The 16th Moment of the Non–Singlet Structure Functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ to $O(\alpha_3^3)$

J. Blümlein$^a$ and J.A.M. Vermaseren$^b$

$^a$DESY Zeuthen
Platanenallee 6, D–15738 Zeuthen, Germany

$^b$NIKHEF Theory Group
Kruislaan 409, 1098 SJ Amsterdam, The Netherlands

Abstract
We present the results of an analytic next–to–next–to leading order QCD calculation of the non–singlet anomalous dimension $\gamma_{NS}^2(N)$ and the coefficient functions $C_{2,L}(N)$ associated to the deeply inelastic structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ for the Mellin moment $N = 16$. Comparisons are made with results in the literature.
1 Introduction

One of the most precise methods to determine the strong coupling constant \( \alpha_s(Q^2) \) is to measure it from the scaling violations of the structure functions in deeply inelastic scattering. The present level of experimental accuracy is of order \( \delta \alpha_s(M_Z^2) \approx \pm 0.001 \) [1]. Since the theory errors due to factorization and renormalization scale uncertainties are of \( \delta \alpha_s(M_Z^2)_{sc} \approx \pm 0.005 \) [2] in next–to–leading order the knowledge of the 3–loop anomalous dimensions is required to reduce this error to the level reached by experiment.\(^1\)

The one–loop anomalous dimensions were calculated in [5] and the one–loop coefficient functions were obtained in a series of subsequent calculations, see Ref. [6] for a summary of the results. Later the two–loop anomalous dimensions [7] and the two–loop coefficient functions [8] were evaluated applying different methods. The calculation of the anomalous dimensions and Wilson coefficients in three loop order is a very difficult task. The first calculations were performed for a series of sum rules and the individual moments for \( N = 2, 4, 6, 8 \) [9] in the non–singlet and singlet case as well as for \( N = 10 \) in the non–singlet case using the symbolic manipulation program FORM [10]. With growing computational power the moments \( N = 10, 12 \) in the singlet case and \( N = 12, 14 \) in the non–singlet case as well as the moments \( N = 3, 5, 7, 9, 11, 13 \) associated to the structure function \( x F_3(x, Q^2) \) could be calculated [11]. Very recently, the 3–loop non–singlet [12] and singlet [13] anomalous dimensions were calculated for deep–inelastic scattering off unpolarized nucleons.

In this paper we calculate the 16th moment of the non–singlet splitting function, \( \gamma_{NS}^{(16),+}(N, Q^2) \), and the Wilson coefficients of the structure functions \( F_2(x, Q^2) \) and \( F_L(x, Q^2) \), \( C_{NS} (N, Q^2) \), for the case of pure photon exchange to 3–loop order using the MINCER formalism [14]. This calculation was started before complete 3–loop results became available and serves as an independent check of the result for the non–singlet anomalous dimension given in Ref. [12] and predicts the 16th moment for the non–singlet coefficient function, [15, 16]. The paper is organized as follows. In section 2 we give an outline of the basic formalism. The renormalization of the hadronic matrix element is described in section 3. The results of the calculation are given in section 4 and section 5 contains the conclusions.

2 Basic formalism

The hadronic tensor for deeply inelastic scattering in the case of pure photon exchange is given by :

\[
W_{\mu\nu}(x, Q^2) = \frac{1}{2\pi} \text{Im} \ T_{\mu\nu}(p, q) \\
= \frac{1}{4\pi} \int d^4x e^{iqx} \langle P|J_\mu(x) J_\nu(0)|P\rangle \\
= - \left[ g_{\mu\nu} + \frac{2x}{q^2} (p_\mu q_\nu + p_\nu q_\mu) + \frac{4x^2}{q^2} p_\mu p_\nu \right] \frac{1}{2x} F_2(x, Q^2) \\
+ \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{2x} F_L(x, Q^2) .
\]

(1)

