On single scalar field cosmology

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Abstract
Observations suggest, that there may be periods in the history of the universe, including the present one, in which its evolution is driven by scalar fields. This paper is concerned with the solution of the evolution equations for a spatially flat universe driven by a single scalar field. Some general theorems relevant to the cosmology of these models are presented, and several approaches to solve the equations are discussed. For some potentials special exact solutions can be found, and for the case of exponential potentials the complete solution is rederived in a new parametrization. For the general case solutions are constructed in terms of a power series expansion in the field. The issue of double-valuedness of such a series expansion in case of oscillating fields with turning points is addressed and resolved.
1 Cosmic scalar fields

It is widely accepted that scalar fields can drive the cosmic expansion. More in particular, scalar fields can possibly account both for an early period of inflation to explain the large-scale homogeneity and isotropy of the universe [1, 2], and for the observed accelerated expansion of the universe in more recent times [3, 4]. One of the most widely studied scenarios for inflation is the minimal chaotic inflation model [5], in which a scalar field moving in a potential creates a dynamical form of dark energy that makes the universe expand. In a similar way cosmic scalar fields can be used to model dark energy [6, 7] which drives the accelerated expansion of the universe deduced from supernova observations. In view of such potential applications the dynamics of scalar fields in a cosmological context is a physically relevant subject of investigation [8, 9].

In this paper we investigate simple cosmological models in which a single dynamical scalar is minimally coupled to gravity, and the gravitational field is taken to be of the Friedmann-Lemaître-Robertson-Walker type. More in particular, motivated by the observations of the cosmic microwave background we take the spatial part of the metric to be flat. Thus the line element describing the proper-time for a comoving particle in the universe is

\[-d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) dx^2.\] (1)

Here \(a(t)\) is the scale factor, whilst \(N(t)\) is the lapse function in the ADM formulation of General Relativity [10], allowing one to keep local time reparametrizations as an invariance in the description of space-time geometry. Usually one chooses \(t\) to be cosmic time such that \(N(t) = 1\). This will also be our preferred choice. However, we find it useful to keep \(N(t)\) free in the derivation of the relevant field equations for reasons to become clear soon.

The action for a real scalar field \(\varphi\) minimally coupled to gravity in General Relativity is

\[S = \int d^4x \sqrt{-g} \left( -\frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V[\varphi] \right),\] (2)

where \(R\) is the Riemann scalar and \(V[\varphi]\) is the scalar potential. Taking the metric of the form (1) and the scalar field to be spatially constant in this frame: \(\nabla \varphi = 0, \varphi(x^\mu) = \varphi(t)\), the effective action per unit of co-ordinate volume takes the form

\[\Sigma = \int dt \left( -\frac{3}{N} a \ddot{a} + \frac{a^3}{2N} \dot{\varphi}^2 - a^3 NV[\varphi] \right).\] (3)

Variation of this action provides the relevant equations of motion for the evolution of this isotropic and homogeneous model universe. First, the dynamical equation for the scale factor is

\[\frac{1}{3aN} \delta \Sigma \delta a = \frac{2}{aN} \frac{d}{dt} \left( \frac{1}{N} \frac{da}{dt} \right) + \frac{1}{aN} \frac{da}{dt} \left( \frac{1}{N} \frac{da}{dt} \right)^2 + \frac{1}{2} \left( \frac{1}{N} \frac{d\varphi}{dt} \right)^2 - V[\varphi] = 0.\] (4)

Next, the dynamical equation for the scalar field is

\[-\frac{1}{a^3N} \delta \Sigma \delta \varphi = \frac{1}{N} \frac{d}{dt} \left( \frac{3}{aN} \frac{d\varphi}{dt} \right) + \left( \frac{3}{aN} \frac{d\varphi}{dt} \right) \frac{1}{N} \frac{d\varphi}{dt} + V'[\varphi] = 0,\] (5)

\(^1\text{In most of this paper we employ Planck units such that } c = \hbar = 8\pi G = 1.\)
where the prime denotes a derivative w.r.t. the field $\varphi$. Finally, there is a constraint imposed by variation of the arbitrary lapse function $N$:

$$\frac{1}{a^3} \frac{\delta \Sigma}{\delta N} = 3 \left( \frac{1}{aN} \frac{da}{dt} \right)^2 - \frac{1}{2} \left( \frac{1}{N} \frac{d\varphi}{dt} \right)^2 - V[\varphi] = 0. \quad (6)$$

In this form, both the effective action and the equations of motion are manifestly invariant under time reparametrizations $t \to t'$, with $N(t)$ transforming as

$$N'(t')dt' = N(t)dt, \quad (7)$$

whilst $a(t)$ and $\varphi(t)$ behave as scalars:

$$a'(t') = a(t), \quad \varphi'(t') = \varphi(t). \quad (8)$$

This reparametrization invariance allows us to choose a gauge $N = 1$, such that the equations of motion become

$$2\dot{H} + 3H^2 + \frac{1}{2} \dot{\varphi}^2 - V = 0,$n
$$\ddot{\varphi} + 3H \dot{\varphi} + V' = 0, \quad (9)$$

$$-3H^2 + \frac{1}{2} \dot{\varphi}^2 + V = 0,$n

where $H = \dot{a}/a$ is the usual Hubble parameter. It is well-known that these equations are redundant to the extent that for $H \neq 0$ the last two equations imply the first one; indeed, for $H \neq 0$ differentiation of the last equation leads –upon use of the middle one– to

$$6H\ddot{H} + 3H\dot{\varphi}^2 = 0 \Rightarrow 2\dot{H} = -\dot{\varphi}^2. \quad (10)$$

Adding this to the last equation (9) gives back the first one. However, when $H = 0$ the proof fails, and the first equation has to be considered as a separate condition. Actually, the last equation (9) obtained from the variation of $N$ is just the reduced hamiltonian constraint of general relativity in FLRW space-times (also known as the Wheeler-DeWitt equation) which restricts the set of allowed solutions of the equations of motion to those for which the hamiltonian vanishes.

In this context we observe, that the effective action (3) can be written in a more familiar form by defining new dynamical variables $(X^0, X^1)$ and a time parameter $\tau(t)$ by

$$X^0 = \sqrt{6} \ln a, \quad X^1 = \varphi, \quad \frac{d\tau}{dt} = \frac{1}{a^3} = e^{-\sqrt{\frac{3}{2}}X^0}.$$n

In terms of these variables the effective the effective action becomes [12]

$$\Sigma = \int d\tau \left( -\frac{1}{2N} \left( \frac{dX^0}{d\tau} \right)^2 + \frac{1}{2N} \left( \frac{dX^1}{d\tau} \right)^2 - NU[X^0, X^1] \right), \quad (12)$$

where

$$U[X^0, X^1] = e^{\sqrt{\pi}X^0}V[X^1] = a^6V[\varphi]. \quad (13)$$

Thus the cosmological model (3) is mathematically equivalent to that of a relativistic particle in Minkowski space moving in a time-dependent scalar potential $U$. The action (12) is a
convenient starting point for a canonical (and quantum) treatment of mini-superspace cos-
omology. In this formulation the hamiltonian constraint in the gauge $N = 1$ takes the simple form

$$-\frac{1}{2} \left( \frac{dX^0}{d\tau} \right)^2 + \frac{1}{2} \left( \frac{dX^1}{d\tau} \right)^2 + U[X^0, X^1] = 0. \quad (14)$$

2 Dynamics: general considerations

Returning to the classical cosmology model described by eqs. (9), we observe that the hamiltonian constraint represents a first integral of motion for the system $(a, \varphi)$, but one which can not take arbitrary values: being a first-class constraint the right-hand side of the third equation (9) necessarily vanishes, even though the other two equations would be consistent with any constant value $E$ such that

$$a^3 \left( -3H^2 + \frac{1}{2} \dot{\varphi}^2 + V \right) = E. \quad (15)$$

The constraint $E = 0$ is the result of local time-reparametrization invariance imposed by the gauge variable $N$. It follows that one can not impose arbitrary initial conditions for the variables $(a(t), \varphi(t))$ and their velocities $(\dot{a}(t), \dot{\varphi}(t))$: any set of initial values is constrained by $E = 0$. Keeping this in mind, the complete classical dynamics can be derived from the hamiltonian constraint and the Klein-Gordon equation for $\varphi$, i.e. the second eq. (9).

