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Quantum Field Theory*

A.N. Schellekens

NIKHEF, Postbus 41882, 1009 DB Amsterdam, The Netherlands

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1. Introduction

In the first quarter of this century three important revolutions took place in Physics: Special Relativity, Quantum mechanics and General Relativity. It took another quarter century to formulate a theoretical framework that successfully combines the first two concepts, and this is called "Relativistic Quantum Field Theory" (often the first word and sometimes also the second one is dropped, and we simply call it "Field Theory"). Initially, field theory was applied mainly, but with great success, to the theory of photons and electrons, "Quantum Electrodynamics" (QED), but during the third quarter of the century this was extended to the weak and strong interactions, and field theory became the language in which the "standard model" was written. (Perhaps one day we will look back at the last quarter of this century as the epoch during which General relativity was successfully combined with quantum mechanics via "string theory", but that's another story.)

As the name suggests, Relativistic Quantum Field Theory rests really on three pillars, Special Relativity, Quantum Mechanics and Field Theory. There are two distinct logical paths one can follow to arrive at the same goal. The first is to start with quantum mechanics and make it "relativistic". One obvious step is to replace non-relativistic kinematics by relativistic kinematics, but that is not enough. The famous relation $E = mc^2$ allows mass to be converted to energy, which in its turn can be converted to masses of other particles. This allows the creation of particles in high energy collisions. Anyone who has seen the results of collisions in particle accelerators will agree that particle creation is not just kinematically allowed, but is happening abundantly. The extension of quantum mechanics to allow creation and destruction of particles, combined with Lorentz invariance and a few other principles ("unitarity", "locality", "causality") leads almost inevitably to quantum field theory.

The second path is to start with relativistic field theory. This is the logical starting point in electrodynamics, because classical electrodynamics is a theory of fields (the electric field, the magnetic field, and more importantly the vector potential), which in fact is relativistic. Quantizing these fields in the standard way leads also to quantum field theory.

In fact these two approaches are equivalent. Perhaps in some cases the most "natural" starting point is particles (*e.g.* for quarks and leptons) and in other cases fields are the more natural concept (*e.g.* in electrodynamics) but in the end both are on the same footing: we will introduce fields describing quarks and leptons, and we will see that the quanta of the vector potential can be interpreted as particles, the photons.

All particles in the standard model correspond to some field in a quantum field theory. Our task is to understand how this works, how to describe interactions of these particles using quantum field theory, and how to compute various processes. We begin by making all these words a little bit more precise. We start with a very brief reminder of special relativity.

1.1. Special relativity

Special relativity can be derived from the assumption that the speed of light is the same for all observers, even if they are in relative motion. To reconcile that with momentum conservation one needs a modification of the relation between momentum and velocity,

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(\frac{\vec{v}}{c})^2}} \ .$$

One also obtains a new relation between energy and momentum

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4} \ ,$$

which when expanded gives

$$E = mc^2 + \tfrac{1}{2}\frac{\vec{p}^2}{m} + \ldots = mc^2 + \tfrac{1}{2}m\vec{v}^2 + \mathcal{O}\left(m\vec{v}^2(\frac{\vec{v}}{c})^2\right) \ .$$

For a particle at rest this implies a relation between mass and energy, $E = mc^2$. The second term is the well-known classical kinetic energy of a particle in motion, and the higher terms are relativistic corrections, which are important when the particles move at velocities close to the speed of light. This is usually the case in high energy physics.

The simplest way to write down theories that are consistent with special relativity (*i.e.* that look the same from the point of view of observers in relative motion) is to write them in a manifestly Lorentz invariant way. To do so one combines space and time variables $\vec{x}$ and t into a four-vector $x^\mu = (ct, \vec{x})$. The index μ takes the values $0, 1, 2, 3$, where 0 is the time component. To refer to the space-components we use indices $i, j, k, \ldots$. From now one we will usually set $c = 1$.

