A no-go for no-go theorems prohibiting cosmic acceleration in extra dimensional models

Rik Koster\textsuperscript{a,b} and Marieke Postma\textsuperscript{a}

\textsuperscript{a} Nikhef, Science Park 105, 1098 XG Amsterdam, The Netherlands.
\textsuperscript{b} Department of Physics and Astronomy, VU University, De Boelelaan 1081, 1081 HV Amsterdam

Abstract: A four-dimensional effective theory that arises as the low-energy limit of some extra-dimensional model is constrained by the higher dimensional Einstein equations. Steinhardt & Wesley use this to show that accelerated expansion in our four large dimensions can only be transient in a large class of Kaluza-Klein models that satisfy the (higher dimensional) null energy condition \cite{1}. We point out that these no-go theorems are based on a rather ad-hoc assumption on the metric, without which no strong statements can be made.
1. Introduction

There is compelling evidence that both the early universe as well as our present-day universe went or is going through a phase of accelerated expansion, referred to as inflation and dark energy respectively. The evolution of the universe is sourced by the energy-density in it; acceleration requires the energy-density to be dominated by vacuum energy. This cannot be achieved by ordinary matter, however, an (effective) scalar field whose kinetic energy is negligible will work. In the context of extra-dimensional models such a four-dimensional (4d) scalar can either arise from higher dimensional matter, or as components of the extra-dimensional metric (called moduli fields). The higher dimensional fields should satisfy the higher dimensional Einstein equations. This puts constraints on the matter properties of the low-energy effective four-dimensional theory.

In Ref. [1] (and also in the earlier Refs. [2, 3]) Steinhardt and Wesley use these constraints to derive stringent no-go theorems on four-dimensional accelerated expansion in models with extra dimensions. It was for example shown that if the higher dimensional theory satisfies the null energy condition (NEC), accelerated expansion can only last for a couple of e-folds.
Models that do violate the NEC can yield accelerated expansion, but the time and spatial 
dependence of the NEC violating elements is heavily constrained. To derive these theorems,
it is assumed that both the higher dimensional and the 4d compactified theory are described 
by general relativity (up to small corrections). The metric is taken block-diagonal, with the 
extra-dimensional part Ricci flat or conformal Ricci flat. This form is motivated by many 
models that exist in the literature, such as the original Kaluza-Klein model, Randall-Sundrum 
models [5, 6], and string compactifications on (warped) Calabi-Yau manifolds [7, 8, 9, 10]. 

The no-go theorems derived in Ref. [1] seem to forbid dark energy and inflation\(^1\) in 
many motivated extra-dimensional models. However, every no-go theorem is as good as the 
assumptions that go into it. In this paper we point out some of these assumptions, which 
are non-trivial and key ingredients in the derivation of the no-go theorems. None of these 
are mentioned explicitly in Ref. [1], although they can be found in some form in the earlier 
works [2, 3]. 

First of all, the theorems derived in Ref. [1, 2, 3] only apply to Kaluza-Klein (KK) type 
compactifications, and not to a set-up with branes or, more generally, localized matter sources. 
There are two instances where brane sources may violate the assumptions made. First, the 
thereoms are formulated using a partial integration, where it is assumed the boundary term 
vanishes: 

\[
\int_{\partial \mathcal{M}} e^{(2+A)} \Omega (\vec{\nabla} \Omega) \cdot d\vec{\Sigma} = 0.
\]

Here \(\Omega\) is the warp factor, \(\vec{\nabla}\) the gradient constructed from the extra-dimensional metric, \(A\) 
a constant, and \(\vec{\Sigma}\) the oriented boundary of compactified space \(\mathcal{M}\). This is automatic if \(\mathcal{M}\) 
is closed and has no boundaries (for orbifold compactification, if the covering space has no 
boundaries), as in KK compactifications. However, if \(\mathcal{M}\) is bounded by branes, boundary 
terms as the above are generically non-zero, as they are sourced by the brane localized energy-
momentum, and the no-go theorems do not apply [4]. Second, it is assumed that the metric 
of our universe \(g_{\mu\nu}^{\text{vis}}\) equals the metric \(g_{\mu\nu}^{E}\) that brings the dimensionally reduced 4d Einstein-
Hilbert action in canonical form. This is not automatic in brane world models. Indeed, if 
the standard model matter is localized on a brane instead of living in the bulk the induced 
metric on the brane, which is the metric of our universe, is 

\[
g_{\mu\nu}^{\text{vis}} = e^{2\Omega(y_{br},t)} g_{\mu\nu}^{E},
\]

with the warp factor evaluated at the position of the brane \(y_{br}\). We can normalize this to 
\(\Omega(y_{br},0) = 1\) by rescaling the the 4d coordinates. Only if the warp factor is time-independent 
the two metrics coincide at all times. Otherwise accelerated expansion in our universe is not 
the same as accelerated expansion in the canonical metric. The two metrics enter because 
matter is localized to the brane, whereas gravity lives in the bulk. 

Secondly, even for KK type compactifications the theorems are not as general as suggested 
in Ref. [1], as an additional restriction is put on the metric, which relates the warp factor to

\(^1\)If during inflation the corrections to the Friedmann equation in the dimensionally reduced theory are large, 
the theorems may be avoided.
the volume modulus of the extra dimensional space. Although it is presented as a necessary condition (which is enforced by hand) in earlier work \cite{2, 3}, it should really be viewed as an additional assumption. This metric restriction is motivated in that an extra constraint is needed to eliminate modes with a wrong-sign kinetic term in the four-dimensional effective action. However, the off-diagonal Einstein equations, which are not taken into account in the derivation of the no-go theorems, can provide exactly such constraints. Another argument given is that it is merely a gauge choice in the “moduli space approximation” \cite{4}. But this approximation breaks down in a time-dependent set-up, which is exactly what we are interested in when looking for 4d accelerated expansion. That the metric restriction is indeed unnecessary is confirmed by an explicit counter example: the solution of the five-dimensional Einstein equations found in Ref. \cite{11} violates it.