It is clear that the middle term of the KG equation linear in the velocity $\dot{\varphi}$ represents the transfer of energy from the scalar field to the scale factor, or vice versa. Note however, that it does not imply the breaking of time reversal invariance: under time reversal $t \to -t$ both $\dot{\varphi}$ and $H$ change sign, with the effect that the equation itself is time-reversal invariant. As a result the energy density $E/a^3$ remains constant (and vanishes), and there is no dissipation of energy for the combined system of scalar and gravitational degrees of freedom as such. To include dissipation by e.g. particle creation, the equations (9) would have to be modified [8].

A related conclusion is, that consistent non-degenerate evolution of the system does not necessarily require the scalar potential $V$ to be bounded below: there is already a negative contribution to the hamiltonian from gravity, witness the term $-3H^2$ in eq. (15), but the constraint $E = 0$ serves to stabilize the system.

As observed earlier the dynamics of gravity, as described by the first equation (9), only follows from the other two equations if $H \neq 0$; therefore in solving for $a$ and $\varphi$ from the KG equation and the constraint we always have to consider the case $H = 0$ corresponding to flat space-time separately. The case of Minkowski space-time $H = \dot{H} = 0$ is quite straightforward: the equations (9) reduce to

$$\frac{1}{2} \dot{\varphi}^2 - V = \frac{1}{2} \dot{\varphi}^2 + V = 0, \quad \ddot{\varphi} + V' = 0. \quad (16)$$

It follows that the kinetic and potential energy have to vanish separately:

$$\dot{\varphi} = V = 0, \quad (17)$$

and as a result also

$$\ddot{\varphi} = V' = 0. \quad (18)$$
This is possible only if the potential has a stationary point which is also a zero: \( V = V' = 0 \). In most potentials this will not apply. Note, that of course the evolution of the universe can pass through a flat point, where \( H = 0 \) but \( \dot{H} \neq 0 \). At such a point

\[
\dot{H} = V = - \frac{1}{2} \dot{\varphi}^2 \leq 0,
\]

i.e. the universal expansion goes through a maximum there.

In all other cases (\( H \neq 0 \)), it suffices to consider the two equations

\[
\ddot{\varphi} + 3H \dot{\varphi} + V' = 0, \quad \frac{1}{2} \dot{\varphi}^2 + V = 3H^2.
\]

Now assuming a solution to exist, this solution must at least for finite stretches of time define a one-to-one map from \( t \) to \( \varphi \). During such a period one can write [12, 13]:

\[
H(t) = H[\varphi(t)], \quad \dot{H} = H' \dot{\varphi},
\]

where the overdot denotes a derivative w.r.t. cosmic time \( t \), and the prime a derivative w.r.t. the scalar field \( \varphi \). Eq. (10) then implies

\[
\dot{\varphi}^2 = -2H' \dot{\varphi},
\]

hence either \( \dot{\varphi} = 0 \) or

\[
\dot{\varphi} = -2H', \quad \ddot{\varphi} = 4H''H'.
\]

After this is substituted back into the equations of motion we find [11] [12] [13]

\[
V = 3H^2 - 2H'^2, \quad V' = 2H' (3H - 2H'') \cdot
\]

It follows, that \( \dot{\varphi} \) and \( H' \) can never vanish when \( V < 0 \), and conversely a configuration in which \( \dot{\varphi} = H' = 0 \) can be reached only in regions where \( V \geq 0 \). It also follows, that stationary points of \( V \) impose a particular constraint on \( H[\varphi] \):

\[
V' = 0 \Rightarrow H' = 0 \quad \text{or} \quad 2H'' = 3H.
\]

In contrast, a stationary point of \( H \) will occur at a stationary point of \( V \), unless \( H'' \) is singular there and \( H'H'' \) is finite and non-zero. In the latter case, eqs. (22) imply that \( \dot{\varphi} = 0, \ddot{\varphi} \neq 0 \); hence \( \varphi \) reaches an extremum and its trajectory in the \((t, \varphi)\)-plane exhibits a turning point there.

## 3 Explicit examples of scalar cosmology

There are several ways to construct solutions for the equations of scalar cosmology, depending on the problem to be addressed. For example, one can search for models and initial conditions allowing for a specific type of solution, or one can try to find all solutions allowed by a given scalar potential and initial conditions. In this section we discuss examples of the first type; in the later sections we construct more generic solutions for specific potentials.

Various cosmological scenarios, such as purely matter or radiation dominated universes, are described by a simple power law:

\[
a(t) = \left( \frac{t}{\tau} \right)^n,
\]
where \( \tau \) is some fixed reference time. Such types of behaviour can also be achieved in scalar field cosmology [14]; it follows from (25) that
\[
H = \frac{\dot{a}}{a} = \frac{n}{t},
\]
and the second equation (10) becomes
\[
\dot{\varphi}^2 = -2\dot{H} = \frac{2n}{t^2} \quad \Rightarrow \quad \dot{\varphi} = \pm \frac{\sqrt{2n}}{t}. \tag{27}
\]
Clearly a power-law solution (25) is possible in the context of a regular scalar field model only for \( n \geq 0 \), i.e. non-contracting universes. Introducing a constant of integration \( \tau \), the solution of eq. (27) is [14, 15]
\[
\varphi(t) = \varphi(\tau) \pm \sqrt{2n \ln \frac{t}{\tau}}. \tag{28}
\]
Substitution of these results into the third eq. (9) now leads to
\[
V = 3H^2 - \frac{1}{2} \dot{\varphi}^2 = \frac{n(3n-1)}{t^2} = V_0 e^{\mp \sqrt{2} \varphi}, \tag{29}
\]
with \( V_0 \) fixed by the requirement
\[
V_0 e^{\mp \sqrt{2} \varphi(\tau)} = \frac{n(3n-1)}{\tau^2}. \tag{30}
\]
Observe, that for \( 0 < n < 1/3 \) the potential is negative definite, whereas for \( n > 1/3 \) it is positive definite. We conclude, that both positive and negative exponential potentials can allow for power-law solutions of the type (25), but only in specific domains of non-negative powers \( n \). We will show later, that flat contracting universes can arise in scalar cosmology for other types of potentials, although not with a simple power-law (25) for the scale factor. A different question to be addressed later is, what other solutions exist for exponential potentials.

It goes without saying, that a constant Hubble parameter \( H_0 \) results from constant \( \varphi = \varphi_0 \) at an extremum of the potential:
\[
\dot{\varphi} = 0, \quad V'(\varphi_0) = 0. \tag{31}
\]
In scalar cosmology this necessarily represents a positive cosmological constant \( V = 3H_0^2 \). Of course, a solution (31), as well as the Minkowski limit \( H_0 = 0 \), can also arise as the final stationary state in a scenario in which the scalar field evolves dynamically from a higher value to end up at an extremum of the potential.

