Low-energy structure of six-dimensional open-string vacua

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Abstract

This dissertation reviews some properties of the low-energy effective actions for six dimensional open-string models. The first chapter is a pedagogical introduction about supergravity theories. In the second chapter closed strings are analyzed, with particular emphasis on type IIB, whose orientifold projection, in order to build type-I models, is the subject of the third chapter. Original results are reported in chapters 4 and 5. In chapter 4 we describe the complete coupling of (1,0) six-dimensional supergravity to tensor, vector and hypermultiplets. The generalized Green-Schwarz mechanism implies that the resulting theory embodies factorized gauge and supersymmetry anomalies, to be disposed of by fermion loops. Consequently, the low-energy theory is determined by the Wess-Zumino consistency conditions, rather than by the requirement of supersymmetry, and this procedure does not fix a quartic coupling for the gauginos. In chapter 5 we describe the low-energy effective actions for type-I models with brane supersymmetry breaking, resulting from the simultaneous presence of supersymmetric bulks, with one or more gravitinos, and non-supersymmetric combinations of BPS branes. The consistency of the resulting gravitino couplings implies that local supersymmetry is non-linearly realized on some branes. We analyze in detail the ten-dimensional $USp(32)$ model and the six-dimensional (1,0) models.
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Introduction

String theory is an extremely powerful tool for the quantization of gravity and the unification of fundamental interactions. In the spectrum of excitations of closed strings, a massless spin 2 field is always present, and its interactions are described at low-energies by the Einstein-Hilbert action. Typically, requiring that in the low-energy limit gravitational interactions be regulated by Newton’s constant fixes the string scale to be of the order of the Planck scale. This explains why particles behave as pointlike objects at low energies, and if one restricts the world-sheet action of the string to be supersymmetric, consistency selects the space-time dimension to be equal to 10. Moreover, in ten dimensions there are five superstring theories, i.e. five theories that have a supersymmetric spectrum. In order to obtain models that are phenomenologically interesting, one has then to suitably compactify them, so that the resulting vacua be four-dimensional. Although these theories have different perturbative spectra, at the non-perturbative level they are related by dualities, and the picture that emerges is that the (unique and unknown) complete theory behind them is described by different string theories in different regimes.

While four of these five theories contain only closed strings at the perturbative level, type-I string theory contains open strings as well. As we will see throughout this thesis, this peculiarity of type-I strings will give rise to several interesting physical phenomena, that in closed string theories correspond to non-perturbative effects. More precisely, the hyper-surfaces on which the open strings end (D-branes) are dynamical objects whose excitations are open-string modes, and they correspond to non-perturbative states of the closed string theories. It is then interesting to study compactifications of type-I models, and in this respect the analysis of type-I vacua with minimal supersymmetry in six-dimensions turns out to be particularly rich, and
will be the main topic of this dissertation.

Before I started my PhD, during the period of my INFN Pre-Doctoral Fellowship, I analyzed minimal six-dimensional supergravity in collaboration with S. Ferrara and A. Sagnotti [69], concentrating in particular on the couplings of supergravity to tensor multiplets and to non-abelian vector multiplets. The result we ended up with is that this theory is not completely determined by supersymmetry, since consistency conditions leave a quartic gaugino coupling undetermined. Subsequently, in [70, 75, 77], we also came to a better understanding of the properties of these models. This dissertation is partly a continuation of these results: during the first part of my PhD I completed the low-energy effective action of six-dimensional supergravity coupled to tensor multiplets and abelian vector multiplets [71], showing that in the abelian case additional couplings can be added, and then in [76] I obtained the complete action of supergravity coupled to vector, tensor and hypermultiplets. These results are the subject of Sections (4.4) and (4.5) of this thesis.

Another peculiar feature of type-I strings is that one can consider brane-world scenarios, in which our universe is confined on some coincident branes, where the standard model lives, while gravity invades the whole bulk. In this scenarios, one can naturally lower the string scale to the order of the supersymmetry scale, since hierarchy is generated by compactification, and thus the Planck scale is obtained dynamically. Moreover, one can consistently obtain “brane supersymmetry breaking” models, in which supersymmetry is realized in the bulk, at least to lowest order, while it is broken on some branes. During the last part of my PhD, I have analyzed in collaboration with G. Pradisi the low-energy effective action corresponding to type-I “brane supersymmetry breaking” models [104]. These results are collected in Chapter 5.

The thesis is organized as follows. Chapter 1 is a pedagogical introduction about supergravity theories, centered on topics that will become useful for the following, and in particular for Chapter 4. In Chapter 2 I analyze closed strings, with particular emphasis on type IIB, while its orientifold projection, in order to build type-I models, will be the subject of Chapter 3. Chapter 4 is devoted to a detailed description of minimal supergravity in six dimensions. In Chapter 5 I derive to lowest order in the Fermi fields the effective action for type-I brane supersymmetry breaking models. Finally, the last chapter is devoted to the conclusions, while the Appendix contains some conventions and useful identities.
Chapter 1

Generalities about supergravity theories

Supersymmetry is a space-time symmetry that combines bosonic and fermionic fields. The simplest example of a model invariant under supersymmetry transformations is the four-dimensional Wess-Zumino model \([\mathbb{P}]\), consisting of a complex scalar and a Weyl fermion. The lagrangian of the model,

\[
\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi
\]

is invariant under the supersymmetry transformations

\[
\delta \phi = \bar{\epsilon} \psi \quad , \quad \delta \psi = -i \gamma^\mu \epsilon \partial_\mu \phi
\]

where \(\epsilon\) is a constant Weyl spinor with opposite chirality with respect to \(\psi\). There are two important properties that this model shares with all other supersymmetric theories: the first is the fact that the commutator of two supersymmetry transformations on the bosonic field generates a translation,

\[
[\delta_1, \delta_2] \phi = -2i (\bar{\epsilon}_2 \gamma^\mu \epsilon_1) \partial_\mu \phi
\]

the second is the fact that on the spinor, the same algebra is realized only on-shell. The first result is a general property of the supersymmetry algebra, and corresponds to the fact that the anticommutator of two supersymmetry generators gives the momentum, the generator of translations. The second result is due to the fact that the
matching of bosonic and fermionic degrees of freedom actually only holds on-shell. In order to have the same matching also off-shell, we should add suitable auxiliary fields, and then, in the complete set of fields, the supersymmetry algebra will close exactly. We will not consider the (partly unsolved) problem of finding the off-shell representations of supersymmetry throughout this thesis, and in fact in Chapter 4 we will use the property that the supersymmetry algebra closes only on shell in order to determine the field equations for the fermionic fields of six-dimensional supergravity.

An additional step is to try to make this symmetry local. There is a field that can naturally be regarded as the gauge field of supersymmetry, the Rarita-Schwinger field \( \psi_\mu \), whose linearized equation

\[
i\gamma^\mu_\nu_\rho \partial_\nu \psi_\rho = 0
\]

is invariant under

\[
\delta \psi_\mu = \partial_\mu \epsilon
\]

From the supersymmetry algebra we learn also that if the parameter of supersymmetry is local (i.e. space-time dependent), then the algebra generates a translation with a local parameter, a general coordinate transformation. If a field theory is invariant under general coordinate transformations, it must contain general relativity. So the important lesson that we learn is that a theory invariant under local supersymmetry, containing the field \( \psi_\mu \), must contain gravity as well. The field \( \psi_\mu \) is then the supersymmetric partner of the graviton, and for this reason it is called gravitino.

In this chapter we want to give a general introduction to supergravity theories. We begin in Section 1 with a brief discussion about torsion in general relativity, and in Section 2 we describe the \( \mathcal{N} = 1 \) four-dimensional model, that contains only a graviton and a gravitino, but already reveals some of the subtleties common to other supergravity models. In Section 3 we consider D=11 supergravity and finally in Section 4 we briefly describe the various supergravity theories that arise in ten dimensions.

## 1.1 Torsion in General Relativity

In this section we report some known results on general relativity \(^2\) that are essential for the following, in particular for Chapter 4. First of all, we fix the notations that we will follow throughout the thesis. The metric has signature \((+, -, ..., -)\), the
covariant derivative for a vector has the form
\[ \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha , \] (1.1)
where \( \Gamma \) is the Christoffel connection, and this covariant derivative can be rewritten as
\[ D_\mu V^m = \partial_\mu V^m + \omega^m_{\mu n} V^n , \] (1.2)
where \( \omega \) is the spin connection, in terms of
\[ V^m = e_\mu^m V^\mu \] (1.3)
whose indices are on the locally tangent space. The covariant derivative for a spinor is then
\[ D_\mu \chi = \partial_\mu \chi + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \chi , \] (1.4)
and these two covariant derivatives are consistent because the vierbein is covariantly constant:
\[ \nabla_\mu e_\nu^m = \partial_\mu e_\nu^m - \Gamma^\rho_{\mu\nu} e_\rho^m + \omega^m_{\mu n} e_\nu^n = 0 . \] (1.5)
A priori, the Christoffel connection has no determined symmetry with respect to its lower indices. Nevertheless, it is important to observe that if the connection is chosen to be symmetric, then it is completely determined by the condition that the metric be covariantly constant. Denoting with \( \{ \} \) this connection, the result is
\[ \{ _\mu^\nu \} = \frac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) . \] (1.6)
Correspondingly, the spin connection in this case is also completely determined in terms of the vielbein, and
\[ e_\nu^m e_\rho^n \omega^0_{\mu mn} = \frac{1}{2} \{ e_\rho^p(\partial_\mu e_\nu^p - \partial_\nu e_\mu^p) - e_\mu^p(\partial_\rho e_\nu^p - \partial_\nu e_\rho^p) + e_\nu^p(\partial_\rho e_\mu^p - \partial_\mu e_\rho^p) \} . \] (1.7)
In general, one can consider a Christoffel connection with an antisymmetric part, that is called torsion,
\[ \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} + S^\rho_{\mu\nu} , \] (1.8)
where \( \Gamma_0 \) is symmetric and \( S \) is antisymmetric under the interchange of the lower indices. It is important to observe that this different choice of connection simply corresponds to the addition of covariant tensors to the minimal choice \( \{ \} \). In fact, it
can be shown that the torsion is a covariant tensor, and the symmetric part of the connection becomes
\[ \Gamma^0_{\mu\nu} = \{^\rho_{\mu\nu} \} + S^\rho_{\mu} + S^\rho_{\nu} \]  
(1.9)

In the presence of torsion, the spin connection is also modified by the addition of covariant terms:
\[ \omega_{\mu mn} = \omega^0_{\mu mn} + S^m_{\mu mn} + S^m_{\mu nmn} + S^m_{nmn} \]  
(1.10)

while the condition that the metric and the vierbein be covariantly constant is independent of the torsion.

In our notations, the Riemann tensor has the form
\[ R^\alpha_{\mu\nu\rho}(\Gamma) = \partial_\nu \Gamma^\alpha_{\rho\mu} - \partial_\rho \Gamma^\alpha_{\nu\mu} + \Gamma^\beta_{\rho\mu} \Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\nu\mu} \Gamma^\alpha_{\rho\beta} \]  
(1.11)
in terms of the Christoffel connection, or
\[ R^mn_{\mu\nu}(\omega) = \partial_\mu \omega^m_{\nu n} - \partial_\nu \omega^m_{\mu n} + \omega^m_{\mu p} \omega^p_{\nu n} - \omega^m_{\nu p} \omega^p_{\mu n} \]  
(1.12)
in terms of the spin connection, and the fact that the metric is covariantly constant implies that these two curvature tensors are equivalent, so that
\[ e^a_m R_{\sigma \tau mn}(\omega) = R^a_{\mu\sigma}(\Gamma) e_{am} \]  
(1.13)

Consequently, the Ricci tensor is
\[ R_{\mu\nu}(\Gamma) = e^a_m e^\tau_m R_{\tau \nu mn}(\omega) \]  
(1.14)

and the Ricci scalar is
\[ R(\Gamma) = e^\sigma_m e^\tau_n R_{\sigma \tau mn}(\omega) \]  
(1.15)

This curvature tensor is the object that naturally appears in the commutator of two covariant derivatives, while the presence of torsion corresponds to the addition in the commutators of a term containing the covariant derivative of the vector, so that
\[ [\nabla_\mu, \nabla_\nu]V_\mu = R^a_{\mu \nu \rho}(\Gamma) V_\alpha - 2S^\alpha_{\mu \rho} \nabla_\alpha V_\mu \]  
(1.16)

One of the consequences of torsion is the fact that the Ricci tensor is no longer symmetric, and precisely
\[ R_{\mu\nu} - R_{\nu\mu} = 2\nabla_\alpha S^\alpha_{\nu \mu} - 2\nabla_\nu S^\alpha_{\alpha \mu} + 2\nabla_\mu S^\alpha_{\alpha \nu} - 4S^\beta_{\nu \mu} S^\alpha_{\alpha \beta} \]  
(1.17)
1.1 Torsion in general relativity

while it can also be shown that the relation

$$\Gamma^\alpha_{\mu\nu} = \partial_\mu \ln \sqrt{-g}$$

is still true in the presence of torsion.

There are three different lagrangian formulations of general relativity. In the second order formulation one assumes from the start that the torsion vanishes, and considers the theory as a function of the metric $g_{\mu\nu}$ only. The connections are determined in terms of the metric by the requirement that the torsion be zero. In the first order (Palatini) formulation, the metric and the connection are assumed to be independent. It is also possible to consider a pure connection formulation, in which the metric is obtained as a composite field. We now show that for pure gravity, the first order and the second order formulations coincide. Fixing Newton’s constant appropriately, the Einstein-Hilbert action takes the form

$$S = -\frac{1}{4} \int d^D x \sqrt{-g} R$$

The variation of the Riemann tensor is

$$\delta R_{\mu\nu} = \nabla_\alpha \delta \Gamma^\alpha_{\nu\mu} - \nabla_\nu \delta \Gamma^\alpha_{\alpha\mu} + 2 \delta_{\alpha\nu} \delta \Gamma^\alpha_{\beta\mu}$$

where the first two terms do not contribute to the field equation, and as a consequence, one sees that the second order formulation, where the absence of torsion is required as a condition, gives the same equation as the first order formulation, where the absence of torsion comes from the equation of the connection. The field equation is then completely determined by the variation of the metric, and is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

The same result is obtained if one expresses the Riemann tensor in terms of the spin connection, and indeed varying the action

$$S = -\frac{1}{4} \int d^D x e_m e^n R^{mn}_{\mu\nu}(\omega)$$

with respect to the spin connection $\omega$ gives

$$\delta S = \frac{3}{2} \int d^D x e_m e^n e^p \delta \omega^{mn}_{\mu\nu}$$

where the covariant derivative $D$ contains only the spin connection. The field equation for $\omega$ then becomes $D[\mu e^n_{\nu}] = 0$, that implies the absence of torsion. Now suppose that
matter is present, and consider the simple case of a single spinor, whose lagrangian is

$$\mathcal{L} = \frac{i}{2} \bar{\chi} \gamma^{\mu} D_{\mu} \chi$$ \hspace{1cm} (1.24)

where

$$D_{\mu} \chi = \partial_{\mu} \chi + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \chi$$ \hspace{1cm} (1.25)

The equation for \( \omega \) now implies that the spin connection is

$$\omega_{\mu mn} = \omega^{0}_{\mu mn} - \frac{i}{4} \bar{\chi} \gamma_{\mu mn} \chi$$ \hspace{1cm} (1.26)

and from eq. (1.10) this corresponds to the presence of the torsion

$$S_{\mu \nu \rho} = \frac{i}{4} \bar{\chi} \gamma_{\mu \nu \rho} \chi$$ \hspace{1cm} (1.27)

In supergravity theories it is necessary to consider the spin connection rather than the Christoffel connection in the lagrangian, since these theories contain spinors. This is also natural, since these theories typically contain \( p \)-forms, antisymmetric tensors with \( p \) space-time indices, that generalize vectors in higher dimensions. The field strength for a \( p \)-form \( A \) is \( F = dA \), that in components means

$$F_{\mu_{1} \ldots \mu_{p}} = p \partial_{[\mu_{1}} A_{\mu_{2} \ldots \mu_{p}]}$$ \hspace{1cm} (1.28)

and the fact that the indices are completely antisymmetrized implies that this object is covariant without the addition of any connection. The gauge invariance of \( F \) under

$$\delta A_{\mu} = d\Lambda_{\mu}$$ \hspace{1cm} (1.29)

is preserved only if no torsion term is present. Moreover, the Rarita-Schwinger action is covariant in terms of the covariant derivative \( D \), with only the spin connection, and thus in general Christoffel and spin connection enter very differently the relevant couplings.

The simplest supergravity theory is \( N = 1 \) supergravity in four dimensions \[3\], that contains only the graviton and a gravitino. The invariance of this model under supersymmetry was originally shown in the second order formalism in \[3\] and then in the first order formalism in \[4\]. One could actually make things simpler, and combine the advantages of both formalisms, considering the spin connection as an independent field, but always imposing that it satisfy its field equations (see Ref. \[5\]). This formalism, known as the 1.5 formalism, is the most natural way to formulate supergravity theories, and is the one we will use in the next sections and in Chapter 4. In the next section we will explicitly see how all this works for minimal supergravity in four dimensions, while in Section 3 we will consider 11-dimensional supergravity.
1.2 Minimal supergravity in four dimensions

The $\mathcal{N} = 1$ four-dimensional supergravity multiplet contains just the graviton and a Majorana gravitino. In this section we show how the lagrangian for this model is constructed, showing that the lagrangian

$$\mathcal{L} = -\frac{1}{4} e R - \frac{i e}{2} \bar{\psi}_\mu \gamma^{\mu \nu \rho} D_\nu \psi_\rho$$

(1.30)

is invariant under the supersymmetry transformations

$$\delta e^m_\mu = -i e \gamma^m \psi_\mu \ ,$$

$$\delta \psi_\mu = D_\mu e \ .$$

(1.31)

First of all, we prove that supersymmetry holds to lowest (i.e. quadratic) order in the fermions. This means that we have to consider the equation for the metric without gravitino terms, and the equation for the gravitino without cubic terms in the gravitino itself. At this level the spin connection does not contain torsion, that as we saw in the previous section is quadratic in the spinors. To lowest order, the variation of the lagrangian is

$$\delta \mathcal{L} = -\frac{i}{2} e (\bar{\epsilon} \gamma_\mu \psi_\nu) [R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R] - i e (D_\mu \bar{\epsilon} \gamma^{\mu \nu \rho} D_\nu \psi_\rho) \ ,$$

(1.32)

where the covariant derivative contains the torsion-free spin connection. The second term, integrated by parts, gives

$$\frac{i}{8} e \bar{\epsilon} \gamma^{\mu \nu \rho} \gamma^{m n} \psi_\mu R_{\rho \mu \nu m n} \ ,$$

(1.33)

and using the properties of the Clifford algebra and the cyclic identity for the Riemann tensor one can show that it cancels with the variation of the Einstein-Hilbert action, so that supersymmetry holds to lowest order in the Fermi fields. We emphasize that this result applies in any dimension, although the actual coefficient in front of the Rarita-Schwinger action depends on the reality properties of the spinor.

We now want to show that supersymmetry holds to all orders in the fermi fields, with a suitable definition of $\omega$. In eq. (1.31), the variation of the vielbein does not receive any further correction, while one could add terms proportional to $\psi^2 e$ to the variation of $\psi$. This is a general feature of supersymmetric models: as we will see also in other cases, the transformations of the bosonic fields are quadratic in the fermions, and are competely determined at the lowest order, while the transformation of the
gravitino contains a derivative term and terms cubic in the fermions. For this reason the lagrangian can contain two-derivative terms, one-derivative terms contracted with fermionic bilinears and four-fermi terms. Any higher order term in the fermions would correspond by supersymmetry to higher derivative terms.

The spin connection that satisfies its field equation is in this case

\[ \omega_{\mu mn} = \omega_{\mu mn}^0 - \frac{i}{2} \epsilon^\nu \epsilon^n \epsilon^m \left[ (\bar{\psi}_\mu \gamma_\nu \gamma_\rho \psi_\rho) + (\bar{\psi}_\nu \gamma_\rho \psi_\mu) + (\bar{\psi}_\nu \gamma_\mu \psi_\rho) \right] , \tag{1.34} \]

and consequently the coupling of gravity with the gravitino in four dimensions induces the torsion

\[ S_{\mu \nu \rho} = -\frac{i}{2} (\bar{\psi}_\nu \gamma_\mu \psi_\rho) \tag{1.35} \]

As we will see in the next section, these two relations will be modified by a bilinear in the gravitino with five \( \gamma \)-matrices in dimensions greater than 4. Observe that the connection defined in eq. (1.34) is supercovariant, since its supersymmetry variation does not contain terms proportional to the derivative of \( \epsilon \). Only for minimal supergravity in four dimensions the supercovariant spin connection is the one that satisfies its field equation, while in more complicated models additional fermions contribute to the torsion and to the connection as well. The concept of supercovariance is very useful in order to construct supergravity theories, and we will also use it in Chapter 4, when we will determine minimal six-dimensional supergravity theories to all orders in the Fermi fields. Working in the 1.5 formalism, we do not consider the variation of the connection (1.34).

Because of the Fierz identity

\[ \gamma_m \gamma_{\mu} \bar{\psi}_\nu \gamma^m \psi_\rho = 0 \tag{1.36} \]

that holds in four dimensions as a consequence of \( \gamma_m \gamma^{np} \gamma^m = 0 \), one can show that the gravitino field equation resulting from the variation of the complete lagrangian in the 1.5 formalism,

\[ -i \gamma^{np} D_\nu \psi_\rho = 0 \tag{1.37} \]

is supercovariant, with the connection (1.34). Using similar relations one can then show that the lagrangian of eq. (1.30) is invariant under the variations (1.31), with the spin connection defined in eq. (1.34). In this model supercovariance completely determines the supersymmetry transformation and the field equation for the gravitino, and one can show using the Fierz relations collected in the Appendix that the complete variation of the action vanishes.
We now show that the closure of the supersymmetry algebra gives exactly the same gravitino equation, from which one can recover the complete lagrangian (1.30). Given a supergravity model, the commutator of two supersymmetry transformations on any field in the multiplet gives all the possible local transformations for that field (supersymmetry, general coordinate, local Lorentz and gauge transformations). In addition, for fermionic fields, the algebra closes on shell. For instance, the commutator of two supersymmetry transformations acting on the vielbein is

$$\left[\delta_{c_1}, \delta_{c_2}\right]e^m_\mu = \delta_{gct}e^m_\mu + \delta_{lL}e^m_\mu + \delta_{susy}e^m_\mu ,$$

(1.38)

where the parameters of general coordinate, local Lorentz and supersymmetry transformations are

$$\xi_\mu = i(\bar{\epsilon}_2 \gamma_\mu \epsilon_1) \quad , \quad \Omega^{mn} = -\xi^\nu \omega^{mn}_\nu \quad , \quad \zeta = \xi^\nu \psi_\nu .$$

(1.39)

Performing the same commutator on $\psi$, one can extract from it all the local symmetries, with the same parameters as in eq. (1.39). The terms that are left must be zero on-shell, and thus from them one can read the field equation for $\psi$. From the field equation for the gravitino, one can determine the lagrangian completely. The end result is

$$\left[\delta_{c_1}, \delta_{c_2}\right]e^a_\mu = \delta_{gct}\psi + \delta_{lL}\psi + \delta_{susy}\psi$$

$$+ 3i\xi_\nu \gamma^\nu [(eq.\psi)_\mu - \frac{1}{3} \gamma_\mu (\gamma - \text{trace})] - 2i\xi_\nu \gamma_\mu [(eq.\psi)_\nu - \frac{1}{2} \gamma_\nu (\gamma - \text{trace})]$$

$$- \frac{1}{4} (\bar{\epsilon}_1 \gamma^\rho \epsilon_2) \gamma^\rho \gamma_\mu [(eq.\psi)_\nu - \frac{1}{2} \gamma_\nu (\gamma - \text{trace})] ,$$

(1.40)

where with $\gamma - \text{trace}$ we mean the gravitino equation contracted with a $\gamma$ matrix, i.e.

$$-2i\gamma^{\nu\rho}D_\nu \psi_\rho = 0 .$$

(1.41)

### 1.3 Eleven-dimensional supergravity

The number of components of the supersymmetry charge for minimal supersymmetry in four dimensions is 4. The chiral multiplet that we described in the introductory section, the gravity multiplet and the vector multiplet, containing a vector and a Weyl spinor, are all the multiplets with minimal supersymmetry in four dimensions. For

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1. Observe that the connection must not be kept fixed when computing the commutator, since it is fixed in the lagrangian just because its variation generates the field equation.
all these multiplets, the number of bosonic and fermionic on-shell degrees of freedom is 2. In four dimensions one can also consider theories with extended supersymmetry. For instance the $\mathcal{N} = 2$ gravity multiplet contains the graviton, two gravitinos and an abelian vector (the graviphoton). The on-shell matching of Bose and Fermi degrees of freedom is straightforward, and the number of supercharges is in this case 8. Continuing this way, one can show that the maximal number of supercharges compatible with representations of ordinary (i.e. spin $\leq 2$) fields is 32. Moreover, in this case one has a single representation of the supersymmetry algebra that contains fields with spin $\leq 2$, that is the $\mathcal{N} = 8$ gravity multiplet [6]. The corresponding lagrangian can be obtained by dimensional reduction from eleven-dimensional supergravity. Indeed, analyzing the properties of the spinors in various dimensions (see the Appendix), one can show that in $D = 11$ the only possibility is to have 32 supercharges [7], and the corresponding multiplet is the gravity multiplet, containing the graviton (44 on-shell degrees of freedom), an antisymmetric 3-form (84 on-shell degrees of freedom) and a Majorana gravitino (128 on-shell degrees of freedom). Dimensional reduction of this theory gives maximal supergravity in any dimension. As we will see in the next section and in Chapter 4, one can also have theories with 16 supercharges starting from ten dimensions, and theories with 8 supercharges starting from six dimensions [8]. $D = 11$ is the maximal dimension for which one can realize supersymmetry in terms of an ordinary supergravity theory.

Returning to the eleven-dimensional supergravity [8], to lowest order, the lagrangian

$$\mathcal{L} = -\frac{e}{4} R - \frac{ie}{2} (\bar{\psi}_\mu \gamma^{\mu\nu} D_\nu \psi_\rho) - \frac{e}{48} F_{\mu\nu\rho\sigma} \gamma^{\mu\nu}$$

$$+ \frac{e}{96} F_{\rho\sigma\delta\tau} (\bar{\psi} \gamma^{\mu\nu\rho\sigma\delta\tau} \psi_\nu) + \frac{e}{8} F_{\mu\nu\rho\sigma} (\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\sigma)$$

$$+ \frac{1}{144 \cdot 72} \epsilon^{\alpha_1 \alpha_4 \beta_1 \beta_4 \mu \nu \rho \sigma} F_{\alpha_1 \cdots \alpha_4} F_{\beta_1 \cdots \beta_4} A_{\mu \nu \rho \sigma}$$

(1.42)

is invariant under the supersymmetry transformations

$$\delta e_\mu^m = -i (\epsilon^{m} \gamma^\mu \psi_\mu)$$

$$\delta \psi_\mu = D_\mu \epsilon + \frac{i}{144} F^{\nu\rho\sigma\tau} \gamma_{\mu\nu\rho\sigma\tau} \epsilon - \frac{i}{18} F_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma} \epsilon$$

$$\delta A_{\mu \nu \rho} = \frac{3}{2} (\bar{\epsilon} \gamma_{\mu \nu} \psi_\rho)$$

(1.43)

where the field strength

$$F_{\mu_1 \mu_2 \mu_3 \mu_4} = 4 \partial_{[\mu_1} A_{\mu_2 \mu_3 \mu_4]}$$

(1.44)
is invariant under the gauge transformation \( \delta A_{\mu
u\rho} = 3\partial_{[\mu}A_{\nu\rho]} \). In order to prove the invariance of the lagrangian, observe that the first term in the variation of the gravitino produces eq. (1.33), while this term cancels as in \( D = 4 \) against the variation of the Einstein-Hilbert action; the additional term containing five antisymmetrized \( \gamma \) matrices, absent in \( D = 4 \), vanishes because of the cyclic identity

\[
R^\alpha_{\mu
u\rho} = 0.
\]  

(1.45)

The cancellation of the terms containing \( F \) and a derivative or \( F^2 \) can be proved using the relation

\[
\gamma_{\mu_1...\mu_n} = \frac{i(-1)^{[n/2]}}{e(11-n)!}e^{\mu_1...\mu_n\nu_1...\nu_{11-n}}\gamma_{\nu_1...\nu_{11-n}}
\]

(1.46)

derived from the similar ten-dimensional relation \( (A.27) \) obtained in the Appendix. Observe the presence of the Wess-Zumino term \( A \wedge F \wedge F \) in the last line of eq. (1.50). Similar terms typically appear also in lower-dimensional supergravity theories, and in the next chapters we will see their implications, in particular for anomaly cancellations.

We now want to prove supersymmetry to all orders in the Fermi fields, working in the 1.5 order formalism. The spin connection that satisfies its field equation is

\[
\omega_{\mu n} = \omega_{\mu n}^0 - \frac{i}{2} e^\nu m e^\rho \left[ (\bar{\psi}_\mu \gamma_\nu \psi_\rho) + (\bar{\psi}_\nu \gamma_\rho \psi_\mu) + (\bar{\psi}_\nu \gamma_\mu \psi_\rho) \right]
\]

\[
- \frac{i}{4} (\bar{\psi}_\nu \gamma_{\mu mn \nu} \psi_\rho)
\]

(1.47)

and differs from the supercovariant spin connection, that is given as in \( D = 4 \) by

\[
\hat{\omega}_{\mu n} = \omega_{\mu n}^0 - \frac{i}{2} e^\nu m e^\rho n \left[ (\bar{\psi}_\mu \gamma_\nu \psi_\rho) + (\bar{\psi}_\nu \gamma_\rho \psi_\mu) + (\bar{\psi}_\nu \gamma_\mu \psi_\rho) \right].
\]

(1.48)

Similarly, one can define the supercovariant 4-form field strength

\[
\hat{F}_{\mu\rho\sigma} = F_{\mu\rho\sigma} - 3(\bar{\psi}_{[\mu} \gamma_{\rho\sigma]} \psi_\chi)
\]

(1.49)

and as in four dimensions, one can determine the lagrangian that gives a supercovariant gravitino field equation. The result is

\[
\mathcal{L} = -\frac{e}{4} R - \frac{i e}{2} \bar{\psi}_\mu \gamma_{\mu\rho\sigma} D_\nu \left( \frac{1}{2} \omega + \hat{\omega}_\nu \right) \psi_\rho - \frac{e}{48} F_{\mu\rho\sigma} F^{\mu\rho\sigma}
\]

\[
+ \frac{e}{192} (F + \hat{F})_{\rho\sigma\delta\tau} (\bar{\psi}_\mu \gamma_{\rho\sigma\delta\tau} \psi_\nu) + \frac{e}{16} (F + \hat{F})_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma_{\rho\sigma} \psi_\nu)
\]

\[
+ \frac{1}{144 \cdot 72} \epsilon^{\alpha_1...\alpha_4 \beta_1...\beta_4 \mu \rho \sigma} F_{\alpha_1...\alpha_4} F_{\beta_1...\beta_4} A_{\mu \rho \sigma}
\]

(1.50)
One can show that the resulting gravitino equation is

\[-ie^{\gamma_{\mu\nu}}D_{\nu}(\bar{\psi})\psi_{\mu} + \frac{e}{48}\hat{F}_{\rho\sigma\delta\tau}\gamma^{\mu\rho\sigma\delta\tau}\psi_{\nu} + \frac{e}{4}\hat{F}_{\mu\nu\rho\sigma}\gamma_{\mu\nu}\psi_{\sigma} = 0 \quad (1.51)\]

using the Fierz identity

\[
\frac{1}{8}\gamma^{\mu\nu\alpha\beta\gamma\delta}(\bar{\psi}_{\alpha}\gamma_{\beta}^{\gamma}\psi_{\delta}) - \frac{1}{8}\gamma_{\beta}\gamma^{\nu}(\bar{\psi}_{\alpha}\gamma_{\mu\alpha\beta\gamma}\psi_{\delta}) \\
- \frac{1}{4}\gamma^{\mu\alpha\beta\gamma}\psi_{\nu}(\bar{\psi}_{\alpha}\gamma_{\beta}\psi_{\gamma}) + \frac{1}{4}\gamma_{\beta}(\bar{\psi}_{\alpha}\gamma_{\mu\alpha\beta\gamma}\psi_{\gamma}) \\
+ \gamma^{\alpha\beta}\gamma_{\beta}(\bar{\psi}_{\alpha}\gamma_{\mu}\psi_{\gamma}) - 2\gamma^{\mu\alpha\beta}\psi_{\beta}(\bar{\psi}_{\alpha}\gamma_{\gamma}\psi_{\gamma}) + \gamma^{\mu\alpha}\psi_{\gamma}(\bar{\psi}_{\alpha}\gamma_{\beta}\psi_{\gamma}) = 0 \quad . (1.52)
\]

Using similar Fierz identities one can then show that the lagrangian (1.50) is invariant under the supersymmetry transformations

\[
\delta e_{m}^{\mu} = -i(\bar{e}\gamma^{m}\psi_{\mu}) \\
\delta \psi_{\mu} = D_{\mu}(\bar{\omega})\epsilon + \frac{i}{144}\hat{F}^{\nu\rho\sigma\tau}\gamma_{\mu\nu\rho\sigma\tau}\epsilon - \frac{i}{18}\hat{F}_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon \\
\delta A_{\mu\nu} = \frac{3}{2}(\bar{e}\gamma_{[\mu\nu]}\psi_{\rho]}). \quad (1.53)
\]

Observe again that both the field equation and the supersymmetry transformation of the gravitino are supercovariant.

As we showed in the four dimensional case, one can arrive at the same result imposing the closure of the supersymmetry algebra. Indeed, the commutator of two supersymmetry transformations gives local Lorentz, general coordinate and supersymmetry transformations on the vielbein, while in the case of the 3-form gives an additional gauge transformation and in the case of the gravitino additional terms proportional to its field equation. The parameters of general coordinate, local Lorentz, supersymmetry and 3-form gauge transformations are

\[
\xi_{\mu} = i(\bar{e}_{2}\gamma_{\mu}\epsilon_{1}) \quad , \\
\Omega_{\mu\nu} = -\xi^{\nu}\omega_{\nu}^{\mu} + \frac{1}{72}\hat{F}_{\mu\nu\rho}(\bar{e}_{2}\gamma_{\mu\nu\rho}\epsilon_{1}) + \frac{1}{3}\hat{F}_{\mu\nu\rho}(\bar{e}_{2}\gamma_{\mu\nu}\epsilon_{1}) \quad , \\
\zeta = \xi^{\nu}\psi_{\nu} \quad , \\
\Lambda_{\mu\nu} = \frac{1}{2}(\bar{e}_{2}\gamma_{\mu\nu}\epsilon_{1}) + \xi^{\rho}A_{\mu\nu\rho}. \quad (1.54)
\]

Performing the same commutator on \(\psi\), and extracting from it all the local symmetries, with the same parameters as in eq. (1.54), one can read the field equation for \(\psi\), that results to be the supercovariant equation (1.51). Once the field equation for the gravitino is known, integrating it one determines the lagrangian completely.
1.4 Ten-dimensional supergravities

By dimensional analysis, one can reinsert Newton’s constant in eqs. \((1.50)\) and \((1.53)\), obtaining

\[
\mathcal{L} = -\frac{e}{4\kappa^2} R - \frac{ie}{2} (\bar{\psi}_\mu \gamma^{\mu\nu} D_\nu (\omega + \dot{\omega}) \psi_\rho) - \frac{e}{48} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \\
+ \frac{e\kappa}{192} (F + \hat{F})_{\rho\sigma\delta\tau} (\bar{\psi}_\gamma \gamma^{\mu\rho\sigma\delta\tau} \psi_\nu) + \frac{e}{16} (F + \hat{F})_{\mu\nu\rho\sigma} (\bar{\psi}_\mu \gamma^{\nu\rho\sigma} \psi_\sigma) \\
+ \frac{\kappa}{144 \cdot 72} \epsilon_{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} F_{\alpha_1 \ldots \alpha_4} F_{\beta_1 \ldots \beta_4} A_{\mu\nu\rho} 
\tag{1.55}
\]

and

\[
\delta e_\mu^m = -i\kappa (\bar{e}_\gamma^m \psi_\mu) \\
\delta \bar{\psi}_\mu = \frac{1}{\kappa} D_\mu \epsilon + \frac{i}{144} \hat{F}^{\nu\rho\sigma\tau} \gamma_{\mu\nu\rho\sigma\tau} \epsilon - \frac{i}{18} \hat{F}_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma} \epsilon \\
\delta A_{\mu\nu\rho} = \frac{3}{2} (\bar{e}_\gamma_{[\mu\nu]} \psi_\rho) . 
\tag{1.56}
\]

1.4 Ten-dimensional supergravities

In this section we shortly describe supergravity theories in ten dimensions. There are two supergravity theories with 32 supercharges, the \(\mathcal{N} = 2a\) and \(\mathcal{N} = 2b\) supergravities, and one theory with 16 supercharges, the \(\mathcal{N} = 1\) supergravity, that can be coupled to a Yang-Mills vector multiplet.

1.4.1 \(\mathcal{N} = 2a\) supergravity

Dimensional reduction of eleven-dimensional supergravity gives \(\mathcal{N} = 2a\) (or \(\mathcal{N} = (1,1)\)) supergravity in ten dimensions (the notation \((1,1)\) means that the supersymmetry charge is a non-chiral Majorana spinor, corresponding to 16 left-handed and 16 right-handed supercharges). It is straightforward to derive the field content of the multiplet, given by the graviton, a 3-form, a 2-form, a vector and a scalar in the bosonic sector and by a Majorana gravitino and a Majorana spinor (both non-chiral) in the fermionic sector. One can verify that the number of bosonic and fermionic on-shell degrees of freedom coincide.

In order to compactify the lagrangian \((1.53)\) on a circle of radius \(r\) (in eleven-dimensional units), we make the ansatz

\[
\hat{e}_M^A = \begin{pmatrix} \Phi^\alpha \epsilon^\alpha_\mu & \Phi A_\mu \\ 0 & \Phi \end{pmatrix} ,
\tag{1.57}
\]
for the vielbein, that corresponds to the ansatz
\[
ds^2 = \Phi^{2\alpha} g_{\mu\nu} dx^\mu dx^\nu - (\Phi A_\mu dx^\mu + \Phi dx_{10})^2
\]
for the metric. We are then interested in the massless sector of the resulting ten-dimensional theory. Denoting with \( \Delta \) the compact dimension, the original eleven-dimensional general coordinate transformation for \( A_{MNP} \) becomes for \( A_\mu \) a ten-dimensional general coordinate transformation plus an additional gauge transformation with respect to the gauge field \( A_\mu \) defined in eq. (1.57). More precisely, defining
\[
F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]} \\
F_{\mu\nu} = 3\partial_{[\mu} A_{\nu]} ,
\]
where \( A_{\mu\nu} = A_{\mu\nu}\Delta \), one obtains that the 4-form
\[
H_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma} + 4A_{[\mu} F_{\nu\rho\sigma]} ,
\]
is gauge invariant. With these definitions, and denoting with \( \kappa_{11} \) the eleven-dimensional Newton constant, it can be verified that the dimensional reduction of the bosonic sector of eleven-dimensional supergravity gives the ten-dimensional lagrangian
\[
L_{2a} = -\frac{e}{4\kappa^2} \Phi^{8\alpha+1} R - \frac{9\alpha(8\alpha + 2)e}{4\kappa^2} \Phi^{8\alpha-1} \partial_\mu \Phi \partial^\mu \Phi - \frac{e}{16\kappa^2} \Phi^{6\alpha+3} F_{\mu\nu} F^{\mu\nu} \\
- \frac{(2\pi r)e}{48} \Phi^{2\alpha+1} H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} + \frac{(2\pi r)e}{12} \Phi^{4\alpha-1} F_{\mu\nu\rho} F^{\mu\nu\rho} \\
+ \frac{(2\pi r)^2 \kappa}{8 \cdot 144} \epsilon^{\alpha_1...\alpha_4\beta_1...\beta_4\mu\nu} F_{\alpha_1...\alpha_4} F_{\beta_1...\beta_4} A_{\mu\nu} ,
\]
where \( \kappa^2 = \kappa_{11}^2/2\pi r \) defines the ten-dimensional Newton’s constant.

The fermionic terms and the supersymmetry transformations can be derived directly by dimensional reduction, and will not be considered here. Rather, we want to emphasize that different choices of \( \alpha \), corresponding to Weyl rescalings, give the same theory in different frames. The choice \( \alpha = -1/8 \) corresponds to the Einstein frame, in which the Einstein-Hilbert term becomes no more dependent on the scalar, and the action assumes the form (setting for simplicity \( \kappa = 2\pi r = 1 \))
\[
L = -\frac{e}{4} R + \frac{e}{2} \partial_\mu \phi \partial^\mu \phi - \frac{e}{16} e^{3\phi} F_{\mu\nu} F^{\mu\nu} \\
- \frac{e}{48} e^\phi H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} + \frac{e}{12} e^{-2\phi} F_{\mu\nu\rho} F^{\mu\nu\rho} \\
+ \frac{1}{8 \cdot 144} \epsilon^{\alpha_1...\alpha_4\beta_1...\beta_4\mu\nu} F_{\alpha_1...\alpha_4} F_{\beta_1...\beta_4} A_{\mu\nu} ,
\]
where $\Phi = e^{\frac{\Phi}{3}}$. We will see in the next chapter that it is interesting to consider the same lagrangian in the *string frame*, in which the dependence on the scalar vanishes in both the kinetic terms for the 1-form and for the 3-form. This corresponds to putting $\alpha = -1/2$, and the resulting lagrangian is

$$L = e^{-2\phi}[-\frac{1}{4}R - \partial_\mu \phi \partial^\mu \phi + \frac{1}{12} F_{\mu \nu \rho} F^{\mu \nu \rho}]$$

$$- \frac{e}{16} F_{\mu \nu} F^{\mu \nu} - \frac{e}{48} H_{\mu \nu \rho \sigma} H^{\mu \nu \rho \sigma}$$

$$+ \frac{1}{8 \cdot 144} e^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu \nu} F_{\alpha_1 \ldots \alpha_4} F_{\beta_1 \ldots \beta_4} A_{\mu \nu} ,$$

where $\Phi = e^{\frac{\Phi}{3}}$.

### 1.4.2 $\mathcal{N} = 2b$ supergravity

In this subsection we describe the lowest order terms of $\mathcal{N} = 2b$ (or $\mathcal{N} = (2,0)$) supergravity [10, 11]. This theory can not be obtained by reduction from a higher-dimensional lagrangian, and contains the graviton, two scalars, two 2-forms and a self-dual 4-form in the bosonic sector, together with a complex left-handed gravitino and a complex right-handed spinor in the fermionic sector.

In [10] the field equations for this model were derived to lowest order in the Fermi fields requiring the closure of the supersymmetry algebra. We will see that all these equations, with the exception of the self-duality condition for the field strength of the 4-form,

$$F^{\mu_1 \ldots \mu_5} = \frac{1}{5!} \epsilon^{\mu_1 \ldots \mu_5 \nu_1 \ldots \nu_5} F_{\nu_1 \ldots \nu_5} ,$$

can be derived from a lagrangian, imposing eq. (1.64) only after varying. More recently, a lagrangian formulation for self dual forms has been developed by Pasti, Sorokin and Tonin [12], and then applied in [13] to the ten-dimensional $\mathcal{N} = 2b$ supergravity. This PST method corresponds to the introduction of an additional scalar auxiliary field, and the self-duality condition results from the gauge fixing (that can not be imposed directly on the action) of additional (PST) local symmetries. It will be used in Chapters 4 and 5 in order to derive the action for six-dimensional supergravity theories, while here we only assume that we have already fixed the PST gauge so that eq. (1.64) holds on-shell.

Let us now summarize the field content of the theory. The two scalars parametrize the coset $SU(1, 1)/U(1)$, that can be described in terms of the $SU(1, 1)$ matrix

$$U = ( \begin{pmatrix} V^\alpha & V_+^\alpha \end{pmatrix} ) ,$$

(1.65)
Chapter 1. Generalities about supergravity theories

satisfying the constraint

\[ V_\alpha V_\beta^\alpha - V_\beta V_\alpha^\beta = \epsilon_\alpha^\beta , \]  

(1.66)

with \((V_\alpha)^* = \epsilon_\alpha^\beta V_\beta^\alpha\), where \(\alpha = 1, 2\) is an \(SU(1, 1)\) index and + and − denote the \(U(1)\) charge. From the left-invariant 1-form

\[ U^{-1} \partial U = \left( \begin{array}{cc} -iQ_\mu & P_\mu^- \\ P_\mu^+ & iQ_\mu \end{array} \right) \]  

(1.67)

one reads the \(U(1)\)-covariant quantity

\[ P_\mu = \epsilon_\alpha^\beta V_\beta^\alpha \partial_\mu V_\alpha^- , \]  

(1.68)

that has charge 2, and the \(U(1)\) connection

\[ Q_\mu = i\epsilon_\alpha^\beta V_\beta^\alpha \partial_\mu V_\alpha^- . \]  

(1.69)

The 2-forms are collected in an \(SU(1, 1)\) doublet \(A_{\mu\nu}^\alpha\) satisfying the constraint

\[ (A_{\mu\nu}^\alpha)^* = \epsilon_\alpha^\beta A_{\mu\nu}^\beta , \]  

(1.70)

while the 4-form is invariant under \(SU(1, 1)\), and varies as

\[ \delta A_{\mu\nu\rho\sigma} = -\frac{i}{4} \epsilon_\alpha^\beta A_{[\mu}^\alpha A_{\nu\rho\sigma]}^\beta \]  

(1.71)

under 2-form gauge transformations, where \(\delta A_{\mu\nu}^\alpha = 2\partial_{[\mu} A_{\nu]}^\alpha\) and \(F_{\mu\nu\rho}^\alpha = 3\partial_{[\mu} A_{\nu\rho]}^\alpha\), so that the proper gauge-invariant 5-form field-strength is

\[ F_{\mu\nu\rho\sigma\tau} = 5\partial_{[\mu} A_{\nu\rho\sigma\tau]} + \frac{5i}{8} \epsilon_\alpha^\beta A_{[\mu\nu}^\alpha F_{\rho\sigma\tau]}^\beta . \]  

(1.72)

This 5-form satisfies the self-duality condition (1.64). It is convenient to define the complex 3-form

\[ G_{\mu\nu\rho} = -\epsilon_\alpha^\beta V_\alpha^\beta F_{\mu\nu\rho}^\beta , \]  

(1.73)

that is an \(SU(1, 1)\) singlet with \(U(1)\) charge 1, and finally the gravitino has \(U(1)\) charge 1/2, while the spinor has \(U(1)\) charge 3/2 [10].

The lagrangian

\[ \mathcal{L} = -\frac{e}{4} R + \frac{e}{2} F_\mu^\rho P_\mu + \frac{e}{48} G_{\mu\nu\rho} G^{\mu\nu\rho} 
\]  

\[ + \frac{e}{5!} F_{\mu_1\ldots\mu_5} F^{\mu_1\ldots\mu_5} + \frac{i}{4 \cdot 123} \epsilon_\alpha^\beta \epsilon^{\mu_1\ldots\mu_{10}} A_{\mu_1\ldots\mu_4} F_{\mu_5\ldots\mu_7}^\alpha F_{\mu_8\ldots\mu_{10}}^\beta 
\]  

\[- i e (\bar{\psi}_\mu \gamma^\mu \gamma D_\nu \psi) + i e (\bar{\chi} \gamma^\mu D_\mu \chi) \]
\[ \begin{align*}
- \frac{e}{12} F_{\mu_1...\mu_5} (\psi^{\mu_1\gamma_2 \mu_3 \mu_4 \psi^{\mu_5} + \frac{e}{2 \cdot 5!} F_{\mu_1...\mu_5} (\bar{\psi}_\mu \gamma^{\mu \nu \mu_1...\mu_5} \psi_\nu) \\
+ \left\{ \frac{ie}{2 \sqrt{2}} G_{\mu \nu} (\bar{\psi}_\mu \gamma^\nu \psi^\nu_C) - \frac{ie}{12 \sqrt{2}} G_{\mu \nu \rho \sigma} (\bar{\psi}_\sigma \gamma^{\mu \nu \rho \sigma} \psi^\sigma_C) \\
- \frac{e}{\sqrt{2}} P_\mu (\bar{\psi}_\mu \gamma^\mu \chi C) + \frac{e}{2 \cdot 5!} G_{\mu \nu}^\rho (\bar{\psi}_\rho \gamma^{\mu \nu \rho \sigma} \chi) + \text{h. c.} \right\} \\
- \frac{e}{5!} F_{\mu_1...\mu_5} (\bar{\chi} \gamma^{\mu_1...\mu_5} \chi) \right) \tag{1.74}
\end{align*} \]

is invariant under the supersymmetry transformations

\[ \begin{align*}
\delta \epsilon^\alpha_\mu &= -i (\bar{\epsilon}^\alpha \gamma^\mu \psi_\mu) + \text{h. c.} \\
\delta V^\alpha_+ &= \sqrt{2} V^-_\alpha (\bar{\epsilon} C \chi) \\
\delta V^-_\alpha &= \sqrt{2} V^\alpha_+ (\bar{\epsilon} C) \\
\delta A^\alpha_{\mu \nu} &= -\frac{1}{2} V^-_\alpha (\bar{\epsilon} \gamma_{\mu \nu} \chi) - i \sqrt{2} V^\alpha_+ (\bar{\epsilon} C \gamma_{\mu \nu} \psi_\rho) + \text{h. c.} \\
\delta A_{\mu \rho \sigma} &= 4 (\bar{\epsilon} \gamma_{\mu \rho \sigma} \psi_\rho) + \text{h. c.} - \frac{3i}{8} \epsilon_{\alpha \beta} A^\alpha_{[\mu \nu} \delta A^\beta_{\rho \sigma]} \\
\delta \psi_\mu &= D_\mu \epsilon - \frac{i}{48} F_{\mu \nu \rho \sigma} \gamma^{\mu_1...\mu_4} \epsilon \\
&+ \frac{1}{24 \sqrt{2}} G^{\mu \rho \sigma} \gamma_{\mu \rho \sigma} \epsilon C - \frac{3}{8 \sqrt{2}} G_{\mu \rho \sigma} \gamma^{\mu \rho} \epsilon C \\
\delta \chi &= -\frac{i}{\sqrt{2}} P_\mu \gamma^\mu C - \frac{i}{12} G_{\mu \rho} \gamma^{\mu \rho} \epsilon \tag{1.75}
\end{align*} \]

provided one imposes the self-duality condition of eq. (1.64) after varying.

It is interesting to study in more detail the kinetic term for the scalar fields. The complex variable

\[ z = \frac{V^+_1}{V^+_2} \tag{1.76} \]

is invariant under local \( U(1) \) transformations, and so it is a good coordinate for the scalar manifold. Under the \( SU(1,1) \) transformation

\[ \begin{pmatrix} V^+_1 \\ V^+_2 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} V^+_1 \\ V^+_2 \end{pmatrix} \tag{1.77} \]

that is an isometry of the scalar manifold, \( z \) transforms as

\[ z \rightarrow \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}. \tag{1.78} \]

The variable \( z \) parametrizes the unit disc, \(|z| < 1\), and the kinetic term assumes the form

\[ L_{\text{scalar}} = \frac{e}{2} \frac{\partial_\mu z \partial^\mu \bar{z}}{(1 - z \bar{z})^2} \tag{1.79} \]
The further change of variables

\[ z = \frac{w - i}{w + i} \quad (1.80) \]

maps the disc in the complex upper-half plane, \( \text{Im}\ w > 0 \), and in terms of \( w \) the transformations (1.77) become

\[ w \rightarrow \frac{aw + b}{cw + d} \quad , \quad (1.81) \]

where

\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, R) \quad , \quad (1.82) \]

while the scalar lagrangian takes the form

\[ \mathcal{L}_{\text{scalar}} = \frac{e}{8} \frac{\partial \mu w \partial \mu \bar{w}}{(\text{Im} w)^2} \quad . \quad (1.83) \]

We now want to see how these redefinitions modify the form of the bosonic part of the action. First of all, we define \( F_{\mu\nu\rho} \) such that

\[ G_{\mu\nu\rho} = -V_+^1 F_{\mu\nu\rho}^* + V_+^2 F_{\mu\nu\rho} \quad , \quad (1.84) \]

and

\[ F_{\mu\nu\rho} = (\mathcal{F} + i\mathcal{G})_{\mu\nu\rho} \quad , \quad (1.85) \]

with \( \mathcal{F} \) and \( \mathcal{G} \) real. We also define

\[ \mathcal{F}' = \mathcal{F} + \text{RewG} \quad . \quad (1.86) \]

Writing \( w = \rho + ie^\phi \), the bosonic part of the action becomes

\[ \mathcal{L} = -\frac{e}{4} R + \frac{e}{8} e^{-2\phi} (\partial_\mu \phi)^2 + \frac{e}{8} (\partial_\mu \phi)^2 + \frac{e}{48} e^{-\phi} (\mathcal{F}'_{\mu\nu\rho})^2 
+ \frac{e}{48} e^\phi (\mathcal{G}_{\mu\nu\rho})^2 + \frac{e}{5!} (F_{\mu_1 \cdots \mu_5})^2 
+ \frac{i}{6 \cdot 24^2} e^{\mu_1 \cdots \mu_{10}} A_{\mu_1 \cdots \mu_4} F_{\mu_5 \cdots \mu_7} F^{*}_{\mu_8 \cdots \mu_{10}} \quad . \quad (1.87) \]

If we now perform the Weyl rescaling \( g_{\mu\nu} \rightarrow e^{\phi/2} g_{\mu\nu} \), we end up with 2b supergravity in the string frame,

\[ \mathcal{L} = e^{-2\phi} \left[ -\frac{e}{4} R + \frac{e}{8} (\partial_\mu \phi)^2 + \frac{e}{48} e^\phi (\mathcal{G}_{\mu\nu\rho})^2 \right] 
+ \frac{e}{8} e^{-2\phi} (\partial_\mu \phi)^2 + \frac{e}{48} e^{-\phi} (\mathcal{F}'_{\mu\nu\rho})^2 + \frac{e}{5!} (F_{\mu_1 \cdots \mu_5})^2 
+ \frac{i}{6 \cdot 24^2} e^{\mu_1 \cdots \mu_{10}} A_{\mu_1 \cdots \mu_4} F_{\mu_5 \cdots \mu_7} F^{*}_{\mu_8 \cdots \mu_{10}} \quad . \quad (1.88) \]

Observe that in this frame the \( SL(2, R) \simeq SU(1, 1) \) symmetry is no longer manifest. We will see in the next chapter the implications of this in the context of string theory.
1.4 10-dimensional supergravities

1.4.3 $\mathcal{N} = 1$ supergravity

The representations of the supersymmetry algebra with 16 supercharges in ten dimensions are the gravity multiplet, containing the graviton, a 2-form, a left-handed Majorana gravitino and a right-handed Majorana spinor, and the Yang-Mills multiplet, containing a gauge vector and a left-handed Majorana gaugino.