Here \( x = Q^2/(2p.q) \) denotes the Bjorken variable, \( P \) is the nucleon momentum, \( q \) denotes the space–like momentum transfer from the leptons to the nucleon with \( q^2 = -Q^2 \) and the Compton

\(^1\text{Cf. Ref. [3, 4] for recent comparisons of the measured value of } \alpha_s(M_Z^2).\)
amplitude reads
\[
T_{\mu\nu}(p, q) = i \int d^4z e^{iqz} \langle P | T [J_\mu(z)J_\nu(0)] | P \rangle.
\]
We now consider the light–cone expansion of the (forward) Compton amplitude [17]
\[
i \int d^4z e^{iqz} T [J_{\nu_1}(z)J_{\nu_2}(0)]_{\text{NS}} = \sum_N \left( \frac{1}{Q^2} \right)^N \left[ \left( g_{\nu_1\nu_2} - \frac{q_{\nu_1}q_{\nu_2}}{q^2} \right) q_{\mu_1} q_{\mu_2} C_{L,N}^{\text{NS}} \left( \frac{Q^2}{\mu^2}, a_s \right) - \left( g_{\nu_1\mu_1} g_{\nu_2\mu_2} q^2 - g_{\nu_1\mu_1} q_{\nu_2} q_{\mu_2} - g_{\nu_2\mu_2} q_{\nu_1} q_{\mu_1} + g_{\nu_1\nu_2} q_{\mu_1} q_{\mu_2} \right) \times C_{2,N}^{\text{NS}} \left( \frac{Q^2}{\mu^2}, a_s \right) \right] q_{\mu_3} \ldots q_{\mu_N} O^{\text{NS}, \{\mu_1, \ldots, \mu_N\}}(0) + \text{higher twists}.
\]
The twist–2 contributions to the local non–singlet operators \( O^{\text{NS}}(0) \) are linear combinations out of the operators
\[
O^{\alpha, \mu_1, \ldots, \mu_N} = \bar{\psi} \lambda^{\alpha \gamma} \{ \mu_1 D^\mu_2 \ldots D^\mu_N \} \psi, \quad \alpha = 1, 2, \ldots, (N_F^2 - 1),
\]
\( \mu^2 \) denotes the factorization scale and \( a_s = \alpha_s/(4\pi) = g^2/(4\pi)^2 \), with \( g \) the strong coupling constant. The functions \( C_{(L,2),N}^{\alpha} \) are the Wilson coefficients associated to the moment index \( N \). The Mellin moments of the structure functions \( F_{2,l}^{\text{NS}}(x, Q^2) \) are given by
\[
M_{k,N-2} = \int_0^1 dx x^{N-2} F_{k,N}^{\text{NS}}(x, Q^2) = C_{k,N}^{\text{NS}} \left( \frac{Q^2}{\mu^2}, a_s \right) A_{\text{nucl},N}^{\text{NS}} \cdot k = 2, L.
\]
For pure electromagnetic interactions \( T_{\mu\nu} \) is even in \( x \). The corresponding crossing relation has the consequence that only the even moments contribute to (3). The nucleon matrix elements are
\[
\langle P | O^{\text{NS}, \{\mu_1, \ldots, \mu_N\}} | P \rangle = P^{\{\mu_1 \ldots \mu_N\}} A_{\text{nucl},N}^{\text{NS}} \left( \frac{P^2}{\mu^2} \right).
\]
The scale–dependence of the non–singlet coefficient function is governed by the renormalization group :
\[
\left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} - \gamma_N^{\text{NS}} \right] C_{i,N}^{\text{NS}} \left( \frac{Q^2}{\mu^2}, a_s \right) = 0, \quad i = 2, L.
\]
The \( \beta \)-function rules the scale dependence of the coupling constant \( a_s(\mu^2) \)
\[
\mu^2 \frac{\partial a_s(\mu^2)}{\partial \mu^2} = \beta(a_s) = - \sum_{l=0}^{\infty} \beta_l a_s^{l+2}(\mu^2)
\]
on the renormalization scale \( \mu = \mu_R \), where \( \beta_l \) are the expansion coefficients. To 3–loop order the constants \( \beta_l \) for \( SU(N) \) [18]
\[
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F N_F \quad (9)
\]
\[
\beta_1 = \frac{34}{3} C_A^2 - 4 C_T T_F N_F - \frac{20}{3} C_A T_F N_F \quad (10)
\]
\[
\beta_2 = \frac{2857}{54} C_A^3 + 2 C_T^2 T_F N_F - \frac{205}{9} C_F C_A T_F N_F - \frac{1415}{27} C_A T_F^2 N_F + \frac{44}{9} C_F T_F^2 N_F^2 + \frac{158}{27} C_A T_F^2 N_F^2 \quad (11)
\]
contribute, with \( C_A = N_c, C_F = (N_c^2 - 1)/(2N_c) \), and \( T_F = 1/2 \) and \( N_c = 3 \). \( N_F \) denotes the number of flavors. In the QCD–improved parton model the forward Compton amplitude (2), valid for nucleon states \( |P⟩ \), reduces to that for photon–quark scattering, since only one initial state quark participates in the scattering process, with \( p \) the quark 4–momentum,