No-go theorems forbidding a 4d de Sitter (dS) space already exist for a long time \cite{12, 13, 14}. These initial works showed that string theory or supergravity compactifications on a smooth time-independent extra-dimensional manifold cannot give a 4d cosmological constant if the original theory satisfies the strong energy condition. Since then much work has been done to generalize and extend these results. Negatively curved extra-dimensions are discussed in \cite{15}. Theorems forbidding dS solutions in string theory constructions with orientifold planes, which break the null energy conditions, have recently been formulated in \cite{16, 17}. In \cite{18, 19, 20, 21, 22} it was shown that supergravity compactifications to dS often are unstable as they have a tachyon in the spectrum. Theorems restricting a period of 4d inflation can be found in \cite{23, 24}. All these theorems are formulated in a string theory or supergravity context, and supersymmetry and/or knowledge of the matter content in string theory (e.g. on the form of the Kähler potential) is used to derive them. The approach of \cite{1} is orthogonal in a way, as it does not rely on supersymmetry, nor on the form of the matter in the theory except that it satisfies the null energy condition. Given that it is stated so general, it may be no surprise that extra assumptions, such as the metric assumption mentioned above, are needed to derive no-go theorems.

This paper is organized as follows. We start in the next section reviewing the no-go theorems of \cite{1}. To do so we use Einstein’s equations to relate the NEC in the higher-dimensional theory to the metric moduli and four-dimensional scale factor. In section 3 we take a step back, and list all the assumption that went into the derivation. We discuss in some detail how brane world models generically evade one or more of the assumptions. In section 4 we discuss the restriction put on the metric in Ref. \cite{1, 2, 3}, that relates the warp factor to the volume modulus of the extra dimensional space. We argue that it should be viewed as an assumption rather than a necessary restriction. We further show that without this assumption no general no-go theorems that can be derived. We end with some concluding remarks.

2. Einstein’s equations and the null energy condition

In this section we review the no-go theorems of \cite{1}. As a preliminary we introduce the metric
and define the 4d Einstein frame, and define an averaging procedure. We then use the Einstein equations to relate the average NEC violation in the higher dimensional theory to the 4d scale factor and moduli fields. We use a slightly different gauge than \[1\], setting $\phi = 0$ in their equations, to make the derivations as transparent as possible.

### 2.1 The metric

We assume the metric to be of the block-diagonal form
\[
ds^2 = g_{MN}^{(D)} dX^M dX^N = e^{2\bar{\Omega}(t,u)} g^{E}_{\mu\nu}(t) dx^\mu dx^\nu + g_{mn}^{(d)}(t,y) dy^m dy^n,
\]
with coordinates $X^M = (x^\mu, y^m)$. Here $(M = 0, ..., D - 1)$ runs over all $D$ dimensions, whereas $(\mu = 0, ..., 3)$ labels our four large space-time dimensions and $(m = 4, ..., D - 1)$ the $d = D - 4$ extra dimensions. The four-dimensional Einstein frame metric $g_{E\mu\nu}$ is of the Friedmann-Robertson-Walker (FRW) form
\[
g^{E}_{\mu\nu}(t) dx^\mu dx^\nu = -n(t)^2 dt^2 + a^2(t) d\vec{x}^2,
\]
appropriate to describe (accelerated) expansion in a homogeneous and isotropic universe. We assume that the universe is flat, in agreement with observations \[31\]. In this metric $n(t)$ is the lapse function and $a(t)$ the scale factor. Accelerated expansion is defined by (taking $a > 0$)
\[
\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0
\]
with $H = \dot{a}/a$ the Hubble constant. The extra-dimensional metric is taken of the form
\[
g_{mn}^{(d)}(t,y) = e^{-2\bar{\Omega}(t,y)} \bar{g}_{mn}^{(d)}(t,y),
\]
with the scalar curvature constructed from the barred metric vanishing $R[\bar{g}^{(d)}_{mn}] = 0$. The metric is Ricci flat (RF) if $\bar{\Omega}$ is constant (can be set to zero by a redefinition of $\bar{g}^{(d)}_{mn}$), and conformal Ricci flat (CRF) if $\bar{\Omega} = \Omega$ equals the warp factor. The Ricci scalar of the extra-dimensional space is $R[g^{(d)}_{mn}] = 0$ for a RF metric, and
\[
R[g^{(d)}_{mn}] = 2(d - 1) \nabla^2 \bar{\Omega} + (d - 1)(d - 2)(\nabla_m \bar{\Omega})^2,
\]
for CRF. Here $\nabla^2 = \nabla_m \nabla^m$, with $\nabla_m$ the covariant derivative constructed from $g^{(d)}_{mn}$.

A generic coordinate transformation $X^M \rightarrow X^M(X^{IV})$ will alter the form of the metric, in particular it will generate off-diagonal terms $g_{\mu m} dx^\mu dy^m$. Hence the block-diagonal form of the metric \[2.7\] fixes most of the gauge. Residual invariance is left of the form $t \rightarrow t(t')$, $\vec{x} \rightarrow \vec{x}(\vec{x}')$, $y^m \rightarrow y^m(y'^m)$. The first transformation can be used to set $n(t) = 1$, which we will do in the following. The second can be used to fix the scale factor at a particular time, it is customary to set the scale factor today to unity $a(t_{\text{now}}) = 1$. We discuss the third below.

We decompose the (time-derivative) of the extra dimensional metric according to \[1\]
\[
\frac{1}{2} \partial_t g_{mn}^{(d)}(t,y) = \frac{1}{d} \xi(t,y) g_{mn}^{(d)}(t,y) + \sigma_{mn}(t,y),
\]

with $g^{(d)mn} \sigma_{mn} = 0$ traceless. The trace factor $\xi$ parametrizes the change in volume: $\partial_t \ln \sqrt{g^{(d)}} = \xi$. Under a time-independent coordinate transformation $y^m \to y^m(y^n)$, which keeps the metric in block-diagonal form, $\xi$ is invariant whereas $\sigma_{mn}$ transforms as a tensor.

The non-invariance of $\sigma_{mn}$ will not be important for the derivation of the no-go theorems, as this quantity only appears in a covariant contraction in the equations. However, if we want to solve the constraint equations arising from the off-diagonal components of Einstein’s equations, we do have to worry about gauge invariance.

