 4, 5] and $H$ and $H^T$ [9, 6] we will make explicit use of the well known quantum mechanical formula for second order perturbation theory

$$\epsilon_\alpha[H + V] = \epsilon_\alpha[H] + \langle \alpha|V|\alpha \rangle + \sum_{\alpha \neq \beta} \frac{|\langle \alpha|V|\beta \rangle|^2}{\epsilon_\alpha[H] - \epsilon_\beta[H]} + \ldots$$

The results of the comparison are summarized in table 3 and will be discussed below. One notices that only the treatment of the vector mesons in spirit of the present paper fulfills the aforementioned criteria of validity.

7.1 Tokyo Prescription

In this case [7] a direct regularization of the energy in Minkowski space is performed. As a consequence gauge invariance is explicitly lost since the sea energy would depend on a constant $\omega$ field. It is only recovered in the infinite cut-off limit which is properly reproduced. The regularization function reads

$$f_{\text{Tokyo}}(\epsilon_\alpha, \epsilon_\beta, \tau) = -\tau(1 - \delta_{\alpha\beta})\frac{\epsilon_\alpha e^{-\tau\beta} - \epsilon_\beta e^{-\tau\alpha}}{\epsilon_\beta - \epsilon_\alpha} - \frac{1}{2} \delta_{\alpha\beta} \tau^2 (e^{-\tau\beta} + e^{-\tau\alpha})$$

$$+ \delta_{\alpha\beta} \tau^2 (\epsilon_\alpha e^{-\tau\beta} + \epsilon_\beta e^{-\tau\alpha})$$

(7.3)

It differs from (7.1). Altogether the method fulfills only two of the four criteria of table 3.

7.2 Bochum Prescription

The energy is regularized in Euclidean space and then a Wick rotation back to Minkowski space is performed on the level of single particle eigenvalues and eigenvectors [8, 3, 4, 5]. The sea energy is independent of a constant $\omega$ field as required by gauge invariance. In the infinite cut-off limit the regularized Minkowski energy is reproduced. The regularization function is

$$f_{\text{Bochum}}(\epsilon_\alpha, \epsilon_\beta, \tau) = \frac{\epsilon_\alpha e^{-\tau\beta} - \epsilon_\beta e^{-\tau\alpha}}{\epsilon_\alpha - \epsilon_\beta}$$

(7.4)

Due to some non-analyticity features of the approach this result is different from (7.2). Thus this method fulfills only three of the four criteria mentioned above.

7.3 Tübingen Prescription

The energy is regularized in Euclidean space but no Wick rotation back to Minkowski space is performed [9, 6]. Also in this approach the sea energy does not depend on the absolute value of the $\omega$ field in agreement with gauge invariance. However, the infinite cut-off limit does not reproduce the unregularized Minkowski energy. An expansion up to second order in the vector field gives

$$f_{\text{Tübingen}}(\epsilon_\alpha, \epsilon_\beta, \tau) = -f_{\text{Bochum}}(\epsilon_\alpha, \epsilon_\beta, \tau)$$

(7.5)

i.e. it is exactly the opposite sign as in the Bochum prescription (as it should be due to the opposite hermiticity of the $\omega$ field). This method fulfills only two of the four criteria of table 3.

7.4 Numerical Comparison

It would be desirable to compare, for illustration, the numerical results of the present approach with those of previous calculations. However, the way the parameters have been fixed in those works is not always the same. Whereas in [7] and [9] a heat kernel determination has been used, in [4] and in the present work a full momentum dependent treatment has been considered. This is not a minor detail since already in the case with no $\omega$ meson (but with $\rho$ and $A$) a completely different behaviour for the self-consistent energy has been observed [8]. Thus, we
compare our results in the present approach with our previous results, as it can be seen in table 4. To do so we compare the energy in both approaches for the same profiles, namely the self-consistent solutions of [8], both in the Proper-Time and in the Pauli-Villars regularization, in dependence of the constituent quark mass $M$. Clearly, for increasing $M$ the differences range from 15% to 40% in the sea energy stemming from the real Euclidean action. The part coming from the imaginary Euclidean action is the same, provided the valence levels $\epsilon_\text{val}$ and $\epsilon_\text{val}$ have not changed sign.

8. Conclusions and Perspectives

In the present paper we have considered the solitonic sector of the Nambu-Jona-Lasinio model with $\sigma\pi\rho\omega$ and $\omega$ mesons. A way to make the path integral well defined and to be able to apply an Euclidean Proper-Time regularization is to go to Euclidean space. As a consequence the effective action becomes a complex number. The theory is however formulated originally in Minkowski space and physical observables can only be defined in that space. We have proposed a prescription for performing the analytical continuation of the fermion determinant from Euclidean to Minkowski space. In contrast to all previous approaches in the literature [7, 9, 6, 8, 3, 4, 5], our new method reproduces by construction the perturbative expansion in powers of the $\omega$ field. This is a very important check since perturbation theory is required to fix the parameters of the model by fitting the on-shell meson masses. In other words, the description of mesons and baryons are based in an unique action. Of course, for finite but not too large vector coupling constants the perturbative expansion may break down. This clearly defines a convergence radius for a region of analyticity. Beyond this region it is not clear what would be the proper definition since the analytical continuation is problematic. This may be the case, among other possibilities, if the valence quark eigenvalues $\epsilon_\text{val}$ change sign. Actually, after the present investigation was completed we became aware of Ref. 18. There, the authors have found similar results by somewhat different methods. Our conclusions are in qualitative agreement with theirs.