\[
T_{\mu\nu}^{\gamma\gamma} = i \int d^4z \, e^{i q z} \langle p|T[J_\mu(z)J_\nu(0)]|p\rangle .
\] (12)

One applies the projector

\[
P_N = \left[ q^{\{\mu_1...\mu_N\}} N! \frac{\partial^N}{\partial p^{\mu_1}...\partial p^{\mu_N}} \right]_{p^2=0}
\] \hspace{1cm} (13)

onto the Mellin moments \( N \) and the Lorentz projectors

\[
P_{\mu\nu}^L = -\frac{q^2}{(p.q)^2} p^\nu p^\mu
\] \hspace{1cm} (14)

\[
P_2^{\mu\nu} = -\left( \frac{3 - 2\varepsilon}{2(1-\varepsilon)} \frac{q^2}{(p.q)^2} p^\nu p^\mu + \frac{1}{2(1-\varepsilon)} g^{\mu\nu} \right),
\] \hspace{1cm} (15)

valid in \( D = 4 - 2\varepsilon \) dimensions. These projections lead to the following moments of the Compton amplitude

\[
T_{k,N}^{\gamma\gamma} \left( \frac{Q^2}{\mu^2}, a_s, \varepsilon \right) = C_{NS}^{k,N} \left( \frac{Q^2}{\mu^2}, a_s, \varepsilon \right) Z_N^{NS} \left( a_s, \frac{1}{\varepsilon} \right) A_{q,N}^{NS,tree}(\varepsilon), \quad k = 2, L .
\] \hspace{1cm} (16)

3 Renormalization

The coefficient functions \( C \) and \( Z \)–factors in (16) obey the following representations:

\[
C(a_0, \varepsilon) = \delta + a_0 \left( C_{10} + \varepsilon C_{11} + \varepsilon^2 C_{12} \right) + a_0^2 \left( C_{20} + \varepsilon C_{21} + a_0^3 C_{30} + O(a_0^4) \right),
\] \hspace{1cm} (17)

\[
Z(a, \varepsilon) = 1 + a_0 \left( Z_{11} + \varepsilon Z_{22} + \varepsilon^2 Z_{21} \right) + a_0^2 \left( Z_{32} + Z_{33} + Z_{31} + \varepsilon^2 \right) + O(a_0^4),
\] \hspace{1cm} (18)

with \( \delta = 1 \) for \( C_2 \) and \( \delta = 0 \) for \( C_L \). Here, \( a_0 \) denotes the bare coupling constant, which is related to the running coupling by

\[
a_0 = a - \frac{\beta_0}{\varepsilon} a^2 + \left( \frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2 \varepsilon} \right) a^3 + O(a^4),
\] \hspace{1cm} (19)