### 2.2 A-averaging and the Einstein frame

It will be useful to define an A-averaging procedure via

$$Q_A(t) \equiv \langle Q(t, y) \rangle_A = \frac{\int Q e^{A} \sqrt{g^{(d)} d^d y}}{\int e^{A} \sqrt{g^{(d)} d^d y}}, \quad (2.7)$$

where the weight of the average is set by $A$. Using this average any quantity can be split in a $y$-independent zero-mode piece and the fluctuations around it

$$Q(t, y) = Q_A(t) + Q_{A\perp}(t, y) \quad (2.8)$$

with $\langle Q_{A\perp} \rangle_A = 0$. Splits using different values of $A$ only differ by a time-dependent function: $Q_A(t) = Q_B(t) + f_{AB}(t)$. The averaging procedure is time-dependent, and in general the time-derivative of an average differs from the average of a time-derivative:

$$\partial_t \langle Q \rangle_A = \langle \dot{Q} \rangle_A + \langle Q(A \dot{\Omega} + \xi)_{A\perp} \rangle_A. \quad (2.9)$$

The Einstein frame is found by integrating the action over the extra dimensions

$$S = \frac{1}{2 \kappa^2_D} \int d^D X \sqrt{-g^{(D)}} R[g^{(D)}_{MN}] = \frac{1}{2 \kappa^2_D} \int d^4 x \sqrt{-g^E} \left( \int e^{2\Omega} \sqrt{g^{(d)} d^d y} \right) R[g^E_{\mu\nu}] + ... \quad (2.10)$$

with $\kappa^{-2}_D = M^{-2}_D$ with $M_D$ the $D$-dimensional Planck scale. For $g^E_{\mu\nu}$ to be the four-dimensional Einstein metric we need

$$\frac{1}{\kappa^2_D} \left( \int e^{2\Omega} \sqrt{g^{(d)} d^d y} \right) = \frac{1}{\kappa^2_4}, \quad (2.11)$$

with $\kappa^{-2}_4 = m^2_p$. By definition, the Planck mass is constant in the Einstein frame. Taking the time-derivative of the above equation implies

$$\langle 2\dot{\Omega} + \xi \rangle_2 = 0. \quad (2.12)$$

Note that the above relation is only valid for what we will call the “canonical average”, with $A = 2$. Demanding $g^E_{\mu\nu}$ to be the Einstein frame metric completely fixes $\Omega$ via (2.11). There is no freedom left to rescale $\Omega \to \Omega + c(t)$ and $g^E_{\mu\nu} \to e^{-2c(t)} g^E_{\mu\nu}$. 


2.3 Einstein’s equations

Both the higher and four-dimensional energy and momentum are defined in their respective Einstein frames $g_{\mu\nu}^{(D)}$ and $g_{\mu\nu}^E$. The definition of the D-dimensional energy density and pressure are

$$\kappa_D^2 \rho = -g^{00} G_{00}, \quad \kappa_D^2 p_3 = \frac{1}{3} g^{ij} G_{ij}, \quad \kappa_D^2 p_d = \frac{1}{d} g^{\mu\nu} G_{\mu\nu}$$

with as before $D = 4 + d$ the total number of dimensions. A possible higher-dimensional cosmological constant is absorbed in the energy and pressure. The extra dimensions do not need to be homogeneous or isotropic, and thus the extra dimensional part of the energy-momentum tensor may have off-diagonal entries. Only the trace is used for the theorems, but all components are needed to fully solve Einstein’s equations. The off-diagonal equations, which provide constraints, are not used in the derivation of the theorems of Ref. [1].

We can dimensionally reduce the theory to four dimensions. If the system is to describe our universe today, this should yield general relativity plus corrections that encode its extra-dimensional origin. Hence, we get the usual Friedmann equations relating the scale factor $a(t)$ to the effective energy and pressure plus corrections that are small in the low energy limit:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa_4^2}{3} \rho^E + \ldots, \quad \left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa_4^2}{6} p^E (1 + 3w^E) + \ldots,$$

where we introduced the equation of state parameter $p^E = w^E \rho^E$. The energy and pressure is the total four-dimensional energy and pressure, which includes the dimensional reduction of any matter added in the higher dimensional theory (possibly localized on e.g. a brane), as well as the energy and pressure of the metric moduli fields.

At low energies, the scales being probed are large compared to the typical size of the extra dimensions, and the corrections to the FRW equations (denoted by the ellipses) are small. We know from observations that during nucleosynthesis, when the temperature in the universe was about an MeV, the corrections to the Friedmann equations should be small. Extrapolating, they are completely negligible today. It then follows that accelerated expansion requires dark energy with an equation of state parameter $w_E < 1/3$. However, this is not necessary the case for inflation, as this period of accelerated expansion takes place well before nucleosynthesis where the corrections to the Friedmann equations may be large. For example, in RS models there is an additional $\rho^2$ term on the right-hand-side of the first Friedmann equation which may play a rôle during inflation [32, 33, 34, 35]. In this regime statements on the scale factor or Hubble parameter (as in (2.3)) cannot be straightforwardly translated in a statement on the equation of state parameter. For this reason we will state our theorems in terms of the scale factor, rather than the equation of state parameter.

The D-dimensional null energy condition (NEC) states that $T_{MN} n^M n^N \geq 0$ for any null vector $n^M$. The NEC is satisfied by all unitary two-derivative quantum field theories. Although NEC violating theories in general suffer from difficulties with unitarity, superluminal expansion or instabilities [36, 37, 38, 39, 40], NEC violating objects such as negative tension branes or orientifold planes can be introduced without such problems. In terms of energy and
momentum, NEC is violated if $\rho + p_3 < 0$ or $\rho + p_d < 0$ at any time or point in space. Equally well one could use the $A$-average of the above equations to probe NEC violation [1, 3]. Using the D-dimensional Einstein equations (2.13) we can then relate NEC violation to the metric moduli and the 4d Hubble parameter. We further split the fields in a zero mode plus perturbations to simplify the equations. If no further restrictions are put on the metric, the above equation vanishes. Here $\langle \vec{\sigma} \rangle$ is the A-average of the above equations to probe NEC violation [1, 2]. Using the freedom allows to derive stronger no-go theorems. We further split the fields in a zero mode plus perturbations $Q = Q_A + Q_{A\perp}$ as in (2.8). The time derivatives of the metric function can be rewritten with (2.9). The Ricci scalar of the extra-dimensional manifold is given in (2.3). The result is:

$$\left\langle 2\dot{\Omega} + \xi \right\rangle_A = 0, \quad \text{with} \quad \left\{ \begin{array}{ll}
A = 2, & \text{(no restriction),} \\
\forall A, & \text{(metric restriction),}
\end{array} \right. \quad (2.17)$$

to simplify the equations. If no further restrictions are put on the metric, the above equation is valid only for the canonical average $A = 2$, and follows from the definition of the Planck mass (2.12). However, with the extra metric restriction (2.23) discussed in section II the equation is valid for all values of $A$. Using this freedom allows to derive stronger no-go theorems. To obtain the last term we performed a partial integration

$$\left\langle e^{2\Omega} \nabla^2 \Omega \right\rangle_A = -2(2 + A)\left\langle e^{2\Omega} (\nabla \Omega)^2 \right\rangle_A + B.T. \quad (2.20)$$