Instead of specifying a certain evolution of the scale factor, one can also start from a specification of the time-dependence of the scalar field. As an example, we construct a solution with an oscillating scalar field
\[
\varphi(t) = \varphi_0 \cos \omega t. \tag{32}
\]
The rate of change of the field is
\[
\dot{\varphi} = -\omega \varphi_0 \sin \omega t = -\omega \sqrt{\varphi_0^2 - \varphi^2}. \tag{33}
\]
The first eq. (22) then becomes
\[
H' = \frac{\omega}{2} \sqrt{\varphi_0^2 - \varphi^2}, \tag{34}
\]
with the solution

\[ H = H_0 - \frac{\omega \varphi_0^2}{4} \arccos \frac{\varphi}{\varphi_0} + \frac{1}{4} \omega \varphi \sqrt{\varphi_0^2 - \varphi^2}. \]  

(35)

Here \( H_0 \) is the initial value of \( H \) when \( \varphi = \varphi_0 \); indeed, substitution of (32) gives the time dependence of \( H \) as

\[ H(t) = H_0 - \frac{1}{4} \varphi_0^2 \omega^2 t + \frac{1}{8} \varphi_0^2 \omega \sin 2\omega t. \]  

(36)

The corresponding solution for the scale factor reads

\[ a(t) = a_0 e^{H_0 t - \frac{1}{2} \varphi_0^2 \omega^2 t^2 + \frac{1}{4} \varphi_0^2 (1 - \cos 2\omega t)}. \]  

(37)

This represents a universe growing from very small size at large negative times, to a finite size around \( t = 4H_0/(\omega \varphi_0)^2 \), when it shows some oscillating behaviour, to contract again to arbitrarily small size for very large positive times. Finally, we can compute the scalar potential from which such behaviour follows:

\[ V = 3H^2 - 2H^2 \]

\[ = 3 \left( H_0 - \frac{\omega \varphi_0^2}{4} \arccos \frac{\varphi}{\varphi_0} + \frac{1}{4} \omega \varphi \sqrt{\varphi_0^2 - \varphi^2} \right)^2 - \frac{\omega^2}{2} (\varphi_0^2 - \varphi^2). \]

(38)

It is rather remarkable that such a complicated scalar potential can give rise to a simple periodic solution for the field \( \varphi(t) \), and allows a complete solution for the scale factor. More importantly, we observe that this solution of the scalar cosmology equations clearly shows essentially reversible behaviour, and illustrates explicitly that the back reaction of the space-time curvature on the scalar field in the Klein-Gordon equation can not be interpreted off-hand as a dissipative friction term.

### 4 More on exact solutions

So far we have constructed potentials starting from a prescribed time-dependence of the scale factor \( a(t) \) or the scalar field \( \varphi(t) \). There is yet another way of finding solutions for scalar cosmology models, allowing to construct particular solutions as well as generic ones. This method starts not from a prescribed time behaviour of the cosmological degrees of freedom, but from postulating the relation \( H[\varphi] \) in eq. (20). In this section we only consider particular cases with simple analytic solutions. The construction of generic solutions is discussed later.

We start with the simplest non-trivial example, in which \( H \) is linear in the field \( \varphi \):

\[ H = h_0 + h_1 \varphi, \quad H' = h_1. \]

(39)

It follows directly, that

\[ V = 3 (h_0 + h_1 \varphi)^2 - 2h_1^2. \]

(40)

To diagonalize the mass term, we need to make the shift

\[ \psi = \varphi + \frac{h_0}{h_1} \Rightarrow V = -2h_1^2 + 3h_1^2 \psi^2, \quad H = h_1 \psi. \]

(41)
Now \( h_1 \) is directly proportional to the mass:

\[
m^2 = 6h_1^2 \implies V = -\frac{1}{3} m^2 + \frac{1}{2} m^2 \psi^2, \quad H = \frac{m}{\sqrt{6}} \psi. \tag{42}
\]

The corresponding solutions for the field and scale factor are easily found:

\[
\dot{\psi} = -2H \psi \implies \psi(t) = -\frac{2m}{\sqrt{6}} (t - t_0),
\]

\[
H = -\frac{m^2}{3} (t - t_0) \implies a(t) = a_0 e^{-\frac{m^2}{3}(t-t_0)^2}. \tag{43}
\]

This is not the only solution for the quadratic potential \( V \) in (42), in fact it is a very special one: the only solution which exists for all times; but it pays to consider it a bit more in detail.

A first observation is, that the rate of change of the scalar field \( \dot{\psi} \) is constant over the whole time domain \(( -\infty, +\infty )\). Thus there is no dissipation of kinetic energy, in spite of the fact that \( H \) does not vanish except at \( t = t_0 \). Secondly we observe, that the Hubble parameter \( H(t) \) is negative for times \( t > t_0 \); indeed this universe expands only during the epoch \( t < t_0 \) from arbitrarily small scales to a maximal size when \( a(t) = a_0 \) for \( t = t_0 \), and contracts again to vanishingly small size at large positive time \( t > t_0 \).

Actually, this behaviour for large times is generic for potentials with a negative minimum: \( V_{\text{min}} < 0 \). This follows from two general observations. First, eq. (10):

\[
\dot{H} = -\frac{1}{2} \dot{\varphi}^2 \leq 0,
\]

implies that for regular kinetic terms of the scalar field the Hubble parameter is a non-increasing function of time, and is constant only at stationary points of the field evolution: \( \dot{\varphi} = 0 \). Second, as eq. (23) shows, any point where the potential is negative must satisfy

\[
V < 0 \implies 2H'^2 > 3H^2 > 0, \tag{44}
\]

and therefore \( \dot{\varphi} = -2H' \) can never vanish in the range where the potential is negative. As a result a negative minimum of the potential \( V \), even if it is the absolute minimum, can never represent a stationary point of the dynamics; we conclude that the field never comes to rest at a negative value of the potential, and the Hubble parameter is a monotonically decreasing function of time as long as \( V \) is negative. This conclusion is in agreement with the general result of ref. [16].

It is not difficult to construct more examples of exact solutions for polynomial potentials by a similar procedure. For example, taking

\[
H = h_0 + h_2 \varphi^2, \quad H' = 2h_2 \varphi, \tag{45}
\]

we get a quartic potential

\[
V = V_0 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4, \tag{46}
\]

with

\[
V_0 = 3h_0^2, \quad m^2 = 12h_0 h_2 - 16h_2^2, \quad \lambda = 12h_2^2, \tag{47}
\]
which implies

\[ V_0 = \frac{4\lambda}{9} \left( 1 + \frac{3m^2}{4\lambda} \right)^2. \]

The corresponding particular time-dependent solutions of the Friedmann and Klein-Gordon equations are

\[ \varphi(t) = \varphi(0) e^{-\omega t}, \quad a(t) = a(0) e^{\sqrt{\frac{V_0}{3}} t + \frac{1}{2} \omega^2 (1 - e^{-2\omega t})}, \] (48)

with \( \omega^2 = 4\lambda/3 \). The cosmology of this model was discussed in detail in ref. [13].

5 On a square root of the Friedmann equation

In some cases it is possible to use the above procedure to construct the complete set of solutions in closed form. To simplify the discussion it is convenient to rescale the scalar field and define

\[ u(t) = \sqrt{\frac{3}{2}} \varphi(t). \] (49)

With this change of variable, the Friedmann equation (23) and the first field equation (22) become

\[ H^2 - H_u^2 = \frac{1}{3} V, \quad \dot{u} = -3H_u, \] (50)

employing the notation \( H_u = dH/du \). Now suppose the potential is positive definite: \( V > 0 \), in some domain of values of \( u \). We can then introduce a function \( K(u) \) defined by

\[ H = \pm \sqrt{\frac{V}{3}} \cosh K, \quad H_u = \pm \sqrt{\frac{V}{3}} \left( K_u \sinh K + \frac{V_u}{2V} \cosh K \right). \] (51)

Observe, that \( H \) can have either sign, but once the sign is fixed it cannot change anymore during the subsequent evolution of the universe; in the following we focus on positive \( H \) so as to describe an expanding universe.