cf. (8). We identified the scales \( \mu^2 = Q^2 \). Yet a separation between the contributions to the anomalous dimension and the coefficient functions is possible as outlined in the following. The anomalous dimension and the Wilson coefficient are

\[
\gamma(a, N) = \sum_{k=0}^{\infty} a^{k+1} \gamma_k
\] \hspace{1cm} (20)

\[
c(a, N) = \delta + \sum_{k=1}^{\infty} a^k c_{k0}.
\] \hspace{1cm} (21)
The respective coefficients $\gamma_k$ and $c_{k0}$ are determined as follows. We denote by $C(\xi)$ the coefficient of type $\xi$ in $T(16)$. Identifying the corresponding powers in $\varepsilon$ one obtains to $O(a^3)$:

$$\gamma_0 = C\left(\frac{a}{\varepsilon}\right)$$

$$\gamma_1 = 2\left[C\left(\frac{a^2}{\varepsilon}\right) - \gamma_0 c_{10}\right]$$

$$\gamma_2 = 3\left[C\left(\frac{a^3}{\varepsilon}\right) - c_{10} Z_{21} - C_{11} Z_{22} - \gamma_0 c_{20}\right]$$

$$c_{10} = C\left(a \varepsilon^0\right)$$

$$c_{20} = C\left(a^2 \varepsilon^0\right) - \gamma_0 C_{11}$$

$$c_{30} = C\left(a^3 \varepsilon^0\right) - C_{11} Z_{21} - C_{12} Z_{22} - \gamma_0 C_{21},$$

with

$$Z_{21} = C\left(\frac{a^2}{\varepsilon}\right) - \gamma_0 c_{10}$$

$$Z_{22} = C\left(\frac{a^2}{\varepsilon^2}\right)$$

$$C_{11} = C\left(a \varepsilon\right)$$

$$C_{12} = C\left(a^2 \varepsilon\right)$$

$$C_{21} = C\left(a^2 \varepsilon\right) - \gamma_0 C_{12}. $$

Furthermore the relations

$$Z_{31} + Z_{11} c_{20} + Z_{22} C_{11} + Z_{21} c_{10} = 0$$

$$Z_{32} + c_{10} Z_{22} = 0$$

hold. The above relations yield the anomalous dimensions and the Wilson coefficients to $O(a^3)$.

### 4 Results

The 16th moment of the 3–loop non–singlet anomalous dimension $\gamma_{NS}^{16, +}$, which describes the evolution of the combination

$$q_{NS}^{+}(x, Q^2) = \left(q_j(x, Q^2) + \overline{q}_j(x, Q^2)\right) - \left(q_k(x, Q^2) + \overline{q}_k(x, Q^2)\right)$$

of quark densities, and the coefficient functions $C_{NS}^{NS, L, 16}$ were calculated using the MINCER algorithm [14]. The calculation was performed majorly using two dual–processor 32bit PC’s (3 and 2.6 GHz). Several diagrams were run on an Opteron 64bit PC. Due to the large disk–space required by a series of diagrams a 4.2 Tbyte raid system was linked to the two 32bit PC’s to store intermediary results. In the calculation the moment (16) is determined in terms of an $\varepsilon$–expansion for $a_s = a_0$. The anomalous dimension and moments of the coefficient functions are determined as described in the previous section.

Up to 3–loop order 388 diagrams contribute effectively, if symmetry relations between diagrams are used. The calculation using the above system needed about 560 CPU days. The CPU
time distribution over the individual diagrams is shown in Figure 1. Several diagrams required computation times of a month, in one case of $O(60)$ days. Despite of these long computation times the system ran completely stable. The use of a parallel facility running FORM [19] would be highly desirable for computations of similar size in the future.