In the derivation of the no-go theorems it is assumed that the boundary term

$$B.T. = \left\langle 1 \right\rangle^{-1} \int_{\partial M} \sqrt{g^{(d)} e^{(2 + A)\Omega}} \vec{\Delta} \Omega \cdot d\vec{\mathcal{S}} \quad (2.21)$$

vanishes. Here $\vec{\mathcal{S}}$ is a directed $(d - 1)$-area element of the boundary surface $\partial M$. The constant in (2.19) is

$$C_1(A, d) = \frac{1}{d} \times \left\{ \begin{array}{ll}
A(d - 4) - 2d - 4, & \text{(RF),} \\
3A(d - 2) - d^2 + 5d - 10, & \text{(CRF),}
\end{array} \right. \quad (2.22)$$
depending on whether the extra-dimensional space is Ricci flat (RF) or conformal Ricci flat (CRF). The equations are written in such a form that it is straightforward to specialize to the two cases of interest, namely \( A = 2 \) which is valid for a generic metric, and \( \langle 2\Omega + \xi \rangle_A = 0 \) and generic \( A \) for the metric with extra restriction.

### 2.4 No-go theorem

To derive the no-go theorem of [1] we assume the “metric restriction” (its motivation and validity is discussed in section (4)):

\[
2(\dot{\Omega} + \xi)_{\perp} = 0, \tag{2.23}
\]

Combining with the requirement of a constant Planck mass (2.12) gives \( (2\dot{\Omega} + \xi) = 0 \), and thus also (2.17) is trivially satisfied for all \( A \). The equations (2.18, 2.19) simplify

\[
\kappa_2^2 (e^{2\Omega}(\rho + p_3))_A = -2\dot{H} - |X|,
\]

\[
\kappa_2^2 (e^{2\Omega}(\rho + p_d))_A = -3(\dot{H} + H^2) - |X| - C_1(A, d)(e^{2\Omega}(\nabla \Omega)^2)_A - C_2(A, d)(\xi^2_A)_{\perp}
\]

\[
+ \frac{(2 + d)}{2d} \partial_t (a^3 \xi_A), \tag{2.25}
\]

with \( |X| = \langle \sigma^2 \rangle_A + \frac{(2+d)}{2d} \xi^2_A \geq 0 \), \( C_1 \) given by (2.22), and \( C_2 = \frac{(2+d)}{4d} (4 - A) \). The D-dimensional matter violates the null energy condition if the left-hand-side of one of the above equations is negative. Depending on the number of extra dimensions and whether the metric is RF or CRF, we can choose \( A \) such that the constants \( C_i \) are non-negative and derive strong no-go theorems. In a nutshell, 4d accelerated expansion (2.3) implies that the first term on the right-hand-side of (2.25) is negative; NEC can only be satisfied if this is compensated by the last term and \( \partial_t \xi > 0 \). However, the growth of \( \xi \) is limited, as otherwise \( |X| \) grows so large that the right-hand-side of (2.24) becomes negative and NEC is violated after all. Note that \( -\dot{H} \geq 0 \) for a scale factor that grows as a power law or exponentially.

The arguments can be made more precise. If only \( C_1 \geq 0 \) and \( C_2 \) arbitrary, then a four-dimensional de Sitter universe with \( \dot{H} = 0 \) is incompatible with the NEC. Indeed, it follows from (2.24) \( \dot{H} = 0 \) is only possible for static extra dimensions \( \xi, \sigma = 0 \). But this leads to a contradiction in (2.25) if \( C_1 \geq 0 \), which can only be resolved if NEC is violated. Choosing an appropriate value for \( A \), \( C_1 \) can always made positive or zero except for \( d = 4 \) respectively \( d = 2 \) in RF and CRF. This gives the theorem:

- **Theorem 1**: Given (1) NEC, and (2) \( d \neq 4 \) (2) for RF (CRF), compactification to 4d dS space is impossible.

Stronger constraints can be derived if \( C_2 \geq 0 \) as well, for example the duration of 4d accelerated expansion can be bounded. This requires \( d < 4 \) or \( d \geq 10 \) in RF, while for CRF only \( d = 2 \) and \( d > 14 \) is excluded. Accelerated expansion corresponds to \( H^2 + \dot{H} \geq 0 \) (2.3). Then (2.24) bounds \( \xi^2_A < H^2 \). The first term of (2.25) is negative, which can only be positive by a positive time-derivative

\[
a^{-3}\partial_t (a^3 \xi_A) \gtrsim \dot{H} + H^2 \tag{2.26}
\]
In the limit that $\dot{H} + H^2 \approx 0$ positive but arbitrarily small, only a slow growth of $\xi_A$ may be sufficient for the right-hand-side of (2.23) to be positive, and it may take many e-folds before $\xi_A$ exceeds $H$ and NEC is violated. Hence, the period of accelerated can only be significantly constrained if $\dot{H} + H^2 \sim H^2$ differs significantly from zero. If the 4d theory is general relativity to a good approximation (as is the case when discussing dark energy, but not necessarily during inflation — see the discussion below (2.14)), this translates to the statement that strong bounds can be derived only when the equation of state parameter is considerably smaller than $-1/3$ but not in the limit $w^E \to -1/3$. Assuming the r.h.s. of (2.24) is $O(H^2)$, we can integrate this equation to get: $\xi_A \gtrsim H^2 t + O(H \dot{H} t)$. Plugging in (2.24) it follows that after a time $t \sim H^{-1}$ the NEC is violated. We can thus formulate the following theorem:

- Theorem 2: (1) Given NEC, (2) $\dot{H} + H^2 \sim H^2$, and (3) $d < 4$ or $d \geq 10$ for RF and $d \neq 2$ and $d < 15$ for CRF, 4d accelerated expansion is only possible for a limited $O(1)$ number of e-folds.

