Generalized resummation of QCD final-state observables

Andrea Banfi
NIKHEF Theory Group, P.O. Box 4188, 1009 DB Amsterdam, The Netherlands and Dipartimento di Fisica, Università di Milano-Bicocca and INFN, Sezione di Milano, Italy.

Gavin P. Salam
LPTHE, Universities of Paris VI and VII and CNRS UMR 7589, Paris, France.

Giulia Zanderighi
IPPP, Department of Physics, University of Durham, Durham DH1 3LE, UK.

The resummation of logarithmically-enhanced terms to all perturbative orders is a prerequisite for many studies of QCD final-states. Until now such resummations have always been performed by hand, for a single observable at a time. In this letter we present a general ‘master’ resummation formula (and applicability conditions), suitable for a large class of observables. This makes it possible for next-to-leading logarithmic resummations to be carried out automatically given only a computer routine for the observable. To illustrate the method we present the first next-to-leading logarithmic resummed prediction for an event shape in hadronic dijet production.

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QCD is unique among the theories of the standard model in that both strong and weak coupling regimes are relevant to modern collider experiments. This manifests itself most dramatically in hadronic final states of high-energy collisions, whose branching pattern is sensitive to physics spanning the whole range of scales from the (perturbative) hard collision virtuality down to (non-perturbative) hadronic masses. Accordingly final states are a privileged laboratory for QCD studies: perturbative investigations have for example led to many measurements of the strong coupling, \( \alpha_s \), and to tests of the underlying SU(3) group structure of the theory \([1]\); and final-states are also proving to be a rich source of information on the poorly understood relation between perturbative, partonic predictions and the non-perturbative, hadronic degrees of freedom observed in practice \([2,3]\).

Among the most widely studied final-state properties are measures \((v)\) of the extent to which the geometric properties of an event’s energy-momentum flow differ from that of a Born event. Fixed-order perturbative calculations, which involve a small number of additional partons, are suitable for describing large departures from the Born-event energy flow pattern, in which the extra partons are energetic and at large angles. Such configurations are however rare, their likelihood being suppressed by powers of the perturbative coupling.

The most common events are instead those in which the departure from the Born energy-flow pattern is small, \( v \ll 1 \), with any extra partons being soft and/or collinear to the original Born-event partons. This poses a problem for fixed-order studies because each power of the coupling is then accompanied by up to two powers of the large logarithm \( \ln 1/v \), associated with soft and collinear divergences. As a result, the perturbative series involves terms \( (\alpha_s \ln^n 1/v)^n \), and must be resummed to all orders.

Today’s state of the art calculations exploit the fact that for many measures (‘observables’), the dominant all-orders perturbative contribution can be written as an exponential of leading-logarithmic (LL) terms \( \alpha_s \ln^{n+1} 1/v \). Furthermore the next-to-leading logarithmic (NLL) terms, \( \alpha_s^n \ln^n 1/v \), factorize and can be calculated to all orders \([4]\). But to obtain this NLL accuracy one needs a detailed understanding of the observable’s analytical properties and of the corresponding phase-space integrals. Thus it is usual for an entire paper to be dedicated to the resummation, in a single process, of just one or two observables.

The purpose of this letter is to present a novel approach to final-state resummation, based on a ‘master’ formula appropriate for a large class of observables in a range processes, including \( e^+e^- \) to 2 or 3 jets, DIS to 1 or 2 jets, Drell-Yan (or \( \gamma, W^\pm, \text{Higgs}, \ldots \)) plus a jet, and hadronic dijet production. The master formula is accompanied by a set of conditions that must be satisfied by the observable in order for the approach to be valid.

Let us start by examining these conditions. We consider an \( n+1 \)-jet observable, one that measures how the energy flow departs from that of an \( n \)-parton Born event, and resum in the \( n \)-jet limit. In addition to satisfying the basic requirement of infrared and collinear (IRC) safety, the observable should

1. be positive for events with more than \( n \) partons and go smoothly to zero in \( n \)-jet limit, \( i.e. \) as the \( n+1 \)th parton is made soft or collinear to one of the \( n \) Born partons (in our counting, where relevant, \( n \) includes incoming partons/legs labeled \( 1\ldots n_i \));

2. be recursively IRC safe, meaning that, given an ensemble of arbitrarily soft and collinear emissions, the addition of a relatively much softer or more
collinear emission should not significantly alter the value of the observable, the condition required for exponentiation of the leading logarithms.

3. be continuously global — this means that for a single soft emission, the observable’s parametric dependence on the emission’s transverse momentum (with respect to the nearest leg) should be independent of the emission direction. This is perhaps the most restrictive of the conditions.

