Avoiding the dangers of a soft-wall singularity

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ABSTRACT: We critically analyse the nature of the infrared singularity in Randall-Sundrum soft-wall models, where the extra dimension is dynamically compactified by the formation of a curvature singularity. Due to the Israel junction conditions, this singularity can only be shielded by a time-independent black-hole horizon if there is ghost matter on the UV brane. For this construction the spectrum of 4D states is shown to be similar to the original soft-wall case. We point out, however, that no such shielding is needed, as the singularity satisfies unitary boundary conditions.

KEYWORDS: Field Theories in Higher Dimensions
1. Introduction

Soft-wall models are a recent modification to the Randall-Sundrum (RS) scenario, where a compact extra dimension is warped in order to solve the electroweak hierarchy problem \cite{RS}. In soft-wall models, the warp factor exponent diverges in the infrared (IR) and the negative tension IR brane is replaced by a curvature singularity. The original motivation for such warping was to obtain Kaluza-Klein (KK) modes with mass squared that is linear in KK number, with the aim to better model excited mesons in QCD using the AdS/CFT correspondence \cite{AdS}. Since this initial work, soft walls have been used to construct extra-dimensional extensions of the standard model \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, as well as a holographic dual description of unparticle models \cite{12, 13}. One of the main motivations of a soft-wall background is that it allows a lower KK energy scale without violating electroweak observables which are problematic in the original RS framework \cite{14, 15, 16}.

The minimal soft-wall spacetime is supported by a dilaton field which also diverges in the IR, and the distance to the singularity is stabilised by a ultraviolet (UV) localised potential for the dilaton \cite{9, 11}. Alternatively, the UV brane can be replaced by a domain-wall and the set-up stabilised by parity and an appropriate bulk potential \cite{17, 18}. With suitable scalar potentials, soft-wall models can solve the hierarchy problem in the same way as in RS \cite{9}. A basic feature of these models which remains to be analysed in detail is the nature of the singularity itself, and this shall be the focus of the current paper. The soft-wall singularity is allowed if it does not lead to pathological behaviour, such as loss of unitarity or instabilities.
Cosmic censorship conjectures that all naked singularities (such as those in the soft-wall set-up) are protected by an event horizon. In the soft-wall literature to date, it has been assumed that such a horizon can be constructed to shield the bulk from the singularity. This assumption is based on the early work of Gubser [19], where it is shown that a physical spacetime can be constructed with a singularity and an event horizon, provided the scalar potential is bounded above in the solution. The Hawking temperature of the horizon can be identified with the finite temperature in the dual field theory which serves as an IR cutoff. Spacetime is asymptotically AdS in a direction away from the horizon. To apply this result to the soft-wall scenario requires patching two truncated versions of the Gubser spacetime back-to-back, so that the singularities bound the (almost) AdS interior. We shall show in this paper that such a patching cannot be accomplished without putting ghost matter at the junction. These results follow simply from the inability to match the Israel junction conditions when the 5D metric is generalised to allow for a horizon. In such a case, the relevant Einstein’s equations require ghost matter to bend the metric components in the right way.

Having ghost matter on the UV brane is particularly unsettling. An obvious way around this is to look for background configurations with a horizon and no ghosts, but which are time dependent in order to satisfy the Israel junction conditions and Einstein’s equations. If such solutions have a life-time longer than the age of the universe, they provide a viable background. It is however questionable whether this can be achieved. If the soft-wall model is to solve the hierarchy problem, a much larger decay rate of the order of the electroweak scale seems natural (as, besides the Planck scale, there is no other scale in the system).

Thus shielding the soft wall singularity by a horizon cannot be achieved in a straightforward manner. As we shall point out in this paper, there is actually no need to do so. The soft-wall singularity is actually not “visible”, and one can thus argue that cosmic censorship does not apply. Even though the singularity is at a finite distance in physical coordinates, there are no plane wave modes in the spectrum that can travel to it in a finite time. The KK spectrum consists of discrete modes that all have an extra-dimensional profile that dies off rapidly towards the singularity.\(^1\) As a consequence, the system satisfies unitary boundary conditions, which guarantee that no conserved charge, such as energy and (angular) momentum, can leak into the singularity [20, 21, 22]. Without this leakage there are no pathologies, and unitarity is automatically conserved.

The paper is organised as follows. In Section 2 we start with a derivation of the classical background solution. We generalise the metric of the soft-wall set-up to allow for a horizon shielding the singularity; we call this the “black-wall” configuration. The rest of this section is devoted to a discussion of the original soft-wall model with a naked singularity. We also briefly comment on the spectrum of KK states. In Section 3 we consider the problems that arise if one tries to shield the singularity by a black wall, followed in Section 4 by a discussion of the perturbative spectrum of a black wall. In Section 5 we argue that the soft-

\(^1\)In the special case that the spectrum consists of unparticles, that is, the spectrum is continuous but with a mass gap, the continuous modes still die off rapidly towards the singularity.
wall singularity is not “visible”, as the KK modes all satisfy unitary boundary conditions. Hence, no shielding by a horizon is needed. We conclude in Section 3.

2. The soft-wall model

In this section we give a concise review of the soft wall-model, focusing on those features that later enter the discussion on the nature of the singularity. We begin by deriving the equations of motion and boundary conditions for a metric that is general enough to include an event horizon shielding the singularity. From these equations one can obtain the original soft-wall background solution, to be discussed in the remainder of this section, as well as the black-wall solution discussed in the following section.

2.1 The background equations

There are three main ingredients for a soft-wall model: something to warp the metric at the origin (the UV brane or domain wall), something to diverge the metric at the edge (the dilaton), and some stabilisation mechanism (brane potential terms). In the minimal set-up we discuss here, a UV brane with a dilaton suffices. Stabilisation is ensured by a brane-potential for the dilaton, fixing its BC there; see [9]. The nature of the singularity only depends on the IR details of the model, in particular on the asymptotic behaviour of the bulk potential \( V \) with respect to the field playing the role of the dilaton.