Allowing for explicit NEC violation does not imply that accelerated expansion is automatic. Additional theorems can be derived that constrain the spatial and/or temporal distribution of the NEC violating component in the energy-momentum tensor, see Ref. [1] for details.

3. Assumptions

Any no-go theorem is as good as the assumptions that go into it. Let us therefore step back for a moment and list the assumptions that went into the derivation of the no-go theorems in section 2.4. They are:

(i) The higher dimensional theory is described by general relativity.

(ii) The metric (2.1) is block-diagonal, with the higher dimensional manifold either $\mathcal{R}$-flat (RF) or conformal $\mathcal{R}$-flat (CRF).

(iii) The 4d metric in the Einstein frame is of the FRW form with zero spatial curvature.

(iv) The metric satisfies the restriction (2.23).

(v) The boundary term (2.21) vanishes.

(vi) The A-average is finite for all finite $A$.

(vii) The Einstein metric that brings the 4d Einstein-Hilbert action in canonical form is the metric of our universe.

Condition [8] states that we do not consider higher order curvature invariants in the higher dimensional action (or consider them negligibly small). It also excludes possible other
modifications of gravity. We note that dS solutions violating this assumption have been constructed, for example, in [25, 26, 27, 28].

It is impossible to derive no-go theorems for generic metrics. Assumption (i) simplifies the metric considerably. The restriction to RF or CRF metrics allows to determine the sign and size of the Ricci-scalar in (2.16). In general, a positive/negative extra-dimensional curvature will reduce the right-hand-side of (2.16), making it easier/harder to satisfy the NEC. Many models discussed in the literature have a metric of the form (2.1). The original Kaluza-Klein (KK) model, Randall-Sundrum (RS) models [5, 6] and all five-dimensional models are RF, flux compactifications on conformal Calabi-Yau manifolds in string theory [7, 8, 9, 10] are CRF. We note that dS solutions in set-ups with a more general extra-dimensional metric exists, see for example [29, 30].

Condition (iii) is motivated by observations. Our universe is nearly homogeneous and isotropic on large scales, and indistinguishable from flat [31]. It excludes set-ups in which 4d Lorentz symmetry is (weakly) violated. In [1] the additional assumption was made that the dimensionally reduced 4d theory is described by general relativity with negligible small corrections. This assumptions is only needed to express the theorems in terms of the equation of state parameter \( \omega_E \) rather than directly in terms of the Hubble constant, such as we did in Theorem 2, and as such can be lifted.

The metric is still too general to derive no-go theorems and further assumptions, such as (iv) are needed. This assumption is discussed in detail in section 4 below. As discussed in the next subsection, the theorems only apply to Kaluza-Klein compactifications, as brane world models generically violate one or more of the assumptions (i)-(vi). Moreover, models with negative tension branes or orientifold planes, such as RS1 and flux compactifications, violate the higher dimensional NEC.\(^2\)

3.1 Brane worlds

To arrive at the Einstein equation (2.19) a partial integration is done, where it is assumed that the boundary term (2.21) vanishes, which is assumption (i). This is automatic if \( \mathcal{M} \) is compact and closed, and thus has no boundaries (for orbifold compactification, if the covering space has no boundaries), as in KK compactifications. However, if \( \mathcal{M} \) is bounded by branes, boundary terms as the above are generically non-zero, as they are sourced by the brane localized energy-momentum. The partial integration is a crucial step in deriving the no-go theorems of section (2.4). First of all, by doing so one d.o.f. is removed as \( \nabla^2 \Omega \)-terms (whose sign is undetermined) are eliminated from the equations. Secondly, the coefficient in front of the \( (\nabla \Omega)^2 \)-term becomes \( A \)-dependent, and can be made negative choosing a suitable \( A \).

Without the partial integration, there is no such freedom.

In co-dimension one brane worlds, for which the brane extends in \( D - 1 \) space-time dimensions and the warp factor only depends on the remaining coordinate, the boundary

\(^2\)If NEC violation only occurs at the boundary of spacetime, as in RS1, one could still try to apply the theorems to bulk matter only. However, this approach fails in general because the boundary term is non-zero and assumption (i) is violated.
terms are given by the Israel-junction conditions, and can be calculated explicitly. For a higher number of co-dimensions it is much harder to formulate the boundary conditions, and explicit solutions to the Einstein and matter equations only exist for co-dimension two brane worlds. Recently the matching conditions for co-dimension two sources in arbitrary dimensions were formulated \cite{1, 14}. Specific examples of co-dimension one and two brane worlds are 5d RS1 models and 6d “football shaped” models \cite{1, 12, 13, 14} respectively. In both cases, the boundary terms can be related to the energy-momentum localized at the boundary branes, and the boundary term appearing in the Einstein equation (2.19) is positive $B.T. > 0$. Hence, the boundary term can compensate negative terms on the right-hand-side of this equations, thereby invalidating attempts at deriving a no-go theorem. Although these results are for particular set-ups, it is clear that in models where space is bounded by branes (or more generally, localized sources of energy-momentum) the boundary terms generically do not vanish, and its value is very model dependent; consequently no general no-go theorems can be derived.

As a side remark, in the older Ref. \cite{4} no-go theorems were derived under the “boundedness” assumption, which requires e.g. the $A$-average $\langle e^{2\Omega} R \rangle_A$ to be bounded in the $A \to \pm \infty$ limit (the sign depending on the number of dimensions). This is in place of the RF or CRF assumption used in \cite{1}. As was already noted in \cite{4} boundedness is violated by the 6d football shaped models which have the property that the curvature diverges at the brane position. We here note that neglecting the boundary term (2.21) is another assumption that fails for these models, and that was used in the derivation of the theorems. This loophole in the no-go theorems was addressed for co-dimension two branes in \cite{4).