Furthermore, when considering a Born event with momenta $p_1 \ldots p_n$, into which one inserts an asymptotically soft and collinear emission $k$, the observable’s value (a function $V$ of the final-state momenta) must go as

$$V\{\{\hat{p}\}\}, k) = d_{\ell} \left( \frac{k_{\perp}}{Q} \right)^{a_{\ell}} e^{-b_{\ell} \eta} g_{\ell}(\phi). \tag{1}$$

Here $Q$ is a hard scale of the problem; $\{\hat{p}\}$ represents the Born momenta after recoil from the emission, defined by the index $\ell$ of the Born leg to which it is collinear, by its transverse momentum $k_{\perp}$ and rapidity $\eta$, with respect to that leg, and by an azimuthal angle $\phi$ relative to a Born event plane (where relevant). The coefficients $b_{\ell}$ and $d_{\ell}$ and the function $g_{\ell}(\phi)$ (normalized to be $g_{\ell}(\pi/2) = 1$) may depend on the leg $\ell$, while continuous globalness necessitates $a_1 = \ldots = a_n \equiv a$. Infrared and collinear safety implies $a > 0$ and $b_{\ell} > -a$ (a similar condition has been pointed out in a more restricted case also in 3).

Once an observable has been established as suitable, one can use the following NLL master resummation formula for the probability $\Sigma(v)$ that the observable’s value is less than $v$

$$\ln \Sigma(v) = -\sum_{\ell=1}^{n} C_{\ell} \left[ r_{\ell}(L) + r'_{\ell}(L) \left( \ln d_{\ell} - b_{\ell} \ln \frac{2E_{\ell}}{Q} \right) \right]$$
$$+ B_{\ell} T \left( \frac{L}{a + b_{\ell}} \right) + \sum_{\ell=1}^{n} \ln \left( f_{\ell}(x_{\ell}, v + \eta_{\ell}, \mu_{T}^2) \right)$$
$$+ \ln S(T(L/a)) + \ln \mathcal{F}(C_{r_1}, \ldots, C_{r_n}, \ell), \tag{2}$$

where $L = \ln \frac{1}{v}$, $C_{\ell}$ is the color factor associated with Born leg $\ell$, and $E_{\ell}$ is its energy, $B_{\ell}$ is $-3/4$ for quarks and $-((11C_A - 4T R_{Q} \eta_{\ell})/(12C_A))$ for gluons, $\ln d_{\ell} = \ln d_{\ell} + \int_{0}^{2\pi} d_{\phi} \ln g_{\ell}(\phi)$, and for incoming legs, the $f_{\ell}$ are the appropriate (Born flavor) parton densities. We note that 3 is independent of the frame in which one determines the $d_{\ell}$ and (to NLL accuracy) of the choice of hard scale $Q$.

The functions $r_{\ell}(L)$ contain all the LL (and some NLL) terms and are defined by

$$r_{\ell}(L) = \frac{2}{v_{\ell}^{2} Q^2} \int_{v_{\ell}^{2} Q^2}^{\infty} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{\alpha_s(k_{\perp})}{\pi} \ln \left( \frac{k_{\perp}}{v_{\ell}/a Q} \right)^{a_{\ell}/b_{\ell}} + \frac{Q^2}{v_{\ell}^{2} Q^2} \int_{v_{\ell}^{2} Q^2}^{\infty} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{\alpha_s(k_{\perp})}{\pi} \ln \left( \frac{Q}{k_{\perp}} \right), \tag{3}$$

where $\alpha_s$ runs at two-loop order and is to be taken in the Bremstrahlung scheme 4. Exponentiation guarantees that the LL terms of $r_{\ell}$ are in the class $\alpha_s^{n} L^{n+1}$.

The NLL function $T(L)$ is

$$T(L) = \int_{e^{-2L Q^2}}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{\alpha_s(k_{\perp})}{\pi} \ln \left( \frac{k_{\perp}}{v_{\ell}/a Q} \right)^{a_{\ell}/b_{\ell}} + \frac{Q^2}{k_{\perp}^2} \ln \frac{Q}{k_{\perp}} , \tag{4}$$

and the $r'_{\ell}$ are also relevant only at NLL level.

The process dependence associated with large-angle soft radiation is contained in $S(T(L/a))$, whose form depends on the number of legs:

$$n = 2: \ln S(t) = -t \cdot 2C_F \ln \frac{Q_{gg'}}{Q} ,$$

$$n = 3: \ln S(t) = -t \left[ C_A \ln \frac{Q_{gg'g}}{Q_{gg'} + 2C_F \ln \frac{Q_{gg'}}{Q} } \right] ,$$

$$n = 4: \ln S(t) = -t \sum_{\ell} C_{\ell} \ln \frac{Q_{12}}{Q} + \ln \left( \frac{Tr(H e^{-t \Gamma / 2} M e^{-t \Gamma / 2})}{Tr(H M)} \right) ,$$

where $Q_{ab} = 2p_a.p_b$ and $q$, $q'$ and $g$ denote the (anti-)quarks and gluon. The $n = 2, 3, 4$ formulas apply to $e^{+}e^{-}$, DIS and Drell-Yan production, while a process such as $gg \rightarrow$ Higgs $+ g$ would simply involve different color factors. The $n = 4$ formula applies to hadronic dijet production, and 1 and 2 label the incoming legs. The quantities $H$, $M$ and $\Gamma$ are the hard, soft and anomalous dimension matrices of 10 (modulo normalizations and our explicit extraction of the factor $t$ from $\Gamma$, see 5).