1.2. Lorentz invariance

To write down Lorentz invariants one introduces the *metric* tensor, $g_{\mu\nu}$, which (numerically) is a diagonal 4×4 matrix with diagonal matrix elements $1, -1, -1, -1$. Then the bilinear

$$x^2 \equiv x^\mu g_{\mu\nu} x^\nu \equiv \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} x^\mu g_{\mu\nu} x^\nu \tag{1.1}$$

is invariant under Lorentz transformations.

These transformations act in the following way on the space-time coordinates

$$x'^\mu = L^\mu{}_\nu x^\nu \ , \tag{1.2}$$

where L is the Lorentz transformation matrix. The invariance of (1.1) follows because Lorentz-transformations by definition leave the metric invariant:

$$g_{\rho\sigma} L^\rho{}_\mu L^\sigma{}_\nu = g_{\mu\nu} \ . \tag{1.3}$$

In writing these relations we have made a distinction between upper and lower indices, and we defined x^μ with an upper index, and $L^\mu{}_\nu$ with one upper and one lower index. This should be regarded as the definition of these objects. Given such a definition, one may define quantities with lowered indices using the metric:

$$x_\mu \equiv g_{\mu\nu} x^\nu \ .$$

To raise indices we define a metric $g^{\mu\nu}$ which as a matrix is the inverse of $g_{\mu\nu}$ (and hence is numerically equal to $g_{\mu\nu}$). Therefore

$$x_\nu x^\nu \equiv x^\mu g_{\mu\nu} x^\nu \equiv x_\mu g^{\mu\nu} x_\nu \ .$$

Rather than writing the metric explicitly in equations, we just make sure that lower indices are always contracted with upper indices. Then Lorentz invariance is automatic.

To respect this convention it is a good idea to define upper and lower indices for the Lorentz transformations as above. However, if one is only interested in the values of L and g one may ignore this and simply read (1.3) as a matrix relation,

$$L^T g L = g \ . \tag{1.4}$$

The matrices L include as a subset the space-time rotation matrices. They also include of course the Lorentz boosts, for example (remember that $c = 1$)

$$\begin{aligned} z' &= \frac{z - vt}{\sqrt{1 - \vec{v}^2}} \ , \\ t' &= \frac{t - vz}{\sqrt{1 - \vec{v}^2}} \ . \end{aligned} \tag{1.5}$$

Mathematically, the matrices L are said to form a group $SO(3,1)$, the Lorentz group.

1.3. Tensors

A Lorentz tensor is a quantity with a certain number of upper and lower indices that transforms as the indices suggest. Note that quantities with upper and lower indices transform differently under Lorentz transformations:

$$x'_\mu = L_\mu{}^\nu x_\nu \ . \tag{1.6}$$

The difference between $L_\mu{}^\nu$ and $L^\mu{}_\nu$ is just raising and lowering an index, which amounts to a few sign changes. A tensor

$$T^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_p}$$

transforms as

$$T'^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_p} = L^{\mu_1}{}_{\rho_1} \ldots L^{\mu_k}{}_{\rho_k} L_{\nu_1}{}^{\sigma_1} \ldots L_{\nu_p}{}^{\sigma_p} T^{\rho_1,\ldots,\rho_k}{}_{\sigma_1,\ldots,\sigma_p} \ .$$

1.4. Derivatives

Derivatives require a bit of attention. We have of course

$$\frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}) \ .$$

If we compute the derivative of a Lorentz invariant quantity

$$\frac{\partial}{\partial x^\mu} x^\nu y_\nu \ ,$$

the result is y_μ. Hence we should regard the derivative as an object with a *lower* index:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ ,$$

with the property

$$\partial_\mu x^\nu = \delta^\nu_\mu \ ,$$

where δ is numerically equal to a Kronecker delta. One may also check that the derivative transforms as shown in (1.6). If we have a function $\phi(x)$ which depends only on x via the Lorentz invariant $x_\mu x^\mu$, then the combination

$$\partial_\mu \partial^\mu \phi(x)$$

is Lorentz invariant.

