Particle motion
in electro-magnetic and gravitational pp-waves

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Abstract

We discuss the motion of neutral and charged particles in a plane electro-magnetic wave and its accompanying gravitational field.

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1 Introduction

The existence of plane-fronted parallel wave solutions in general relativity has been known since a long time [1]-[4]. These solutions are not only interesting by themselves, but also because similar solutions accompany massless fields of lower spin, such as electro-magnetic, scalar and spinor fields [4], [5]-[7].

Gravitational pp-waves can be described by space-time metrics

\[ g_{\mu\nu}dx^\mu dx^\nu = -du dv - K(u, x, y)du^2 + dx^2 + dy^2, \]  

where the light-cone co-ordinates \((u, v)\), transverse to the \(x-y\)-plane, are related to time- and longitudinal co-ordinates \((t, z)\) by:

\[ u = ct - z, \quad v = ct + z. \]

There are similar solutions with the roles of \(v\) and \(u\) interchanged, which we will not discuss explicitly here. With the metric (1), the only non-vanishing connection co-efficients are:

\[ \Gamma^v_{uu} = K_{,u}, \quad \Gamma^x_{uu} = \frac{1}{2} \Gamma^v_{xu} = \frac{1}{2} K_{,x}, \quad \Gamma^y_{uu} = \frac{1}{2} \Gamma^v_{yu} = \frac{1}{2} K_{,y}. \]

The free gravitational pp-waves are solutions of the free Einstein equations, which for the metrics (1) simplify to

\[ \Delta_{\text{trans}} K = K_{,xx} + K_{,yy} = 0. \]

The simplest non-trivial one is

\[ K(u, x, y) = \frac{1}{2} \kappa_+ (u)(x^2 - y^2) + \kappa_\times (u) xy. \]

Here \(\kappa_{(+,\times)}(u)\) are the amplitudes of the two different polarization states, as appropriate to quadrupole-type waves propagating at the speed of light.

Metrics of the type (1) are also found in the solution of the coupled Maxwell-Einstein, massless Klein-Gordon-Einstein and massless Dirac-Einstein equations. As a physically interesting example, consider the gravitational field of a light wave propagating along the \(z\)-axis. The light-wave is characterized completely by a vector potential 1-form

\[ A = \sin ku (a_x dx + a_y dy), \]

with \((a_x, a_y)\) the components of the constant transverse polarization vector. The corresponding electric and magnetic fields are

\[ E(u) = \omega a \cos ku, \quad B(u) = k \times a \cos ku, \]

where \(\omega = kc\). These fields provide a solution of the coupled Maxwell-Einstein with a metric of the form (1), with

\[ K(u, x, y) = \frac{2\pi \varepsilon_0 G}{c^4} \left( E^2(u) + c^2 B^2(u) \right) \left( x^2 + y^2 \right) \]

\[ = \frac{2\pi \varepsilon_0 G}{c^2} k^2 a^2 (1 + \cos 2ku) \left( x^2 + y^2 \right). \]
The motion of a particle of mass \( m \) and charge \( q \) propagating in gravitational and electro-magnetic background fields is governed, in the limit of neglecting radiation reaction terms, by the covariant Lorentz force

\[ \ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\nu} \dot{x}^{\lambda} \dot{x}^{\nu} = \frac{q}{m} F^{\mu\nu} \dot{x}^{\nu}. \]

(9)

Here the overdot denotes a derivative w.r.t. proper time, defined in the usual way from the universal conserved value of the total four-velocity:

\[ u_{\mu}^{2} = g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\mu}}{d\tau} = -c^2. \]

(10)

We apply these equations to study the motion of a particle in the background of the waves fields (1,6,7). Recalling the results of ref.[7], the light-cone co-ordinate \( u \) can be used as a proper-time variable via the relation

\[ \dot{u} = \gamma = \text{const.} \Rightarrow u = \gamma \tau. \]

(11)

The expression (10) for the total four-velocity then leads to the following relation between proper time and laboratory time:

\[ \frac{dt}{d\tau} = \gamma \frac{dt}{du} = \frac{\sqrt{1 - \gamma^2 K/c^2}}{\sqrt{1 - \gamma^2/c^2}}. \]