After substitution into eq. (50) and taking a square root, the function \( K \) is then seen to satisfy

\[ K_u + \frac{V_u}{2V} \cotanh K = \pm 1. \] (52)

Observe that the equation is odd in \( K \), hence in this equation the two sign choices are related by \( K \rightarrow -K \). Therefore it is sufficient to consider only the case with +1 on the right-hand side. An example is provided by the exponential potential \( [20, 15, 22]: \)

\[ V = V_0 e^{\lambda u}, \quad V_0 > 0, \]

leading to a very simple equation for \( K \):

\[ K_u + \frac{\lambda}{2} \cotanh K = 1. \] (53)

First consider the special case\(^2\) \( \lambda = 2 \), for which

\[ K_u = 1 - \cotanh K. \] (54)

\(^2\)The case \( \lambda = -2 \) is obtained directly by the transformation \( u \rightarrow -u \) and \( K \rightarrow -K \).
Observe, that $K_u$ cannot vanish anywhere, except in the limit $K \to \pm \infty$. The general solution of these equations is given implicitly by

$$2K - e^{2K} = 4(u - u_0), \tag{55}$$

for some constant of integration $u_0$. The Hubble parameter for an expanding universe is then determined by

$$H = \frac{1}{6} \sqrt{3V_0 e^{2u_0}} \left( e^{2K} + 1 \right) e^{-\frac{1}{4}(2K + e^{2K})}. \tag{56}$$

The explicit time dependence can be obtained from the relation

$$\dot{K} = \sqrt{3V_0 e^{2u_0}} e^{-\frac{1}{4}(2K + e^{2K})} \tag{57}.$$

The two equations (56) and (57) can be combined to write

$$3H = \frac{1}{2} \left( e^{2K} + 1 \right) \dot{K} = \dot{K} - \dot{u}. \tag{58}$$

It follows that there is a direct relation between $K$, $u$ and $a$:

$$a^3 e^{u-K} = \text{constant}. \tag{59}$$

The constant defines a reference scale $a_0$ such that

$$e^K = e^u \left( \frac{a}{a_0} \right)^3. \tag{60}$$

Using this result one can eliminate $K$ in terms of $a$ and $u$. In addition, it also allows us to calculate the total expansion factor in some period of evolution, as expressed by the number of $e$-folds:

$$N = \ln \frac{a_2}{a_1} = \frac{1}{3} (K_2 - K_1 - u_2 + u_1) = \frac{1}{12} \left( 2K_2 - 2K_1 + e^{2K_2} - e^{2K_1} \right).$$

Similar results can be derived for $\lambda = -2$.

Having disposed of the cases for which $\lambda^2 = 4$, we next turn to the generic case $\lambda^2 \neq 4$. In terms of the initial condition $K_0 = K(u_0)$ such that

$$e^{-2K_0/\lambda} = \left( 1 + \frac{\lambda}{2} \right) e^{-K_0} - \left( 1 - \frac{\lambda}{2} \right) e^{K_0}, \tag{61}$$

the full solution is then given by

$$K + \frac{\lambda}{2} \ln \left| \left( 1 + \frac{\lambda}{2} \right) e^{-K} - \left( 1 - \frac{\lambda}{2} \right) e^K \right| = \left( 1 - \frac{\lambda^2}{4} \right)(u - u_0). \tag{62}$$

Equivalently,

$$e^{(\lambda - \frac{\lambda^2}{4})(u-u_0)} = e^{2K/\lambda} \left| \left( 1 + \frac{\lambda}{2} \right) e^{-K} - \left( 1 - \frac{\lambda}{2} \right) e^K \right|. \tag{63}$$

The corresponding expression for the Hubble parameter is

$$H = \frac{1}{6} \sqrt{3V_0 e^{\lambda u_0}} \left( e^K + e^{-K} \right) \left[ e^{2K/\lambda} \left| \left( 1 + \frac{\lambda}{2} \right) e^{-K} - \left( 1 - \frac{\lambda}{2} \right) e^K \right| \right]^\frac{1}{3}. \tag{64}$$
The pair of equations (62) and (64) represent the parametrized general solutions for \((u, H)\), with time eliminated in favor of the parameter \(K\). A well-known special solution of this kind is one for which \(K\) is constant:

\[
\cotanh K = \frac{2}{\lambda}, \quad K_u = 0, \tag{65}
\]

which requires \(|\lambda| < 2\). The Hubble parameter is then given by

\[
H = \pm \sqrt{\frac{V_0}{3}} \frac{e^{\lambda u/2}}{\sqrt{1 - \lambda^2/4}}, \tag{66}
\]

with opposite signs for an expanding or contracting universe. It is easy to check directly by substitution that this is a solution of the Friedmann equation. To construct the dynamics explicitly, observe that eq. (52) implies that

\[
H_u = \sqrt{\frac{V}{3}} \sinh K, \quad \dot{u} = -3H_u = -\sqrt{3V} \sinh K. \tag{67}
\]

For the special solution (66) this leads to the results

\[
\dot{u} = \pm \sqrt{\frac{3\lambda^2 V_0}{4 - \lambda^2}} e^{\lambda u/2}, \quad H = \frac{4}{3\lambda^2} \frac{1}{t - t_0}. \tag{68}
\]

In the general case, by using eq. (62) one finds

\[
\frac{2\dot{K}}{\sqrt{3V_0 e^{\lambda u_0}}} = \left[ \left( 1 + \frac{\lambda}{2} \right) e^{-K} - \left( 1 - \frac{\lambda}{2} \right) e^K \right] \times
\]

\[
\left[ e^{2K/\lambda} \left( 1 + \frac{\lambda}{2} \right) e^{-K} - \left( 1 - \frac{\lambda}{2} \right) e^K \right]^{1 - \frac{1}{\lambda^2}}. \tag{69}
\]

This result can be used again to derive a direct relation between \(a, u\) and \(K\). Indeed, eqs. (64) en (69) together imply

\[
3H = \frac{(e^{-K} + e^K) \dot{K}}{(1 + \frac{\lambda}{2}) e^{-K} - (1 - \frac{\lambda}{2}) e^K} = \frac{2}{\lambda} \left( \dot{K} - \dot{u} \right). \tag{70}
\]

For \(\lambda \to \pm 2\) this reproduces the results (58). The relation (59) now generalizes to

\[
a^3 e^{\frac{2}{\lambda} (u - K)} = \text{constant} \quad \Rightarrow \quad e^K = e^u \left( \frac{a}{a_0} \right)^{3\lambda/2}. \tag{71}
\]

It is straightforward to extend the construction above to potentials which are negative definite: \(V \leq 0\). This allows us to parametrize \(H\) as

\[
H = \sqrt{\frac{|V|}{3}} \sinh Q, \quad H_u = \sqrt{\frac{|V|}{3}} \left( Q_u \cosh Q + \frac{V_u}{2V} \sinh Q \right), \tag{72}
\]

for some function \(Q(u)\). In contrast to the previous case, eq. (51), here there is no need of a sign choice, as it can be absorbed in the sign of \(Q\). Moreover, \(H\) can change sign during
the evolution of the universe, in case $Q$ switches sign. We have noted before, that this is a fundamental difference between strictly non-negative potentials and potentials taking negative values in some domain of scalar field values.