1.5. The mass shell condition

Energy and momentum are also combined into a 4-vector: $p^\mu = (E, \vec{p})$. With our choice of metric the "square" of a four-momentum is equal to

$$p^2 \equiv p_\mu p^\mu = E^2 - \vec{p}^2 = m^2 \ . \tag{1.7}$$

This is often called the "mass-shell condition" for a particle.

Some people use a different choice for the metric, namely diag $(-1, 1, 1, 1)$. This does not affect the final results, but it does affect most formulas. For example, the mass shell condition is $p^2 = -m^2$ in that metric.

1.6. Hamiltonians and Lagrangians

Consider a particle whose position is given by a function $x(t)$. Suppose this particle is moving in a potential $V(x)$. The classical (non-relativistic, non-quantum) equations of motion are simply Newton's law,

$$F = m\ddot{x} \ .$$

The force is the derivative of the potential,

$$F = -\frac{d}{dx}V(x) \ .$$

The equation of motion is thus

$$m\ddot{x} = -\frac{d}{dx}V(x) \ . \tag{1.8}$$

We now introduce the *Lagrangian* L of the system:

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - V(x) \ . \tag{1.9}$$

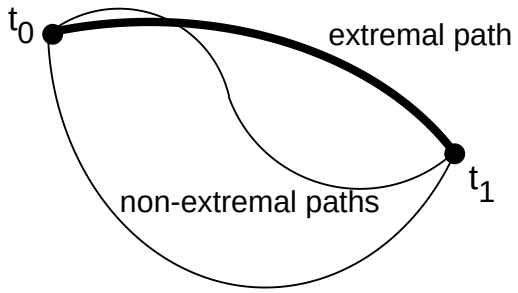
Then it is easy to see that (1.8) is given by the *Euler-Lagrange* equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \ .$$

This may seem a rather pointless operation, but the nice feature of this Lagrangian formulation is Hamilton's action principle. Suppose a particle moves from x_0 at an initial time t_0 to a point x_1 at a final time t_1. It can move between these points along some arbitrary trajectory $x(t)$, with $x(t_0) = x_0$ and $x(t_1) = x_1$. Now we define the action as

$$S(x(t)) = \int_{t_0}^{t_1} L(x, \dot{x})dt$$

Note that S gives us a number for every function $x(t)$.



The equations of motion do not allow an arbitrary trajectory, but a very specific one.[⋆] It turns out that the function $x(t)$ that corresponds to the actual, classical path is an extremum of the function S:

$$\delta S(x(t)) = 0 \ . \tag{1.10}$$

The Euler-Lagrange equations can be derived from this principle.

This principle may be taken as a very general definition of a classical mechanical system. In general one has a set of dynamical variables $q_i(t)$ (for example the positions of many particles) and a Lagrangian $L(q_i, \dot{q}_i)$. The equations of motion of this system are the Euler Lagrange equations (following from Hamilton's principle):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \ .$$

Another important notion is that of the *canonical momentum*, defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \ .$$

For example for Lagrangian (1.9) one gets $p = m\dot{x}$, as one might have expected.

⋆ One may be more used to a situation where the initial position and velocity are specified, and the equations of motion determine $x(t)$. Here we specify the initial and final position, but that is the same amount of data.

Finally we introduce the Hamiltonian of the system as

$$H = \sum_i \dot{q}_i p_i - L \ .$$

Using the equations of motion one may show that this quantity is conserved, $\frac{dH}{dt} = 0$ (assuming L does not depend explicitly on time, *i.e.* $\frac{\partial L}{\partial t} = 0$). This quantity has the dimension of energy, and since it is conserved it cannot be anything else than the total energy of the system. In the example we started with we have indeed

$$H = \tfrac{1}{2}mv^2 + V(x) \ .$$

1.7. Quantum Mechanics

The quantization of a classical system described by a Lagrangian (and without constraints) is very simple: one replaces the dynamical variables q_i and their canonical momenta p_i by operators in a Hilbert space, and imposes the canonical commutation relations

$$[q_i, p_j] = i\hbar\delta_{ij} \ . \tag{1.11}$$

The concept of the position of a particle is replaced by that of a wave function. The time-dependence of the wave-function is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(q_i, t) = H(\hat{p}_i, \hat{q}_i)\psi(q_i, t) \ ,$$

where $H(\hat{p}_i, \hat{q}_i)$ is the classical Hamiltonian with p_i and q_i replaced by operators $\hat{p}_i$ and $\hat{q}_i$.