Consider then 5D spacetime with gravity, a single real scalar field \( \phi \) in the bulk (the dilaton), and a fundamental 3-brane with localised matter placed at \( y_{br} = 0 \). Using normal Gaussian coordinates, the most general static metric is of the form \[ (2.1) \]

\[ d\sigma^2 = e^{-2\sigma(y)}(-f(y)dt^2 + dx^2) + \frac{dy^2}{f(y)}. \]

We could redefine \( y \) to absorb \( f \) in the \( dy^2 \) term, but the current form of the metric allows us to clearly see the formation of an event horizon. Indeed, this occurs at \( y_h \) where \( f(y_h) = 0 \). Since all metric functions depend only on \( y \), we can rescale all coordinates by a constant to set \( f(0) = 1 \) and \( \sigma(0) = 0 \). The 5D action consists of a bulk and brane contribution:

\[ S = \int d^5x \sqrt{-g_5} \left[ \frac{R}{6\kappa^2} - \frac{1}{2} g_5^{MN} (\partial_M \phi)(\partial_N \phi) - V(\phi) \right] \]

\[ + \int d^5x \sqrt{-g_{br}} [-\lambda(\phi) + L_{br}(\psi, X)] \delta(y - y_{br}), \]

where \((g_5)_{MN}\) is the 5D metric and \( \kappa^2 \) is proportional to the 5D Newton’s constant. The induced metric on the brane is defined via

\[ (g_{br})_{\mu\nu} = \frac{\partial x^M}{\partial z^\mu} \frac{\partial x^N}{\partial z^\nu} (g_5)_{MN} = \delta^M_{\mu} \delta^N_{\nu} (g_5)_{MN}, \]

\[ (2.3) \]

\(^2\)For \( f(y) \) not constant 4D Lorentz invariance is broken if bulk fields have different 5D profiles. For \( f \) nearly constant, except close to the horizon/singularity, these effects can presumably be made arbitrarily small. See [23] for further details of such Lorentz violation.
where \( z^\mu \) are the coordinates on the brane and \( x^M \) the coordinates in the bulk. The bulk scalar \( \phi(x^\mu, y) \) has a brane-localised potential \( \lambda(\phi) \), and we allow for additional brane matter \( \psi(x^\mu) \) through the Lagrangian \( \mathcal{L}_{\text{br}} \). This Lagrangian is generically a function of the kinetic term \( X = -(1/2)g_{\text{br}}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \); for canonical kinetic terms the brane Lagrangian is \( \mathcal{L}_{\text{br}} = X - V_{\text{br}}(\psi) \).

The energy momentum tensor for the action \((2.2)\) is \( T_{MN} = (-2/\sqrt{-g_5})\partial S/\partial g_5^{MN} \), and takes the explicit form

\[
T_{MN} = (g_5)_{MN} \left(-\frac{1}{2}g_5^{PQ}(\partial_P\phi)(\partial_Q\phi) - V(\phi) + (\partial_M\phi)(\partial_N\phi) \right) + \frac{\sqrt{-g_5}}{\sqrt{-g_5}} \delta_M^\mu \delta_N^\nu \delta(y - y_{\text{br}}) \left[ (g_{\text{br}})_{\mu\nu}(-\lambda(\phi) + \mathcal{L}_{\text{br}}) + \frac{\partial \mathcal{L}_{\text{br}}}{\partial X}(\partial_\mu\psi)(\partial_\nu\psi) \right]. \tag{2.4}
\]

Splitting the energy-momentum tensor into pieces that do and do not depend on the brane matter field \( \psi \), we can identify the piece that does depend on \( \psi \) as a cosmological fluid localised to the brane, with \( T_{MN} = \delta(y - y_{\text{br}}) \) \( \text{diag}(-\rho_{\text{br}}, p_{\text{br}}, p_{\text{br}}, 0) \). The brane energy and pressure components then read

\[
\rho_{\text{br}} = -\mathcal{L}_{\text{br}} + \frac{\partial \mathcal{L}_{\text{br}}}{\partial X} \dot{\psi}^2 = \frac{1}{2} \dot{\psi}^2 + V_{\text{br}}, \tag{2.5}
\]
\[
p_{\text{br}} = \mathcal{L}_{\text{br}} = \frac{1}{2} \dot{\psi}^2 - V_{\text{br}}. \tag{2.6}
\]

We have used \( f(0) = e^{\sigma(0)} = 1 \), and also the off-diagonal Einstein’s equations to obtain \( \nabla \psi = 0 \). The last equality in the above definitions is only for canonically normalised brane kinetic terms such that \( \partial \mathcal{L}_{\text{br}}/\partial X = 1 \). We shall express all \( \psi \) quantities in terms of \( \rho_{\text{br}} \) and \( p_{\text{br}} \), and treat it as an arbitrary cosmological source localised to the brane. When we need to study the form of the energy and pressure in more detail (such as the condition for ghosts) we shall use the above definitions.

The equations of motion for the system are Einstein’s equations \( G^M_N = 3\kappa^2 T^M_N \), and the Euler-Lagrange equations. The bulk off-diagonal Einstein’s equations enforce \( \partial_\mu \phi = 0 \), a consequence of our diagonal and isotropic metric ansatz. Then, the Euler-Lagrange equation and the \((00) - (ii)\), \((00) - (55)\) and \((55)\) Einstein’s equations read:

\[
f \phi'' - 4f'\phi' + f'\phi' - \partial_\phi V = \delta(y - y_{\text{br}})\partial_\phi \lambda, \tag{2.7}
f'' - 4f'\sigma' = 6\kappa^2 \delta(y - y_{\text{br}})(\rho_{\text{br}} + p_{\text{br}}), \tag{2.8}
f (\sigma'' - \kappa^2 \phi'^2) = \kappa^2 \delta(y - y_{\text{br}})(\lambda + \rho_{\text{br}}), \tag{2.9}
4f \sigma'^2 - f' \sigma' - \kappa^2 (f \phi'^2 - 2V) = 0. \tag{2.10}
\]

Only three of these four equations are independent. In addition there is the brane localised Euler-Lagrange equation for the brane matter \( \psi \), which will not be important for our purposes.