By taking a square root, the Friedmann equation becomes

$$Q_u + \frac{V_u}{2V} \tanh Q = \pm 1,$$

(73)

where again the two sign choices are related by $Q \to -Q$ and we can restrict ourselves to the positive sign without loss of generality. Using the example of the exponential potential (29), with negative amplitude:

$$V = V_0 e^{\lambda u}, \quad V_0 < 0,$$

eq (73) becomes

$$Q_u + \frac{\lambda}{2} \tanh Q = 1.$$  

(74)

For $\lambda^2 \neq 4$ the solution is

$$Q + \frac{\lambda}{2} \ln \left| \left( 1 + \frac{\lambda}{2} \right) e^{-Q} + \left( 1 - \frac{\lambda}{2} \right) e^Q \right| = \left( 1 - \frac{\lambda^2}{4} \right) (u - u_0),$$

(75)

or equivalently

$$e^{\left( \frac{\lambda^2}{4} - \frac{\lambda}{2} \right) (u - u_0)} = e^{2Q/\lambda} \left| \left( 1 + \frac{\lambda}{2} \right) e^{-Q} + \left( 1 - \frac{\lambda}{2} \right) e^Q \right|.$$  

(76)

In this way we again construct a parametrized solution for the pair $(u, H)$. For $|\lambda| > 2$ there exists another special simple solution, with constant $Q$:

$$\tanh Q = \frac{2}{\lambda}, \quad Q_u = 0.$$  

(77)

The corresponding Hubble parameter is

$$H = \pm \sqrt{\frac{V_0}{3}} \frac{e^{\lambda u/2}}{\sqrt{\frac{\lambda^2}{4} - 1}}.$$  

(78)

For the general solution

$$H = \sqrt{\frac{V}{3}} \sinh Q, \quad \dot{u} = -\sqrt{|3V|} \cosh Q,$$  

(79)

and from (74):

$$\dot{Q} = \dot{u} Q_u = \sqrt{|3V|} \left( \frac{\lambda}{2} \sinh Q - \cosh Q \right) = \frac{3\lambda}{2} H + \dot{u}.$$  

(80)

This implies a relation between $Q$, $u$ and $a$ similar to (71):

$$e^Q = e^u \left( \frac{a}{a_0} \right)^{3\lambda/2}.$$  

(81)
6 Series expansions: the regular case

For general potentials, even if one cannot produce exact solutions, one can always construct solutions for scalar cosmology based on a series expansion method. In this section we consider the regular case, in which \( \varphi(t) \) is a single-valued function of time in some finite time domain. We have already observed before, that in single-scalar cosmology \( H(t) \) is a non-increasing function of time, and therefore it can be represented by a well-behaved function \( H[\varphi(t)] \) in the time domain considered. Later we will also consider solutions in a time domain in which \( \varphi(t) \) has a turning point, and the equation for \( H[\varphi] \) becomes double-valued.

As in sect. 5, it is convenient to work with a rescaled scalar field \( u = \sqrt{3/2} \varphi \), and a Hubble parameter \( H[u] \), satisfying the equations (50). Let \( u_0 \) be a point in the regular domain; then we can develop \( H \) in a power series

\[
H = \sum_{n \geq 0} h_n (u - u_0)^n = h_0 + h_1 (u - u_0) + h_2 (u - u_0)^2 + ...
\]

\( H_u = \sum_{n \geq 0} (n + 1) h_{n+1} (u - u_0)^n = h_1 + h_2 (u - u_0) + ...
\]

Clearly, in this case

\[
V_u = 6H_u (H - H_{uu}) = 0 \iff H_u = 0.
\]

i.e., \( H \) can have a stationary point only at an extremum of the potential; in all other points \( H[u] \) is a strictly monotonically decreasing function of time, hence a monotonic function of \( u \), decreasing or increasing for positive or negative slope of \( u(t) \), respectively:

\[
V_u \neq 0 \implies \begin{cases} 
\dot{u} > 0 & \implies H_u < 0, \\
\dot{u} < 0 & \implies H_u > 0. 
\end{cases}
\]

For example, a quadratic potential

\[
V = \varepsilon + \frac{m^2}{3} u^2,
\]

has a single minimum at \( u = 0 \). Hence \( H \) can have a stationary point \( H_u = 0 \) only at \( u = 0 \), and

\[
H_u = 0 \iff H^2 = \frac{\varepsilon}{3}.
\]

This condition can only be fulfilled if \( V(0) = \varepsilon \geq 0 \), and the stationary solution is a Minkowski space for

\[
u = 0, \quad \varepsilon = H = 0,
\]

whilst it becomes a de Sitter space if

\[
u = 0, \quad \varepsilon = 3H^2 > 0.
\]

There is no stationary solution for \( \varepsilon < 0 \). Note, that all other solutions for any \( \varepsilon \) are non-stationary, with \( \dot{u} = -3H_u \neq 0 \).

Returning to the series expansion (82), and a similar expansion for the potential:

\[
V = \sum_{n \geq 0} v_n (u - u_0)^n = v_0 + v_1 (u - u_0) + v_2 (u - u_0)^2 + ..., \]

(89)
eqs. (50) give rise to the infinite set of relations
\[
\sum_{k=0}^{n} [h_k h_{n-k} - (k + 1)(n - k + 1)h_{k+1} h_{n-k+1}] = \frac{v_n}{3},
\]
for all non-negative integers \(n = 0, 1, 2, \ldots\), and to
\[
\dot{u} = -3 \sum_{n \geq 0} (n + 1)h_{n+1}(u - u_0)^n.
\]

The first few relations (90) in explicit form are
\[
\begin{align*}
 n &= 0 : \quad v_0 = 3h_0^2 - 3h_1^2, \\
 n &= 1 : \quad v_1 = 6h_1(h_0 - 2h_2), \\
 n &= 2 : \quad v_2 = 3h_1(h_1 - 6h_3) + 6h_2(h_0 - 2h_2), \\
 n &= 3 : \quad v_3 = 6h_1(h_2 - 4h_4) + 6h_3(h_0 - 6h_2).
\end{align*}
\]

It follows that either \(h_1 = H_u(u_0) = 0\), which can happen only if \(v_1 = V_u(u_0) = 0\) and \(\dot{u}(u_0) = 0\), or \(h_1 \neq 0\) and
\[
h_0^2 = h_1^2 + \frac{v_0}{3}, \quad h_2 = \frac{h_0}{2} \left( 1 - \frac{v_1}{6h_0h_1} \right),
\]
\[
h_3 = \frac{h_1}{6} \left( 1 + \frac{h_0v_1}{6h_3^3} - \frac{v_1^2}{36h_1^4} - \frac{v_2}{3h_1^2} \right),
\]
\[
h_4 = \frac{h_0}{24} - \frac{v_3}{24h_1} - \frac{h_0}{72h_1^2} \left( 1 - \frac{v_1}{4h_0h_1} \right) \left( \frac{h_0v_1}{h_1} - \frac{v_1^2}{6h_1^2} - 2v_2 \right).
\]

Using these last results, the equation for the scalar field becomes
\[
- \frac{\dot{u}}{3h_1} = 1 + \left( \frac{h_0}{h_1} - \frac{v_1}{6h_1^2} \right) (u - u_0) + \frac{1}{2} \left( 1 + \frac{h_0v_1}{6h_3^3} - \frac{v_1^2}{36h_1^4} - \frac{v_2}{3h_1^2} \right) (u - u_0)^2 + \ldots
\]

Thus the complete solution is given in terms of two parameters \(h_0 = H(u_0)\) and \(h_1 = H_u(u_0)\), representing the initial conditions of the cosmology.