The lagrangian for $\mathcal{N} = 1$ supergravity coupled to vector multiplets in terms of these fields is [14, 15]

$$
e^{-1} \mathcal{L} = - \frac{1}{4} R + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{6} e^{-2\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2} e^{-\phi} \text{tr}(F_{\mu\nu} F^{\mu\nu})$$

$$- \frac{i}{2} (\bar{\psi}_{\mu} \gamma^{\mu\nu\rho} D_{\nu} \psi_{\rho}) + \frac{i}{2} (\bar{\chi} \gamma^{\mu} D_{\mu} \chi) + \frac{1}{\sqrt{2}} (\bar{\psi}_{\mu} \gamma^{\nu} \gamma^{\lambda} \chi) \partial_{\nu} \phi$$

$$- \frac{i}{12 \sqrt{2}} e^{-\phi} H_{\mu\nu\rho} (\bar{\psi}_{\sigma} \gamma^{\sigma} \delta_{\mu\nu\rho} \psi_{\delta}) + \frac{i}{2 \sqrt{2}} e^{-\phi} H_{\mu\nu\rho} (\bar{\psi}_{\mu} \gamma^{\nu} \psi_{\rho})$$

$$+ \frac{1}{12} e^{-\phi} H_{\mu\nu\rho} (\bar{\psi}_{\sigma} \gamma^{\mu\nu\rho} \gamma^{\sigma} \chi) + i \text{tr}(\bar{\lambda} \gamma^{\mu} D_{\mu} \lambda)$$

$$- \frac{1}{2} e^{-\frac{\phi}{2}} \text{tr}[F^{\mu\nu} (\bar{\lambda} \gamma_{\mu\nu} \chi)] + \frac{i}{\sqrt{2}} e^{-\frac{\phi}{2}} \text{tr}[F^{\mu\nu} (\bar{\lambda} \gamma_{\mu\nu} \psi_{\rho})]$$

$$- \frac{i}{6 \sqrt{2}} e^{-\phi} H_{\mu\nu\rho} \text{tr}(\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) , \quad (1.89)$$

up to quartic terms in the fermions. The 3-form $H_{\mu\nu\rho}$ includes a Chern-Simons coupling, so that

$$H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} + \sqrt{2} \omega_{\mu\nu\rho} , \quad (1.90)$$

where $\omega_{\mu\nu\rho}$ is the Chern-Simons 3-form defined as

$$\omega = AdA - \frac{2i}{3} A^{3} , \quad (1.91)$$

while the supersymmetry transformations are

$$\delta e_{\mu}^{m} = -i (\bar{e} \gamma^{m} \psi_{\mu}) ,$$

$$\delta \phi = - \frac{1}{\sqrt{2}} (\bar{\epsilon} \chi) ,$$

$$\delta B_{\mu\nu} = - \frac{i}{\sqrt{2}} e^{\phi} (\bar{e} \gamma_{[\mu} \psi_{\nu]}) - \frac{1}{4} e^{\phi} (\bar{e} \gamma_{\mu\nu} \chi) + 2 \sqrt{2} \text{tr}(A_{[\mu} \delta A_{\nu]}) ,$$

$$\delta A_{\mu} = - \frac{i}{\sqrt{2}} e^{\frac{3}{2} \phi} (\bar{e} \gamma_{\mu} \lambda) ,$$

$$\delta \psi_{\mu} = D_{\mu} \epsilon + \frac{1}{24 \sqrt{2}} e^{-\phi} H^{\mu\rho\sigma} \gamma_{\mu\rho[\sigma} \epsilon - \frac{3}{8 \sqrt{2}} e^{-\phi} H_{\mu\rho\sigma} \gamma^{\nu\rho} \epsilon ,$$

$$\delta \lambda = \frac{1}{24 \sqrt{2}} e^{-\phi} H_{\mu\rho\sigma} \gamma^{\nu\rho} \epsilon ,$$

$$\delta \chi = \frac{1}{24 \sqrt{2}} e^{-\phi} H^{\mu\rho\sigma} \gamma_{\mu\rho[\sigma} \chi .$$
\[ \delta \chi = -\frac{i}{\sqrt{2}} \partial_\mu \phi \gamma^\mu \epsilon - \frac{i}{12} e^{-\phi} H^{\mu \nu \rho} \gamma_{\mu \nu \rho} \epsilon, \]
\[ \delta \lambda = -\frac{1}{2\sqrt{2}} e^{-\frac{1}{2} \phi} F^{\mu \nu} \gamma_{\mu \nu} \epsilon, \] (1.92)

and gauge invariance of \( H \) requires that under vector gauge transformations \( B \) transform as
\[ \delta B = -\sqrt{2} \text{tr}(\Lambda dA). \] (1.93)

This lagrangian was initially written in [16] in terms of the 6-form dual to the 2-form. We will return to this point in Chapter 5.

In concluding this chapter, we observe that one can perform a Weyl rescaling, in order to map eq. (5.14) to a lagrangian in a different frame. In particular, the bosonic part of the action is
\[ \mathcal{L} = e^{-2\phi} \left[ -\frac{e}{4} R - e(\partial_\mu \phi)^2 + \frac{e}{12} H^{\mu \nu \rho} - \frac{e}{2} F^{\mu \nu} \right] \] (1.94)
in the \textit{heterotic-string frame}, and
\[ \mathcal{L} = e^{-2\phi} \left[ -\frac{e}{4} R - e(\partial_\mu \phi)^2 + \frac{e}{12} H^{\mu \nu \rho} - \frac{e}{2} e^{-\phi} F^{\mu \nu} \right] \] (1.95)
in the \textit{type I-string frame}. These two lagrangians are mapped one in the other by the relations
\[ g_{H, \mu \nu} = e^{-\phi_I} g_{I, \mu \nu}, \]
\[ B_{H, \mu \nu} = B_{I, \mu \nu}, \]
\[ A_{H, \mu} = A_{I, \mu}, \]
\[ \phi_H = -\phi_I. \] (1.96)

We will see that these and analogous relations in supergravity theories correspond to dualities between different string theories. In particular, the last relation in eq. (1.96) is a manifestation of a strong-weak coupling duality between the \textit{SO}(32) heterotic string and the type-I string in ten dimension.
Chapter 2

Closed strings

This chapter is a brief introduction to oriented closed strings. Most of the analysis is devoted to the derivation of one-loop vacuum amplitudes, and the results obtained will be applied in the next chapter to the derivation of type-I models. As we will see, the consistency of the models is guaranteed requiring modular invariance for the one-loop vacuum amplitudes, and this naturally implies anomaly cancellation in the low-energy effective action. We will also comment on the non-perturbative states of these models, that will turn out to have a role in perturbative type-I vacua. The picture one ends up with, after the analysis of this and the next chapter, is that perturbative type-I models embody some peculiar features in their dynamics, that typically are revealed in the closed-string setting only at the non-perturbative level.

The content of this chapter is the following. In Section 1 we shortly describe how to build the spectrum of closed strings, while Section 2 is devoted to the partition function of various superstring theories. The rules for writing partition functions are then applied to type IIB compactified on the $T^4/Z_2$ orbifold in Section 3. We will use these results in the next chapter when we discuss orientifolds of type IIB. In Section 4 we discuss gauge and gravitational anomalies in field theory, showing that two-dimensional consistency (i.e. modular invariance) of closed superstring theories always gives rise to low-energy effective actions that are anomaly-free. Finally, in Section 5 we introduce the concept of D-branes and describe how the various supersymmetric string theories can be connected at the non-perturbative level by dualities.
2.1 The spectrum of closed oriented superstrings

Before considering superstrings, we present an introduction to the bosonic string \[\text{[17, 18]}\]. The action for a bosonic string in flat space-time is

\[
S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu},
\]

where \(X^\mu(\xi)\) are the coordinates of the string in D space-time dimensions and \(\xi^\alpha (\alpha = 1, 2)\) parametrize the world-sheet, whose metric is \(g_{\alpha\beta}\), and \(\alpha'\) is a dimensionful constant, related to the string tension by \(T = \frac{1}{2\pi\alpha'}\). The field equation for the metric implies that the energy momentum tensor

\[
T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} \partial^\gamma X^\mu \partial_\gamma X_\mu
\]

vanishes, while the field equation for \(X^\mu\) is

\[
\partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta X^\mu) = 0.
\]

In two dimensions, one can use reparametrization invariance to prove that every metric is conformally equivalent to the flat metric. With \(\xi^\alpha = (\tau, \sigma)\), eq. (2.3) reduces to the standard wave equation

\[
\left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu = 0,
\]

and for a closed oriented string, with the periodicity condition \(X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)\), this equation has the solution

\[
X^\mu = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left[ \frac{\alpha^\mu_n}{n} e^{-in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)} \right].
\]

The quantization condition for \(X\) results in the commutation relations

\[
[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{m+n,0} \eta^{\mu\nu}, \quad [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\delta_{m+n,0} \eta^{\mu\nu}
\]

for the expansion modes, that thus behave like creation and annihilation operators.

In fact, the original invariance under Weyl rescalings and reparametrizations leaves a residual gauge symmetry that corresponds to arbitrary (anti)analytic reparametrizations, and can be used to eliminate the oscillators in the + direction, where
The spectrum of closed oriented superstrings

In this light cone gauge, the Virasoro operators, i.e. the modes of the energy-momentum tensor, are written in terms of the transverse oscillators as

\[
L_m = \frac{1}{2} : \sum_n \alpha^i_{m-n} \alpha^i_n : ,
\]

\[
\tilde{L}_m = \frac{1}{2} : \sum_n \tilde{\alpha}^i_{m-n} \tilde{\alpha}^i_n : .
\] (2.7)

The \(L_m\) and \(\tilde{L}_m\) are mutually commuting, and they satisfy the Virasoro algebra with central charge \(c = D - 2\):

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{D-2}{12}m(m^2-1)\delta_{m+n,0} .
\] (2.8)

The zero modes \(L_0\) and \(\tilde{L}_0\) define the mass-shell condition for physical states,

\[
M^2 = \frac{2}{\alpha'} \left( L_0 + \tilde{L}_0 - \frac{D-2}{24} \right) ,
\] (2.9)

where the shift originates from the normal ordering of \(L_0\) and \(\tilde{L}_0\), together with the level-matching condition \(L_0 = \tilde{L}_0\). This relation determines the dimension of space-time: a Lorentz invariant spectrum is obtained with \(D = 26\), with a massless first excited level, \(\alpha^i_{-1} \tilde{\alpha}^j_{-1} |0\rangle\), corresponding to a metric fluctuation \(h_{\mu\nu}\), an antisymmetric tensor \(B_{\mu\nu}\) and a scalar \(\phi\), called the dilaton, whose vacuum expectation value weights the perturbative expansion. The ground state of this model is a tachyon.

We can now analyze the supersymmetric version of the string action,

\[
S = -\frac{1}{4\pi\alpha'} \int d^2 \xi \left( \partial^\alpha X^\mu \partial_\alpha X^\nu \eta_{\mu\nu} + i \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \right) ,
\] (2.10)

where the \(\psi\)'s are Majorana spinors whose mode expansion is

\[
\psi^\mu = \frac{1}{\sqrt{2}} \sum_r \left[ \lambda_r^\mu e^{-i(r-\sigma)} + \tilde{\lambda}_r^\mu e^{-i(r+\sigma)} \right] ,
\] (2.11)

and where the oscillators \(\lambda\) satisfy the anticommutation relations

\[
\{ \lambda^\mu_r, \lambda^\nu_s \} = -\eta^{\mu\nu} \delta_{r+s} .
\] (2.12)

Actually, the periodicity of the currents of space-time symmetries is guaranteed if \(\psi\) is periodic or antiperiodic, i.e. if \(r\) is integer or half-integer. The Ramond (R) sector corresponds to integer \(r\), and the anticommutation relations for the zero mode \(\lambda_0\) give rise to the Clifford algebra, resulting in a fermionic vacuum. The Neveu-Schwarz (NS) sector, on the other hand, corresponds to half-integer \(r\), so that there are no
zero modes, and the vacuum is a scalar. The dimension of space-time is determined to be $D = 10$ requiring Lorentz invariance of the spectrum, while the super-Virasoro modes assume the form

$$L_m = \frac{1}{2} \sum_n \alpha_m^n \alpha_n^i + \frac{1}{2} \sum_r (r - m) \lambda_r^i \lambda_r^i + \delta_{m,0} \Delta,$$  

where the normal-ordering shift is $-\frac{1}{2}$ in the NS sector and vanishes in the R sector.

In the next section we will determine the spectrum of superstring theories from the analysis of their partition functions. Consistency at the one loop level imposes a suitable projection of the states, so that the spectrum one ends up with is supersymmetric and free of tachyons. This projection selects two different ten-dimensional theories of oriented closed strings, type IIA and type IIB, whose massless sectors correspond to the field contents of $2a$ and $2b$ supergravities, respectively. There is another consistent model that can be built, the heterotic string, a closed oriented string whose left movers are bosonic and right movers are supersymmetric. The spectrum of the theory has $N = 1$ supersymmetry, and at the massless level it contains the supergravity multiplet plus vector multiplets, while consistency imposes that the gauge group be $SO(32)$ or $E_8 \times E_8$. Actually, we will see that consistent non-supersymmetric models can be generated as well, but the corresponding spectra contain tachyons.

The perturbative spectra of these theories consist of a finite number of massless particles, as well as an infinite tower of massive excitations, with masses proportional to the string tension. Since this is the only scale in the theory, and the massless degrees of freedom always contain a graviton, this string scale must be naturally of the order of the Planck scale. The effective theory for the massless modes results from integrating all the massive ones, and since the expansion in derivatives is suppressed by powers of $E/M$, where $E$ is the energy scale we are considering and $M$ is the string scale, we expect that at low energy only two-derivative terms are relevant.

There are two ways to compute the effective action for the massless modes. The first consists in computing the S-matrix elements in string theory, and then extracting the low-energy limit. The second consists in studying string propagation in a curved background, and determining the equations for the background fields imposing conformal invariance of the world-sheet action. In theories with 32 or 16 supercharges, the low-energy effective action is also completely constrained by supersymmetry. The end result is that the low-energy effective actions of type IIA and type IIB superstrings are respectively $2a$ and $2b$ supergravity in the string frame, while the low-energy effective action for the heterotic string is $N = 1$ supergravity coupled to $SO(32)$ or
$E_8 \times E_8$ vector multiplets in the heterotic string frame (see Section (1.4)). In the case of type I, that we will analyze in the next chapter, the low-energy effective action is $\mathcal{N} = 1$ supergravity coupled to $SO(32)$ vector multiplets in the type I string frame. The reason for the dilaton dependence of the massless string fields in the various models is that the coupling constant regulating the loop expansion in string theory is the exponential of the vacuum expectation value of the dilaton.

2.2 Partition function for closed oriented strings

The one-loop partition function measures the vacuum energy of a given theory. For the simple case of a free scalar field of mass $M$ in $D$ dimensions, it is given by

$$e^{-\Gamma} = \int [D\phi]e^{-S_E} \sim \det^{-\frac{1}{2}}(-\Delta + M^2)$$

where $S_E$ is the euclidean free-field action. The $M$ dependence of $\Gamma$ can be extracted from the identity

$$\log \det A = -\int_\epsilon^\infty \frac{dt}{t} \text{tr}(e^{-tA})$$

where $\epsilon$ is an ultraviolet cutoff and $t$ is a Schwinger parameter. The result is

$$\Gamma = \frac{V}{2} \int_\epsilon^\infty \frac{dt}{t} e^{-tM^2} \int \frac{d^D p}{(2\pi)^D} e^{-tp^2},$$

where $V$ denotes the volume of space-time, and performing the gaussian integral yields the final expression

$$\Gamma = \frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} e^{-tM^2}.$$  \hspace{1cm} (2.17)

Generalizing this result in order to include generic Bose and Fermi fields, one obtains

$$\Gamma_{tot} = \frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \text{Str} \frac{dt}{t^{D/2+1}} e^{-tM^2},$$  \hspace{1cm} (2.18)

where $\text{Str}$ counts the multiplicities of the fields, with a minus sign in the case of fermions, on account of the Grassmann nature of the fermionic path integral. From eq. (2.18) it follows that $\Gamma$ vanishes identically for supersymmetric models. Nevertheless, also in this case one can read from its integrand the masses and multiplicities of Bose and Fermi fields independently.

We now want to derive the vacuum amplitude for oriented closed strings. As a starting point, consider the closed bosonic string in $D = 26$, whose spectrum is encoded in

$$M^2 = \frac{2}{\alpha'}(L_0 + \bar{L}_0 - 2)$$  \hspace{1cm} (2.19)
together with the level-matching condition
\[ L_0 = \bar{L}_0 \quad . \] (2.20)

Applying eq. (2.18) in this case gives
\[ \Gamma = \frac{V}{2(2\pi)^{13}} \int_{-\infty}^{\infty} ds \int_{\epsilon}^{\infty} \frac{dt}{t^{14}} \text{tr} (e^{-\frac{2}{\pi}(L_0 + \bar{L}_0 - 2)t} e^{2\pi i(L_0 - \bar{L}_0)s}) \quad . \] (2.21)

Defining the “complex” Schwinger parameter
\[ \tau = \tau_1 + i\tau_2 = s + \frac{it}{\alpha' \pi} \quad , \] (2.22)
and denoting
\[ q = e^{2\pi i \tau} \quad , \] (2.23)
eq (2.21) then takes the form
\[ \Gamma = \frac{V}{2(4\pi^2 \alpha')^{13}} \int_{-\infty}^{\infty} d\tau_1 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{14}} \text{tr} q^{L_0 - 1} q^{\bar{L}_0 - 1} \quad . \] (2.24)

In order to compute this vacuum amplitude, we should recall that \( L_0 \) and \( \bar{L}_0 \) are effectively number operators for two infinite sets of harmonic oscillators, and in terms of conventionally normalized creation and annihilation operators, for each transverse space-time dimension we have
\[ L_0 = \sum_k k a_k^+ a_k \quad . \] (2.25)

Thus, for any mass-level \( k \), we have
\[ \text{tr} q^{k a_k^+ a_k} = 1 + q^k + q^{2k} + ... = \frac{1}{1 - q^k} \quad , \] (2.26)
and putting all the contributions together, in the light-cone gauge, one gets
\[ \Gamma = \int_{C_+} \frac{d^2 \tau}{\tau_2} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} \quad , \] (2.27)
where the integral is performed over the complex upper-half plane \( C_+ \) and \( \eta \) is the Dedekind function
\[ \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad . \] (2.28)

However, a more careful analysis of the amplitude of eq. (2.27) shows that it diverges, since the integrand is invariant under the modular transformations
\[ \tau \rightarrow \frac{a \tau + b}{c \tau + d} \quad , \] (2.29)
2.2 Partition function for closed oriented strings

with \( ad - bc = 1 \), and \((a, b, c, d) \in \mathbb{Z}\). These transformations connect different points in the upper-half plane, and in order to get a finite integral we have to mod it out. We now want to explain the origin of this invariance.

The one-loop diagram for a closed oriented string is a torus, whose points are in correspondence with the points in a cell of the periodic lattice of fig. 2.1. The parameter \( \tau \) in the figure is the complex modulus (Teichmuller parameter) of the torus, and periodicity of the lattice means that the modulus is invariant under the \( \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\mathbb{Z}_2 \) modular group, whose action on \( \tau \) is exactly like in eq. (2.29). This group is generated by the two transformations

\[
T : \tau \to \tau + 1 \quad , \quad S : \tau \to -\frac{1}{\tau} ,
\]

that satisfy the relation

\[
S^2 = (ST)^3 .
\] (2.31)

As a result, all different values of \( \tau \) can be mapped by a modular transformation to the values within a fundamental region, for instance the region

\[
\mathcal{F} = \{ -1/2 < \tau_1 \leq 1/2, |\tau| \geq 1 \} \quad (2.32)
\]

of fig. (2.2).

Coming back to our amplitude, using the transformation properties of the \( \eta \) function under \( S \) and \( T \),

\[
T : \eta(\tau + 1) = e^{\frac{i\pi}{12}}\eta(\tau) \quad , \quad S : \eta(-1/\tau) = \sqrt{-i\tau_2}\eta(\tau) \quad ,
\] (2.33)
one can prove that the integrand of eq. (2.27) is modular invariant. The Schwinger parameter $\tau$ is the complex modulus of the torus, and the torus amplitude is obtained restricting the integration region to a fundamental region, for instance $F$:

$$\mathcal{T} = \int_{F} \frac{d^{2}\tau}{\tau_{2}^{2}} \frac{1}{\eta(\tau)|^{48}} .$$

(2.34)

From the amplitude, one could read the spectrum just expanding the integrand of eq. (2.34) in powers of $q$ and $\bar{q}$, and taking only the terms containing $qq$ because of the level matching condition. The factor $\tau_{2}^{-12}$ comes from the integral over the momentum, so we have just to expand

$$\frac{1}{|\eta(\tau)|^{48}} \simeq \frac{1}{q\bar{q}}[1 + 24^{2}(qq) + ...] .$$

(2.35)

The first term corresponds to the tachyon, while the second corresponds to the graviton, the 2-form and the scalar that form the massless spectrum of the bosonic string.

Our next task is to write the torus partition function for IIA and IIB superstrings in ten dimensions. As we have seen in the previous section, the Virasoro generators in the light cone gauge have the form

$$L_{m} = \frac{1}{2} :\sum_{n} \alpha_{m-n}^{i} \alpha_{n}^{i} : + \frac{1}{2} :\sum_{r}(r - m)\lambda_{m-r}^{i} \lambda_{r}^{i} : + \delta_{m,0} \Delta ,$$

(2.36)

where $r$ is half-odd integer in the NS sector and integer in the R sector. The normal-ordering shift is determined by the following rule: each fermionic coordinate contributes $-\frac{1}{24}$ in the NS sector and $\frac{1}{12}$ in the R sector while, as for the bosonic string,
2.2 Partition function for closed oriented strings

Each bosonic coordinate contributes \(-\frac{1}{12}\). As a result, the total shift in ten dimensions is \(-\frac{1}{2}\) in the NS sector and vanishes in the R sector.

Let us first consider the NS sector. In this case the (antiperiodic) transverse fermions \(\lambda^i\) do not have zero modes, and so the vacuum is a scalar. The vacuum amplitude receives contributions from these fermionic oscillators, and since

\[
\text{tr}(q^{\sum_r r \lambda^i \lambda^r}) = \prod_r \text{tr}(q^{r \lambda^r \lambda^r}) = \prod_r (1 + q^r)^8 ,
\]

including the contribution of the bosonic oscillators, in this sector we have

\[
\text{tr}(q^{L_0}) = \frac{\prod_{m=1}^{\infty} (1 + q^m)^8}{q^{1/2} \prod_{m=1}^{\infty} (1 - q^m)^8} .
\]

In the R sector, the zero modes of the \(\lambda^i\) imply that the vacuum carries a 16-dimensional representation of the \(SO(8)\) Clifford algebra, and is thus a space-time spinor, like all its excitations. Consequently, the analogue of eq. (2.38) is

\[
\text{tr}(q^{L_0}) = 16 \prod_{m=1}^{\infty} (1 + q^m)^8 \prod_{m=1}^{\infty} (1 - q^m)^8 .
\]

The factor \(q^{1/2}\) is now absent in the denominator since, as we have seen, the R sector has \(\Delta = 0\), and so it starts with massless modes.

In order to build modular-invariant quantities from these expressions, one has to suitably project the spectrum. The simplest possibility is then to project out all states created by even numbers of fermionic oscillators. This prescription, originally proposed by Gliozzi, Scherk and Olive (GSO) \[19\], has the virtue of removing the tachyon, giving rise to models that are supersymmetric in the target space. The corresponding GSO-projected NS sector is described by

\[
\text{tr} \left( \frac{1 - (-1)^F}{2} q^{L_0} \right) = \frac{1}{2} \frac{\prod_{m=1}^{\infty} (1 + q^{m-1/2})^8 - \prod_{m=1}^{\infty} (1 - q^{m-1/2})^8}{q^{1/2} \prod_{m=1}^{\infty} (1 - q^m)^8} .
\]

In the R sector, the GSO-projection consists in taking space-time fermions of a given chirality at each mass-level. Precisely, the term \(q^{L_0}\) is projected by the operator

\[
\frac{1}{2} (1 \pm \gamma_{11} (-1)^{\sum_r \lambda^r \lambda^r}) .
\]

As we will see, one can consider the case in which the left and right R vacua have the same chirality (type IIB) or opposite chirality (type IIA). Since \(\gamma_{11}\) is traceless, the second term does not contribute to the vacuum energy. This simply means that the trace has numerically the same value if the vacuum is a left or a right spinor.
We now want to prove that the properly projected partition function is modular invariant. With this in mind, let us introduce the Jacobi theta functions
\[
\theta_{[\alpha]}^\beta(z|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i (n+\alpha)(z + \beta)},
\]
that transform under $S$ and $T$ as
\[
\theta_{[\alpha]}^\beta(z|\tau + 1) = e^{-i\pi \alpha(\alpha - 1)} \theta_{[\beta + \alpha - 1/2]}^\alpha(z|\tau),
\]
\[
\theta_{[\alpha]}^\beta\left(\frac{z}{\tau} - \frac{1}{\tau}\right) = e^{2i\pi \alpha \beta + i\pi \beta^2/\tau} (-i\tau)^{1/2} \theta_{[-\beta]}^{-\beta}(z|\tau).
\]
Since the fermions $\lambda^i$ are (anti)periodic, we are interested in theta functions with vanishing argument $z$ and with $\alpha$ and $\beta$ equal to 0 and $\frac{1}{2}$. Denoting
\[
\theta_1 = \theta_{[1/2]}^{[1/2]} , \quad \theta_2 = \theta_{[0]}^{[1/2]} , \quad \theta_3 = \theta_{[0]}^{[0]} , \quad \theta_4 = \theta_{[1/2]}^{[0]},
\]
one can then define the $so(2n)$ characters
\[
O_{2n} = \frac{\theta_3^n + \theta_4^n}{2\eta^n},
\]
\[
V_{2n} = \frac{\theta_3^n - \theta_4^n}{2\eta^n},
\]
\[
S_{2n} = \frac{\theta_2^n + i^n \theta_1^n}{2\eta^n},
\]
\[
C_{2n} = \frac{\theta_2^n - i^n \theta_1^n}{2\eta^n},
\]
whose transformation properties under $T$ and $S$ transformations are summarized by
\[
T_{2n} = e^{-in\pi/12} \text{diag}(1, -1, e^{in\pi/4}, e^{in\pi/4})
\]
and
\[
S_{2n} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & 1 & i^n & i^{-n} \\
1 & 1 & -i^n & i^{-n}
\end{pmatrix}.
\]
One can also prove that the theta functions with argument $z = 0$ have the product expansions
\[
\frac{\theta_2^2(0|\tau)}{\eta^{12}(\tau)} = 16 \prod_{m=1}^{\infty} (1 + q^m)^8, \quad \frac{\theta_2^4(0|\tau)}{\eta^{12}(\tau)} = \prod_{m=1}^{\infty} (1 + q^{m-1/2})^8, \quad \frac{\theta_4^2(0|\tau)}{\eta^{12}(\tau)} = q^{1/2} \prod_{m=1}^{\infty} (1 - q^{m})^8, \quad \frac{\theta_4^4(0|\tau)}{\eta^{12}(\tau)} = q^{1/2} \prod_{m=1}^{\infty} (1 - q^{m-1/2})^8,
\]
while \( \theta_1 \) vanishes when evaluated at \( z = 0 \).

Collecting all these results, one obtains that the GSO-projected torus partition functions

\[
T_{\text{IIA}} = \frac{V}{2(4\pi^2\alpha')^5} \int_\mathcal{F} \frac{d^2 \tau}{\tau_2^2} \frac{(\bar{V}_8 - \bar{S}_8)(V_8 - C_8)}{|\eta(\tau)|^{16}},
\]

\[
T_{\text{IIB}} = \frac{V}{2(4\pi^2\alpha')^5} \int_\mathcal{F} \frac{d^2 \tau}{\tau_2^2} \frac{|\bar{V}_8 - \bar{S}_8|^2}{|\eta(\tau)|^{16}},
\]

are modular-invariant. Leaving the modular integration and the bosonic contribution implicit, one can write

\[
T_{\text{IIA}} = (\bar{V}_8 - \bar{S}_8)(V_8 - C_8),
\]

\[
T_{\text{IIB}} = |\bar{V}_8 - \bar{S}_8|^2.
\]

As anticipated, the IIA amplitude is obtained by opposite chiral projections in left and right R sectors, while the IIB amplitude is obtained by the same chiral projection in the two sectors. As in the bosonic case, we can read the spectrum expanding the argument of the integral in power of \( q \). The \( O_8 \) character starts at the lowest mass level with the tachyon and, in group theoretical language, corresponds to the conjugacy class of the singlet in the weight lattice. \( V_8 \) starts with the massless vector and corresponds to the conjugacy class of the vector in the weight lattice. Finally, \( S_8 \) and \( C_8 \) start at the lowest mass level with massless left and right-handed spinors, respectively. It is then straightforward to see from eqs. (2.50) that IIA and IIB have no tachyon, while their massless spectra exactly coincide with the field content of 2a and 2b ten-dimensional supergravities. Numerically, both these amplitudes vanish because of supersymmetry, as can be verified using the famous Jacobi’s \textit{aequatio identica satis abstrusa},

\[
\theta_3^4 - \theta_4^4 - \theta_2^4 = 0.
\]

Actually, there are other two modular invariant partition functions one can write in ten dimensions, namely \cite{20}

\[
T_{0A} = |O_8|^2 + |V_8|^2 + \bar{S}_8C_8 + \bar{C}_8S_8,
\]

\[
T_{0B} = |O_8|^2 + |V_8|^2 + |S_8|^2 + |C_8|^2.
\]

Both these theories are tachyonic and non-supersymmetric (the spectrum is in fact purely bosonic). We will not analyze these theories, but here we only mention that a suitable orientifold projection of the 0B model gives rise to a spectrum without
tachyons in the closed sector. Moreover, the tachyon in the open sector can be projected out by a proper choice of the gauge group \[21\]. The details about how to build orientifold models are the subject of the next chapter. We conclude by writing also the partition functions corresponding to the heterotic strings in ten dimensions \[22\],

\[
T_{SO(32)} = (\tilde{V}_8 - \tilde{S}_8)(O_{32} + S_{32})
\]

for the \(SO(32)\) case and

\[
T_{E_8 \times E_8} = (\tilde{V}_8 - \tilde{S}_8)(O_{16} + S_{16})(O_{16} + S_{16})
\]

for the \(E_8 \times E_8\) case. A suitable projection of the latter model gives a tachyon-free string theory without space-time supersymmetry, whose low-energy limit is a ten-dimensional anomaly-free chiral \(O(16) \times O(16)\) gauge theory coupled to gravity \[23\].

### 2.3 IIB superstring on \(T^4/Z_2\)

Type IIB string theory compactified on \(K3\) gives rise to a six-dimensional model with \((2,0)\) supersymmetry, whose low-energy effective action corresponds to \((2,0)\) supergravity coupled to 21 tensor multiplets. The \((2,0)\) gravity multiplet contains the metric, two left-handed gravitinos and five self-dual 2-forms, while the tensor multiplet contains an antiself-dual 2-form, two right-handed spinors and five scalars. The couplings of supergravity to tensor multiplets were described in \[24\] to lowest order in the Fermi fields, and then completed in \[25\] to all orders in the Fermi fields. In this section we describe IIB string theory compactified on a four-dimensional orbifold, a singular limit of \(K3\), in which all the (infinite) curvature is localized on the fixed points. In particular, we will consider the orbifold \(T^4/Z_2\), where \(Z_2\) changes the sign of all the coordinates of \(T^4\). This orbifold has 16 fixed points. We derive the torus partition function for this model, and in the next chapter we will use these results to derive the partition function for some six-dimensional IIB orientifolds.

In order to describe the orbifold compactification, we first derive the partition function for IIB on \(T^4\). When a closed string is compactified on a circle, the spectrum includes, in addition to the usual Kaluza-Klein momentum modes, an infinity of topologically distinct sectors, associated to closed strings wrapped \(n\) times around the circle. The mode expansion for the coordinate along the string then reads

\[
X = x + \alpha' \frac{m}{R} + nR\sigma + \text{(oscillators)}
\]
where \( m \) is the KK momentum and \( n \) is the winding number. In terms of left and right oscillators the same expansion becomes

\[
X_{L,R} = \frac{1}{2}x + \frac{\alpha'}{2} p_{L,R}(\tau \pm \sigma) + (\text{oscillators})_{L,R},
\]

(2.56)

where

\[
p_{L,R} = \frac{m}{R} \pm \frac{nR}{\alpha'}. \tag{2.57}
\]

Thus, if a non-compact coordinate is replaced with a compact one, the continuous integration over internal momenta is replaced by a lattice sum, so that

\[
\frac{1}{\tau_2 |\eta(q)|^8} \to \sum_{m,n \in \mathbb{Z}} \frac{q^{\alpha\beta} p_L^2/4 q^{\alpha'\beta'} p_R^2/4}{\eta(q)|\eta(q)|}. \tag{2.58}
\]

In order to generalize this result to \( T^4 \), we shall consider for simplicity the case in which the torus is a product of circles of equal radii and the internal NS \( B \) field vanishes. In the partition function, the continuous integration over internal momenta is now replaced by

\[
\frac{1}{\tau_2 |\eta(q)|^8} \to \sum_{m,n \in \mathbb{Z}^4} \frac{q^{\alpha\beta} p_L^2/4 q^{\alpha'\beta'} p_R^2/4}{|\eta(q)|}, \tag{2.59}
\]

where \( p_{L,R} \) is a 4-vector with components

\[
p^a_{L,R} = \frac{m^a}{R} \pm \frac{n^a R}{\alpha'}. \tag{2.60}
\]

Finally, we need to analyze the effect of the compactification on the fermionic oscillators. From eq. (2.43) it turns out that the decomposition of the \( SO(8) \) characters in terms of \( SO(4) \times SO(4) \) is

\[
V_8 = V_4 O_4 + O_4 V_4, \quad S_8 = S_4 S_4 + C_4 C_4, \quad O_8 = O_4 O_4 + V_4 V_4, \quad C_8 = S_4 C_4 + C_4 S_4, \tag{2.61}
\]

where the second \( SO(4) \) is associated to the four compact dimensions. Collecting all the results, the torus partition function for IIB compactified on a flat \( T^4 \) is (again neglecting the measure and the non-compact bosonic oscillators)

\[
T_{T^4} = |V_4 O_4 + O_4 V_4 - S_4 S_4 - C_4 C_4|^2 \sum_{m,n \in \mathbb{Z}^4} \frac{q^{\alpha\beta} p_L^2/4 q^{\alpha'\beta'} p_R^2/4}{|\eta(q)|^8}. \tag{2.62}
\]
Applying the rules we derived in the ten dimensional case, it is straightforward to determine the massless spectrum: it consists of the metric, 16 vectors, 5 antisymmetric tensors and 25 scalars in the bosonic sector, and 2 Dirac gravitinos and 10 Dirac spinors in the fermionic sector. This is the field content of maximal six-dimensional supergravity, whose scalars parametrize the coset manifold $SO(5,5)/SO(5) \times SO(5)$.

Starting from this result, we want to project out of (2.62) the states that are not invariant under the orbifold projection. It is convenient to define the supersymmetric combination of characters

\[
Q_o = V_4 O_4 - C_4 C_4 , \quad Q_v = O_4 V_4 - S_4 S_4 , \\
Q_s = O_4 C_4 - S_4 O_4 , \quad Q_c = V_4 S_4 - C_4 V_4 .
\] (2.63)

Since the internal $V_4$ and one of the two spinorial characters, e.g. $C_4$, change sign under the orbifold projection, the $Q$'s are eigenvectors of the $Z_2$ generator. Moreover, at the massless level, $|Q_o|^2$ contains the $(2,0)$ gravity multiplet plus one tensor multiplet, $|Q_v|^2$ contains four tensor multiplets and $|Q_s|^2$ contains one tensor multiplet. The result of the projection is then

\[
\frac{1}{2} \left[ |Q_o + Q_v|^2 \sum_{m,n \in \mathbb{Z}^4} \frac{q^\alpha \nu_1^2 q^\beta \nu_2^2}{|\eta|^8} + |Q_o - Q_v|^2 \frac{|2\eta|^4}{\bar{\theta}_2} \right] .
\] (2.64)

A straightforward analysis, using eqs. (2.46) and (2.47), reveals that this amplitude is not modular invariant. In other words, we are missing some states. More precisely, what we have done is just to take the untwisted states, the subset of the original closed string states that are invariant under the orbifold projection. But other states can be added, namely the ones that correspond to a string closing up to an orbifold transformation. These twisted states are exactly needed to restore modular invariance [26], and the modular invariant torus amplitude is thus

\[
\frac{1}{2} \left( |Q_o + Q_v|^2 \sum_{m,n \in \mathbb{Z}^4} \frac{q^\alpha \nu_1^2 q^\beta \nu_2^2}{|\eta|^8} + |Q_o - Q_v|^2 \frac{|2\eta|^4}{\bar{\theta}_2} \right) + 16 \left[ |Q_s + Q_c|^2 \left| \frac{\eta}{\bar{\theta}_4} \right|^4 + |Q_s - Q_c|^2 \left| \frac{\eta}{\bar{\theta}_3} \right|^4 \right] .
\] (2.65)

Observe that the twisted sector has a multiplicity equal to the number of fixed points. Using the standard technique, we can then read the massless sector of this amplitude. The relevant terms are

\[
|Q_o|^2 + |Q_v|^2 + 16|Q_s|^2 ,
\] (2.66)
and so the massless spectrum corresponds to supergravity with 5 tensor multiplets from the untwisted sector and 16 additional tensor multiplets from the twisted sector. As in general for $K3$ compactifications of type IIB, we thus obtain that the low-energy effective action of type IIB on $T^4/Z_2$ is $(2,0)$ supergravity coupled to 21 tensor multiplets.

### 2.4 Anomaly cancellation

An anomaly is a breakdown of a classical symmetry by quantum corrections, originated by diagrams that do not admit a regulator compatible with simultaneous conservation of all the attached currents. Here we are interested only in anomalies associated to local symmetries. Given a classically gauge invariant theory, consider the effective action $\Gamma(A_\mu)$ defined by

$$e^{-\Gamma} = \int D\psi D\bar{\psi} e^{-S},$$  \hspace{1cm} (2.67)

obtained integrating over the matter fields $(\psi, \bar{\psi})$ in the theory. Under an infinitesimal gauge transformation $\delta A_\mu = D_\mu \Lambda$ one obtains

$$\delta \Gamma = -\text{tr} \int d^D x \Lambda D_\mu \frac{\delta \Gamma}{\delta A_\mu},$$ \hspace{1cm} (2.68)

and from eq. (2.67) we thus read that

$$\frac{\delta \Gamma}{\delta A_\mu} = < J^\mu >.$$ \hspace{1cm} (2.69)

Therefore eq. (2.68) becomes

$$\mathcal{A}_\Lambda = \delta \Gamma = -\text{tr} \int d^D x \Lambda D_\mu < J^\mu >,$$ \hspace{1cm} (2.70)

and a lack of conservation of the expectation value of the current implies that the effective action is not invariant under infinitesimal gauge transformations.

Anomalous diagrams can occur only when chiral fermions (or (anti)self-dual bosons) are circulating in the loop, otherwise one can always construct a gauge invariant mass term that can be used as a Pauli-Villars regulator. As a consequence, we expect to find purely gravitational anomalies (anomalies corresponding to amplitudes whose external fields are only gravitons) only in $2k + 2$ dimensions (with signature $(1, 2k + 1)$), since in these dimensions the charge conjugated of a chiral spinor is a
spinor of the same chirality (see Appendix (A.1) for details). In general, we expect to find anomalies only if the dimension of space-time is even. Denoting with $D = 2n$ the dimension of space-time, the first parity violating amplitude that is potentially anomalous is a loop with $n + 1$ external legs. In $D = 10$ this corresponds to the hexagon diagram in fig. (2.3).

![Hexagon diagram](image)

Figure 2.3: Hexagon diagram responsible for ten-dimensional anomalies.

We now resume the techniques for deriving anomalies in various dimensions (see Refs. [27] for reviews). The index of an operator $\mathcal{O}$ is defined as the difference between the dimension of the kernel of $\mathcal{O}$ and that of its adjoint. It can be proved that the anomaly produced by a chiral fermion is in correspondence with the index of the $(2n + 2)$-dimensional Weyl operator $D_+ = \gamma^\mu D_\mu P_+$, where $P_+$ is the chirality projector. The Atiyah-Singer index theorem [28] relates the index of $D_+$ on a manifold $M$ to topological invariants of the bundle:

$$\text{ind} \ iD_+ = \int_M [\hat{A}(M) ch(F)]_{\text{vol}} ,$$

where $\hat{A}(M)$ is the Dirac genus (or A-roof genus) of $M$ and $ch(F)$ is the Chern character. They are defined by

$$\hat{A}(M) = \prod_a \frac{x_a/2}{\sinh(x_a/2)} ,$$

$$ch(F) = \text{tr} e^{\frac{i}{2} F} ,$$

where $x_a$ are the skew-eigenvalues of the curvature on $M$. Analogously, the index of the Rarita-Schwinger operator is given by

$$\text{ind} \ iD_{3/2} = \int_M [\hat{A}(M)(ch(R) - 1)ch(F)]_{\text{vol}} ,$$
while the index for self-dual antisymmetric tensors is given by

\[
\text{ind} \ iD_S = \frac{1}{4} \int_M [L(M)]_{\text{vol}} ,
\]

where \( L(M) \) is the Hirzebruch polynomial, defined by

\[
L(M) = 2^n \prod_a \frac{x_a/2}{\tanh(x_a/2)} .
\]

we will need in the following the expansion of \( \hat{A} \) and \( L \) with respect to the traces of the curvature:

\[
\hat{A}(M) = 1 + \frac{1}{(4\pi)^2} \frac{1}{12} \text{tr} R^2 + \frac{1}{(4\pi)^4} \left[ \frac{1}{288} (\text{tr} R^2)^2 + \frac{1}{360} \text{tr} R^4 \right]
\]
\[+ \frac{1}{(4\pi)^6} \left[ \frac{1}{128 \cdot 81} (\text{tr} R^2)^3 + \frac{1}{16 \cdot 270} \text{tr} R^2 \text{tr} R^4 + \frac{1}{90 \cdot 63} \text{tr} R^6 \right] + ... \]

\[
L(M) = 1 - \frac{1}{(2\pi)^2} \frac{1}{6} \text{tr} R^2 + \frac{1}{(2\pi)^4} \left[ \frac{1}{72} (\text{tr} R^2)^2 - \frac{7}{180} \text{tr} R^4 \right]
\]
\[+ \frac{1}{(2\pi)^6} \left[ -\frac{1}{1296} (\text{tr} R^2)^3 + \frac{7}{1080} \text{tr} R^2 \text{tr} R^4 - \frac{31}{2835} \text{tr} R^6 \right] + ... .
\]

The anomaly polynomials

\[
I_{1/2}(R, F) = [\hat{A}(M)ch(F)]_{D+2} ,
\]

\[
I_{3/2}(R) = [\hat{A}(M)(ch(R) - 1)]_{D+2} ,
\]

\[
I_S(R) = -\frac{1}{8} [L(M)]_{D+2}
\]

are closed \((D+2)\)-forms, and locally determine a \((D+1)\)-form whose gauge transformation is exact:

\[
I_{D+2} = dI_{D+1} ,
\]

\[
\delta_{\Lambda} I_{D+1} = dI_{D}^{\Lambda} .
\]

The anomaly is finally determined as the \(D\)-dimensional integral of the form \(I_D^\Lambda\).

In ten dimensions, the anomaly originates from a 12-form, and an analysis of the previous results shows that the anomaly polynomial exactly cancels for a theory containing a complex left-handed gravitino, a complex right-handed spinor and a self-dual 4-form. This is exactly the chiral content of 2\(b\) supergravity, so that the effective action of type-II\(B\) string theory turns out to be anomaly-free [29].

We now consider \((2, 0)\) supersymmetric models in six dimensions. In this case the chiral content corresponds to two left-handed gravitinos and five self-dual 2-forms
from the gravity multiplet, as well as \( n \) antiself-dual 2-forms and \( 2n \) right-handed spinors from \( n \) tensor multiplets. The cancellation of the anomaly polynomial 8-form implies \( n = 21 \) \cite{24}, that as we have seen in the previous section is exactly what one obtains compactifying type IIB on \( K3 \).

Now we consider ten-dimensional \( (1,0) \) supergravity coupled to Yang-Mills vector multiplets. In this case the chiral content of the theory is a left-handed Majorana gravitino and a right-handed Majorana spinor from the gravity multiplet, and a left-handed Majorana gaugino in the adjoint of the gauge group from the vector multiplet. The 12-form anomaly polynomial is proportional to

\[
\frac{-496 - n}{128 \cdot 2835} \text{tr} R^6 - \frac{224 + n}{256 \cdot 1080} \text{tr} R^2 \text{tr} R^4 - \frac{64 - n}{512 \cdot 1296} \text{(tr} R^2\text{)^3}
\]

\[
- \frac{1}{720} \text{Tr} F^6 + \frac{1}{24 \cdot 48} \text{Tr} F^4 \text{tr} R^2 - \frac{1}{256 \cdot 45} \text{Tr} F^2 \text{tr} R^4 - \frac{1}{192} \text{Tr} F^2 (\text{tr} R^2)^2 \tag{2.79}
\]

where \( \text{Tr} \) is the trace in the adjoint representation and \( n \) is the dimension of the gauge group. Imposing \( n = 496 \) cancels the irreducible term \( \text{tr} R^6 \), and we are left with the residual anomaly polynomial

\[
\frac{1}{8} \text{tr} R^2 \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^3 - \frac{1}{15} \text{Tr} F^6
\]

\[
+ \frac{1}{24} \text{Tr} F^4 \text{tr} R^2 - \frac{1}{240} \text{Tr} F^2 \text{tr} R^4 - \frac{1}{192} \text{Tr} F^2 (\text{tr} R^2)^2 \tag{2.80}
\]

The result of \cite{30} is that for the gauge groups \( SO(32) \) and \( E_8 \times E_8 \) this residual anomaly polynomial factorizes into the product of a 4-form and an 8-form. In the \( SO(32) \) case the result is

\[
(\text{tr} R^2 - \text{tr} F^2)(\text{tr} F^4 - \frac{1}{8} \text{tr} R^2 \text{tr} F^2 + \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2) \tag{2.81}
\]

Adding to the low-energy effective action the term

\[
B \wedge (\text{tr} F^4 - \frac{1}{8} \text{tr} R^2 \text{tr} F^2 + \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2) \tag{2.82}
\]

and demanding that under gauge and Lorentz transformations \( B \) transform as

\[
\delta B = \omega_{2,YM} - \omega_{2,L} \tag{2.83}
\]

where \( \omega_2 \) is determined from the Chern-Simons 3-form by \( d\omega_2 = \delta \omega_{CS} \), the total anomaly exactly cancels. This is the celebrated Green-Schwarz mechanism \cite{30}, that

\[1\text{The first term on the right-hand side of eq. (2.83) is already determined by supersymmetry, as was shown in the first chapter.}\]
guarantees consistency of the low-energy effective action for both heterotic and type I in ten dimensions. It corresponds to the cancellation of part of the anomaly resulting from the hexagon diagram of fig. (2.3) against the anomalous contribution coming from the tree-level amplitude of fig. (2.4).

![Diagram](image)

Figure 2.4: Anomalous diagram responsible for the Green-Schwarz mechanism

We finally consider $N = (1, 0)$ supersymmetry in six dimensions, whose multiplets are the gravity multiplet, containing the metric, a self-dual 2-form and a left-handed gravitino, the tensor multiplet, containing an antiself-dual 2-form, a scalar and a right-handed spinor, the vector multiplet, containing a vector and a left-handed gaugino, and the hypermultiplet, containing four scalars and a right-handed spinor (we will analyze in detail the couplings of these multiplets in Chapter 4). From eq. (2.77) we can read the 8-form anomaly polynomial corresponding to supergravity coupled to $n_T$ tensor multiplets, $n_V$ vector multiplets and $n_H$ hypermultiplets, and the cancellation of the coefficient of the irreducible term $\text{tr} R^4$ implies the condition

$$273 - 29n_T + n_V - n_H = 0.$$  

(2.84)

In the next chapter we will analyze the reducible part of the anomaly polynomial both for the heterotic string compactified on $K3$ and for six-dimensional type-IIB orientifolds.

## 2.5 D-branes and dualities

All the theories of oriented closed strings we have analyzed so far have in the massless spectrum a 2-form coming from the NS sector. Oriented closed strings, in fact, are precisely the sources for this field, as electrically charged particles are sources for vector potentials. One can then ask which are the objects that are charged with respect to the R-R forms present in type IIA and type IIB. The nature of these objects is non-perturbative, since there are no states belonging to the perturbative spectrum of string theories that are charged under R-R forms, and they result to be
defined as hyper-planes on which open strings can end \[32\]. More precisely, the excitations of these p-dimensional extended objects are open strings that satisfy Dirichlet boundary conditions in the 9-p directions orthogonal to them, and for these reason these objects are called D-branes \[32\]. As a consequence type IIA and type IIB, that at a perturbative level are theories of oriented closed strings only, admit non-perturbative open-string excitations. Moreover, analyzing the spectrum of the two theories, one sees that type IIA contains D-branes with an even number of spatial directions, while type IIB contains D-branes with an odd number of spatial directions. The non-perturbative origin of these branes is manifest if one consider that in the string frame they have tensions inversely proportional to the string coupling constant, and so they disappear from the spectrum at weak coupling. Finally, their supersymmetric nature corresponds to the fact that these objects correspond to BPS solutions of the low-energy supergravities, while their supersymmetric world-sheet action contains the Born-Infeld action and Wess-Zumino terms of the type

\[
C^{(p+1)} + F \wedge C^{(p-1)} + \ldots
\]

that naturally couple the brane to the R-R potentials. In the low-energy limit, these objects decouple from gravity, and the resulting action is supersymmetric Yang-Mills theory. In the case of \(N\) coincident D-branes, the resulting gauge group is \(U(N)\) (we will see in the next chapter that if the open strings are not oriented, the gauge group can be orthogonal or symplectic \[33, 34, 35\]).

