Worldline deviations and epicycles

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Abstract
In general relativity, only relative acceleration has an observer-independent meaning: curvature and non-gravitational forces determine the rate at which world lines of test bodies diverge or converge. We derive the equations governing both in the conventional geometric formalism as well as using the background field method. This allows us to generalize the results to test bodies with charge and/or spin. The application of the equations to the motion of particles in a central field results in an elegant, fully relativistic version of the Ptolemaean epicycle scheme.
1. World line deviation equations

According to the equivalence principle, structureless test bodies (sometimes referred to as point masses) in a gravitational field move on geodesics of space-time. Their worldline $x^\mu(\tau)$ is a solution of the geodesic equation

$$\frac{D^2 x^\mu}{D\tau^2} = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0,$$  \hspace{1cm} (1)

where the world-line parameter $\tau$ is to be taken as proper time. Introducing the four-velocity as the time-like tangent unit vector to the worldline:

$$u^\mu = \frac{dx^\mu}{d\tau},$$

the equation can be written in geometrical language as

$$u \cdot \nabla u = 0, \quad u^2 = -1.$$  \hspace{1cm} (2)

with $\nabla$ the covariant derivative. It is easily observed from eq.(1), that the proper acceleration $a^\mu = \frac{d^2 x^\mu}{d\tau^2}$ is not a covariant object. In particular, its vanishing or non-vanishing has no observer-independent meaning.

In contrast, the relative acceleration between worldlines is a covariant quantity, and its vanishing or non-vanishing does not depend on the frame of reference [1]. We recall the argument. Consider a one-parameter congruence of geodesics $x^\mu(\tau; \lambda)$, where $\lambda$ labels the geodesics and $\tau$ is the proper-time parameter along the geodesic. We suppose the parametrization to be smooth, hence we can construct the tangent vector fields $u^\mu = \frac{\partial x^\mu}{\partial \tau}$, and $n^\mu = \frac{\partial x^\mu}{\partial \lambda}$. It is straightforwardly established that

$$(u \cdot \nabla n)^\mu = \frac{\partial^2 x^\mu}{\partial \tau \partial \lambda} + \Gamma^\mu_{\lambda\nu} \frac{\partial x^\lambda}{\partial \tau} \frac{\partial x^\nu}{\partial \lambda} = (n \cdot \nabla u)^\mu.$$  \hspace{1cm} (3)

As a corollary, we obtain

$$u \cdot \nabla (u \cdot \nabla n) = u \cdot \nabla (n \cdot \nabla u) = (u \cdot \nabla n) \cdot \nabla u + u^\mu n^\nu (\nabla_\mu \nabla_\nu u)$$

$$= (n \cdot \nabla u) \cdot \nabla u + u^\mu n^\nu (\nabla_\mu \nabla_\nu u)$$

$$= n \cdot \nabla (u \cdot \nabla u) + u^\mu n^\nu [\nabla_\mu, \nabla_\nu] u = u^\mu n^\nu R_{\mu\nu}[u, \cdot].$$  \hspace{1cm} (4)

In component notation this reads

$$\frac{D^2 n^\mu}{D\tau^2} = R^\mu_{\kappa\lambda\nu} u^\kappa u^\nu n^\lambda.$$  \hspace{1cm} (5)

The interest in the deviation vector $n^\mu$ obviously derives from the fact that, if $x^\mu_0(\tau) = x^\mu(\tau; \lambda_0)$ is a solution of the geodesic equation (1), then to first order $x^\mu_1 = x^\mu_0 + n^\mu \Delta \lambda$ is a solution as well:

$$x^\mu(\tau; \lambda_1) = x^\mu(\tau; \lambda_0) + \Delta \lambda \frac{\partial x^\mu}{\partial \lambda}(\tau, \lambda_0) \approx x^\mu(\tau; \lambda_0 + \Delta \lambda).$$  \hspace{1cm} (6)

It follows, that eq.(3) describes the covariant relative acceleration between these world lines. Of course, $n^\mu$ is only a first approximation to the neighboring geodesic at $\lambda_1 = \lambda_0 + \Delta \lambda$. To increase the precision of the approximation,
one has to compute higher-order derivatives w.r.t. \( \lambda \), by solving higher-order versions of eq. (5), involving not only the Riemann curvature tensor, but its derivatives as well. A systematic procedure of this type has been developed in ref. [2]. Here I pursue the first-order equation (5) and study some generalizations and applications.