(12)

showing that there is both a gravitational and a kinematic time dilation. Substitution back into eq.(10) allows one to cast it into a true conservation law of the form

\[ h \equiv K + \frac{1 - \gamma^2/c^2}{(1 - v_z/c)^2} = \frac{c^2}{\gamma^2}. \]

(13)

Two independent equations of motion remain, which we take to be those for the transverse co-ordinates \( \xi = (x, y) \). The Lorentz-Einstein equation (9) for these components becomes

\[ \frac{d^2\xi}{du^2} = -\frac{2\pi\varepsilon_0 G}{c^2} k^2 a^2 (1 + \cos 2ku) \xi - \frac{q}{m\gamma} k a \cos ku. \]

(14)

We can define the components w.r.t. the polarization of the light wave as

\[ \xi = \xi_{\parallel} + \xi_{\perp}, \quad \xi_{\parallel} = \frac{\xi \cdot a}{|a|^2} a, \quad \xi_{\perp} = \frac{\xi \times a}{|a|}. \]

(15)

Under this decomposition eq.(14) splits into a homogeneous and an inhomogeneous equation

\[ \frac{d^2\xi_{\perp}}{du^2} + \frac{2\pi\varepsilon_0 G}{c^2} k^2 a^2 (1 + \cos 2ku) \xi_{\perp} = 0, \]

(16)

\[ \frac{d^2\xi_{\parallel}}{du^2} + \frac{2\pi\varepsilon_0 G}{c^2} k^2 a^2 (1 + \cos 2ku) \xi_{\parallel} = -\frac{q}{m\gamma} k a \cos ku. \]
In refs. [6, 7] it was observed, that in the static limit $k \to 0$ taken such that $E_0 = cB_0 = ck|a| = \text{constant}$, one obtains a simple harmonic motion with frequency $\nu$ (in Hz) given by:

$$2\pi\nu = \sqrt{\frac{4\pi\varepsilon_0 G}{c^2} k^2 |a|} 0 = 0.3 \times 10^{-18} E_0 \text{ (V/m)}. \quad (17)$$

In this paper we discuss the motion in the non-static case $k \neq 0$.

## 2 Transverse motion

We first focus on the case $q = 0$, describing a neutral massive test particle in an electro-magnetic and gravitational $pp$-wave. According to (16), the transverse coordinates ($x, y$) all obey the same homogeneous Mathieu equation. Besides, we choose to restrict our study to the case where the only contribution to the metric co-efficient $K$ is the electromagnetic one, given by (8). In other words, no free gravitational $pp$-waves are taken to be present: $\kappa_{+,x}(u)=0$. It then follows that the motion of the particle is invariant under the exchange of the transverse coordinates and rotations in the transverse plane. The equation for $\xi = (x, y)$ now reads (with $a^2 = a^2$):

$$\frac{d^2 \xi}{du^2} + \frac{2\pi\varepsilon_0 G}{c^2} k^2 a^2 (1 + \cos 2ku) \xi = 0, \quad (18)$$

A solution procedure for eq. (18) in the weak-coupling regime was presented in ref. [6]. Here we discuss the construction of the full solution in terms of a generalized Fourier series expansion. Because of the particular form of the equation (18), we can construct a series in terms of either even or odd multiples of the fundamental wave factor $k$. In this section we consider the generalized odd solutions, which can be parametrized as

$$\xi(u) = e^{\sigma ku} \sum_{n=0}^{\infty} \left( \xi_n \cos(2n + 1)ku + \eta_n \sin(2n + 1)ku \right), \quad (19)$$

with real Fourier components $(\xi_n, \eta_n)$. Note that for $\sigma^2 \leq 0$ (and taking the real part) one obtains solutions which are purely of trigonometric type; for $\sigma^2 > 0$ the solutions describe oscillations enhanced by parametric resonance. The other type of solutions having even Fourier part is discussed in the appendix. The substitution (19) converts the Mathieu equation into an infinite-dimensional linear matrix equation