In analyzing IIB supergravity, we showed that the scalar manifold has an \(SL(2, R)\) isometry, that is a symmetry of the low-energy theory. In the full non-perturbative string theory the discrete subgroup \(SL(2, Z)\) survives \[36\], and the theory manifests an S-duality, acting on the scalars (see eq. \((1.81)\)) as

\[
w \rightarrow \frac{aw + b}{cw + d}
\]

with \(w = \rho + i e^\phi\), where \(\phi\) is the dilaton and \(\rho\) the R-R scalar. This discrete symmetry is non-perturbative, as can be seen from the fact that it inverts the coupling, and this is why it is not manifest if we express the low-energy action in the string frame (see Section \((1.4)\)). For instance, this symmetry justifies the presence of a D1-brane in the non-perturbative spectrum of type IIB: in the dual description, this D1-string becomes fundamental. Another remark concerning type IIB has to be made: in the non-perturbative spectrum, a space-time filling D9-brane is also present. This object has no dynamics, but his role will appear to be relevant in the next chapter.
Having shown that type IIB is self-dual, so that the theory has the same form at strong and at weak coupling, we want to see what happens to type IIA at strong coupling. In Section (1.4) we have shown that dimensional reduction of eleven-dimensional supergravity gives rise to IIA supergravity. In particular, in the string frame the relation between the compactification radius and the string coupling constant is

\[ R = g_A^{2/3}, \quad (2.87) \]

and this relation shows that, for generic values of the dilaton, the type-IIA string manifests a perturbatively hidden eleventh dimension \cite{37}. The strong coupling limit develops this extra dimension, resulting in an eleven-dimensional theory, called \textit{M-theory} \cite{37,38}. The low-energy action of M-theory is eleven-dimensional supergravity, and the presence of a 3-form potential suggests that the complete theory describes an M2-brane and a dual M5-brane. Even if this theory is not known, some higher derivative couplings can be deduced by supersymmetry and by anomaly considerations. Consider as an example the Wess-Zumino term \( A \wedge F \wedge F \) of eleven-dimensional supergravity. This term is gauge invariant in the absence of M5-branes, but if an M5-brane, a magnetic source for the 4-form field strength, is present, the Bianchi identity \( dF = 0 \) must be replaced by \( dF = \delta^5(x - x') \), where \( x' \)'s are the directions orthogonal to the M5-brane, and the Wess-Zumino term is no more gauge invariant. The resulting anomaly produced in the bulk gravity action exactly cancels the anomalous contribution from the Wess-Zumino term (analogous to eq. (2.85)) in the M5-brane action. This \textit{anomaly inflow} mechanism \cite{39}, in which the anomaly in the bulk generated by a source is canceled by the anomaly on the brane \cite{40}, also applies to higher derivative gravitational anomalies, and can be used to determine higher derivative couplings in the low-energy supergravity action. The Kaluza-Klein modes of the compactification, that with respect to the eleven-dimensional metric have masses proportional to \( 1/R \), in the string metric have masses of the order of \( e^{-1/3\phi}/R = 1/g_A \) \cite{37}, and are D0-branes in the ten-dimensional theory. An M2-brane wrapped around the circle gives a string in ten dimensions, and it can be shown that its tension in the string frame does not depend on the dilaton, as pertains to a fundamental object, while an unwrapped M2-brane is a D2-brane, whose tension again scales like \( 1/g_A \) in the string frame. Finally, M-theory compactified on \( S^1/Z_2 \) corresponds to non perturbative \( E_8 \times E_8 \) heterotic string, where again the ten-dimensional dilaton is related to the length of the interval \cite{41}. We will return to this identification in the next chapter.
This chain of dualities relating various theories in different regimes becomes richer and richer in lower dimensions. In this respect, all known (supersymmetric) string theories represent different charts of a moduli space of an unknown more fundamental theory, while the duality relations are transition functions between different charts. We will not describe all these duality relations in this Thesis, see for instance [12, 13] for reviews.
Chapter 3

Open strings

In this chapter we want to describe how to construct type-I models. The basic idea is that these models are generalized orbifolds (orientifolds) of type IIB superstrings \([44, 45]\). More precisely, if we denote with \(\Omega\) the operation that exchanges left and right oscillators, the basic relation is

\[
\text{Type } \ I = (\text{Type } \ IIB)/\Omega.
\]

In the last chapter we have seen how to write the one-loop partition function for closed oriented superstring theories. The \(\Omega\) operation is implemented on the torus partition function projecting out of the spectrum the states that are not invariant under left-right exchange. This corresponds to substituting to the torus amplitude, that is a closed oriented string loop, the halved sum of the torus amplitude and the Klein bottle amplitude, in which a closed string flips its orientation before closing the loop. We have seen in the previous chapter that in order to restore modular invariance in orbifold compactifications of closed oriented strings, one has to add to the untwisted sector (corresponding to the closed spectrum projected by the orbifold operation) the twisted sector, corresponding to the spectrum of strings that close only after the orbifold operation. In the construction of type-I models, one encounters a similar problem: the unoriented closed spectrum is typically inconsistent, and the inconsistency manifests itself by the appearance of tadpoles, that correspond to gauge and gravitational anomalies in the low-energy effective action. In order to cure this problem, one has to add a suitable open sector.
In Section 1, we will derive the one-loop partition function for the ten-dimensional $SO(32)$ type-I superstring. In Section 2 we will see that a different orientifold projection of type IIB in ten dimensions results in a consistent non-supersymmetric model which is free of tachyons, and whose gauge group is $USp(32)$. Actually, the closed spectrum of this model is supersymmetric, and supersymmetry is only broken in the open sector. We will then describe in Section 3 how to obtain six-dimensional type-I models as $T^4/Z_2$ orientifolds of type IIB. Also in six dimensions one can construct consistent non-tachyonic models in which supersymmetry is broken in the open sector. Section 4 is devoted to a discussion of anomaly cancellation in type-I models, and finally in Section 5 we will discuss of general features of six-dimensional models that will be relevant for the following chapters.

### 3.1 The ten-dimensional $SO(32)$ type-I superstring

In this section we describe how to obtain the $SO(32)$ type-I model as an orientifold projection of type IIB. Since type IIB is invariant under $\Omega$, the operation that exchanges left and right moving oscillators, we can project out all states that are not invariant. This projection is realized at the level of the partition function by

$$T \rightarrow \frac{1}{2}(T + K) \ ,$$

where $K$ is the Klein bottle amplitude, corresponding to a closed string that flips its orientation before closing the loop (see fig (3.1)). The Klein bottle is obtained from the torus by the anti-conformal involution

$$z \rightarrow 1 - \bar{z} + i\tau_2 \ ,$$

compatible with the periodicity of the lattice only if $\tau$ is purely imaginary. The final result is that the Klein bottle amplitude (we omit the overall constants)

$$K = \frac{1}{2} \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^8} \frac{(V_8 - S_8)(2i\tau_2)}{\eta^8(2i\tau_2)}$$

depends naturally on $2i\tau_2$, the modulus of the doubly covering torus of fig. (3.1). From the torus and Klein bottle amplitudes we can read the spectrum: at the massless level, in the NS-NS sector we are projecting out the 2-form, while in the RR sector we are projecting out the self-dual 4-form and the scalar. We are thus left with the metric, the dilaton and the RR 2-form. In the fermionic NS-R and R-NS sector,
the Klein bottle does not introduce new states, so the projection just halves the number of spinors, and the end result is the field content of $\mathcal{N} = 1$ ten dimensional supergravity, discussed in Section (1.4).

While the torus amplitude has no ultraviolet divergence, since modular invariance removes the dangerous region from the integral, the Klein bottle is not modular invariant, and it manifests a divergence for small values of $\tau_2$ that we want to analyze. Performing an $S$ transformation $i\tau_2 \rightarrow 1/\tau_2$, where $i\tau_2$ is the complex parameter of the doubly covering torus, the one loop amplitude is mapped to a tree level amplitude, in which a closed string bounces between two crosscaps\footnote{A crosscap can be considered as a boundary with opposite points identified.}. The end result is that eq. (3.3) becomes the transverse-channel amplitude

$$\tilde{\mathcal{K}} = \frac{2^5}{2} \int_0^\infty dl \frac{(V_8 - S_8)(il)}{\eta^8(il)}, \quad (3.4)$$

while the ultraviolet divergence in the direct channel becomes an infrared divergence in the transverse channel, generated by massless closed-string exchanges between the two crosscaps. From a target space point of view, this can be interpreted as closed-string tree-level scattering between two orientifold planes, extended objects that invade all the nine spatial dimensions, changing the orientation of the strings they interact with. In this sense, if the massless particle that bounces between the two O9-planes is the graviton or the dilaton, the transverse amplitude is proportional to the square of the tension of the O9-plane, while if the particle is the RR 10-form (a 9-plane can be charged with respect to a 10-form) this amplitude is proportional to the square of its charge. The overall tension and charge determine the NS and RR tadpoles, whose values can be read from (3.4). In particular, the presence of a RR tadpole makes the model inconsistent, since it corresponds to an overall charge that fills the whole space.

In order to cure this divergence, one adds D9-branes, i.e. the open sector. The one-loop annulus amplitude is

$$\mathcal{A} = \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(V_8 - S_8)(i\tau_2/2)}{\eta^8(i\tau_2/2)}, \quad (3.5)$$

where the multiplicity $N$ of the Chan-Paton charge is associated to the number of D9-branes, and $i\tau_2/2$ is the modulus of the doubly covering torus of fig. (3.1). Performing the $\Omega$ projection, we then obtain the Möbius strip amplitude, corresponding to an
open string that flips its orientation before closing the loop. The resulting amplitude,

$$\mathcal{M} = \frac{\epsilon N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(\hat{V}_8 - \hat{S}_8)(i\tau_2/2 + 1/2)}{\hat{\eta}^8(i\tau_2/2 + 1/2)},$$  \hspace{1cm} (3.6)

is written in terms of the “hatted” characters and $\epsilon = \pm 1$. The argument $i\tau_2/2 + 1/2$ is again the modulus of the doubly covering torus, as can be deduced from fig. (3.1).

![Diagram](image)

Figure 3.1: Möbius strip, Klein bottle and annulus

We now want to write the same amplitudes in the transverse channel. In the case of the Möbius amplitude, it is important to observe that since the parameter of the doubly covering torus has a real part, the modular transformation connecting direct and transverse Möbius amplitudes is no longer an $S$ transformation, but the

Given the character $\chi(q) = q^{h-c/24} k \sum_k q^k$, one introduces a real hatted character $\hat{\chi}(i\tau_2 + 1/2) = q^{h-c/24} \sum_k (-1)^k k \hat{q}^k$, where $q = e^{-2\pi \tau_2}$, that differs from $\chi(i\tau_2 + 1/2)$ by the phase $e^{-\pi(h-c/24)}$. 
transformation

\[ P : \frac{1}{2} + i \frac{\tau_2}{2} \rightarrow \frac{1}{2} + i \frac{1}{2\tau_2} \quad . \]

The operator that realizes this transformation on the hatted characters is then

\[ P = T^{1/2}ST^2ST^{1/2} \quad , \]

and the end result is

\[ \tilde{A} = \frac{2^{-5} N^2}{2} \int_0^\infty dl \frac{(V_8 - S_8)(il)}{\eta^8(il)} \quad , \]

\[ \tilde{M} = 2\epsilon N \int_0^\infty dl \frac{(\tilde{V}_8 - \tilde{S}_8)(il + 1/2)}{\tilde{\eta}^8(il + 1/2)} \quad . \]

We interpret these transverse amplitudes as tree-level amplitudes for closed strings bouncing between two boundaries or between a boundary and a crosscap. The request that the orientifold plane and the D-brane have opposite charge corresponds then to \( \epsilon = -1 \), while the consistency of the model is finally guaranteed imposing the tadpole cancellation condition (fig. \( \text{[B.2]} \)) [46]

\[ 2^5 + 2^{-5} N^2 - 2N = 2^{-5}(N - 32)^2 = 0 \quad , \]

that selects \( N = 32 \). This condition actually cancels both the RR and the NS tadpoles, so that the resulting configuration has zero charge and zero tension. This corresponds to the fact that the orientifold plane involved, \( O_- \), has negative tension and negative charge. From the amplitudes \( \text{[B.3]} \) and \( \text{[B.0]} \) in the direct channel we
can finally read the spectrum: at the massless level, we have $N(N-1)/2$ vectors and $N(N-1)/2$ spinors, corresponding to the ten-dimensional $\mathcal{N} = 1$ vector multiplet of gauge group $SO(32)$. From the results of the previous chapter we can then conclude that tadpole cancellation reflects the absence of anomalies in the low-energy effective action $[46]$.

### 3.2 Ten-dimensional $USp(32)$ type-I string

The $SO(32)$ model can be modified adding brane-antibrane pairs. Denoting with $N_+$ and $N_-$ the number of D9 and D$\bar{9}$ branes, we obtain the transverse amplitudes

\[ \hat{A} = \frac{2^{-5}}{2} [(N_+ + N_-)^2 V_8 - (N_+ - N_-)^2 S_8] \]
\[ \hat{M} = -\frac{2}{2} [(N_+ + N_-) \hat{V}_8 - (N_+ - N_-) \hat{S}_8] \]  

(3.11)

RR tadpole cancellation requires

\[ N_+ = 32 + N_- \]  

(3.12)

so that this model corresponds to adding $N_-$ brane-antibrane pairs to the stable configuration of 32 branes. From the direct channel amplitudes

\[ A = \frac{1}{2} [(N_+^2 + N_-^2) (V_8 - S_8) + N_+ N_- (O_8 - C_8)] \]
\[ M = -\frac{1}{2} [(N_+ + N_-) \hat{V}_8 - (N_+ - N_-) \hat{S}_8] \]  

(3.13)

we derive the open spectrum, that corresponds to the gauge group $SO(32 + N_-) \times SO(N_-)$, where the first gauge factor is supported on the branes and the second on the anti-branes. The spinors in the 9-9 sector are in the adjoint representation of $SO(32 + N_-)$, while the spinors in the $\bar{9}$-$\bar{9}$ sector are in the symmetric (reducible) representation of $SO(N_-)$. The presence of the tachyon reflects the instability of the vacuum $[47]$.

One can actually consider a different orientifold projection, induced by $O_+$ planes, with positive tension and positive charge. This is obtained changing the sign of the transverse M"obius amplitude in eq. (3.11), while RR tadpole cancellation implies in this case

\[ N_- = 32 + N_+ \]  

(3.14)
From the amplitudes in the direct channel, we then read the massless spectrum: the gauge group is $USp(32 + N_+) \times USp(N_+)$, while the spinors in the 9-9 sector are in the adjoint representation of $USp(N_+)$ and the spinors in the 9-9 sector are in the antisymmetric (reducible) representation of $USp(32 + N_+)$ \[48\]. Observe that this model is consistent only for an even number of brane-antibrane pairs, since only in this case a non-degenerate symplectic form exists. For the particular value $N_+ = 0$, the spectrum does not contain a tachyon, and so the model could be considered stable from this one-loop analysis, even if the NS tadpole, corresponding to a dilaton potential, changes the vacuum. The resulting open spectrum is non-supersymmetric. We will analyze the low-energy action for this model in Chapter 5. Finally, it is important to observe that all the models mentioned in this section are anomaly-free \[48\], as can be checked analyzing their chiral field content.

### 3.3 Six-dimensional type-I models

In Section (2.3) we have derived the partition function for type-IIB theory compactified on $T^4/Z_2$. Here we want to analyze the corresponding type-I models.

Six-dimensional orientifold projections result in general in the introduction of $O5$-planes, in addition to the $O9$-planes present in the ten-dimensional models. In supersymmetric models, both the tension and the charge of the $O9_-$ and $O5_-$ planes are canceled by a suitable configuration of $D9$ and $D5$ branes. In $Z_2$ orbifold compactifications, and in general for orbifolds with order-two group generators, one has the option of antisymmetrizing some twisted sectors or, equivalently, of inverting tensions and charges for the corresponding $O5$-planes. This requires that $D9$ branes be accompanied by suitable stacks of $D5$ branes, with a resulting brane supersymmetry breaking \[49\]. Concentrating on the torus partition function of eq. (2.65), describing IIB compactified on $T^4/Z_2$, two choices for the unoriented projection compatible with the crosscap constraint of \[50\] are described by

$$K = \frac{1}{4} \left\{ (Q_o + Q_v) (P + W) + 2\epsilon \times 16 (Q_s + Q_c) \left( \frac{n}{\theta_4} \right)^2 \right\}, \quad (3.15)$$

where $P$ ($W$) indicates the momentum (winding) lattice sum and $\epsilon = \pm 1$. At the massless level, $\epsilon = 1$ gives $N = (1, 0)$ supergravity with 1 tensor multiplet and 20 hypermultiplets, while $\epsilon = -1$ gives $N = (1, 0)$ supergravity with 17 tensor multiplets and 4 hypermultiplets. These closed spectra are both supersymmetric, but the latter projection introduces $O9_+$ and $O5_-$ planes, and this leads to an open sector with
brane supersymmetry breaking. This is clearly spelled by the massless contributions to the transverse Klein-bottle amplitude, that can be read from

\[
\tilde{K}_0 = \frac{2^5}{4} \left\{ Q_o \left( \sqrt{v} + \frac{1}{\sqrt{v}} \right)^2 + Q_v \left( \sqrt{v} - \frac{1}{\sqrt{v}} \right)^2 \right\}, \tag{3.16}
\]

where \( v = \sqrt{\det g/(a')^4} \) is the internal volume. The reflection coefficients are interchanged in the two cases: \( \epsilon = 1 \) corresponds to the introduction of \( O9_- \) and \( O5_- \) planes, both with negative tension and negative charge, while \( \epsilon = -1 \) corresponds to the introduction of \( O9_- \) and \( O5_+ \) planes, where the \( O5_+ \) planes have positive tension and positive charge. We now want to describe the open sector that gives rise to RR tadpole cancellation.

For \( \epsilon = 1 \), the simplest choice corresponds to placing all branes at a single fixed point. The corresponding annulus amplitude is

\[
A = \frac{1}{4} \left\{ (Q_o + Q_v) \left( N^2 P + D^2 W \right) + (R_N^2 + R_D^2) \left( \frac{2\eta}{\theta_2} \right)^2 \right. \\
+ 2ND(Q_s + Q_c) \left( \frac{\eta}{\theta_4} \right)^2 + 2R_NR_D(Q_s - Q_c) \left( \frac{\eta}{\theta_4} \right)^2 \right\}, \tag{3.17}
\]

where \( N \) and \( D \) count the multiplicities of the string ends with Neumann and Dirichlet boundary conditions, and \( R_N \) and \( R_D \) define the orbifold action on the Chan-Paton charges. In the present example, these are associated to the \( D9 \) and \( D5 \) branes that have to be present to cancel the RR tadpoles. The massless contributions to the transverse annulus amplitude can be read from

\[
\tilde{A}_0 = \frac{2^5}{4} \left\{ Q_o \left( N\sqrt{v} + \frac{D}{\sqrt{v}} \right)^2 + Q_v \left( N\sqrt{v} - \frac{D}{\sqrt{v}} \right)^2 \right. \\
+ Q_s[15R_N^2 + (R_N - 4R_D)^2] + Q_c[15R_N^2 + (R_N + 4R_D)^2] \right\}. \tag{3.18}
\]

Analyzing this expression, one can deduce that, for the untwisted contributions, the CP multiplicities \( N \) and \( D \) determine the overall numbers of \( D9 \) and \( D5 \) branes. The structure of these terms matches precisely the corresponding contributions of \( O9 \) and \( O5 \) planes in eq. (3.10) with \( \epsilon = 1 \). The additional terms are associated to the exchange of twisted closed-string modes, and encode the geometry of the branes sitting at the fixed points. In this case, where all the \( D5 \) branes are at the same fixed point, these tadpole terms account precisely for the 15 fixed points seen only by the space-filling \( D9 \) branes, as well as for the additional single fixed point where also \( D5 \)
branes are present. From eqs. (3.16) and (3.18) one can deduce the massless part of
the transverse Möbius amplitude,
\[
\tilde{\mathcal{M}}_0 = -\frac{2}{4} \left\{ \hat{Q}_o \left( \sqrt{v} + \epsilon \frac{1}{\sqrt{v}} \right) \left( N \sqrt{v} + \frac{D}{\sqrt{v}} \right) + \hat{Q}_v \left( \sqrt{v} - \epsilon \frac{1}{\sqrt{v}} \right) \left( N \sqrt{v} - \frac{D}{\sqrt{v}} \right) \right\} , \tag{3.19}
\]
and tadpole cancellation requires \( N = D = 32 \) and \( R_N = R_D = 0 \). In order to
read the open spectrum, we have finally to derive the Möbius amplitude in the direct
channel. This is obtained performing a \( P \) modular transformation on the complete
transverse Möbius amplitude, and the final result is
\[
\tilde{\mathcal{M}} = -\frac{1}{4} \left\{ (\hat{Q}_o + \hat{Q}_v) (NP + DW) - (N + D)(\hat{Q}_o - \hat{Q}_v) \left( \frac{2\eta}{\theta} \right) \right\} . \tag{3.20}
\]
The proper parametrization of the Chan-Paton multiplicities,
\[
N = n + \bar{n} \quad , \quad D = d + \bar{d} \quad , \quad R_N = i(n - \bar{n}) \quad , \quad R_D = i(d - \bar{d}) \quad , \tag{3.21}
\]
with \( n = d = 16 \), identifies the gauge group \( U(16) \times U(16) \) \([15, 51]\), where one
gauge factor lives on the \( D9 \) branes and the other on the \( D5 \) branes. Together with
the vector multiplets associated to this gauge group, the massless open spectrum
contains charged hypermultiplets in the \( (120 + \bar{120}, 1) \) and \( (1, 120 + \bar{120}) \) coming
from \( DD \) and \( NN \) strings, and charged hypermultiplets in the \( (16, \bar{16}) \) coming from
\( ND \) strings. As in the ten-dimensional case, the RR tadpole cancellation implies, as a
consequence of supersymmetry, that the NS tadpole cancels as well. A more general
case, where the \( D5 \) branes are distributed in the 16 fixed points, can be analyzed
following the same technique. One can also consider a situation in which pairs of
image \( D5 \) branes are moved away from the fixed points \([51]\). While the \( D5 \) branes
sitting at the fixed points lead to unitary gauge groups whose rank is determined
by their total number, the remaining \( D5 \) branes lead to symplectic gauge groups
whose rank is determined by the number of displaced pairs. Other models can be
obtained turning on an internal quantized \( B_{ab} \), or by non-geometric orbifolds: these
models typically have several tensor multiplets in the closed spectrum \([15, 52, 58]\), a
characteristic that makes these orbifold constructions quite peculiar with respect to
heterotic models, where the spectrum can only contain a single tensor multiplet at
the perturbative level.
Chapter 3. Open strings

Coming back to the Klein-bottle amplitude of eq. (3.15), for $\epsilon = -1$ all R-R tadpoles can be canceled by $32$ $D9$ branes and $32$ $\bar{D}5$ branes, since the latter indeed revert all the 9-5 R-R contributions to the transverse-channel annulus amplitude. The resulting transverse annulus amplitude is

$$\tilde{A} = \frac{2^{-5}}{4} \left\{ \left( V_4 O_4 + O_4 V_4 - S_4 S_4 - C_4 C_4 \right) \left( N^2 v W + \frac{D^2 P}{v} \right) \right. + 2ND \left( V_4 O_4 - O_4 V_4 - S_4 S_4 + C_4 C_4 \right) \left( \frac{2\eta}{\theta_2} \right)^2 \\
+ 16 \left( O_4 C_4 + V_4 S_4 - S_4 O_4 - C_4 V_4 \right) \left( R_N^2 + R_D^2 \right) \left( \frac{\eta}{\theta_4} \right)^2 \\
+ 8R_N R_D \left( V_4 S_4 - O_4 C_4 - S_4 O_4 + C_4 V_4 \right) \left( \frac{\eta}{\theta_3} \right)^2 \left\} , \quad (3.22)$$

and from $\tilde{K}$ and $\tilde{A}$, by standard methods, it is straightforward to obtain the open spectra, encoded in the direct-channel amplitudes

$$A = \frac{1}{4} \left\{ \left( V_4 O_4 + O_4 V_4 - S_4 S_4 - C_4 C_4 \right) \left( N^2 P + D^2 W \right) \right. + 2ND \left( O_4 S_4 + V_4 C_4 - C_4 C_4 - S_4 V_4 \right) \left( \frac{\eta}{\theta_4} \right)^2 \\
+ \left( R_N^2 + R_D^2 \right) \left( V_4 O_4 - O_4 V_4 + S_4 S_4 - C_4 C_4 \right) \left( \frac{2\eta}{\theta_2} \right)^2 \\
+ 2R_N R_D \left( V_4 C_4 - O_4 S_4 + S_4 V_4 - C_4 O_4 \right) \left( \frac{\eta}{\theta_3} \right)^2 \left\} \quad (3.23)$$

and

$$M = - \frac{1}{4} \left\{ NP \left( \hat{O}_4 \hat{V}_4 + \hat{V}_4 \hat{O}_4 - \hat{S}_4 \hat{S}_4 - \hat{C}_4 \hat{C}_4 \right) \\
- DW \left( \hat{O}_4 \hat{V}_4 + \hat{V}_4 \hat{O}_4 + \hat{S}_4 \hat{S}_4 + \hat{C}_4 \hat{C}_4 \right) \\
- N \left( \hat{O}_4 \hat{V}_4 - \hat{V}_4 \hat{O}_4 - \hat{S}_4 \hat{S}_4 + \hat{C}_4 \hat{C}_4 \right) \left( \frac{2\eta}{\theta_2} \right)^2 \\
+ D \left( \hat{O}_4 \hat{V}_4 - \hat{V}_4 \hat{O}_4 + \hat{S}_4 \hat{S}_4 - \hat{C}_4 \hat{C}_4 \right) \left( \frac{2\eta}{\theta_2} \right)^2 \left\} . \quad (3.24)$$

The R-R tadpole cancellation conditions require

$$N = D = 32 \quad , \quad R_N = R_D = 0 \quad , \quad (3.25)$$

and allow a parametrization in terms of real Chan-Paton multiplicities of the form $N = n_1 + n_2$, $D = d_1 + d_2$, $R_N = n_1 - n_2$ and $R_D = d_1 - d_2$, with $n_1 = n_2 = d_1 =$
$d_2 = 16$. The massless spectrum can be extracted from

$$A_0 + M_0 = \left[ \frac{n_1(n_1 - 1)}{2} + \frac{n_2(n_2 - 1)}{2} + \frac{d_1(d_1 + 1)}{2} + \frac{d_2(d_2 + 1)}{2} \right] V_4 O_4$$
$$- \left[ \frac{n_1(n_1 - 1)}{2} + \frac{n_2(n_2 - 1)}{2} + \frac{d_1(d_1 + 1)}{2} + \frac{d_2(d_2 + 1)}{2} \right] C_4 C_4$$
$$+ (n_1 d_2 + n_2 d_1) O_4 S_4 - (n_1 d_1 + n_2 d_2) C_4 O_4$$
$$+ (n_1 n_2 + d_1 d_2) (O_4 V_4 - S_4 S_4), \quad (3.26)$$

and the gauge group is thus $[SO(16) \times SO(16)]_9 \times [USp(16) \times USp(16)]_5$. Supersymmetry is realized in the 9-9 sector, where the vector multiplets of the two $SO(16)$ are accompanied by a hypermultiplet in the $(16, 16, 1, 1)$, while it is broken on the $D5$ branes, where the gauge vectors of the two $USp(16)$ are in the adjoint representation while the left-handed Weyl fermions are again in reducible antisymmetric representations, now the $(1, 1, 120, 1)$ and the $(1, 1, 1, 120)$. In addition, there are four scalars and two right-handed Weyl fermions in the $(1, 1, 16, 16)$, as well as two scalars in the $(16, 1, 1, 16)$, two scalars in the $(1, 16, 16, 1)$ and symplectic Majorana-Weyl fermions in the $(16, 1, 16, 1)$ and $(1, 16, 1, 16)$. This model is free of gauge and gravitational anomalies and provides an example of type-I vacuum with a stable non-BPS configuration of BPS branes. As in the ten-dimensional $USp(32)$ model, the breaking of supersymmetry yields a tree-level potential for the NS-NS moduli related to the uncanceled tadpoles, the dilaton and the internal volume in this case, that reflects the positive tension resulting from the $O5_-$ planes and the anti-branes.

Models with brane supersymmetry breaking exhibit in their spectra a gauge singlet on the non supersymmetric branes, with the right quantum numbers to be a goldstino. As we shall see in the last chapter, these goldstinos play the role of Volkov-Akulov fields that allow consistent couplings of the gravitinos to the non supersymmetric matter. Supersymmetry is thus linearly realized in the bulk and on some branes, while it is non-linearly realized on other (anti)branes.

### 3.4 Anomaly analysis of six-dimensional models

In the previous chapter we have analyzed the conditions of anomaly cancellations for models of oriented closed strings. We have seen that in the case of type IIB, both the ten-dimensional model and the chiral six-dimensional model, obtained by $K3$ compactification, are anomaly-free. Moreover, the consistency of the heterotic $SO(32)$ and $E_8 \times E_8$ models is guaranteed, at the level of the low-energy effective
action, by the presence of the Wess-Zumino term $B \wedge F^4$, whose anomaly exactly cancels the one produced by fermion loops. Basically, modular invariance is the guiding principle for writing consistent models for oriented closed strings. In this chapter we have seen that for open and unoriented closed strings modular invariance is not a property of all amplitudes, and consistency at the one loop level is granted in this case by R-R tadpole cancellation. As we will see in this section, at the level of the low-energy action this corresponds to the cancellation of gauge and gravitational anomalies.

If we focus for the moment on gauge anomalies in ten dimensions, the potentially anomalous one-loop string amplitudes are the planar (fig. (3.3)), non-orientable (fig. (3.4)) and non-planar (fig. (3.5)) amplitudes, with six vectors inserted on the boundaries.

In the first two cases, the vectors are all inserted on one boundary of the annulus or on the single boundary of the Möbius strip, and thus the Chan-Paton factor is $\text{tr} F^6$, while in the third case the vectors are inserted in both boundaries of the annulus, and the resulting amplitude is proportional to $\text{tr} F^2 \text{tr} F^4$ (all traces here are in the fundamental representation). Thus, the first two amplitudes give the irreducible part of the gauge anomaly, that is canceled if the gauge group is $SO(32)$ in the supersymmetric model \cite{31} or $USp(32)$ in the Sugimoto model \cite{48}. The third amplitude, that should in principle give the reducible part of the gauge anomaly, is in fact non-anomalous,
since it can be regulated by the momumentum flowing in the tube $^{30}$. The apparent contradiction between this result and the field theory analysis, in which the reducible part of the hexagon gauge anomaly (fig (2.3)) is canceled by the anomalous exchange of the B field (fig. (2.4)), is explained because of the open-closed string duality of the annulus diagram: the amplitude of fig. 3.5, that in the direct channel is a one-loop open-string amplitude, in the transverse channel becomes a tree-level closed-string amplitude. The two interpretations are valid in two different regions of the integration variable $\tau$, and the anomalous contributions from these regions exactly cancel.

Turning to $\mathcal{N} = (1, 0)$ six-dimensional models, we have seen that the condition for the cancellation of the irreducible term $\text{tr}R^4$ in the anomaly polynomial is

$$n_H - n_V + 29n_T = 273, \quad (3.27)$$

where $n_T$, $n_V$ and $n_H$ are the number of tensor, vector and hypermultiplets, respectively. Perturbative heterotic models, obtained by reduction on $K3$ of the ten-dimensional heterotic strings, always contain a single tensor multiplet. The antiself-dual tensor in this tensor multiplet adds to the self-dual tensor of the gravity multiplet to form an unconstrained tensor $B$. In these models with $n_T = 1$, the residual anomaly polynomial always factorizes, as in the ten-dimensional case, assuming the form

$$I_8 = c^z \bar{c}^{\bar{z}} \text{tr}_z F^2 \text{tr}_{\bar{z}} F^2, \quad (3.28)$$

where $c^z$’s and $\bar{c}^{\bar{z}}$’s are constants and the index $z$ runs over the various factors of the gauge group and over the Lorentz group. The consistency of the model, that is granted at the string level because of modular invariance, from the point of view of the low-energy effective action corresponds to the addition of the Wess-Zumino term

$$B \wedge c^z \text{tr}_z (F \wedge F). \quad (3.29)$$

As in the ten-dimensional case this term is anomalous, since gauge-invariance of the field strength

$$H = dB - \bar{c}^{\bar{z}} \omega^z \quad (3.30)$$

requires

$$\delta B = \bar{c}^{\bar{z}} \text{tr}_z (\Lambda dA). \quad (3.31)$$

Here we denote with $\omega$ the Chern-Simons 3-form (for all the notations we remind the reader to the next chapter). The anomaly resulting from eq. (3.29) has exactly
the form of the one originating from the anomaly polynomial \((3.28)\), and so can exactly cancel it, thus resulting in a Green-Schwarz mechanism exactly as in the ten-dimensional case. The difference, here, is that the Wess-Zumino term has two derivatives, and thus is already present in the low-energy effective action. In the next chapter we shall see a number of peculiar consequences of this result.

We now want to perform an anomaly analysis for six-dimensional type-I models. We refer in particular to supersymmetric models, even if the same conclusions apply to six-dimensional brane supersymmetry breaking vacua, in which supersymmetry is realized on the bulk and broken on some branes (we will see in the last chapter that in these models supersymmetry is non-linearly realized on these branes). An important feature of these models, with respect to the heterotic case, is that one can obtain vacua with several tensor multiplets. In all cases, the cancellation of R-R tadpoles corresponds to the absence of irreducible anomalies. Analyzing the chiral content of all these six-dimensional vacua, and using the results of the previous chapter, one can write down the residual anomaly polynomial, that in general does not factorize, but can be written in the form \([54]\)

\[
\eta_{rs} c^r c^{s'} \text{tr}_z F^2 \text{tr}_{z'} F^2 ,
\]

(3.32)

where \(\eta\) is the Minkowski metric for \(SO(1, n_T)\) \((r, s = 0, ..., n_T)\), the \(c\)'s are constant and the index \(z\) runs over the various simple factors of the gauge group and over the Lorentz group. The resulting anomaly is canceled since several tensors take part in a generalized Green-Schwarz mechanism \([54]\), corresponding to the inclusion in the low-energy effective action of the term

\[
c^r B_r \wedge \text{tr}_z F^2 ,
\]

(3.33)

that is anomalous since the fields \(B^r\) transform as

\[
\delta B^r = c^{r z} \text{tr}_z (\Lambda dA)
\]

(3.34)

under gauge and Lorentz transformations\(^4\).

More precisely, only the tensors connected to the characters that are present in the transverse annulus amplitude with reflection coefficients not identically vanishing take part to the anomaly cancellation. Among the various terms in the anomaly polynomial, only one contains the Riemann curvature, and this factor, always present, is canceled by the antisymmetric tensor in the gravity multiplet. For the type-I models analyzed in \([54]\), the \(c\)'s are related to the rows of the matrix \(S\).

\(^4\)The connection associated to Lorentz transformations is of course the spin-connection.
### 3.5 Six-dimensional vacua and dualities

At the end of the first chapter we have considered ten-dimensional $\mathcal{N} = 1$ supergravity coupled to vector multiplets, and we have derived the field redefinitions that map the theory in the heterotic frame to the same theory in the type-I frame. Since the heterotic $SO(32)$ and the supersymmetric type-I ten-dimensional strings have the same massless field content, this field redefinition relates their low-energy effective actions. This relation is the low-energy realization of the duality between the two complete theories \[55\], that maps the strong coupling regime of one into the weak coupling regime of the other, as can be argued from the relation

$$\phi_H = -\phi_I \quad ,$$

that corresponds to

$$g_H = g_I^{-1} \quad .$$

The fact that this duality is a strong-weak coupling duality is in accordance with the fact that, at a perturbative level, these two string theories have vastly different spectra of massive states. Since the heterotic 2-form is mapped to the type-I RR 2-form, the fundamental heterotic string, that is charged with respect to this form, is mapped to the D1-string in the type-I model.

Let us now consider these two theories compactified to lower dimensions. Denoting with $h_{10-d}$ the metric of the internal manifold, the dilaton in $d$ dimensions is

$$\phi_d = \phi_{10} - \frac{1}{4} \log \det h_{10-d} \quad .$$

Combining this relation with the duality transformations of the dilaton and the metric in eq. (1.96), one obtains \[56\]

$$\phi_{I,d} = \frac{6 - d}{4} \phi_{H,d} + \frac{2 - d}{16} \log \det h_{H,10-d} \quad .$$

This relation shows that the duality is a strong-weak coupling duality only if $d > 6$, while it becomes a perturbative duality for $d < 6$ (a first pair of four-dimensional heterotic and type-I vacua that are perturbatively equivalent was also displayed in \[56\]). Moreover, in six dimensions the dilaton of one theory is purely geometric if expressed in terms of the fields in the dual theory. This is consistent with the fact that in six-dimensional $\mathcal{N} = (1,0)$ vacua the dilaton belongs to a hypermultiplet in the type-I case and to the (unique) tensor multiplet in the heterotic case. Perturbative
Chapter 3. **Open strings**

Type-I models without tensor multiplets [57] correspond, according to this relation, to non-perturbative heterotic vacua without dilaton fluctuations.

It is then of interest to understand what kind of non-perturbative picture corresponds on the heterotic side to perturbative type-I models with different numbers of tensor multiplets. Starting from the perturbative heterotic \( SO(32) \) model compactified on \( K3 \), with total instanton number \( k = 24 \), one develops an extra \( SU(2) \) gauge symmetry when an instanton shrinks to zero size [58]. In the type-I picture, this corresponds to the appearance of a pair of D5-branes, with an \( USp(2) \) Chan-Paton group, whose world-volume fills the six uncompactified dimensions. The condition \( k = 24 \) is replaced by \( k = 24 - n \), where \( n \) is the number of instantons shrinking to zero size. When \( r \) of them coincide at a given point of \( K3 \), the non-perturbative gauge symmetry is enhanced to \( USp(2r) \), and this naturally corresponds to the appearance of \( 2r \) coinciding D5-branes, while the perturbative gauge group \( SO(32) \) is broken by a background with a lower instanton number, \( k = 24 - n \). Vacua with several tensor multiplets can be obtained compactifying the ten-dimensional theory on \( K3 \) manifolds with singularities: small instantons on these singularities give rise to Coulomb phases parametrized by the real scalars in the tensor multiplets [59, 60, 61].

Now we consider six-dimensional vacua of the \( E8 \times E8 \) heterotic theory on \( K3 \). In this case, perturbative vacua correspond to the condition \( k_1 + k_2 = 24 \), where \( k_1 \) and \( k_2 \) are the instanton numbers corresponding to the two \( E8 \) factors. As already anticipated in Section (2.5), the non-perturbative ten-dimensional \( E8 \times E8 \) theory corresponds to M-theory compactified on the interval \( S^1/Z_2 \), with a length proportional to the heterotic string coupling [11], while the gauge fields of the two \( E8 \) factors are on the two different “end of the world” 9-branes. Further compatifications on \( K3 \) are obtained embedding \( k_1 \) instantons in one \( E8 \) factor and \( k_2 \) in the other, but another possibility is left, the addition of \( n \) M5-branes, located at points in \( K3 \times S^1/Z_2 \), and filling the six non-compact dimensions. The condition \( k_1 + k_2 = 24 \) is then replaced by \( k_1 + k_2 + n = 24 \), and the corresponding six-dimensional vacua have \( n + 1 \) tensor multiplets. An instanton shrinking to zero size corresponds to the appearance of an M5-brane stuck to the end of the world 9-brane. When the 5-brane leaves the boundary, a tensor multiplet appears in the low-energy action. The M5-brane can travel to the other boundary and be reabsorbed, so that all possible values of \( (k_1, k_2) \) are connected [62]. The presence of an antiself-dual tensor in the effective field theory of the M5-brane means that M2-branes can end on the M5-brane, with one direction tangent to it [63]. The corresponding string on the world-volume of the M5-brane is
antiself-dual. Since the tension of the string is proportional to the distance between the M5-brane and the boundary, the singularity associated to a shrinking instanton signals the appearance of tensionless strings. This phenomenon actually manifests itself in the low-energy effective action because of the appearance of a singularity in the gauge kinetic term, corresponding to an infinite gauge coupling constant \([54]\). As an example, consider the case in which a single tensor multiplet is present. In this case, the kinetic term in the Einstein frame is proportional to \(ae^\phi + be^{-\phi}\), where \(a\) and \(b\) are constants and \(\phi\) is the scalar in the tensor multiplet. If \(a\) and \(b\) have opposite sign, the kinetic term diverges for a particular value of the scalar, and singularities of the same type appear if several tensor multiplets are present. We have now seen in the \(E_8 \times E_8\) picture that these singularities signal a new kind of phase transition \([64]\), reflecting the presence in the vacuum of string excitations with vanishing tension \([62, 65]\). The singularity appears for particular values of the scalars in the tensor multiplets, since these scalars parametrize the distance between the M5-brane and the 9-brane.

An analogous geometrical picture of this low-energy phenomenon can be given in the type-I theory. In this case the starting points are the duality between heterotic theory on \(T^4\) and type IIA on \(K3\), and the T-duality between type IIA and type IIB. Since the former implies that type IIA have enhanced gauge symmetry at certain points in \(K3\) moduli space, one can ask how type IIB behaves at the corresponding points. If one further compactifies IIA and IIB theories on T-dual circles to five dimensions, T-duality implies

\[
g_{A,6} = \frac{g_{B,6}}{R_B} ,
\]

and at a distance \(\epsilon\) from a point at which IIA gets enhanced gauge symmetry, one has W-bosons of mass \(\epsilon/g_{A,6}\). In five dimensions, in IIB units this is then a W-boson of mass

\[
M_W = \frac{\epsilon R_B}{g_{B,6}} ,
\]

that corresponds to a selfdual string in six-dimensions wrapped on a circle of radius \(R_B\). This selfdual string is in fact a selfdual D3-brane of type IIB wrapped on a collapsing 2-sphere in \(K3\) \([66]\). After the orientifold projection, these D-strings manifest themselves as singularities in the low-energy effective action (see \([53]\) for a review).

Following \([62]\), we want to make a final comment about what kind of physics one should expect at the singularity. In six dimensions, every conventional gauge
theory is free in the infrared. This is because the gauge coupling has the dimension of a length, and consequently for scales larger than this length the theory is usually expected to be free. When one approaches the singularity, however, this length scale becomes larger and larger, and at the singularity it diverges, so that the theory can indeed be non-free in the infrared.
Chapter 4

Minimal six-dimensional supergravity

As we have seen in the previous chapters, one of the most striking features of perturbative superstring theory in ten dimensions is the absence of anomalies. In type-IIB theory this is realized by miraculous cancellations between various contributions \[29\], while in the type-I and heterotic theories the Green-Schwarz mechanism \[30\] generates anomalous couplings that exactly cancel the contributions of fermion loops, once one restricts the gauge group to be \(SO(32)\) for the type I theory and \(SO(32)\) or \(E_8 \times E_8\) for the heterotic theory. All these \(N = 1\) theories are very interesting, since they can be naturally compactified to \(N = 1\) theories in four dimensions. In this context, an interesting intermediate step is the study of \((1,0)\) vacua in six dimensions, since in these compactifications the absence of anomalies is a very strong restriction on the low-energy physics.

The massless representations of \((1,0)\) supersymmetry in six dimensions are the gravity multiplet \((e^a_\mu, \psi_L, B^a_{\mu
u})\), the tensor multiplet \((B^-_{\mu
u}, \chi_R, \varphi)\), the vector multiplet \((A_\mu, \lambda_L)\) and the hypermultiplet \((4\phi, \Psi_R)\). We have seen in the last chapter that, letting \(n_T\), \(n_V\) and \(n_H\) denote the numbers of tensor, vector and hypermultiplets, the condition that the term \(\text{tr} R^4\) be absent in the anomaly polynomial is \[31\]

\[n_H - n_V + 29n_T = 273\]

and compared to the ten dimensional case this allows a large number of possible vacua. Perturbative heterotic vacua in six dimensions can be obtained by orbifold
compactifications or by compactifications on smooth \(K3\) manifolds with instanton backgrounds. Anomaly cancellation requires that the total instanton number be 24, and these vacua include a single tensor multiplet, as one can easily see reducing the ten-dimensional low-energy theory. Repeating what we said in the last chapter, we would like to stress that the situation is quite different in perturbative six-dimensional type I vacua since, as suggested in \[44\], these models are determined by a parameter space orbifold (orientifold) construction, that naturally allows several tensor multiplets \[45, 52\]. In general the residual anomaly polynomial does not factorize, and several antisymmetric tensors contribute to the cancellation via a generalized Green-Schwarz mechanism \[74\].

The anomaly polynomial of eq. (3.32) reveals an important difference with respect to ten dimensions: while the ten-dimensional Green-Schwarz coupling, \(B \wedge (F^4 - R^4)\), is a higher derivative term, the gauge portion of the corresponding six-dimensional coupling, \(B \wedge (F^2 - R^2)\), belongs to the low-energy effective action. One is thus facing a case of unprecedented complexity in supergravity constructions, whereby the model is determined by Wess-Zumino conditions \[67\], rather than by the usual requirement of local supersymmetry. In this chapter, we shall see that one important consequence of this is that the resulting equations are not unique, since a quartic coupling for the gauginos is undetermined, and the construction is consistent for any choice of this coupling. Correspondingly, the commutator of two supersymmetry transformations on the gauginos contains an extension, that plays a crucial role in ensuring that the Wess-Zumino consistency conditions close on-shell. Moreover, in this model the divergence of the energy-momentum tensor is non-vanishing, as is properly the case for a theory that has gauge anomalies but no gravitational anomalies (gravitational anomalies could be accounted for introducing higher-derivative couplings) \[68, 69, 70\]. Once more, the low-energy couplings are obtained by consistency once one includes the Green-Schwarz term in the low-energy theory, but the complete theory, supersymmetric and gauge-invariant, would also include additional non-local couplings arising from fermion loops. Let us stress that this is exactly as in the ten-dimensional case, but in these six-dimensional models the anomalous terms belong to the low-energy effective action.

One can even consider a slight modification of these couplings, resulting from the inclusion in the low-energy Lagrangian of more general Green-Schwarz couplings to
abelian vectors of the form [71]

\[ B^r c_{r}^{ab} F^a F^b \]

where the indices \( a, b \) run over the different \( U(1) \) gauge groups, while the symmetric matrices \( c^r \) may not be simultaneously diagonalized. This naturally corresponds to the inclusion of non-diagonal Chern-Simons couplings [72], and in this case the residual anomaly polynomial has the more general form

\[ c_{ab}^{cr} c_{cd}^{es} \eta_{rs} F^a \wedge F^b \wedge F^c \wedge F^d \]

The corresponding kinetic terms of vectors and gauginos are also non-diagonal, compatibly with the abelian gauge invariances.

At the end of the last chapter we remarked that another interesting feature of these models is the fact that they exhibit singularities in the moduli space of tensor multiplets, corresponding to infinite gauge coupling constants [54]. As an example, consider the case in which a single tensor multiplet is present. In this case the kinetic term is proportional to \( ae^{\phi} + be^{-\phi} \), where \( a \) and \( b \) are constants and \( \phi \) is the scalar in the tensor multiplet, and if \( a \) and \( b \) have opposite sign, the kinetic term diverges for a particular value of the scalar. The same singularities appear if several tensor multiplets are present. These singularities signal a new kind of phase transition [64], reflecting the presence in the vacuum of string excitations with vanishing tension [62].

The coupling of \((1,0)\) supergravity to \( n \) tensor multiplets was originally studied in [24] to lowest order in the Fermi fields, while [23] considered the coupling to a single tensor multiplet and to vector and hypermultiplets to all orders in the Fermi fields. As we anticipated, in this case the kinetic term is generally proportional to \( ae^{\phi} + be^{-\phi} \) [24], while [23] actually deals with the particular case \( a = 0 \), in which the anomaly polynomial vanishes and no tensionless string transition occurs. The general coupling to non-abelian vectors and self-dual tensors was worked out to lowest order in the Fermi fields in [54]. In this covariant formulation the requirement of supersymmetry gives non-integrable equations, while the divergence of the vector equation gives the covariant anomaly. The same model was then reconsidered, again to lowest order in the Fermi fields, in [68] in the consistent formulation, requiring the closure of the Wess-Zumino conditions, that relate the consistent gauge anomaly to the supersymmetry anomaly. Additional couplings, as well as the inclusion of hyper-multiplets, were then considered in [74]. The complete coupling to non-abelian vector and tensor multiplets was then obtained in [69] in the consistent formulation.
and in [75] in the covariant formulation. [71] contains the most general couplings for the case in which also abelian vectors are present, and finally in [76] supergravity coupled to vector, tensor and hypermultiplets was constructed to all orders in the Fermi fields.

In this chapter we construct the complete \((1,0)\) supergravity coupled to tensor, vector and hypermultiplets. Since all the subtleties of the construction are already present in the absence of hypermultiplets, in Section 2 we construct minimal supergravity coupled to non-abelian vector and \(n_T\) tensor multiplets. In Section 3 we will apply to this model the PST construction [77]. Section 4 considers the case in which abelian vectors are present, and finally Section 5 contains the complete coupling of minimal six-dimensional supergravity to tensor, vector and hypermultiplets.

4.1 Supergravity coupled to tensor and vector multiplets

In this section we describe minimal \((1,0)\) six-dimensional supergravity coupled to \(n\) tensor multiplets and non-abelian vector multiplets [69]. Simple supersymmetry in six dimensions is generated by a \(USp(2)\) doublet of chiral spinorial charges \(Q^A\) \((A = 1,2)\) obeying the symplectic Majorana condition

\[
Q^A = e^{AB}C\bar{Q}^T_B ,
\]

where \(e^{AB}\) is the \(USp(2)\) antisymmetric invariant tensor. Since all Fermi fields are \(USp(2)\) doublets, in this section we will mostly write \(\psi\) to denote a doublet \(\psi^A\).

4.1.1 Supergravity coupled to tensor multiplets

We start our analysis from the simpler case in which only tensor multiplets are present. Let us begin by reviewing the work of Romans [24]. The theory includes the vielbein \(e_\mu^m\), a left-handed gravitino \(\psi_\mu\), \((n+1)\) antisymmetric tensors \(B_{\mu\nu}^r\) \((r = 0, \ldots, n)\) obeying (anti)self-duality conditions, \(n\) right-handed “tensorinos” \(\chi^M\) \((M = 1, \ldots, n)\), and \(n\) scalars. The scalars parameterize the coset space \(SO(1,n)/SO(n)\), and are thus associated to the \(SO(1,n)\) matrix \((r = 0, \ldots n)\)

\[
\begin{pmatrix}
\nu_r \\
\chi_r^M \\
x_r^M
\end{pmatrix},
\]

\[(4.2)\]
whose elements satisfy the constraints
\begin{align}
v^r v_r &= 1 \\
v_r v_s - x^M_r x^M_s &= \eta_{rs} \\
v^r x^M_r &= 0
\end{align}

(4.3)

Defining
\[G_{rs} = v_r v_s + x^M_r x^M_s \]

(4.4)

the tensor (anti)self-duality conditions can be succinctly written
\[G_{rs} H^{\mu \nu \rho} = \frac{1}{6} \epsilon^{\mu \nu \rho \alpha \beta \gamma} H_{\alpha \beta \gamma} \]

(4.5)

where \(H^{\mu \nu \rho} = 3 \partial_{[\mu} B_{\nu \rho]}^r\). These relations only hold to lowest order in the Fermi fields, and imply that \(v_r H^{\mu \nu \rho}_r\) is self dual, while the \(n\) tensors \(x^M_r H^{\mu \nu \rho}_r\) are antiself dual, as one can see using eqs. (4.3). The divergence of eq. (4.5) yields the second-order tensor equation
\[\nabla_\mu (G_{rs} H^{\mu \nu \rho}) = 0 \]

(4.6)

while, to lowest order, the fermionic equations are
\[-i \gamma^{\mu \nu \rho} D_\nu \psi_\rho - iv_r H^{\mu \nu \rho} \gamma_\nu \psi_\rho - \frac{1}{2} x^M_r H^{\mu \nu \rho} \gamma_\nu \chi^M + \frac{1}{2} x^M_r \partial_\nu v^r \gamma_\nu \gamma_\mu \chi^M = 0 \]

(4.7)

and
\[i \gamma^\mu D_\mu \chi^M - \frac{i}{12} v_r H^{\mu \nu \rho} \gamma_{\mu \nu \rho} \chi^M + \frac{1}{2} x^M_r H^{\mu \nu \rho} \gamma_{\mu \nu \rho} \psi_\rho + \frac{1}{2} x^M_r \partial_\nu v^r \gamma_\nu \gamma_\mu \psi_\rho = 0 \]

(4.8)

Varying the Fermi fields in them with the supersymmetry transformations
\begin{align}
\delta e^m_\mu &= -i (\bar{\epsilon} \gamma^m \psi_\mu) \\
\delta B_{\mu \nu}^r &= i v^r (\bar{\psi}_\mu \gamma_\nu \epsilon) + \frac{1}{2} x^M_r (\bar{\chi}^M \gamma_{\mu \nu} \epsilon) \\
\delta v_r &= x^M_r (\bar{\epsilon} \chi^M) \\
\delta \psi_\mu &= D_\mu \epsilon + \frac{1}{4} v_r H^{r}_{\mu \nu \rho} \gamma_{\mu \nu \rho} \epsilon \\
\delta \chi^M &= \frac{i}{2} x^M_r \partial_\mu v^r \gamma_\mu \epsilon + \frac{i}{12} x^M_r H^{r}_{\mu \nu \rho} \gamma_{\mu \nu \rho} \epsilon
\end{align}

(4.9)

generates the bosonic equations, using also eqs. (4.3) and (4.6). Thus, the scalar field equation is
\[x^M_r D_\mu (\partial^\mu v^r) + \frac{2}{3} x^M_r v_\mu H^{r}_{\alpha \beta \gamma} H^{s \alpha \beta \gamma} = 0 \]

(4.10)
while the Einstein equation is
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \partial_\mu v^r \partial_\nu v_r - \frac{1}{2} g_{\mu\nu} \partial_\alpha v^r \partial_\alpha v_r - G^{r}_{rs} H^{s}_{\mu\alpha\beta} H^{r}_{\nu\alpha\beta} = 0 \quad . \] (4.11)

To this order, this amounts to a proof of supersymmetry, and it is also possible to show that the commutator of two supersymmetry transformations on the bosonic fields closes on the local symmetries:
\[ [\delta_1, \delta_2] = \delta_{\text{gct}}(\xi^\mu = -i(\xi_1 \gamma^\mu \xi_2)) + \delta_{\text{tens}}(A^r_\mu = -\frac{1}{2} v^r \xi_\mu - \xi^\nu B^r_{\mu\nu}) \]
\[ + \delta_{\text{SO}(n)}(A^{M\bar{N}} = \xi^\mu x^M_{\mu\bar{N}}(\partial_{\mu} x^{N}_{\bar{R}})) \]
\[ + \delta_{\text{Lorentz}}(\Omega^{mn} = -\xi_\mu (\omega^{\mu mn} - v_r H^{rmn})) \]. (4.12)

To this order, one can not see the local supersymmetry transformation in the gauge algebra, since the expected parameter, \( \xi^\mu \psi_\mu \), is generated by bosonic variations. As usual, the spin connection satisfies its equation of motion, that to lowest order in the Fermi fields is
\[ D_\mu e^m_\nu - D_\nu e^m_\mu = 0 \quad , \] (4.13)
and implies the absence of torsion.