We first observe, that eq. (5) is linear and homogeneous in \( n^{\mu} \). It is therefore not very difficult to construct an action from which it can be derived. The lagrangean of interest reads

\[
L(n) = \frac{1}{2} g_{\mu\nu} \frac{D n^\mu}{D\tau} \frac{D n^\nu}{D\tau} + \frac{1}{2} R_{\mu\nu\lambda\kappa} u^\kappa u^\lambda n^\mu n^\nu. \tag{7}
\]

In this lagrangean the metric, connection and curvature are those on the given reference geodesic \( x^\mu_0(\tau) \), with \( u^\mu(\tau) = \dot{x}^\mu_0 \) representing the four-velocity along this same geodesic. These quantities act as background variables. Only the \( n^{\mu}(\tau) \) are independent lagrangean generalized coordinates which are to be varied in the action.

The action (7) can be derived independently by starting from the geodesic lagrangean

\[
L(x) = \frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \tag{8}
\]

and expanding \( x^{\mu}(\tau) \) near the given background geodesic solution in the form \( x^{\mu} = x^{\mu}_0 + n^{\mu} \Delta \lambda \). The term independent of \( \Delta \lambda \) does not contain \( n^{\mu} \), and contributes a constant to the action. Next all terms linear in \( \Delta \lambda \) drop out of the result because \( x_0 \) is a solution of the geodesic equation. Finally, the terms quadratic in \( \Delta \lambda \) reproduce the expression (7), up to a total proper-time derivative and terms which vanish because of the geodesic equation for \( x^{\mu}_0(\tau) \). Thus the lagrangean (7) represents the lowest-order non-trivial term in a systematic expansion:

\[
S[x] = m \int d\tau L(x_0) + m(\Delta \lambda)^2 \int d\tau L(n) + O[(\Delta \lambda)^3] \tag{9}
\]

Obviously, the higher-order approximations can also be derived in this way.

Clearly, as the variation of \( L(n) \) w.r.t. \( n^{\mu} \) reproduces the geodesic deviation equation (5), the background-field method provides an alternative derivation of this equation.

The above procedures can be generalized quite straightforwardly to cases in which test bodies are not completely structureless point masses, but carry e.g. charge and/or spin. In these cases particles do not move on geodesics, but on more general world lines \( \bar{x}^{\mu}(\tau) \). For the case of charged particles in a combined electro-magnetic and gravitational field, the resulting world line deviation equation was derived along the lines of eqs. (3)-(5) in ref. [3]. An alternative derivation using the background field method starts from the action

\[
S_q[x] = \int d\tau \left( \frac{m}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + q A_\mu(x) \dot{x}^\mu \right), \tag{10}
\]
with the overdot the usual short-hand for proper-time derivatives. The world-lines given by the stationary points of this action are solutions of the Einstein-Lorentz equation

\[ \frac{D^2x^\mu}{D\tau^2} = \frac{q}{m} F^\mu_{\nu} \frac{dx^\nu}{d\tau}. \]  

(11)

Now given a solution \( x^\mu_0(\tau) \) of this equation, and expanding the path in \( S_q[x] \) as

\[ x^\mu(\tau) = x^\mu_0(\tau) + \Delta \lambda n^\mu(\tau), \]

(12)

the action can be expanded to second order in \( \Delta \lambda \) as

\[ S_q[x] = S_q[x_0] + \frac{(\Delta \lambda)^2}{2} \int d\tau \left[ m \left( g^\mu_{\nu} \frac{Dn^\mu}{D\tau} \frac{Dn^\nu}{D\tau} + R^\mu_{\nu\lambda\kappa} u^\lambda u^\kappa n^\nu \right) + q \left( F^\mu_{\nu\lambda} u^\nu n^\lambda + \nabla_\mu F^\nu_{\lambda\nu} \right) \right] + O[(\Delta \lambda)^3]. \]

(13)

To this order we then find that other solutions of the world-line equation (11), close to \( x^\mu_0(\tau) \), are given by (12), with \( n^\mu \) the solution of the world-line deviation equation \[ \frac{D^2n^\mu}{D\tau^2} = R^\mu_{\nu\lambda\kappa} u^\lambda u^\kappa n^\nu + \frac{q}{m} F^\mu_{\nu\lambda} \frac{Dn^\nu}{D\tau} + \frac{q}{m} \nabla_\lambda F^\nu_{\nu\lambda} u^\nu n^\lambda. \]  

(14)

The alternative interpretation of \( n^\mu \), as parametrizing the distance between two particles on neighboring world lines, holds in this case as well, provided the particles have the same charge-to-mass ratio \( q/m \). Interestingly, this observation is not contradicted by the fact that one can obtain the Einstein-Lorentz equation as well as the world-line deviation equation (14) from reduction of the geodesic equation and geodesic deviation equation in five-dimensional space-time, as particles with different charge-to-mass ratio in four dimensions correspond to particles with different momentum in five-dimensional space-time [6].