The 16th moments of the non–singlet anomalous dimension and coefficient functions for unpolarized nucleons are:

\[
\gamma_{NS}^{16,(0),+} = \frac{64419601}{6126120} C_F = 14.02075071
\]

\[
\gamma_{NS}^{16,(1),+} = \frac{21546159166129889}{484994628518400} C_F C_A - \frac{3689024452928781382877}{459818557352009856000} C_F^2
\]

\[- \frac{1176525373840303}{112588038763200} C_F N_F
\]

\[= 163.4395247 - 13.93310085 N_F\]

\[
\gamma_{NS}^{16,(2),+} = -\left( \frac{58552930270652300886778705063429867}{345133797061245253431709673280000} - \frac{59290512768143}{1563722760600} \right) C_F^3 
\]

\[+ \left( \frac{167042372808398420787825467}{6488959481351563087872000} + \frac{59290512768143}{3127445521200} \right) C_F C_A 
\]

\[+ \left( - \frac{1229794646000775781127856064477}{303358855731855743580160000} - \frac{59290512768143}{1042481840400} \right) C_F^2 C_A 
\]

\[+ \left( \frac{71543599677985155342551355451}{93896788685509820634624000} + \frac{64419601}{765765} \right) C_F^2 N_F 
\]

\[+ \left( \frac{15018421824060388659436559}{579371382263532418560000} - \frac{64419601}{765765} \right) C_F C_A N_F 
\]

\[+ \frac{5559466349834573157251}{20691835080844352000} C_F N_F^2 
\]

\[= 2849.5632736921273714 - 463.86001156801831223 N_F 
\]

\[-3.5823897546153993659 N_F^2 .\]

\[
C_{NS,16}^{2}(a_s) = 1 + \frac{4047739719}{190590400} C_F a_s 
\]

\[+ \left[ \frac{44426674163044428879366970127}{3219318469217479564615680000} + \frac{24439538}{255255} \right] C_F^2 
\]

\[+ \left( \frac{17918308408498294222783087}{59422705873182812160000} - \frac{113298677}{1021020} \right) C_F C_A 
\]

\[- \frac{143568372761907472111177}{2758911344112059136000} C_F N_F \right] a_s^2 
\]

\[+ \left[ \frac{303681339759959725084677293842505976559161689}{8034458016040775933421647863403347968000000} 
\]

\[+ 1 + \frac{4047739719}{190590400} C_F a_s 
\]

\[+ \left[ \frac{44426674163044428879366970127}{3219318469217479564615680000} + \frac{24439538}{255255} \right] C_F^2 
\]

\[+ \left( \frac{17918308408498294222783087}{59422705873182812160000} - \frac{113298677}{1021020} \right) C_F C_A 
\]

\[- \frac{143568372761907472111177}{2758911344112059136000} C_F N_F \right] a_s^2 
\]