Completing these equations will require terms cubic in the Fermi fields in the fermionic equations, and terms quadratic in the Fermi fields in their supersymmetry transformations. Supersymmetry will then determine corresponding modifications of the bosonic equations, and the (anti)self-duality conditions (4.5) will also be modified by terms quadratic in the Fermi fields. Supercovariance actually fixes all terms containing the gravitino in the first-order equations and in the supersymmetry variations of Fermi fields.

The supercovariant forms
\[ \hat{\omega}_{\mu\nu\rho} = \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \quad , \] (4.14)
\[ \hat{H}_{\mu\nu}^r = H_{\mu\nu}^r - \frac{1}{2} x^M_\nu (\bar{X}_\mu^m \gamma_\nu \psi_\rho + \bar{X}_\mu^M \gamma_\rho \psi_\nu + \bar{X}_\mu^M \gamma_\mu \psi_\nu) \]
\[ - \frac{i}{2} v^r (\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \quad , \] (4.15)
\[ \partial_\mu \hat{v}^r = \partial_\mu v^r - x^M_\nu (\bar{X}_\mu^M \psi_\mu) \quad , \] (4.16)
where
\[ \omega^0_{\mu\nu\rho} = \frac{1}{2} \epsilon_{pm}(\partial_\mu e^m_\nu - \partial_\nu e^m_\mu) - \frac{1}{2} \epsilon_{\mu m}(\partial_\nu e^m_\rho - \partial_\rho e^m_\nu) + \frac{1}{2} \epsilon_{\nu m}(\partial_\rho e^m_\mu - \partial_\mu e^m_\rho) \quad . \] (4.17)
is the standard spin connection in the absence of torsion, do not generate derivatives of the parameter under supersymmetry. In the same spirit, one can consider the supercovariant transformations

$$\delta \psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} v_\nu \hat{H}^{\nu}_{\mu \rho} \gamma^{\rho \epsilon}$$

$$\delta \chi^M = \frac{i}{2} x^M_\nu (\partial_\mu v^\nu) \gamma^\mu \epsilon + \frac{i}{12} x^M_\nu \hat{H}^{\nu}_{\mu \rho} \gamma^{\rho \epsilon} \quad .$$

(4.18)

The tensorino transformation is complete, while the gravitino transformation could include additional terms quadratic in the tensorinos. On the other hand, one does not expect modifications of the bosonic transformations in the complete theory.

The algebra (4.12) has been obtained varying only the Fermi fields in the bosonic supersymmetry transformations. The next step is to compute the commutator varying the bosonic fields as well. There is no important novelty in the complete commutator on $v^\nu$ and on the vielbein $e^m_\mu$. However, the local Lorentz parameter is modified and takes the form

$$\Omega^{mn} = -\xi^\mu (\hat{\omega}^m_\mu - v_\nu \hat{H}^{mn}_\mu)$$

(4.19)

while, as anticipated, the supersymmetry parameter is

$$\zeta = \xi^\mu \psi_\mu \quad .$$

(4.20)

These results are obtained using the torsion equation for $\hat{\omega}$,

$$\hat{D}_\mu e^m_\nu - \hat{D}_\nu e^m_\mu = 2 S^m_\mu \nu = -i (\bar{\psi}_\mu \gamma^m \psi_\nu) \quad .$$

(4.21)

One can also compute the commutator on $x^M_\nu$. Eqs. (4.13) determine its supersymmetry variation

$$\delta x^M_\nu = v_\nu (\bar{\epsilon} \chi^M) \quad ,$$

(4.22)

and the resulting commutator includes a local $SO(n)$ transformation of parameter

$$A^{MN} = \xi^\mu x^M_\nu (\partial_\mu x^N_\nu) + (\bar{\chi}^M_\epsilon_2 \epsilon_1) - (\bar{\chi}^N_\epsilon_1 \epsilon_2) \quad .$$

(4.23)

New results come from the complete commutator on $B^s_{\mu \nu}$, where one needs to use the (anti)self-duality conditions. Supercovariantization is at work here, since these conditions are first-order equations, that become

$$G_{rs} \hat{H}^s_{\mu \rho} = \frac{1}{6} \epsilon_{\mu \rho \beta \gamma} \hat{H}^{\alpha \beta \gamma}_r \quad .$$

(4.24)
It is actually possible to alter these conditions demanding that the modified tensor

$$\hat{H}_{\mu\nu}^r = \hat{H}_{\mu\nu}^r + i\alpha v^r (\bar{\chi}^M \gamma_{\mu\nu} \chi^M)$$

(4.25)
satisfy (anti)self-duality conditions as in eq. (4.24). Using eqs. (4.3), one can see that the new $\chi^2$ terms contribute only to the self-duality condition, while the tensors $x_r^M \hat{H}_{\mu\nu}^r$ remain antiseft dual without extra $\chi^2$ terms. Consequently, since the commutator on $B_{\mu\nu}^r$ uses only the antiself-duality conditions, the result does not contain terms proportional to $\alpha$. The commutator on the tensor fields generates all local symmetries in the proper form, aside from the extra terms

$$[\delta_1, \delta_2]_{\text{extra}} B_{\mu\nu}^r = \frac{1}{2} v^r (\bar{\epsilon}_1 \chi^M)(\bar{\chi}^M \gamma_{\mu\nu} \epsilon_2) - \frac{1}{2} v^r (\bar{\epsilon}_2 \chi^M)(\bar{\chi}^M \gamma_{\mu\nu} \epsilon_1) \ ,$$

(4.26)

that may be canceled adding $\chi^2$ terms to the transformation of the gravitino. The most general expression one can add is

$$\delta' \psi_\mu = i a \gamma_\mu \chi^M (\bar{\epsilon}_1 \chi^M) + i b \gamma_\nu \chi^M (\bar{\epsilon}_1 \gamma_\nu \chi^M) + i c \gamma_{\mu\nu} \chi^M (\bar{\epsilon}_1 \gamma_{\nu\rho} \chi^M) \ ,$$

(4.27)

with $a$, $b$ and $c$ real coefficients, and the total commutator on $B_{\mu\nu}^r$ then leads to the relations

$$a + b = -\frac{1}{2} \ , \quad b + 2c = 0 \ .$$

(4.28)

The commutator on $e_\mu^m$ now closes with a local Lorentz parameter modified by the addition of

$$\Delta \Omega^{mn} = -\frac{1}{2} [ (\bar{\chi}^M \epsilon_1)(\bar{\epsilon}_2 \gamma^{mn} \chi^M) - (\bar{\chi}^M \epsilon_2)(\bar{\epsilon}_1 \gamma^{mn} \chi^M) ] \ ,$$

(4.29)

while the commutators on the scalar fields are not modified.

One can now start to compute the commutators on Fermi fields, that as usual close only on shell. Following (10), we will actually use this result to derive the complete fermionic equations. Let us begin with the commutator on the tensorinos, using eq. (4.18). This fixes the free parameter in the gravitino variation and the parameter $\alpha$ in eq. (4.25), so that

$$a = -\frac{3}{8} \ , \quad b = -\frac{1}{8} \ , \quad c = \frac{1}{16} \ , \quad \alpha = -\frac{1}{8} \ .$$

(4.30)

Supercovariance determines the field equation of the tensorinos up to a term proportional to $\chi^3$. Closure of the algebra fixes this additional term, and the end result is

$$i \gamma^\mu \tilde{D}_\mu \chi^M - \frac{i}{12} v^r (\hat{H}_{\mu\nu}^s \gamma_{\mu\nu} \chi^M) + \frac{1}{2} \bar{\epsilon}_r \hat{H}^{r\mu\nu\rho} \gamma_{\mu\nu} \psi^\rho + \frac{1}{2} \gamma^\mu \chi^N (\bar{\chi}^N \gamma_\mu \chi^M) = 0 \ .$$

(4.31)
4.1 Supergravity coupled to tensor and vector multiplets

The complete commutator of two supersymmetry transformations on the tensorinos is then

\[ [\delta_1, \delta_2] \chi^M = \delta_{\text{gct}} \chi^M + \delta_{\text{Lorentz}} \chi^M + \delta_{\text{SO}(n)} \chi^M + \delta_{\text{susy}} \chi^M - \frac{i}{4} \gamma^\mu \xi_\mu \text{[eq. } \chi^M \text{]} . \]  

(4.32)

A similar result can be obtained for the gravitino. In this case the complete equation,

\[-i \gamma^\mu \psi^\rho \bar{D}_\mu \psi_\rho - i \frac{1}{4} v^r \hat{H}_\nu^{\mu \tau} \chi^\rho \gamma^\nu \gamma^\tau \psi_\rho - \frac{1}{2} x^M \hat{H}^{\mu \nu \rho} \gamma_{\nu \rho} \chi^M + \frac{1}{2} x^M (\partial_\nu v^r) \gamma^\nu \gamma^\mu \chi^M \]

\[ + \frac{3}{2} \gamma^\mu \chi^M (\bar{\chi}^M \psi_\nu) - \frac{1}{4} \gamma^\mu \chi^M (\bar{\chi}^M \gamma_{\nu \rho} \psi^\rho) + \frac{1}{4} \gamma_{\nu \rho} \chi^M (\bar{\chi}^M \gamma^\mu \psi^\rho) \]

\[ - \frac{1}{2} \chi^M (\bar{\chi}^M \gamma^\mu \psi^\rho) = 0 \]  

is fixed by supercovariance, and the commutator closes up to terms proportional to a particular combination of eq. (4.33) and its $\gamma$-trace. Moreover, a non-trivial symplectic structure makes its first appearance in a commutator, so that the final result is (see the Appendix for the notations)

\[ [\delta_1, \delta_2] \psi^A_\mu = \delta_{\text{gct}} \psi^A_\mu + \delta_{\text{Lorentz}} \psi^A_\mu + \delta_{\text{susy}} \psi^A_\mu \]

\[ + \frac{3i}{8} \gamma_\mu \mu (\text{[eq. } \psi_\mu \text{]} - \frac{1}{4} \gamma_\mu [\gamma - \text{trace}])^A \]

\[ + \frac{i}{96} \sigma^A B \gamma^\mu \rho \gamma^\nu \rho \text{[eq. } \psi_\mu \text{]} - \frac{1}{4} \gamma_\mu [\gamma - \text{trace}] B \]  

(4.34)

where

\[ \xi^i_{\mu \rho} = -i [\bar{\xi}_1 \gamma_{\mu \rho} \epsilon_2]^i \]  

(4.35)

Summarizing, from the algebra we have obtained the complete fermionic equations of (1, 0) six-dimensional supergravity coupled to $n$ tensor multiplets. In addition, the modified 3-form

\[ \hat{\mathcal{H}}^{\mu \nu \rho} = \hat{H}^{\mu \nu \rho} - i \frac{1}{8} v^r (\bar{\chi}^M \gamma_{\mu \rho} \chi^M) \]  

(4.36)

satisfies the (anti)sself-duality conditions

\[ G_{\tau \lambda} \hat{\mathcal{H}}^{\tau \lambda} = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma \delta \tau} \hat{\mathcal{H}}^{\mu \nu \rho \sigma \delta \tau} \]  

(4.37)

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

\[ \delta e_{\mu}^m = -i (\bar{\epsilon} \gamma^m \psi_\mu) \]  

\[ \delta B^r_{\mu \nu} = i v^r (\bar{\psi}_\mu \gamma_\nu \epsilon) + \frac{1}{2} x^M v^r (\bar{\chi}^M \gamma_{\mu \nu} \epsilon) \]  

.
\[ \delta v_r = x^{M}_r (\bar{\chi}^M \epsilon) , \]
\[ \delta \psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} v_r \hat{H}^r_{\mu \rho \gamma} \gamma^{\mu \rho} \epsilon - \frac{3i}{8} \bar{\gamma}_\mu \chi^M (\bar{\epsilon} \chi^M) \]
\[ - \frac{i}{8} \bar{\gamma}_\mu \chi^M (\bar{\epsilon} \gamma_{\mu \rho} \chi^M) + \frac{i}{16} \gamma_{\mu \rho} \chi^M (\bar{\epsilon} \gamma^{\mu \rho} \chi^M) , \]
\[ \delta \chi^M = \frac{i}{2} x^{M}_r (\partial_\mu v^r) \gamma^\mu \epsilon + \frac{i}{12} x^{M}_r \hat{H}^{r \rho \gamma} \gamma^{\mu \rho} \epsilon . \] (4.38)

In order to obtain the bosonic equations, it is convenient to associate the fermionic equations to the Lagrangian

\[ e^{-1} L_{fer} = - \frac{i}{2} \bar{\psi}_\mu \gamma^{\mu \rho} D_\mu [\frac{1}{2} (\omega + \bar{\omega})] \psi_\rho - \frac{i}{8} v_r [H + \hat{H}]^{\mu \rho} (\bar{\psi}_\mu \gamma_\rho \psi_\rho) \]
\[ + \frac{i}{48} v_r [H + \hat{H}]^{\mu \rho} (\bar{\psi}_\mu \gamma_\rho \psi_\rho) \bar{\psi}_\mu \gamma^{\mu \rho} \chi^M \]
\[ - \frac{i}{24} v_r \hat{H}^{r \rho} (\bar{\chi}^M \gamma^{\mu \rho} \chi^M) + \frac{1}{4} x^M_r [\partial_\rho v^r + \partial_\rho \hat{v}^r] (\bar{\psi}_\mu \gamma^\nu \gamma^{\rho} \chi^M) \]
\[ - \frac{1}{8} x^M_r [H + \hat{H}]^{\mu \rho} (\bar{\psi}_\mu \gamma_\rho \chi^M) + \frac{1}{24} x^M_r [H + \hat{H}]^{\mu \rho} (\bar{\psi}_\mu \gamma_\rho \chi^M) \]
\[ + \frac{1}{8} (\bar{\chi}^M \gamma^{\mu \rho} \chi^M) (\bar{\psi}_\mu \gamma_\rho \psi_\rho) - \frac{1}{8} (\bar{\chi}^M \gamma^{\mu \rho} \chi^M) (\bar{\chi}^M \gamma_\mu \chi^N) , \] (4.39)

where, in the 1.5 order formalism, the spin connection

\[ \omega_{\mu \rho} = \omega^0_{\mu \rho} - \frac{i}{2} (\bar{\psi}_\mu \gamma_\rho \psi_\rho + \bar{\psi}_\rho \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\rho) \]
\[ - \frac{i}{4} (\bar{\psi}_\rho \gamma^{\mu \rho} \psi^\tau) - \frac{i}{4} (\bar{\chi}^M \gamma^{\mu \rho} \chi^M) \] (4.40)
satisfies its equation of motion, and is thus kept fixed in all variations.

In order to derive the bosonic equations, one can add to (4.39)

\[ e^{-1} L_{base} = - \frac{1}{4} R + \frac{1}{12} G_{rs} H^{s \mu \rho} H^r_{\mu \rho} - \frac{1}{4} \partial_\rho v^r \partial^\rho v_r . \] (4.41)

One can then obtain from \( L_{fer} + L_{base} \) the equations for the vielbein and the scalars, with the prescription that the (anti)self-duality conditions be used only after varying. Actually, ignoring momentarily eq. (4.37) and varying \( L_{fer} + L_{base} \) with respect to the antisymmetric tensor \( B^r_{\mu \nu} \), yields the second-order tensor equation, the divergence of eq. (4.37),

\[ \nabla_\mu (G_{rs} \hat{H}^{s \mu \rho}) = \frac{1}{2} \nabla_\mu [x^M_r (\bar{\chi}^M \gamma^{\mu \rho} \psi_\sigma)] \]
\[ - \frac{i}{4} \nabla_\mu [v_r (\bar{\psi}_\sigma \gamma^{\tau \mu \rho} \psi^\tau)] + \frac{i}{4} \nabla_\mu [v_r (\bar{\chi}^M \gamma^{\mu \rho} \chi^M)] . \] (4.42)
In a similar fashion, the scalar equation is
\[ x^M_r \left[ \frac{1}{2} D_\mu (\partial^\mu r) + \frac{1}{3} \nu_s H^{\mu \nu \rho} H^s_{\mu \nu \rho} - \frac{i}{4} H^{\mu \nu \rho} (\bar{\psi}_\mu \gamma_\nu \psi_\rho) + \frac{i}{24} H^r_{\rho \sigma \tau} (\bar{\psi}_\mu \gamma^{\mu \rho \sigma \tau} \psi_\nu) \right. \]
\[ - \frac{i}{24} H^{\mu \nu \rho} (\bar{\chi}^N \gamma^{\mu \nu \rho} \chi^N) - \frac{1}{2} \nu_\nu (\bar{\psi}_\mu \gamma_\nu \chi^N) \]
\[ + v_r \left[ - \frac{1}{4} H^{\mu \nu \rho} (\bar{\psi}_\mu \gamma_\nu \chi^M) + \frac{1}{12} H^{\mu \nu \rho} (\bar{\psi}_\sigma \gamma_{\sigma \mu \nu \rho} \chi^M) \right] = 0 \tag{4.43} \]

while the Einstein equation is
\[ \frac{1}{2} \epsilon^r_{\mu \nu} [R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R - G_{\tau s} H^{\tau \mu \rho \sigma} H_{\rho \sigma \tau s} + \frac{1}{6} g^{\mu \nu} G_{\tau s} H^r_{\rho \sigma \tau s} H^{\rho \sigma \tau} \]
\[ + \partial^\mu u^\rho \partial^\nu v_r - \frac{1}{2} g^{\mu \nu} \partial_\rho u^\sigma \partial^\rho v_r] - \frac{i}{2} \epsilon^r_{\mu \nu} (\bar{\psi}_\rho \gamma^{\mu \rho \sigma \tau} D_\sigma \psi_\tau) + \frac{i}{2} (\bar{\psi}_m \gamma^{\mu \rho \sigma} \hat{D}_\nu \psi_\rho) \]
\[ - \frac{i}{2} (\bar{\psi}_\nu \gamma^{\mu \rho \sigma} \hat{D}_m \psi_\rho) + \frac{i}{2} (\bar{\psi}_\nu \gamma^{\mu \rho \sigma} \hat{D}_\rho \psi_m) - \frac{i}{4} \epsilon^r_{\mu \nu} v_r \hat{H}^{\tau \rho \sigma} (\bar{\psi}_\sigma \gamma^{\mu \rho \sigma \tau} \psi_\tau) \]
\[ + \frac{i}{4} v_r \hat{H}_{\mu \nu \rho \sigma} (\bar{\psi}_\nu \gamma^{\mu \rho \sigma} \psi_\sigma) + \frac{i}{2} v_r \hat{H}^{\tau \rho \sigma} (\bar{\psi}_\sigma \gamma^{\mu \rho \sigma \tau} \psi_\tau) - \frac{i}{24} \epsilon^r_{\mu \nu} m (\bar{\chi}^M \gamma^M \hat{D}_m \chi^M) - \frac{i}{2} (\bar{\chi}^M \gamma^M \hat{D}_m \chi^M) \]
\[ + \frac{i}{4} \epsilon^r_{\mu \nu} m (\bar{\chi}^M \gamma^M \hat{D}_m \chi^M) + \frac{1}{2} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^r) (\bar{\psi}_\mu \gamma^r \chi^M) \]
\[ - \frac{i}{2} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^r) (\bar{\psi}_\mu \gamma^r \chi^M) - \frac{1}{2} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^\nu) (\bar{\psi}_\mu \gamma^\nu \chi^M) \]
\[ - \frac{1}{2} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^\nu) (\bar{\psi}_\mu \gamma^\nu \chi^M) - \frac{1}{4} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^\nu) (\bar{\psi}_\mu \gamma^\nu \chi^M) \]
\[ + \frac{1}{12} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^\nu) (\bar{\psi}_\mu \gamma^\nu \chi^M) - \frac{1}{12} \epsilon^r_{\mu \nu} m (\partial_\nu \psi^\nu) (\bar{\psi}_\mu \gamma^\nu \chi^M) \]
\[ = 0 \tag{4.44} \]

For the sake of brevity, a number of quartic fermionic couplings, fully determined by the lagrangian of eqs. (4.39) and (4.41), are not written explicitly. It then takes a direct, if somewhat tedious, calculation to prove local supersymmetry, showing that
\[ \delta F \frac{\delta L}{\delta F} + \delta B \frac{\delta L}{\delta B} = 0 \tag{4.45} \]

where \( F \) and \( B \) denote collectively the Fermi and Bose fields aside from the antisymmetric tensors.
A (1, 0) Yang-Mills multiplet in six dimensions comprises gauge vectors $A_\mu$ and pairs of left-handed spinors $\lambda^A$ satisfying a symplectic Majorana condition, all in the adjoint representation of the gauge group. In this subsection we write the complete field equations for $N = 1$ supergravity coupled to $n$ tensor multiplets and to vector multiplets. This setting plays a crucial role in six-dimensional perturbative type-I vacua, that naturally include a number of tensor multiplets [45], and more generally in the context of string dualities relating these to non-perturbative vacua of other strings and to $M$ theory [64]. In all these cases, the anomaly polynomial comprises in principle an irreducible part, that in perturbative type-I vacua is removed by tadpole conditions, and a residual reducible part of the form

$$I_8 = -\sum_{x,y} c_x^r c_y^s \eta_{rs} \operatorname{tr}_x F^2 \operatorname{tr}_y F^2 ,$$

with the $c$’s a collection of constants and $\eta$ the Minkowski metric for $SO(1, n)$. In general, this residual anomaly should also include gravitational and mixed contributions, but we leave them aside, since they would contribute higher-derivative couplings not part of the low-energy effective supergravity.

The antisymmetric tensors are not inert under vector gauge transformations, as demanded by the Chern-Simons couplings

$$H^r = dB^r - c^{rz} \omega_z ,$$

where the index $z$ runs over the various factors of the gauge group. Gauge invariance of $H^r$ indeed requires that $B^{r}_{\mu\nu}$ transform under vector gauge transformations according to

$$\delta B^r = c^{rz} \operatorname{tr}_z (\text{Ad} A),$$

To lowest order, the (anti)self-duality conditions (4.5) are not affected, while their divergence becomes

$$\nabla_\mu (G_{rs} H^{s\mu\rho}) = -\frac{1}{4} \varepsilon^{\mu\rho\alpha\beta\gamma\delta} c_r^z \operatorname{tr}_z (F_{\alpha\beta} F_{\gamma\delta}).$$

In a similar fashion, the fermionic equations become

$$-i \gamma^{\mu\rho} D_\rho \psi_\mu - i v_r H^{r\mu\rho} \gamma_\nu \psi_\rho - \frac{1}{2} x^M_r H^{r\mu\rho} \gamma_\nu \psi_\rho - \frac{1}{2} x^M_r H^{r\mu\rho} \gamma_\nu \chi^M = 0,$$

$$+ \frac{1}{2} x^M_r \partial_\nu (v_r \gamma^\nu \gamma^\mu \chi^M) + \frac{i}{\sqrt{2}} v_r c^{rz} \operatorname{tr}_z (F_{\nu\rho} \gamma^{\nu\rho} \gamma^\mu \chi^M) = 0$$
for the gravitino,

\[ i\gamma^\mu D_\mu \chi^M - \frac{i}{12} v_r H^{\mu\nu\rho} \gamma_{\mu\nu\rho} \chi^M + \frac{1}{2} x_r^M H^{\mu\nu\rho} \gamma_{\mu\nu\rho} \psi_\rho + \frac{1}{2} x_r^M \partial_\nu v^r \gamma^\mu \gamma^\nu \psi_\mu + \frac{1}{2} x_r^M c^{rz} \text{tr}_z (F_{\mu\nu}) \gamma^{\mu\nu} \chi = 0 \]  

(4.51)

for the tensorinos and

\[ 2iv_r c^{rz} \gamma^\mu D_\mu \lambda + iv_\mu v_r c^{rz} \gamma^\mu \lambda + \frac{i}{\sqrt{2}} v_r c^{rz} F_{\mu\rho} \gamma^{\mu\rho} \psi_\mu - \frac{1}{2} x_r^M c^{rz} F^{\mu\nu} \gamma^{\mu\nu} \chi^M - \frac{i}{6} c^{rz} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \lambda = 0 \]  

(4.52)

for the gauginos. The supersymmetry transformations of the vector multiplet are

\[ \delta A_\mu = -\frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda) , \]

\[ \delta \lambda = -\frac{1}{2\sqrt{2} F_{\mu\nu}} \gamma^{\mu\nu} \epsilon , \]  

(4.53)

while the tensor transformation becomes

\[ \delta B_{\mu\nu} = iv^r (\bar{\psi} \gamma_{[\mu} v_{\nu]} \epsilon) + \frac{1}{2} x_r^M (\bar{\chi}^{\lambda M} g_{\nu\rho} \epsilon) - 2c^{rz} \text{tr}_z (A_{[\mu} \delta A_{\nu]} ) . \]  

(4.54)

The other transformations are not modified, aside from the change induced by \((4.47)\) in the definition of \(H^r\). Varying the Fermi fields in the fermionic equations then gives the bosonic equations

\[ x_r^M D_\mu (\partial^r v^r) + \frac{2}{3} x_r^M v_\lambda H^{\mu\nu\rho}_{\mu\nu\rho} - x_r^M c^{rz} \text{tr}_z (F_{\mu\nu} F^{\mu\nu}) = 0 \]  

(4.55)

for the scalar,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \partial_\mu v^r \partial_\nu v_r - \frac{1}{2} g_{\mu\nu} \partial_\rho v^r \partial^\rho v_r - G_{\mu\nu} H^{\rho\lambda}_{\mu\nu} H^{\nu\lambda}_\rho = 0 \]  

(4.56)

for the metric, and

\[ D_\mu (v_\nu c^{rz} F^{\mu\nu}) - c^{rz} G_{\nu\lambda} H^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \]  

(4.57)

for the vectors. The commutator of two supersymmetry transformations now includes a gauge transformation of parameter

\[ \Lambda = \xi^\mu A_\mu . \]  

(4.58)
The novelty here is the non-vanishing divergence of eq. \((4.57)\)

\[
D_\mu J^\mu = -\frac{1}{2e} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c^r c^{r'} F_{\mu\nu} tr_z (F_{\alpha\beta} F_{\gamma\delta}) \, , \tag{4.59}
\]

that reflects the presence of the residual gauge anomaly \([54, 68]\). In particular, eq. \((4.129)\) gives the covariant anomaly. Leaving aside momentarily the (anti)self-duality conditions, one might expect to derive eq. \((4.57)\) from

\[
e^{-1} \mathcal{L} = -\frac{1}{2} v_r c^{r'} tr_z F_{\mu\nu} F_{\mu\nu} + \frac{1}{12} G_{rs} H^{\mu\nu\rho} H_{\mu\nu\rho} \, , \tag{4.60}
\]

but this is actually not the case. In fact, eq. \((4.57)\) is not integrable, while the inclusion of a Wess-Zumino term

\[
e^{-1} \mathcal{L} = -\frac{1}{2} v_r c^{r'} tr_z F_{\mu\nu} F_{\mu\nu} + \frac{1}{12} G_{rs} H^{\mu\nu\rho} H_{\mu\nu\rho}
- \frac{1}{8e} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z B_{\mu\nu} tr_z (F_{\alpha\beta} F_{\gamma\delta}) \ , \tag{4.61}
\]

turns the vector equation into

\[
D_\mu (v_r c^{r'} F_{\mu\nu}) - G_{rs} H^{\mu\nu\rho} c^{r'} F_{\rho\sigma} - \frac{1}{8e} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z A_\rho c^{r'} tr_z (F_{\alpha\beta} F_{\gamma\delta})
- \frac{1}{12e} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z F_{\rho\sigma} c^{r'} \omega_{\beta\gamma\delta} = 0 \ , \tag{4.62}
\]

and now the divergence of the gauge current is the consistent anomaly \([68]\)

\[
\mathcal{A}_\Lambda = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{r'} tr_z (\Lambda \partial_\mu A_\nu) tr_z (F_{\alpha\beta} F_{\gamma\delta}) \ . \tag{4.63}
\]

As an aside, one can observe that, ignoring the (anti)self-duality conditions, eq. \((4.61)\) yields the second-order tensor equations \((4.49)\) when varied with respect to the antisymmetric fields.

The Wess-Zumino consistency condition \([67]\)

\[
\delta_\Lambda A_\epsilon = \delta_\epsilon A_\Lambda \tag{4.64}
\]

now implies the presence of a supersymmetry anomaly of the form

\[
\mathcal{A}_\epsilon = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{r'} tr_z (\delta_\epsilon A_\mu A_\nu) tr_z (F_{\alpha\beta} F_{\gamma\delta})
- \frac{1}{6} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{r'} tr_z (\delta_\epsilon A_\mu F_{\nu\alpha}) \omega_{\beta\gamma\delta}^z \ , \tag{4.65}
\]

and indeed the supersymmetry variation of the lagrangian is exactly eq. \((4.65)\). Moreover, the divergence of the gravitino field equation, proportional to eq. \((4.65)\),
reflected the presence of the induced supersymmetry anomaly. We shall now complete
this construction to all orders in the Fermi fields.
Let us begin by noting that the supercovariant Yang-Mills field strength is
\[ \hat{F}_{\mu
u} = F_{\mu
u} + \frac{i}{\sqrt{2}} (\bar{\lambda} \gamma_\mu \psi_\nu) - \frac{i}{\sqrt{2}} (\bar{\lambda} \gamma_\nu \psi_\mu) \; , \] (4.66)
while the other supercovariant fields are not modified. The supersymmetry transformations
\[ \delta e_\mu^m = -i (\bar{\epsilon} \gamma^m \psi_\mu) \; , \]
\[ \delta B_{\mu
u}^r = iv^r (\bar{\psi}_{[\mu} \gamma_{\nu]} \epsilon) + \frac{1}{2} x^M (\bar{\psi}_M \gamma_{\mu
u} \epsilon) - 2 c^r \epsilon \text{tr}_z (A_{[\mu} \delta A_{\nu]}) \; , \]
\[ \delta v_r = x^M (\bar{\psi}_M \epsilon) \; , \]
\[ \delta A_\mu = -\frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda) \; , \]
\[ \delta \psi_\mu = D_\mu \epsilon + \frac{1}{4} v_r \hat{H}_{\mu
u\rho} \bar{\gamma}^{\nu\rho} \epsilon - \frac{3i}{8} \gamma_\mu \lambda (\bar{\epsilon} \chi^M) \]
\[ - \frac{i}{8} \bar{\gamma}^M \chi^M (\bar{\epsilon} \gamma_{\mu\nu} \chi^M) + \frac{i}{16} \gamma_{\mu\nu \rho} \chi^M (\bar{\epsilon} \gamma_{\rho} \chi^M) \; , \]
\[ \delta \chi^M = \frac{i}{2} x^M (\bar{\psi}_M \gamma^r \epsilon) + \frac{i}{12} x^M \hat{H}_{\mu
u\rho} \gamma^{\mu\rho} \epsilon \; , \]
\[ \delta \lambda = -\frac{1}{2 \sqrt{2}} \hat{F}_{\mu
u} \gamma^{\mu\nu} \epsilon \; , \] (4.67)
could in principle include additional terms proportional to \( \lambda^2 \). To be precise, one
could add to \( \delta \psi \) a term proportional to \( v_r c^r \epsilon \text{tr}_z (\lambda^2 \epsilon) \), and to \( \delta \chi \) a term proportional
to \( x^M c^r \epsilon \text{tr}_z (\lambda^2 \epsilon) \). Moreover, the (anti)self-duality conditions could be modified by a
self-dual term of the form \( c^r \epsilon \text{tr}_z (\bar{\lambda} \gamma_{\mu\nu \rho} \lambda) \).

Let us proceed to study the supersymmetry algebra completely. On the scalar,
the vielbein and the gauge field, the algebra closes with no subtleties, while additional
information comes from the algebra on the tensor fields. Using the (anti)self-duality
conditions satisfied by the 3-forms in eq. (4.25), one can show that the algebra on
\( B_r^\ast \) closes up to the extra terms
\[ [\delta_1, \delta_2]_{\text{extra}} B_{\mu
u}^r = c^r \epsilon \text{tr}_z [(\bar{\epsilon}_1 \gamma_\mu \lambda)(\bar{\epsilon}_2 \gamma_\nu \lambda) - (\bar{\epsilon}_1 \gamma_\nu \lambda)(\bar{\epsilon}_2 \gamma_\mu \lambda)] \; . \] (4.68)
These can be canceled modifying the transformations of the gravitino and of the
tensorinos according to
\[ \delta' \psi_\mu = iv_r c^r \{ a \epsilon \text{tr}_z [\lambda (\bar{\epsilon}_1 \gamma_\mu \lambda)] + b \epsilon \text{tr}_z [\gamma_{\mu\nu} \lambda (\bar{\epsilon}_2 \gamma_\nu \lambda)] + c \epsilon \text{tr}_z [\gamma^{\mu\rho} \lambda (\bar{\epsilon}_2 \gamma_{\mu\rho} \lambda)] \} \; , \]
\[ \delta' \chi^M = d x^M c^r \epsilon \text{tr}_z [\gamma_\mu \lambda (\bar{\epsilon} \gamma^\mu \lambda)] \; , \] (4.69)
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and requiring that the modified 3-form

\[ \hat{H}_{\mu
u\rho}^r = \hat{H}_{\mu
u\rho}^r - \frac{i}{8} v^r (\chi^M (\gamma_{\mu
u\rho} \chi^M) + i \alpha \ c^r \text{tr}_z (\tilde{\lambda} \gamma_{\mu
u\rho} \lambda) \]  

satisfy the (anti)self-duality conditions

\[ G_{rs} \hat{H}^s_{\mu
u\rho} = \frac{1}{6\epsilon} \epsilon_{\mu
u\rho\alpha\beta\gamma} \hat{H}_{\alpha\beta\gamma}^r . \]

It should be appreciated that this change in the definition of the field strengths only affects the antiself-duality conditions, since \((\tilde{\lambda} \gamma_{\mu
u\rho} \lambda)\) is self-dual.

Requiring closure of the algebra on \(B^r\) then implies the conditions

\[ \alpha = \frac{1}{4} \ , \quad d = \frac{1}{2} \ , \quad a + b = -1 \ , \quad b + 2c = 0 \ , \quad \]  

and only one of the parameters is still undetermined. These terms have no effect for the scalars and the vectors, while the commutator on \(e^m_\mu\) shows that the local Lorentz parameter is modified by the addition of

\[ \Delta' \Omega^{mn} = v_r c^r \text{tr}_z [(\bar{\epsilon}_1 \gamma^m \lambda) (\bar{\epsilon}_2 \gamma^n \lambda) - (\bar{\epsilon}_2 \gamma^m \lambda) (\bar{\epsilon}_1 \gamma^n \lambda)] \ . \]

Turning to the Fermi fields, the commutator on the tensorini \(\chi^M\) involves techniques already met in the case with tensor multiplets only, and fixes the last free parameter in eqs. (4.72), so that

\[ a = -\frac{9}{8} \ , \quad b = \frac{1}{8} \ , \quad c = -\frac{1}{16} \ . \]

It closes on the field equation

\[ i\gamma^\mu \hat{D}_\mu \chi^M - \frac{i}{12} v_r \hat{H}_{\mu
u\rho}^r \gamma_{\mu
u\rho} \chi^M + \frac{1}{12} x_r^M \hat{H}^{r\mu
u\rho} \gamma_\sigma \gamma_{\mu
u\rho} \psi^\sigma + \frac{1}{2} x_r^M (\partial_{\nu} v^r) \gamma^{\mu} \gamma_\nu \psi_\mu 
+ \frac{1}{\sqrt{2}} x_r^M c^r \text{tr}_z (\hat{F}_{\mu
u}^r \gamma^{\mu\nu} \lambda) - \frac{i}{2} x_r^M c^r \text{tr}_z [\gamma^{\mu} \gamma_\nu \lambda (\bar{\psi}_\mu \gamma_{\nu} \lambda)] + \frac{1}{2} \gamma^\mu \chi^N (\bar{\chi}^N \gamma_\mu \chi^M) 
- \frac{3}{8} v_r c^r \text{tr}_z [(\bar{\chi}^M \gamma_{\mu\nu} \lambda) \gamma_{\mu\nu} \lambda] - \frac{1}{4} v_r c^r \text{tr}_z [(\bar{\chi}^M \lambda) \lambda] = 0 \ , \]

where all terms containing the gravitino are exactly determined by supercovariance. Moreover, the field equation appears in the commutator as in the theory without gauge fields:

\[ [\delta_1, \delta_2] \chi^m = \delta_{\text{gct}} \chi^m + \delta_{\text{Lorentz}} \chi^m + \delta_{\text{SO}(n)} \chi^m + \delta_{\text{susy}} \chi^m - \frac{i}{4} \gamma^\mu \xi_\mu [\text{eq. } \chi^m] \ . \]
Using similar techniques, one can compute the commutator on the gauginos \( \lambda \). Here, however, the transformation

\[
\delta \lambda = -\frac{1}{2\sqrt{2}} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \epsilon \quad (4.77)
\]

can not produce the terms proportional to \( x_r^M \) already present at the lowest order, and the only way to generate them is to modify eq. (4.77) by terms of the form

\[
\frac{x_r^M c r z}{\nu s c sz} \chi^M \lambda \epsilon \quad (4.78)
\]

Singular couplings of this type were previously introduced in [74]. We therefore add all possible extra terms, that modulo Fierz identities are

\[
\delta' \lambda = \frac{x_r^M c r z}{\nu s c sz} [a(\chi^M \lambda) \epsilon + b(\chi^M \gamma_{\mu\nu} \lambda) \gamma^{\mu\nu} \epsilon + c(\chi^M \epsilon) \lambda + d(\chi^M \gamma_{\mu\nu} \epsilon) \gamma^{\mu\nu} \lambda] \quad , (4.79)
\]

and determine their coefficients from the algebra. Eq. (4.79) should not affect the vector (and, a fortiori, the tensor) commutator, and thus the coefficients are to obey the three equations

\[
a - 2c = 0 \quad , \quad b = 0 \quad , \quad c + 2d = 0 \quad . \quad (4.80)
\]

The other conditions,

\[
a + 2b = -\frac{1}{2} \quad , \quad c + 2d + 4b = 0 \quad , \quad 2d + \frac{1}{8}a + \frac{1}{4}b = \frac{3}{16} \quad , \quad (4.81)
\]

are obtained from the algebra on the gauginos, for instance tracking the terms generated by eq. (4.79) and proportional to \( \partial v \). Combining eqs. (4.80) and (4.81), one finally obtains

\[
a = -\frac{1}{2} \quad , \quad c = -\frac{1}{4} \quad , \quad d = \frac{1}{8} \quad . \quad (4.82)
\]

As was the case for the gravitino already without vector multiplets, here the algebra generates the field equation with a non trivial symplectic structure,

\[
-\frac{3i}{16} \gamma^\mu \xi_\mu [\text{eq.}\chi^A] - \frac{i}{192} \gamma^{\mu\nu\rho} \sigma^A_B \xi^i \xi^{i A}_{\mu\nu\rho} [\text{eq.}\lambda^B] \quad , \quad (4.83)
\]

where \( \xi^i_{\mu\nu\rho} \) is defined in eq. (4.35).

Eq. (4.79) also affects the algebra on the tensorinos, whose field equation now includes two additional terms, and becomes

\[
\frac{i}{12} \gamma^\mu \hat{D}_\mu \chi^M - \frac{i}{12} v_r \hat{H}_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi^M + \frac{1}{12} x_r^M \hat{H}^{\mu\nu\rho} \gamma_\sigma \gamma_{\mu\nu\rho} \psi^\sigma + \frac{1}{2} x_r^M (\hat{D}_\nu \psi^\mu - \hat{D}_\mu \psi^\nu) \quad . \quad (4.84)
\]
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\[ + \frac{1}{\sqrt{2}} x^M \epsilon^{\alpha \beta} \text{tr}_x (F_{\mu \nu} \gamma^{\mu \nu} \lambda) - \frac{i}{2} x^M \epsilon^{\alpha \beta} \text{tr}_x [\gamma^\mu \gamma^\nu \lambda (\bar{\psi}_\mu \gamma_\nu \lambda)] + \frac{1}{2} \gamma^\mu \chi^N (\bar{\chi}^N \gamma_\mu \chi) \]

\[ - \frac{3}{8} v_r \epsilon^{\alpha \beta} x^M [\bar{\chi}^N \gamma_\mu \lambda] - \frac{1}{4} v_r \epsilon^{\alpha \beta} \text{tr}_x [\bar{\chi}^N \lambda] \]

\[ - \frac{3}{2} x^M \epsilon^{\alpha \beta} x^N \epsilon^{\gamma \delta} \text{tr}_x (\bar{\chi}^N \lambda) + \frac{1}{4} x^M \epsilon^{\alpha \beta} x^N \epsilon^{\gamma \delta} \text{tr}_x [\bar{\chi}^N \gamma_\mu \lambda] = 0 \quad (4.84) \]

In the commutator of two supersymmetry transformations on the gauginos, these terms complete the algebra and let it close on the field equation, that now includes \( \chi^2 \lambda \) terms corresponding to the \( \lambda^2 \chi \) terms in the equation for the tensorinos. In addition, the \( \lambda^3 \) terms comprise two groups: those proportional to \( v_r v_s \) and those proportional to \( \eta_{rs} \) (recall, from eqs. (4.3), that \( x^M x^N = v_r v_s - \eta_{rs} \)). The former generate local Lorentz transformations according to eq. (4.73) and the term

\[ i v_r v_s \epsilon^{\alpha \beta} \epsilon^{\gamma \delta} \text{tr}_x [(\bar{\lambda} \gamma_\mu \lambda') \gamma^{\mu \lambda}] \quad (4.85) \]

in the field equation, while the latter are

\[ [\delta_1, \delta_2]_{\text{extra} \lambda} = \frac{c^2 \epsilon^{\alpha \beta} \epsilon^{\gamma \delta}}{v_r \epsilon^{\gamma \delta}} \text{tr}_x [\frac{-1}{4} (\bar{\epsilon}_1 \gamma_\mu \lambda') (\bar{\epsilon}_2 \gamma_\nu \lambda') \gamma^{\mu \nu} \lambda + \frac{1}{4} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\mu \lambda') \bar{\epsilon}_2 - (1 \leftrightarrow 2)] \]

\[ + \frac{1}{16} (\bar{\epsilon}_1 \gamma_\mu \lambda') (\bar{\lambda} \gamma^{\mu \rho} \lambda') \gamma^{\rho \lambda}] \quad (4.86) \]

In general, one could allow for a modified field equation including the \( \lambda^3 \) term

\[ 2 \alpha c^2 \epsilon^{\alpha \beta} \epsilon^{\gamma \delta} \text{tr}_x [(\bar{\lambda} \gamma_\mu \lambda') \gamma^{\mu \lambda}] \quad (4.87) \]

with \( \alpha \) an arbitrary parameter. Although the choice \( \alpha = 1 \) could seem the preferred one on account of the rigid limit, since the supersymmetric Yang-Mills theory in six dimensions does not contain such a \( \lambda^3 \) term, the \((1, 0)\) supergravity is actually consistent for an arbitrary value of \( \alpha \), with the corresponding residual terms

\[ \delta_{\text{extra}(\alpha) \lambda} \equiv [\delta_1, \delta_2]_{\text{extra}(\alpha) \lambda} = \frac{c^2 \epsilon^{\alpha \beta} \epsilon^{\gamma \delta}}{v_r \epsilon^{\gamma \delta}} \text{tr}_x [-\frac{1}{4} (\bar{\epsilon}_1 \gamma_\mu \lambda') (\bar{\epsilon}_2 \gamma_\nu \lambda') \gamma^{\mu \nu} \lambda \]

\[ - \frac{\alpha}{2} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\nu \lambda') \gamma^{\mu \nu} \bar{\epsilon}_2 + \frac{\alpha}{16} (\bar{\lambda} \gamma^{\mu \rho} \lambda') (\bar{\epsilon}_1 \gamma_\rho \lambda') \gamma^{\mu \rho} \bar{\epsilon}_2 \]

\[ + \frac{\alpha}{16} (\bar{\lambda} \gamma^{\mu \rho} \lambda') (\bar{\epsilon}_1 \gamma^{\mu \rho} \lambda') \gamma^{\mu \rho} \bar{\epsilon}_2 - (1 \leftrightarrow 2)] \]

\[ + \frac{1}{16} (\bar{\epsilon}_1 \gamma_\mu \lambda') (\bar{\lambda} \gamma^{\mu \rho} \lambda') \gamma^{\rho \lambda}] \quad (4.88) \]

in the commutator of two supersymmetry transformations on the gauginos. It should be appreciated that no choice of \( \alpha \) can eliminate all these terms, that play the role
of a central charge felt only by the gauginos. The Jacobi identity for this charge is properly satisfied for any value of $\alpha$, and thus we are effectively discovering a 2-cocycle in our problem. It has long been known that, in general, anomalies in current conservations are accompanied by related anomalies in current commutators \[78\], but it is amusing to see how this “classically anomalous” model displays all these intricacies.

The complete algebra

$$[\delta_1, \delta_2] \lambda^A = \delta_{gct} \lambda^A + \delta_{\text{Lorentz}} \lambda^A + \delta_{\text{susy}} \lambda^A + \delta_{\text{gauge}} \lambda^A + \delta_{\text{extra}(\alpha)} \lambda^A$$

$$- \frac{3i}{16} \gamma^\mu \xi_{[\mu \nu \rho \sigma]} \lambda^{A} [c.q. \lambda^A]_{(a)} - \frac{i}{192} \gamma^{\mu \rho} \sigma_B \xi_{\rho \mu \nu} [c.q. \lambda^B]_{(a)}$$

(4.89)

determines the complete field equation of the gauginos

$$2iv_r c^{r x} \gamma^\mu \lambda \delta_r v_r c^{r x} \gamma^\nu \lambda + \frac{i}{\sqrt{2}} v_r c^{r x} \hat{F}_{\nu \rho} \gamma^{\mu \nu} \psi_{\rho} - \frac{1}{\sqrt{2}} x_r^M c^{r x} \hat{F}_{\mu \nu} \gamma^{\mu \nu} \chi^M$$

$$+ \frac{i}{6} x_r^M c^{r x} \hat{H}_{\mu \rho \sigma} \gamma^{\mu \nu} \rho \lambda - \frac{i}{2} x_r^M c^{r x} (\chi^M \gamma_{\nu \rho} \psi_{\rho}) \gamma^\nu \lambda - \frac{1}{4} x_r^M c^{r x} (\chi^M \gamma_{\nu \rho} \psi_{\rho}) \gamma^\nu \lambda$$

$$- \frac{3}{8} v_r c^{r x} (\bar{\lambda} \gamma_{\mu \rho} \chi^M) \gamma^{\mu \nu} \lambda^M - \frac{3}{2} x_r^M c^{r x} \chi^N c^{s z} \left(\frac{\bar{\lambda} \gamma_{\mu \rho} \chi^M}{v_r c^{r x} \chi^N} \right) \lambda^M$$

$$+ \frac{1}{4} v_r c^{r x} \chi^N c^{s z} \left(\bar{\lambda} \gamma_{\mu \rho} \chi^M \right) \gamma^{\mu \nu} \lambda^N - 2v_r v_r c^{r x} c^{s z} \frac{\gamma^\rho \lambda^M}{v_r c^{r x} \chi^N} \lambda^N$$

$$+ 2\alpha c^{r x} c^{s z} \lambda^M \lambda^N = 0$$

(4.90)

where, again, all terms containing the gravitino are fixed by supercovariance, while the $\chi^2 \lambda$ terms are precisely as demanded by the $\lambda^2 \chi$ terms in the field equations of the tensorinos. At last, one can study the algebra on the gravitino, thus obtaining the field equation

$$-i\gamma^{\mu \rho} \hat{D}_\nu \psi_{\rho} - \frac{i}{4} v_r \hat{H}_\nu v_{\nu \mu \sigma} \gamma^{\rho \sigma} \gamma^\mu \lambda^M + \frac{1}{12} x_r^M (\partial_v v_{\nu \rho \sigma} \gamma^{\rho \sigma} \gamma^\mu \lambda^M + \frac{1}{2} x_r^M \gamma^{\mu \nu} \lambda^M)$$

$$+ \frac{3}{2} \gamma^{\mu \nu} \chi^M (\chi^M \gamma_{\nu \rho} \psi_{\rho}) + \frac{1}{4} \gamma^{\mu \nu} \chi^M (\chi^M \gamma_{\nu \rho} \psi_{\rho}) - \frac{1}{2} \chi^M (\chi^M \gamma^\mu \psi_{\rho})$$

$$- i v_r c^{r x} \lambda^M \frac{1}{\sqrt{2}} \gamma^{\rho \mu} \lambda^M \bar{\psi}_{\nu} \gamma_{\nu \rho} \lambda - \frac{i}{2} \gamma^\mu \lambda^M (\bar{\psi}_{\nu} \gamma^\nu \lambda)$$

$$+ \frac{i}{4} \gamma_r (\bar{\psi}_{\rho} \gamma^{\mu \rho} \lambda^M) - \frac{i}{2} v_r c^{r x} \lambda^M (\bar{\psi}_{\rho} \gamma^{\mu \rho} \lambda^M) = 0$$

(4.91)

that enters the supersymmetry algebra as in eq. (4.34). Once more, all terms containing the gravitino are fixed by supercovariance, while the other $\lambda^2 \chi$ terms are
precisely as demanded by the $\lambda^2 \psi$ terms in the tensorino equation and by the $\lambda \psi \chi$ terms in the equations of the gauginos.