Finally, the NLL term $\mathcal{F}$ accounts for the details of the observable’s dependence on multiple emissions:

$$\mathcal{F}(R'_{1}, \ldots, R'_{n}) = \left\{ \exp \left\{ -R' \ln \frac{V\{\{\bar{p}\}, k_1, \ldots, k_m\}}{\max\{V\{\{\bar{p}\}, k_1, \ldots, V\{\{\bar{p}\}, k_m\}\}} \right\} \right\} , \tag{5}$$

where $R' = \sum_{\ell} R'_{\ell}$, $R'_{\ell} = C_{\ell} r'_{\ell}$. Schematically, the average is carried out over an ensemble of configurations tailored to the observable and to the values of the $R'_{\ell}$. First one specifies the value of the maximum of the $V\{\{\bar{p}\}, k_1, \ldots, k_m\}$ and $v_{\max}$, given this condition, all emissions are distributed according to an independent emission pattern uniform in $\ln k_{\perp}$, $\eta$ and $\phi$ such that the density in $\ln V\{\{\bar{p}\}, k\}$ of emissions along leg $\ell$ is $R'_{\ell}$. To ensure a result containing only NLL terms, one takes the result in the limit $v_{\max} \rightarrow 0$. The full details, including the derivation and a treatment of subtleties associated with the running of the coupling and the recoil momenta, $\{\bar{p}\}$ (determined anew for each set of emitted momenta), are given elsewhere 8 11.

Given the above elements, one could imagine a procedure whereby the applicability conditions and the parameters of 11 are established by hand, analytically, with
only the $\mathcal{F}$ being determined numerically. A related approach was presented in [11], though instead of using a master formula, we had to analytically carry out a resummation for a ‘simplified’ version of the full observable — new results were obtained there for three observables in $e^+e^- \rightarrow 2$ jets. This was already a considerable improvement over the traditional, entirely manual resummation approach, which requires a painstaking analysis of the observable’s dependence on arbitrary numbers of emissions followed by involved mathematical procedures to obtain a result which quite often cannot even be expressed in closed form (see [12] for a tortuous example).

However the introduction of a master formula makes it possible to implement a fundamentally new approach. Given a subroutine that calculates the observable for an arbitrary set of four momenta, a computer program can carry out the entire resummation: it first establishes whether the applicability conditions hold true and determines for each leg $\ell$ the parameters and functions of $a_\ell$, $b_\ell$, $d_\ell$ and $g_\ell(\phi)$. This is achieved by probing the observable with randomly chosen test configurations of soft and collinear emissions, taking the asymptotic limit with the help of high precision arithmetic (using Bailey’s portable multiple-precision package [13]).

If any of the applicability conditions fail to hold (e.g. for the Jade 3-jet resolution parameter in $e^+e^-$, which is not recursively IRC safe and so does not exponentiate [14]), the program does not proceed, i.e. a resumed answer is provided only when the correctness of the result is guaranteed to NLL accuracy.

The function $\mathcal{F}$ is then determined for relevant $R_\ell$ values by Monte Carlo averaging over appropriate configurations. The resulting information is inserted into [2] to give the resummed distribution. Since the prediction is in semi-analytical form, it is easily expanded to give the fixed-order coefficients needed when matching with fixed-order calculations.

We have verified that our approach reproduces the analytically known results in $e^+e^-$ and DIS (e.g. [5, 4, 12]). Here, to demonstrate its feasibility more generally, we show the first resummed result for an event shape in hadronic dijet production. Rapid progress is currently being made on measurements [15] and fixed-order predictions [16] for such observables. We shall examine the (global) transverse thrust (as opposed to DO’s discontinuously global variant [15]), defined as:

$$T_\perp \equiv \max_{\vec{n}_\perp} \frac{\sum_i |\vec{p}_\perp i \cdot \vec{n}_\perp i|}{\sum_i p_\perp i},$$

where the sum runs over all particles in the final state, $p_\perp i$ is the momentum transverse to the beam direction (rather than to a given leg, denoted by $p_\ell$) and $\vec{n}_\perp i$ is the unit transverse vector that maximizes the projection.