\[+ \left[ \frac{303681339759959725084677293842505976559161689}{8034458016040775933421647863403347968000000} 
\]
\[ C_{L,\text{NS}}^{16}(a_s) = \frac{4}{17} C_F a_s \]
\[ + \left[ \left( \frac{29393927457809}{44659922042736} + \frac{96}{17} \zeta_3 \right) C_F^2 + \left( \frac{55969347000169}{8209544493150} - \frac{48}{17} \zeta_3 \right) C_F C_A \right] a_s^2 \]
\[ - \frac{39366889}{39054015} C_F N_F \]
\[ + \left[ \left( -\frac{750828182127677148126447290110919}{1364789823543885242924259840000} - \frac{196256899828170631}{133698296031300} \zeta_3 \right) \right] a_s^3 \]
\[ + \left( \frac{39360}{17} \zeta_5 \right) C_F^3 \]
\[ + \left( \frac{296045501010133565322039207159677}{93662046713796046083037400000} + \frac{2253147763389895}{1188429298056} \zeta_3 \right) \]
\[ + \left( \frac{1494341926940450865387403}{5956740402061276800} \zeta_3 + \frac{59290512768143}{3127445521200} \zeta_4 - \frac{27643576}{21879} \zeta_5 \right) C_F^3 \]
\[ + \left( \frac{26286537783475726558800935515033190333}{566468058525038486710210437120000000} - \frac{15355050469171482313}{4991403051835200} \zeta_3 \right) \]
\[ + \frac{59290512768143}{625491042400} \zeta_4 + \frac{47187263}{51051} \zeta_5 \right) C_F C_A^2 \]
\[ + \left( \frac{7750026627118768752845091760890051465242741}{165250062032924227343102588716646400000} \right. \]
\[ - \frac{2849482004138921491531}{6741167121672984000} \zeta_3 \]
\[ - \frac{59290512768143}{2084963680800} \zeta_4 + \frac{983963}{21879} \zeta_5 \right) C_F N_F \]
\[ + \left( \frac{722738493599967031321318789884999}{76056398835262954714045440000} + \frac{64419601}{20675665} \zeta_3 \right) C_F N_F^2 \]
\[ + \left( \frac{705894258514655486993}{3248429831350740000} + \frac{38404365803}{1533061530} \zeta_3 - \frac{14560}{51} \zeta_5 \right) \frac{d_{abc}^2 N_F}{N_c} a_s^3 \]
\[ = 1 + 28.31719904 a_s + (1122.549565 - 69.38406971 N_F) a_s^2 \]
\[ + (50309.36422 - 6651.875513 N_F + 131.6959033 N_F^2 - 216.0757466 \langle e \rangle N_F) a_s^3 \]
\[- \frac{40160}{17} \zeta_5 \bigg) C_A C_F^2 \\
+ \left( \frac{1460792499427100139493280371}{8256042197255336964480000} - \frac{1634895686765221}{2673965920626} \zeta_3 \right)
+ \frac{10240}{17} \zeta_5 \bigg) C_A^2 C_F \\
+ \left( \frac{3529137346321170453160463}{136796020812222932160000} - \frac{44651224}{765765} \zeta_3 \right) C_F^2 N_F \\
+ \left( - \frac{4495805144658565385501573689}{57792295380787358751360000} + \frac{43594330672}{1249937325} \zeta_3 \right) C_A C_F N_F \\
+ \frac{895967716232}{209134250325} C_F N_F^2 \\
+ \langle e \rangle \left( \frac{1798450729620489619601}{18272417801347710000} - \frac{28854977192}{547521975} \zeta_3 + \frac{2560}{17} \zeta_5 \right) \frac{d_{abc}^2 N_F}{N_c} \bigg] a_s^3 \]

\begin{align}
\langle e \rangle &= \frac{3}{N_F} \sum_{k=1}^{N_F} e_f \\
\text{and } d_{abc}^2 / N_c &= 40/9 \text{ for } SU(3)_c. \text{ These results agree with the complete 3–loop results for the anomalous dimension in [12]. The moments for the } N_F^2 \text{–terms [20] and } N_F \text{–terms [21] were known before. For the coefficient functions the moments agree with the very recent complete results [15] and an upcoming paper [16].} \end{align}

5 Conclusions

We calculated the hitherto unknown 16th moments for the 3–loop non–singlet anomalous dimension $\gamma_{16,+}^{\text{NS}}$ and the non–singlet coefficient functions $C_{2L}(x, Q^2)$ for pure photon exchange. The computation was performed using the MINCER algorithm, which is different from the algorithms used in the recent complete calculations. In view of the rather long CPU time of about 560 days spent for the calculation the reliability of the formula manipulation program FORM has been tested intensely as a by–product. The results agree with recent and upcoming complete results. The computation of the 16th moment provides a non–trivial test of these computations.

Acknowledgment. We would like to thank S. Moch for useful discussions, U. Gensch for support of the project and S. Wiesand, P. Wegner and C. Spiering for their technical support. This paper was supported in part by DFG Sonderforschungsbereich Transregio 9, Computergestützte
Figure 1: Execution time profile for the set of diagrams. Full line: $g_{\mu\nu}$ projection; dash-dotted line: $P_\mu P_\nu$ projection.
References