The extra dimension do not need to be compact and closed for the boundary term to vanish. It may also be that the extra dimensions are infinite or end in a singularity, such as in RS2 and soft-wall models respectively \cite{46} (where branes/domain walls are localized inside the bulk, but not at the boundary), but that the boundary terms vanish because of the strong warping. However, in this case we have to be careful that the $A$-average remains finite, and assumption (vi) is satisfied. This is automatic for the canonical $A = 2$ average, but not for arbitrary (but finite) $A > 2$. Compactifying to an effective four-dimensional theory only makes sense if the effective 4d Planck mass (2.11) is finite, which is assured if the warped extra-dimensional volume — the integral appearing in (2.11) — is bounded. The Planck mass is defined in terms of the canonical $A = 2$ average (2.7). If we consider arbitrary $A$-average, this only makes sense if the integrals are finite $\langle Q \rangle_A < \infty$. This is only assured for values $A > 2$ (or $A > 0$ if any amount of warping is sufficient to kill off the boundary terms — but this is a model dependent question).

Assumption (vii) states that the metric of our universe $g^\text{vis}_{\mu\nu}$ equals the metric $g^E_{\mu\nu}$ that brings the dimensionally reduced 4d Einstein-Hilbert action in canonical form. This is not automatic in brane world models. Indeed, if the standard model matter is localized on a brane instead of living in the bulk the induced metric on the brane, which is the metric of
our universe, is

\[ g_{\mu\nu}^{\text{vis}} = e^{2\Omega(y_{br}, t)} g_{\mu\nu}^E \] (3.1)

with the warp factor evaluated at the position of the brane \( y_{br} \). The two metrics enter because matter and gravity “see” a different number of dimensions. We can normalize this to \( \Omega(y_{br}, 0) = 1 \) by rescaling the 4d coordinates. Only if the warp factor is time-independent the two metrics coincide at all times. Otherwise accelerated expansion in our universe is not the same as accelerated expansion in the canonical metric. Writing \( a^{\text{vis}} = e^{\Omega(t)} a^E \), then \( H^{\text{vis}} = H^E + \dot{\Omega} \) and \( \dot{H}^{\text{vis}} = \dot{H}^E + \ddot{\Omega} \) in the Einstein equations (2.18, 2.19). The extra terms can make the r.h.s. of these equations positive, thereby circumventing the need for NEC violation.

Finally, we would like to mention that in co-dimension one brane worlds, it follows from the Israel-Junction conditions that if a brane or domain wall is located somewhere in the bulk, and if it carries energy-momentum other than a pure brane tension, 4d Lorentz symmetry is necessarily broken [47, 48]. If the Lorentz symmetry breaking is small, the model may still be phenomenologically acceptable. However, since assumption (iii) is broken, the no go theorems do not apply. It may be interesting to see how general this phenomenon of 4d Lorentz symmetry breaking is in general brane world models.

Given all the considerations of this section, it may be hard if not impossible, to extend the no-go theorems to also include (classes of ) brane world models.

4. The metric restriction

The “metric restriction” (iv) is not listed as an explicit assumption in Ref. [1], but it should be viewed as such. Combining (2.23) with the requirement of a constant Planck mass (2.12) gives \( 2\dot{\Omega} + \xi = 0 \), and thus (2.17) is trivially satisfied for all \( A \). The restriction simplifies Einstein’s equations, as it removes one degree of freedom. Moreover, it allows to write the equations unambiguously in terms of the generalized \( A \)-average, which provides additional power as a suitable \( A \) can be chosen to make certain terms negative.

Ref. [1] argues that the metric restriction is a consequence of coordinate invariance “in the adiabatic limit”. It is unclear how to interpret this statement, as \( \xi \) and \( \dot{\Omega} \) are time-derivatives of the metric functions, and thus also vanish in the adiabatic limit (which makes the restriction trivially satisfied). Since the goal is to derive no-go theorems that constrain time-dependent metrics, to be precise metrics that give accelerated expansion in 4d, the adiabatic limit does not seem applicable. In the full time-dependent set-up the restriction cannot be enforced merely by gauge invariance, as there is in general no coordinate transformation that can set \( (2\dot{\Omega} + \xi)_{2,1} = 0 \).\(^3\) While keeping the metric in the original block-diagonal form (2.1).

Another argument given for the metric restriction is that it removes one linear combination of the moduli fields which in the compactified theory has a wrong sign kinetic term. The

\(^3\)This generically requires a time-dependent transformation \( y^m \to y^m + \delta y^m(t, y^n) \), which introduces an off-diagonal \( \propto dy dt \) term in the metric.
dimensionally reduced Einstein-Hilbert plus matter action has kinetic terms of the form

\[
S_{\text{kin}} = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left[ -6H^2 + \frac{d+2}{2d} \xi_0^2 - \frac{d-1}{d} \langle \xi_\perp^2 \rangle_2 - 6\langle \xi_\perp \dot{\Omega}_\perp \rangle_2 - 6\langle \dot{\Omega}_\perp^2 \rangle_2 \right] + \ldots \quad (4.1)
\]

where the average is the canonical \( A = 2 \) average (2.7). The ellipses denote possible matter kinetic terms. Diagonalizing the kinetic terms, there is one linear combination of \( \dot{\Omega}_\perp \) and \( \dot{\xi}_\perp \) that has a negative kinetic term, corresponding to a ghost mode. A constraint equation relating the fields will remove one degree of freedom, and thus may eliminate the ghost mode. However, it is by no means clear that the restriction (2.23), which is not unique and is imposed by hand, is really needed to remove the ghost. Indeed, a theory that is well defined in the extra dimensions cannot develop a pathology merely by compactifying, that is, by integrating over the extra dimensions. The off-diagonal components of the Einstein equations make this happen. They provide the necessary constraints on the system — they contain no second order time derivatives of the metric functions, and thus must be imposed on the system at all times. Only for special metrics will the restriction (2.23) be a solution of the full Einstein equations.

For the metric (2.1) the non-zero off-diagonal Einstein equations are the \( \{mn\} \)-equation, and \( \{0m\} \):

\[
\kappa_D^2 T_{0m} = \nabla_p \sigma_m^p + \frac{(1-d)}{d} \nabla_m \xi + \frac{(d+2)}{d} \xi (\nabla_m \Omega) - 3\partial_t \nabla_m \Omega + 2(\nabla_p \Omega) \sigma_m^p. \quad (4.2)
\]

The number of degrees of freedom left after gauge fixing (or alternatively, rewriting the equations in explicitly gauge invariant form) and applying the constraint equations depends on the symmetries of the extra dimensional manifold. The \( \{0m\} \)-equation equation will always remove one linear combination of \( \Omega, \xi, \sigma \). Actually, as soon as \( \Omega \) and/or \( \xi \) depend on the extra-dimensional coordinates, they necessarily mix with some combination of \( \sigma \) and possible additional matter fields, and the restriction is more complicated than (2.23).