In the well-known case of a single particle in a classical potential the operators can be realized explicitly as $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, where the Hilbert space is the space of functions of x, and the operator $\hat{x}$ is simply multiplication by x (here the hats serve to distinguish quantum operators from their classical analogues). By considering a basis of wave functions of the

form

$$\psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

one obtains then the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}(\frac{\partial}{\partial x})^2 + V(x)\right]\psi(x) = E\psi(x) \ .$$

In the simplest possible case, that of a free particle ($V(x) = 0$), the solutions of this equation are plane waves:

$$\psi_k(x) = e^{ikx} \ ,$$

with $E = \frac{p^2}{2m}$, with $p = \hbar k$. Plane waves have a definite momentum. One can make a linear combination to make a wave packet, introducing a function $\psi(k)$. Generalizing to three space dimensions we get

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot\vec{x}-i\omega t}\psi(\vec{k}) \ , \tag{1.12}$$

where $\omega = \frac{p^2}{2m} = \frac{E}{\hbar}$ and $\psi(\vec{k}) \equiv \langle k|\psi\rangle$ is the momentum space wave function. The pre-factor is conventional.

1.8. Relativistic Quantum Mechanics

Let us try to make the foregoing system relativistic. We replace the Hamiltonian by the relativistic expression for the energy, with momentum replaced by an operator:

$$H = \sqrt{-\hbar^2(\frac{\partial}{\partial\vec{x}})^2 + m^2}$$

We can still solve the Schrödinger equation in terms of plane waves:

$$\psi_{\vec{k}}(\vec{x},t) = e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}$$

with $\vec{p} = \hbar\vec{k}$, $E = \hbar\omega$ and

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \ .$$

From here on we will also set $\hbar = 1$.

As in the non-relativistic case we can make wave-packets

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega} e^{i\vec{k}\cdot\vec{x}-i\omega t}\psi(\vec{k}) \ . \tag{1.13}$$

The only differences with the non-relativistic wave-packet (1.12) are the relation between ω and k and the factor 2ω. The latter is present because otherwise the infinitesimal volume element d^3k is not Lorentz-invariant.

The combination $d^3k/2\omega$ can be written as

$$\frac{d^3k}{2\omega(\vec{k})} = d^4k\ \delta(k^2-m^2)\theta(k^0) \ ,$$

which is Lorentz invariant (note that k on the left-hand side denotes the space momentum $\vec{k}$, whereas k on the right hand side is the 4-momentum k^μ). The equality is shown by integrating both sides; on the left hand side the integration over k^0 is performed, removing the δ-function that imposes the mass-shell constraint (1.7)*. The θ-function selects one of the two roots of the argument of the δ-function.*

The equation we are considering so far,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[\sqrt{-(\frac{\partial}{\partial\vec{x}})^2+m^2}\right]\psi(x,t)$$

certainly doesn't look relativistic, since t and $\vec{x}$ play rather asymmetric roles. However, note that any solution to the previous equation will also satisfy

$$\left[(\frac{\partial}{\partial t})^2-(\frac{\partial}{\partial\vec{x}})^2+m^2\right]\psi(x^\mu) = (\partial_\mu\partial^\mu+m^2)\psi(x^\mu) = 0 \ . \tag{1.14}$$

This equation is known as the Klein-Gordon equation. However, this equation introduces additional solutions, with ω replaced by $-\omega$. Such additional solutions have negative total energy (where energy includes the rest mass!) and are clearly unacceptable. This is a first indication that a different treatment will be necessary.