Our prescription is suitable for numerical investigations in the soliton sector.  

We have computed the total energy in Minkowski space both in the Proper-Time and in the Pauli-Villars regularization for given meson profiles. In some cases we have determined the convergence radius of the perturbative expansion. In no case have we attempted to go beyond the corresponding critical value. It is natural to ask whether further continuation can be carried out or whether there is an upper limit for the vector coupling constant. At present we do not know how to tackle this problem.

The detailed analytical structure depends quantitatively on the particular profiles. In this sense the numerical results in the present paper can only be considered illustrative but not definitive. Therefore we have attempted a self-consistent treatment within the present approach and using the usual iterative method. The resulting equations of motion are rather involved since they contain double sums with respect to the grand spin basis. Unfortunately, the needed computer time increases with respect to previous solitonic calculations in the Nambu-Jona-Lasinio model with vector mesons by a factor of thirty. This makes practical calculations almost impossible. In this sense it would be desirable to find an alternative numerical method. We envisage two possible ways. The first might be a direct minimization of the energy functional with respect to the values of the fields on a numerical grid as considered in the scalar model [17], hence avoiding the double sums. Another interesting possibility might be, as suggested very recently [18], to consider a perturbative expansion in the $\omega$ field up to second order. Although such a method is very appealing because it simplifies tremendously the calculational effort, it cannot tell us about the convergence rate nor can help us to decide if one is beyond the analyticity region. Furthermore, one does not know if for a soliton the expansion in second order is sufficient.

The previous results lead naturally to the question whether one can perform the whole calculation directly in Minkowski space without going to Euclidean space as an intermediate step. In fact the path integral can also be made well defined by introducing $\pi$ mass terms in all dynamical fields. We have formulated the problem in Minkowski space in the Pauli-Villars regularization scheme. The total energy in Minkowski energy is real if the vector coupling constant is not too large. The
resulting formulas can be made to coincide with our perturbative continuation by rotating the frequency integration contour from the real to the imaginary axis.

ACKNOWLEDGEMENTS

One of us (E.R.A.) acknowledges NIKHEF for hospitality and G. Ripka for useful correspondence.

This work has been partially supported by the KFA-Jülich (COSY-project), the Bundesministerium für Forschung und Technologie, Bonn (Contract 06-BO-702), the DGICYT under contract PB92-0927, the Junta de Andalucía (Spain) and FOM and NWO (The Netherlands).
### Table 1

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### Table 3

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**FIGURE CAPTIONS**

**Figure 1** Low lying real part of the spectrum of the operator $\mathcal{R}(\nu)$ (defined in the text) in dependence on the integration variable $\nu$ in units of the constituent quark mass squared $M^2$ for the particular value $M = 340$ MeV. We plot the real part of the would-be valence squared eigenvalue $\lambda_{\text{val}}^R(\nu) (\lambda_{\text{val}}(\nu = 0) = \lambda_{\text{val}}^2)$, together with some continuum states.

**Figure 2** The real part of the would-be valence squared eigenvalue $\lambda_{\text{val}}^R(\nu = 0)$ versus the constituent quark mass $M$, for the particular case $g_\rho = 0$ (only $\omega$ meson). The parameters have been fixed to reproduce the physical $\omega$ on-shell meson mass.

**Figure 3** The same as figure 2 but with $g_\rho \neq 0$, i.e. all vector mesons $\rho, A$ and $\omega$ included.

**Figure 4** The integrand $f(\nu)$ (defined in the text) as a function of the integration variable $\nu$, in units of the constituent quark mass and for the particular value $M = 340$ MeV. The dotted line corresponds to $g_\omega = 0$ and the solid line denotes the complete model with all vector mesons $\rho, A$ and $\omega$ included.

**TABLE CAPTIONS**

**Table 1** The total physical soliton energy in Minkowski space, together with the contributions to the sea energy stemming from the real and imaginary Euclidean parts, as well as the valence quark contribution. The Proper-Time regularization has been used.

**Table 2** The same as table 1 but for the Pauli-Villars regularization scheme.

**Table 3** Consistency criteria required for the final expression for the total energy of the NJL model in the presence of the $\omega$ meson. The headings mean 1.) limiting case $\omega = 0$, 2.) infinite cut-off limit, 3.) gauge invariance, and 4.) perturbation theory. Tokyo-group is Ref. 7, Tübingen-group means Ref. 9, 6 and Bochum-group Ref. 8, 3, 4, 5.

**Table 4** Comparison of the total soliton energy of old prescription [4] with the new prescription (see main text) for several values of the constituent quark mass $M$. The parameters are fixed by the propagator method (see Ref. 4) and the Pauli-Villars regularization is used.
REFERENCES


fig. 1

M = 340 MeV

fig. 2

$v = 0$