We can check whether the metric restriction (2.23) applies to explicit time-dependent solutions of extra dimensional models that exist in the literature. Einstein’s equations are non-linear partial differential equations, and explicit solutions are known only for simple metrics. Most solutions assume a separable form of the metric, in which all metric components can be written in the form \( f(t,y) = f_1(t)f_2(y) \). This greatly simplifies the system, as it transforms the Einstein equations into ordinary differential equations. If in addition there is only one extra dimension the equations can be solved relatively straightforwardly. Consider then a 5d metric, which is of the general form (2.1),

\[
ds^2 = e^{2\Omega(t,y)} g^{E}_{\mu\nu}(t) dx^\mu dx^\nu + b(t,y)^2 dy^2. \quad (4.3)
\]

Translating to the notation of (2.4) gives \( \xi = \dot{b}/b \). Ref. [49] tries to find solutions for \( \Omega = \Omega(y) \) and \( b = b(t) \). Since \( \dot{\Omega} = 0 \) vanishes by assumption, and one can only define the Einstein frame at all times if \( \xi = 0 \) as well (2.12), the metric restriction (2.23) is satisfied trivially. Also for
the solutions in Ref. [50, 51], which has $\Omega = \Omega(y)$ and $b = b(y)$, the restriction is automatic. The solution found in [48] violates assumption (iii) as 4d Lorentz symmetry is broken.

We have found only one explicit solution in the literature in which the metric components have a non-separable and non-trivial form [11]. In this case the off-diagonal Einstein equation (4.2) and the metric restriction (2.23) are inconsistent, providing an explicit counter example to the validity of the metric restriction. Hence (2.23) should indeed be viewed as an extra assumption rather than a necessary condition. Ref. [11] considers a metric of the form

$$
\begin{align*}
\text{d} s^2 &= c^2(y, \tau)(-\text{d}\tau^2 + \text{d}\vec{x}^2) + b^2(y, \tau)\text{d}y^2 \\
&= e^{2\Omega(y, t)}(-\text{d}t^2 + a^2(t)\text{d}\vec{x}^2) + b^2(y, t)\text{d}y^2.
\end{align*}
$$

(4.4)

with $a \text{d}\tau = \text{d}t$ with $\tau$ conformal time. The split of $c^2 = e^{2\Omega}a(t)^2$ is chosen such that $\Omega$ satisfies (2.12), and $a$ is the Einstein frame scale factor. In addition the model has a bulk scalar with an appropriate potential, such that all Einstein’s equations, including the constraint equation (4.2), are satisfied. A family of solutions is found

$$
b^2 = \varepsilon^2 c^2, \quad c(\tau, y) = [(\lambda(y + \tau))^{2k} + 1]^{-1/(2k)},
$$

(4.5)

with $\varepsilon, \lambda, k$ constants that enter the potential for the bulk field. For this metric [4.4] $\xi = \dot{b}/b = \dot{c}/c$, and $\hat{\Omega} = \dot{c}/c - \dot{a}/a$. Since $a$ depends only on time, it contributes to the zero mode $\langle \hat{\Omega} \rangle_2$, but drops out of the fluctuations. We find

$$
\langle \hat{\Omega} \rangle_2 = \xi_2 = \left(\frac{\dot{c}}{c}\right)_2 \quad \Rightarrow \quad \langle \hat{\Omega} - \xi \rangle_2 = 0.
$$

(4.6)

And thus the restriction (2.23) is not satisfied for this solution. We checked that the Planck mass (2.11) for this solution is finite, and thus a sensible 4d theory can be defined. Since pathologies cannot arise merely by integrating over the 5th dimension (unless some invalid approximations are made), the ghost mode has been removed by the constraint equation (4.2).

4.1 Lifting the metric restriction

The strong no-go theorems derived in section 2.4 depend on the metric restriction (2.23) in two essential ways. First of all, the extra constraint on the metric removes one degree of freedom. Secondly, the power of generalized $A$-averaging can be used to choose a suitable $A$ which sets $C_i \geq 0$. Without the metric restriction it is not possible to derive powerful no-go theorems, except for some simplified forms of the metric.

Lifting the metric restriction of assumption (iii) the Einstein equations are only valid for $A = 2$; they become:

$$
\begin{align*}
\kappa^2_D \langle e^{2\Omega}(\rho + p_3) \rangle &= -2\dot{H} - |X| + \frac{(d - 1)}{d} \langle \xi_1^2 \rangle + 6 \langle \hat{\Omega} \langle \hat{\Omega} + \xi \rangle \rangle, \\
\kappa^2_D \langle e^{2\Omega}(\rho + p_d) \rangle &= -3(\dot{H} + H^2) - |X| + \frac{(d - 1)}{d} \langle \xi_1^2 \rangle + 6 \langle \hat{\Omega} \langle \hat{\Omega} + \xi \rangle \rangle \\
&+ \frac{(d + 2)}{2d} \frac{\partial_t (a^3 \xi_0)}{a^3} - C_1 \langle e^{2\Omega}(\nabla \Omega)^2 \rangle.
\end{align*}
$$

(4.7)
where we dropped the subscript $A = 2$ from the averages, and introduced the notation $Q_0 = \langle Q \rangle$. As before $|X| = (\sigma^2) + (d + \xi_0^2)$. Using $A = 2$ in (2.22) gives $C_1 = -12/d$ for RF and $C_1 = (-d^2 + 11d - 22)/d$ for CRF; only in the latter case and for $3 \leq d \leq 8$ is $C_1 \geq 0$ non-negative.

No general no-go theorems can be derived. NEC requires the left-hand-side of the above equations to be positive. Accelerated expansion implies the first term on the right-hand-side of (4.8) is negative; but this can now always be compensated by large $\dot{\Omega}$ or $\xi_\perp$ contributions, and for RF also by the $(\nabla \Omega)^2$ term. We can only derive theorems if we make additional assumptions on the metric, such that these contributions are limited.

