Loops and Mathematics

Johannes Blümlein

- The First Loop
- 2 Loops
- More Loops

The First Loop

7.A.1 Nuclear Physics B44 (1972) 189–213. North-Holland Publishing Comp

REGULARIZATION AND RENORMALIZATION OF GAUGE FIELDS

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Abstract: A new regularization and renormalization procedure is presented. It is particularly well suited for the treatment of gauge theories. The method works for theories that v known to be renormalizable as well as for Yang-Mills type theories. Overlapping dive gencies are disentangled. The procedure respects unitarity, causality and allows shifts integration variables. In non-anomalous cases also Ward identities are satisfied at all st It is transparent when anomalies, such as the Bell-Jackiw-Adler anomaly, may occur.

Nuclear Physics B153 (1979) 365–401

SCALAR ONE-LOOP INTEGRALS

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The completely general one-loop scalar one-, two-, three- and four-point functions are studied. Also an integral occurring in connection with soft bremsstrahlung is considered. Formulas in terms of Spence functions are given. An expansion for Spence functions with complex argument is presented. The whole forms a basis for the calculation of one-loop radiative corrections in the general case, including unstable particles and particles with spin.

$\Longrightarrow$ establish a complete system of integrals: 2, 3, 4-point functions

$$
\begin{aligned}
B(k, m_1, m_2) &= \int \frac{d^n q}{(q^2 + m_1^2 - i0)((q+k)^2 + m_2^2 - i0)} \\
&= -i\pi^2 \left[\frac{2}{n-4} + \ln(-k^2 - i0)\right] + \sum_j \left\{\ln(1 - x_j) - x_j \ln\left(\frac{x_j - 1}{x_j}\right) - 1\right\}
\end{aligned}
$$

The 3- and 4-point functions

$$C(p_j, m_i) = \int \frac{d^n q}{(q^2+m_1^2)((q+p_1)^2+m_2^2)((q+p_1+p_2)^2+m_3^2)}$$

$$D(p_j, m_i) = \int \frac{d^n q}{(q^2+m_1^2)((q+p_1)^2+m_2^2)((q+p_1+p_2)^2+m_3^2)((q+p_1+p_2+p_3)^2+m_4^2)}$$

$$\frac{C}{i\pi^2} = \int_0^1 dy \frac{1}{(c+2\alpha b)y+d+e\alpha+2a+c\alpha}[\ln\{by^2+(c+e)y+a+d+f\} - \ln\{by_1^2+(c+e)y_1+a+d+f\}]$$
$$-\int_0^1 dy \frac{1-\alpha}{(c+2ab)(1-\alpha)y+d+e\alpha}[\ln\{(a+b+c)y^2+(e+d)y+f\} - \ln\{(a+b+c)y_2^2+(e+d)y_2+f\}]$$
$$-\int_0^1 dy \frac{\alpha}{-(c+2\alpha b)\alpha y+d+e\alpha}[\ln\{ay^2+dy+f\}-\ln\{ay_3^2+dy_3+f\}]. \qquad (5.6)$$

$$Li_2(z) = -\int_0^z \frac{dt}{t}\ln(1-t)$$

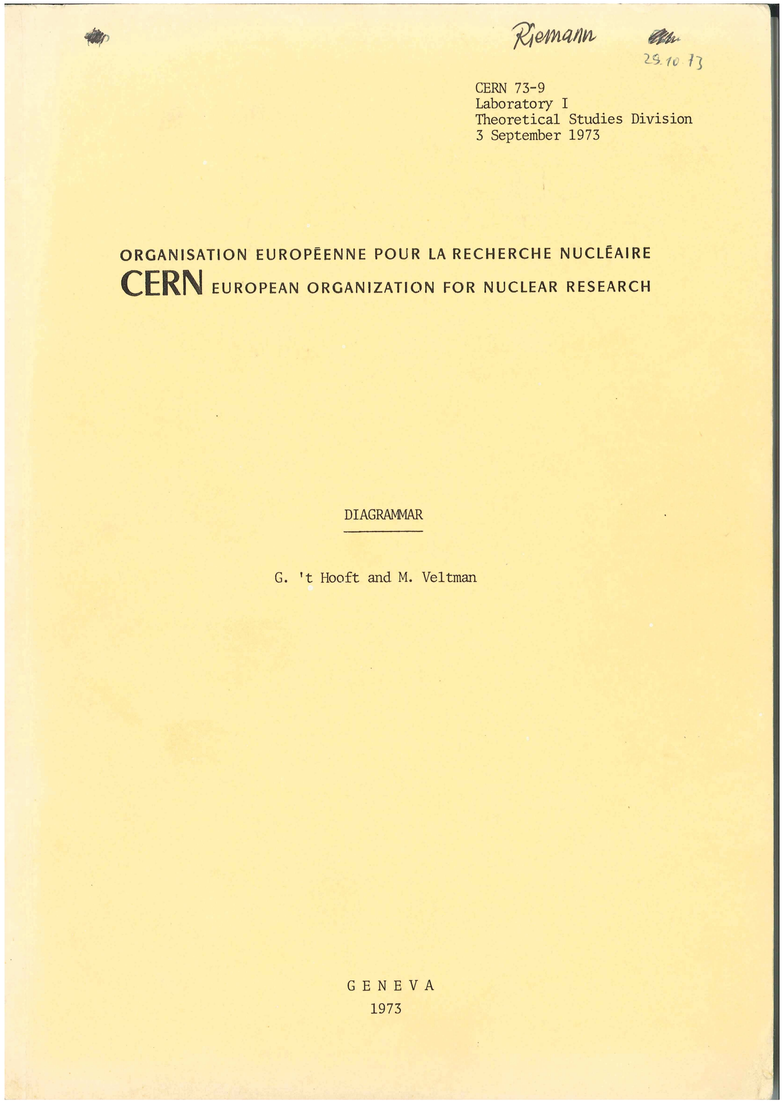
the following result obtains:

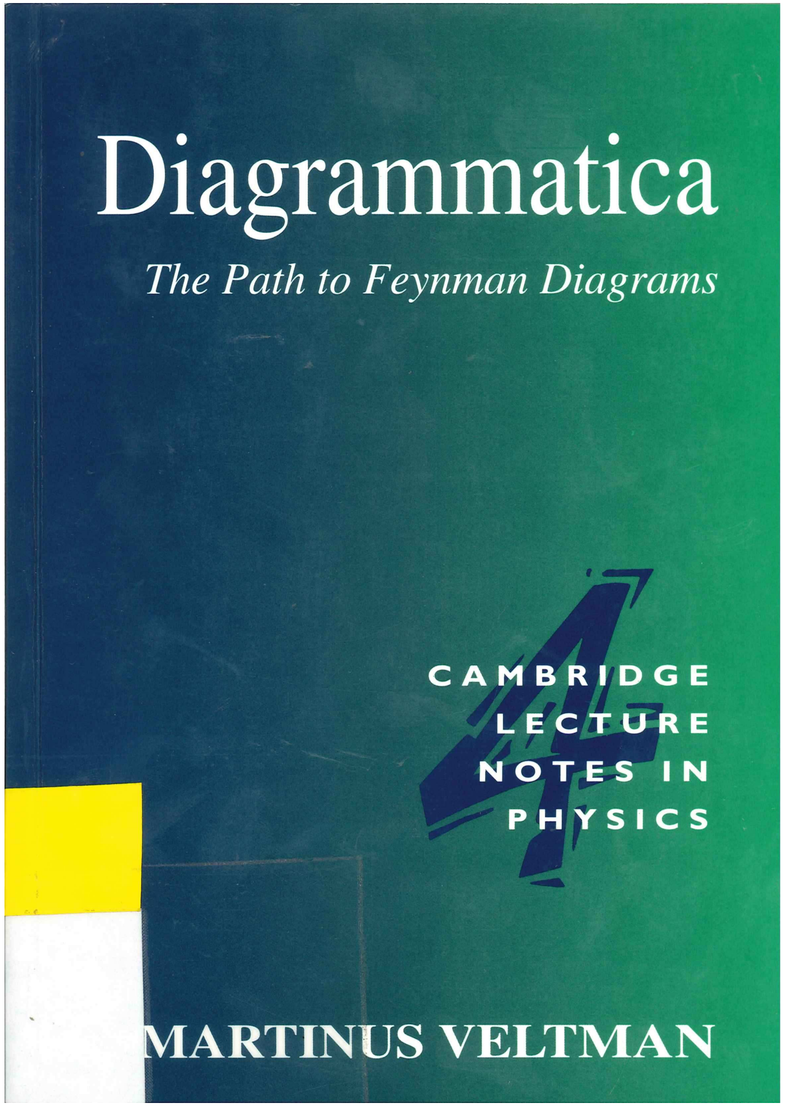
$$\frac{D}{i\pi^2} = \frac{A_1A_2A_3A_4}{k}$$
$$\times\Bigg[-\int_0^1 dy \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y+d+e\alpha}\{\ln L_{24}(y)-\ln L_{24}(y_1)\}-R_{24}^1$$
$$-\int_0^1 dy \frac{\alpha}{-(c+2\alpha\beta)\alpha y+d+e\alpha}\{\ln L_{34}(y)-\ln L_{34}(y_2)\}+R_{34}^2$$
$$+\int_0^1 dy \frac{1}{(c+2\alpha b)y+d+e\alpha+c\alpha+2a}\{\ln L_{23}(y)-\ln L_{23}(y_3)\}+R_{23}^3$$
$$+\int_0^1 dy \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y+d+(e+k)\alpha}\{\ln L_{14}(y)-\ln L_{14}(y_4)\}+R_{14}^4$$
$$+\int_0^1 dy \frac{\alpha}{-(c+2\alpha b)\alpha y+d+(e+k)\alpha}\{\ln L_{34}(y)-\ln L_{34}(y_5)\}-R_{34}^5$$
$$-\int_0^1 dy \frac{1}{(c+2\alpha b)y+d+(e+k)\alpha+c\alpha+2a}\{\ln L_{13}(y)-\ln L_{13}(y_6)\}-R_{13}^6$$
$$+\theta(-A_1A_2)S\Bigg], \qquad (6.12)$$