A solution with \(h_1 = 0\) exists only if \(v_1 = 0\) and \(u_0\) is an extremum of \(V\); then eqs. (92) reduce to
\[
\begin{align*}
v_0 &= 3h_0^2, & v_1 &= 0, \\
v_2 &= 6h_0h_2 - 12h_2^2, & v_3 &= 6h_0h_3 - 36h_2h_3, \quad (95) \\
v_4 &= 6h_0h_4 - 48h_2h_4 + 3h_2^2 - 27h_3^2, & \ldots
\end{align*}
\]

Therefore, if \(v_2 \neq 0\) we have \(H_{uu} = 2h_2 \neq 0\), and the point \(u_0\) is a point of inflection of \(H[u]\), where the field comes to rest (either momentarily or permanently). Obviously such solutions are very special, if they exist at all for some given potential. It requires a trajectory \(H[u]\)
to reach a local extremum of the potential at zero velocity. An example is provided by the special solution of the quartic potential in eq. (45) and following, which comes to rest at the minimum $\varphi_0 = 0$ of the potential [46].

From the results above one can also estimate the total expansion factor of the universe between two times $(t_1, t_2)$, as given by the number of e-folds. The central result is, that

$$N = \int_1^{t_2} \frac{H}{3H} du = -\frac{1}{3} \int_1^{t_2} du \left( h_0 + h_1(u - u_0) + h_2(u - u_0)^2 + \ldots \right)$$

For the generic case $h_1 \neq 0$ the result is again a power series expansion

$$N = \sum_{k \geq 1} n_k(u - u_0)^k = \left[ n_1(u - u_0) + n_2(u - u_0)^2 + n_3(u - u_0)^3 + \ldots \right]_1^2,$$

with coefficients

$$n_1 = -\frac{1}{3} \frac{h_0}{h_1}, \quad n_2 = -\frac{1}{6} + \frac{1}{3} \frac{h_0 h_2}{h_1^2},$$

$$n_3 = \frac{1}{3} \frac{h_2}{h_1} + \frac{1}{3} \frac{h_0 h_3}{h_1^2} - \frac{4}{9} \frac{h_0 h_2^2}{h_1^3},$$

$$n_4 = \frac{1}{6} \frac{h_3}{h_1} + \frac{1}{3} \frac{h_0 h_4}{h_1^2} - \frac{1}{6} \frac{h_2^2}{h_1^2} - \frac{h_0 h_2 h_3}{h_1^3} + \frac{2}{3} \frac{h_0 h_2^3}{h_1^4}.$$

For the special case $h_1 = 0$ one gets in stead an expansion

$$N = \left[ n_0 \ln(u - u_0) + n_1(u - u_0) + n_2(u - u_0)^2 + n_3(u - u_0)^3 + \ldots \right]_1^2,$$

with

$$n_0 = -\frac{1}{6} \frac{h_0}{h_2}, \quad n_1 = \frac{1}{4} \frac{h_0 h_3}{h_2^2},$$

$$n_2 = -\frac{1}{12} + \frac{1}{6} \frac{h_0 h_4}{h_2^2} - \frac{3}{16} \frac{h_0 h_2^2}{h_2^3},$$

$$n_3 = \frac{1}{36} \frac{h_3}{h_2} + \frac{5}{36} \frac{h_0 h_5}{h_2^2} - \frac{1}{3} \frac{h_0 h_3 h_4}{h_2^3} + \frac{3}{16} \frac{h_0 h_2^3}{h_2^4}.$$

7 Applications

In this section we apply the general results above to simple quadratic potentials [85]. The simplest models are those with $\varepsilon = 0$, which have a Minkowski minimum $H = 0$ at $u = 0$. It is most convenient to expand around the minimum $u_0 = 0$. Then there is only one non-vanishing term in the potential

$$v_2 = \frac{m^2}{3}, \quad v_n = 0, \quad n = 0, 1, 3, \ldots$$

As a result we get for the non-stationary solutions which all have $h_1 \neq 0$:

$$\frac{h_0}{h_1} = \pm 1, \quad \frac{h_2}{h_1} = \pm \frac{1}{2}, \quad \frac{h_3}{h_1} = \frac{1}{6} \left( 1 - \frac{m^2}{9h_1^2} \right), \quad \frac{h_4}{h_1} = \pm \frac{1}{24} \left( 1 + \frac{2m^2}{9h_1^2} \right), \ldots$$
The power series expansion for $H[u]$ then takes the form

$$
H = \pm h_1 \left[ 1 \pm u + \frac{1}{2} u^2 \pm \frac{1}{6} \left( 1 - \frac{m^2}{9h_1^2} \right) u^3 + \frac{1}{24} \left( 1 + \frac{2m^2}{9h_1^2} \right) u^4 + \ldots \right] 
$$

(103)

$$
= \pm h_1 \left[ e^{\pm u} + O \left( \frac{m^2}{h_1^2} \right) \right].
$$

This result was to be expected, as in the limit $m^2 \to 0$ the potential vanishes and the solutions of the Friedmann equation (50) become pure exponentials. For the equation of motion of the scalar field we get similarly

$$
-\frac{\dot{u}}{3h_1} = 1 \pm u + \frac{1}{2} \left( 1 - \frac{m^2}{9h_1^2} \right) u^2 + \ldots = e^{\pm u} + O \left( \frac{m^2}{h_1^2} \right).
$$

(104)

![Fig. 1: Physical domain in the $(u, H)$-plane for $\varepsilon = 0$ as determined by eq. (105).](image)

The domain of validity of this series expansion is restricted by the requirement that $H[u]$ is single valued. Now $u(t)$ can have a turning point only where $\dot{u} = -3H_u = 0$. Therefore the locus of potential turning points is

$$
H^2 = \frac{V}{3} = \left( \frac{mu}{3} \right)^2,
$$

(105)

for the case at hand. Thus the $(u, H)$-plane is divided in four sectors by straight lines solving eq. (105), and only the upper and lower quadrants in fig. 1 are allowed regions for the solutions (103):

$$
- \left| \frac{mu}{3} \right| \leq H[u] \leq \left| \frac{mu}{3} \right|.
$$

(106)

There is no solution crossing from the upper to the lower quadrant. Indeed, at the point $u = H = 0$ where the lines cross, $\dot{u} = H_u = 0$ and any solution passing through this point must have $h_0 = h_1 = 0$, hence $H_{uu}(0) = 2h_2 = 0$. It follows that no solution can pass from positive to negative $H$, and the two sets of solutions are strictly separated. The only exceptional case is the Minkowski solution represented by the point at the origin.
Next we turn to the case $\varepsilon > 0$, which has a stationary solution $u = \dot{u} = 0$ representing de Sitter space, with a cosmological constant given by eq. (86). The solution for $H$ is now slightly modified to

$$
\begin{align*}
\frac{h_0}{h_1} &= \pm \sqrt{1 + \frac{\varepsilon}{3h_1^2}}, & \frac{h_2}{h_1} &= \frac{h_0}{2h_1} = \pm \frac{1}{2} \sqrt{1 + \frac{\varepsilon}{3h_1^2}}, \\
\frac{h_3}{h_1} &= \frac{1}{6} \left(1 - \frac{m^2}{9h_1^2}\right), & \frac{h_4}{h_1} &= \pm \frac{1}{24} \sqrt{1 + \frac{\varepsilon}{3h_1^2}} \left(1 + \frac{2m^2}{9h_1^2}\right), \ldots
\end{align*}
$$