Summarizing, from the algebra we have obtained the complete fermionic equations of (1, 0) six-dimensional supergravity coupled to vector and tensor multiplets. In addition, the modified 3-form

$$\mathcal{H}^r_{\mu \nu \rho} = \hat{H}^r_{\mu \nu \rho} - \frac{i}{8} v^r (\hat{\chi}^m \gamma_{\mu \nu \rho} \lambda^m) + \frac{i}{4} c^{rz} tr_z (\hat{\lambda} \gamma_{\mu \nu \rho} \lambda)$$

(4.92)

satisfies the (anti)self-duality conditions

$$G_{rs} \mathcal{H}^s_{\mu \nu \rho} = \frac{1}{6 e} \epsilon_{\mu \nu \rho \sigma \beta \gamma} \mathcal{H}^{\sigma \beta \gamma}_{r}.$$  

(4.93)

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

$$\delta \epsilon^{\mu \nu} = -i (\hat{\epsilon} \nu_{\mu} \psi_{\nu}) ,$$

$$\delta B^r_{\mu \nu} = i v^r (\hat{\psi}_{[\nu} \gamma_{\mu]} \epsilon) + \frac{1}{2} x^M r (\hat{\chi}^M \gamma_{\mu \nu} \epsilon) - 2 c^{rz} tr_z (A_{[\mu} \delta A_{\nu]}),$$

$$\delta v_{r} = x^M r (\hat{\chi}^M \epsilon) ,$$

$$\delta A_{\mu} = -\frac{i}{\sqrt{2}} (\epsilon_{\mu} \lambda) ,$$

$$\delta \psi_{\mu} = \hat{D}_{\mu} \epsilon + \frac{1}{4} v_{r} \hat{H}_{\mu \nu \rho} \gamma^{\nu \rho} \epsilon - \frac{3i}{8} \gamma_{\mu} \lambda (\hat{\epsilon} \chi^M) - i \frac{3}{8} \gamma^M (\hat{\epsilon} \gamma_{\mu \nu} \lambda^M)$$

$$+ \frac{i}{16} \gamma_{\mu \nu} \lambda^M (\hat{\epsilon} \gamma_{\mu} \chi^M) - \frac{9i}{8} v_{r} c^{rz} tr_z [\lambda (\hat{\epsilon} \mu \lambda)]$$

$$+ \frac{i}{8} v_{r} c^{rz} tr_z [\gamma_{\mu \nu} \lambda (\hat{\epsilon} \nu_{\lambda})] - \frac{i}{16} v_{r} c^{rz} tr_z [\gamma \nu_{\lambda} (\hat{\epsilon} \nu_{\lambda})] ,$$

$$\delta \chi^M = \frac{i}{2} x^M r (\hat{\partial} \nu \nu) \gamma^{\nu \mu} \epsilon + \frac{i}{12} x^M r \hat{H}_{\mu \nu \rho} \gamma^{\nu \rho} \epsilon + \frac{1}{2} x^M r c^{rz} tr_z [\gamma_{\mu} \lambda (\hat{\epsilon} \nu_{\lambda})] ,$$

$$\delta \lambda = -\frac{1}{2 \sqrt{2}} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \epsilon - \frac{1}{2 v_{\alpha \epsilon \sigma \zeta} (\hat{\chi}^M \lambda) \epsilon} - 4 v_{\alpha \epsilon \sigma \zeta} (\hat{\chi}^M \epsilon) \lambda$$

$$+ \frac{1}{8} v_{r} c^{rz} tr_z (\hat{\chi}^M \gamma_{\mu \nu} \epsilon) \gamma^{\mu \nu} \lambda .$$

(4.94)

Proceeding as in the previous subsection, the bosonic equations can be derived from a lagrangian, with the prescription of using the tensor (anti)self-duality conditions only after varying. The lagrangian is obtained supplementing $\mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{bose}}$ of eqs. (4.39) and (4.41) with the terms

$$-\frac{1}{2} v_{r} c^{rz} tr_z (F_{\mu \nu} F^{\mu \nu}) - \frac{1}{8 e} \epsilon^{\mu \nu \rho \sigma \delta} \lambda^r c^{rz} tr_z (F_{\alpha \beta} F_{\gamma \delta})$$
The complete scalar equation is obtained adding to eq. (4.43) the terms
\[ x^M_r c^{rz} \text{tr}_z [(F + \tilde{F})_{\sigma\delta} (\bar{\psi}_\mu \gamma^\sigma \gamma^\delta \gamma^\mu \lambda)] + \frac{1}{\sqrt{2}} x^M_r c^{rz} \text{tr}_z [(\bar{\chi}^M \gamma^{\mu \nu} \lambda) \tilde{F}_{\mu \nu}] \]
\[ + i v_r c^{rz} \text{tr}_z [(\lambda \gamma^m \tilde{D}_\mu \lambda) + \frac{i}{12} x^M_s \tilde{H}_{\mu \rho \sigma} c^{sz} \text{tr}_z (\lambda \gamma^{\mu \rho \sigma} \lambda)] \]
\[ + \frac{1}{16} v_r c^{rz} \text{tr}_z (\lambda \gamma_{\mu \rho \sigma} \lambda)(\bar{\chi}^M \gamma^{\mu \nu} \lambda) - \frac{i}{8} (\bar{\chi}^M \gamma_{\mu \nu} \psi_\rho) x^M_r c^{rz} \text{tr}_z (\lambda \gamma^{\mu \rho} \lambda) \]
\[ - \frac{i}{2} x^M_r c^{rz} \text{tr}_z [(\bar{\chi}^M \gamma^{\mu \nu} \lambda)(\bar{\psi}_\mu \gamma_\nu \lambda)] - \frac{3}{16} v_r c^{rz} \text{tr}_z [(\bar{\chi}^M \gamma_{\mu \nu} \lambda)(\bar{\chi}^M \gamma^{\mu \nu} \lambda)] \]
\[ - \frac{1}{8} v_r c^{rz} \text{tr}_z (\bar{\chi}^M \lambda)(\bar{\chi}^M \lambda) - \frac{3 x^M_r c^{rz} x^N_s c^{sz}}{4} \text{tr}_z [(\bar{\chi}^M \lambda)(\bar{\chi}^N \lambda)] \]
\[ + \frac{1}{8} v_r c^{rz} \text{tr}_z [(\bar{\chi}^M \gamma_{\mu \nu} \lambda)(\bar{\chi}^N \gamma^{\mu \nu} \lambda)] + \frac{1}{4} (\bar{\psi}_\mu \gamma_\nu \psi_\rho)(\lambda \gamma^{\mu \rho} \lambda) \]
\[ - \frac{1}{2} v_r s c^{rz} c^{sz} \text{tr}_{z',z''} [(\lambda \gamma_\mu \lambda')(\lambda \gamma^\mu \lambda'') + \frac{\alpha}{2} c^{rz} c^{sz} \text{tr}_{z',z''} [(\lambda \gamma_\mu \lambda')(\lambda \gamma^\mu \lambda'')] \] , (4.95)

and the 1.5 order formalism requires that the spin connection \( \omega_{\mu \nu \rho} \) now include the additional term
\[ \omega_{(\mu \nu \rho)}^{(\lambda)} = - \frac{i}{2} v_r c^{rz} \text{tr}_z (\lambda \gamma_{\mu \rho} \lambda) \] . (4.96)

With the new definition of \( \omega \), eqs. (4.39), (4.41) and (1.93) then yield the Fermi equations. Moreover, varying with respect to \( B_{\mu \nu}^r \) yields the second-order tensor equations, the divergence of the (anti)self-duality conditions. The vector equation is covariant, aside from the anomalous couplings introduced by the Wess-Zumino term in eq. (4.61). The complete residual gauge anomaly is thus given in eq. (1.63). As we shall see, it solves the Wess-Zumino consistency conditions even in the presence of supersymmetry.

The complete vector field equation is
\[ c^{rz} D_\nu [v_r c^{\nu \mu} - G_{rs} \tilde{H}^{\mu \rho \sigma} c^{sz} F_{\rho \sigma}] - \frac{1}{12} c^{\nu \rho \sigma \gamma} c^z \text{tr}_z (F_{\rho \sigma} F_{\gamma \tau}) - \frac{1}{8} (\bar{\psi}_\sigma \gamma^{\mu \rho \sigma \tau} \psi_\tau) \]
\[ - \frac{i}{4} v_r c^{sz} F_{\nu \rho} (\bar{\chi}^M \gamma^{\mu \rho \sigma} \chi^M) - \frac{i}{4} v_r c^{sz} F_{\nu \rho} (\bar{\chi}^M \gamma^{\mu \rho \sigma} \chi^M) - \frac{i}{2} x^M_r x^N_s c^{sz} c^{rz} \text{tr}_z (\lambda \gamma^{\mu \rho} \lambda) \]
\[ + \frac{i}{\sqrt{2}} c^{rz} D_\nu [v_r (\bar{\psi}_\rho \gamma^{\mu \rho \gamma} \lambda)] + \frac{1}{\sqrt{2}} c^{rz} D_\nu [x^M_r (\bar{\chi}^M \gamma^{\mu \nu} \lambda)] = 0 \] , (4.97)

the complete scalar equation is obtained adding to eq. (4.43) the terms
\[ x^M_r \left( \frac{1}{32} c^{rz} \text{tr}_z (\lambda \gamma_{\rho \sigma \tau} \lambda)(\bar{\psi}_\rho \gamma^{\mu \rho \gamma} \gamma^\sigma \gamma^\tau \psi_\rho) + \frac{i}{2 \sqrt{2}} c^{rz} \text{tr}_z [(\bar{\psi}_\rho \gamma^{\mu \rho} \gamma^\lambda \lambda)(F + \tilde{F})_{\nu \rho}] \right) \]
\[ + i c^{rz} \text{tr}_z (\lambda \gamma^\mu \tilde{D}_\mu \lambda) - \frac{3}{16} c^{rz} \text{tr}_z [(\bar{\chi}^N \gamma^{\mu \nu} \lambda)(\bar{\chi}^N \gamma_{\mu \nu} \lambda)] - \frac{1}{8} c^{rz} \text{tr}_z [(\bar{\chi}^N \lambda)(\bar{\chi}^N \lambda)] \]
\[ + \frac{1}{4} c^{su} \text{tr}_z (\bar{\lambda} \gamma^{\mu \rho} \lambda) (\bar{\psi}_\mu \gamma_\rho \psi_\rho) - v_s c^{su} c^{su'} \text{tr}_{z,z'} [(\bar{\lambda} \gamma^\mu \lambda') (\bar{\lambda} \gamma^\rho \lambda')] \\
- \frac{1}{8} x_N^N c^{sz} x_N^P c^{s_1} c^{s_1'} \text{tr}_z [(\bar{\chi} N \gamma^\mu \lambda') (\bar{\chi} P \gamma^\rho \lambda)] + \frac{3}{4} x_s^N c^{sz} x_s^P c^{s_1} c^{s_1'} \text{tr}_z [(\bar{\chi} N \lambda) (\bar{\chi} P \lambda)] \]
\[ + i \frac{1}{\sqrt{2}} c^{sz} \text{tr}_z [(\bar{\chi} M \gamma^\mu \lambda) \tilde{F}_{\mu \nu} ] + \frac{i}{12} x_s^M \tilde{H}_{\mu \rho \lambda} \text{tr}_z (\bar{\lambda} \gamma^\mu \lambda) \]
\[ + \frac{i}{12} x_s^M \tilde{H}^{\mu \rho \lambda} e^{sz} \text{tr}_z (\bar{\lambda} \gamma^\mu \lambda) - i \frac{v_s}{8} c^{sz} \text{tr}_z (\bar{\lambda} \gamma^\mu \lambda) (\bar{\lambda} M \gamma^\rho \lambda) \]
\[ - \frac{i}{2} c^{sz} \text{tr}_z [(\bar{\lambda} M \gamma^\mu \lambda) (\bar{\psi}_\mu \gamma_\lambda \lambda)] + \frac{1}{x_s^N c^{sz}} \text{tr}_z [(\bar{\chi} M \gamma^\mu \lambda) (\bar{\chi} N \gamma^\rho \lambda)] \]
\[ - \frac{3}{2} x_s^N c^{sz} \text{tr}_z [(\bar{\chi} M \lambda) (\bar{\chi} N \lambda)] \]

\[ \text{and the Einstein equation is obtained adding to eq. (4.44) the terms} \]
\[ c^{sz} \text{tr}_z \left\{ 2 e_{\nu m} v_\nu (F^\mu F^\nu - \frac{1}{2} g^{\mu \nu} F^\rho F^\rho) + \frac{i}{\sqrt{2}} e^\mu_m v_\nu (\bar{\psi}_\nu \gamma^\rho \lambda) F^\nu \right\} \]
\[ - \frac{2i}{\sqrt{2}} v_\nu (\bar{\lambda} \gamma^\mu \lambda) F^\mu - \frac{i}{\sqrt{2}} v_\nu (\bar{\psi}_\nu \gamma^\rho \gamma^\mu \lambda) F^\nu + \frac{1}{x_s^M} e^\mu_m (\bar{\chi} M \gamma^\mu \lambda) F^\nu \]
\[ - \sqrt{2} x_s^M (\bar{\lambda} \gamma^\nu \lambda) F^\nu - i e^\mu_m v_\nu (\bar{\chi} M \gamma^\mu \lambda) - i v_\nu (\bar{\chi} M \gamma^\mu \lambda) \]
\[ + \frac{i}{4} e^\mu_n x_s^M H_{\nu \sigma} (\bar{\lambda} \gamma^\mu \lambda) - \frac{i}{4} c^{sz} x_s^M H_{\nu \sigma} (\bar{\lambda} \gamma^\mu \lambda) \]
\[ - \frac{i}{4} e^s_m D_\epsilon [v_\nu (\bar{\lambda} \gamma^\mu \lambda)] + (\text{fermion})^4 \]
4.1 Supergravity coupled to tensor and vector multiplets

\[ + \frac{e x_M c^{zz'}}{vtc^{zz'}} \left[ \frac{3i}{2\sqrt{2}} \delta_\mu A_\mu (\bar{x}^\gamma \mu \lambda') (\bar{x}'^M) - \frac{i}{4\sqrt{2}} \delta_\mu A_\mu (\bar{x}^\gamma \mu \nu \lambda') (\bar{x}'^\nu \lambda^M) \right. \]

\[ - \left. \frac{i}{2\sqrt{2}} \delta_\epsilon A_\mu (\bar{x}^\gamma \mu \lambda') (\bar{x}'^\gamma \lambda^M) \right] \text{ , (4.101)} \]

while including the last term in eq. (4.95) would give the additional contribution

\[ \Delta A_\epsilon = \delta_\epsilon \mathcal{L}_\lambda^4 \text{ , (4.102)} \]

where

\[ \mathcal{L}_\lambda^4 = \frac{e \alpha}{2} c_{\lambda} c^{zz'} tr_{z,z'} [(\bar{x}^\gamma \alpha \lambda')(\bar{x}^\gamma \alpha \lambda') \text{ . (4.103)}] \]

In verifying the supersymmetry anomaly, the equations for the fermi fields and for the vector field are presented here must be rescaled by suitable overall factors that may be simply identified.

We now turn to show that \( A_\epsilon \) satisfies the complete Wess-Zumino consistency conditions.

4.1.3 Wess-Zumino Consistency Conditions

In general, the Wess-Zumino consistency conditions follow from the requirement that the symmetry algebra be realized on the effective action. For locally supersymmetric theories this implies

\[ \delta_{\Lambda_1} A_{\Lambda_2} - \delta_{\Lambda_2} A_{\Lambda_1} = A_{[\Lambda_1, \Lambda_2]} \text{ , } \]

\[ \delta_\epsilon A_\lambda = \delta_\lambda A_\epsilon \text{ , } \]

\[ \delta_\epsilon A_{\epsilon_2} - \delta_{\epsilon_2} A_\epsilon = A_\tilde{\epsilon} + \tilde{A}_\lambda \text{ . (4.104)} \]

where only gauge and supersymmetry anomalies are considered, and where \( \epsilon \) and \( \tilde{\lambda} \) are the parameters of supersymmetry and gauge transformations determined by the supersymmetry algebra.

In global supersymmetry the analysis is somewhat simpler, since the r.h.s. of the last of eqs. (4.104) does not contain the (global) supersymmetry anomaly. Let us therefore begin by reviewing the case of supersymmetric Yang-Mills theory in four dimensions . From the 6-form anomaly polynomial

\[ I_6 = \text{tr} F^3 \text{ , (4.105)} \]

in the language of forms, one obtains the four-dimensional gauge anomaly

\[ A_\lambda^{(4)} = \text{tr} [\Lambda (dA)^2 + \frac{ig}{2} dA A^3] \text{ , (4.106)} \]
and from eqs. (4.104) one can determine the form of the global supersymmetry anomaly. With the classical lagrangian

$$L_{SYM} = \text{tr} \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2i \bar{\lambda} \gamma^\mu D_\mu \lambda \right] ,$$  

(4.107)

and $\lambda$ a right-handed Weyl spinor, the supersymmetry transformations are

$$\delta A_\mu = \frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon) ,$$

$$\delta \lambda = \frac{1}{2 \sqrt{2}} F_{\mu\nu} \gamma^{\mu\nu} \epsilon .$$  

(4.108)

The second of eqs. (4.104) (with $A_\xi$ absent in this global case), then determines the supersymmetry anomaly up to terms cubic in $\lambda$,

$$\mathcal{A}_6^{(4)} = \text{tr}[\delta_\epsilon AA(dA) + \delta_\xi (dA)A] - \frac{3i g}{2} \delta_\epsilon AA^3 ] ,$$  

(4.109)

and indeed

$$\delta_{\epsilon_2} \mathcal{A}_6^{(4)} - \delta_{\epsilon_1} \mathcal{A}_6^{(4)} = \mathcal{A}_6^{(3)} + 3 \text{tr}[\delta_{\epsilon_1} A \delta_{\epsilon_2} AF - \delta_{\epsilon_2} A \delta_{\epsilon_1} AF] .$$  

(4.110)

In order to compensate the second term in eq. (4.110), one is to add to $\mathcal{A}_6^{(4)}$ the gauge-invariant term

$$\Delta \mathcal{A}_6^{(4)} = -\frac{i}{2} \text{tr}[\delta_\epsilon A \bar{\lambda} \gamma^{(3)} \lambda + \bar{\lambda} \delta_\epsilon A \gamma^{(3)} \lambda] ,$$  

(4.111)

so that $\mathcal{A}_6^{(4)} + \Delta \mathcal{A}_6^{(4)}$ is the proper global supersymmetry anomaly. Although the supersymmetry algebra closes only on the field equation of $\lambda$, in four dimensions a simple dimensional counting shows that eqs. (4.104) can not generate a term proportional to $\gamma^\mu D_\mu \lambda$. Therefore, in this case the Wess-Zumino consistency conditions close accidentally even off-shell, as pointed out in [73].

The situation is quite different in six dimensions. In this case, in the spirit of the previous section, let us restrict our attention to the 8-form residual anomaly polynomial

$$I_8 = -c^r c^{r'} \text{tr}_z (F^2) \text{tr}_{z'} (F^2) ,$$  

(4.112)

where the sums are left implicit, so that the gauge anomaly is

$$\mathcal{A}_6^{(6)} = -c^r c^{r'} \text{tr}_z (AdA) \text{tr}_{z'} (F^2) .$$  

(4.113)

Then, from the second of eqs. (4.104),

$$\mathcal{A}_6^{(6)} = -c^r c^{r'} \left[ \text{tr}_z (\delta_\epsilon AA) \text{tr}_{z'} (F^2) + 2 \text{tr}_z (\delta_\epsilon AF) \omega_{3}^{z'} \right] ,$$  

(4.114)
but there are residual terms, so that
\[
\left( \delta_{e_1} A^{(6)}_{e_2} - \delta_{e_2} A^{(6)}_{e_1} \right)_{\text{extra}} = -4c^\tau c^{\tau'} \left[ \text{tr}_z (\delta_{e_1} A \delta_{e_1} A) \text{tr}_{z'} (F^2) \right] + 2 \text{tr}_z (\delta_{e_2} A F) \text{tr}_{z'} (\delta_{e_1} A F) \right].
\]
Consequently, eq. (4.114) is to be modified by terms cubic in the gauginos, and the complete result, written in component notation, is finally
\[
A^{(6)} = -\frac{1}{4} \epsilon^{\mu \nu \alpha \beta \gamma \delta} c^\tau c^{\tau'} \text{tr}_z (\delta_e A_\mu A_\nu) \text{tr}_{z'} (F'_{\alpha \beta} F'_{\gamma \delta})
- \frac{1}{6} \epsilon^{\mu \nu \alpha \beta \gamma \delta} c^\tau c^{\tau'} \text{tr}_z (\delta_e A_\mu F_{\nu \alpha}) \omega_{\beta \gamma \delta}^{z'}
+ A c^\tau c^{\tau'} \text{tr}_z (\delta_e A_\mu F_{\nu \rho}) \text{tr}_{z'} (\lambda'_{\gamma} F_\mu F_{\rho}^{\prime})
+ B c^\tau c^{\tau'} \text{tr}_z (\delta_e A_\mu \overline{\lambda}) \gamma_{\mu \nu} \text{tr}_{z'} (\lambda' F_{\nu \rho}^{\prime})
+ C c^\tau c^{\tau'} \text{tr}_z (\delta_e A_\mu \overline{\lambda}) \gamma_{\nu} \text{tr}_{z'} (\lambda' F_\mu^{\prime \rho}) \ ,
\]
where the coefficients \( A, B \) and \( C \) satisfy the relations
\[
A + B = i, \\
C = 4A - 2B \ .
\]
These leave one undetermined parameter, in agreement with the well-known fact that anomalies are defined up to the variation of local functionals. Indeed, adding to the supersymmetry anomaly the term
\[
\left( \text{eq: } 0 \right) \left( \text{eq: } 0 \right)
\]
corresponds to adding terms like the last three in eq. (4.116) with coefficients satisfying the relations \( A + B = 0 \) and \( C = 4A - 2B \), that thus preserve eqs. (4.117). One can then show that the last of eqs. (4.104) generates terms containing one derivative and four gauginos, that cancel using the Dirac equation \( \gamma^\mu D_\mu \lambda = 0 \). Naturally, something similar also happens in six-dimensional supergravity, as we are about to verify.

Returning to the supersymmetry anomaly of eq. (4.101), one can observe that the coefficients of the third, fourth and fifth terms are consistent with eqs. (4.117). Moreover, demanding that the last of eqs. (4.104) be satisfied fixes the other gauge-invariant terms to give exactly the anomaly in eq. (4.101). Finally, the Wess-Zumino condition is satisfied only on-shell, and one obtains
\[
(\delta_{e_1} A_{e_2} - \delta_{e_2} A_{e_1}) = A_{\tilde{e}} + A_{\tilde{\lambda}} + c^\tau c^{\tau'} \text{tr}_{z,z'} \left\{ \frac{i e}{16} \xi_\sigma (\overline{\lambda} \gamma_\sigma \lambda') (\overline{\lambda} \gamma_\tau F_{\mu \nu}^{\prime} (\text{eq: } 0)_{(a=0)})
- \frac{i e}{32} \xi_{\alpha \delta \tau} \{ [\overline{\lambda} \gamma^\tau \lambda']_i (\overline{\lambda} \gamma_\sigma \text{[eq: } \lambda')(\text{eq: } 0)_{(a=0)}) + [\overline{\lambda} \gamma^\tau \lambda]_i (\overline{\lambda} \gamma_\sigma \text{[eq: } \lambda](a=0)) \} \right\} \ ,
\]
where we have stressed that the corresponding field equation for the gaugini is determined by eq. (4.95) with \( \alpha = 0 \). To reiterate, the anomaly obtained for \( \alpha = 0 \) naturally closes on the corresponding field equation for \( \lambda \). Still, the identity

\[
\frac{i e}{16} \xi_{\sigma}[\hat{\lambda} \gamma^\sigma \lambda'](\hat{\lambda} \gamma'^{\sigma}[\text{eq.}\lambda'][(\alpha)] - \frac{i e}{32} \xi_{\sigma\delta}[\hat{\lambda} \gamma^\sigma \lambda'][(\alpha)] - \frac{i \bar{\lambda} \gamma^\sigma \lambda'[\text{eq.}\lambda][\alpha = 0)] + \frac{i \bar{\lambda} \gamma^\sigma \lambda'[\text{eq.}\lambda'][\alpha = 0)])
\]

implies that the last term should somehow be generated in the anomaly, if the Wess-Zumino condition is to close for any value of \( \alpha \). In the presence of \( L_{\Lambda} \), however, the anomaly is modified by eq. (4.102), and applying the last of eqs. (4.104) to this term gives

\[
\frac{i e}{8} \xi_{\rho\sigma}[\hat{\lambda} \gamma^\rho \lambda'][\hat{\lambda} \gamma^\sigma \lambda'][\alpha = 0)] + 2 e \alpha \xi_{\rho\sigma}[\hat{\lambda} \gamma^\rho \lambda'][\hat{\lambda} \gamma^\sigma \lambda'][\alpha = 0])
\]

The commutator in eq. (4.121) is fully known: in particular, the coordinate transformation in the second term combines with the commutator on \( e \) to give a total divergence, while gauge and local Lorentz transformations give a vanishing result. Moreover, the field equation is obtained from eq. (4.95). The charge in eq. (4.88) thus plays a crucial role: it generates in eq. (4.121) precisely

\[
\frac{i e \alpha}{8} \xi_{\rho\sigma}[\hat{\lambda} \gamma^\rho \lambda'][\hat{\lambda} \gamma^\sigma \lambda'][\alpha = 0])
\]

as needed for consistency. Thus, one can understand the rationale behind the occurrence of the extension in the algebra on the gauginos: it lets the Wess-Zumino conditions close precisely on the field equations determined by the algebra. Since the Wess-Zumino conditions close on the equation of the gauginos, only these fields perceive the additional transformation.

4.1.4 The energy-momentum tensor

The gauge anomaly \( A_\Lambda = \delta_\Lambda L \) naturally satisfies the condition

\[
A_\Lambda = -tr(\Lambda D_\mu J^\mu)
\]

The energy-momentum tensor
where $J^\mu = 0$ is the complete field equation of the vector field. One can similarly show that the supersymmetry anomaly is related to the field equation of the gravitino, that we write succinctly $\mathcal{J}^\mu = 0$, according to

$$\mathcal{A}_\xi = -\langle \epsilon D_\mu \mathcal{J}^\mu \rangle . \quad (4.124)$$

We would like to stress that the Noether identities (4.123) and (4.124) relate the anomalies to the equations of the fields whose transformations contain derivatives. This observation has a natural application to gravitational anomalies, that we would now like to elucidate. In fact, in analogy with the previous cases one would expect that

$$\mathcal{A}_\xi = \delta \xi \mathcal{L} = 2 \xi_\mu D_\nu T^{\mu\nu} , \quad (4.125)$$

where the variation of the metric under general coordinate transformations is

$$\delta g_{\mu\nu} = -\xi^\alpha \partial_\alpha g_{\mu\nu} - g_{\alpha\nu} \partial_\mu \xi^\alpha - g_{\mu\alpha} \partial_\nu \xi^\alpha . \quad (4.126)$$

Thus, for models without gravitational anomalies one would expect that the divergence of the energy-momentum tensor vanish. Actually, this is no longer true if other anomalies are present, since all fields, not only the metric, have derivative variations under coordinate transformations. For instance, in a theory with gauge and supersymmetry anomalies, the gravitational anomaly is actually

$$\mathcal{A}_\xi = \delta \xi \mathcal{L} = 2 \xi_\nu D_\mu T^{\mu\nu} + \xi_\nu \text{tr}(A^\nu D_\mu J^\mu) + \xi_\nu (\bar{\psi}^\nu D_\mu \mathcal{J}^\mu) . \quad (4.127)$$

In particular, in our case we are not accounting for gravitational anomalies, that would result in higher-derivative couplings, and indeed one can verify that the divergence of the energy-momentum tensor does not vanish, but satisfies the relation

$$D_\mu T^{\mu\nu} = -\frac{1}{2} \text{tr}(A^\nu D_\mu J^\mu) - \frac{1}{2} \langle \bar{\psi}^\nu D_\mu \mathcal{J}^\mu \rangle . \quad (4.128)$$

### 4.2 Covariant field equations and anomalies

It is well known that consistent and covariant gauge anomalies are related by the divergence of a local functional \[81\]. In six dimensions the residual covariant gauge anomaly is \[53\]

$$\mathcal{A}_{\Lambda}^{\text{cov}} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta\gamma\delta} \epsilon^{rz} \epsilon^{x'y'} \text{tr}_z (\Lambda F_{\mu\nu}) \text{tr}_{x'} (F'_{\alpha\beta} F'_{\gamma\delta}) . \quad (4.129)$$
and is related to the consistent anomaly by a local counterterm,

\[
\mathcal{A}_\text{cons} + \text{tr}[\Lambda D_\mu f^\mu] = \mathcal{A}_\text{cov},
\]  

(4.130)

where

\[
f^\mu = e^{\nu}_{\rho} e^{\rho}_{\sigma} \left\{ -\frac{1}{4} \epsilon^{\mu \nu \alpha \beta \gamma \delta} A_\nu \text{tr}_{z'}(F'_{\alpha \beta} F'_{\gamma \delta}) - \frac{1}{6} \epsilon^{\mu \nu \alpha \beta \gamma \delta} F'_{\nu \alpha} \omega'_{\beta \gamma \delta} \right\}.
\]  

(4.131)

Comparing eq. (4.131) with eq. (4.101) one can see that, to lowest order in the Fermi fields,

\[
\mathcal{A}_e = \text{tr}(\delta_e A_\mu f^\mu),
\]  

(4.132)

and this implies that the transition from consistent to covariant anomalies turns a model with a supersymmetry anomaly into another without any [54, 68]. Indeed, six-dimensional supergravity coupled to vector and tensor multiplets was originally formulated in this fashion in [54] to lowest order in the Fermi fields, extending the results of Romans [24]. The resulting vector equation is not integrable. Moreover, the corresponding gauge anomaly is not the gauge variation of a local functional and does not satisfy Wess-Zumino consistency conditions.

This result can be generalized naturally, if somewhat tediously, to include terms of all orders in the Fermi fields [71]. The complete supersymmetry anomaly of eq. (4.101) has the form

\[
\mathcal{A}_e = \text{tr}(\delta_e A_\mu f^\mu) + \delta_e e^a_{\mu} g^\mu_{\alpha},
\]  

(4.133)

where to lowest order \( f^\mu \) is defined in eq. (4.131). Modifying the vector equation so that

\[
(eq. A^\mu)_{(\text{cov})} \equiv J^\mu_{(\text{cov})} = \frac{\delta \mathcal{L}}{\delta A_\mu} - f^\mu,
\]  

(4.134)

and similarly for the Einstein equation, the resulting theory is supersymmetric but no longer integrable. The covariant vector field equation is

\[
2D_\nu(v_\nu F^{\mu \nu}) - 2G_{\nu \rho} \hat{H}^{\rho \mu \nu \rho} F_{\nu \rho} - i v_\nu (\bar{\psi}_\alpha \gamma^\alpha \gamma^{\mu \nu \rho} \psi_\beta) F_{\nu \rho} + i \sqrt{2} D_\nu[\bar{\psi}_\nu (\bar{\lambda} M \gamma_{\mu \nu \rho} \lambda)] + \sqrt{2} D_\nu[\bar{\lambda} M (\bar{\lambda} M \gamma_{\mu})]
\]

\[
+ \frac{i}{2} F_{\nu \rho} e^{\nu}_{\rho} \text{tr}_{z'}(\bar{\lambda} M \gamma^{\mu \nu \rho} \lambda') F_{\nu \rho} + i \sqrt{2} e^{\nu}_{\rho} \text{tr}_{z'}[\bar{\lambda} M \gamma^{\mu \nu \rho} \lambda'] F_{\nu \rho} - i e^{\nu}_{\rho} \text{tr}_{z'}[(\bar{\lambda} M \gamma^{\mu \nu \rho} \lambda') F_{\nu \rho}]
\]

\[
- \frac{1}{2} \sqrt{2} e^{\nu}_{\rho} \text{tr}_{z'}[(\bar{\lambda} M \gamma^{\mu \nu \rho} \lambda') (\bar{\lambda} M \gamma_{\nu \rho})]
\]
and completes the results in [4] to all orders in the Fermi fields. Its divergence satisfies
\[
\text{tr}(\Lambda D_{\mu}J_{(\text{cov})}^{\mu}) = -A_{A}^{\text{cov}}
\]
where \(A_{A}^{\text{cov}}\) contains higher-order Fermi terms:
\[
\begin{align*}
A_{A}^{\text{cov}} &= \sigma^x_c \sigma^z_c \text{tr}_{z',z}[\frac{1}{2} e_{\mu\nu\alpha\beta\gamma\delta}(\Lambda F_{\nu\mu})(F'_{\alpha\beta}F'_{\gamma\delta}) \\
&+ i e\Lambda F_{\nu\mu}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')D_{\mu}\lambda' + ie\Lambda D_{\mu}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')F'_{\nu\mu} + ie\Lambda D_{\mu}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')F'_{\nu\mu} \\
&- \frac{e}{2\sqrt{2}}\Lambda D_{\mu}[(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')] \\
&+ e\Lambda D_{\mu}\{\sigma^x_c \sigma^z_c \text{tr}_{z',z'}[\frac{3i}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}') - i\frac{i}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')F'_{\nu\mu} \\
&- \frac{e}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')]\} \\
&+ e\Lambda D_{\mu}\{\sigma^x_c \sigma^z_c \text{tr}_{z',z'}[-i\frac{3}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')F'_{\nu\mu} \\
&+ e\Lambda D_{\mu}\{\sigma^x_c \sigma^z_c \text{tr}_{z',z'}[\frac{i}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}') - i\frac{3}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')F'_{\nu\mu} \\
&+ \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}'_{\gamma\mu\nu\rho\lambda}')(\lambda'_{\gamma\nu\nu\rho\lambda}')]\}\}.
\end{align*}
\]

Finally, one can study the divergence of the Rarita-Schwinger and Einstein equations in the covariant model. To this end, let us begin by stating that the derivation of Noether identities for a system of non-integrable equations does not present difficulties of principle, since these involve only first variations. Indeed, the only difference with respect to the standard case of integrable equations is that now \(\delta L\) is not an
exact differential in field space. Still, all invariance principles reflect themselves in linear dependencies of the field equations. Thus, for instance, with the covariant equations obtained from the consistent ones by the redefinition of eq. \((4.134)\) and by

\[
(eq. e^\mu_m)_{(cov)} = \frac{\delta \mathcal{L}}{\delta e^\mu_m} - g^\mu_m ,
\]

the total \(\delta_\epsilon \mathcal{L}\) vanishes by construction. The usual procedure then proves that the divergence of the Rarita-Schwinger equation vanishes for any value of the parameter \(\alpha\). On the other hand, the divergence of the energy-momentum tensor presents some subtleties, already anticipated in the previous section, that we would now like to describe. In particular, it vanishes to lowest order in the Fermi couplings, while it gives a covariant non-vanishing result if all fermion couplings are taken into account. The subtlety has to do with the behavior of the vector under general coordinate transformations,

\[
\delta_\xi A_\mu = -\xi^\alpha \partial_\alpha A_\mu - \partial_\mu \xi^\alpha A_\alpha ,
\]

and with the corresponding full (off-shell) form of the identity of eq. \((4.127)\). Starting again from the consistent equations, one finds

\[
A_\xi = \delta_\xi \mathcal{L} = 2\xi_\mu D_\mu T^{\mu\nu} + \xi_\nu tr(A^\nu D_\mu J^\mu) + \xi_\nu tr(F^{\mu\nu} J^\mu) + \xi_\nu (\bar{\psi}^\nu D_\mu J^\mu) .
\]

Reverting to the covariant form eliminates the divergence of the Rarita-Schwinger equation and alters the vector equation, so that the third term has to be retained. The final result is then

\[
D_\mu T^{\mu\nu}_{(cov)} = -\frac{1}{2} tr(A^\nu D_\mu J^\mu_{(cov)}) - \frac{1}{2} tr(f_\mu F^{\mu\nu}) - \frac{1}{2} tr(A^\nu D_\mu f^\mu) - \frac{1}{2} e^{\nu m} D_\mu g^\mu_m ,
\]

and is nicely verified by our equations. In particular, this implies that, to lowest order in the Fermi couplings, the divergence of \(T^{\mu\nu}_{(cov)}\) vanishes.

### 4.3 PST construction

In the previous section we have reviewed a number of properties of six-dimensional \((1,0)\) supergravity coupled to vector and tensor multiplets \([54, 68, 69, 70, 73]\). We have always confined our attention to the field equations, thus evading the traditional difficulties met with the action principles for (anti)self-dual tensor fields. In this section we would like to complete our discussion, presenting an action principle for the consistent field equations. What follows is an application \([74]\) of a general
method introduced by Pasti, Sorokin and Tonin (PST), that have shown how to obtain Lorentz-covariant Lagrangians for (anti)self-dual tensors with a single auxiliary field \[12\]. Alternative constructions \[81\], some of which preceded the work of PST, need an infinite number of auxiliary fields, and bear a closer relationship to the BRST formulation of closed-string spectra \[82\]. This method has already been applied to a number of systems, including \((1,0)\) six-dimensional supergravity coupled to tensor multiplets \[83\] and type IIB ten-dimensional supergravity \[13\], whose (local) gravitational anomaly has been shown to reproduce \[84\] the well-known results \[29\] of Alvarez-Gaumé and Witten. Still, an action principle for the consistent equations reviewed in the previous section is of some interest since, as we have seen, these six-dimensional models have a number of unfamiliar properties.

Let us begin by considering a single 2-form with a self-dual field strength in six-dimensional Minkowski space. The PST lagrangian \[12\]
\[
\mathcal{L} = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4(\partial \Xi)^2} \partial^{\mu} \Xi H^{-\rho}_{\mu\nu\rho} H^{-\sigma\mu\rho} \partial^{\sigma} \Xi ,
\]
where \(H = dB\) and \(H^{-} = H - *H\), is invariant under the gauge transformation \(\delta B = d\Lambda\), as well as under the additional gauge transformations
\[
\delta B = (d\Xi)\Lambda' \quad (4.143)
\]
and
\[
\delta \Xi = \Lambda'' ,
\]
\[
\delta B_{\mu\nu} = \frac{\Lambda''}{(\partial \Xi)^2} H^{-\rho}_{\mu\nu\rho} \partial^{\rho} \Xi .
\]
(4.144)

The last two types of gauge transformations can be used to recover the usual field equation of a self-dual 2-form. Indeed, the scalar equation results from the tensor equation contracted with
\[
\frac{H^{-\rho}_{\mu\nu\rho} \partial^{\rho} \Xi}{(\partial \Xi)^2} ,
\]
and consequently does not introduce any additional degrees of freedom. The invariance of eq. \((4.144)\) can then be used to eliminate the scalar field. This field, however, can not be set to zero, since this choice would clearly make the Lagrangian of eq. \((4.143)\) inconsistent. With this proviso, using eq. \((4.143)\) one can see that the only solution of the tensor equation is precisely the self-duality condition for its field strength.
We now want to apply this construction to six-dimensional supergravity coupled to vector and tensor multiplets. The theory describes a single self-dual 2-form

$$\tilde{H}_{\mu\nu} = v_r \tilde{H}^r_{\mu\nu} - i 8 (\bar{\chi}^m \gamma_{\mu\nu} \chi^m)$$  \hspace{1cm} (4.146)

and \( n \) antiself-dual 2-forms

$$\tilde{H}^M_{\mu\nu} = x_r^M \tilde{H}^r_{\mu\nu} + \frac{i}{4} x_r^M e^{r z} t z (\bar{\lambda} \gamma_{\mu\nu} \lambda)$$  \hspace{1cm} (4.147)

The complete Lagrangian is obtained adding the term

$$-\frac{1}{4} \frac{\partial^\mu \Xi \partial^\nu \Xi}{(\partial \Xi)^2} [\tilde{H}_{\mu\nu} \tilde{H}^{-}_{\sigma\nu} + \tilde{H}^M_{\mu\nu} \tilde{H}^{M+}_{\sigma\nu}]$$  \hspace{1cm} (4.148)

to the lagrangian derived in Section (4.2). It can be shown that the 3-form

$$\hat{K}_{\mu\nu} = \hat{\tilde{H}}_{\mu\nu} - 3 \frac{\partial_\mu \Xi \partial_\nu \Xi}{(\partial \Xi)^2} \hat{\tilde{H}}_{\mu\nu}$$  \hspace{1cm} (4.149)

is identically self-dual, while the 3-forms

$$\hat{K}^M_{\mu\nu} = \hat{H}^M_{\mu\nu} - 3 \frac{\partial_\mu \Xi \partial_\nu \Xi}{(\partial \Xi)^2} \hat{H}^M_{\mu\nu}$$  \hspace{1cm} (4.150)

are identically antiself-dual. With these definitions, we can display rather simply the complete supersymmetry transformations of the fields. Actually, only the transformations of the gravitino and of the tensorinos are affected, and become

$$\delta \psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} \hat{K}_{\mu\nu} \gamma^{\nu} \epsilon + \frac{i}{32} (\bar{\chi}^M \gamma_{\mu\nu} \chi^M) \gamma^{\nu} \epsilon - \frac{3i}{8} (\bar{\chi}^M \gamma_{\mu} \chi^M)$$

$$- \frac{i}{8} (\bar{\epsilon} \gamma_{\mu} \chi^M) \gamma^{\mu} \chi^M + \frac{i}{16} (\bar{\epsilon} \gamma^{\mu} \chi^M) \gamma_{\mu\nu} \chi^M - \frac{9i}{8} v_r e^{r z} t z [(\bar{\epsilon} \gamma_{\mu} \lambda)]$$

$$+ \frac{i}{8} v_r e^{r z} t z [(\bar{\epsilon} \gamma^{\mu} \lambda) \gamma_{\mu\nu} \lambda] - \frac{i}{16} v_r e^{r z} t z [(\bar{\epsilon} \gamma_{\mu\nu} \lambda) \gamma^{\mu} \lambda]$$

$$\delta \chi^M = \frac{i}{2} x_r^M \partial_\mu \psi_\nu \gamma^{\nu} \epsilon + \frac{i}{12} \hat{K}^M_{\mu\nu} \gamma^{\nu} \epsilon + \frac{1}{2} x_r^M e^{r z} t z [(\bar{\epsilon} \gamma_{\mu} \lambda) \gamma^{\mu} \lambda]$$  \hspace{1cm} (4.151)

while the scalar field \( \Xi \) is invariant under supersymmetry. It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions.

We now turn to describe the corresponding modifications of the supersymmetry algebra. In addition to general coordinate, gauge and supersymmetry transformations, the commutator of two supersymmetry transformations on \( B_{\mu\nu}^r \) now generates two local PST transformations with parameters

$$\Lambda'_{\mu} = \frac{\partial^{\rho} \Xi}{(\partial \Xi)^2} (v_r \hat{\tilde{H}}_{\sigma\mu} - x_r^M \hat{H}^{M+}_{\sigma\mu}) \xi^{\rho}$$

$$\Lambda = \xi^{\mu} \partial_\mu \Xi$$  \hspace{1cm} (4.152)
The transformation of eq. (5.84) on the scalar field $\phi$ is opposite to its coordinate transformation, and this gives an interpretation of the corresponding commutator \[83^3\]
\[ [\delta_1, \delta_2] \Xi = \delta g_{ab} \Xi + \delta_{PST} \Xi = 0 \ , \quad (4.153) \]
that vanishes consistently with the invariance of $\Xi$ under supersymmetry. Finally, the commutator on the vielbein determines the parameter of the local Lorentz transformation, that is now
\[ \Omega_{mn} = -\xi^\nu(\omega_{\nu mn} - \tilde{K}_{\nu mn} - \frac{i}{8}(\lambda^M \gamma_{\nu mn} \chi^M)) \]
\[ + \frac{1}{2}(\bar{\chi}^M \epsilon_1)(\bar{\chi}^M \gamma_{\nu mn} \epsilon_2) - \frac{1}{2}(\lambda^M \epsilon_2)(\lambda^M \gamma_{mn} \epsilon_1) \]
\[ + v_r c^{rz} \text{tr}_z[(\epsilon_1 \gamma_m \lambda)(\bar{\epsilon}_2 \gamma_n \lambda) - (\bar{\epsilon}_2 \gamma_m \lambda)(\epsilon_1 \gamma_n \lambda)] \ . \quad (4.154) \]

All other parameters remain unchanged while, aside from the extension \[69^3\], the algebra closes on-shell on the modified field equations of the Fermi fields. Of course, the resulting field equations reduce to those of Section 1 once one fixes the PST gauge invariances in order to recover the conventional equations for (anti)self-dual tensor fields.

For completeness, we conclude by displaying the lagrangian of six-dimensional supergravity coupled to vector and tensor multiplets with the inclusion of the PST term,

\[ e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{12} G_{rs} H^{\rho \mu \nu} H_{\mu \rho \nu} - \frac{1}{4} \partial_\mu v^r \partial^\mu v_r - \frac{1}{2} v_r c^{rz} \text{tr}_z (F_{\mu \nu} F^{\mu \nu}) \]
\[ - \frac{1}{8} e^{\rho \alpha \beta \gamma \delta} e_r B^{r}_{\mu \nu} \text{tr}_z (F_{\alpha \beta} F_{\gamma \delta}) - \frac{i}{2} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) - \frac{i}{2} (\psi_\rho \gamma_\mu \nu \bar{\psi}_\rho) \]
\[ - \frac{i}{8} v_r [H + \hat{\mathcal{H}}]^{\rho \mu \nu} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) + \frac{i}{48} v_r [H + \hat{\mathcal{H}}]^{\rho \mu \nu} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) \]
\[ + \frac{i}{2} (\bar{\chi}^M \gamma^\mu D_\mu (\bar{\omega}) \chi^M) - \frac{i}{24} v_r \hat{D}_{\mu \rho \nu} (\bar{\chi}^M \gamma^{\rho \mu \nu} \chi^M) \]
\[ + \frac{1}{4} x_r^M [\partial_\mu v^r + \partial^r v_\mu] (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) - \frac{1}{8} x_r^M [H + \hat{\mathcal{H}}]^{\rho \mu \nu} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) \]
\[ + \frac{1}{24} x_r^M [H + \hat{\mathcal{H}}]^{\rho \mu \nu} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho) \]
\[ + \frac{i}{2\sqrt{2}} v_r c^{rz} \text{tr}_z [((F + \hat{\mathcal{F}})_{\nu \rho} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho))] \]
\[ + \frac{i}{12} x_r^M x_r^M \hat{H}_{\mu \rho \nu} c^{sz} \text{tr}_z (\bar{\chi}^M \gamma^{\rho \mu \nu} \lambda) - \frac{i}{2} x_r^M c^{rz} \text{tr}_z [\bar{\chi}^M \gamma^{\rho \mu \nu} (\bar{\psi}_\rho \gamma_\mu \nu \psi_\rho)] \]
\[ + \frac{1}{16} x_r^M x_r^M \hat{H}_{\mu \rho \nu} c^{sz} \text{tr}_z (\bar{\chi}^M \gamma^{\rho \mu \nu} \lambda) + \frac{1}{\sqrt{2}} x_r^M c^{rz} \text{tr}_z [\bar{\chi}^M \gamma^{\rho \mu \nu} \lambda \hat{F}_{\mu \rho}] \]
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\[ - \frac{i}{8} (\bar{\chi}^M \gamma_{\mu\nu} \psi_{\rho}) x_r^M c^r z (\lambda \gamma^{\mu\nu} \lambda) \]

\[ - \frac{3}{16} v_r c^r z [\bar{\chi}^M \gamma_{\mu\nu} \lambda (\bar{\chi}^M \gamma^{\mu\nu} \lambda)] - \frac{1}{8} v_r c^r z [\bar{\chi}^M \gamma_{\mu\nu} \lambda (\bar{\chi}^M \lambda)] \]

\[ - \frac{3}{4} x_r^M c^r z x_s^N c^s z \frac{1}{v_l c^{l z}} \frac{1}{2} \frac{1}{v_r c^r z} [\bar{\chi}^M (\bar{\chi}^N \lambda) + \frac{1}{8} (\bar{\chi}^M \gamma_{\mu\nu} \chi^{\mu\nu}) (\bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho})] \]

\[ + \frac{1}{8} \frac{1}{v_l c^{l z}} [\bar{\chi}^M (\bar{\chi}^N \gamma_{\mu\nu} \lambda)] - \frac{1}{8} (\bar{\chi}^M \gamma_{\mu\nu} \chi^{\mu\nu}) (\bar{\chi}^M \gamma_{\mu\nu} \lambda) \]

\[ + \frac{1}{4} (\bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho}) v_r c^r z z (\bar{\lambda} \gamma_{\mu\nu} \lambda) - \frac{1}{2} v_r v_s c^r z c^s z [\bar{\lambda} \gamma_{\mu\nu} \lambda] \]

\[ + \frac{\alpha}{2} c^r z c^s z \bar{\lambda} \gamma_{\mu\nu} \lambda] \]

\[ - \frac{\partial^\mu \xi^r \partial^\nu \xi^z}{4 (\partial \xi^z)^2} [\hat{\mathcal{H}}_{\mu\nu} \hat{\mathcal{H}}_{-\nu\rho} + \hat{\mathcal{H}}_{\mu\rho} \hat{\mathcal{H}}_{-\mu\nu}] \quad (4.155) \]

where \( \alpha \) is the (undetermined) coefficient of the quartic coupling for the gauginos, and the corresponding supersymmetry transformations

\[ \delta e_{\mu}^m = -i (\bar{\epsilon}^m \lambda) \]

\[ \delta B_{\mu}^r = i r^r (\bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho}) + \frac{i}{2} c^r z (\bar{\chi}^M \gamma_{\mu\nu} \lambda) - 2 c^r z (A_{\mu} \delta A_{\nu}) \]

\[ \delta v_r = x_r^M (\bar{\chi}^M \lambda) \]

\[ \delta A_{\mu} = - \frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_{\mu} \lambda) \]

\[ \delta \psi_{\mu} = \bar{D}_{\mu} \lambda + \frac{i}{4} \bar{K}_{\mu\rho\nu} \chi^{\mu\rho} \lambda + \frac{i}{32} (\bar{\chi}^M \gamma_{\mu\rho} \chi^{\mu\nu} \lambda) \gamma^{\mu\rho} \epsilon - \frac{3i}{8} (\bar{\epsilon} \gamma^M \gamma_{\mu\nu} \lambda) \lambda \]

\[ - \frac{i}{8} (\bar{\epsilon} \gamma_{\mu\nu} \lambda) \lambda^{\mu\nu} \lambda + \frac{i}{16} (\bar{\epsilon} \gamma^{\mu\nu} \lambda) \gamma_{\mu\rho} \lambda + \frac{9i}{8} v_r c^r z [\bar{\epsilon} \gamma_{\mu} \lambda] \]

\[ + \frac{1}{4} v_r c^r z \gamma_{\nu} \gamma_{\lambda} \lambda - \frac{i}{16} v_r c^r z [\bar{\epsilon} \gamma_{\mu\rho} \lambda] \gamma^{\mu\nu} \lambda \]

\[ \delta \chi^M = \frac{i}{2} x_r^M \partial_{\nu} \bar{v} \gamma^\mu \epsilon + \frac{i}{12} \bar{K}_{\mu\rho\nu} \chi^{\mu\rho} \lambda + \frac{1}{2} c^r z x_r^M \bar{v} \gamma_{\nu} \gamma_{\lambda} \lambda \]

\[ \delta \lambda = - \frac{1}{2 \sqrt{2}} \bar{F}_{\mu\nu} \lambda^\mu \epsilon - \frac{1}{2} \frac{1}{v_s c^s z} (\bar{\chi}^M \lambda) \epsilon - \frac{1}{2} \frac{1}{v_s c^s z} (\bar{\chi}^M \epsilon) \lambda \]

\[ + \frac{1}{8} \frac{1}{v_s c^s z} (\bar{\chi}^M \gamma_{\mu\rho} \lambda) \quad (4.156) \]

One further comment is in order. Kavalov and Mkrtchyan \[85\] obtained long ago a complete action for pure d=6 (1,0) supergravity in terms of a single tensor auxiliary field. Their work may be connected to this special case of our result via an ansatz relating their tensor to the PST scalar. Still, the PST formulation has the virtue of simplicity and makes it manifest that the extra degrees of freedom may be locally eliminated via additional gauge transformations.
4.4 Inclusion of abelian vector multiplets

In this section we construct the general coupling of \((1,0)\) six-dimensional supergravity to \(n\) tensor multiplets and \textit{abelian} vector multiplets \([71]\). We will see the inclusion of abelian vectors allows the presence of more general couplings, with respect to the ones we have derived so far.

In this case, indeed, the field strengths of the 2-forms include generalized Chern-Simons 3-forms of the vector fields \([72]\) according to

\[
H_{\mu \nu \rho}^r = 3\partial_{[\mu} B_{\nu \rho]}^r - 3 c^{r a b} A_{[\mu}^a \partial_{\nu}^b A_{\rho]}^b ,
\]

where the \(c^{r a b}\) are the constants that determine the gauge part of the residual anomaly polynomial. In the complete theory, the anomaly induced by this term would cancel against the contribution of fermion loops, while the irreducible part of the anomaly polynomial is directly absent in consistent models \([31, 54]\).

The model can be constructed using the same method as before: the completion to all orders in the Fermi fields of the equations of motion is obtained requiring the closure of the commutator of two supersymmetry transformations on the fermionic field equations. All the resulting equations may be conveniently derived from the Lagrangian

\[
e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{12} G_{r s} H_{\mu \nu \rho}^r H_{\mu \nu \rho}^s - \frac{1}{4} \partial_{[\mu} B_{\nu \rho]}^r \partial_{\mu} B_{\nu \rho]}^r - \frac{1}{4} v_r c^{r a b} F_{[\mu}^a F_{\nu \rho]}^b + \frac{1}{16} \epsilon^{\mu \alpha \beta \gamma} c_r^a B_{\mu}^a F_{\alpha \beta}^b F_{\gamma \delta}^c - \frac{i}{2} \bar{\psi}_{\mu} \gamma^{\mu \nu \rho} D_\nu \left[ \frac{1}{2} (\omega + \hat{\omega}) \right] \psi_{\rho} + \frac{i}{8} v_r [H + \hat{H}] r^{\mu \nu \rho} (\bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho}) + \frac{i}{48} v_r [H + \hat{H}]^r_{\alpha \beta \gamma} (\bar{\psi}_{\mu} \gamma^{\mu \alpha \beta} \gamma_{\gamma} \psi_{\nu}) + \frac{i}{2} \chi^M \gamma^\mu D_\mu (\hat{\omega}) \chi^M - \frac{i}{24} v_r \hat{H}_{\mu \nu \rho} (\bar{\chi}^M \gamma^{\mu \nu \rho} \chi^M) + \frac{1}{4} x_r^M [\partial_{\mu} B^r + \partial_{\nu} B^r] (\bar{\psi}_{\mu} \gamma^\nu \gamma^\mu \chi^M) - \frac{1}{8} x_r^M [H + \hat{H}] r^{\mu \nu \rho} (\bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho} \chi^M) + \frac{1}{24} x_r^M [H + \hat{H}] r^{\mu \nu \rho} (\bar{\psi}_{\mu} \gamma_{\nu} \gamma^\mu \gamma_{\rho} \chi^M) + \frac{1}{8} (\bar{\chi}^M \gamma^\mu \chi^N) (\bar{\chi}^M \gamma^\mu \chi^N) + \frac{i}{4 \sqrt{2}} v_r c^{r a b} (F + \hat{F})_{[\mu}^a (\bar{\psi}_{\nu} \gamma^\mu \gamma^\nu \gamma_{\rho} \chi^b) + \frac{1}{2 \sqrt{2}} v_r c^{r a b} (\bar{\chi}^M \gamma^\mu \gamma^a \chi^b) + \frac{i}{24} x_r^M [H + \hat{H}]^r_{\mu \nu \rho} c^{r a b} (\bar{\chi}^a \gamma^{\mu \nu \rho} \chi^b) + \frac{1}{32} v_r c^{r a b} (\bar{\chi}^a \gamma^{\mu \nu \rho} \chi^b) (\bar{\chi}^M \gamma^{\mu \nu \rho} \chi^M) - \frac{i}{16} (\bar{\chi}^M \gamma_{\mu} \psi_{\rho}) x_r^M c^{r a b} (\bar{\chi}^a \gamma^{\mu \nu \rho} \chi^b) - \frac{i}{4} x_r^M c^{r a b} (\bar{\chi}^M \gamma^{\mu \nu \rho} \chi^a) (\bar{\psi}_{\mu} \gamma_{\nu} \chi^b)
\]
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\[- \frac{1}{16} v_r c^{ab} (\bar{\chi}^M \lambda^a) (\bar{\chi}^M \lambda^b) - \frac{3}{32} v_r c^{ab} (\bar{\chi}^M \gamma_{\mu \nu} \lambda^a) (\bar{\chi}^M \gamma_{\mu \nu} \lambda^b) + \frac{1}{(x^M \cdot c)(x^N \cdot c)^{-1}} \left\{ - \frac{1}{4} (\bar{\chi}^M \lambda^a) (\bar{\chi}^N \lambda^b) + \frac{1}{16} (\bar{\chi}^N \gamma_{\mu \nu} \lambda^a) (\bar{\chi}^M \gamma_{\mu \nu} \lambda^b) \right\} \]

\[- \frac{1}{8} (\bar{\chi}^N \lambda^a) (\bar{\chi}^M \lambda^b) + \frac{1}{8} v_r c^{ab} (\bar{\psi}_* \gamma_\mu \psi^\dagger_*) (\bar{\lambda}^a \gamma_{\mu \nu} \lambda^b) \]

\[- \frac{1}{8} v_r v_s c^{cd} (\bar{\lambda}^a \gamma_\mu \lambda^c) (\bar{\lambda}^b \gamma_\mu \lambda^d) + \frac{\alpha}{8} c^{cd} (\bar{\lambda}^a \gamma_\mu \lambda^c) (\bar{\lambda}^b \gamma_\mu \lambda^d) \]  

(4.158)

after imposing the (anti)self duality conditions. The last term, proportional to the arbitrary parameter \( \alpha \), vanishes identically in the case of a single abelian vector multiplet. Since the kinetic terms of the vector fields are non-diagonal, this generalization is only possible in the abelian case.