1.9. The hydrogen atom

To get an idea of the importance of relative corrections it is instructive to study the hydrogen atom. In non-relativistic quantum mechanics the Hamiltonian is

$$H = -\frac{1}{2m}(\frac{\partial}{\partial \vec{x}})^2 - \frac{\alpha}{r} \ ,$$

and the energy levels are

$$E_n = -\tfrac{1}{2}\frac{\alpha^2}{n^2}m \ ,$$

where m is the electron mass and α the fine-structure constant. For given n there are degenerate states with orbital angular momentum $l = 0, \ldots, n-1$.

If instead of the non-relativistic Hamiltonian ones uses the Klein-Gordon equation (with an additional term to include the Coulomb potential), and one simply ignores the negative energy states, one gets

$$E_{n,l} = \frac{m}{\sqrt{1 + \frac{\alpha^2}{\left[n-l-\frac{1}{2}-\sqrt{(l+\frac{1}{2})^2-\alpha^2}\right]^2}}} \ , \tag{1.15}$$

which can be expanded as

$$E_{n,l} = m(1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{n^3}\left(\frac{1}{l+\frac{1}{2}} - \frac{3}{4n}\right) + \ldots) \tag{1.16}$$

The relativistic correction is seen to give a small correction to the energy levels, which splits the orbital angular momentum degeneracy.

However this answer is in a sense less good than the non-relativistic one since it misses a contribution that is equally important as the relativistic correction. Up to now we have ignored the spin of the electron, which contributes via the spin-orbit coupling, which is of order $\vec{S} \cdot \vec{L}/r^3$. Taking this into account has the effect of replacing l by the total angular momentum j in (1.16). The fact that the effect of spin enters in a similar way as the first relativistic correction suggests that these two effects should be dealt with simultaneously: the Klein-Gordon equation is not the appropriate one, and we need a new equation for relativistic spin-$\frac{1}{2}$ particles. This equation is the Dirac-equation, to be discussed later. The result of using the Dirac equation to compute the energy levels is again (1.15), but now with l replaced by j everywhere, rather than just in the first correction.

The Dirac equation does not solve the problem of the extra negative energy solutions we found for the Klein-Gordon equation: it has negative energy solutions as well. These negative energy solutions are a first hint of a very general feature predicted by quantum field theory: the existence of anti-particles. Properly dealing with that requires more than just relativistic quantum mechanics in the sense of this section. There is another phenomenon that is not taken into account so far. All equations written down so far describe the wave function of a single particle in a potential. The effects of particle creation are not included. Because of the uncertainty relation there is even a more subtle effect: a particle and anti-particle pair can be created from the vacuum and annihilate again, temporarily "borrowing" some energy from the vacuum. Such particles, which do not appear in the initial and final states of a process, are called *virtual* particles.

In the case of the hydrogen atom the effect of taking into account the lowest order quantum field theory correction is a splitting in the degenerate doublets of the solution to the Dirac equation: states with different orbital angular momentum coupling to the same value of j. At the second level the states with $j = \frac{1}{2}$ and $l = 0$ resp. $l = 1$ (usually denoted as $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ are separated by 1057.845(9) MHz (about 4.38×10^{-6} eV). The theoretical prediction from quantum electrodynamics is 1057.857(12) MHz. The effects of full quantum field theory as compared to relativistic quantum mechanics are small in atomic physics, because the kinetic and potential energy of the electron in a hydrogen atom is much smaller than its rest mass. This is not the case in particle physics, and this is why we cannot be satisfied with just writing down relativistic wave equations.

2. Quantum Field Theory

2.1. Multi-particle Quantum mechanics

For the reasons explained above we need to extend our single particle Hilbert space. We introduce a multi-particle Hilbert space which is called a Fock space. It consists of states labelled by the momenta (and later also spins, as well as any other quantum numbers) of an arbitrary number of particles.

First of all there is a zero-particle state $|0\rangle$. Its normalization is $\langle 0 \,|0\rangle = 1$. Then we introduce one-particle states. We choose them as eigenstates of momentum, and denote them as $|\vec{k}\rangle$.