Similarly, pseudo-classical spinning particles can be described by the supersymmetric lagrangean \[ L_{spin}(x, \psi) = \frac{1}{2} g^\mu_{\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} \psi^a \frac{D\psi^a}{D\tau}, \]

(15)

with \( \psi^a \) an anti-commuting tangent-space vector such that the pseudo-classical spin is described by \( S^{ab} = -i\psi^a \psi^b \). The corresponding equations of motion for spinning particles can be written as

\[ \frac{D^2x^\mu}{D\tau^2} = \frac{1}{2} S^{ab} R^\mu_{\nu ab} u^\nu, \quad DS^{ab}{D\tau} = 0. \]

(16)

Starting from a one-parameter congruence of solutions \( (x^\mu(\tau; \lambda), \psi^a(\tau; \lambda)) \) we define the deviation vectors

\[ n^\mu = \frac{\partial x^\mu}{\partial \lambda}, \quad \xi^a = \frac{D\psi^a}{D\lambda} = \frac{\partial \psi^a}{\partial \lambda} - n^\mu \omega^a_{\mu b} \psi^b, \]

(17)

\textsuperscript{1}The transition between base-space and tangent-space vectors is made as usual by the vierbein \( e^a_{\mu} \) and its inverse.
where $\omega^a_{\mu b}$ is the spin connection. The covariant change in the spin-tensor is then

$$J^{ab} = \frac{DS^{ab}}{D\lambda} = -i \left( \psi^a \xi^b + \xi^a \psi^b \right).$$  \hspace{1cm} (18)

These vectors satisfy the world-line deviation equations

$$\frac{D^2 n^\mu}{D\tau^2} = R_{\kappa\nu\lambda}^{\mu} u^\kappa u^\lambda n^\nu + \frac{1}{2} S^{ab} R_{\nu\lambda}^{\mu} Dn^\nu + \frac{1}{2} \left( S^{ab} \nabla^\lambda R_{\nu\lambda}^{\mu} u^\nu n^\lambda + J^{ab} R_{\nu\lambda}^{\mu} u^\nu \right).$$  \hspace{1cm} (19)

They define the stationary points of the quadratic deviation action

$$L_{\text{spin}}(n, \xi) = \frac{1}{2} g_{\mu\nu} Dn^\mu Dn^\nu + \frac{i}{2} \frac{D\xi^a}{D\tau} + \frac{1}{2} R_{\mu\nu\lambda\kappa} u^\kappa n^\lambda u^\mu n^\nu$$

$$- \frac{i}{4} \psi^a \psi^b \left( R_{\mu\nu\lambda\kappa} n^\mu Dn^\nu + \nabla^\lambda R_{\nu\lambda}^{\mu} u^\nu n^\lambda \right)$$

$$- i R_{\mu\nu\lambda\kappa} n^\mu u^\nu \xi^a \psi^b.$$  \hspace{1cm} (20)

2. Application: the Coulomb-Reissner-Nordstrom field

World-line deviation equations can be used to compute the relative motion between particles in given background fields, or to obtain an approximation to solutions for orbits close to a known one. We illustrate the general results with an application to the study of the motion of charged particles in a central gravitational and electric Coulomb-Reissner-Nordstrom field.

The vector potential and electric field strength for the Coulomb part of this solution of the Einstein-Maxwell equations are given by the one- and two-forms

$$A = -\frac{Q}{4\pi r} dt,$$

$$F = dA = \frac{Q}{4\pi r^2} dr \wedge dt,$$  \hspace{1cm} (21)

whilst the metric for the gravitational field can be taken as

$$-d\tau^2 = -B(r) dt^2 + \frac{1}{B(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),$$  \hspace{1cm} (22)

where $B(r) = 1 - (2M/r) + (Q^2/r^2)$; $Q$ and $M$ are the charge and mass of the central body which is the source of the field.