The variation of this Lagrangian with respect to gauge transformations gives the gauge anomaly

\[ A_\Lambda = - \frac{1}{32} e^{\mu \nu \alpha \beta \gamma \delta} c^{\nu \rho} c^{\rho \sigma} A^a F^b_{\mu \nu} F^c_{\alpha \beta} F^d_{\gamma \delta} \]  

(4.159)

while the variation with respect to the supersymmetry transformations

\[ \delta e^m_\mu = - i (\bar{\epsilon} \gamma^m \psi_\mu) \]

\[ \delta B^r_{\mu \nu} = i v^r (\bar{\psi}_* \gamma_\mu \gamma_\nu \epsilon) + \frac{1}{2} x^M (\bar{\chi}^M \gamma_\mu \epsilon) - c^{ab} (A^a_\mu \delta A^b_\nu) \]

\[ \delta v_r = x^r (\bar{\chi}^M \epsilon) \]

\[ \delta A^a_\mu = - \frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda^a) \]

\[ \delta \psi_\mu = \bar{D}_\mu \epsilon + \frac{1}{4} v_r H^r_{\mu \rho \sigma} \gamma^{\mu \rho \sigma} - \frac{3i}{8} \bar{\gamma}_\mu \epsilon \bar{\chi}^M (\bar{\epsilon} \chi^M) - \frac{3i}{8} \bar{\gamma}_\mu \epsilon \bar{\chi}^M (\bar{\epsilon} \gamma_\mu \chi^M) + \frac{i}{16} \bar{\gamma}_\mu \gamma^a (\bar{\epsilon} \gamma_\mu \lambda^a) + \frac{i}{16} \bar{\gamma}_\mu \gamma^a (\bar{\epsilon} \gamma_\mu \lambda^a) \]

\[ \delta A^a_\mu = \frac{i}{2} x^M (\bar{\epsilon} \gamma^r \gamma_\mu \epsilon) + \frac{i}{12} x^M \bar{H}^r_{\mu \rho \sigma} \gamma^{\mu \rho \sigma} + \frac{1}{4} x^M c^{\nu \rho} \gamma_\mu \lambda^a (\bar{\epsilon} \gamma_\nu \lambda^a) \]

\[ \delta \lambda^a = - \frac{1}{2 \sqrt{2}} \bar{F}^a_{\mu \nu} \gamma^{\mu \nu} \epsilon + \left\{ (v \cdot c)^{-1} (x^M \cdot c)^{-1} \right\} \left\{ \frac{1}{2} (\bar{\chi}^M \gamma_\mu \epsilon) - \frac{1}{4} (\bar{\chi}^M \epsilon) \right\} \]

(4.160)

gives the supersymmetry anomaly

\[ A_\epsilon = c^{ab} c^{cd} \left\{ - \frac{1}{16} e^{\mu \nu \alpha \beta \gamma \delta} \delta A^a_\mu F^b_{\alpha \beta} F^c_{\gamma \delta} - \frac{1}{8} e^{\mu \nu \alpha \beta \gamma \delta} \delta A^a_\mu F^b_{\nu \alpha} A^c_{\beta \gamma \delta} \right\} + \frac{i e}{8} \delta A^a_{\mu \nu} (\bar{\lambda}^b \gamma_\mu \lambda^d) + \frac{i e}{8} \delta A^a_{\mu} (\bar{\lambda}^b \gamma_\mu \lambda^c) F^d_{\nu \rho} + \frac{i e}{4} \delta A^a_{\mu} (\bar{\lambda}^b \gamma_\mu \lambda^c) F^d_{\nu \rho} \]
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\[
- \frac{ie}{128} (\gamma^a \psi_\mu) (\tilde{\lambda}^a \gamma^{\mu \rho} \lambda^b) (\tilde{\lambda}^c \gamma_{\nu \rho} \lambda^d) - \frac{e}{8 \sqrt{2}} \delta \epsilon A^a_\mu (\tilde{\lambda}^b \gamma^\nu \gamma^\rho \lambda^c) (\tilde{\lambda}^d \gamma_\nu \psi_\mu)
\]

\[
+ c^a_{\mu} e (v \cdot c)^{-1} (x^M \cdot c)^{cd} \left\{ - \frac{i}{4 \sqrt{2}} \delta \epsilon A^a_\mu (\tilde{\lambda}^b \gamma^\mu \lambda^c) (\tilde{\lambda}^d \chi^M) \right\}
\]

\[
+ \frac{i}{16 \sqrt{2}} \delta \epsilon A^a_\mu (\tilde{\lambda}^b \gamma^\mu \gamma^\nu \lambda^d) (\tilde{\chi}^M \gamma_\mu \lambda^c) - \frac{i}{8 \sqrt{2}} \delta \epsilon A^a_\mu (\tilde{\lambda}^b \gamma^\mu \lambda^d) (\tilde{\chi}^M \lambda^c)
\]

\[
+ \frac{\alpha}{8} c^a_{\mu} e^{r c d} (\epsilon (\tilde{\lambda}^a \gamma_\mu \lambda^c) (\tilde{\lambda}^b \gamma^\mu \lambda^d)) \right\} .
\]

(4.161)

Once again, the complete theory would contain additional non-local couplings induced by fermion loops, whose variation would cancel the anomalous contribution of the contact terms. Thus, the low-energy couplings that we are displaying are properly neither gauge-invariant nor supersymmetric. However, gauge and supersymmetry anomalies are related by Wess-Zumino consistency conditions, and this grants the coherence of the construction. The presence of the arbitrary parameter \(\alpha\) reflects the freedom of adding to the anomaly the variation of a local functional, consistently with all Wess-Zumino conditions. We have already seen that this anomalous behavior of the low-energy Lagrangian is related to another remarkable property of these models [69]: aside from local symmetry transformations and the equation of motion, the commutator of two supersymmetry transformations on the gauginos generates the two-cocycle

\[
\delta (\alpha) \lambda^a = \left[ (v \cdot c)^{-1} e^a_{\mu} e^{r c d} \right] \left[ - \frac{1}{8} (\tilde{\epsilon}_1 \gamma_\mu \lambda^c) (\tilde{\epsilon}_2 \gamma_\nu \lambda^d) \gamma^\mu \lambda^b - \frac{\alpha}{4} (\tilde{\lambda}^b \gamma_\mu \lambda^c) (\tilde{\epsilon}_1 \gamma_\mu \lambda^d) \gamma^\mu \epsilon_2 
\right.
\]

\[
+ \frac{\alpha}{32} (\tilde{\lambda}^b \gamma_\mu \lambda^c) (\tilde{\epsilon}_1 \gamma^\mu \lambda^d) \gamma^\mu \epsilon_2 + \frac{\alpha}{32} (\tilde{\lambda}^b \gamma_\mu \lambda^c) (\tilde{\epsilon}_1 \gamma^\mu \lambda^d) \gamma^\mu \epsilon_2 
\]

\[
+ \frac{1 - \alpha}{8} (\tilde{\epsilon}_1 \gamma^\mu \epsilon_2) (\tilde{\lambda}^c \gamma_\mu \lambda^d) \gamma^\nu \lambda^b \left. \right] .
\]

(4.162)

different from zero for any value of \(\alpha\). In six dimensions the Wess-Zumino conditions close only on the field equations of the gauginos, and this two-cocycle actually makes these conditions close for any value of \(\alpha\). In the case of a single vector multiplet, in which the term of the Lagrangian proportional to \(\lambda^4\) disappears, the two-cocycle is still present, although it is properly independent of \(\alpha\).

Of course, it is possible to apply the construction of Pasti, Sorokin and Tonin (PST) [12] also in this case, following the results of the previous section. Since the theory describes a single self-dual 2-form

\[
\hat{H}_{\mu \nu \rho} = v_\tau \hat{H}^\tau_{\mu \nu \rho} - \frac{i}{8} (\tilde{\chi}^M \gamma_{\mu \nu \rho} \lambda^M)
\]

(4.163)
and $n$ antiself-dual 2-forms
\[ \hat{H}^{M}_{\mu\nu\rho} = x^{M}_{r} \hat{H}^{r}_{\mu\nu\rho} + \frac{i}{8} x^{M}_{r} c^{rab}(\bar{\chi}^{a}_{r\gamma} \gamma^{b\mu\nu\rho}) , \tag{4.164} \]
the complete Lagrangian is obtained by the addition of the term
\[ -\frac{1}{4} \frac{\partial^{\mu} \Xi \partial^{\nu} \Xi}{(\partial \Xi)^{2}} [\hat{H}^{\mu\nu\rho}_{\mu\nu\rho} \hat{H}^{\nu_{\mu\nu\rho}} + \hat{H}^{M}_{\mu\nu\rho} \hat{H}^{M+\nu_{\rho}}] , \tag{4.165} \]
where $\Xi$ is an auxiliary scalar field $[12, 53]$ and $H^{\pm} = H \pm *H$. This lagrangian has PST gauge invariances needed to cancel the additional degrees of freedom. As before, once the transformations of the gravitino and of the tensorinos are properly modified, the supersymmetry algebra generates also the PST gauge transformations $[83]$. The field equations obtained from the complete Lagrangian reduce to those obtained from the Lagrangian without the PST term, once these gauge invariances are fixed.

As supersymmetry does not constrain the values of the coefficients $c^{r}$, we have obtained a class of models whose anomaly polynomials can contain odd powers of the individual field strengths $F^{a}$. It is interesting to compare these results with $[72]$. Although for generic values of the $c$'s the $SO(1, n)$ global symmetry is broken, the authors of $[72]$ consider the amusing case of $n = 2$ with two abelian vector multiplets transforming in the spinorial representation of $SO(1, 2)$. Identifying this group with the one that transforms the tensor fields, one obtains an $SO(1, 2)$-invariant Lagrangian if $c^{r} = \gamma^{0} \gamma^{r}$. In particular the results of $[72]$ correspond to the Majorana representation of $SO(1, 2)$:
\[ \gamma^{0} = \sigma_{2} , \quad \gamma^{1} = i \sigma_{1} , \quad \gamma^{2} = i \sigma_{3} \tag{4.166} \]
and for this choice the anomaly polynomial vanishes identically.

The transition to tensionless strings corresponds to values of the scalar fields for which the gauge coupling vanishes $[52, 53]$. In our Lagrangian, this would correspond to the vanishing of some eigenvalues of the matrix $v_{r}c^{rab}$. In the case of $[72]$ the moduli space is a two-dimensional hyperboloid, described by the equation $v_{0}^{2} - v_{1}^{2} - v_{2}^{2} = 1$, and one can show that the eigenvalues of the matrix $v_{r}c^{rab}$ are both positive for $v_{0} \geq 1$ and both negative for $v_{0} \leq -1$, so that the transition is not reached.

### 4.5 Inclusion of hypermultiplets

In this section we describe the full coupling of six-dimensional supergravity to vector, tensor and hypermultiplets $[76]$. In the description of the coupling to hypermultiplets
we will follow the notation of [73]. Some details about our conventions are contained in the Appendix.

The spinors in the theory are the left-handed gravitino $\psi^A$, $n_T$ right-handed tensorinos $\chi^{MA}$, the left-handed gauginos $\lambda^A$ the right-handed hyperinos $\Psi^a$, where $a = 1, \ldots, 2n_H$. The index $A = 1, 2$ is in the fundamental representation of $USp(2)$, and the gravitino, the tensorinos and the gauginos are $USp(2)$ doublets satisfying the symplectic-Majorana condition

$$\psi^A = \epsilon^{AB} C \bar{\psi}_B^T$$

(4.167)

The index $a$, instead, is a $USp(2n_H)$ index, and the hyperinos satisfy the symplectic-Majorana condition

$$\Psi^a = \Omega^{ab} C \bar{\Psi}_b^T$$

(4.168)

where $\Omega^{ab}$ is the antisymmetric invariant tensor of $USp(2n_H)$ (see the appendix for more details). The hyper-scalars $\phi^\alpha$, $\alpha = 1, \ldots, 4n_H$, are coordinates of a quaternionic manifold, that is a manifold whose holonomy group is contained in $USp(2) \times USp(2n_H)$.

If the quaternionic manifold parametrized by the hyper-scalars has isometries, these correspond to global symmetries of the supergravity theory. Then the global symmetry group, or a subgroup thereof, can be gauged. We recall the notations used to describe the scalars in the hypermultiplets. We denote by $V^a_A(\phi)$ the vielbein of the quaternionic manifold, where the index structure corresponds to the requirement that the holonomy be contained in $USp(2) \times USp(2n_H)$. The internal $USp(2)$ and $USp(2n_H)$ connections are then denoted, respectively, by $A^A_{\alpha\beta}$ and $A^a_{\alpha\beta}$, that in our conventions are anti-hermitian matrices. The index $a = 1, \ldots, 4n_H$ is a curved index on the quaternionic manifold. The field-strengths of the connections are

$$F^A_{\alpha\beta} = \partial_\alpha A^A_{\beta\beta} - \partial_\beta A^A_{\alpha\beta} + [A_\alpha, A_\beta]^A_{\alpha\beta}$$
$$F^a_{\alpha\beta} = \partial_\alpha A^a_{\beta\beta} - \partial_\beta A^a_{\alpha\beta} + [A_\alpha, A_\beta]^a_{\alpha\beta}$$

(4.169)

where $\partial_\alpha = \partial / \partial \phi^\alpha$. The request that the vielbein $V^a_A(\phi)$ be covariantly constant gives the following relations [80]:

$$V^a_A V^\beta_B g_{\alpha\beta} = \Omega_{ab} \epsilon_{AB}$$
$$V^a_A V^{\beta A} + V^a_A V^{\alpha A} = \frac{1}{n_H} g^{\alpha\beta} \delta^A_B$$
$$V^a_A V^{\beta A} + V^a_A V^{\alpha A} = g^{\alpha\beta} \delta^A_B$$

(4.170)
where $\Omega_{ab}$ is the antisymmetric invariant tensor of $USp(2n_H)$. The raising and lowering conventions are collected in the appendix. The field-strength of the $USp(2)$ connection $A_{\alpha B}^A$ is naturally constructed in terms of $V_{\alpha}^{aA}$ by the relation:

$$F_{\alpha\beta AB} = V_{\alpha A} V_{\beta B}^a + V_{\alpha B} V_{\beta A}^a ,$$

and then the cyclic identity for the internal curvature tensor implies that the field-strength of the $USp(2n_H)$ connection $A_{ab}^A$ has the form

$$F_{\alpha\beta ab} = V_{\alpha A} V_{\beta B}^A + V_{\alpha B} V_{\beta A}^A + \Omega_{abcd} V_{\alpha dA} V_{\beta c}^A ,$$

where $\Omega_{abcd}$ is totally symmetric in its indices [86].

In order to describe the gauging of a subgroup of the isometry group, we denote the gauge fields of this group by $A_{\mu}^i$, where $i$ takes values in the adjoint representation, and the corresponding field-strengths are

$$F^{i}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^{ijk} A^j_\mu A^k_\nu ,$$

where $f^{ijk}$ are the structure constants of the gauge group. Under the gauge transformation

$$\delta A^i_\mu = D_\mu \Lambda^i$$

the scalars transform as

$$\delta \phi^\alpha = \Lambda^i \xi^{\alpha i} ,$$

where $\xi^{\alpha i}$ are the Killing vectors corresponding to the isometries that we are gauging. The covariant derivative for the scalars is then

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - A^i_\mu \xi^{\alpha i} .$$

One can correspondingly define the covariant derivatives for the spinors in a natural way, adding the composite connections $D_\mu \phi^\alpha A_\alpha$. For instance, the covariant derivative for the hyperinos $\Psi^a$ will contain the connections $D_\mu \phi^\alpha A_{ab}^\alpha$, while the covariant derivative for the gravitino and the tensorinos will contain the connections $D_\mu \phi^\alpha A_{\alpha B}^A$. The covariant derivatives for the gauginos $\lambda^{iA}$ are

$$D_\mu \lambda^{iA} = \partial_\mu \lambda^{iA} + \frac{1}{4} \omega^{nmn} \gamma^{mn} \lambda^{iA} + D_\mu \phi^\alpha A_{\alpha B}^A \lambda^{iB} + f^{ijk} A^j_\mu \lambda^{kA} .$$

Notice that the gravitino, the tensorinos and the hyperinos are not coupled to the gauge vectors through terms that do not contain the hyper-scalars.
We now proceed to the construction of the model. We assume that the gauge group has the form $G = \prod G_z$, with $G_z$ semi-simple. The scalars in the hypermultiplets are charged with respect to $G_1$. To lowest order in the Fermi fields, we reproduce the construction of Section 1, adding the hypermultiplet couplings. The equations for all fields, with the exception of the 2-forms, can be obtained from the lagrangian

$$
e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{12} G_{\tau\mu\nu\rho} H_{\mu\nu}^s \psi_{\tau} - \frac{1}{4} \partial_\mu v^\tau \partial^\mu v_\tau - \frac{1}{2} v_\tau c^{\tau z} v_\tau (F_{\mu\nu} F_{\mu\nu}^*)$$

$$- \frac{1}{8 e} \varepsilon^{\mu
u\rho\sigma} B_{\mu\nu} ^* c^{\tau z} v_\tau (F_{\rho\sigma} F_{\sigma \tau}) + \frac{1}{2} g_{\alpha\beta} (\phi) D_\mu \phi^\alpha D^\mu \phi^\beta + \frac{1}{4 v_\tau c^{\tau z}} A_\alpha^{A} B_\beta^{A} \xi^{\alpha\iota} \xi^{\beta j}$$

$$- \frac{i}{2} (\bar{\psi}_\mu \gamma^{\mu\nu} D_\nu \psi_\mu) - \frac{i}{2} v_\tau H_{\mu\nu\rho} (\bar{\psi}_\mu \gamma^{\nu} \gamma^{\mu} \chi^M) + \frac{i}{2} (\bar{\chi}^M \gamma^{\mu} D_\mu \chi^M)$$

$$- \frac{i}{24} v_\tau H_{\mu\nu\rho} (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) + \frac{1}{2} v_\tau \partial_\mu v^\tau (\bar{\psi}_\mu \gamma^{\nu} \gamma^{\mu} \chi^M) - \frac{1}{2} x_\tau^M H_{\mu\nu\rho} (\bar{\psi}_\mu \gamma^{\nu\rho} \chi^M)$$

$$+ \frac{i}{2} (\bar{\Psi}_a \gamma^{\mu} D_\mu \Psi^a) + \frac{i}{24} v_\tau H_{\mu\nu\rho} (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) - V_\alpha^A D_\nu \phi^\alpha (\bar{\psi}_\mu \gamma^{\mu} \psi_\mu)$$

$$+ i v_\tau c^{\tau z} v_\tau (\bar{\chi}^M \gamma^{\mu} D_\mu \lambda) + \frac{i}{\sqrt{2}} v_\tau c^{\tau z} v_\tau (F_{\mu\nu} (\bar{\psi}_\mu \gamma^{\mu} \gamma^{\nu} \gamma^{\mu} \chi^M))$$

$$+ \frac{1}{\sqrt{2}} x_\tau^M c^{\tau z} v_\tau (F_{\mu\nu} (\bar{\chi}^M \gamma^{\mu\nu} \chi^M)) - \frac{i}{12} c^{\tau z} v_\tau (\bar{\chi}^M \gamma^{\mu\nu} \lambda)$$

$$- \sqrt{2} V_\alpha^A \xi^{\alpha\iota} (\bar{\lambda}_A \gamma^{\mu} \Psi^a) + \frac{1}{\sqrt{2}} A_\alpha^{A B} \left[ i \xi^{\alpha\iota} (\bar{\lambda}_A \gamma^{\mu} \psi^a_B) + \frac{x_\tau^M c^{\tau z} \gamma^{\mu\nu\rho}}{v_\tau c^{\tau z} \gamma^{\mu\nu\rho}} \xi^{\alpha\iota} (\bar{\lambda}_A \gamma^{\mu\nu\rho} \chi^M) \right],$$

(4.178) after imposing the (anti)self-duality conditions. With this prescription, its variation under the supersymmetry transformations

$$\delta e_\mu^m = -i (\varepsilon^{m\nu} \psi_\mu),$$

$$\delta B_{\mu\nu}^r = i v^r (\bar{\psi}_{\mu \nu} \varepsilon) + \frac{1}{2} x_\tau^M (\bar{\chi}_M \gamma^{\mu\nu} \varepsilon) - 2 c^{\tau z} v_\tau (A_{\mu \nu} \delta A_{\mu \nu})$$

$$\delta v_\tau = x_\tau^M (\bar{\varepsilon}_M),$$

$$\delta \phi^\alpha = V_\alpha^A (\bar{\epsilon}^A \Psi^a),$$

$$\delta A_\mu = -\frac{i}{\sqrt{2}} (\bar{\varepsilon} \gamma_\mu \lambda),$$

$$\delta \psi_\mu^A = D_\mu \epsilon^A + \frac{1}{4} v_\tau H_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon^A,$$

$$\delta \chi_\mu^M A = \frac{i}{2} x_\tau^M \partial_\mu v^r \gamma^{\nu} \epsilon^A + \frac{i}{12} x_\tau^M H_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon^A,$$

$$\delta \Psi^a = i \gamma^{\mu} \epsilon^A \chi_\mu^M A \delta \phi^\alpha,$$

$$\delta \chi^A = -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon^A (z \neq 1).$$
\[ \delta \lambda^i = -\frac{1}{2\sqrt{2}} F^i_{\mu \nu} \gamma^{\mu \nu} \epsilon^A - \frac{1}{\sqrt{2} v_c c^1} A^A_{\alpha B} \xi^{\alpha i} \epsilon^B \]  

(4.179)

gives the supersymmetry anomaly

\[ A_\epsilon = -\frac{1}{4} \epsilon^{\mu \nu \rho \delta \tau} c_\epsilon^z c^r s^z \epsilon^z \epsilon^r \epsilon^s \epsilon^t \epsilon^q \epsilon^{r s t} \]  

(4.180)

related by the Wess-Zumino conditions to the consistent gauge anomaly

\[ A_A = -\frac{1}{4} \epsilon^{\mu \nu \rho \delta \tau} c^r c^{s t} \epsilon^z \epsilon^r \epsilon^s \epsilon^t \epsilon^q \epsilon^{r s t} \]  

(4.181)

Notice the presence in the lagrangian of the scalar potential

\[ V(\phi) = -\frac{1}{4 v_c c^1} A^A_{\alpha B} A^B_{\beta A} \xi^{\alpha i} \xi^{\beta i} \]  

(4.182)

As in rather more conventional gauged models, the potential contains interesting informations, and it may be very instructive to study its extrema in special cases.

We now want to extend the results to all orders in the Fermi fields. First of all, we define the supercovariant quantities

\[ \hat{\omega}_{\mu \nu} = \omega_{\mu \nu} - \frac{i}{2} (\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\mu \gamma_\nu \psi_\rho) \]  

\[ \hat{H}_{\mu \nu}^r = H_{\mu \nu}^r - \frac{1}{2} x^{r m} (\bar{X}^M \gamma_{\mu \nu} \psi_\rho + \bar{X}^M \gamma_\nu \gamma_\mu \psi_\rho + \bar{X}^M \gamma_\mu \gamma_\nu \psi_\rho) - \frac{i}{2} v^r (\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\mu \psi_\rho + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \]  

\[ \partial_\mu \hat{v}^r = \partial_\mu v^r - x^{r m} (\bar{X}^M \psi_\mu) \]  

\[ D_\mu \hat{\phi}^\alpha = D_\mu \phi^\alpha - V_\alpha^A (\bar{\psi}_\mu \Psi^A) \]  

\[ \hat{F}_{\mu \nu} = F_{\mu \nu} + \frac{i}{\sqrt{2}} (\bar{\lambda} \gamma_\mu \psi_\nu) - \frac{i}{\sqrt{2}} (\bar{\lambda} \gamma_\nu \psi_\mu) \]  

(4.183)

and require that the transformation rules for the Fermi fields be supercovariant.

All fermionic terms in the supersymmetry transformations of the Fermi fields that are not determined by supercovariance are then obtained requiring the closure of the supersymmetry algebra on Bose and Fermi fields. Moreover, since the supersymmetry algebra on the Fermi fields closes only on-shell, in this way one can determine the complete fermionic field equations, and from these the complete lagrangian, up to some subtleties related to the (anti)self-dual forms, that will be described in section 4.
4.5 Inclusion of hypermultiplets

The complete supersymmetry transformations of the Fermi fields are

$$
\delta \psi^A_\mu = D_\mu (\hat{\omega}) \epsilon^A + \frac{1}{4} v_r H^r_{\mu \rho \sigma} \gamma^{\rho \sigma} \epsilon^A - \frac{3i}{8} \gamma^\mu \chi^{MA} (\tilde{\epsilon}^A \chi^M) - \frac{i}{8} \gamma^\nu \chi^{MA} (\tilde{\epsilon}^A \gamma^\nu \chi^M) + \frac{i}{16} \gamma_{\mu \nu \rho} \chi^{MA} (\tilde{\epsilon}^A \gamma^{\mu \nu} \chi^M) - \frac{9i}{8} v_r c^{rz} \text{tr}_z [\lambda^A (\tilde{\epsilon}^A \gamma^r \lambda)] + \frac{i}{8} v_r c^{rz} \text{tr}_z [\gamma_{\mu \nu} \chi^{A} (\tilde{\epsilon}^A \gamma^r \lambda)] - \frac{i}{16} v_r c^{rz} \text{tr}_z [\gamma_{\mu \nu} \chi^{A} (\tilde{\epsilon}^A \gamma^r \lambda)] - \delta \phi^\alpha A^A_{\alpha B} \psi^B_\mu ,
$$

$$
\delta \chi^{MA} = \frac{i}{2} x^r_\mu (\partial_\mu \hat{\omega}) \gamma^r \epsilon^A + \frac{i}{12} x^r_\mu \hat{H}^r_{\mu \rho \sigma} \gamma^{\rho \sigma} \epsilon^A + \frac{1}{x^r_\mu c^{rz}} \text{tr}_z [\gamma_{\mu} \chi^{A} (\tilde{\epsilon}^A \gamma^r \lambda)] - \delta \phi^\alpha A^A_{\alpha B} \lambda^{MB} ,
$$

$$
\delta \Psi^a = i \gamma^\mu \epsilon^A V^a A D_\mu \hat{\omega} - \delta \phi^\alpha A^A_{\alpha \beta} \Psi^b ,
$$

$$
\delta \lambda^A = - \frac{1}{2 \sqrt{2}} \hat{F}^i_\mu \gamma^{i \nu} \epsilon^A - \frac{x^r M c^{rz}}{2 v_s c^{8 s}} (\tilde{\chi}^M \lambda) \epsilon^A - \frac{x^r M c^{rz}}{4 v_s c^{8 s}} (\chi^M \epsilon) \lambda^A + \frac{x^r M c^{rz}}{8 v_s c^{8 s}} (\tilde{\chi}^M \gamma_{\mu \nu \epsilon}) \gamma^{\mu \nu} \lambda^A - \delta \phi^\alpha A^A_{\alpha B} \lambda^B (z \neq 1) ,
$$

$$
\delta \lambda^{IA} = - \frac{1}{2 \sqrt{2}} \hat{F}^i_\mu \gamma^{i \nu} \epsilon^A - \frac{x^r M c^{1 \nu}}{2 v_s c^{8 s}} (\tilde{\chi}^M \lambda) \epsilon^A - \frac{x^r M c^{1 \nu}}{4 v_s c^{8 s}} (\chi^M \epsilon) \lambda^{IA} + \frac{x^r M c^{1 \nu}}{8 v_s c^{8 s}} (\tilde{\chi}^M \gamma_{\mu \nu \epsilon}) \gamma^{\mu \nu} \lambda^{IA} - \delta \phi^\alpha A^A_{\alpha B} \lambda^{IB} - \frac{1}{\sqrt{2} x^r c^{1 \nu}} A^A_{\alpha B} \xi \epsilon^B .
$$

One can compute the commutators of two supersymmetry transformations on the Bose fields using these relations, and show that they generate the local symmetries:

$$
[\delta_1, \delta_2] = \delta_{\text{gen}} + \delta_{\text{Lorentz}} + \delta_{\text{susy}} + \delta_{\text{tens}} + \delta_{\text{gauge}} + \delta_{\text{SO(n)}} ,
$$

where the parameters of generic coordinate, local Lorentz, supersymmetry, tensor gauge, vector gauge and composite $SO(n)$ transformations are respectively

$$
\xi^\mu = - i (\tilde{\epsilon}_1 \gamma^\mu \epsilon_2) ,
$$

$$
\Omega^{mn} = - i \xi^\mu (\tilde{\omega}^m_\mu \gamma^n - v_r \hat{H}^{r mn}) - \frac{1}{2} [(\tilde{\chi}^M \epsilon_1) (\epsilon_2 \gamma^{mn} \chi^M) - (\tilde{\chi}^M \epsilon_2) (\epsilon_1 \gamma^{mn} \chi^M)] - v_r c^{rz} \text{tr}_z [((\tilde{\chi}^M \epsilon_1) (\epsilon_2 \gamma^{mn} \lambda) - (\tilde{\chi}^M \epsilon_2) (\epsilon_1 \gamma^{mn} \lambda))] ,
$$

$$
\zeta^A = \xi^\mu A^A_{\mu \nu} + V_{\alpha c} A^A_{\alpha B} \epsilon_{2 B} (\tilde{\epsilon}_1 \Psi^a) - V_{\alpha c} A^A_{\alpha B} \epsilon_{1 A} (\tilde{\epsilon}_2 \Psi^a) ,
$$

$$
\lambda^r = - \frac{1}{2} v_r \xi^\mu - \xi^\nu B^r_\nu ,
$$

$$
\Lambda = \xi^\mu A_\mu ,
$$

$$
A^{MN} = \xi^\mu x^{Mr} (\partial_\mu x^N) + (\tilde{\chi}^M \epsilon_2) (\tilde{\chi}^N \epsilon_1) - (\tilde{\chi}^M \epsilon_1) (\tilde{\chi}^N \epsilon_2) .
$$

In order to prove this result, one has to use the (anti)self-duality condition for the tensor fields, that to all orders in the Fermi fields is

$$
G_{rs} \hat{H}^{s \rho}_{\mu \nu} = \frac{1}{6 \epsilon} \delta_{\mu \rho \sigma \nu} \hat{H}^{s \delta \tau} .
$$
in terms of the 3-forms $[73]$

\[ \hat{H}_{\mu
u} = \hat{H}_{\mu
u} - \frac{i}{8} \hat{H}_{\mu
u} \bar{\psi}_a \gamma_{\mu\nu} \chi^a + \frac{i}{8} \hat{H}_{\mu\nu} \bar{\psi}_a \gamma_{\mu\nu} \gamma^5 \chi^a + \frac{i}{4} \hat{c}^{\alpha z} \text{tr}_z (\bar{\chi}_r \gamma_{\mu\nu} \lambda) \ . \]  

(4.188)

Requiring that the commutator of two supersymmetry transformations on the Fermi fields close on-shell then determines the complete Fermi field equations. The equations obtained in this way are

\[ -i \gamma^\mu \gamma^\beta D_\nu (\bar{\omega}) \psi^A_\rho - \frac{i}{4} v_r \hat{H}_{\nu\alpha} \gamma_{\mu\nu} \gamma^\alpha \gamma^\beta \psi^A_\rho - \frac{1}{12} x_r \hat{H}_{\nu\rho} \gamma_\nu \gamma_\rho \gamma^\mu \chi^{MA} \\
+ \frac{1}{2} x_r (\partial_\nu \bar{\omega}) \gamma^\mu \gamma^\rho \chi^{MA} + \frac{3}{2} \gamma_{\mu\nu} \chi^{MA} (\bar{\chi}_M \psi_\nu) - \frac{1}{4} \gamma_{\mu\nu} \chi^{MA} (\bar{\chi}_M \gamma_{\mu\nu} \psi_\rho) \\
+ \frac{1}{4} \gamma_{\mu\nu} \chi^{MA} (\bar{\chi}_M \gamma_{\mu\nu} \psi_\rho) - \frac{1}{2} \chi^M (\bar{\chi}_M \gamma_{\mu\nu} \psi_\nu) - i v_r c^{r z} \text{tr}_z [\frac{1}{\sqrt{2}} \gamma_{\mu\nu} \gamma^\mu \chi^A] \\
+ \frac{3i}{4} \gamma_{\mu\nu} \chi^A (\bar{\psi}_\nu \gamma_{\mu\nu} \lambda) - \frac{i}{2} \gamma_{\mu\nu} \chi^A (\bar{\psi}_\nu \gamma_{\mu\nu} \lambda) + \frac{i}{2} \gamma_{\mu\nu} \chi^A (\bar{\psi}_\nu \gamma_{\mu\nu} \lambda) + \frac{i}{4} \gamma_{\mu\nu} \lambda (\bar{\psi}_\nu \gamma_{\mu\nu} \lambda) \\
- \frac{i}{2} x_r c^{r z} \text{tr}_z [\gamma_{\nu \rho \lambda} (\bar{\chi}_M \gamma^\nu \lambda)] - V^a_{\alpha} D_\nu \phi^a \gamma_\nu \gamma^\mu \psi_a \\
+ \frac{i}{2} A^a \chi^A \lambda^B = 0 \]  

(4.189)

for the gravitino,

\[ i \gamma^\mu D_\mu (\bar{\omega}) \chi^{MA} - \frac{i}{12} v_r \hat{H}_{\mu\rho} \gamma_{\mu\rho} \chi^{MA} + \frac{1}{12} x_r \hat{H}_{\mu\rho} \gamma_{\mu\rho} \psi^A_\rho + \frac{1}{2} x_r (\partial_\nu \bar{\omega}) \gamma^\mu \gamma^\nu \chi^A \\
+ \frac{1}{2} x_r c^{r z} \text{tr}_z (\bar{F}_{\mu\nu} \gamma_{\mu\nu} \lambda^A) - \frac{i}{2} x_r c^{r z} \text{tr}_z [\gamma_{\nu \rho \lambda} (\bar{\psi}_\nu \gamma_{\mu\nu} \lambda)] + \frac{1}{2} \gamma_{\nu \rho \lambda} (\bar{\psi}_\nu \gamma_{\mu\rho} \lambda) \\
- \frac{3}{8} v_r c^{r z} \text{tr}_z [(\bar{\chi}_M \gamma_{\mu\nu} \lambda) \gamma_{\mu\nu} \lambda^A] - \frac{1}{4} v_r c^{r z} \text{tr}_z [(\bar{\chi}_M \lambda) \lambda^A] \\
- \frac{3}{2} x_r c^{r z} x^N_{M} c_{sz} \text{tr}_z [(\bar{\chi}_N \gamma_{\mu\nu} \lambda) \gamma_{\mu\nu} \lambda^A] + \frac{1}{4} x_r c^{r z} x^N_{M} c_{sz} \text{tr}_z [(\bar{\chi}_N \gamma_{\mu\nu} \lambda) \gamma_{\mu\nu} \lambda^A] \\
- \frac{x_r c^{r z}}{\sqrt{2} v_{sz}} A^A_{\alpha} B^A_{\alpha \lambda^B} = 0 \]  

(4.190)

for the tensorinos,and

\[ i \gamma^\mu D_\mu (\bar{\omega}) \Psi^a + \frac{i}{12} v_r \hat{H}_{\mu\rho} \gamma_{\mu\rho} \Psi^a + \gamma_{\mu \beta} F_{\mu\nu} \gamma^\nu \Psi_a + \gamma_{\mu \beta} \gamma^\nu \gamma_\nu \chi^A \phi^a - \frac{1}{48} v_r c^{r z} \text{tr}_z (\bar{\chi}_{\mu\rho} \lambda) \gamma_{\mu\rho} \Psi^a \\
+ \frac{1}{12} \Omega_{abcd} \chi^a \Psi_b (\bar{\chi}_c \gamma_{\mu\nu} \Psi_d) + \sqrt{2} V^a_{\alpha} \xi^A_{\alpha i} \lambda^A_i = 0 \]  

(4.191)

for the hyperinos. As usual, more care is needed in order to derive the equations for the gauginos, since the $c^{r z} c^{r z'}$ terms in the commutator of two supersymmetry
transformations are
\[ c^z c^{z'} \frac{v_s c^{sz}}{v_s c^{sz}} \text{tr}_{z'} \left[ -\frac{1}{4} (\bar{\epsilon}_1 \gamma_\mu \lambda')(\bar{\epsilon}_2 \gamma_\nu \lambda') \gamma^{\mu\nu} \lambda^A + \frac{1}{4} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\mu \lambda') \epsilon^2_2 - (1 \leftrightarrow 2) \right. \\
+ \left. \frac{1}{16} (\bar{\epsilon}_1 \gamma_\mu \epsilon_2) (\bar{\lambda} \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \lambda^A \right]. \] (4.192)

If one allows for the term
\[ \alpha c^z c^{z'} \text{tr}_{z'} [(\bar{\lambda} \gamma_\mu \lambda') \gamma^{\mu} \lambda^A] \] (4.193)
in the gaugino field equation, then what remains of eq. (4.192) is
\[ \delta_{\text{extra}(\alpha)} \lambda^A = c^z c^{z'} \frac{v_s c^{sz}}{v_s c^{sz}} \text{tr}_{z'} \left[ -\frac{1}{4} (\bar{\epsilon}_1 \gamma_\mu \lambda')(\bar{\epsilon}_2 \gamma_\nu \lambda') \gamma^{\mu\nu} \lambda^A \\
+ \frac{\alpha}{2} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\mu \lambda') \gamma^{\mu} \epsilon^2_2 + \frac{\alpha}{16} (\bar{\lambda} \gamma_{\mu\nu} \lambda') (\bar{\epsilon}_1 \gamma_\rho \lambda') \gamma^{\mu\nu} \epsilon^2_2 \\
+ \frac{\alpha}{16} (\bar{\lambda} \gamma_\mu \lambda')(\bar{\epsilon}_1 \gamma^{\mu\nu} \lambda') \gamma_{\mu\nu} \epsilon^2_2 + \frac{1 - \alpha}{4} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\mu \lambda') \epsilon^2_2 - (1 \leftrightarrow 2) \\
+ \frac{1 - \alpha}{16} (\bar{\epsilon}_1 \gamma_\rho \epsilon_2) (\bar{\lambda} \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \lambda^A \right]. \] (4.194)

As explained in Section (4.1), no choice of \( \alpha \) can eliminate all these terms, that play the role of a central charge felt only by the gauginos. This is the “classical” realization of a general feature: anomalies in current conservations are accompanied by related anomalies in current commutators \[ \hat{c} \]. When this is properly taken into account, the field equations for the gauginos are
\[ i v_r c^{sz} \gamma_\mu D_\mu (\hat{\omega}) \lambda^A + \frac{i}{2} (\hat{\partial}_\mu v_r) c^{sz} \gamma_\mu \lambda^A + \frac{i}{2} v_r c^{sz} \hat{F}_\mu \gamma^{\mu\nu} \psi^A \\
- \frac{1}{2\sqrt{2}} x_r^M c^{sz} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \chi^A + \frac{i}{12} x_r^M c^{sz} x_s^M \hat{H}^s_{\mu\nu} \gamma^{\mu\nu} \lambda^A + \frac{i}{2} x_r^M c^{sz} (\bar{\chi}^M \gamma_\mu \lambda') \gamma^{\mu} \psi^A \\
+ \frac{i}{4} x_r^M c^{sz} (\bar{\chi}^M \gamma_\mu \lambda') \gamma^{\mu} \lambda^A - \frac{i}{8} x_r^M c^{sz} (\bar{\chi}^M \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \lambda^A - \frac{i}{4} x_r^M c^{sz} (\bar{\chi}^M \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \chi^A \\
- \frac{1}{8} x_r^M c^{sz} x_s^N c^{sz} (\bar{\lambda} \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \chi^A - \frac{3}{16} x_r^M c^{sz} x_s^N c^{sz} \lambda^A - \frac{3}{4} x_r^M c^{sz} x_s^N c^{sz} (\bar{\chi}^M \gamma_{\mu\nu} \psi^A) \gamma^{\mu\nu} \chi^A \\
+ \frac{1}{8} x_r^M c^{sz} x_s^N c^{sz} (\bar{\lambda} \gamma_{\mu\nu} \lambda') \gamma^{\mu\nu} \chi^A - \frac{1}{96} (\bar{\Psi}_a \gamma_{\mu\nu} \Psi^a) \gamma^{\mu\nu} \chi^A. \] (4.195)

Actually to the left-hand side of this equation, valid for the case \( z \neq 1 \), one has to add the terms
\[ -\sqrt{2} v_a \xi^a \xi B^a + \frac{i}{\sqrt{2}} A_{\alpha B} \xi^a \gamma_\mu \psi^A + \frac{x_r^M c^1}{2 v_a c^1} A_{\alpha B} \xi^a \chi^A \] (4.196)
in the remaining case, i.e. for $\lambda^i$.

Having obtained the complete fermionic field equations, one can add to eq. (4.178) all the terms quartic in the Fermi fields, thus obtaining the complete lagrangian

$$e^{-1}{\cal L} = -\frac{1}{4} R + \frac{1}{12} G_{\mu
u\rho\sigma} H_{\mu
u}^{\rho\sigma} - \frac{1}{4} \partial_{\mu} u^r \partial^\mu v_r + \frac{1}{2} g_{\alpha\beta}(\phi) D_\mu \phi^\alpha D^\mu \phi^\beta$$

$$- \frac{1}{2} v_r c^{rz} \text{tr}_z (F_{\mu\rho} F^{\mu\sigma}) - \frac{1}{8 e} \epsilon_{\mu\rho\sigma\delta} c^r B_{\mu\nu} \text{tr}_z (F_{\rho\sigma} F_{\delta\tau}) + \frac{1}{4 v_r c^{a1}} A^A_B A^A_B A^A_{\alpha\beta} L_i^A \chi_{\beta}$$

$$- \frac{i}{2} (\bar{\psi}_\mu \gamma_{\mu\rho\sigma} D_\nu [\frac{1}{2} (\omega + \bar{\omega})] \psi_\rho) - \frac{i}{8 v_r} [H + \tilde{H}] \gamma_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma_\rho \psi_\sigma)$$

$$+ \frac{i}{48} v_r [H + \tilde{H}] \gamma_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma_\rho \gamma_{\mu\nu} \psi_\nu) + \frac{i}{2} (\bar{\chi}_\mu \gamma_\nu \chi_\mu)$$

$$- \frac{i}{24} x_r^M [H + \tilde{H}] \gamma_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma_\rho \gamma_{\mu\nu} \chi_\nu) + \frac{i}{24} x_r^M [H + \tilde{H}] \gamma_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma_\rho \gamma_{\mu\nu} \chi_\nu)$$

$$+ \frac{i}{2} (\bar{\Psi}_a \gamma^\mu D_\mu (\omega) \phi^a) + \frac{i}{2} v_r \tilde{H}_{\rho\mu\nu} (\bar{\Psi}_a \gamma_{\mu\nu} \psi^a)$$

$$- \frac{1}{2} v_r c^{rz} \text{tr}_z (\bar{\lambda} \gamma_\mu D_\mu (\omega) \lambda) + \frac{i}{12} v_r c^{rz} \text{tr}_z ([F + \tilde{F}]_{\nu\rho\sigma} (\bar{\psi}_\mu \gamma_{\mu\rho\sigma} \lambda)) + \frac{1}{2} c^{rz} \text{tr}_z [\bar{\lambda} \gamma^\mu \lambda]$$

$$- \frac{\sqrt{2} v_r c^{rz} \text{tr}_z [(F + \tilde{F})_{\nu\rho\sigma} (\bar{\psi}_\mu \gamma_{\mu\rho\sigma} \lambda))] + \sqrt{2} v_r c^{rz} \text{tr}_z [\bar{\lambda} \gamma^\mu \lambda]$$

$$- \frac{1}{8} (\bar{\chi}_\mu \gamma_{\mu\nu} \chi_\nu) - \frac{1}{8} (\bar{\chi}_\mu \gamma_{\mu\nu} \chi_\nu)$$

$$+ \frac{1}{8} (\bar{\psi}_a \gamma^\mu \psi^a) + \frac{1}{8} \Omega_{abcd} (\bar{\psi}_a \gamma^\mu \psi^b) (\bar{\psi}_c \gamma^\mu \psi^d)$$

From this lagrangian, in the 1.5 order formalism and using the (anti)self-duality con-
4.5 Inclusion of hypermultiplets

ditions of eqs. (4.187) and (4.188), one can obtain the remaining complete bosonic field equations. Once more, it is important to notice that this lagrangian in neither gauge invariant nor supersymmetric: its variation under gauge transformations produces the gauge anomaly of eq. (4.181), while its variation under the complete supersymmetry transformations produces the complete supersymmetry anomaly

\[ A_\alpha = e^{2z_s} tr_{z_s} \left\{ -\frac{1}{4} \epsilon^{\mu \nu \rho \sigma} \delta \epsilon A_\mu A_\nu F_{\rho \sigma} F_{\delta \tau} - \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \delta \epsilon A_\mu F_{\rho \sigma} \right\} \]

The presence of a term proportional to the parameter \( \alpha \) in eq. (4.197) reflects the general fact that anomalies are defined up to the variation of a local functional. Gauge and supersymmetry anomalies are in general related by the Wess-Zumino consistency conditions [67]

\[ A_\alpha = A_\alpha + A_\alpha \]

So the inclusion of hypermultiplets does not alter the peculiarity of these six dimensional models: the second condition closes only on-shell, and precisely on the gaugino field equations [69]. Since the inclusion of the term proportional to \( \alpha \) in the lagrangian modifies both these equations and the supersymmetry anomaly, there must be some extra terms that permit the Wess-Zumino conditions to close on-shell for every value of \( \alpha \). This is precisely the role of the terms in eq. (4.194) in the commutator of two supersymmetry transformations on the gauginos, that thus can be seen as a transformation needed in order to close the Wess-Zumino conditions precisely on the field equations determined by the algebra. Since the Wess-Zumino conditions need only the equation of the gauginos, only these fields sense the additional transformation.

For completion, we observe that the PST method can be naturally applied here
following the results of Section (4.3): our theory describes a single self-dual 3-form
\[ \mathcal{H}_{\mu
u} = v_\nu \mathcal{H}^\nu_{\mu\rho} - \frac{i}{8} (\tilde{X}^M \gamma_{\mu\nu\rho} \lambda^M) + \frac{i}{8} (\bar{\Psi}_a \gamma_{\mu\nu\rho} \psi^a) \] (4.200)
and \( n_T \) antiself-dual 3-forms
\[ \mathcal{H}^M_{\mu\nu\rho} = x^M_\nu \mathcal{H}^\nu_{\mu\rho} + \frac{i}{4} x^M_\rho \epsilon^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \] . (4.201)
The complete Lagrangian is obtained adding to eq. (4.197) the term
\[ - \frac{\partial^\mu \Xi \partial^\sigma \Xi}{4(\partial \Xi)^2} \left[ \mathcal{H}^{-\mu\nu\rho} \mathcal{H}^{-\nu\rho} + \mathcal{H}^{M+\mu\nu\rho} \mathcal{H}^{M+\nu\rho} \right] , \] (4.202)
where \( \Xi \) is an auxiliary field and \( H^\pm = H \pm *H \). The resulting lagrangian is invariant under the additional gauge transformations
\[ \delta B^r_{\mu\nu} = (\partial_{\mu} \Xi) \Lambda^r_{\nu} - (\partial_{\nu} \Xi) \Lambda^r_{\mu} \] (4.203)
and
\[ \delta \Xi = \Lambda , \quad \delta B^r_{\mu\nu} = \frac{\Lambda}{(\partial \Xi)^2} \left[ \partial^r \mathcal{H}^{-\mu\nu\rho} - x^M_\nu \mathcal{H}^{M+\nu\rho} \right] \partial^\rho \Xi \] , (4.204)
used to recover the usual field equations for (anti)self-dual forms. The 3-form
\[ \mathcal{K}_{\mu\nu\rho} = \mathcal{H}_{\mu\nu\rho} - 3 \frac{\partial_{\nu} \Xi \partial^\sigma \Xi}{(\partial \Xi)^2} \mathcal{H}^{-\nu\rho}_{\sigma} \] (4.205)
is identically self-dual, while the 3-forms
\[ \mathcal{K}^M_{\mu\nu\rho} = \mathcal{H}^M_{\mu\nu\rho} - 3 \frac{\partial_{\nu} \Xi \partial^\sigma \Xi}{(\partial \Xi)^2} \mathcal{H}^{M+\nu\rho}_{\sigma} \] (4.206)
are identically antiself-dual \[83\]. In order to obtain the complete supersymmetry transformations, we have to substitute \( \mathcal{H} \) with \( \hat{K} \) in the transformation of the gravitino and \( \mathcal{H}^M \) with \( \hat{K}^M \) in the transformations of the tensorinos. Moreover, the auxiliary scalar is invariant under supersymmetry \[83, 13\]. It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions. The commutator of two supersymmetry transformations on \( B^r_{\mu\nu} \) now generates the local PST transformations with parameters
\[ \Lambda_{r\mu} = \frac{\partial^\rho \Xi}{(\partial \Xi)^2} \left( \partial_{\nu} \mathcal{H}^{-\nu\rho}_{\sigma} - x^M_\nu \mathcal{H}^{M+\nu\rho}_{\sigma} \right) \xi^\nu \] , \[ \Lambda = \xi^\mu \partial_{\mu} \Xi \] , (4.207)
while in the parameter of the local Lorentz transformation the term \( \mathcal{H} \) is replaced by \( \hat{K} \). All other parameters remain unchanged.
4.5 Inclusion of hypermultiplets

Following the results of Section (4.4), one can also generalize these results to the case in which abelian vectors are present. We will then consider the gauging with respect to abelian subgroups of the isometry group. There are no subtleties when the symmetric matrices $c_{rI}^I$ are diagonal (or simultaneously diagonalizable), since in this situation the previous results can be straightforwardly applied. We are thus interested in the case in which the $c_{rI}^I$ can not be simultaneously diagonalized. To this end, we will consider a model in which only these abelian gauge groups are present. The most general situation can be obtained combining the following results with those obtained previously.