$$\begin{pmatrix} A_0 & B & 0 & \ldots \\ B & A_1 & B & \ldots \\ 0 & B & A_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \vdots \end{pmatrix} = 0, \quad (20)$$
where the blocks $A_m$ and $B$ are $2 \times 2$ matrices given by

$$A_0 = \begin{pmatrix} \sigma^2 - 1 + \frac{3\sigma^2}{4} & 2\sigma \\ -2\sigma & \sigma^2 - 1 + \frac{\kappa^2}{4} \end{pmatrix}, \quad A_m = \begin{pmatrix} \sigma^2 - (2m + 1)^2 + \frac{\kappa^2}{2} & 2(2m + 1)\sigma \\ -2(2m + 1)\sigma & \sigma^2 - (2m + 1)^2 + \frac{\kappa^2}{4} \end{pmatrix}, \quad (m \geq 1), \quad (21)$$

and

$$B = \frac{\kappa^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_m = \begin{pmatrix} \xi_m \\ \eta_m \end{pmatrix}. \quad (22)$$

A solution of the system (21) exists provided the linear matrix operator has zero-modes. To determine their existence we cast the equation into upper-triangular form:

$$\begin{pmatrix} a_0 & B & 0 & \ldots \\ 0 & a_1 & B & \ldots \\ 0 & 0 & a_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \bar{\rho}_0 \\ \bar{\rho}_1 \\ \bar{\rho}_2 \\ \ldots \end{pmatrix} = 0. \quad (23)$$

The $2 \times 2$ matrices $a_m$ can be expressed in terms of continuing fractions as

$$a_m = A_m - B^2 \left[ A_{m+1} - B^2 \left[ A_{m+2} - \ldots \right]^{-1} \right]^{-1}, \quad (24)$$

and the basis of vectors has been transformed by $\bar{\rho}_m = \rho_m$, and

$$\bar{\rho}_m = \rho_m + a_m^{-1} B \rho_{m-1}, \quad (m \geq 1). \quad (25)$$

This transformation is singular if and only if any of the $a_m$ $(m \geq 1)$ has zero-modes, which is precisely the condition for the Mathieu equation to have non-trivial solutions.

Zero-modes of $a_0$ represent the lowest-frequency solutions of the Mathieu equation; they exhibit parametric resonance for $\sigma > 0$. The existence of zero-modes implies that the determinant of $a_0$ vanishes. This determinant can be computed in successive approximations by truncating the infinite fraction (24). The zeroth-order approximation is

$$a_0 = A_0 \Rightarrow \left| a_0^{(0)} \right| = (\sigma^2 + 1)^2 + \kappa^2(\sigma^2 - 1) + \frac{3\kappa^4}{16}. \quad (26)$$

The first- and higher-order approximations are of the form

$$a_0 = \begin{pmatrix} \sigma^2 - 1 + \frac{3\sigma^2}{4} + \kappa^4 \Delta_{11} & 2\sigma + \kappa^4 \Delta_{12} \\ -(2\sigma + \kappa^4 \Delta_{12}) & \sigma^2 - 1 + \frac{\kappa^2}{4} + \kappa^4 \Delta_{11} \end{pmatrix} + O(\kappa^6), \quad (27)$$

where the quantities $\Delta_{ij}(\sigma)$ are rational functions of $\sigma^2$; the determinant then is

$$\left| a_0 \right| = (\sigma^2 + 1)^2 + \kappa^2(\sigma^2 - 1) + \kappa^4 \left( \frac{3}{16} - \frac{\Delta}{8} \right) + O(\kappa^6). \quad (28)$$
In first order $a_0^{(1)} = A_0 - B^2 A_1^{-1}$, with

$$
\Delta_{11}^{(1)} = \frac{1}{48} \frac{9 - \sigma^2}{27 + 6\sigma^2 + \sigma^4/3}, \quad \Delta_{12}^{(1)} = \frac{1}{8} \frac{1}{27 + 6\sigma^2 + \sigma^4/3},
$$

and

$$
\Delta^{(1)} = \frac{\sigma^4 - 22\sigma^2 + 9}{\sigma^4 + 18\sigma^2 + 81}.
$$