The orbits of particles with mass $m$ and charge $q$ in this background can be computed in closed form in terms of elliptic integrals \[3\]. More precisely, the orbits are given by

$$r(\varphi) = \frac{r_0}{1 + e \cos y(\varphi)},$$  \hspace{1cm} (23)

where $y(\varphi)$ is the solution of the differential equation

$$\frac{dy}{d\varphi} = \sqrt{A + B \cos y + C \cos^2 y},$$  \hspace{1cm} (24)
with coefficients of given by

\[ A = 1 + \frac{Q^2}{\ell^2} \left( 1 - \left( \frac{q}{4\pi m} \right)^2 \right) - \frac{6M}{r_0} + \frac{Q^2}{r_0} (6 + e^2), \]

\[ B = -\frac{2e}{r_0} \left( M - \frac{2Q^2}{r_0} \right), \tag{25} \]

\[ C = \frac{e^2Q^2}{r_0}. \]

Here \( \ell \) is the constant angular momentum per unit of mass. As the periastra of the orbit are at the points \( y(\varphi) = 2\pi n \), one can now compute the angular distance \( \Delta \varphi \) between successive periastra. Writing \( \Delta \varphi = 2\pi + \delta \varphi \), it follows that the periastron shift per orbit is

\[ \delta \varphi = 2\pi \left( \frac{3M}{r_0} - \frac{Q^2}{2Mr_0} \right) + ... \tag{26} \]

the dots denoting terms of higher order in \( e, M/r_0 \) or \( Q/r_0 \).

Eqs. (23) and (24) describe a general orbit in the exterior region of the central body. However, they do not provide all information about the orbit. In particular, as the time coordinate has been eliminated from these equations, the solution does not tell us where in its orbit the test particle is at any moment. Such information can be relevant for some important applications, e.g. to compute estimates of the amount of electro-magnetic and gravitational radiation emitted by the system. The method of world-line deviations is useful to obtain parametrized expressions of orbits \( (r(t), \varphi(t)) \).

As the reference orbit, the zeroth order approximation to the real orbit, we take a circular one with constant radial coordinate \( R \). Constants of motion on all orbits are the angular momentum per unit of mass, \( \ell = \omega R^2 \), with \( \omega = \dot{\varphi} \) the angular velocity, and the energy per unit of mass \( \varepsilon \), defined by

\[ \frac{dt}{d\tau} = \varepsilon - \frac{qQ/4\pi mR}{1 - 2M/R + Q^2/R^2}. \tag{27} \]

On circular orbits the constants \( R, \ell \) and \( \varepsilon \) are then related by

\[ \left( \varepsilon - \frac{qQ}{4\pi mR} \right)^2 = \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) \left( 1 + \frac{\ell^2}{R^2} \right), \tag{28} \]

and

\[ \left[ \frac{\ell^2}{R} - M \left( 1 + \frac{3\ell^2}{R^2} \right) + \frac{Q^2}{R} \left( 1 + \frac{2\ell^2}{R^2} \right) \right]^2 = \left( \frac{qQ}{4\pi m} \right)^2 \left( 1 + \frac{\ell^2}{R^2} \right) \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right). \tag{29} \]

As all orbits are planar, we can always choose the orientation of the coordinate system such that \( \theta = \pi/2 \) for the reference orbit. For orbits tilted w.r.t. this one, we then find from eq. (24) that

\[ \ddot{n}^\theta + \omega^2 n^\theta = 0, \tag{30} \]
from which it follows, as the physics dictates, that the distance perpendicular to the plane of the reference orbit oscillates with the period of the circular orbit \( T = \frac{2\pi}{\omega} = \frac{2\pi R^2}{\ell} \).

Considering orbits in the plane of the reference orbit, the world-line deviation equations (14) for the other components \( n^i = (n^t, n^r, n^\varphi) \) become

\[
\ddot{n}^i + \gamma_j^i \dot{n}^j + m_j^i n^j = 0,
\]

where the coefficient matrices take the form

\[
\gamma = \begin{pmatrix}
0 & \gamma_t^t & 0 \\
\gamma_r^t & 0 & \gamma_r^\varphi \\
0 & \gamma_\varphi^r & 0
\end{pmatrix}, \quad m = \begin{pmatrix}
0 & 0 & 0 \\
0 & m_r^r & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(32)

This represents a system of coupled linear oscillators, which has solutions

\[
n^t(\tau) = n_0^t \sin \omega_1 \tau, \quad n^r(\tau) = n_0^r \cos \omega_1 \tau, \quad n^\varphi(\tau) = n_0^\varphi \sin \omega_1 \tau,
\]

(33)

where \( \omega_1 \) is the solution of the characteristic equation for (31). The detailed form of this equation, using explicit expressions for the elements of the matrices \( \gamma \) and \( m \) were given in [5]. The resulting expression for the characteristic frequency is

\[
\omega_1 = \omega \left( 1 - \frac{3M}{R} + \frac{Q^2}{2MR} + \ldots \right),
\]