The above system has five independent integration constants. As mentioned before, \( f(0) = 1 \), \( \sigma(0) = 0 \) can be fixed by reparameterisation of the coordinates. The remaining three are \( \{f'(0_+), \sigma'(0_+), \phi'(0_+)\} \). Once these constants are known, \( \phi(0) \) is fixed by the
constraint equation (2.10) evaluated at $y = 0$ [the values of all functions must be continuous over the brane, so $\phi(0) = \phi(0_{\pm})$]. The three derivative integration constants are all fixed by the boundary conditions, that is, by the matter on the brane. Integrating the above equations over the brane, and assuming a $Z_2$-symmetry, these boundary conditions are

\[
\begin{align*}
  f'(0_{\pm}) &= 3\kappa^2 (\rho_{br} + p_{br}), \\
  \sigma'(0_{\pm}) &= \frac{1}{2}\kappa^2 (\lambda + \rho_{br}), \\
  \phi'(0_{\pm}) &= \frac{1}{2}\partial_\phi \lambda.
\end{align*}
\] (2.11)

Given a choice of matter content on the brane, and brane potential terms, the model is stabilised precisely because there are no choices for the integration constants.

One important consistency check on the theory is that the effective 4D action has a vanishing cosmological constant. This is because 4D slices of our metric ansatz are time-independent. The 4D action is obtained by integrating the 5D action (2.2) over the extra dimension. The 5D Ricci scalar is

\[
R = 9f'\sigma' - 20f'\sigma'^2 - f'' + 8f\sigma'', \quad \sqrt{-g_5} = e^{-4\sigma}.
\]

Use the equations of motion (2.8-2.10) to eliminate $V, f''$ and $\phi'$. The result, including the brane term, is then

\[
S_5 = \int d^4x \int dy \left\{ \frac{1}{6}\kappa^2 e^{-4\sigma} (2f'\sigma' - 8f\sigma'^2 + 2f\sigma'') + \delta(y - y_{br}) (-p_{br} + L_{br}) \right\}
\]

\[
= \int d^4x \left\{ \frac{1}{3}\kappa^2 \left[ f\sigma' e^{-4\sigma} \right]_{y_s}^{y_{br}} \right\}.
\] (2.12)

To get flat 4D Minkowski spacetime this boundary term should vanish, and its vanishing depends on the model.

### 2.2 The soft-wall solution

In the usual soft-wall set-up the background solution has $f(y) = 1$ (see for example [9]). The boundary conditions can be satisfied in this case by the choice of vanishing brane matter, $\rho_{br} = p_{br} = 0$ (a cosmological constant $\rho_{br} + p_{br} = 0$ is already accounted for by $\lambda$). Having a constant $f$ and no brane matter means equation (2.8) is satisfied identically, and 4D slices of the metric at constant $y$ are Poincaré. The slope $\sigma'(0_{\pm})$ is positive for a positive tension brane ($\lambda > 0$) and energy scales are warped down towards the IR as one moves out into the bulk. In this case the model can solve the electroweak hierarchy problem as detailed in [9].

To see the formation of a singularity in the IR region, and a corresponding dynamical truncation of the extra dimension, we must solve the background equations of motion. In Ref. [1] the fake supergravity formalism [24, 25] is used to rewrite the equations of motion in first order form, allowing the construction of an explicit model. This approach requires the introduction of superpotential $W(\phi)$ such that the full potential can be written as $V = (1/2)(\partial_\phi W)^2 - 2\kappa^2 W^2$. Then the bulk equations of motion are satisfied if

\[
\sigma' = \kappa^2 W, \quad \phi' = \partial_\phi W,
\] (2.13)

while the boundary conditions read

\[
W(\phi_0) = \frac{1}{2}\lambda(\phi_0), \quad \partial_\phi W(\phi_0) = \frac{1}{2}\partial_\phi \lambda(\phi_0),
\] (2.14)
where \( \phi_0 = \phi(0) \). Consider then the class of superpotentials with asymptotic behaviour

\[
\lim_{\phi \to \infty} W = c(\kappa \phi)^\beta e^{\nu \kappa \phi}, \quad (\nu > 0).
\]  

(2.15)

We take \( \nu \neq 0 \) as polynomial functions cannot solve the hierarchy problem. The field \( \phi \) (and consequently also the metric function \( \sigma \) and the Ricci scalar) blows up at

\[
y_s = \int_{\phi_0}^{\infty} \frac{d\phi}{\partial_\phi W},
\]

(2.16)

which occurs at a finite and positive value if \( \nu > 0 \) and \( c > 0 \) (the equations of motion are solved on the interval \( 0 < y < y_s \)). The extra dimension is cut off dynamically by a singularity. This is the soft-wall.

The solution gives rise to a 4D Minkowski spacetime provided the surface terms (2.12) vanish for \( f(y) = 1 \). At the singularity, \( y \to y_s \), the field \( \phi \to \infty \), and the boundary term only depends on the asymptotic form of the superpotential, which we parameterise as before (2.15). It follows that in this limit \( \sigma \to \kappa \phi/\nu - (\beta/\nu^2) \ln(\nu \kappa \phi) \), and thus

\[
\lim_{y \to y_s} \sigma' e^{-4\nu} \propto (\kappa \phi)^{(1+4/\nu^2)} e^{\nu \kappa \phi(1-4/\nu^2)} \to 0 \quad \text{iff} \quad \nu < 2.
\]  

(2.17)

Furthermore, in order to have a phenomenologically viable model with a mass gap in the spectrum of KK states, one requires \( \nu \geq 1 \). Therefore, a viable soft-wall model that solves hierarchy problem is only possible for \( 1 \leq \nu < 2 \) [9].