Similarly, in second order with

$$
a_0 = \left( A_0 A_1 A_2 - B^2 (A_0 + A_2) \right) \left( A_1 A_2 - B^2 \right)^{-1},
$$

one finds up to terms $O(\kappa^6)$:

$$
-16(\sigma^2 - 15) \Delta_{11}^{(2)} = \frac{5}{8} + \frac{3}{8} \frac{(\sigma^2 + 25)(\sigma^4 - 94\sigma^2 + 225)}{(\sigma^4 - 94\sigma^2 + 225)^2 + 256\sigma^2(\sigma^2 - 15)^2},
$$

$$
\Delta_{12}^{(2)} = \frac{3}{8} \frac{(\sigma^2 + 25)^2}{(\sigma^4 - 94\sigma^2 + 225)^2 + 256\sigma^2(\sigma^2 - 15)^2}.
$$

Then in the determinant

$$
\Delta^{(2)} = \frac{(\sigma^2 - 1)(\sigma^2 - 25)(\sigma^4 - 94\sigma^2 + 225) + 160\sigma^2(\sigma^2 - 15)}{(\sigma^4 - 94\sigma^2 + 225)^2 + 256\sigma^2(\sigma^2 - 15)^2}.
$$

Observe that in both first and second order approximation

$$
\lim_{\sigma^2 \to 0} \Delta^{(k)} = \frac{1}{9}, \quad \lim_{\sigma^2 \to \infty} \Delta^{(k)} = 1, \quad k = (1, 2).
$$

From the above analysis it follows, that up to terms of order $\kappa^6$ a solution for $\sigma^2$ exists for which the determinant vanishes: $|a_0| = 0$, as given by the expression (28). This condition has approximate solutions

$$
\sigma^2 = -\left(1 + \frac{\kappa^2}{2}\right) \pm \sqrt{2\kappa^2 \left(1 + \frac{\kappa^2}{32} (1 + 2\Delta)\right)}.
$$

Positive non-zero values of $\sigma^2$ exist only for the positive square root, provided

$$
2\kappa^2 \left(1 + \frac{\kappa^2}{32} (1 + 2\Delta)\right) > \left(1 + \frac{\kappa^2}{2}\right)^2.
$$

or

$$
\left(1 - \frac{\kappa^2}{2} + \frac{\kappa^2}{4} \sqrt{1 + 2\Delta}\right) \left(1 - \frac{\kappa^2}{2} - \frac{\kappa^2}{4} \sqrt{1 + 2\Delta}\right) < 0.
$$

Hence $\kappa^2$ must be in the range

$$
\frac{1}{2 + \sqrt{1 + 2\Delta}} < \frac{\kappa^2}{4} < \frac{1}{2 - \sqrt{1 + 2\Delta}}.
$$
Purely periodic solutions of the Mathieu equation exist for a discrete set of values of $\kappa$ for which $\sigma = 0$; for these values the inequality (36) turns into and equality. For all other values outside the range (38) the solutions for $\sigma$ are imaginary; the series (19) then is purely trigonometric, but for non-rational $\sigma$ the result is not periodic.

From the inequality (38) it follows, that the smallest possible value of $\kappa^2$ for which parametric resonance occurs is of the order unity: $\kappa^2 \geq 1$. As $\kappa^2$ represents the ratio of field intensity and frequency, it follows that one needs very strong fields at very low frequencies ($2\pi \nu = c\kappa$):

$$\kappa^2 = \left( \frac{0.3 \times 10^{-18} E_0 \text{ (V/m)}}{2\pi \nu \text{ (Hz)}} \right)^2 \geq 1. \quad (39)$$

An upper limit to the fields strength of the order of $10^{19}$ V/m arises because of pair creation processes which destroy the electric field at higher intensities [8]. Therefore we obtain an upper limit of the order of $\sim 1$ Hz for the frequencies which can give rise to parametric resonance. For lower fields and higher frequencies one finds standard periodic behavior described by solutions with $\sigma = 0$ or imaginary.

An important characteristic of the motion of a chargeless particle in the background of the fields (1,6,7) is that a particle initially at the origin in the transverse plane before the arrival of the wave, remains at the origin at all later times. This allows one to translate directly the geodesic motion of a test particle in terms of the relative rate of acceleration of this particle w.r.t. another one located in the origin: $\xi = 0$.