$$L_{ij}(y) = (-l_{ij}A_iA_j+m_i^2A_i^2+m_j^2A_j^2)y^2+(l_{ij}A_iA_j-2m_j^2A_j^2)y+m_j^2A_j^2-i\varepsilon. \qquad (6.13)$$

The One-Loop Problem: Solved Completely

The Basis for Many Calculations :





Integration = Anti-Differentiation

- Integration is easy if one knows the complete target space.
- Try to construct the target space for the Feynman integrals, forming certain algebras in general.

Important examples:

• Poincaré-iterated integrals

$$L(a_1, ...a_k; y) = \int_0^y \frac{dy_1}{y_1 - a_1} \int_0^{y_1} \frac{dy_2}{y_2 - a_2} ... \int_0^{y_{k-1}} \frac{dy_k}{y_k - a_k}$$

Work out all iterations for the alphabet $\mathfrak{A} = \{a_1, ..., a_k\}, \ a_i \in \mathbb{R}$
e.g. $a_1 = 0, a_2 = 1, a_3 = -1, ...$ polylogs, Nielsen functions, HPL's, ...

• Nested generalized harmonic sums

$$S_{n_1,...n_k}(a_1, ...a_k; N) = \sum_{i_1=1}^{N} \frac{a_1^{i_1}}{i_1^{n_1}} \sum_{i_2=1}^{i_1} \frac{a_2^{i_2}}{i_2^{n_2}} ... \sum_{i_k=1}^{i_{k-1}} \frac{a_k^{i_k}}{i_k^{n_k}}$$

The representations are related by a Mellin transform

$$\mathbf{M}\left[f(x)\right](N) = \int_0^1 dx x^{N-1} [f(x)]_{(+)}$$

Both types of functions form (quasi) shuffle algebras and further obey structural relations.

$$\text{e.g.}: \; L(a_3; y) L(a_1, a_2; y) = L(a_3, a_1, a_2; y) + L(a_1, a_3, a_2; y) + L(a_1, a_2, a_3; y)$$

How do these structures emerge ?

Feynman parameter integrals massless or with one mass as regulator map into multi Mellin Barnes integrals

$$\frac{1}{(2\pi i)^l} \prod_l \int_{\gamma_l - i\infty}^{\gamma_l + i\infty} d\sigma_l \frac{\Gamma(a_1 N + b_1(\sigma_l) + r_1 \varepsilon) ... \Gamma(a_k N + b_k(\sigma_l) + r_k \varepsilon)}{\Gamma(c_1 N + d_1(\sigma_l) + q_1 \varepsilon) ... \Gamma(c_k N + d_k(\sigma_l) + q_k \varepsilon)}$$

These functions define (generalized) hypergeometric functions, Meijer G-functions, and generalizations thereof, which obey (multiple) integral and (multiple) sum representations.

The ε-expansion turns

$$\frac{\Gamma(z+r\varepsilon)}{\Gamma(z)} = 1 + \psi(z)\varepsilon r + \frac{1}{2}\left[\psi'(z) + \psi^2(z)\right]\varepsilon^2 r^2 ...,$$

i.e. into product of single harmonic sums $S_k(z)$ and ζ-values.
The multiple sums, e.g. via z, introduce nesting which can be solved by modern summation methods SIGMA, C. Schneider 2000–

Two Loops

- Very many important calculations all over the world; notably in The Netherlands: LEP, HERA, pp-colliders ...