### 2.3 The Kaluza-Klein spectrum

For the original soft-wall model, we parameterise the complete set of metric perturbations as

\[
ds^2 = e^{-2\sigma} (1 - 2F) \left[ -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right] + (1 + 4F)dy^2
\]

(2.18)

and the dilaton perturbation as \( \phi(x,y) = \phi(y) + \delta \phi(x,y) \). \( F = F(x,y) \) and \( h_{ij}(x,y) \) is transverse and traceless. As before \( \sigma(y), \phi(y) \) are the background solutions. The scalar \( (\delta \phi, F) \) and tensor perturbations \( (h_{ij}) \) decouple at first order and can be analysed separately. Einstein’s equations yield a constraint equations for the scalar perturbations

\[
\kappa^2 \phi' \delta \phi = F' - 2\sigma' F,
\]

(2.19)

leaving one scalar degree of freedom. There are two tensor degrees of freedom corresponding to the two polarisations of the graviton. We perform separation of variables for the scalar perturbations and tensor perturbations:

\[
F(x^\mu, y) = F(y) \rho(x^\mu), \quad \delta \phi(x^\mu, y) = \delta \phi(y) \rho(x^\mu), \quad h_{ij}(x^\mu, y) = h(y) \epsilon_{ij}(x^\mu).
\]  

(2.20)

The 4D perturbations satisfy \( \Box \rho = m^2 \rho \) and \( \Box \epsilon_{ij} = m^2 \epsilon_{ij} \). It is now convenient to go to conformally flat coordinates \( (dy = e^{-\sigma} dz) \), and rescale the perturbations, to obtain a Schrödinger equation for the profile functions

\[
-\partial_z^2 \tilde{q} + V_q \tilde{q} = m^2 \tilde{q}
\]

(2.21)
with $q = \{ \tilde{F}, \tilde{h} \} = e^{-3\sigma/2} \{ F/(\partial_z \phi), h \}$. This equation is solved to find the KK spectrum of modes, including the profile functions. Substituting these solutions into the original 5D action, expanding to second order, and integrating out the extra dimension yields the effective 4D action. This procedure requires partial integration, and surface terms are obtained. Demanding that these surface terms vanish, so to obtain 4D Minkowski space, gives the boundary equations for the tensor mode

$$e^{-4\sigma} h \partial_z h |_{y_s} = 0.$$  \hfill (2.22)

For the scalar perturbation there is a boundary condition first order in the perturbations as well (see [18] for a derivation)

$$e^{-4\sigma} (F' - 6\sigma' F) |_{y_s} = 0,$$

$$e^{-4\sigma} \left( \frac{1}{3} FF' - 10\sigma' F^2 + \frac{k^2}{2} \delta \phi \delta \phi' \right) |_{y_s} = 0. \hfill (2.23)$$

These are the first and second order extensions of the zeroth order result (2.12). Using the constraint equation (2.19), and the asymptotic behaviour of the background near the singularity;\(^3\) we find that all first and second order terms above (such as $e^{-4\sigma} FF'$) should vanish separately. In essence, there can be no cancellations of badly-divergent behaviour among the perturbations $F$ and $\delta \phi$ in the two expressions above, at least in the parameter range of interest, $1 \leq \nu < 2$. In conformally flat coordinates the boundary condition for, for example, the tensor perturbation translates to

$$e^{-3\sigma} \partial_z \tilde{h} |_{z_0, z_s} = \tilde{h} (\partial_z \tilde{h} + \frac{3}{2} \partial_z \sigma \tilde{h}) |_{z_0, z_s} = 0 \hfill (2.24)$$

In addition the mode functions should be normalisable: $\int \tilde{q}^2 dz < \infty$.

The potential $V_q$ depends on the background solutions, and is given explicitly in [9]. The spectrum is discrete with a mass gap for $\nu > 1$ and continuous with a mass gap for the boundary value $\nu = 1$. The mass gap is of order $m \sim k(y_s)^{-1/\nu^2} e^{-k y_s}$, with $k$ of the order of the 5D Planck mass. This yields electroweak scale masses for $k y_s \sim 30$, just as in RS.

### 3. The black-wall solution

The soft-wall solution has a naked singularity at finite coordinate distance, so matter can potentially reach it in a finite time. This may lead to problems with loss of unitarity or unknown quantum gravity corrections. The usual assumption in the soft-wall literature to date is that the singularity can be shielded by a black-wall horizon, which builds on earlier work by Gubser [19]. This is in line with the cosmic censorship conjecture. The black-wall solution corresponds to a non-trivial function $f$ in the metric (2.1) which becomes zero at some finite coordinate distance $f(y_h) = 0$, with $y_h < y_s$ the position of the black-wall horizon. At the horizon there is infinite time dilation, and it takes matter an infinite time

\(^3\)Near the singularity we have $\sigma \sim (-1/\nu^2) \ln k(y_s - y)$ and $\kappa \phi \sim (-1/\nu) \ln \nu^2(y_s - y)$.  

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to reach it (as seen by an asymptotic observer). The singularity at \( y = y_s \) is therefore shielded by this event horizon.

By way of background, we first discuss the construction of Gubser [19]. There, singular spacetimes were considered arising in theories with bulk matter, and which were asymptotically AdS in one half of the extra dimension. This corresponds to our set-up (with \( k > 0 \)) if the boundary conditions are modified: the brane at the origin is removed, and one half of the space is AdS, while the other half ends at a singularity at \( y = y_s \). The bulk solutions to the equations of motion are the same as in our case, but there are no longer additional boundary conditions to satisfy at the brane position. Asymptotically, as \( y \to -\infty \), spacetime approaches pure AdS, \( \sigma' \to k \) a constant, and \( f' \to 0 \). In [19] it is shown that provided the potential \( V \) is bounded from above in the solution, a family of black hole solutions shielding the singularity exist, suggesting cosmic censorship is at work in such a spacetime.

However, these conclusions do not have a straightforward application to the soft-wall model with a brane and \( Z_2 \) orbifold symmetry, as now there are also the boundary conditions at the brane position to satisfy (2.11). The equations of motion (2.8) and (2.9) can be solved in the bulk by quadrature [19]:

\[
\begin{align*}
\sigma &= ky + \kappa^2 \int_0^y \int_0^{y_1} dy_2 \phi'(y_2)^2, \\
f &= 1 + A|k| \int_0^y dy_1 e^{4\sigma(y_1)},
\end{align*}
\]

(normalised such that \( \sigma(0) \equiv \sigma_0 = 0 \) and \( f(0) \equiv f_0 = 1 \), and with \( \sigma'_0 = k \) and \( f'_0 = A|k| \). The dimensionless constant \( A \) controls the location of the horizon of the black wall. Except close to the singularity, spacetime is near AdS with \( \sigma \approx ky \). Only near to the singularity does the second term in (3.1) become important. Note this term is always positive and \( \lim_{\phi \to \infty} \sigma = +\infty \). For \( k < 0 \) this implies the sign of \( \sigma' \) flips. (For completeness, in the following discussion we consider both positive and negative \( k \).)