For charged particles this situation changes. If $q \neq 0$ and we take the transverse co-ordinate $\xi = \xi_\parallel$ parallel to the electric field, the equation of transverse motion is:

$$\frac{d^2 \xi}{du^2} + \frac{\kappa^2 k^2}{2} (1 + \cos 2ku) \xi = -\varepsilon \cos ku, \quad (40)$$

with

$$\varepsilon = \frac{qka}{mc^2 \gamma} = \frac{qE_0}{mc^2 \gamma}. \quad (41)$$

In the inhomogeneous case the absence of an exponential term in the driving force implies that any special solution of the inhomogeneous equation can have an expansion (19) only with $\sigma = 0$; for these special solutions the inhomogeneous Mathieu equation becomes

$$\begin{pmatrix} a_0 & B & 0 & \ldots \\ 0 & a_1 & B & \ldots \\ 0 & 0 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\sigma=0} \begin{pmatrix} \tilde{\rho}_0 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \theta \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \theta = \frac{\varepsilon}{k^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (42)$$

Here we are interested especially in obtaining a particular solution of the previous type: $\tilde{\rho}_0 \neq 0$, $\tilde{\rho}_n = 0$, $n \geq 1$. The non-zero component $\tilde{\rho}_0$ is a solution of the inhomogeneous equation

$$a_0 \tilde{\rho}_0 = \theta. \quad (43)$$
Now \( a_0 \) is of the form (27), with \( \sigma = 0 \):

\[
a_0 = \left( \begin{array}{cc}
-1 + \frac{3\kappa^2}{4} + \Delta_{11}(0) \kappa^4 & 0 \\
0 & -1 + \frac{3\kappa^2}{4} + \Delta_{11}(0) \kappa^4
\end{array} \right) + \mathcal{O}(\kappa^6),
\]

(44)

where beyond the lowest-order approximation we find \( \Delta_{11}(\sigma = 0) = \frac{1}{144} \). As a result a special solution of the inhomogeneous equation is given by

\[
\tilde{\rho}_0 = \frac{-\varepsilon}{k^2(1 - \frac{3\kappa^2}{4} - \Delta_{11}(0) \kappa^4 + \ldots)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\rho}_n = 0, \quad n \geq 1.
\]

(45)

The general solution of this system is the sum of a particular solution plus a solution of the homogeneous equation, which can have \( \sigma \neq 0 \). An important difference with the previous case is that a particle initially at the origin does not remain at the origin, but is accelerated by the electric field. Thus the study of the relative acceleration between particles on different world lines becomes more complicated; for a general discussion we refer to ref. [10].

3 QED corrections

As the magnitude of the electric field in the region of parametric resonance is necessarily extremely high, QED effects such as vacuum polarization can in principle not be ignored. However, as we now show they do not affect the classical wave solutions we have studied in the previous paragraphs.

The non-linear effects of quantum fluctuations on the propagation of light in flat (Minkowski) space-time were first studied by Euler and Heisenberg [9]. The effects are summerized by a well-known effective lagrangean incorporating the corrections due to the electromagnetic higher-order terms; its generally covariant form reads:

\[
\mathcal{L} = \varepsilon_0 c^2 \sqrt{-g} \left( -\frac{1}{4} F^2 + \beta \left[ (F^2)^2 + \frac{7}{4} \phi^2 \right] \right)
\]

(46)

where:

\[
F^2 = g^{\mu\kappa} g^{\nu\lambda} F_{\mu\nu} F_{\kappa\lambda},
\]

(47)

\[
\phi = F_{\mu\nu} \tilde{F}^{\mu\nu},
\]

and \( \tilde{F}_{\mu\nu} \) is the dual electromagnetic tensor defined by:

\[
\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.
\]

(48)

The coupling constant \( \beta \) is given by

\[
\beta = \frac{\alpha^2 \hbar^3 \varepsilon_0}{90m^4 c^4}.
\]

(49)