We denote with $A_I^I, I = 1, \ldots, m$, the set of abelian vectors, and the gauginos are correspondingly denoted by $\lambda^I$. We collect here only the final results, since the construction follows the same lines as in the non-abelian case. All the field equations may then be derived from the lagrangian

$$e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{12} G_{rs} H^{\tau \rho \mu \nu} H_{\mu \nu}^s - \frac{1}{4} \partial_{\rho} v^r \partial^\rho v_r$$

$$- \frac{1}{4} v_r c^{rI} F^I_{\mu \nu} F^{I \mu \nu} - \frac{1}{16} \epsilon_{\mu \nu \rho \delta} c_{rI}^{IJ} B_{\mu \nu} F^r_{\rho \delta}$$

$$+ \frac{1}{2} g_{\alpha \beta} (\phi) D_{\mu} \phi^\alpha D_{\nu} \phi^\beta + \frac{1}{4} [(v \cdot c)^{-1}]^{I J} A^A_{\alpha B} A^B_{\beta A} \xi^I \xi^J$$

$$- \frac{i}{2} (\bar{\psi}_\mu \gamma^{\mu \nu \rho \delta} D_{\nu} [\omega + \hat{\omega}] \psi_\rho) - \frac{i}{8} v_r [H + \hat{H}]^{\rho \mu \nu \sigma} (\bar{\psi}_\mu \gamma^{\rho \sigma} \gamma^{\nu \lambda} \lambda^M)$$

$$+ \frac{i}{8} v_r \hat{H}_{\mu \nu} (\bar{\psi}_\mu \gamma^{\mu \nu \rho \delta} \lambda^M) + \frac{1}{4} x^M_{\rho \sigma} (\bar{\psi}_\mu \gamma^{\rho \sigma} \gamma^{\nu \lambda} \lambda^M)$$

$$- \frac{1}{2} \bar{\psi}_a \gamma^\mu D_{\mu} (\hat{\omega}) \Psi^a + \frac{i}{24} v_r \hat{H}_{\mu \nu} (\bar{\psi}_a \gamma^{\mu \nu \rho \delta} \lambda^M)$$

$$- \frac{1}{2} \bar{\psi}_a \gamma^\mu D_{\mu} (\hat{\omega}) \Psi^a$$

$$- \frac{1}{2} \bar{\psi}_a (D_{\rho} \phi^\rho + \hat{\phi}^\rho \phi^\rho)(\bar{\psi}_a \gamma^{\mu \nu \rho \delta} \lambda^M)$$

$$+ \frac{i}{24} v_r c^{rI} (\bar{\lambda}^I \gamma^M D_{\mu} (\hat{\omega}) \lambda^I) + \frac{i}{24} x^M_{\mu \nu} \hat{H}^{\mu \nu} (\bar{\lambda}^I \gamma^M \lambda^I)$$

$$+ \frac{i}{4 \sqrt{2}} v_r c^{rI} (F + \hat{F})^{I \mu \nu} (\bar{\psi}_\mu \gamma^{\rho \sigma} \gamma^\nu \lambda^J) + \frac{1}{2 \sqrt{2}} x^M_{\mu \nu} c^{IJ} (\bar{\lambda}^I \gamma^M \gamma^\nu \lambda^J)$$

$$- \frac{1}{2} \sqrt{2} A^A_{\alpha B} \xi^a (\bar{\lambda}^I \Psi^a) + \frac{i}{2} A^A_{\alpha B} \xi^a (\bar{\lambda}^I \gamma^M \psi^a)$$

$$+ \frac{1}{\sqrt{2}} [(v \cdot c)^{-1}(x^M \cdot c)]^{I J} A^A_{\alpha B} \xi^I (\bar{\lambda}^J \lambda^M \psi^B)$$
+ \frac{1}{8}(\tilde{\chi}^M \gamma^\rho \chi^M)(\tilde{\psi}_\mu \gamma_\rho \psi_\rho) - \frac{1}{8}(\tilde{\chi}^M \gamma^\nu \chi^N) - \frac{1}{8}(\tilde{\chi}^M \gamma^\nu \chi^N) \\
+ \frac{1}{8}(\tilde{\psi}_a \gamma^\mu \tilde{\psi}_a)(\tilde{\psi}_\mu \gamma_\nu \psi_\rho) + \frac{1}{48} \Omega^{abcd}(\tilde{\psi}_a \gamma_\mu \tilde{\psi}_b)(\tilde{\psi}_c \gamma_\mu \psi_d) \\
+ \frac{1}{32} v_r c^{IJ}(\tilde{\chi}^I \gamma_\mu \chi^J)(\tilde{\chi}^M \gamma^\nu \chi^M) - \frac{i}{16}(\tilde{\chi}^M \gamma_\mu \psi_\mu) x_r^M c^{IJ}(\tilde{\chi}^I \gamma_\mu \chi^J) \\
- \frac{i}{4} x_r^M c^{IJ}(\tilde{\chi}^M \gamma_\mu \chi^J)(\tilde{\psi}_\mu \gamma_\nu \psi_\rho) + \frac{1}{8}(\tilde{\psi}_\mu \gamma_\nu \psi_\rho) v_r c^{IJ}(\tilde{\chi}^I \gamma_\mu \chi^J) \\
- \frac{1}{16} v_r c^{IJ}(\tilde{\chi}^M \chi^I)(\tilde{\chi}^M \chi^J) - \frac{3}{32} v_r c^{IJ}(\tilde{\chi}^M \gamma_\mu \chi^J) - \frac{3}{32} v_r c^{IJ}(\tilde{\chi}^M \gamma_\mu \chi^J) \\
+ [(x^M \cdot c)(v \cdot c)^{-1}(x^N \cdot c)]I^J[-\frac{1}{4}(\tilde{\chi}^M \chi^I)(\tilde{\chi}^N \chi^J) \\
+ \frac{1}{16}(\tilde{\chi}^N \gamma_\mu \chi^I)(\tilde{\chi}^M \gamma_\mu \chi^J) - \frac{1}{8}(\tilde{\chi}^N \chi^I)(\tilde{\chi}^M \chi^J)] \\
+ \frac{5}{192} v_r c^{IJ}(\tilde{\chi}^I \gamma_\mu \chi^J)(\tilde{\psi}_a \gamma_\mu \psi^a) - \frac{1}{8} v_r v_s c^{KL}(\tilde{\chi}^I \gamma_\mu \chi^K)(\tilde{\chi}^J \gamma_\mu \chi^L) \\
+ \frac{\alpha}{8} c^{IJ} c^{KL}(\tilde{\chi}^I \gamma_\mu \chi^K)(\tilde{\chi}^J \gamma_\mu \chi^L)] . \tag{4.208}

The variation of this lagrangian with respect to gauge transformations gives the abelian gauge anomaly

$$A_\lambda = -\frac{1}{32} \varepsilon^{\mu \rho \sigma \tau} e^{IJ} c^{KL} A_I^{\mu} F_{\rho \sigma}^{J} F_{\delta \tau}^{L} , \tag{4.209}$$

while its variation with respect to the supersymmetry transformations

$$\delta e_\mu^m = -i(\bar{\epsilon} \gamma_\mu \psi_\mu) ,$$

$$\delta B_{\mu \nu} = iv_r(\bar{\psi}_\mu \gamma_\nu \psi_\rho) + \frac{1}{2} \varepsilon^M(\tilde{\chi}^M \gamma_{\mu \nu} \epsilon) + 2 e^{IJ} A_{\mu}^I \delta A_{\nu}^J ,$$

$$\delta v_r = x_r^M(\bar{\epsilon} \chi^M) , \quad \delta x_r^M = v_r(\bar{\epsilon} \chi^M) ,$$

$$\delta \phi^a = V_{aA}(\bar{e}^A \psi^a) ,$$

$$\delta A_\mu^I = -i \frac{1}{\sqrt{2}}(\bar{\epsilon} \gamma_\mu \lambda^I) ,$$

$$\delta \psi_a^A = D_\mu(\bar{\omega}) e^A + \frac{i}{4} v_r \hat{H}_{\mu \rho}^I \chi_\mu \chi^A - \frac{3i}{8} \gamma_\mu \chi^M(\bar{\epsilon} \chi^M) - \frac{i}{8} \gamma_\mu \chi^M(\bar{\epsilon} \chi^M) \\
+ \frac{i}{16} \gamma_{\mu \nu} \chi^M(\bar{e} \gamma_\nu \chi^M) - \frac{9i}{16} v_r c^{IJ} \lambda^I A(\bar{\epsilon} \gamma_\mu \lambda^J) + \frac{i}{16} v_r c^{IJ} \gamma_\mu \lambda^IA(\bar{\epsilon} \gamma^J) \\
- \frac{i}{32} v_r c^{IJ} \gamma_\mu \lambda^I(\bar{\epsilon} \gamma_\mu \lambda^J) - \delta \phi^a A_{aB}^A \psi_\mu^B ,$$

$$\delta \chi^M = \bar{\chi}^M(\partial_\mu v^\rho) \gamma_\mu e^A + \frac{i}{12} \bar{\chi}^M(\hat{H}_{\mu \rho}^I \gamma_\mu \psi_\rho) e^A \\
+ \frac{1}{4} x_r^M c^{IJ} \gamma_\mu \lambda^IA(\bar{\epsilon} \gamma_\mu \lambda^J) - \delta \phi^a A_{aB}^A \chi^M ,$$

$$\delta \psi^a = i \gamma_\mu \epsilon A_{aA} D_\mu \phi^a - \delta \phi^a A_{aB}^a \Psi^a .$$
\[
\delta \lambda^{IA} = -\frac{1}{2\sqrt{2}} \dot{F}^{IJ}_{\mu} \gamma^{I\mu} \epsilon^{A} + [(v \cdot c)^{-1}(x^{M} \cdot c)]^{IJ} \left[-\frac{1}{2} (\chi^{M} \lambda^{J}) \epsilon^{A} - \frac{1}{4} (\chi^{M} \epsilon) \lambda^{JA} \right]
+ \frac{1}{8} (\chi^{M} \gamma^{I\mu} \epsilon) \gamma^{\mu\nu} \lambda^{JA} - \delta \phi^{A} \alpha_{\alpha}^{A} \lambda^{JB} - \frac{1}{\sqrt{2}} [(v \cdot c)^{-1}]^{IJ} A_{\alpha}^{A} \xi^{\alpha} \epsilon^{B} \quad \text{(4.210)}
\]
gives the supersymmetry anomaly
\[
A_{\epsilon} = c_{\epsilon}^{I} c^{KL} \left\{ -\frac{1}{16} \epsilon^{\mu\nu} \phi \delta_{\epsilon} A_{\mu}^{J} \gamma^{I} F^{K}_{\mu} F^{L} \delta_{\epsilon} + \frac{i e}{8} \delta_{\epsilon} A_{\mu}^{I} F_{\mu \nu} (\bar{\lambda}^{K} \gamma^{I} \gamma^{\mu} \lambda^{K}) F^{L} + \frac{i e}{2} \delta_{\epsilon} A_{\mu}^{I} (\bar{\lambda}^{J} \gamma_{\mu} \lambda^{K}) F^{IJ} \right. \\
+ \frac{e}{128} \delta_{\epsilon} e^{m} (\bar{\lambda}^{L} \gamma^{I} \gamma^{\mu} \lambda^{J}) (\bar{\lambda}^{K} \gamma_{m} \lambda^{L}) - \frac{e}{4} \delta_{\epsilon} A_{\mu}^{I} (\bar{\lambda}^{J} \gamma_{\mu} \gamma^{I} \lambda^{K})(\bar{\lambda}^{L} \gamma_{\nu} \psi_{\nu}) \left. \right\}
+ i c_{\epsilon}^{I} (v \cdot c)^{-1} (x^{M} \cdot c) c^{KL} \left\{ -\frac{i}{4 \sqrt{2}} \delta_{\epsilon} A_{\mu}^{I} (\bar{\lambda}^{J} \gamma_{\mu} \lambda^{K})(\bar{\lambda}^{M} \lambda^{L}) \right.
+ \frac{i}{16 \sqrt{2}} \delta_{\epsilon} A_{\mu}^{I} (\bar{\lambda}^{J} \gamma_{\mu} \lambda_{L})(\bar{\lambda}^{M} \gamma_{\nu} \lambda^{K}) - \frac{i}{4 \sqrt{2}} \delta_{\epsilon} A_{\mu}^{I} (\bar{\lambda}^{J} \gamma_{\mu} \lambda_{L})(\bar{\lambda}^{M} \lambda^{K}) \left. \right\}
- \frac{i e}{4} c_{\epsilon}^{I} (v \cdot c)^{-1} c^{KL} \delta_{\epsilon} A_{\mu}^{I} A_{\alpha}^{B} \xi^{\alpha} \delta_{\epsilon} (\bar{\lambda}^{J} \gamma_{L} \lambda^{K})(\bar{\lambda}^{M} \gamma^{I} \lambda^{L}) \right. \\
+ \frac{\alpha}{8} c_{\epsilon}^{I} c^{KL} \delta_{\epsilon} (\bar{\lambda}^{I} \gamma_{L} \lambda^{K})(\bar{\lambda}^{J} \gamma^{I} \lambda^{L}) \left. \right\}. \quad \text{(4.211)}
\]

Once again, in the case of the gauginos, aside from local symmetry transformations and field equations, the commutator of two supersymmetry transformations generates the additional two-cocycle
\[
\delta_{(\alpha)} \lambda^{I} = [(v \cdot c)^{-1} c_{\epsilon}^{IJ}] c^{KL} \left\{ -\frac{1}{8} (\bar{\epsilon}_{1} \gamma_{I} \lambda^{K})(\bar{\epsilon}_{2} \gamma_{L} \lambda^{L}) \gamma^{\mu \nu} \lambda^{I} - \frac{\alpha}{4} (\bar{\lambda}^{I} \gamma_{I} \lambda^{K})(\bar{\epsilon}_{1} \gamma_{L} \lambda^{L}) \gamma^{\mu \nu} \epsilon_{2} \right. \\
+ \frac{\alpha}{32} (\bar{\lambda}^{J} \gamma_{L} \lambda^{K})(\bar{\epsilon}_{1} \gamma^{I} \lambda^{L}) \gamma^{\mu \nu} \epsilon_{2} + \frac{\alpha}{32} (\bar{\lambda}^{I} \gamma_{I} \lambda^{K})(\bar{\epsilon}_{1} \gamma^{I} \lambda^{L}) \gamma^{\mu \nu} \epsilon_{2} \\
+ \frac{1 - \alpha}{8} (\bar{\lambda}^{I} \gamma_{I} \lambda^{K})(\bar{\epsilon}_{1} \gamma_{L} \lambda^{L}) \epsilon_{2} - (1 \leftrightarrow 2) \\
+ \frac{1 - \alpha}{32} (\bar{\epsilon}_{1} \gamma^{I} \lambda^{L})(\bar{\lambda}^{K} \gamma_{I} \lambda^{L}) \gamma^{\mu \nu} \lambda^{I} \right. \left. \quad \text{(4.212)}
\right. \\
\]

All the observations made for the non-abelian case are naturally valid also here: the theory is obtained by the requirement that the Wess-Zumino conditions close on-shell, and, as we have already shown, it is determined up to an arbitrary quartic coupling for the gauginos. In the case of a single vector multiplet, in which this quartic coupling vanishes, the two-cocycle of eq. \textbf{(4.212)} is still present, although it is properly independent of \(\alpha\). The tensionless string phase transition point in the moduli space of the scalars in the tensor multiplets now would correspond to the vanishing of some of the eigenvalues of the matrix \((v \cdot c)^{IJ}\) \[71\].
4.6 Discussion

In the previous Sections we have completed the coupling of $(1,0)$ six-dimensional supergravity to tensor, vector and hypermultiplets. The coupling to tensor multiplets only, initiated by Romans [24], is of a more conventional nature, and parallels similar constructions in other supergravity models. On the other hand, the coupling to vector multiplets [54], originally suggested by perturbative type-I vacua [45], is of a more unconventional nature, since it is induced by the residual anomaly polynomial left over after tadpole conditions are imposed,

\[ I_8 = - \sum_{x,y} c_x^r c_y^s \eta_{rs} \text{tr}_x F^2 \text{tr}_y F^2 . \]  

(4.213)

The corresponding Chern-Simons couplings of the two-forms,

\[ H^r = dB^r - c^{rz} \omega_z , \]  

(4.214)

involve the constants $c_x^r$ and determine related couplings of the other fields. In particular, the Yang-Mills currents are not conserved, and the consistent residual gauge anomaly is accompanied by a corresponding anomaly in the supersymmetry current [68]. In completing these results to all orders in the Fermi fields, we have come to terms with another peculiar feature of anomalies, neatly displayed by these “classical” field equations: anomalous divergences of gauge currents are typically accompanied by corresponding anomalies in current commutators [78]. Indeed, we have discovered an amusing extension of the supersymmetry algebra on the gaugini, and we have linked its presence to an ambiguity in the definition of the supergravity model via Wess-Zumino consistency conditions. Whereas typical supergravity constructions yield a unique result, here one is free to add to the theory a quartic coupling for the gaugini

\[ \mathcal{L}_\lambda^4 = \frac{e \alpha}{2} c^r c^{r'} \text{tr}_{z,z'} [(\tilde{\lambda} \gamma^\alpha \lambda') (\tilde{\lambda} \gamma^\alpha \lambda')] \]  

(4.215)

whose presence affects only the supersymmetry anomaly. The Wess-Zumino conditions for six-dimensional supergravity close only on the field equation of the gaugini, and are consistent for any choice of $\alpha$ only thanks to the presence of the extension, as discussed in Section 3.3. Finally, we should mention that the singular gauge couplings $v^r c_x^r$ of ref. [54] are accompanied by corresponding divergent fermionic couplings, while the inclusion of charged hypermultiplets gives an additional contribution to the supersymmetry anomaly.
It would be interesting to study in some detail the vacua of the lagrangian (4.197), analyzing the extrema of the potential (4.182). As a simple example, consider the model without hypermultiplets, in which one can gauge the global R-symmetry group $USp(2)$ of the theory. Formally, the gauged theory without hypermultiplets is obtained from the theory described previously putting $n_H = 0$ and making the identification

$$A^A_{\alpha B} \xi^{\alpha i} \rightarrow -i T^{i\alpha A}_B ,$$

where $T^i$ are the hermitian generators of $USp(2)$. This corresponds to the replacement of the previous couplings between gauge fields and spinors, dressed by the scalars in the case $n_H \neq 0$, with ordinary minimal couplings:

$$D_\mu \phi^\alpha A^A_{\alpha B} \epsilon^B \rightarrow i A^A_{\mu B} \epsilon^B .$$

Implementing this identification gives in this case the positive-definite potential

$$V = \frac{3}{8v_r c^1}$$

for the scalars in the tensor multiplets. One would thus expect that in these models supersymmetry be spontaneously broken. Notice that this potential diverges at the tensionless string phase transition point. Similarly, one could try to study explicitly the behavior of the potential in simple models containing charged hypermultiplets. Their dimensional reduction gives N=2 supergravity coupled to vector and hypermultiplets in five dimensions, and in the context of the AdS/CFT correspondence and its generalizations [87] there is a renewed interest in studying the explicit gauging of these five-dimensional models (see, for instance, [88] and references therein). Notice that in five dimensions the anomaly that results from the dimensional reduction of our model can be canceled by a local counterterm, and thus the low-energy effective action does not present the subtleties of the six-dimensional case [89].

The couplings we have derived here are the most general couplings of $(1,0)$ six-dimensional supergravity to vector, tensor and hypermultiplets. One may wonder if one had the option to gauge a subgroup of $SO(1, n_T)$, the isometry group of the scalars in the tensor multiplets. Of course, we do not know how to write a gauge covariant field-strength for interacting antisymmetric tensor fields, but there is a more direct reason why this gauging is not expected to work, namely the fact that once we couple vector and tensor multiplets, the $SO(1, n_T)$ transformations are no longer global symmetries of the theory, because of the presence of the matrices $c^r$. 

4.6 Discussion

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Chapter 5

Low-energy actions for brane supersymmetry breaking

In Chapter 3 we have shown that orientifold vacua \[44, 89\] allow the simultaneous presence of supersymmetric bulks, with one or more gravitinos, and non-supersymmetric combinations of BPS branes. The resulting “brane supersymmetry breaking” can be realized in stable configurations, in ten dimensions with only anti-$D9$ ($\bar{D}9$) branes \[48\], and in six and four dimensions, up to T-dualities, with tachyon-free combinations of $D9$ branes and anti-$D5$ ($\bar{D}5$) branes \[49\]. Since this is only one of the options offered by this class of models for the breaking of supersymmetry, a fundamental issue in attempting to relate string theory to low-energy physics, it is instructive to briefly review our current knowledge in this respect.

In perturbative string vacua, one has actually four options for supersymmetry breaking. The first is to break supersymmetry from the start, so that no gravitinos are present, and the resulting models, descendants of the type-0 models of [20], have in general tachyonic modes [13], although a special Klein-bottle projection, suggested by the WZW constructions of [50], leads to the $0'B$ model [21], that is free of tachyons, a property shared by its compactifications [70] and neatly rooted in its brane content [91]. The second option is the Scherk-Schwarz mechanism [92], in which the breaking, induced by deformed harmonic expansions in the internal space, is at the compactification scale. In this setting, widely studied in the context of models of oriented closed strings [13], the presence of branes allows the new option of correlating
the Scherk-Schwarz deformations to the brane geometry, giving rise, in particular, to the phenomenon of “brane supersymmetry”, whereby one or more residual global supersymmetries are left, to lowest order, for the brane modes [4]. The third option, magnetic deformations [95], resorts to the different magnetic moments of the various fields to induce supersymmetry breaking [96], again at the compactification scale, but the resulting vacua, that have also T-dual descriptions in terms of branes at angles [97], generally contain tachyons [98], aside from some special instanton-like stable configurations that recover supersymmetry, albeit with gauge groups of reduced rank [99]. Finally, one has the option of brane supersymmetry breaking [48, 49], made possible by the presence of two types of $O$-planes. Together with the conventional $O^-$, with negative tension and negative R-R charge, there are indeed additional BPS objects, the $O^+$ planes, with positive tension and positive R-R charge, and while the two can coexist in supersymmetric Klein-bottle projections, the saturation of the $O^+$ charge requires the presence of anti-branes, with the result that supersymmetry is broken on the latter at the string scale. It is the rigidity of the breaking scale, together with some special features of the resulting low-energy effective field theories, that typically do not allow a gravitino mass term, that makes the explicit construction of the goldstino couplings quite interesting in this case.

Dudas and Mourad [100] have shown that, in the simplest model with brane supersymmetry breaking, the $USp(32)$ model of [48], the low-energy gravitino couplings reflect a non-linear realization of local supersymmetry à la Volkov-Akulov, along the lines of [101], and their work is the starting point for our considerations. Let us stress that, while all branes, including the supersymmetric ones, result in the non-linear realization à la Born-Infeld of the supersymmetries broken by their presence, here one arrives at a complete breaking, and the peculiarity with respect to lower-dimensional settings for the super-Higgs mechanism is the absence of a gravitino mass term. This feature is common to the case analyzed in [100] and to the lower-dimensional models of [49], that we shall also discuss in this Chapter [104]. Actually, all these configurations, even the supersymmetric ones, can accommodate additional brane-antibrane pairs of identical dimensions, that are to be spatially separated in order to lift the resulting tachyons. These additional pairs provide in their own right additional ways to realize brane supersymmetry breaking, but have clearly potential tachyon instabilities [102] for their geometric moduli, in view of the mutual attraction of identical branes and antibranes. We shall thus confine our attention to the “minimal” configurations of [48, 49] demanded by tadpole cancellation, although the other pairs could
be described along similar lines. Still, we should mention that non-minimal brane-antibrane configurations are also quite interesting, and are currently the object of a considerable activity as a string setting for brane-world extensions of the Standard Model \[98, 103\].

All models with brane supersymmetry breaking contain a candidate goldstino among their brane modes, and in \[100\] Dudas and Mourad indeed constructed the low-energy couplings of the goldstino for the ten-dimensional $USp(32)$ Sugimoto model of \[48\] up to quartic fermionic terms. These were all shown to be of a geometric nature, being induced by the dressing of bulk fields with additional terms depending on the goldstino in all their couplings to the non-supersymmetric brane matter, aside from some Wess-Zumino-like terms resulting from the supersymmetrization of the Chern-Simons couplings. The geometric nature of the dressing implies that non-linear supersymmetries of the matter sector take the form of gaugino-dependent general coordinate transformations. In this chapter, following the results of \[104\], we extend the work of \[100\], showing that, up to quartic fermionic terms, the whole low energy effective Lagrangian of the Sugimoto model, including the Chern-Simons couplings, has a geometric nature when expressed in terms of the dual 6-form gauge field, rather than of the more familiar 2-form. The starting point in this case is thus the low-energy supergravity built long ago by Chamseddine \[16\], rather than the model of \[14, 15\]. The ten-dimensional Chern-Simons terms become in this way higher derivative couplings that, as such, do not appear in the low-energy effective action, while a Wess-Zumino term must be added, and this can be simply “geometrized” dressing the six-form along the lines of \[100\]. We also extend the analysis of the low-energy effective action to six dimensional models with brane supersymmetry breaking. The starting point in this case is provided by the low-energy (1,0) effective actions of \[54, 58, 59, 70, 73, 74\]. These, however, include both Wess-Zumino and Chern-Simons couplings for the gauge fields, and as a result have the subtle feature of embodying reducible gauge anomalies to be canceled by fermion loops. This peculiar feature, not present in the earlier constructions of \[73\] motivated by perturbative heterotic strings, links these constructions to the Wess-Zumino conditions for the anomalies, with the end result that many familiar properties of current algebra find in this case an explicit local realization. In order to write a covariant action for the resulting (anti)self-dual 3-forms, we shall resort to the method of Pasti-Sorokin-Tonin \[12\]. The remaining couplings are determined requiring that supersymmetry be non-linearly realized as in the ten-dimensional case, but the simultaneous presence of Chern-Simons and Wess-
Zumino terms produces a novel effect. Indeed, while the action is still determined by the underlying geometrical structure, as is often the case with Wess-Zumino terms, only the field equations are geometrical in this case, aside from anomalous terms that arise in the presence of vectors from both supersymmetric and non-supersymmetric sectors.

The chapter is organized as follows. In Section 1 we review the low-energy effective couplings built by Dudas and Mourad \cite{100} for the ten-dimensional model and exhibit their geometric nature in terms of the 6-form potential. Section 2 is devoted to the six-dimensional non-linear realizations, and finally, Section 3 contains a discussion of the results.

### 5.1 Low-energy couplings for the Sugimoto model

This section builds on \cite{100}, where the low-energy effective action for the $USp(32)$ model was constructed, to lowest order in the Fermi fields, requiring that supersymmetry be non-linearly realized on the $D9$ branes, and thus obtaining consistent couplings for the gravitino. Our aim is to show how all the couplings of \cite{100} can be written in a geometric form \cite{104}.

Let us briefly review how a single spinor can be treated à la Volkov-Akulov \cite{101} as a goldstino of global supersymmetry. Let us restrict our attention to the ten dimensional case, considering a Majorana-Weyl fermion $\theta$ with the supersymmetry transformation

$$
\delta \theta = \epsilon - \frac{i}{2} (\bar{\epsilon} \gamma^\mu \theta) \partial_\mu \theta.
$$

(5.1)
The commutator of two such transformations is a translation,

$$[\delta_1, \delta_2] \theta = -i(\epsilon_2 \gamma^\mu \epsilon_1) \partial_\mu \theta \quad ,$$  

(5.2)

and thus eq. (5.1) provides a realization of supersymmetry. In order to write a Lagrangian for \( \theta \) invariant under eq. (5.1), let us define the 1-form

$$e_\mu^m = \delta_\mu^m - \frac{i}{2}(\bar{\theta} \gamma^m \partial_\mu \theta) \quad ,$$  

(5.3)

whose supersymmetry transformation is

$$\delta e^m = -L_\xi e^m \quad ,$$  

(5.4)

with \( L_\xi \) the Lie derivative with respect to

$$\xi_\mu = -\frac{i}{2}(\bar{\theta} \gamma^\mu \epsilon) \quad .$$  

(5.5)

The action of supersymmetry on \( e \) is thus a general coordinate transformation, with a parameter depending on \( \theta \), and therefore

$$\mathcal{L} = -\det e$$  

(5.6)

is clearly an invariant Lagrangian. Expanding the determinant, one can see that the energy has a positive vacuum expectation value, and supersymmetry is thus spontaneously broken. Using the same technique, for a generic field \( A \) that transforms under supersymmetry as

$$\delta A = -L_\xi A \quad ,$$  

(5.7)

defining the induced metric as \( g_{\mu\nu} = e_\mu^m e_{\nu m} \), a supersymmetric Lagrangian in flat space is determined by the substitution

$$\mathcal{L}(\eta, A) \rightarrow e\mathcal{L}(g, A) \quad .$$  

(5.8)

We can now review the results of [100], and to this end we begin by considering the Lagrangian for the closed sector

$$e^{-1} \mathcal{L}_{\text{closed}} = -\frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{6} e^{-2\phi} H_{\mu\nu\rho} H^{\mu\nu\rho}$$  

$$- \frac{i}{2}(\bar{\psi} \gamma^{\mu\nu\rho} D_\nu \psi) + \frac{i}{2}(\bar{\chi} \gamma^\mu D_\mu \chi) + \frac{1}{\sqrt{2}}(\bar{\psi} \gamma^\mu \gamma^\mu \chi) \partial_\nu \phi$$  

$$- \frac{i}{12 \sqrt{2}} e^{-\phi} H_{\mu\nu\rho}(\bar{\psi} \gamma^{\sigma\delta\mu\nu\rho} \psi) + \frac{i}{2 \sqrt{2}} e^{-\phi} H_{\mu\nu\rho}(\bar{\psi} \gamma^\nu \psi^\rho)$$  

$$+ \frac{1}{12} e^{-\phi} H_{\mu\nu\rho}(\bar{\psi} \gamma^{\mu\nu\rho} \gamma^\sigma \chi) \quad .$$  

(5.9)
that provides a linear realization of the minimal (1,0) ten-dimensional supersymmetry, and is thus invariant under the local supersymmetry transformations

$$
\delta e_\mu^m = -i(\bar{e}\gamma^m \psi_\mu),
$$
$$
\delta B_{\mu\nu} = -\frac{i}{\sqrt{2}} e^\phi (\bar{e} \gamma_{[\mu} \psi_{\nu]}),
$$
$$
\delta \phi = -\frac{1}{\sqrt{2}} (\bar{e} \chi),
$$
$$
\delta \psi_\mu = D_\mu \epsilon + \frac{1}{24\sqrt{2}} e^{-\phi} H^{\mu\rho\sigma} \gamma_{\mu\rho\sigma} \epsilon - \frac{3}{8\sqrt{2}} e^{-\phi} H_{\mu\rho\sigma} \gamma^{\mu\rho} \epsilon,
$$
$$
\delta \chi = -\frac{i}{\sqrt{2}} \partial_\mu \phi \gamma^\mu \epsilon - \frac{i}{12} e^{-\phi} H^{\mu\rho\sigma} \gamma_{\mu\rho\sigma} \epsilon.
$$

(5.10)

In the supersymmetric case, when these bulk modes are coupled to a gauge multiplet supported on the 9-branes and containing a vector $A_\mu$ and a left-handed gaugino $\lambda$ both in the adjoint representation of $SO(32)$, supersymmetry requires that the 3-form $H_{\mu\rho\sigma}$ include a Chern-Simons coupling, so that

$$
H_{\mu\rho\sigma} = 3 \partial_{[\mu} B_{\nu\rho]} + \sqrt{2} \omega_{\mu\rho\sigma},
$$

(5.11)

where $\omega_{\mu\rho\sigma}$ is the Chern-Simons 3-form defined as

$$
\omega = AdA - \frac{2i}{3} A^3,
$$

(5.12)

and this leads to the modified Bianchi identity

$$
\partial_{[\mu} H_{\nu\rho\sigma]} = \frac{3}{\sqrt{2}} \text{tr}(F_{[\mu\nu} F_{\rho\sigma]}).
$$

(5.13)

The Lagrangian for supergravity coupled to vector multiplets is then [14, 13] (see Section (1.4))

$$
e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{6} e^{-2\phi} H_{\mu\rho\sigma} H^{\mu\rho\sigma} - \frac{1}{2} e^{-\phi} \text{tr}(F_{\mu\nu} F^{\mu\nu})
$$
$$
- i \frac{1}{2} (\bar{\psi}_\mu \gamma^{\mu\rho\sigma} D_\nu \psi_\rho) + i \frac{1}{2} (\bar{\chi} \gamma^\mu D_\mu \chi) + \frac{1}{\sqrt{2}} (\bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi) \partial_{\nu} \phi
$$
$$
- \frac{i}{12\sqrt{2}} e^{-\phi} H_{\mu\rho\sigma} (\bar{\psi}_\sigma \gamma^{\rho\mu\sigma} \psi_\delta) + \frac{i}{2\sqrt{2}} e^{-\phi} H_{\mu\rho\sigma} (\bar{\psi}_\mu \gamma^\nu \psi^\rho)
$$
$$
+ \frac{1}{12} e^{-\phi} H_{\mu\rho\sigma} (\bar{\psi}_\sigma \gamma^{\mu\rho\sigma} \gamma^\mu \chi) + i \text{tr}(\bar{\lambda} \gamma^\mu D_\mu \lambda)
$$
$$
- \frac{1}{2} e^{-\frac{1}{2} \phi} \text{tr}[F^{\mu\nu} (\bar{\lambda} \gamma_{\mu\nu} \chi)] + \frac{i}{\sqrt{2}} e^{-\frac{1}{2} \phi} \text{tr}[F^{\mu\nu} (\bar{\lambda} \gamma_{\rho} \gamma_{\mu\nu} \psi^\rho)]
$$
$$
- \frac{i}{6\sqrt{2}} e^{-\phi} H^{\mu\rho\sigma} \text{tr}(\bar{\lambda} \gamma_{\mu\rho} \lambda),
$$

(5.14)
up to quartic terms in the fermions. The supersymmetry transformations of the bulk fields $e^m_\mu$, $\phi$, $\psi_\mu$ and $\chi$ are as before, while for the gauge multiplet

$$\delta A_\mu = - \frac{i}{\sqrt{2}} e^{\frac{i}{2} \phi} \bar{\epsilon} \gamma_\mu \lambda ,$$
$$\delta \lambda = - \frac{1}{2 \sqrt{2}} e^{-\frac{i}{2} \phi} F^{\mu \nu} \gamma_{\mu \nu} \epsilon . \quad (5.15)$$

Gauge invariance of $H$ requires that under vector gauge transformations $B$ transform as

$$\delta B = - \sqrt{2} \text{tr} (A dA) , \quad (5.16)$$
and in order that gauge and supersymmetry transformations commute, up to a tensor gauge transformation, one has to add a term to the supersymmetry variation of $B_{\mu \nu}$, obtaining

$$\delta B_{\mu \nu} = - \frac{i}{\sqrt{2}} e^{\phi} \bar{\epsilon} \gamma_{\mu \nu} \psi_\eta \right) - \frac{1}{24 \sqrt{2}} e^{\phi} (\bar{\epsilon} \gamma_{\mu \nu} \chi) + 2 \sqrt{2} \text{tr} (A_\mu \delta A_\nu) . \quad (5.17)$$

In order to couple the Lagrangian (5.9) to non-supersymmetric matter, one must construct from the fields of the supergravity multiplet quantities whose supersymmetry variations are general coordinate transformations with the parameter $\xi_\mu$ of eq. (5.5). We thus define

$$\hat{\phi} = \phi + \frac{1}{\sqrt{2}} (\bar{\theta} \chi) + \frac{i}{24 \sqrt{2}} e^{\phi} (\bar{\gamma}_{\mu \nu \rho} \theta) H^{\mu \nu \rho} , \quad (5.18)$$
so that

$$\delta \hat{\phi} = - \xi^\mu \partial_\mu \hat{\phi} = \delta_{\text{get}} \hat{\phi} \quad (5.19)$$
and

$$\hat{e}^m_\mu = e^m_\mu + i (\bar{\theta} \gamma^m \psi_\mu) - \frac{i}{2} (\bar{\theta} \gamma^m D_\mu \theta) - \frac{i}{48 \sqrt{2}} e^m_\mu e^{-\phi} (\bar{\gamma}_{\mu \nu \rho} \theta) H^{\mu \nu \rho}$$
$$+ \frac{i}{16 \sqrt{2}} e^{-\phi} (\bar{\gamma}_{\mu \nu \rho} \theta) H^{\mu \nu \rho} + \frac{3i}{16 \sqrt{2}} e^{-\phi} (\bar{\gamma}^{\mu \nu \rho} \theta) H_{\mu \nu \rho} \quad (5.20)$$

so that

$$\delta \hat{e}^m_\mu = \delta_{\text{get}} \hat{e}^m_\mu + \Lambda^m_n \hat{e}^n_\mu , \quad (5.21)$$
where the parameter of the local Lorentz transformation is

$$\Lambda^{mn} = \frac{i}{2} (\bar{\theta} \gamma^\rho \epsilon) \omega^{mn}_\rho + \frac{i}{24 \sqrt{2}} e^{-\phi} (\bar{\gamma}^{\mu \nu \rho \sigma} \epsilon) H_{\nu \rho \sigma} + \frac{3i}{4 \sqrt{2}} e^{-\phi} (\bar{\gamma}^\rho \epsilon) H^{\rho mn} . \quad (5.22)$$
In constructing a Lagrangian invariant under non-linear supersymmetry that couples supergravity to non-supersymmetric matter, it is important to notice that eq. (5.11) still holds, because of anomaly cancellation. For the same reason, the variation of $B$ is still given by eq. (5.17), once one uses the new transformation for $A_\mu$,

$$\delta A_\mu = F_{\mu\nu}\xi^\nu \quad .$$

(5.23)

Observe that this covariant expression for $\delta A_\mu$ contains the proper coordinate transformation, together with an additional gauge transformation of parameter

$$\Lambda = \xi^\mu A_\mu \quad .$$

(5.24)

The supersymmetry transformation of the spinor $\lambda$ in the 495 of $USp(32)$ will not be taken into account in this discussion, since it contains higher-order Fermi terms. One can now include the kinetic term for $A_\mu$ and the dilaton tadpole in a Lagrangian that is supersymmetric up to terms quartic in the fermions, considering

$$\mathcal{L} = \mathcal{L}_{\text{closed}} - \frac{1}{2} \partial^\alpha \hat{\theta}^{\alpha\beta} \hat{\gamma}^{\mu\nu} \hat{\gamma}^{\rho\sigma} \text{tr}(F_{\mu\nu} F_{\rho\sigma}) - \Lambda \hat{\theta} \hat{\phi}$$

$$+ \frac{i}{6\sqrt{2}} \text{tr}(\bar{\lambda} \gamma^\mu D_\mu \lambda) - \frac{i}{6\sqrt{2}} \text{tr}(\bar{\lambda} \gamma^\mu D_\mu \lambda) \quad ,$$

(5.25)

where in the Sugimoto model $\Lambda = 64 T_9$, with $T_9$ the anti-brane tension. By string considerations, one can show that the coefficient of the coupling of $H$ to $\lambda^2$, not constrained by supersymmetry at this level, is the same as in the supersymmetric case. Actually, the Lagrangian of eq. (5.25) is still not invariant under supersymmetry, since the inclusion of the Chern-Simons term and the consequent modification of the Bianchi identity for $H$ generate contributions proportional to $F \wedge F$ in the variation of $\mathcal{L}_{\text{closed}}$. Up to higher order fermionic terms, however, these are exactly canceled by the variation of the additional terms

$$\frac{1}{6!\sqrt{2}} \epsilon^{\mu_1 \ldots \mu_6 \nu \rho \sigma} \left[ \frac{3i}{\sqrt{2}} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_5} \psi_{\mu_6}) - 1/4 e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} \chi) \right]$$

$$+ \frac{i}{8\sqrt{2}} e^{-2\phi} \partial_\tau (\bar{\theta} \gamma_{\mu_1 \ldots \mu_5} \gamma^\tau \theta) - \frac{3i}{2\sqrt{2}} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} D_\mu \theta)$$

$$- \frac{3i}{8} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_5} \gamma^\tau \theta) H_{\mu_6} \gamma^\tau \delta - \frac{5i}{2} e^{-2\phi} (\bar{\theta} \gamma_{\mu_1 \mu_2 \mu_3} \theta) H_{\mu_4 \mu_5 \mu_6} \text{tr}(F_{\mu\nu} F_{\rho\sigma}) \quad .$$

(5.26)

To summarize, up to quartic fermionic terms the Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{closed}} - \frac{1}{2} \partial^\alpha \hat{\theta} \hat{\gamma}^{\mu\nu} \hat{\gamma}^{\rho\sigma} \text{tr}(F_{\mu\nu} F_{\rho\sigma}) - \Lambda \hat{\theta} \hat{\phi}$$
5.1 Low-energy couplings for the Sugimoto model

\[ + \frac{i e}{6\sqrt{2}} e^{-\phi} H^{\mu\rho} \text{tr}(\lambda \gamma_5 D_\mu \lambda) \]
\[ + \frac{1}{6!\sqrt{2}} e^{\mu_1...\mu_6 \nu \rho \sigma \tau \delta} e^{-\phi} (\hat{\theta} \gamma_{\mu_1...\mu_5} \psi_{\mu_6}) - \frac{1}{4} e^{-\phi} (\hat{\theta} \gamma_{\mu_1...\mu_6} \chi) \]
\[ + \frac{i}{8\sqrt{2}} e^{-2\phi} \partial_\tau \phi (\hat{\theta} \gamma_{\mu_1...\mu_6} \tau \theta) - \frac{3i}{2\sqrt{2}} e^{-\phi} (\hat{\theta} \gamma_{\mu_1...\mu_5} D_{\mu_6} \theta) \]
\[ - \frac{3i}{8} e^{-2\phi} (\hat{\theta} \gamma_{\mu_1...\mu_5 \tau \delta} \theta) H_{\mu_6} \tau \delta - \frac{5i}{2} e^{-2\phi} (\hat{\theta} \gamma_{\mu_1 \mu_2 \mu_3} \theta) H_{\mu_4 \mu_5 \mu_6} \text{tr}(F_{\mu \nu} F_{\rho \sigma}) \] (5.27)

As noticed in [100], in this formulation a geometric description for the terms in eq. (5.26) is not possible, i.e. it is not possible to rewrite them in terms of properly dressed bulk fields adding fermionic bilinears containing the goldstino. We can now explain why this is the case, and moreover we can also show how a geometric description is possible, after performing a duality transformation to a 6-form gauge field.

Let us again begin with standard results: performing a duality transformation on eq. (5.14), one obtains a new Lagrangian, with a 6-form rather than a 2-form, coupled to vector multiplets [19]. Technically, this is performed starting from the first-order Lagrangian

\[ \mathcal{L} = \frac{1}{6} e^{-2\phi} H_{\mu \nu \rho} H^{\mu \nu \rho} + \frac{1}{3 \cdot 6!} e^{\mu_1...\mu_7} \mu_1 \mu_7 \nu \rho \sigma \tau \delta \partial_\mu_1 \tilde{B}_{\mu_2...\mu_7} H_{\mu_8 \mu_9} \]
\[ + \frac{1}{6!\sqrt{2}} e^{\mu_1...\mu_6 \nu \rho \sigma \tau \delta} \tilde{B}_{\mu_1...\mu_6} \text{tr}(F_{\mu \nu} F_{\rho \sigma}) \] (5.28)

that contains both the 2-form and the 6-form. The field equation for \( \tilde{B}_6 \) is then exactly the Bianchi identity of eq. (5.13) for \( H_3 \), while the field equation for \( H_3 \) is

\[ e^{-\phi} H_3 = e^\phi \ast \tilde{H}_7 \] , (5.29)

where \( \tilde{H}_7 = d\tilde{B}_6 \). The Lagrangian obtained substituting this relation in (5.28) and redefining \( \tilde{H}_7 \rightarrow H_7 \),

\[ \mathcal{L} = \frac{1}{7!} e^{2\phi} H_{\mu_1...\mu_7} H^{\mu_1...\mu_7} + \frac{1}{6!\sqrt{2}} e^{\mu_1...\mu_6 \nu \rho \sigma \tau \delta} B_{\mu_1...\mu_6} \text{tr}(F_{\mu \nu} F_{\rho \sigma}) \] (5.30)

shows how the Chern-Simons term in \( H_3 \) is replaced by the Wess-Zumino term \( B \wedge F \wedge F \).

If one performs this duality transformation in (5.14), one ends up with the Lagrangian originally obtained by Chamseddine [16]:

\[ e^{-1} \mathcal{L} = - \frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{7!} e^{2\phi} H_{\mu_1...\mu_7} H^{\mu_1...\mu_7} - \frac{1}{2} e^{-\phi} \text{tr}(F_{\mu \nu} F^{\mu \nu}) \]
\[ \begin{align*}
&+ \frac{1}{6! \sqrt{2}} \epsilon_{\mu_1 \ldots \mu_6 \nu \rho \sigma} B_{\mu_1 \ldots \mu_6} \text{tr}(F_{\mu \nu} F_{\rho \sigma}) - \frac{i}{2} (\bar{\psi}_\mu \gamma^{\mu \nu} D_\nu \psi_\mu) \\
&+ \frac{i}{2} (\bar{\lambda} \gamma^\mu D_\mu \chi) + \frac{1}{\sqrt{2}} (\bar{\psi}_\mu \gamma^\mu \chi) \partial_\nu \phi + \frac{i}{240 \sqrt{2}} e^\phi H_{\mu_1 \ldots \mu_7} (\bar{\psi}_{\mu_1} \gamma_{\mu_2 \ldots \mu_6} \psi_{\mu_7}) \\
&+ \frac{i}{2 \cdot 7! \sqrt{2}} e^\phi H_{\mu_1 \ldots \mu_7} (\bar{\psi}_{\mu_1} \gamma_{\mu_2 \ldots \mu_7} \psi_{\nu}) - \frac{1}{2 \cdot 7!} e^\phi H_{\mu_1 \ldots \mu_7} (\bar{\psi}_{\rho} \gamma_{\mu_1 \ldots \mu_7} \gamma_\sigma \chi) \\
&+ \text{itr}(\bar{\lambda} \gamma^\mu D_\mu \lambda) - \frac{1}{2} e^{-\psi} \text{tr}(F^{\mu \nu} (\bar{\lambda} \gamma_\mu \phi)) \\
&+ \frac{i}{\sqrt{2}} e^{-\phi} \text{tr}(F^{\mu \nu} (\bar{\lambda} \gamma^\rho \gamma_\mu \psi_\nu)) + \frac{i}{7! \sqrt{2}} e^\phi H_{\mu_1 \ldots \mu_7} \text{tr}(\bar{\lambda} \gamma_{\mu_1 \ldots \mu_7} \lambda) .
\end{align*} \]

The corresponding supersymmetry transformations are obtained from eq. (5.13) performing the redefinition of eq. (5.29) on the variations of \( \psi_\mu \) and \( \chi \), leaving the variations of \( e_{\mu}^m \), \( \phi \), \( A_\mu \) and \( \lambda \) unaffected and replacing the variation of the 2-form with

\[ \delta B_{\mu_1 \ldots \mu_6} = -\frac{3i}{\sqrt{2}} e^{-\phi} (\bar{\psi}_{\gamma_{\mu_1 \ldots \mu_5}} \psi_{\mu_6}) + \frac{1}{4} e^{-\phi} (\bar{\psi}_{\gamma_{\mu_1 \ldots \mu_6}} \chi) . \]

Notice that the supersymmetry variation of the 6-form does not include a term depending on the vector field. This reflects the fact that the 6-form is inert under gauge transformations, since its field-strength does not contain a Chern-Simons form, that in this case would enter higher-derivative couplings not present in the effective supergravity.

One can now couple the supergravity multiplet expressed in terms of the 6-form to non-supersymmetric matter. In order to do this, together with \( \hat{\phi} \) and \( \hat{e}_{\mu}^m \) of eqs. (5.13) and (5.20), one must define the dressed 6-form

\[ \hat{B}_{\mu_1 \ldots \mu_6} = B_{\mu_1 \ldots \mu_6} + \frac{3i}{\sqrt{2}} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_5} \psi_{\mu_6}) - \frac{1}{4} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} \chi) \\
+ \frac{i}{8 \sqrt{2}} e^{-2\phi} \partial_\rho (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} \gamma^\rho) - \frac{3i}{2 \sqrt{2}} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} D_{\mu_6} \phi) \\
- \frac{3i}{8} e^{-2\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_6} \gamma^\rho) H_{\mu_6} \gamma^\rho - \frac{5i}{2} e^{-2\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_3} \theta) H_{\mu_4 \mu_5 \mu_6} \] ,

whose supersymmetry transformation is a coordinate transformation, up to an additional tensor gauge transformation of parameter

\[ \Lambda_{\mu_1 \ldots \mu_5} = -\frac{i}{4 \sqrt{2}} e^{-\phi} (\bar{\theta} \gamma_{\mu_1 \ldots \mu_5} \epsilon) . \]

We have intentionally written the last line of eq. (5.33) in terms of the dual 3-form, using eq. (5.29), so that the similarity with eq. (5.26) be more transparent. The
Lagrangian for the closed sector,

\[
e^{-1} \tilde{\mathcal{L}}_\text{closed} = - \frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{6} e^{2\phi} H_{\mu_1 \cdots \mu_7} H^{\mu_1 \cdots \mu_7} \\
- \frac{i}{2} (\tilde{\psi}_\mu \gamma^{\mu \rho} D_\rho \psi_\mu) + \frac{i}{2} (\bar{\chi} \gamma^\mu D_\mu \chi) + \frac{1}{\sqrt{2}} (\tilde{\psi}_\mu \gamma^\nu \gamma^\mu \chi) \partial_\nu \phi \\
+ \frac{i}{240 \sqrt{2}} e^\phi H^{\mu_1 \cdots \mu_7} (\tilde{\psi}_{\mu_1} \gamma_{\mu_2 \cdots \mu_6} \psi_{\mu_7}) + \frac{i}{2 \sqrt{27!}} e^\phi H^{\mu_1 \cdots \mu_7} (\tilde{\psi}^{\mu \nu} \gamma_{\mu_1 \cdots \mu_7} \psi^{\nu}) \\
+ \frac{1}{2 \cdot 7!} e^\phi H^{\mu_1 \cdots \mu_7} (\tilde{\psi}^\sigma \gamma_{\mu_1 \cdots \mu_7} \gamma^\sigma \chi)
\]

is simply obtained performing the duality transformation in eq. (5.9), while the same duality in eq. (5.27) gives

\[
\mathcal{L} = \tilde{\mathcal{L}}_\text{closed} - \frac{1}{2} \hat{e} e^{-\hat{\phi}} \hat{g}^{\mu \rho} \hat{g}^{\nu \sigma} \text{tr}(F_{\mu \nu} F_{\rho \sigma}) - \Lambda \hat{e} e^{\frac{1}{2} \hat{\phi}} \\
+ i e \text{tr}(\bar{\lambda} \gamma^\mu D_\mu \lambda) + \frac{i e}{7! \sqrt{2}} e^\phi H^{\mu_1 \cdots \mu_7} \text{tr}(\bar{\lambda} \gamma_{\mu_1 \cdots \mu_7} \lambda) \\
+ \frac{1}{6! \sqrt{2}} e^{\mu_1 \cdots \mu_6 \nu \rho \sigma} \hat{B}_{\mu_1 \cdots \mu_6} \text{tr}(F_{\mu \nu} F_{\rho \sigma})
\]

(5.35)

Note that in this Lagrangian all terms containing the goldstino are grouped in redefinitions of the bulk fields, and therefore all couplings are written in a geometric form.

This result concludes this section: for the ten-dimensional \(USp(32)\) model a fully geometric description is possible if one formulates it in terms of the 6-form, since in this case the Chern-Simons term is higher derivative, and thus is not in the low-energy effective action. More precisely, as we have seen, duality maps the Chern-Simons term into the Wess-Zumino term, and this falls simply into a geometric form. The result is still valid in presence of additional brane-antibrane pairs, since the introduction of supersymmetric vectors does not modify the field strength relative to the 6-form potential in the low-energy effective action. In the dual theory, although the field strength of the 2-form is modified, no additional terms containing the goldstino have to be added to the low-energy lagrangian. As we shall see, this feature is common to the six-dimensional case.

5.2 Geometric couplings in six-dimensional models

In this section we construct the low-energy couplings for six-dimensional type-I models with brane supersymmetry breaking \[49\]. All the features of brane supersymmetry
breaking are present in the $T^4/Z_2$ orientifold of [49], where a change of the orientifold projection leads to $D9$ branes and $D5$ branes. The spectrum has $(1,0)$ supersymmetry in the closed and 9-9 sectors, while supersymmetry is broken in the 9-5 and 5-5 sectors. The gauge group is $SO(16) \times SO(16)$ on the $D9$ branes and $USp(16) \times USp(16)$ on the $D5$ branes, if all the $D5$ branes are at a fixed point.

One of the peculiar features of low-energy effective actions for six-dimensional type-I models with minimal supersymmetry is the fact that they embody reducible gauge and supersymmetry anomalies, to be canceled by fermion loops. Consequently, the Lagrangian is determined imposing the closure of the Wess-Zumino consistency conditions, rather than by the requirement of supersymmetry. We use the notations of the previous chapter, and we denote the vector multiplet from the 9-9 sector as $A^{(9)}_\mu$. Denoting with $\Phi^{\a} (\a = 1, ..., n_T)$ the scalars in the tensor multiplets, parametrizing the coset $SO(1, n_T)/SO(n_T)$, the vielbein $V^{(9)}_\alpha$ of the internal manifold is related to $v^r$ and $x^{M r}$ of eq. (4.3) by

$$V^{(9)}_\alpha = v^r \partial_\alpha x^{M r}, \quad (5.37)$$

where $\partial_\alpha = \partial/\partial \Phi^{\a}$. The metric of the internal manifold is $g_{\alpha \beta} = V^{(9)}_\alpha V^{(9)}_\beta$.

Denoting with $A^{(9)i}_\mu$ the gauge fields under which the hypermultiplets are charged (the index $i$ runs in the adjoint of the gauge group), under the gauge transformations

$$\delta A^{(9)i}_\mu = D_\mu \Lambda^{(9)i} \quad (5.38)$$

the scalars transform as

$$\delta \phi^\alpha = \Lambda^{(9)i} \xi^{\alpha i} \quad (5.39)$$

where $\xi^{\alpha i}$ are the Killing vectors corresponding to the isometry that we are gauging.

The covariant derivative for the scalars is then

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - A^{(9)i}_\mu \xi^{\alpha i} \quad (5.40)$$

The covariant derivatives for the gauginos $\Lambda^{(9)iA}$ are

$$D_\mu \Lambda^{(9)iA} = \partial_\mu \Lambda^{(9)iA} + \frac{1}{4} \omega_{i mn} \gamma^{mn} \Lambda^{(9)iA} + D_\mu \phi^\alpha A^A_{\alpha B} \Lambda^{(9)iB} + f^{ijk} A^{(9)j} A^{(9)k} \Lambda^{(9)i} \quad (5.41)$$

where $f^{ijk}$ are the structure constants of the group.