To form a horizon requires \( A < 0 \). Then \( f \) is guaranteed to pass through zero at some point before the singularity, since \( \sigma \to \infty \). Thus \( A < 0 \) is a necessary and sufficient condition to achieve a black-wall solution. If the horizon is close to the singularity, then in the whole bulk region the soft-wall (\( f = 1 \)) and black-wall (\( f \neq 1 \)) solutions are nearly the same, and one might expect that the soft-wall spectrum analysis carries over to the black-wall case. This is confirmed by our analysis of the perturbative spectrum in Section 4. This is a desirable scenario, as then one retains all the merits of the soft-wall set-up. For \( k < 0 \) and \( |A| \lesssim 1/|k|y_s \) one always has \( y_h \approx y_s \), as the second term in (3.2) is only important near the singularity. On the other hand, for the usual case of \( k > 0 \) the black-wall horizon is close to the singularity only for very small \( |A| \lesssim e^{-4ky_s}/ky_s \). Recall the KK scale is \( m \sim ke^{-ky_s} \) with \( k \) of the order of the 5D Planck scale, so the bound on \( A \) gives

\[
|A| \lesssim \left( \frac{m}{k} \right)^4 \left( \ln \frac{k}{m} \right)^{-1} \sim 10^{-62},
\]

where we have used \( m = 1 \) TeV and \( k = 10^{18} \) GeV.
The black-wall solution requires $A |k| = f'_0 < 0$, which, in combination with the boundary conditions (2.11), implies $\rho_{br} + p_{br} < 0$. This brakes the null energy condition, implying some kind of “ghost matter” is needed on the brane. Indeed, for the case of brane-localised scalar matter this implies negative kinetic terms:

$$\rho_{br} + p_{br} = \frac{\partial L_{\text{br}}}{\partial \dot{\psi}} \dot{\psi}^2 < 0.$$  \hspace{1cm} (3.4)

Thus, even though the bulk solution is static, the matter on the brane is not — though it may still be stationary. The second boundary condition in (2.11) gives $(\lambda + \rho_{br}) > (\langle 0)$ for $k > (\langle 0$, which can be easily satisfied, irrespective of $\rho_{br}$, by introducing a suitably large positive (negative) value for $\lambda$; this is effectively the brane tension.

To get an order of magnitude for the required ghost matter density, use the bound on $A$ above (for the case $k > 0$) to get

$$\rho_{br} \lesssim m^4 \left(\frac{M_s}{k}\right)^3 \left(\ln \frac{k}{m}\right)^{-1} \sim m^4,$$  \hspace{1cm} (3.5)

where $M_s = \kappa^{-3/2}$ is the fundamental 5D Planck scale; $M_s \sim 10^{19}$ GeV. Thus, even for a tiny value of $|A| \sim 10^{-62}$ we require a relatively large density $\rho_{br} \sim (1 \text{ TeV})^4$. For cosmological constant-like values of $\rho_{br} \sim (10^{-11} \text{ GeV})^4$, one obtains an extremely small value for $A$: $|A| \sim 10^{-119}$. Note that such a small value still generates a black-wall solution.

To summarise, the bulk solution for $f$ is fixed by equation (3.2) and allows $f$ to pass through zero, and hence form an event horizon, if and only if $A < 0$. In the soft-wall set-up, one must patch together two spacetimes with a singularity (or have reflective boundary conditions), and the resulting Israel junction conditions can only be satisfied when ghost matter is present on the central brane. It is not clear if ghosts, or phantoms [27], can be accommodated in a theory without instabilities, and without violating cosmological (and other) constraints [28].

Is there anyway around this conclusion? One could try to go from 4D Minkowski to some cosmological time-dependent spacetime. But this will not affect the boundary conditions, as they only involve discontinuities across the brane [that is, in the $y$-coordinate; see equation (1.2)]. Adding extra bulk or brane matter will also not change the situation; this can immediately be seen from the fact that the boundary conditions are phrased in terms of total brane energy and pressure, and only depend on the statement that $A < 0$. Note in this respect that the solution (3.2) depends purely on the metric, with the bulk matter entering only indirectly. From this solution for $f$ we can see that there is no way to have $f' = 0$ at some location and then have $f' \neq 0$ at some other location. Similarly we cannot have $f'$ change sign in the bulk.

We have so far assumed a $Z_2$ symmetry across the brane, but this also is not an essential ingredient. As long as there are discontinuities along the extra dimension, boundary conditions exists that can only be satisfied by violating the null energy condition. An analogy can be made here with the warp factor in a compact RS set up. The relevant equation is $\sigma'' = \kappa^2 \phi'^2 + \delta (y - y_{br}) \lambda$, which shows that positive energy matter can only bend $\sigma$ upwards. To construct a compact space, $\sigma$ must bend down at some point in order
to repeat the solution. This requires a negative tension brane, $\lambda < 0$, at the IR fixed point. Similarly, a non-trivial $f$ must always bend either up or down ($A > 0$ or $A < 0$ respectively), and the direction of bending can only be changed by matter with $\rho + p < 0$.

The above concerns static bulk solutions. It may still be possible that there exist stationary or long-lived black-wall solutions. For example, consider placing two Schwarzschild black holes in the same spacetime. This cannot be done with a static metric unless there is some exotic matter between the black holes which shield their gravitational effects from one another. One solution is to put them in orbit around each other. Or, if they are far enough apart, they will only interact very weakly and constitute a very long-lived solution. Either way, one requires time dependence in the metric to find physical solutions. Similarly, two black walls will gravitate unless one arranges the precise form of ghost matter (which behaves like anti-gravity) to shield them from one another.