We normalize them as follows

$$\langle \vec{k}' | \vec{k} \rangle = 2\omega(\vec{k})(2\pi)^3 \delta^3(\vec{k} - \vec{k}') \ .$$

The factor 2ω appears in order to cancel the one in Lorentz-invariant momentum integrals, as in (1.13), so that

$$\frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} \langle \vec{k}' | \vec{k} \rangle = 1 \ .$$

Note that the state $|\vec{k} = 0\rangle$ is not the vacuum: it is a one-particle state of a zero-momentum particle.

Multi-particle states $|\vec{k}_1, \ldots \vec{k}_n\rangle$ are introduced in a similar way. States with different numbers of particles are orthogonal. Among states with the same number of particles there are orthogonality relations like

$$\langle \vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2 \rangle = 2\omega(\vec{k}_1) 2\omega(\vec{k}_2) (2\pi)^6 \left[\delta^3(\vec{k}_1 - \vec{k}_1') \delta^3(\vec{k}_1 - \vec{k}_1') + \delta^3(\vec{k}_1 - \vec{k}_2') \delta^3(\vec{k}_2 - \vec{k}_1') \right] \ ,$$

etc. Note that we are considering identical particles here, so that there are 2 (in general $n!$) possible overlaps.

Up to now the Fock space consists of disjoint parts, each with a fixed number of particles. Now we introduce operators that allow us to change the number of particles, creation and annihilation operators. They can be defined as follows

$$\begin{aligned} |\vec{k}\rangle &= a_{\vec{k}}^\dagger |0\rangle \\ a_{\vec{k}} |0\rangle &= 0 \\ [a_{\vec{k}}, a_{\vec{k}'}^\dagger] &= 2\omega(\vec{k})(2\pi)^3 \delta^3(\vec{k} - \vec{k}') \ . \end{aligned} \tag{2.1}$$

The first of these relations states that the creation operator $a_{\vec{k}}^\dagger$ creates a one particle state from the vacuum; the second one ensures that the annihilation operator $a_{\vec{k}}$ annihilates the vacuum, and the last relation, combined with the first two, ensures that the one-particle states are correctly normalized. One also has $\langle 0 | a_{\vec{k}}^\dagger = 0$. It is the easy to check that also the multi-particle states are properly normalized, and that the action of the creation (annihilation) operator increases (decreases) the number of particles in a state by one.

Cross sections in particle physics will be related to transition amplitudes among certain multi-particle states. These amplitudes are matrix elements of operators that take some set of initial states to a set of final states. Such operators can be built out of the creation and annihilation operators, but this is not a very convenient basis to work with. For example, we want to be able to make properties like Lorentz-invariance manifest. A more useful basis turns out to be the set of quantum fields

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega(\vec{k})}\left(a_{\vec{k}}e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t} + a^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}+i\omega(\vec{k})t}\right) . \tag{2.2}$$

This is nothing but a Fourier-transformation, so nothing is lost: we can always express the creation and annihilation operators back in terms of quantum fields. If we sandwich the quantum field between the vacuum and a state $|\psi\rangle$ we get

$$\langle 0|\phi(\vec{x},t)|\psi\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega(\vec{k})}e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}\langle\vec{k}|\psi\rangle ,$$

which is precisely the relativistic one-particle wave-packet, *cf.* (1.13). The annihilation part is not needed here, but if we left it out we wouldn't have a complete set of operators.

2.2. Classical Field Theory

There is a different way of arriving at quantum fields, namely by quantizing a classical field theory. A classical field is just a function of space and time satisfying an equation of motion. Examples are electric and magnetic fields or the vector potential A_μ in classical electrodynamics. Here we will start with a simpler case, a scalar field. This is just a real function $\varphi(x^\mu)$. Let us consider a scalar field satisfying the following equation of motion

$$(\partial_\mu\partial^\mu + m^2)\varphi(x^\mu) = 0 . \tag{2.3}$$

This is just the Klein-Gordon equation again, but the similarity with (1.14) stops there. In this case φ is a classical object that can be measured directly, whereas in (1.14) ψ is a quantum-mechanical wave function whose absolute value represents a probability. The reason the same equation appears in both cases is that there is just one way to write down a relativistic wave equation with at most two derivatives.