We use the method of Pasti, Sorokin and Tonin (PST) [12] in order to write a covariant action for fields that satisfy self-duality conditions. For a self-dual 3-form in six dimensions the PST action

$$\mathcal{L}_{PST} = \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} - \frac{1}{4} \frac{\partial^\mu \Xi \partial^\nu \Xi}{(\partial \Xi)^2} H^{- \mu \nu \rho} H^{- \mu \nu \rho} \quad (5.42)$$
where \( H^- = H - *H \) and \( \Xi \) is a scalar auxiliary field, is invariant under the standard
gauge transformations for a 2-form,
\[
\delta B = d\Lambda,
\]
and under the additional PST gauge transformations
\[
\delta B_{\mu\nu} = (\partial_\mu \Xi)\Lambda_\nu - (\partial_\nu \Xi)\Lambda_\mu
\]
and
\[
\delta \Xi = \Lambda, \quad \delta B_{\mu\nu} = \frac{\Lambda}{(\partial \Xi)^2} H^-_{\mu\nu\rho} \partial^\rho \Xi.
\]
We have a single self-dual 3-form and \( n_T \) antiself-dual 3-forms, where \( n_T \) is equal to
17 in the \( T^4/Z_2 \) model of [19]. These forms are obtained dressing with the scalars in
the tensor multiplets the 3-forms
\[
H^r = dB^r - c^{rz} \omega^{(9)z},
\]
where the index \( z \) runs over the various semi-simple factors of the gauge group in the
9-9 sector, \( \omega \) is the Chern-Simons 3-form and the \( c \)'s are constants (we denote with
\( z = 1 \) the group under which the hypermultiplets are charged). More precisely, the
combinations
\[
H_{\mu\nu\rho} = v_r H^r_{\mu\nu\rho}
\]
and
\[
H^M_{\mu\nu\rho} = x_r^M H^r_{\mu\nu\rho}
\]
are respectively self-dual and antiself-dual [24], to lowest order in the Fermi fields,
although in the complete lagrangian these conditions are modified by the inclusion
of fermionic bilinears. As in ten dimensions, the gauge invariance of \( H^r \) in eq. (5.46)
implies that \( B^r \) vary as
\[
\delta B^r = c^{rz} \text{tr}_z (\Lambda^{(9)} dA^{(9)})
\]
under gauge transformations.

To lowest order in the Fermi fields, the Lagrangian describing the coupling of the
supergravity multiplet to \( n_T \) tensor multiplets, vector multiplets and \( n_H \) hypermulti-
plet is
\[
e^{-1} \mathcal{L}_{\text{susy}} = - \frac{1}{4} R + \frac{1}{12} G_{\alpha\beta} H^{r\mu\nu\rho} F^s_{\mu\nu\rho}
+ \frac{1}{4} g_{\alpha\beta} \partial_\mu \Phi^\alpha \partial_\nu \Phi^\beta
- \frac{1}{2} v_r c^{rz} \text{tr}_z (F^{(9)}_{\mu\nu} F^{(9)\mu\nu}).
\]
that one can recover varying the Lagrangian of eq. (5.50) under the supersymmetry anomaly 

\[ \delta \epsilon^\mu = -i (\bar{c} \gamma^m \psi_m) \]  

related by the Wess-Zumino conditions to the supersymmetry anomaly 

\[ \mathcal{A}_\epsilon = \epsilon^{\mu \rho \sigma \delta} c^r c^{r'} \left[ -\frac{1}{4} \text{tr}_z (\delta c A^{(9)} \mu A^{(9)} \nu) \text{tr}_{z'} (F^{(9)} \rho \sigma F^{(9)} \delta \tau) - \frac{1}{6} \text{tr}_z (\delta c A^{(9)} \mu A^{(9)} \nu) \omega^{(9)} \sigma \delta \tau \right] , \]  

that one can recover varying the Lagrangian of eq. (5.50) under the supersymmetry

\[ \mathcal{A}_\lambda = -\frac{1}{4} \epsilon^{\mu \rho \sigma \delta} c^r c^{r'} \left[ (\delta c A^{(9)} \mu A^{(9)} \nu) \text{tr}_{z'} (F^{(9)} \rho \sigma F^{(9)} \delta \tau) \right] , \]  

related by the Wess-Zumino conditions to the supersymmetry anomaly 

\[ A_B \left( \frac{\partial}{\partial x} \right) A\chi \]  

satisfy on-shell self-duality and antiself-duality conditions, respectively. Finally, \( \Xi \) is the PST auxiliary field.

Due to eq. (5.49), the Wess-Zumino term \( B \wedge F \wedge F \) is not gauge invariant, and thus the variation of eq. (5.50) under gauge transformations produces the consistent gauge anomaly

\[ \mathcal{A}_\lambda = -\frac{1}{4} \epsilon^{\mu \rho \sigma \delta} c^r c^{r'} \left[ (\delta c A^{(9)} \mu A^{(9)} \nu) \text{tr}_{z'} (F^{(9)} \rho \sigma F^{(9)} \delta \tau) \right] , \]  

related by the Wess-Zumino conditions to the supersymmetry anomaly 

\[ A_B \left( \frac{\partial}{\partial x} \right) A\chi \]  

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related by the Wess-Zumino conditions to the supersymmetry anomaly 

\[ A_B \left( \frac{\partial}{\partial x} \right) A\chi \]  

satisfy on-shell self-duality and antiself-duality conditions, respectively. Finally, \( \Xi \) is the PST auxiliary field.
where

\[ K_{\mu\nu\rho} = \mathcal{H}_{\mu\nu\rho} - \frac{3}{(\partial \Xi)^2} \mathcal{H}_{\sigma\nu\rho} \]  

and

\[ K_{\mu\nu\rho}^M = \mathcal{H}_{\mu\nu\rho}^M - \frac{3}{(\partial \Xi)^2} \mathcal{H}_{\sigma\nu\rho}^{M+} \]  

are identically self-dual and antiself-dual, respectively. In the complete theory, the anomalous terms would be exactly canceled by the anomalous contributions of fermion loops.

Following the same reasoning as for the ten dimensional case of [100], we can describe the couplings to non-supersymmetric matter requiring that local supersymmetry be non-linearly realized on the $D5$-branes, and requiring that the supersymmetry variation of the non-supersymmetric fields be as in eq. (5.7). As explained in [100] and reviewed in the previous section, to lowest order in the fermions the coupling between the supersymmetric sector and the non-supersymmetric one is obtained dressing the bosonic fields in the supersymmetric sector with fermionic bilinears containing the goldstino, whose supersymmetry transformation is $\delta \theta = \epsilon$ to lowest order in the fermionic fields. As a result, the supersymmetry variation of the dressed scalars in the tensor multiplets

\[ \delta \Phi = \Phi - V^{[\alpha}(\bar{\theta} X^{\mu)} + \frac{i}{24} V^{[\alpha} x^{\mu} H^{\nu\rho} (\bar{\theta} \gamma^{\mu\nu\rho} \theta) \]  

(5.55)
is a general coordinate transformation of parameter

\[ \xi_\mu = -\frac{i}{2} (\bar{\theta} \gamma_\mu \epsilon) \]  \hspace{2cm} (5.59)

This definition of \( \Phi \) then induces the corresponding dressing

\[ \tilde{v}^r = v^r - x^{Mr}(\bar{\theta} \chi^M) - \frac{i}{24} H_{\mu\nu\rho}^r (\bar{\theta} \gamma^{\mu\nu\rho} \theta) \]  \hspace{2cm} (5.60)

and, in a similar fashion, the supersymmetry transformation of

\[ \tilde{\phi}^a = \phi^a - V^a_{\alpha\lambda} (\bar{\theta} A^\lambda) - \frac{i}{2} V_{\beta\alpha A} V_{\alpha\alpha B} (\bar{\theta} A^{\lambda\mu} \theta B) D_\mu \phi^B \]  \hspace{2cm} (5.61)

is again a coordinate transformation with the same parameter, together with an additional gauge transformation of parameter

\[ \Lambda^{(9)} = \xi_\mu A^{(9)}_\mu \]  \hspace{2cm} (5.62)

Similarly, the supersymmetry variation of

\[ \hat{e}_m = e_m^\mu + i(\bar{\theta} \gamma^m \psi_\mu) - \frac{i}{2} (\bar{\theta} \gamma^m D_\mu \theta) - \frac{i}{8} v_r H_{\mu\nu\rho}(\bar{\theta} \gamma^{m\nu\rho} \theta) \]  \hspace{2cm} (5.63)

contains also an additional local Lorentz transformation of parameter

\[ \Lambda^{mn} = -\xi_\mu [\omega_\mu^{mn} - v_r H_{\mu}^{mn}] \]  \hspace{2cm} (5.64)

where \( \omega \) denotes the spin connection. Since the scalars in the non-supersymmetric 9-5 sector are charged with respect to the vectors in the 9-9 sector, we define also

\[ \hat{A}^{(9)}_\mu = A^{(9)}_\mu + \frac{i}{\sqrt{2}} (\bar{\theta} \gamma_\mu \lambda^{(9)}) + \frac{i}{8} F^{(9)\mu\rho}(\bar{\theta} \gamma_{\mu\rho} \theta) \quad (z \neq 1) \]  \hspace{2cm} (5.65)

whose supersymmetry transformation is a general coordinate transformation of parameter as in eq. (5.59), aside from a gauge transformation of parameter as in (5.62). If one requires that the supersymmetry variation of the vector \( A^{(5)}_\mu \) from the non-supersymmetric 5-5 sector be

\[ \delta A^{(5)}_\mu = F^{(5)}_{\mu\nu} \xi^\nu \]  \hspace{2cm} (5.66)
namely a general coordinate transformation together with a gauge transformation of parameter
\[ \Lambda^{(5)} = \xi^{\mu} A^{(5)}_{\mu} \]  
(5.67)
one obtains a supersymmetrization of the kinetic term for \( A^{(5)}_{\mu} \) writing
\[ -\frac{1}{2} \hat{\epsilon}^{\nu} e^{w}_{r} \text{tr}_{w}(F^{(5)}_{\mu\nu} F^{(5)}_{\rho\sigma}) \hat{g}^{\mu\rho} \hat{g}^{\nu\sigma} \]  
(5.68)
where
\[ \hat{g}_{\mu\nu} = \hat{e}^{m}_{\mu} \hat{e}^{n}_{\nu} \]  
(5.69)
and the index \( w \) runs over the various semi-simple factors of the gauge group in the 5-5 sector. In analogy with the ten-dimensional case, the uncanceled NS-NS tadpole translates, in the low-energy theory, in the presence of a term
\[ -\Lambda \hat{\epsilon} f(\hat{\Phi}^{A}, \hat{\phi}^{A}) \]  
(5.70)
that depends on the scalars of the closed sector and contains the dilaton, that belongs to a hypermultiplet in type-I vacua. Thus, supersymmetry breaking naturally corresponds in this case also to a breaking of the isometries of the scalar manifolds.

Denoting with \( S \) the scalars in the 9-5 sector, charged with respect to the gauge fields in both the 9-9 and 5-5 sectors, we define their covariant derivative as
\[ \hat{D}_{\mu} S = \partial_{\mu} S - i \hat{A}^{(9)}_{\mu} S - i A^{(5)}_{\mu} S \]  
(5.71)
so that the term
\[ \frac{1}{2} \hat{\epsilon}(\hat{D}_{\mu} S)^{\dagger}(\hat{D}_{\nu} S) \hat{g}^{\mu\nu} \]  
(5.72)
is supersymmetric, if again the supersymmetry transformation of \( S \) is a general coordinate transformation, together with a gauge transformation of the right parameters. As in the ten-dimensional case, if one considers terms up to quartic couplings in the fermionic fields, one does not have to supersymmetrize terms that are quadratic in the additional fermions from the non-supersymmetric 5-5 and 9-5 sectors. Denoting with \( \lambda^{(5)} \) these fermions, the coupling of \( \lambda^{(5)2} \) to the 3-forms is not determined by supersymmetry, and can only be determined by string considerations, as in [100].

The inclusion of additional non-supersymmetric vectors modifies \( H^{r} \), that now includes the Chern-Simons 3-forms corresponding to these fields, so that eq. (5.46) becomes
\[ H^{r} = dB^{r} - e^{rz} \omega^{(9)z} - e^{rw} \omega^{(5)w} \]  
(5.73)
The gauge invariance of $H^r$ then requires that

$$\delta B^r = c^{rw} \text{tr}_w (\Lambda^{(5)} dA^{(5)})$$

(5.74)

under gauge transformations in the $5\bar{5}$ sector. Consequently, the supersymmetry transformation of $B^r$ is also modified, and becomes

$$\delta B_{\mu\nu}^r = iv^r(\bar{\psi}[\mu\nu] \psi) + \frac{1}{2} F^{\alpha\beta}(\chi^M \gamma_{\mu\nu} \epsilon)$$

$$- 2c^{rz} \text{tr}_z (A_{[\mu}^{(9)} \delta A_{\nu]}^{(9)}) - 2c^{rw} \text{tr}_w (A_{[\mu}^{(5)} \delta A_{\nu]}^{(5)}) .$$

(5.75)

The complete reducible gauge anomaly

$$A_A = - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\delta\tau} \{ c^{z} c^{rz} \text{tr}_z (\Lambda^{(9)} \partial_\mu A_\nu^{(9)}) \text{tr}_z(F^{(9)}_{\rho\sigma} F^{(9)}_{\delta\tau})$$

$$+ c^{w} c^{rw} \text{tr}_w (\Lambda^{(9)} \partial_\mu A_\nu^{(9)}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau})$$

$$+ c^{w} c^{wz} \text{tr}_w (\Lambda^{(5)} \partial_\mu A_\nu^{(5)}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau})$$

$$+ c^{w} c^{ww} \text{tr}_w (\Lambda^{(5)} \partial_\mu A_\nu^{(5)}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau}) \} ,$$

(5.76)

related by the Wess-Zumino conditions to the supersymmetry anomaly

$$A_\epsilon = \epsilon^{\mu\nu\rho\sigma\delta\tau} \{ c^{z} c^{rz} [ - \frac{1}{4} \text{tr}_z(\delta_\epsilon A^{(9)}_{\mu} A^{(9)}_{\nu}) \text{tr}_z(F^{(9)}_{\rho\sigma} F^{(9)}_{\delta\tau}) - \frac{1}{6} \text{tr}_z(\delta_\epsilon A^{(9)}_{\mu} F^{(9)}_{\nu\rho}) \omega^{(9)z}_{\sigma\delta\tau}]$$

$$+ c^{w} c^{rw} [ - \frac{1}{4} \text{tr}_w(\delta_\epsilon A^{(9)}_{\mu} A^{(9)}_{\nu}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau}) - \frac{1}{6} \text{tr}_w(\delta_\epsilon A^{(9)}_{\mu} F^{(9)}_{\nu\rho}) \omega^{(9)w}_{\sigma\delta\tau}]$$

$$+ c^{w} c^{wz} [ - \frac{1}{4} \text{tr}_w(\delta_\epsilon A^{(5)}_{\mu} A^{(5)}_{\nu}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau}) - \frac{1}{6} \text{tr}_w(\delta_\epsilon A^{(5)}_{\mu} F^{(5)}_{\nu\rho}) \omega^{(5)w}_{\sigma\delta\tau}]$$

$$+ c^{w} c^{ww} [ - \frac{1}{4} \text{tr}_w(\delta_\epsilon A^{(5)}_{\mu} A^{(5)}_{\nu}) \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau}) - \frac{1}{6} \text{tr}_w(\delta_\epsilon A^{(5)}_{\mu} F^{(5)}_{\nu\rho}) \omega^{(5)w}_{\sigma\delta\tau}] \} ,$$

(5.77)

is induced by the Wess-Zumino term

$$- \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\delta\tau} B^r_{\mu\nu} c^{rw} \text{tr}_w(F^{(5)}_{\rho\sigma} F^{(5)}_{\delta\tau}) .$$

(5.78)

It should be noticed that, as in the case of linearly realized supersymmetry, eqs. (5.76) and (5.77) satisfy the Wess-Zumino condition

$$\delta_A A_\epsilon = \delta_\epsilon A_A$$

(5.79)

since the explicit form of the gauge field supersymmetry variation plays no role in its proof. We expect that, to higher order in the fermions, the supersymmetry anomaly will be modified by gauge-invariant terms as in [69, 70]. From the definition of $H^r$, one can deduce the Bianchi identities

$$\partial_{[\mu} H^r_{\nu\rho\sigma]} = - \frac{3}{2} c^{z} \text{tr}_z(F_{[\mu\nu}^{(9)} F_{\rho\sigma]}^{(9)}) - \frac{3}{2} c^{w} \text{tr}_w(F_{[\mu\nu}^{(9)} F_{\rho\sigma]}^{(9)}) .$$

(5.80)
We now want to determine the terms proportional to $F \wedge F$ containing the goldstino that one has to add for the consistency of the model. Unlike the ten dimensional case, where duality maps the 2-form theory with Chern-Simons couplings to the 6-form theory with Wess-Zumino couplings, in this case the low-energy effective action contains both Chern-Simons and Wess-Zumino couplings. First of all, we observe that for the quantity

$$B^r = B^r_{\mu
u} - iv^r(\bar{\psi}[\gamma_{\mu}\gamma_{\nu}],\theta) - \frac{1}{2}x^M(r\chi(\gamma_{\mu}\gamma_{\nu},\theta) - \frac{2i}{\sqrt{2}}e^{\gamma_{\nu} tr_z[A^{(9)}_{\mu}(\bar{\theta}\gamma_{\nu}\lambda^{(9)})]}
$$

$$+ \frac{i}{8}(\partial_{\mu}v^\nu) (\bar{\theta}\gamma_{\mu\nu}\theta) + \frac{i}{8}x^M H^{M\rho\sigma}(\bar{\theta}\gamma_{[\rho\sigma]}\theta) + \frac{i}{2}v^r(\bar{\theta}\gamma_{[\rho\sigma]}D_{\nu}\theta)
$$

$$- \frac{i}{4}e^{\gamma_{\nu} tr_z[A^{(9)}_{\mu}(F^{(9)}\rho\sigma)](\bar{\theta}\gamma_{\nu\rho\sigma}\theta) - \frac{i}{4}e^{\gamma_{\nu} tr_z[A^{B}_{\alpha B}(x^C A^{(9)}_{\mu}(\bar{\theta}\gamma_{\nu}\theta))^B]}
$$

the supersymmetry variation is a general coordinate transformation of the correct parameter, together with an additional tensor gauge transformation of parameter

$$\Lambda^r_{\mu} = - \frac{1}{2}v^r \xi_{\mu} - \xi^{\mu} B^r_{\mu\nu},
$$

as well as PST gauge transformations of parameters

$$\Lambda^{(PST)}_{\mu} = \frac{\partial^\rho \Xi}{(\partial \Xi)^2}[v^r v_s H_{\rho\mu\nu} - x^M x^M H_{n\mu\nu}]\xi^\rho
$$

and

$$\Lambda^{(PST)} = \xi^\mu \partial_{\mu} \Xi
$$

and gauge transformations of the form $[5.44]$ and $[5.74]$ whose parameters are as in eq. $[5.62]$ and $[5.67]$. We should now consider all the terms proportional to $F \wedge F$ that arise, those directly introduced by the inclusion of the Chern-Simons 3-form for the fields in the 5-5 sector, those originating from the consequent modification of the Bianchi identities, and finally those introduced by the variation of the Wess-Zumino term.

The end result is that the variation of all these contributions gives

$$\delta \mathcal{L} = \epsilon^{\mu\nu\rho\delta} \{ - \frac{i}{4}v^r \bar{\epsilon}\gamma_{\mu\nu},\psi_{\nu} + \frac{1}{8}x^M(\bar{\epsilon}\gamma_{\mu\nu}\chi^M)\}e^{\gamma_{\nu} tr_z[F^{(5)}_{\rho\delta} F^{(5)}_{\delta\mu}]
$$

$$- 2v^r e^{\gamma_{\nu} tr_w(\delta A^{(5)}_{\mu} F^{(5)}_{\nu\rho}) K^{M\nu\rho} + 2x^M e^{\gamma_{\nu} tr_w(\delta A^{(5)}_{\mu} F^{(5)}_{\nu\rho}) K^{M\nu\rho}}.
$$

The first two terms are canceled by the goldstino variation in the additional couplings

$$\mathcal{L'} = \epsilon^{\mu\nu\rho\delta} \{ \frac{i}{4}v^r (\bar{\theta}\gamma_{\mu\nu},\psi_{\nu} - \frac{1}{8}x^M(\bar{\theta}\gamma_{\mu\nu}\chi^M)\}e^{\gamma_{\nu} tr_z[F^{(5)}_{\rho\delta} F^{(5)}_{\delta\mu}]
$$

$\text{(5.86)}$
where, however, the variations of the gravitino and the tensorinos produce additional terms. Some of these cancel the last two terms in eq. (5.87), while the remaining ones are canceled by the goldstino variation in
\[
\mathcal{L}'' = \epsilon^{\mu \nu \rho \sigma} \{ - \frac{i}{32} \partial^\mu \psi \bar{\theta} \gamma_{\mu \rho} \theta - \frac{i}{8} \psi \bar{\theta} \gamma_\mu D_\nu \theta \\
- \frac{i}{32} x^M \alpha \beta \bar{\theta} \gamma_{\nu \alpha} \theta \} c^w tr_w (F^{(5)} \delta^{(5)}_{\delta \tau}).
\]  
(5.87)

If one restricts the attention to terms up to quartic fermion couplings, no further contributions are produced. We can thus conclude that the non-linear realization of supersymmetry is granted by the inclusion of \(L_0\) and \(L_0^0\) in the low-energy effective action. From eq. (5.81) we also see that these two contributions can be written in the compact form
\[
L' + L'' = - \frac{1}{4} \epsilon^{\mu \nu \rho \sigma} B_{\mu \nu}^{extra} c^w tr_w (F^{(5)}_\rho F^{(5)}_{\delta \tau}),
\]  
(5.88)

where
\[
B_{\mu \nu}^{extra} = - i \psi \bar{\theta} \gamma_{\mu \rho} \theta - \frac{1}{2} x^M \gamma_{\mu \nu} \theta - \frac{2i}{\sqrt{2}} c^w tr_w [F^{(9)}_{\mu} (\bar{\theta} \gamma_\nu \lambda^{(9)})]
\]  
\[
+ \frac{i}{8} (\partial_\mu \psi) (\bar{\theta} \gamma_{\nu \rho} \theta) + \frac{i}{8} x^M K^{\rho M \sigma} (\bar{\theta} \gamma_{\nu \rho} \theta) + \frac{i}{2} \psi (\bar{\theta} \gamma_\nu D_\rho \theta)
\]  
(5.89)

coincides with the counterterm of \(B'\) only if no 9-9 vectors are present.

We now want to interpret these non-geometric terms along the lines of Section (5.1). To this end, observe that, if no 9-9 vectors are present, eq. (5.88) is exactly twice the term that one should add to eq. (5.78) in order to geometrize the Wess-Zumino term, substituting \(B\) with \(\hat{B}\). This means, roughly speaking, that half of the contribution in eq. (5.88) comes from the Green-Schwarz term, and half from the Chern-Simons couplings. This interpretation is in perfect agreement with self-duality, and thus in six dimensions there is no duality transformation that can give a fully geometric Lagrangian. If also 9-9 vectors are in the spectrum, no additional terms are produced in the lagrangian, in agreement with the fact that the additional terms of \(\hat{B}'\) in eq. (5.81) are not gauge invariant.

To resume, the Lagrangian for supergravity coupled to tensor multiplets, hypermultiplets and non-supersymmetric vectors is
\[
\mathcal{L} = \mathcal{L}_{susy} - \frac{1}{2} \hat{\epsilon} \bar{\psi} c^w tr_w (F^{(5)}_\mu F^{(5)}_\rho) \hat{\gamma}^w \tilde{\gamma}^w - \Lambda \hat{\epsilon} f(\hat{\Phi}, \hat{\phi})
\]  
\[
+ \frac{1}{2} \hat{\epsilon} (\hat{D}_\mu S)^w (\hat{D}_\nu S) \tilde{g}_{\mu \nu} - \frac{1}{8} \hat{\epsilon} \mu \nu \rho \sigma B_{\nu \rho} c^w tr_w (F^{(5)}_\rho F^{(5)}_{\delta \tau})
\]  
\[
- \frac{1}{4} \hat{\epsilon} \mu \nu \rho \sigma B_{\nu \rho}^{extra} c^w tr_w (F^{(5)}_\rho F^{(5)}_{\delta \tau}).
\]  
(5.90)
5.2 Geometric couplings in six dimensions

Since the supersymmetry transformation of other non-supersymmetric fermions is of higher order in the Fermi field, at this level we can always add them in the construction, while the couplings that can not be determined by supersymmetry could in principle be determined by string inputs, as in [100].

Finally, it is important to observe that without 9-9 vectors, although the Lagrangian \( (5.90) \) is not completely geometric, the corresponding field equations are. Indeed, if one fixes the PST gauge in such a way that the 3-forms satisfy the standard (anti)self-duality conditions, the equation for the vector fields, up to terms quartic in the fermions, is

\[
\hat{e}D_{\nu}[^{5}_{c_r w}F^{(5)}_{\rho \sigma} \hat{g}^{\mu \nu} \hat{g}^{\rho \sigma}] + \frac{1}{6} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} F^{(5)}_{\nu \rho} \hat{H}^{r}_{\sigma \delta \tau}
+ \frac{1}{12} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} F^{(5)}_{\nu \rho} c^{r w} \omega^{(5)w}_{\sigma \delta \tau} + \frac{1}{8} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} A^{(5)}_{\nu} c^{r w} tr_w (F^{(5)}_{\rho \sigma} F^{(5)}_{\delta \tau}) = 0 \tag{5.91}
\]

where

\[
\hat{H}^{r}_{\mu \nu \rho} = 3 \partial_{[\mu} \hat{B}^{r}_{\nu \rho]} - c^{-r}_{w} \omega^{(5)w}_{\mu \nu \rho} \tag{5.92}
\]

and this is nicely of geometric form. It should be noticed that no additional counterterms containing the goldstino have to be added if also 9-9 vectors are present. In fact, all the terms in \( \hat{B}^{r} \) induced by \( A^{(9)} \) are not gauge invariant, and their inclusion in the lagrangian is forbidden because it would modify the gauge anomaly. The resulting equation for \( A^{(5)} \) is then

\[
\hat{e}D_{\nu}[^{5}_{c_r w}F^{(5)}_{\rho \sigma} \hat{g}^{\mu \nu} \hat{g}^{\rho \sigma}] + \frac{1}{6} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} F^{(5)}_{\nu \rho} \hat{H}^{r}_{\sigma \delta \tau}
+ \frac{1}{12} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} F^{(5)}_{\nu \rho} c^{r z} \omega^{(9)z}_{\sigma \delta \tau} + \frac{1}{8} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} A^{(5)}_{\nu} c^{r z} tr_z (F^{(9)}_{\rho \sigma} F^{(9)}_{\delta \tau})
+ \frac{1}{12} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} F^{(5)}_{\nu \rho} c^{r w} \omega^{(5)w}_{\sigma \delta \tau} + \frac{1}{8} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{r w} A^{(5)}_{\nu} c^{r w} tr_w (F^{(5)}_{\rho \sigma} F^{(5)}_{\delta \tau}) = 0 \tag{5.93}
\]

where

\[
\hat{H}^{r}_{\mu \nu \rho} = 3 \partial_{[\mu} B^{r}_{\nu \rho]} + 3 \partial_{[\mu} B^{r}_{\nu \rho]}^{extra} - c^{r z} \omega^{(9)z}_{\mu \nu \rho} - c^{r w} \omega^{(5)w}_{\mu \nu \rho} \tag{5.94}
\]

is geometric up to gauge-invariant terms proportional to \( c^{r z} \). The result is thus in agreement with what expected by anomaly considerations. If gauge and supersymmetry anomalies are absent, the \( A^{(5)} \) equation is mapped into itself by supersymmetry: this is the very reason why this equation is geometric. In the presence of gauge and supersymmetry anomalies, as long as 9-9 vectors are absent, the equation for \( A^{(5)} \) is still geometric, albeit not gauge invariant. The supersymmetry anomaly, in this case,
results from the gauge transformation contained in eq. (5.66). When also 9-9 vectors are present, these arguments do not apply, and thus in eq. (5.93) the geometric structure is violated by terms proportional to $c^r_z c^w_r$.

The consistent formulation described above can be reverted to a supersymmetric formulation in terms of covariant non-integrable field equations [54, 75], that embody the corresponding covariant gauge anomaly

$$ A_{\mu}^{\text{cov}} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma \delta \tau} \left[ c^r_z c^z_r tr_z (\Lambda^{(9)} F^{(9)}_{\mu \nu}) tr_{\tau} (F^{(9)}_{\rho \sigma} F^{(9)}_{\delta \tau}) + c^r_z c^w_r tr_z (\Lambda^{(9)} F^{(9)}_{\mu \nu}) tr_w (F^{(9)}_{\rho \sigma} F^{(9)}_{\delta \tau}) + c^w_r c^r_w tr_w (\Lambda^{(5)} F^{(5)}_{\mu \nu}) tr_z (F^{(9)}_{\rho \sigma} F^{(9)}_{\delta \tau}) + c^w_r c^w_0 tr_w (\Lambda^{(5)} F^{(5)}_{\mu \nu}) tr_w (F^{(9)}_{\rho \sigma} F^{(9)}_{\delta \tau}) \right], \quad (5.95) $$

given by the divergence of the covariant equation for $A^{(5)}_{\mu}$,

$$ \dot{D}_{\nu} [ e^{\mu \nu \rho \sigma \delta \tau} \epsilon_{\rho \sigma} \epsilon_{\delta \tau} \gamma^{\nu \sigma} + \frac{1}{6} \epsilon^{\mu \nu \rho \sigma \delta \tau} c_{\tau z} c_{\rho z} F^{(9)}_{\nu \rho} \tilde{H}_{\alpha \beta \gamma} = 0, \quad (5.96) $$

and the divergence of the covariant equation for $A^{(9)}_{\mu}$ [54]. Without 9-9 vectors, eq. (5.96) is both geometric and gauge-covariant, while, if 9-9 vectors are present, the geometric structure is violated by gauge-covariant terms proportional again to $c^r_z c^w_r$.

### 5.3 Discussion

In this chapter we have reviewed the results of [104], extending the work of [100] on the low-energy effective action for models with brane supersymmetry breaking. In this class of models a supersymmetric bulk is coupled to a non-supersymmetric open sector, and as a result local supersymmetry is non-linearly realized à la Volkov-Akulov. In particular, we have shown that, up to quartic order in the fermions, the low-energy couplings between the supersymmetric bulk and the non-supersymmetric open sector in the ten-dimensional $USp(32)$ model of [48] are all of geometric origin, being induced by the dressing of the bulk fields in terms of the goldstino, provided one turns to the dual 6-form [16] formulation. Thus, in retrospect, the non-geometric terms in [100] are precisely what is needed to geometrize the dual form of the theory, where the (high-derivative) Chern-Simons couplings are absent. We have completed a similar construction for six-dimensional models with brane supersymmetry breaking. Since in this case both Chern-Simons and Wess-Zumino terms are simultaneously present, not all couplings in the Lagrangian can be related to goldstino-dependent
dressings of bulk fields. However, in the absence of supersymmetric vectors, the field equations exhibit this geometric structure, that is naturally violated in the general case by anomalous terms.

It would be interesting to apply the same construction to the four-dimensional brane supersymmetry breaking models of [49], and in general to brane-world scenarios (see [88] and references therein) in which supersymmetry is linearly realized in the gravitational sector and non-linearly realized in the brane universe. However, in four dimensions the gravitino can acquire a Majorana mass through a super-Higgs mechanism [103], and it is this difference with respect to minimal supergravity in ten and six dimensions that makes the models studied in this paper rather peculiar.

A general property of all this class of models is the presence of a dilaton tadpole of positive sign, required in order to have a correct kinetic term for the goldstino [100], and guaranteed by the residual tension of anti-branes and orientifolds. In ten dimensions the tadpole signals the impossibility of having maximally symmetric vacuum configurations [108], and one should try to analyze the same effects in the six-dimensional models discussed in this paper.

Finally, it should be observed that it is always possible, in any supersymmetric theory coupled to a goldstino, to dress the fields in the linear sector by terms containing the goldstino itself. This is a property of the commutator of two supersymmetries: by construction, a supersymmetry transformation on the dressed fields exactly corresponds to the commutator of two supersymmetries on the linear fields, producing general coordinate transformations together with all the other local symmetry transformations. In the six dimensional case discussed in Section 4, this can be explicitly verified: the parameters of eqs. (5.59), (5.62), (5.64), (5.82), (5.83) and (5.84) coincide with those coming from the supersymmetry algebra, provided one substitutes $-\frac{i}{2}(\bar{\theta}\gamma_\mu\epsilon)$ with $-i(\bar{\epsilon}_1\gamma_\mu\epsilon_2)$ [69, 70, 76]. Following this way of reasoning, one could try to generalize the results obtained here to all orders in the Fermi fields.
Conclusions

The main subject of this dissertation has been the analysis of minimal six-dimensional supergravity. As we have seen in Chapter 4, these models are “classically” anomalous, and are thus determined requiring the closure of the Wess-Zumino consistency conditions \[67\] rather than requiring supersymmetry invariance. As a result, the low-energy effective action is not unique, and a quartic gaugino coupling remains undetermined, since the model is consistent for any value of this coupling. An interesting open problem is then to analyze the same coupling at the string level, that would result from an annulus amplitude with the insertion of two gauginos at each boundary. Once one extracts the low-energy limit from this amplitude (see for instance \[107\] and references therein) one should be able to understand whether this coupling is determined by string theory. This result would be quite important, since it would state that string theory is more powerful than supersymmetry in determining the low-energy effective action, even for vacua with 8 supercharges, that a priori should receive string corrections only in the hypermultiplet sector.

Another very interesting open problem related to these six-dimensional vacua is to understand the physics corresponding to the tensionless string phase transition \[32, 35\]. There is an analogous phenomenon in \(\mathcal{N} = (2,0)\) six-dimensional models \[66, 63\], and in \[108\] some recent attempts have been made in order to give a proper definition to interacting superconformal quantum theories in six dimensions. These theories (see e.g. \[109, 110\] for reviews) seem also to give the correct framework for a better understanding of \(\mathcal{N} = 4\) SYM theory in four dimensions. In particular, the fact that \(\mathcal{N} = 4\) four-dimensional theories are obtained by compactification of six-dimensional superconformal quantum theories on \(T^2\) could provide an explanation for \(SL(2,\mathbb{Z})\) S-duality (in this respect, a better understanding of these theories could
also correspond to a deeper comprehension of F-theory \cite{111}).

In the last part of the thesis we have analyzed the low-energy effective action for type-I models with brane supersymmetry breaking, both in ten and six dimensions. We emphasize once more that these models are very interesting from a phenomenological point of view in the context of brane-world scenarios. It would be relevant then to study them in more detail, trying to determine higher order Fermi terms, and also obtaining in specific cases the scalar potential.
Appendix A

Notations and spinor algebra

A.1 Reality properties

In our conventions, the Clifford algebra has the form

\[ \{ \gamma_m, \gamma_n \} = 2 \eta_{mn} 1 \]  \hspace{1cm} (A.1)

and we use the mostly minus convention for the signature of space-time. The matrix \( \gamma^0 \) is hermitian, while the \( \gamma^i \)'s are anti-hermitian, with \((\gamma^0)^2 = 1 \) and \( \gamma^i \gamma^i = -1 \).

We want to resume here the reality properties of the spinors in various dimensions (see for instance [112] and references therein for details). We are interested only in spinors of the Lorentz group \( SO(D-1,1) \). In general it can be shown that for the group \( SO(t,s) \), the reality properties of the spinor only depend on \( |s-t| \mod 8 \). One can define a real spinor if it is possible to construct consistently a charge conjugation matrix \( C \) such that

\[ C \gamma_\mu C = \pm \gamma_\mu^T \]  \hspace{1cm} (A.2)

The charge-conjugated spinor is

\[ \psi_C = C \bar{\psi}^T \]  \hspace{1cm} (A.3)

where \( \bar{\psi} = \psi^\dagger \gamma_0 \), and this definition is consistent if one can consistently define a Majorana spinor, satisfying

\[ \psi = \psi_C \]  \hspace{1cm} (A.4)
This can be obtained it two ways, with either $C$ symmetric satisfying eq. (A.2) with the plus sign, or $C$ antisymmetric satisfying eq. (A.2) with the minus sign. In the former case, if $\psi$ and $\chi$ satisfy eq. (A.4), the bilinear $(\bar{\psi}\chi)$ is odd under Majorana flip and thus is an imaginary number, while in the latter case it is even under Majorana flip and thus is a real number.\footnote{We define the charge conjugation operation on Grassmann variables as $(ab)^* = b^*a^*$, and so if $a$ and $b$ are real, $ab$ is imaginary.} As we will see, the situation is reversed in the case in which the spinors satisfy symplectic Majorana conditions. The properties under flip of the bilinears obtained contracting the spinors with $\gamma$ matrices can be straightforwardly obtained: the result is that if the number of $\gamma$ matrices is odd, the two $C$’s give the same Majorana flip properties, while if the number of $\gamma$ matrices is even the flip properties are opposite. The fact that the two definitions give the same flip properties when an odd number of $\gamma$ matrices is inserted is fundamental in order to close the supersymmetry algebra. Whenever possible, we always choose $C$ such that $(\bar{\psi}\chi)$ is even under Majorana flip.

Let us begin by discussing the two-dimensional case, in which the $\gamma$ matrices are two-dimensional. One possible choice is

\begin{align}
\gamma^0 &= \sigma_1 , \\
\gamma^1 &= i\sigma_2 .
\end{align}

(A.5)

The chirality matrix is $\gamma_3 = \sigma_3$. The symmetric matrix $C_S = \sigma_1$ satisfies (A.2) with the plus sign, while the antisymmetric matrix $C_A = \sigma_2$ satisfies eq. (A.2) with the minus sign, so they are both good definitions for a real spinor. Observe that in our base, the choice $C_S$ corresponds in eq. (A.4) to the condition $\psi = \psi^*$. Both these matrices anticommute with $\gamma_3$, and so both the conditions are compatible with the chirality condition: in two dimensions one can define a Majorana-Weyl spinor.

As usual, one goes from $D = 2n$ to $D = 2n + 1$ adding $\gamma^D = i\gamma$. So in three dimensions we add

\[ \gamma^2 = i\sigma_3 \]

(A.6)

to eq. (A.5). The only possible choice is $C = C_A = \sigma_2$, satisfying eq. (A.2) with the minus sign, and so in three dimensions one can define a Majorana spinor.

In four dimensions the $\gamma$ matrices are $4 \times 4$, and can be written as the tensor product of two Pauli matrices:

\[ \gamma^0 = \sigma_1 \otimes 1 , \]
The chirality matrix is $\gamma_5 = \sigma_3 \otimes \mathbf{1}$. The condition (A.2) is satisfied by $C_1 = \mathbf{1} \otimes \sigma_2$ with the plus sign and by $C_2 = \sigma_3 \otimes \sigma_2$ with the minus sign. Since both these matrices are antisymmetric, only the second gives a consistent reality condition. The fact that this matrix commutes with $\gamma_5$ means that the Majorana and Weyl conditions are not compatible.

In five dimensions we add

$$\gamma^4 = i\sigma_3 \otimes \mathbf{1}$$

to eq. (A.7). Then we are left with only the matrix $C_1 = \mathbf{1} \otimes \sigma_2$, and so it is not possible to impose a Majorana condition on a single spinor. This is analogous to the six-dimensional case, in which the $8 \times 8$ $\gamma$ matrices can be written in the form

$$\begin{align*}
\gamma^0 &= \sigma_1 \otimes \mathbf{1} \otimes \sigma_1, \\
\gamma^1 &= i\sigma_2 \otimes \sigma_1 \otimes \mathbf{1}, \\
\gamma^2 &= i\sigma_2 \otimes \sigma_2 \otimes \mathbf{1}, \\
\gamma^3 &= i\sigma_2 \otimes \sigma_3 \otimes \mathbf{1}, \\
\gamma^4 &= i\sigma_1 \otimes \mathbf{1} \otimes \sigma_2, \\
\gamma^5 &= i\sigma_1 \otimes \mathbf{1} \otimes \sigma_3.
\end{align*}$$

The matrix $C_1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ is antisymmetric and satisfies eq. (A.2) with the plus sign, while the matrix $C_2 = \sigma_1 \otimes \sigma_2 \otimes \sigma_2$ is symmetric and satisfies eq. (A.2) with the minus sign, and neither give a consistent Majorana condition for a single spinor. The seven-dimensional case is analogous to five dimensions.

In five, six and seven dimensions one can define for a $USp(2)$ doublet of spinors the symplectic Majorana condition

$$\psi^A = \epsilon^{AB} C \bar{\psi}^T_B,$$

where

$$\bar{\psi}_A = (\psi^A)^\dagger \gamma_0$$

and $\epsilon^{12} = \epsilon_{12} = 1$. Any bilinear $\bar{\psi}_A \chi^B$ carries a pair of $USp(2)$ indices, and can be decomposed in terms of the identity and of the three Pauli matrices. Indeed, one can
form the bilinears
\[(\bar{\psi}\chi) = \bar{\psi}_A \chi^A , \quad [\bar{\psi}\chi]_i = \sigma^B_{iA} \bar{\psi}_B \chi^A , \quad (A.12)\]
and standard properties imply that
\[\bar{\psi}_A \chi^B = \frac{1}{2} \delta^B_A (\bar{\psi}\chi) + \frac{1}{2} \sigma^B_{iA} [\bar{\psi}\chi]_i , \quad (A.13)\]
Using eq. (A.10), and choosing \(C\) to be symmetric, one can then see that the Fermi bilinear \((\bar{\psi}\chi)\) has standard behavior under Majorana-flip, namely
\[(\bar{\psi}\chi) = (\bar{\chi}\psi) , \quad (A.14)\]
while all three bilinears \([\bar{\psi}\chi]_i\) have the anomalous behavior
\[[\bar{\psi}\chi]_i = -[\bar{\chi}\psi]_i . \quad (A.15)\]
Corresponding relations hold for all Fermi bilinears, that naturally display pairs of opposite behaviors under Majorana flip. In particular, these properties imply that
\[[\bar{\psi}\gamma_{\mu\nu}\psi]_i = 0 , \quad (A.16)\]
a relation often used in deriving the results of Chapter 4. In six dimensions the symplectic Majorana condition and the Weyl condition can be imposed together, since \(C\) anticommutes with the chirality matrix \(\gamma_7 = \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} \).

Following the same arguments as before, one can show that the eight-dimensional case is similar to four dimensions, while the nine-dimensional case is similar to three dimensions. Let us analyze in more detail the ten-dimensional case, that is completely analogous to the two-dimensional case by general arguments. Making the choice
\[
\begin{align*}
\gamma^0 &= \sigma_1 \otimes \mathbf{1} \otimes \sigma_1 \otimes \sigma_1 \otimes \mathbf{1} , \\
\gamma^1 &= i\sigma_2 \otimes \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1 , \\
\gamma^2 &= i\sigma_2 \otimes \sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1 , \\
\gamma^3 &= i\sigma_2 \otimes \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1 , \\
\gamma^4 &= i\sigma_1 \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbf{1} , \\
\gamma^5 &= i\sigma_1 \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbf{1} , \\
\gamma^6 &= i\sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \mathbf{1} , \\
\gamma^7 &= i\sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} , \\
\gamma^8 &= i\sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 , \\
\gamma^9 &= i\sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 , \\
\end{align*}
(A.17)\]
one obtains that eq. (A.2) is satisfied by the symmetric matrix $C_S = \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3$ with the plus sign, and by the antisymmetric matrix $C_A = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3$ with the minus sign, and both these choices are consistent with eq. (A.4). Moreover, both these matrices anticommute with the chirality matrix $\gamma_{11} = \gamma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1$, so that the Majorana condition can be imposed on a Weyl spinor, as in two dimensions. Finally the eleven-dimensional case is completely analogous to three dimensions: the complete set of $\gamma$ matrices can be obtained adding

$$\gamma^{10} = i \gamma_{11} = i \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1$$  \hspace{1cm} (A.18)$$

to eq. (A.17), and from this one obtains the antisymmetric charge-conjugation matrix

$$C_A = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3$$  \hspace{1cm} (A.19)$$
satisfying eq. (A.2) with the minus sign.

\section*{A.2 Fierz identities}

In this section we collect the Fierz relations that are used in this dissertation. We begin with the four dimensional case, in which the product of two spinors has 16 components, and can be expanded in terms of the bilinears obtained contracting the spinors with the identity, $\gamma_5, \gamma_\mu, \gamma_\mu \gamma_5$ and $\gamma_\mu \gamma_\nu$. All the other bilinears are related to these by the relation

$$\gamma^{\mu_1 \cdots \mu_n} = \frac{i(-1)^{[n/2]}}{e(4-n)!} e^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_{4-n}} \gamma_{\nu_1 \cdots \nu_{4-n} \gamma_5}$$  \hspace{1cm} (A.20)$$

where $\gamma^{\mu_1 \cdots \mu_n} = 1$ for $n = 0$, and $e^{0123} = 1$. The result is

$$\psi \bar{X} = -\frac{1}{4} 1 (\bar{X} \psi) - \frac{1}{4} \gamma_5 (\bar{X} \gamma_5 \psi) - \frac{1}{4} \gamma_\mu (\bar{X} \gamma^\mu \psi)$$
$$+ \frac{1}{4} \gamma_\mu \gamma_5 (\bar{X} \gamma^\mu \gamma_5 \psi) + \frac{1}{8} \gamma_{\mu \nu} (\bar{X} \gamma_{\mu \nu} \psi).$$  \hspace{1cm} (A.21)$$

For Majorana spinors, this implies

$$\psi \bar{X} - \bar{\chi} \psi = -\frac{1}{2} \gamma_\mu (\bar{X} \gamma^\mu \psi) + \frac{1}{4} \gamma_{\mu \nu} (\bar{X} \gamma_{\mu \nu} \psi)$$  \hspace{1cm} (A.22)$$

In six dimensions, using $e^{012345} = +1$, one obtains

$$\gamma^{\mu_1 \cdots \mu_n} = -\frac{(-1)^{[n/2]}}{e(6-n)!} e^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_{6-n}} \gamma_{\nu_1 \cdots \nu_{6-n} \gamma_7}$$  \hspace{1cm} (A.23)$$
In particular, eq. [A.23] shows that $\gamma_{\mu\nu\rho}\psi$ is self-dual if $\psi$ is left-handed, i.e. $\gamma_7\psi = \psi$, and antiself-dual if $\psi$ is right-handed. One can study Fierz relations between spinor bilinears using eq. [A.13]. If $\psi$ and $\chi$ have the same chirality
\[
\psi \chi = -\frac{1}{4} \bar{\chi} \gamma^\mu \gamma_\mu + \frac{1}{48} \bar{\chi} \gamma^{\mu\nu\rho} \gamma_\mu \gamma_\nu \gamma_\rho ,
\]
while if they have opposite chirality
\[
\psi \chi = -\frac{1}{4} \bar{\chi} \psi + \frac{1}{8} \bar{\chi} \gamma^{\mu\nu} \gamma_\mu \gamma_\nu .
\]
In the case of spinors satisfying the symplectic Majorana condition [A.10], interesting results are obtained (anti)symmetrizing these relations. In particular, eq. [A.24] implies
\[
\psi^A \chi_B - \chi^A \psi_B = -\frac{1}{4} (\bar{\chi} \gamma^\mu \psi) \delta^A_B \gamma_\mu + \frac{1}{48} [\bar{\chi} \gamma^{\mu\nu\rho} \psi] i \sigma_{iB} A^{\mu\nu\rho} .
\]
Now we consider the ten-dimensional case. From the definition of $\gamma_{11}$ one obtains
\[
\gamma_{\mu_1 \ldots \mu_n} = \frac{(-1)^{[n/2]}}{e(10-n)!} e^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_{10-n} \eta_{\nu_1 \ldots \nu_{10-n} \gamma_{11}}},
\]
that again implies that $\gamma_{\mu_1 \ldots \mu_5} \psi$ is self-dual if $\psi$ is left-handed, and antiself-dual if $\psi$ is right-handed. The Fierz identity is
\[
\psi \bar{\chi} - \bar{\psi} \chi = -\frac{1}{32} \gamma_\mu (\bar{\chi} \gamma^\mu \psi) + \frac{1}{192} \gamma_{\mu\nu\rho} (\bar{\chi} \gamma^{\mu\nu\rho} \psi) - \frac{1}{32 \cdot 120} \gamma_{\mu\nu\rho\sigma\tau} (\bar{\chi} \gamma^{\mu\nu\rho\sigma\tau} \psi)
\]
for spinors of the same chirality and
\[
\psi \bar{\chi} = -\frac{1}{32} (\bar{\psi} \chi) + \frac{1}{64} \gamma_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu} \psi) - \frac{1}{32 \cdot 24} \gamma_{\mu\nu\rho\sigma} (\bar{\chi} \gamma^{\mu\nu\rho\sigma} \psi)
\]
for spinors of opposite chirality.

Finally, we write the Fierz identities for $D = 11$:
\[
\psi \bar{\chi} = -\frac{1}{32} (\bar{\psi} \chi) - \frac{1}{32} (\bar{\chi} \psi) + \frac{1}{64} \gamma_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu} \psi) + \frac{1}{192} \gamma_{\mu\nu\rho} (\bar{\chi} \gamma^{\mu\nu\rho} \psi) - \frac{1}{32 \cdot 24} \gamma_{\mu\nu\rho\sigma\tau} (\bar{\chi} \gamma^{\mu\nu\rho\sigma\tau} \psi)
\]
\[
\bar{\psi} \chi = -\frac{1}{32} (\psi \bar{\chi}) - \frac{1}{32} (\chi \bar{\psi}) + \frac{1}{64} \gamma_{\mu\nu} (\psi \bar{\chi}) + \frac{1}{192} \gamma_{\mu\nu\rho} (\psi \bar{\chi}) - \frac{1}{32 \cdot 24} \gamma_{\mu\nu\rho\sigma\tau} (\psi \bar{\chi})
\]
\[\quad - \frac{1}{32 \cdot 24} \gamma_{\mu\nu\rho\sigma\tau} (\bar{\psi} \chi) - \frac{1}{32 \cdot 120} \gamma_{\mu\nu\rho\sigma\tau} (\bar{\psi} \chi)
\]
\[\quad = \frac{1}{32 \cdot 120} \gamma_{\mu\nu\rho\sigma\tau} (\bar{\psi} \chi) .
\]

### A.3 Six-dimensional conventions

In six dimensions, a 3-form $X_{\mu\nu\rho}$ is (anti)self-dual if
\[
X_{\mu\nu\rho} = (-) \frac{1}{6e} \epsilon_{\mu\nu\rho\alpha\beta\gamma} X^{\alpha\beta\gamma} .
\]
If $X_{\mu\nu\rho}$ and $Y_{\mu\nu\rho}$ are both (anti)self-dual,

$$X_{\mu\nu\rho}Y^{\mu\nu\rho} = 0 \quad (A.32)$$

and

$$X_{\mu\nu}Y^{\mu\nu}_{\beta} - X_{\mu\nu\beta}Y^{\mu\nu}_{\alpha} = 0 \quad , \quad (A.33)$$

while if they have opposite duality properties

$$X_{\mu\nu\beta}Y^{\mu\nu}_{\beta} + X_{\mu\nu\beta}Y^{\mu\nu}_{\alpha} = \frac{1}{3}g_{\alpha\beta}X_{\mu\nu\rho}Y^{\mu\nu\rho} \quad . \quad (A.34)$$

Moreover, an (anti)self-dual antisymmetric tensor $X_{\mu\nu\rho}$ satisfies

$$X^{\mu\nu\rho}X_{\alpha\beta\rho} = \frac{1}{4}[-\delta_{\beta}^{\mu}X_{\alpha\gamma\delta}X^{\nu\gamma\delta} + \delta_{\beta}^{\nu}X_{\alpha\gamma\delta}X^{\mu\gamma\delta} + \delta_{\alpha}^{\mu}X_{\beta\gamma\delta}X^{\nu\gamma\delta} - \delta_{\alpha}^{\nu}X_{\beta\gamma\delta}X^{\mu\gamma\delta}] \quad . \quad (A.35)$$

The indices of $USp(2)$ and $USp(2n_H)$ are raised and lowered by the antisymmetric symplectic invariant tensors $\epsilon^{AB}$ and $\Omega^{ab}$ with the following conventions:

$$V^A = \epsilon^{AB}V_B \quad , \quad V_A = \epsilon_{BA}V_B \quad (\epsilon^{AB}\epsilon_{AC} = \delta^B_C) \quad , \quad (A.36)$$

$$W^a = \Omega^{ab}W_b \quad , \quad W_a = \Omega_{ba}W_b \quad (\Omega^{ab}\Omega_{ac} = \delta^b_c) \quad . \quad (A.36)$$

Spinors with $USp(2n_H)$ indices satisfy the symplectic-Majorana condition

$$\Psi^a = \Omega^{ab}C\Psi_b^T \quad , \quad (A.37)$$

where

$$\bar{\Psi}_a = (\Psi^a)^{\dagger}\gamma_0 \quad . \quad (A.38)$$

From these relations one can deduce the properties of spinor bilinears under Majorana flip. For instance:

$$\bar{\chi}_A\Psi^a = \epsilon_{AB}\Omega^{ab}(\bar{\Psi}_b\chi^B) \quad , \quad (A.39)$$

and similar relations when $\gamma$-matrices are included. In our notations a spinor bilinear with two $USp(2)$ indices contracted is written without explicit indices, i.e.

$$(\bar{\chi}_A\Psi^a) \equiv (\bar{\chi}\Psi) \quad , \quad (A.40)$$

while in all the other bilinears the symplectic indices are explicit.

The connections $A^A_{a\beta}$ and $A^a_{\alpha\beta}$ are anti-hermitian. Belonging to the adjoint representation of a symplectic group, they are symmetric if considered with both upper or both lower indices.
The hermitian gauge-group generators $T^i$ satisfy the commutation relations

$$[T^i, T^j] = i f^{ijk} T^k ,$$  \hspace{1cm} (A.41)

as well as the trace conditions

$$\text{tr}(T^iT^j) = \frac{1}{2} \delta^{ij} .$$  \hspace{1cm} (A.42)
Bibliography


[102] C. Angelantonj, R. Blumenhagen and M. Gaberdiel, in [19].