In analogy with the Schwarzschild black holes, it may be possible to construct a black-wall solution that does not require ghosts, but instead uses a more general, time-dependent metric. The assumption here is that any useful time-dependent solution closely resembles the static black-wall configuration. Finding such a solution is difficult due to the non-linear nature of the equations, and is beyond the scope of this paper. But it would be interesting to see whether such a solution can be constructed, and whether it has a cosmologically long decay time, making it phenomenologically viable. Besides the Planck scale, the electroweak scale is the only other scale in the problem, and one might naively think that it sets the decay rate of the system, leading to rapidly evolving black walls.

4. Perturbations around the black-wall background

In this section we develop the equations governing the scalar perturbations in the black-wall background. Since in the presence of a black-wall four dimensional Lorentz symmetry is broken, this is a non-trivial generalisation of the soft-wall perturbations discussed in Section 2.3. We do not concern ourselves here with the tensor perturbation $h_{ij}$. In the transverse traceless gauge these degrees of freedom appear only with the spatial coordinates $\vec{x}$ and the introduction of the black-wall metric factor $f$ will not alter the components of $h_{ij}$. Nevertheless, the spin-2 spectrum will be modified, but for small $A$ we expect it to be a minimal change, and the change should follow qualitatively what happens in the spin-0 sector.

If one is willing to admit ghost matter in the theory then the black-wall solution is valid and the singularity is shielded by a horizon. In this case the classical stability of such a configuration as a solution to Einstein’s equations must be checked, which requires looking at perturbations of the set up. As per the discussion following (2.11), since there are no free integration constants in the solution, there is no zero-mode in the spectrum. If all KK modes have positive mass, the solution is classically stable. In addition to checking the stability, it is also interesting to see whether the spectrum of KK modes is qualitatively the same as in the soft-wall background. If so, the soft-wall solution to the hierarchy problem straightforwardly carries over to the black-wall case.
Consider the metric
\[ ds^2 = -n^2(t, y)dt^2 + a^2(t, y)dx^2 + b^2(t, y)dy^2. \] (4.1)

For comparison with our previous ansatz (2.1): \( n^2 = f e^{-2\sigma} \), \( a^2 = e^{-2\sigma} \) and \( b^2 = 1/f \). Using the same action as before, Einstein’s equations yield the Israel junction condition
\[ \left[ \frac{n'}{n} - \frac{a'}{a} \right]_{y_{br}} = 3\kappa^2 b(p + p), \] (4.2)
where the jump operator is \([X]_y = \lim_{\epsilon \to 0}[X(y + \epsilon) - X(y - \epsilon)]\). Fix the gauge by choosing \( n(0) = a(0) = 1 \). If \( n'(0) < a'(0) \) (which is what we want for a black-wall solution) then we must have ghost matter. If we instead choose \( n'(0) = a'(0) \) (or \( n'(0) > a'(0) \)) then we do not need ghosts, but instead need the time-dependence in the metric factors to turn \( n \) over in the bulk so that it passes through zero at some horizon value \( y_h \). Determining whether or not this is possible involves solving Einstein’s equations with the general metric ansatz (4.1), something which we do not attempt here. Instead, we assume ghost matter and proceed to compute the KK spectrum.

The scalar perturbations to the metric (4.1) are [29, 30]
\[ g_{AB} = \begin{pmatrix} -n^2(1 - 2F) & naB|_i & nbJ \\ naB|_i & a^2[(1 - 2G)\delta_{ij} + 2E|_{ij}] & abB_{y|i} \\ nbJ & abB_{y|i} & b^2(1 + 2H) \end{pmatrix}. \] (4.3)

A vertical bar denotes a derivative with respect to \( x^i \). Using a gauge transformation we can set \( E = B = B_y = 0 \) (three degrees of freedom by choosing \( \{\delta t, \delta x, \delta y\} \)). With \( f' \neq 0 \), Einstein’s equations require \( J \neq 0 \), and one cannot make a clear identification of a physical mode in the usual sense (that is, a combination of the degrees of freedom that satisfies a Schrödinger-like equation). This is in part due to the fact that separation of variables between 4D and \( y \) does not go through, since different 4D slices at constant \( y \) of the metric (4.1) have different scales for \( t \) and \( \vec{x} \) (although see [23] for a possible way of dealing with this problem).

To proceed we therefore consider long wavelength perturbations such that \( \vec{x} \) derivatives of the perturbations are set to zero, and hence \( g_{AB} \) (4.3) is independent of the 3-spatial coordinates. The gauge freedom is no longer fixed completely. Performing a gauge transformation \( \delta x^M(t, y) \) with \( x^M = \{t, x, y\} \) leaves the metric invariant. This is because \( B, B_y, E \) will change by a \( (t, y) \) dependent function, but since they all appear with spatial derivatives in the metric, the metric remains invariant. We use this gauge freedom to set \( J = 0 \) and \( F = G = H/2 \), which leaves us in the same gauge used for the pure soft-wall case.

Consider then the metric
\[ ds^2 = e^{-2\sigma(z)} \left\{ -f(z)[1 - 2F(t, z)]dt^2 + [1 - 2F(t, z)]dx^2 + f(z)[1 + 4F(t, z)]dz^2 \right\}, \] (4.4)
where the \( z \)-coordinate is chosen (as opposed to the original \( y \) frame) such that the resulting equation of motion for the physical mode is of the Schrödinger form. Further, we perturb
the scalar $\phi(t,z) = \phi(z) + \delta\phi(t,z)$. The equations of motion are solved for

$$\kappa^2 \phi' \delta\phi = F' - 2\sigma' F - \frac{f'}{2f} F,$$

(4.5)

which can be used to eliminate $\delta\phi$. Note that a prime in this section denotes derivative with respect to $z$. Rescaling $F$ via

$$F = \frac{\kappa \phi'}{k} e^{\frac{\sigma}{2} \tilde{F}},$$

(4.6)

the system consists of a first order constraint equation and a Schrödinger equation, which are, respectively,

$$\tilde{F} - \tilde{F}'' + V_F \tilde{F} = 0,$$

(4.7)

and

$$\tilde{F} = F_0 k e^{-\frac{\sigma}{2} \mathcal{F}}.$$

(4.11)

The potential is

$$V_F = \frac{9}{4} \sigma'^2 + \frac{5}{2} \sigma'' - \frac{\sigma'^2}{\phi'} + \frac{2(\phi'')^2}{(\phi')^2} - \frac{\phi'''}{\phi'} + \frac{3\sigma' f'}{f} - \frac{\phi' f'}{\phi f}.$$  

(4.9)

The zeroth order boundary conditions at the brane position are (2.11); at first order we find in addition

$$(\rho_{br} + p_{br}) \tilde{F}(0) = 0.$$  

(4.10)

For $f' = 0$ we recover the soft-wall background, and the above equations yield the correct result: the constraint equation (4.7) is satisfied, $V_F$ is equivalent to the soft-wall form (4.1), and the boundary condition (4.10) is satisfied since $(\rho_{br} + p_{br}) = 0$.