We can solve (2.3) very easily by means of Fourier analysis. Define

$$\varphi(k^\mu) = \int d^4x e^{ik_\mu x^\mu} \varphi(x^\mu)$$

and inversely,

$$\varphi(x^\mu) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik_\mu x^\mu} \varphi(k^\mu) \ .$$

Then the equation becomes

$$(-k^2 + m^2)\varphi(k^\mu) = 0 \ .$$

This is easy to solve: either $\varphi(k^\mu) = 0$ or $k^2 = m^2$. In other words, we may write

$$\varphi(k^\mu) = 2\pi\delta(k^2 - m^2)\ a_{k^\mu} \ .$$

Now we insert this back into the Fourier expansion of $\varphi(x^\mu)$ and we integrate over k^0. It is important to note that the argument of the δ function has two roots, $k^0 = \pm\sqrt{\vec{k}^2 + m^2} = \pm\omega(\vec{k})$. To evaluate the k^0 integral one uses the following property of δ functions

$$\delta(f(x)) = \sum_{\text{roots } x_n} \left|\frac{1}{f'(x_n)}\right| \delta(x - x_n) \ ,$$

where the sum is over all x_n with $f(x_n) = 0$. Using this one arrives at

$$\varphi(x^\mu) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(\vec{k})} \left(e^{i\vec{k}\cdot\vec{x} - i\omega t} a_{(\omega,\vec{k})} + e^{i\vec{k}\cdot\vec{x} + i\omega t} a_{(-\omega,\vec{k})}\right) \ .$$

In the second term we may perform a change of integration variable $\vec{k} \to -\vec{k}$. Finally we use the fact that $\varphi(x)$ is a real field. This clearly implies that

$$a(-k^\mu) = a(k^\mu)^* \ .$$

Since the k^0 component has become a dependent variable, we may write a as a function of $\vec{k}$ alone. The final result is

$$\varphi(x^\mu) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(\vec{k})} \left(e^{i\vec{k}\cdot\vec{x} - i\omega t} a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x} + i\omega t} a^*_{\vec{k}}\right) \ . \tag{2.4}$$

This looks strikingly similar to (2.2), except that $a_{\vec{k}}$ there is an operator, whereas here it is just a complex number. Obviously we should quantize our classical field theory to get a closer relation.

2.3. Canonical quantization of classical field theory

To quantize the classical theory described above we may follow the same procedure as for any other classical theory. We will first attempt to find a Lagrangian whose Euler-Lagrange equations reproduce our field equation (2.3).

We first have to identify the variables q_i of section 1.6. It seems plausible that we should take

$$\varphi(\vec{x},t) \leftrightarrow q_i(t) \ ,$$

in other words we view x as a label of a dynamical variable φ, which plays the role of q_i. Note that space and time are treated in an asymmetric way here. Since $\vec{x}$ is a continuous variable, summing over i becomes integration over x. Differentiation with respect to q_i works as follows:

$$\frac{dq_i}{dq_j} = \delta_{ij} \ .$$

Analogously we define differentiation by $\varphi(\vec{x})$ as*

$$\frac{\delta\varphi(\vec{x})}{\delta\varphi(\vec{y})} = \delta^3(\vec{x}-\vec{y}) \ ,$$

where we used δ instead of ∂ to distinguish the *functional derivative* from an ordinary derivative.

The name can be explained as follows. A functional derivative acts on functionals $F(f)$. A functional assigns to every function f a number. For example the integral of f between certain limits is a a functional of f. Another example is the action, which is a functional of the fields and their derivatives. The functional derivative $\delta F/\delta f$ can be thought of as the derivative of $F(f)$ with respect to its argument, the function f.