Veltman, Berends, Gastmans, Gaemers, Laenen, van Neerven, Smith, Vermaseren, van der Bij, ... and many other groups world wide, 1980 –

$${}_pF_q(a_k; b_l; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(1)_n (b_1)_n ... (b_q)_n} x^n$$

More Loops

... necessarily means less external Legs: currently two.

- Massless and massive results up to $O(\alpha_s^5)$

- g-2: $O(\alpha^3)$ Laporta, Remiddi, 1996
- Running α_s: $O(\alpha_s^4)$ Larin, van Ritbergen, Vermaseren 1997; Czakon, 2005
- Unpol. anomalous dimensions and Wilson coefficients: $O(\alpha_s^3)$ Moch, Vermaseren, Vogt 2004/05
- Unpol. NS anomalous dimension 2nd Moment: $O(\alpha_s^4)$ Baikov, Chetyrkin 2006
- Moments of DIS Heavy Flavor Wilson Coefficients for $Q^2 \gg m^2$: $O(\alpha_s^3)$: Bierenbaum, Blümlein, Klein, 2009
- Contributions to the vacuum polarization : $O(\alpha_s^5)$ Baikov, Chetyrkin, Kühn 2003
- R(s), Adler function, ρ-parameter, Z- and τ-decay ...: $O(\alpha_s^4)$ Baikov, Chetyrkin, Kühn, last 10 years

- Zero scale quantities: up to $O(\alpha_s^{4(5)})$
- Single scale quantities: $O(\alpha_s^3)$

More Loops

Massless(ive) Zero Scale Quantities : $\ln(2);\ \zeta_2;\ \zeta_3;\ \mathrm{Li}_4(1/2);\ (\zeta_5,\ \mathrm{Li}_5(1/2));$
$(\mathrm{Li}_6(1/2),\ \sigma_{-5,-1});(\zeta_7,\ \mathrm{Li}_7(1/2),\ \sigma_{-5,1,1},\ \sigma_{5,-1,-1});\ldots$ ⇑3 loops Vermaseren 1998

Massless Single Scale Quantities : $S_1(N), ..., S_{-2,2,1}(N), ...$
⇑3 loops, weight 5

Massive Single Scale Quantities : in addition $..., S_{-2,2,1}\left(\frac{1}{2}, 1, -2; N\right), ...$
⇑3 loops, weight 5

Massive Zero Scale Quantities (more massive lines): (3- and 4 loop)
cyclotomic zeta values, elliptic functions Broadhurst 1998, Ablinger, JB, Schneider, 2011, Laporta 2008

Systematic Integration

Consider graphs with no poles in $1/\varepsilon$. F. Brown, 2008

Zero Scale Graphs can be integrated directly, if all Feynman parameters enter linearly; first problems at 6 loops in scalar field theory.

More Loops

Topics of contemporary research include:

- To which structures does a single mass lead (fixed and variable moments) ?
- Which combinatoric and summation problems arise ?
- What is needed additionally to treat the large amount of diagrams with poles ?
- At which loop level do new structures occur in QED and QCD ?

Symbioses with Mathematics:

- Advanced discrete algorithms and combinatorics $\longrightarrow$ computer algebra
- Complex analysis of new higher transcendental functions
- Theory of irrational numbers, motivic numbers, periods, algebraic geometry

Advanced Technologies to Evaluate Feynman Diagrams

Some Examples:

- Zero-scale Problems : Euler-Zagier and Multiple Zeta Values
  JB, D. Broadhurst, J. Vermaseren, Comput.Phys.Commun. 181 (2010) 582
  find all relations : $\Longrightarrow$ **Tera-Terms** to be processed
  alternating: all relations up to w = 12 (6-loop level);
  non-alternating: all relations up to w = 22; determined.
  Interesting relations: to w = 30;

- Reconstructing recurrent quantities from Mellin Moments
  JB, M. Kauers, S. Klein, C. Schneider, Comput.Phys.Commun. 180 (2009) 2143
  Can one find the anomalous dimensions and Wilson coefficients to 3-loops just from their moments ? Yes - recurrent quantities in Mellin space.
  $\leq$ 5114 Moments; difference equation fills 440 books
  Complete computation: 5 CPU Months

Tini at Loops and Legs

1996: before the prize

Talk about the physics in momentum space.

Veltman prize $\Longrightarrow$ Ettore Remiddi.

Special Talk: 2000

1999

... after the prize

We owe Tini :

- Various key precision predictions
- Many essential technologies in physics
- The example of a straight character
- Reminding us, what is really essential.

Happy Birthday, Tini! - And many Happy Returns.