When $f' \neq 0$ we have a black-wall background, with event horizon at $z = z_h$ defined by $f(z_h) = 0$. At first sight it may seem that equation (4.7) requires the solution

$$\tilde{F} = F_0 k e^{-\frac{\sigma}{2} \mathcal{F}}.$$  

(4.11)

The boundary condition (4.10) then forces $F_0 = 0$ hence $\tilde{F} = 0$ everywhere. Thus all long wavelength perturbations vanish. But this conclusion is only valid when $f'/f$ is much larger in magnitude than the perturbation $F$.

The general solution for $f$ is given by equation (3.2) and, as previously discussed, $A$ should be kept very small so that the horizon is as close as possible to the singularity. Then it should be that the generic features of the soft-wall (for example the KK spectra) are retained. For perturbations that have magnitude of order, or larger than, $A$, the factor $f'/f = Ae^{3\sigma}$ is now perturbatively small in the region leading up to the horizon. Since the derivation of the perturbation equations for $\tilde{F}$ are only valid to first order, we should therefore not include higher order terms such as $(f'/kf)\tilde{F}$. Furthermore, the ghost matter contribution $(\kappa^2/k)(\rho_{br} + p_{br})$ is also order $A$ and thus perturbatively small.

This means, for a given small $A$, there are three regions in $z$ to consider. The first is where the black-wall contributes only perturbatively, so that $Ae^{3\sigma(z)} \lesssim \tilde{F}$. The second
region is where the black-wall background starts to dominate the solution, and the third region has the black-wall completely washing out the perturbation $\tilde{F}$, defined by $A e^{3\sigma(z)} \gg \tilde{F}$. Note that these regions always exist, no matter how small $A$ is.

In the first region equation (4.7) is second order (or greater) in $\tilde{F}$ and so is identically zero to first order in perturbations (the order to which we are working). Similarly, the boundary condition (4.10) is second order and so provides no constraints. In the potential $V_F$ the last two terms contain the factor $f'/f$ and are subdominant compared with the other terms, hence should be discarded. In this first region, the system therefore reduces to the usual soft-wall case. This is as expected since $f \sim 1$.

The second region is a transition from normal soft-wall behaviour to the dominance of the black-wall horizon, and serves simply to provide a matching of the effective Schrödinger equations valid at the two extremes. The KK spectrum is not expected to depend critically on the middle region.

In $z$ coordinates, the horizon is approached as $z \to \infty$ (it must be infinite, otherwise the black-wall is not black). Thus the third region has the domain $z \in [z_3, \infty)$ for some large but finite $z_3$. For $z$ in this domain, the asymptotic behaviour of the background functions is found to be

\begin{align*}
  f(z) &\to e^{-2m} e^{-2lz} \left(1 + \frac{n}{l} e^{-2lz} \right), \\
  \sigma(z) &\to \frac{1}{3} \ln \left(\frac{-2l}{Ak} \right) + \frac{n}{3l} e^{-2lz}, \\
  \phi(z) &\to \phi_H + \frac{n}{3l\kappa} e^{-2lz},
\end{align*}

where $l, m, n$ are free constants which parameterise the asymptotic behaviour. The horizon value of the scalar, $\phi_H$, is defined by

\begin{align*}
  V(\phi_H) = \frac{2l}{3\kappa^2} e^{2m} \left(\frac{-2l}{Ak} \right)^\frac{7}{4}.
\end{align*}

Given this behaviour for the background we can determine the allowed solutions for the perturbation $\tilde{F}$. The boundary condition at the horizon is $F(y_h) = 0$, which is satisfied for finite $\tilde{F}$ at the horizon due to the $\phi' \sim e^{-2lz}$ factor in equation (4.6).\(^4\) Similarly, one can deduce that $\partial_y F(y_h)$ is finite, $\partial_z F \to 0$, but $\partial_z \tilde{F}$ is again allowed to be finite at the horizon. As for the Schrödinger equation for $\tilde{F}$, the asymptotic form of $V_F$ is

\begin{align*}
  V_F \to 2ln e^{-2lz},
\end{align*}

which goes to zero. Combining this with the above boundary conditions we find that the KK spectrum in region three is simply plane waves. Going back to physical $y$ coordinates, these plane waves become squeezed into a tiny finite region just before the horizon at $y_h$.

To get the full KK spectrum of the black wall we must match region one, the usual soft-wall spectrum, with the plane waves near the horizon. For a KK energy/mass on

\(^4\) $F(y_h) = 0$ only requires $\tilde{F}$ to diverge slower than $e^{-2lz}$. But, as usual, we require $\tilde{F}$ to be finite so that it is plane-wave normalisable at infinity.
resonant with a soft-wall mode, the wavefunction is exponentially small as it enters region three, and the amplitude of the plane wave is almost zero. For off-resonant KK energies, the plane wave has sizeable amplitude and after the wavefunction is properly normalised (which happens in the $z$ coordinate) it has almost zero amplitude in the first region. Thus the black-wall KK spectrum is essentially the same as the soft-wall spectrum, but with an overlay of continuum modes starting from zero energy which are strongly localised toward the horizon. Such behaviour is reminiscent of RS2 domain-wall models [31].

5. Unitary boundary conditions

As seen in Section 3, shielding the singularity by a horizon requires ghost matter. But such a construction may not be necessary. We shall argue in this section that the naked singularity obeys unitary boundary conditions which assure no conserved charges can leak away [20, 21, 22]. Consequently, unitarity is conserved in the theory and a sensible bulk physics can be defined. In a way, the singularity is not “visible”, and thus no shielding by a horizon is needed.