### 4.1.1 Static extra dimensions

From the work of Maldacena and Nuñez [14] we already know that with time-independent extra dimensions the strong energy condition has to be violated to get 4d accelerated expansion. Can we extend this to the weaker NEC condition? For static extra dimensions $|X| = \xi = \sigma_{mn} = 0$. For CRF the warp factor enters the extra-dimensional metric and $\dot{\Omega} = 0$ as well. In this case the metric restriction (2.23) is trivially satisfied, and the Einstein equation for generic $A$ (2.24, 2.25) apply. If $C_1(A, d) > 0$, which only excludes $d = 2$, the right-hand-side of (2.23) is negative for 4d accelerated expansion and NEC is violated.

For RF it is harder to derive theorems as $\dot{\Omega}$ does not need to vanish. The warp factor is constrained by the off-diagonal Einstein equations, which simplify enormously for static extra dimensions. By tuning the energy-momentum sources it may be possible to satisfy these equations even for a time-dependent warp factor. If $T_{0m} = 0$ the warp factor needs to be time-independent $\dot{\Omega} = 0$, the metric restriction applies, and strong theorems can be formulated.

- **Theorem:** Given (1) NEC, (2) static extra dimensions, (3) CRF with $d \neq 2$ or RF with $d \neq 4$ and $\Omega = 0$, accelerated expansion $\ddot{a} > 0$ is impossible.

### 4.1.2 Generalized restriction

Consider metrics for which the warp factor and volume modulus are related via

$$ (\dot{\Omega})_\perp = f \xi_\perp = 0 \quad (4.9) $$

with $f$ a (time-dependent) constant. This includes the restriction imposed by Ref. [1] quoted in (2.23) for which $f = -1/2$, as well as the solution (4.6) for which $f = 1$ [11]. The equations (4.7, 4.8) become

$$ \kappa_D^2 \langle e^{2\Omega}(\rho + p_3) \rangle = -2\dot{H} - |X| - C_3 \langle \xi_\perp^2 \rangle \quad (4.10) $$

$$ \kappa_D^2 \langle e^{2\Omega}(\rho + p_d) \rangle = -3(\dot{H} + \dot{H}) - |X| + \frac{(d + 2)}{2d} \partial_\parallel (a^3 \xi_0) - \frac{C_3 \langle \xi_\perp^2 \rangle - C_1 \langle e^{2\Omega}(\nabla \Omega)^2 \rangle}{a^3} \quad (4.11) $$
with $|X| = \langle \sigma^2 \rangle + \frac{(2 + d)}{2d} \xi_0^2$. The coefficient $C_3 = (2 + d)/(2d) - (3/2)(2f + 1)^2$, and $C_1$ is given as before by (2.22). If both $C_1, C_3 \geq 0$ strong theorems analogous to those in section 2.4 can be derived. This is the case for CRF with $3 \leq d \leq 8$ and

$$-3d - \sqrt{3(d^2 + 2d)} \leq f \leq -3d + \sqrt{3(d^2 + 2d)} \quad (4.12)$$

which includes the case of the metric restriction $f = -1/2$ discussed in section 4. The allowed interval for $f$ ranges from $\{-0.87, -0.13\}$ for $d = 3$ to $\{-0.82, -0.18\}$ for $d = 8$. For RF $C_1 < 0$ independent of the number of dimensions. We can formulate the following theorem:

- **Theorem:** Given (1) NEC, (2) $\dot{H} + H^2 \sim H^2$, (3) extra dimensions that satisfy the generalized restriction (4.9) with (4.12), and (4) are CRF with $d \geq 3$, 4d accelerated expansion can only last for a few $\mathcal{O}(1)$ e-folds.

Of course for the special case of $f = -1/2$ the stronger theorems derived in section 2.4 apply.

### 4.1.3 No warping

The unwarped metric is a special case of the generalized restriction. It corresponds to $\Omega = \Omega(t)$ and thus $\nabla_m \Omega = \dot{\Omega} = 0$. This gives the equations above (4.10, 4.11) with $f = 0$ and the term proportional to $C_1$ vanishes. Strong theorems can be derived if $C_3 \geq 0$, which is only the case for $d = 1$ but not for higher dimensions. The off-diagonal Einstein equations generically also do not help to constrain the system. Hence

- **Theorem:** Given (1) NEC, (2) $\dot{H} + H^2 \sim H^2$, (3) no warping, and (4) 5d ($d = 1$), 4d accelerated expansion $\ddot{a} > 0$ can only last for a few $\mathcal{O}(1)$ e-folds.

### 5. Conclusions

In this work we revisited no-go theorems forbidding accelerated expansion in our four large space-time dimensions in the context of extra-dimensional models. The metric was taken to be of block-diagonal form, with the extra dimensions either Ricci flat or conformal Ricci flat. This metric allows for a non-trivial warp factor, and applies to many of the motivated examples in the literature. The time-evolution of the four-dimensional Einstein metric is constrained, as the full set-up has to satisfy the higher-dimensional Einstein equations.

To derive the no-go theorems some rather technical and innocent looking assumptions were made, such as the vanishing of the boundary term (2.22). Nevertheless they have far reaching consequences. Indeed the theorems only apply to Kaluza-Klein compactifications. As we discussed, brane world models, or more generically set-ups with localized sources of energy-momentum, generically violate one of the assumptions made.

But even for KK compactifications, no-go theorems can only be derived for simple metrics, which satisfy additional restrictions. In Ref. [1] Steinhardt and Wesley (silently) assumed the
a relation between the warp factor and the volume of the extra dimensions. With this metric restriction strong no-go theorems can be derived. For example, if the higher dimensional NEC is satisfied 4d accelerated expansion can only last for a couple of e-folds.

In Ref. [2] the metric restriction was presented as a necessary condition to get a well-defined four-dimensional theory. We argue, though, that it should rather be viewed as an additional assumption on the metric. The restriction is ad-hoc and put in by hand. Moreover, it may be inconsistent with the constraints coming from the off-diagonal Einstein equations. Indeed, the work of Ref. [11] provides an explicit solution to the 5d Einstein equations that does not satisfy the metric restriction.

Any no-go theorem is as strong as the assumptions that go into it. To obtain 4d accelerated expansion requires to violate the assumptions made in Ref. [1]. The easiest way is to add branes or domain walls to the system. For KK compactifications, our work shows that another way around the theorems is to look for solutions that do not satisfy the metric restriction.

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References