(34)

where the dots represent terms of higher order in \( M/R, Q/M \) or \( q/m \). We also observe, that the amplitudes \( n_0^i \) are not all independent: as \( u^2 = -1 \) both on the original orbit and on the displaced world-line, it follows that \( n \) is space-like and \( u \cdot n = 0 \). In the present case this general result translates to the constraint

\[
\left( \varepsilon - \frac{qQ}{4\pi mR} \right) n_0^t - \frac{qQ}{4\pi m \omega_1 R^2} u^t n_0^r - \ell n_0^\varphi = 0.
\]

(35)

Although the components \( n^\mu \) define the direction of the deviation, they do not determine the actual distance between neighboring world lines; this is given by equation (12) as \( \Delta x^\mu = n^\mu \Delta \lambda \). Therefore, for any particular orbit specified by the circular reference orbit (zeroth order approximation) and a world-line deviation vector \( n \) (first order approximation), we must determine in addition the scale factor \( \Delta \lambda \) to be applied. This can be done as follows. Comparing the approximate solution (31) with the exact solution (23), we observe that

\[
r(\varphi) = R + \Delta r \approx R - eR \cos y(\varphi).
\]

(36)

Hence at the periastron, one has

\[
\Delta r = -\Delta \lambda n_0^r = -eR.
\]

(37)

Thus the scale is set by the eccentricity of the orbit.
Finally, we can determine the shift in angular coordinate between successive periastra, i.e., the advance of the periastron per orbital period. First observe, that the periastron occurs at the minima of $n^r(\tau)$, i.e., for $\tau_n = (2n + 1)\pi/\omega_1$. Thus the amount of proper time elapsing between periastra is $\Delta\tau = 2\pi/\omega_1$; the corresponding period of observer time is

$$T = \int_0^{2\pi/\omega_1} d\tau \frac{dt}{d\tau} = \int_0^{2\pi/\omega_1} d\tau (u^t + \dot{n}^t\Delta\lambda) = \frac{2\pi}{\omega_1} u^t.$$ (38)

Here $u^t$ is the rate of change of $t$ per unit of proper time along the circular reference orbit. Next we observe, that at the proper times $\tau_n$ the angular coordinates at the reference orbit and the true orbit coincide: $n^\varphi(\tau_n) = 0$. Hence the change in angular coordinate $\varphi$ between successive periastra is the same as the change of this coordinate along the circular reference orbit after time $T$. This we can easily compute. Defining

$$\delta\varphi = \varphi(t_0 + T) - \varphi(t_0) - 2\pi,$$ (39)

and using for the angular velocity the expression $d\varphi/dt = \varphi d\tau/dt = \omega/u^t$, we find

$$\delta\varphi = \frac{\omega T}{u^t} - 2\pi = 2\pi \left( \frac{\omega}{\omega_1} - 1 \right) \approx 2\pi \left( \frac{3M}{R} - \frac{Q^2}{2MR} \right).$$ (40)

This is in perfect agreement with the expression (26) obtained from the analytical form of the orbit.

It is of interest to consider the geometrical interpretation of the approximation scheme we have used in a little more detail. The zeroth order approximation to the orbit we have constructed is a fully relativistic circular solution of the Einstein-Lorentz equation in a Coulomb-Reissner-Nordstrom field, with period $T_0 = 2\pi/\omega_0$. Included in this result is of course the simpler case of a circular geodesic in a Schwarzschild field. The first-order correction is a geodesic deviation which oscillates in all its components in the same plane with period $T_1 = 2\pi/\omega_1$. Geometrically this represents another circular movement on the
background of the zeroth-order solution, i.e. an *epicycle*, with period slightly different from the zeroth-order approximation. This has two immediate consequences: the orbit becomes eccentric, and the period between extrema of the orbit differs from the period of the average (zeroth order) circular motion. This is in contrast with Newtonian gravity, where the periods are equal. Thus the extrema of the orbit (periastron and apastron) are shifted compared to the Newtonian approximation, by the amount predicted by the analytic description of the orbit.

It can easily be shown [2], that higher-order world-line deviations all satisfy linear harmonic-oscillator type equations. Thus, computing higher-order corrections to our result amounts to the construction of higher-order epicycles. For the case of orbits in a central field, the method of world-line deviations then becomes a fully relativistic version of the Ptolemaean scheme [9], which differs genuinely from the standard post-Newtonian approximation scheme because it uses the eccentricity of the orbit and the quantities $M/R, Q/M$ as expansion parameters, rather than $v/c$. As such this scheme offers an alternative to post-newtonian calculations of binary systems in a different physical regime, e.g. in the calculation of radiative effects.

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**References**