We first derive the Maxwell equations from the Euler-Lagrange equations taking \( A_\nu \) and \( \partial_\nu A_\mu \) as the variables:

\[
\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = 0.
\]

(50)
Combining (46) and (50), we get the following equations:

\[ \nabla_{\mu} \left[ (1 - 8\beta F^2)F^{\mu\nu} - 14\beta \phi \tilde{F}^{\mu\nu} \right] = 0. \]  
\[ (51) \]

The energy-momentum tensor is obtained from the lagrangean (46) by:

\[ T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial L}{\partial g^{\mu\nu}} \]
\[ = \varepsilon_0 c^2 \left[ (1 - 8\beta F^2)F_{\mu\lambda}F^\nu^\lambda - \frac{1}{4} g_{\mu\nu} (F^2 - 4\beta (F^2)^2) - \frac{7\beta}{4} g_{\mu\nu} \phi^2 \right]. \]  
\[ (52) \]

With the metric (1) the Ricci tensor \( R_{\mu\nu} \) has only one non-zero component \( R_{uu} \), the Einstein field equations reduce to:

\[ R_{uu} = -\frac{1}{2} \Delta_{\text{trans}} K = 8\pi G T_{uu}, \quad \text{all other } T_{\mu\nu} = 0. \]  
\[ (53) \]

In our particular case of a plane-wave vector potential of the form (6), the only non-vanishing components of the electromagnetic tensor \( F_{\mu\nu} \) are:

\[ F_{ui} = -F_{ui} = a_i k \cos ku, \quad i = (x,y). \]  
\[ (54) \]

It follows that \( F^2 = \phi = 0 \), and the Maxwell-Einstein field equations reduce to the original form:

\[ \nabla_{\mu} F^{\mu\nu} = 0, \]
\[ T_{uu} = -\varepsilon_0 c^2 F_{u\lambda} F_{\mu}^\lambda. \]  
\[ (55) \]

Thus the quantum corrections do not modify the gravito-electro-magnetic plane wave solution.
Appendix

The solutions of the Mathieu equation are classified as having an odd or even periodic part. Eq. (13) represents the general solution of odd type. The general solution of even type can be expanded as

\[ \xi(u) = e^{\sigma ku} \left( \frac{\mu_0}{2} + \sum_{n=1}^{\infty} [\mu_n \cos 2nku + \nu_n \sin 2nku] \right). \]  

(56)

After multiplication with \( e^{-\sigma ku} \) the homogeneous Mathieu equation becomes

\[ 0 = \left( \frac{1}{2} \sigma^2 + \frac{1}{4} \kappa^2 \right) \mu_0 + \frac{\kappa^2}{4} \mu_1 
+ \sum_{n=1}^{\infty} \left[ \left( \sigma^2 - 4n^2 + \frac{\kappa^2}{2} \right) \mu_n + 4n\sigma \nu_n + \frac{\kappa^2}{4} (\mu_{n-1} + \mu_{n+1}) \right] \cos 2nku
+ \sum_{n=1}^{\infty} \left[ \left( \sigma^2 - 4n^2 + \frac{\kappa^2}{2} \right) \nu_n - 4n\sigma \mu_n \right] \sin 2nku
+ \frac{\kappa^2}{4} \left( \nu_2 \sin 2ku + \sum_{n=2}^{\infty} (\nu_{n-1} + \nu_{n+1}) \sin 2nku \right). \]  

(57)

It follows, that

\[ \mu_0 = -\frac{\kappa^2 \mu_1}{2\sigma^2 + \kappa^2}, \]  

(58)

and

\[ \begin{pmatrix} D_1 & B & 0 & \ldots \\ B & D_2 & B & \ldots \\ 0 & B & D_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_1 & B & 0 & \ldots \\ 0 & d_2 & B & \ldots \\ 0 & 0 & d_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots \\ c_1 & 1 & 0 & \ldots \\ 0 & c_2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]  

(59)

where \( B \) is the 2 \times 2-matrix \([22] \), whilst

\[ D_1 = \begin{pmatrix} \sigma^2 - 4 + \frac{\kappa^2}{2} & -\frac{\kappa^4}{4(2\sigma^2 + \kappa^2)} & 4\sigma \\ -4\sigma & \sigma^2 - 4 + \frac{\kappa^2}{2} \end{pmatrix}, \]  