(107)

with the result

$$
\begin{align*}
H &= \pm h_1 \left[\sqrt{1 + \frac{\varepsilon}{3h_1^2}} \pm u + \frac{1}{2} \sqrt{1 + \frac{\varepsilon}{3h_1^2}} u^2 \pm \frac{1}{6} \left(1 - \frac{m^2}{9h_1^2}\right) u^3 \\
&\quad + \frac{1}{24} \sqrt{1 + \frac{\varepsilon}{3h_1^2}} \left(1 + \frac{2m^2}{9h_1^2}\right) u^4 + \ldots\right], \\
\frac{\dot{u}}{3h_1} &= 1 \pm \sqrt{1 + \frac{\varepsilon}{3h_1^2}} u + \frac{1}{2} \left(1 - \frac{m^2}{9h_1^2}\right) u^2 + \ldots
\end{align*}
$$

(108)

In the $(u, H)$-plane the domain of validity of these approximations is restricted by the branches of the hyperbola

$$
H^2 - \left(\frac{mu}{3}\right)^2 = \frac{\varepsilon}{3},
$$

(109)

shown in fig. 2. On this hyperbola $\dot{u} = H_u = 0$. The two domains of allowed positive and negative $H$-values are separated by a gap of size $\Delta H = 2\sqrt{\varepsilon/3}$. No cross-over is possible, and the solutions describe only permanently expanding or permanently contracting universes.
Finally, we consider the case $\varepsilon < 0$. If there would be a solution $u = \dot{u} = 0$, this would give rise to an anti-de-Sitter space. However, no such solution exists: in the domain $V < 0$ solutions of the Klein-Gordon equation are always dynamical:

$$\dot{u} = -3H_u, \quad H_u^2 = H^2 - \frac{V}{3} > 0, \quad \text{for all } V < 0. \quad (110)$$

In particular

$$h_1^2 = h_0^2 + \frac{|\varepsilon|}{3} > 0, \quad (111)$$

hence it is guaranteed that $h_1 \neq 0$. The solution for $H$ and for $\dot{u}$ is formally the same as in eqs. $\text{(109)}$, except that one has to replace $\varepsilon = -|\varepsilon|$. The restriction imposed by the single-valuedness of $H[u]$ now becomes

$$\left(\frac{mu}{3}\right)^2 - H^2 = \frac{|\varepsilon|}{3}, \quad (112)$$

which is a hyperbola with branches in the left and right quadrants, leaving an opening on the $u$-axis in the interval

$$-\frac{\sqrt{|\varepsilon|}}{m} < u < \frac{\sqrt{|\varepsilon|}}{m}. \quad (113)$$

This hyperbola is shown in fig. 3; passing through the allowed interval on the $u$-axis, $H$ can cross from positive to negative values. As $\dot{H}(t)$ is a non-increasing function, this will eventually happen and an expanding universe will turn into a contracting universe. We have seen this behaviour already in the example given by eqs. $\text{(42)}, \text{(43)}$. We return to this point for a fuller discussion in the next section.

8 Series expansions: turning points

As eq. $\text{(10)}$ shows, in single scalar cosmology $\dot{H} \leq 0$. Now the scalar field $u = \sqrt{3/2} \varphi$ can have turning points where $\dot{u} = H_u = 0$, but $\ddot{u} = -V_u \neq 0$. Adapting eq. $\text{(23)}$ to the present notation

$$V_u = 6H_u (H - H_{uu}), \quad (113)$$
hence such a turning point occurs if $H_{uu}$ is singular in such a way that at this point

$$0 < |H_u H_{uu}| < \infty. \quad (114)$$

As discussed in the previous sections, in the $(u, H)$-plane turning points lie on the curves

$$V(u) = 3H^2(u),$$

which bound the domain of physically allowed values. In fact all points on these boundary curves are turning points, unless $V_u = 0$, i.e. at a local extremum of the potential. In the latter case a solution with $\dot{u} = H_u = 0$ can exist only if this extremum occurs at a non-negative value of $V$.

In the neighborhood of a turning point $u(t)$ takes identical values before and after the turning point; but as $\dot{u} = -3H_u \neq 0$ away from the turning point, and therefore $\dot{H} < 0$ both before and after, $H[u(t)]$ necessarily becomes double-valued there. As a result, the power series expansion studied in sect.\[\] can not be used in this neighborhood.

This double-valuedness can be resolved by a reparametrization of the field. For definiteness it is convenient to consider a point where $u$ reaches a maximum $u_m$; then a new dynamical variable $\eta(t)$ can be introduced such that in a sufficiently small but finite time interval around the turning point

$$u(t) = u_m - \eta^2(t). \quad (115)$$

At the turning point $\eta = 0$, and the evolution of $u(t)$ can be parametrized by a monotonically increasing function $\eta(t)$, the negative and positive values of $\eta$ corresponding to $u$ before and after the turning point, respectively. In case $u_m$ were a minimum, we could similarly define

$$u(t) = u_m + \eta^2(t). \quad (116)$$

However, for our discussion we will assume a maximum and use eq. (115).

We first re-express our dynamical equations in terms of the new field $\eta(t)$. Using the definition (115) and the notation $H_{\eta} = dH/d\eta$, it is straightforward to derive the equations

$$H^2 - \frac{1}{4\eta^2} H_{\eta}^2 = \frac{1}{3} V[u(\eta)], \quad \dot{\eta} = -\frac{3}{4\eta^2} H_{\eta}. \quad (117)$$

These equations can be used to develop a new power series expansion

$$H = \sum_{n \geq 0} g_n \eta^n, \quad H_{\eta} = \sum_{n \geq 0} (n + 1)g_{n+1}\eta^n. \quad (118)$$

Now expressed in $\eta$ the condition (114) at the turning point, where $H_u = 0$, reads

$$H_{\eta}|_{\eta=0} = 0, \quad 0 < \left| \frac{1}{\eta^3} H_{\eta} H_{\eta\eta} \right|_{\eta=0} < \infty. \quad (119)$$

In terms of the expansions (118) this is translated as

$$g_1 = g_2 = 0. \quad (120)$$

As a result

$$H = g_0 + g_3\eta^3 + g_4\eta^4 + 5g_5\eta^5 \ldots, \quad H_{\eta} = 3g_3\eta^2 + 4g_4\eta^3 + 5g_5\eta^4 + \ldots \quad (121)$$
Also, assuming that the potential $V$ has a power series expansion with $u_0 = u_m$, the potential has the expansion
\[
V = \sum_{n \geq 0} (-1)^n v_n \eta^{2n}. \tag{122}
\]

Then eqs. (117) imply for the coefficients $g_n$:
\[
g_0 = \frac{v_0}{3}, \quad g_1 = g_2 = 0, \quad g_3 = \frac{4v_1}{27},
\]
\[
g_4 = \frac{g_0}{3}, \quad g_5 = \frac{g_3}{30v_1} (2v_0 - 9v_2),
\]
\[
g_6 = \frac{2g_0}{405v_1} (-2v_0 + 9v_2), \quad g_7 = \frac{2g_3}{21} \left[ 1 + \frac{9v_3}{4v_1} + \frac{1}{240v_1^2} (2v_0 - 9v_2) (2v_0 + 15v_2) \right], \tag{123}
\]
etc. Observe, that these equations require $v_0 \geq 0$ and $v_1 \geq 0$. The first condition implies that turning points only occur at non-negative values of the potential $V$. The second condition is a direct consequence of our choice to consider a turning point which is a maximum of $u$: $V_u(u_m) = v_1 \geq 0$.