The soft-wall singularity is located at a finite coordinate distance in the phenomenologically interesting parameter range $\nu \geq 1$. Naively one might think this is problematic, as light can travel to it in a finite time. The time it takes light to travel to the boundary follows from the metric (2.1):

$$\Delta t = \int_{0}^{y_s} \frac{e^\sigma}{f} \, dy = \begin{cases} \text{(finite)} + \int_{0}^{\infty} (\kappa \phi)^{-\beta(1+1/\nu^2)} \, e^{-\nu \kappa \phi (1-1/\nu^2)} \, d\phi, & (f = 1), \\ \infty, & (f \neq 1). \end{cases} \quad (5.1)$$

For the soft-wall case with $f = 1$, we have split the integral into the bulk region and the region near the singularity. The bulk always gives a finite contribution, and we can thus focus on the singularity region only. As $y \to y_s$ the field diverges $\phi \to \infty$ and the superpotential is of the asymptotic form (2.15). The integral is finite for $\nu = 1$ and $\beta > 1/2$, and also for $\nu > 1$ with arbitrary $\beta$. For the black-wall case, $f \neq 1$, we must have an infinite time-interval, otherwise light can escape from the horizon, meaning it is not actually black. Thus $e^\sigma/f$ must diverge at least as fast as $(y_h - y)^{-1}$ at the horizon.

The key point, however, is that there are no modes in the spectrum that can travel freely along the bulk direction. The KK spectrum of the soft-wall solution is discrete (for $\nu = 1$ continuous, but the same reasoning applies), with all bound states satisfying the boundary conditions (2.22, 2.23) at the singularity. All perturbative modes have an extra dimensional profile that dies off rapidly at the singularity, and consequently these modes cannot really probe the boundary. As we shall see, this behaviour means that no 4D energy or momentum can leak into the singularity.

The energy-momentum current density is $J^M = T^{MN} \xi_N^{(\mu)}$ with $T^{MN}$ the energy-momentum tensor (2.4), and $\xi_N^{(\mu)} = \delta_N^{\mu}$ the Killing vector generating 4D translations. The labelling is such that the 4D index $\mu$ corresponds to current of energy ($\mu = 0$) and spatial momentum ($\mu = 1, 2, 3$) respectively; the 5D index $M$ labels the direction of the current. The current satisfies the conservation law $\partial_M (\sqrt{\mathcal{g}} J^M) / \sqrt{\mathcal{g}} = 0$, which expresses conservation of 4D energy and momentum. To ensure that energy and momentum remain...
conserved in the presence of a singularity, we demand that the flux into the singularity vanishes

$$\sqrt{g}J^y|_{y_s} = 0. \quad (5.2)$$

For the tensor perturbations we can use the result of \[21, 22\]

$$\sqrt{g}J^y|_{y_s} = \sqrt{g}g^{uy} \frac{1}{2} h'_{kl} \partial_i h_{kl}|_{y_s} \propto e^{-4\sigma} h'_{s}|_{y_s} = 0 \quad (5.3)$$

where in the second step we used separation of variables (2.20). This vanishes exactly for all KK modes because of the boundary condition (2.22). Expanding the energy-momentum tensor in the scalar perturbations defined in Section 2.3 gives

$$\sqrt{g}J^y|_{y_s} = e^{-4\sigma} \left[ \phi' + (\delta \phi' - 10 F \phi') \right] \partial_i \delta \phi|_{y_s} \propto e^{-4\sigma} \left[ \phi' \delta \phi + \left( \delta \phi \delta \phi' - \frac{10}{\kappa^2} (FF' - 2\sigma' F^2) \right) \right]|_{y_s} \quad (5.4)$$

where in the second line we again used separation of variables (2.27). Moreover, we used that near the singularity \(\phi' = \sigma'\), and also the constraint equation relating \(F\) and \(\delta \phi\) (2.19). The terms first and second order in the perturbations all vanish independently due to the boundary conditions (2.23), and hence this flux vanishes at the singularity.

We conclude that, at least in the perturbative regime, no 4D energy or momentum leaks into the singularity, and there should be no problems with loss of unitarity.

### 6. Conclusions

We have carried out a critical analysis of the singularity in soft-wall models. The general idea is to make sure that this singularity is not “visible”, to protect low energy physics from uncontrollable phenomena such as non-unitarity, loss of conserved quantities, or unknown quantum gravity.

In the soft-wall literature to date it has been assumed that a black-wall horizon is needed to hide the singularity. With this in mind, we first set-up the general equations for a black-wall. Working in the fake supergravity formalism, we assumed a superpotential of the asymptotic form

$$\lim_{\phi \to \infty} W \propto c(\kappa \phi)^\beta e^{\nu \kappa \phi}, \quad (6.1)$$

A phenomenologically viable soft-wall solution solving the hierarchy problem can be constructed for \(c > 0\) and \(1 \leq \nu < 2\). In the bulk, this solution can be extended to a black-wall solution with a horizon shielding the singularity. The KK spectrum of such a set-up is qualitatively similar to the soft-wall case, as discussed in Section 4. However, matching the Israel junction conditions at the origin requires ghost matter on the UV brane. Even though the density of this ghost matter can be made tiny, the set-up is rather unsatisfying. To avoid ghost matter one could try to look for a time-dependent but cosmologically long-lived black-wall solutions, although the naive expectation is a decay time of order the electroweak scale.
However, as we further pointed out in this paper, a horizon is not in fact necessary as the singularity is not “visible” to any of the states in the spectrum. Indeed, the extra-dimensional profile of all KK modes decay rapidly near the singularity; this is needed to obtain a vanishing 4D cosmological constant, and thus a phenomenologically viable model. This decay also ensures that the system satisfies unitary boundary conditions. No 4D energy or any other conserved quantity can leak into the singularity, and consequently a well-defined unitary theory can be constructed.

Acknowledgments

This research was supported by the Netherlands Foundation for Fundamental Research of Matter (FOM) and the Netherlands Organisation for Scientific Research (NWO).

References