(60)

\[ D_n = \begin{pmatrix} \sigma^2 - 4n^2 + \frac{\kappa^2}{2} & 4n\sigma \\ -4n\sigma & \sigma^2 - 4n^2 + \frac{\kappa^2}{2} \end{pmatrix}, \quad n \geq 2, \]  

and

\[ \zeta_n = \begin{pmatrix} \mu_n \\ \nu_n \end{pmatrix}. \]  

(61)

To solve this equation, we bring the infinite-dimensional matrix equation \([54] \) into upper triangular form:

\[ \begin{pmatrix} D_1 & B & 0 & \ldots \\ B & D_2 & B & \ldots \\ 0 & B & D_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} d_1 & B & 0 & \ldots \\ 0 & d_2 & B & \ldots \\ 0 & 0 & d_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots \\ c_1 & 1 & 0 & \ldots \\ 0 & c_2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]  

(62)
with $d_n$ and $c_n$ the solutions of the equations
\[ d_n + Bc_n = D_n, \quad d_{n+1}c_n = B \Rightarrow B^{-1}d_n + (B^{-1}d_{n+1})^{-1} = B^{-1}D_n. \] (63)

The solution for $d_n$ takes the form of a continuing fraction
\[ d_n = D_n - B^2 \left[ D_{n+1} - B^2 [D_{n+2} - \ldots]^{-1} \right]^{-1}, \] (64)
where we have used the fact that $B$ is proportional to the unit matrix, hence commutes with all other matrices. If we define
\[ \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \\ \tilde{\zeta}_3 \\ \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ c_1 & 1 & 0 & \ldots \\ 0 & c_2 & 1 & \ldots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \ddots \end{pmatrix}, \] (65)
equivalent with $\tilde{\zeta}_1 = \zeta_1$, and
\[ \tilde{\zeta}_n = \zeta_n + c_{n-1}\zeta_{n-1}, \quad n \geq 2. \] (66)

Now the equation
\[ \begin{pmatrix} d_1 & B & 0 & \ldots \\ 0 & d_2 & B & \ldots \\ 0 & 0 & d_3 & \ldots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \\ \tilde{\zeta}_3 \\ \ddots \end{pmatrix} = 0, \] (67)
has solutions with $\tilde{\zeta}_N \neq 0$, and
\[ d_k \tilde{\zeta}_k = -B \tilde{\zeta}_{k+1}, \quad 1 \leq k \leq N - 1 \quad \text{and} \quad \tilde{\zeta}_n = 0, \quad n > N. \] (68)

The first one of these is the one given by $d_1 \tilde{\zeta}_1 = 0$, $\tilde{\zeta}_n = 0$, $n \geq 2$.

**Approximations.** Solutions of the equation $d_1 \tilde{\zeta}_1 = 0$ exist if $\det d_1 = 0$. To solve this condition, we need to compute the infinite fraction (64). We can do this by successive approximations:
\[ d_1^{(0)} = D_1, \]
\[ d_1^{(1)} = D_1 - B^2D_2^{-1} = \left( D_1D_2 - B^2 \right) D_2^{-1}, \]
\[ d_1^{(2)} = \left( D_1D_2D_3 - B^2D_1 - B^2D_3 \right) \left( D_2D_3 - B^2 \right)^{-1}, \] (69)
\[ \ldots \]

To zeroth order approximation, we can show that $\det D_1 = 0$ has solutions for $8 < \kappa^2 < 16$:
\[ |D_1| = \frac{\kappa^2}{8} \left( \kappa^2 - 8 \right) \left( \kappa^2 - 16 \right) + 2\sigma^2 \left( \sigma^4 + \left( 8 + \frac{3\kappa^2}{2} \right) \sigma^2 + 16 + \frac{5\kappa^2}{8} \right). \] (70)

It suffices to note, that the first term is negative in the range $\kappa^2 \in (8, 16)$, whilst for any $\kappa^2$ the second term can take all positive values between $(0, \infty)$. 

10
References