The equations for $H[\eta]$ and $\eta(t)$ now become
\[
H = \sqrt{\frac{v_0}{3}} \left[ 1 - \frac{2}{3} \sqrt{\frac{v_1}{v_0}} \eta^3 + \frac{1}{3} \eta^4 + \frac{9v_2 - 2v_0}{45\sqrt{v_0v_1}} \eta^5 + ... \right], \tag{124}
\]
\[
\dot{\eta} = \sqrt{\frac{3v_1}{4}} \left[ 1 + \frac{2}{3} \sqrt{\frac{v_0}{v_1}} \eta + \left( \frac{2v_0 - 9v_2}{18v_1} \right) \eta^2 + ... \right]
\]
Note, that in order to get a monotonically increasing $\eta(t)$, we have to take the negative square root for $g_3$:
\[
g_3 = -\frac{2}{3} \sqrt{\frac{v_1}{3}}.
\]

In the present formulation one can derive yet another formula for the total expansion factor in a given time interval $(t_1, t_2)$:
\[
N = \int_{t_1}^{t_2} H \, dt = -\frac{4}{3} \int_{t_1}^{t_2} \frac{\eta^2 H}{H} \, dt
\]
\[
= -\frac{4}{3} \int_{t_1}^{t_2} \frac{d\eta}{g_0 + g_3 \eta^3 + g_4 \eta^4 + g_5 \eta^5 + ...}.
\tag{125}
\]
The result can be written in a series expansion as
\[
N = \left[ \nu_1 \eta + \nu_2 \eta^2 + \nu_3 \eta^3 + \nu_4 \eta^4 + ... \right]^2_1,
\tag{126}
\]
with coefficients given by
\[
\nu_1 = \frac{4g_0}{9g_3}, \quad \nu_2 = \frac{8g_0g_4}{27g_3^2},
\]
\[
\nu_3 = \frac{20g_0g_5}{81g_3^2} - \frac{64g_0g_4^2}{243g_3^2}, \quad \nu_4 = -\frac{1}{9} + \frac{2g_0g_6}{9g_3^2} - \frac{40g_0g_4g_5}{81g_3^3} + \frac{64g_0g_4^3}{243g_3^3}.
\tag{127}
\]
Note again, that by taking the negative value for $g_3$ the first coefficient becomes positive:

$$\nu_1 = \frac{2}{3}\sqrt{\frac{v_0}{v_1}}. \quad (128)$$

## 9 Quadratic potentials

The general description of turning points developed in sect. 8 can be illustrated with the example of quadratic potentials considered earlier

$$V = \varepsilon + \frac{m^2}{3} u^2. \quad (129)$$

We consider a solution $u(t)$ which rolls down the potential from negative values to positive values, reaching a turning point at some positive $u_m > 0$, where $\dot{u}_m = 0$ and $V_u(u_m) > 0$. Using the parametrization (115) the potential is expressed as

$$V = v_0 - v_1 \eta^2 + v_2 \eta^4, \quad v_0 = \varepsilon + \frac{m^2}{3} u_m^2, \quad v_1 = \frac{2m^2}{3} u_m, \quad v_2 = \frac{m^2}{3}. \quad (129)$$

The requirement $v_1 > 0$ is satisfied automatically, whilst the condition $v_0 > 0$ is non-trivial only if $\varepsilon < 0$, requiring $u_m$ to be in the domain of $V(u_m) > 0$. As before we distinguish the cases $\varepsilon \geq 0$ and $\varepsilon < 0$.

For non-negative $\varepsilon$ the coefficients $g_n$ in eq. (123) become

$$g_0 = \frac{m u_m}{3} \sqrt{1 + \frac{3\varepsilon}{m^2 u_m^2}}, \quad g_3 = -\frac{2m}{9} \sqrt{2u_m},$$

$$g_4 = \frac{m u_m}{9} \sqrt{1 + \frac{3\varepsilon}{m^2 u_m^2}}, \quad g_5 = \frac{m}{15\sqrt{2u_m}} \left(1 - \frac{2u_m^2}{9} - \frac{2\varepsilon}{3m^2}\right), \quad (130)$$

$$g_6 = \frac{m}{135} \sqrt{1 + \frac{3\varepsilon}{m^2 u_m^2} \left(1 - \frac{2u_m^2}{9} - \frac{2\varepsilon}{3m^2}\right)}, \ldots$$

The singularity of $g_5$ for $u_m \to 0$ indicates, that the only consistent solution with $\dot{u} = 0$ at the point $u = 0$ is the one with constant $H = g_0 = \sqrt{\varepsilon/3}$, corresponding to a de Sitter space for $\varepsilon > 0$, or a Minkowski space for $\varepsilon = 0$. This is a stationary solution, rather than a turning point.

For negative $\varepsilon$ turning points can occur, and eqs. (130) still hold, if $u_m^2 > 3|\varepsilon|/m^2$. Therefore, if the scalar field starts at a large enough value in the region $V(u) > 0$, it can roll down the potential and oscillate, meeting a number of turning points, until it can no longer escape from the region $V < 0$ and the universe starts to contract back to infinitely small size. The point is, that after each turning point the Hubble parameter will continue to decrease: $\dot{H} < 0$, until it finally crosses over into the region of negative $H$ and the contraction phase starts.
10 Discussion and conclusions

The main concern of this paper is the cosmological evolution of spatially flat universes driven by a single scalar field. Such a scenario may be relevant for the present epoch in the evolution of our accelerating universe, and it may have a bearing on the very early universe going through an epoch of inflation. Observations of the CMB require an inflationary expansion by at least $60-70\, e$-folds, which can happen only if the universe spends a relatively long time in a phase with large Hubble parameter. This requirement is usually expressed by the slow-roll condition; therefore it is of interest to study this condition in the present context of single scalar-field cosmology.

The acceleration of the universe can be expressed as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2 (1 - \epsilon), \quad (131)$$

where the slow-roll parameter is defined by

$$\epsilon = \frac{3H^2}{H^2}. \quad (132)$$

Thus an accelerated expansion requires $0 \leq \epsilon < 1$. Now we can combine the two inequalities

$$H^2 > 3H^2_u, \quad 3H^2 - V = 3H^2_u \geq 0, \quad (133)$$

to translate the slow-roll condition to a double bound on $H^2$:

$$\frac{V}{3} \leq H^2 < \frac{V}{2}. \quad (134)$$

Clearly these bounds can be satisfied only in a region where $V > 0$. Also, the bound is always satisfied at and near a turning point where $H_u = 0$. In terms of our series expansion $[121]$ this is at $\eta = 0$, where $H^2 = V/3$. In order to estimate the total expansion factor in the slow-roll domain $[134]$ near a turning point, we also ought to find a value for $\eta$ at $H^2 = V/2$ for a given specific potential. In general however, we can use the observation that in Planck units $u_m$ and $\eta$ must satisfy

$$\eta^2 < u_m < 1, \quad (135)$$

and as a first approximation

$$N = -\frac{4}{9} \frac{g_0}{g_3} \int_{0}^{\eta_m} d\eta \left(1 - \frac{4}{3} \frac{g_4}{g_3} \eta + \ldots\right) \approx \frac{2}{3} \sqrt{\frac{v_0}{v_1}} \times O(1),$$

where $\eta_m$ is the upper limit for $\eta$ where $H = V/2$. The condition on the number of $e$-folds for inflation then becomes

$$\frac{v_0}{v_1} \sim 10^4.$$ 

However, significant modification of this estimate may result for specific potentials $[23]$.

The methods we have used in this paper to find solutions for the the cosmological equations of a spatially flat universe driven by scalar fields relies heavily on the fact that we have assumed a single field to drive the cosmological expansion. This allows us to replace time by the field as the evolution parameter. Actually eq. $[12]$ suggests another option, taking $X^0 = \sqrt{6} \ln a$ as
evolution parameter; this might be more readily generalizable to the case of many scalar fields
[24][9]. At present such a modification of the methods presented here is under investigation.

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