The transverse thrust has a couple of features worth commenting: firstly, it receives non-negligible contributions from emissions nearly collinear to the beams — thus it will be sensitive to radiation from the ‘underlying event’, making it useful for quantitative studies of non-perturbative effects that are qualitatively new compared to those examined up to now in $e^+e^-$ and DIS. Various other observables will be proposed in forthcoming work [8], a number of which will be less sensitive to radiation from the incoming legs, providing a good degree of complementarity. Secondly, whereas [4] sums over all particles, experiments can only measure up to some maximum rapidity $\eta_{\text{max}}$. In the presence of such a restriction it can be shown that the resummation still remains valid for values of $v \gtrsim e^{-(\alpha+b_{\text{min}})\eta_{\text{max}}}$, where $b_{\text{min}}$ is the smaller of the two incoming leg $b_\ell$ values [5].

Let us now examine the automated resummation itself: the quantity to be resummed is actually $\tau_\perp \equiv 1 - T_\perp$, since it is this that vanishes in the Born limit. The observable passes all the (automated) applicability tests and table I, generated automatically, shows the leg properties for a particular reference Born configuration. The different $b_\ell$ values for incoming and outgoing legs imply different leading logarithmic structures. The azimuthal dependence $g_\ell(\phi)$ is tabulated and integrated numerically, except in the case of certain easily recognizable analytical functions.

It so happens (also established automatically) that $\tau_\perp$ belongs to the special class of additive observables, those satisfying $V(\{\vec{p}\}, k_1, \ldots, k_m \{V(\{\vec{p}\}, k_1) + \ldots + V(\{\vec{p}\}, k_m)\}$. For such (relatively common) observables, $\mathcal{F}$ is known analytically, $e^{-\gamma E^R/\Gamma(1 + R)}$ [8], and the program would normally make use of this information. However so as to demonstrate the feasibility of our whole approach, we shall show results based on a numerically determined $\mathcal{F}$.

One further step is needed before presenting actual distributions: our master formula applies to individual Born configurations, whereas experimental measurements integrate over a range of Born configurations. A priori there is no reason for the leg parameters or $\mathcal{F}$ to be independent of the configuration and it could be necessary to repeat the analysis for a range of configurations. However for most observables, modulo certain permutations of momenta (as can once again be verified automatically), it is only the $d_\ell$ that depend on the configuration and they are easily redetermined as one integrates over Born configurations.

| leg $\ell$ | $a_\ell$ | $b_\ell$ | $g_\ell(\phi)$ | $d_\ell$ | $|\ln g_\ell(\phi)|$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | tabulated | 1.02062 | $-1.85939$ |
| 2 | 1 | 0 | tabulated | 1.02062 | $-1.85939$ |
| 3 | 1 | 1 | $\sin^2 \phi$ | 1.04167 | $-2 \ln(2)$ |
| 4 | 1 | 1 | $\sin^2 \phi$ | 1.04167 | $-2 \ln(2)$ |

TABLE I: Automatically determined leg parameters for $\tau_\perp$ in hadronic dijet production (in a c.o.m. frame with outgoing legs at an angle $\cos \theta = 0.2$).
We select events containing two outgoing jets with $\eta < 1.0$ and use the CTEQ6M parton density set [17], corresponding to $a_s(M_Z) = 0.118$. We have set $Q = \mu_F = \mu_R$ to be the Born partonic c.o.m. energy, though in future work we intend to explore a range of alternative scales. As is to be expected, channels with lower overall color charge have broader distributions. We note that the different shapes of the various channels constitutes information that might be exploitable in fits of parton distributions. Of course detailed phenomenological analyses, both for perturbative and non-perturbative quantities, will also require matching to fixed-order predictions, another step that we leave to future work.

To summarize, in this letter we have provided the elements needed for a novel, automated approach to general NLL resummation, specifically for the case of continuously global, exponentiable $n+1$-jet final-state observables in the $n$-jet limit. Results are obtained simply by specifying the Born process (and the number of hard partons) and providing the definition of the observable to be resummed in the form of a computer routine, similar to the long-established practice for fixed-order calculations, and in contrast to the tedious manual approach that has been used up to now for resummations. The results are provided in semi-analytical form, making it straightforward to obtain the expansions needed for procedures such as matching to fixed-order predictions.

We have demonstrated that the approach is practically feasible by presenting automatically generated predictions for the transverse thrust in hadronic dijet production, the first event shape to be resummed in this important process. Only concerns for brevity prevent us from showing results for a range of other observables and processes, including several new observables in hadronic dijet production and jet rates in $e^+e^-$ and DIS.

An open question is whether such an approach, based on the analysis of classes of observables can be applied in other resummations contexts, or in the search for higher resummation accuracies. We enthusiastically advocate investigations in this direction.

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