For example, the analog of

$$\frac{d}{dq_i}\sum_j q_j^2 = 2q_i$$

now becomes

$$\frac{\delta}{\delta\varphi(\vec{y})}\int d^3x\varphi(\vec{x})^2 = \int d^3x 2\varphi(\vec{x})\delta^3(\vec{x}-\vec{y}) = 2\varphi(y) \ . \tag{2.5}$$

$\star$ We omit the argument "t" in the following.

We are now ready to write down the Lagrangian:

$$L(\varphi(\vec{x}),\dot{\varphi}(\vec{x})) = \int d^3x \tfrac{1}{2}\left[\partial_\mu\varphi(\vec{x},t)\partial^\mu\varphi(\vec{x}) - m^2\varphi(\vec{x})^2\right] \ .$$

This is integrated over all of space. We are implicitly assuming that this integral make sense, which will be the case if the field φ falls off to zero sufficiently fast at spatial infinity. Let us check that this gives the correct equation. First we write time and space derivatives explicitly:

$$L = \int d^3x \tfrac{1}{2}\left[\dot{\varphi}(\vec{x})^2 - (\frac{\partial}{\partial x^i}\varphi(\vec{x}))^2 - m^2\varphi(\vec{x})^2\right] \ .$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\varphi}(\vec{y})}\right) - \frac{\delta L}{\delta\varphi(\vec{y})} = 0 \ .$$

The $\dot{\varphi}$ derivative acts only on the first term in the Lagrangian. Then the first term of the Euler-Lagrange equation yields straightforwardly $\ddot{\varphi}(y,t)$. The φ derivative in the Euler-Lagrange equations acts on the second and third term in the Lagrangian. The action on the second term is somewhat tricky because of the space-derivatives. but they can be moved by integrating by parts:

$$\int d^3x(\frac{\partial}{\partial x^i}\varphi(\vec{x}))^2 = -\int d^3x\varphi(\vec{x})(\frac{\partial}{\partial x^i})^2\varphi(\vec{x}) \ .$$

Note that there are no boundary terms because φ vanishes at infinity. Now we can differentiate the first factor φ using (2.5). To differentiate the second one we move all derivatives to the first factor. The action on the third term was already explained in (2.5). Putting all together we get

$$\ddot{\varphi}(y) - (\frac{\partial}{\partial y^i})^2\varphi(\vec{y}) + m^2\varphi(\vec{y}) = 0 \ ,$$

which is indeed the Klein-Gordon equation.

This derivation doesn't look Lorentz-invariant, but does lead to a Lorentz invariant equation. We can get a Lorentz invariant expression by integrating the Lagrangian over time from

$t = -\infty$ to $t = +\infty$. This leads to the following action

$$S = \int d^4x \tfrac{1}{2} \left[\partial_\mu \varphi(\vec{x}, t) \partial^\mu \varphi(\vec{x}, t) - m^2 \varphi(\vec{x}, t)^2 \right] \ .$$

Since the equations of motion follow from varying the action, it is no surprise that they are Lorentz-invariant as well. We may derive a set of Euler-Lagrange equations that is more suitable for field theory by defining a *Lagrange density* that is the integrand of the action:

$$S = \int d^4x \mathcal{L} \ .$$

In terms of $\mathcal{L}$ the Euler-Lagrange equations are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \ ,$$

where $\mathcal{L}$ is viewed as a function of φ and $\partial_\mu \varphi$. Also in this form the Euler-Lagrange produce of course the Klein-Gordon equation.

The canonical momentum of $\varphi(x)$ is

$$\pi(y) = \frac{\delta L}{\delta \dot{\varphi}(y)} = \dot{\varphi}(y) \ .$$

This can be used to define the Hamiltonian as

$$H = \int d^3x \pi(x) \dot{\varphi}(x) - L$$

Analogous to the Lagrange density, we may define a Hamiltonian density

$$\mathcal{H} = \pi(x) \dot{\varphi}(x) - \mathcal{L}(\varphi(x)) \ ,$$

which has the property that

$$H = \int d^3x \mathcal{H} \ .$$

To quantize the theory we simply impose the commutation relation, analogous to (1.11)

$$[\varphi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \ .$$

This is often called an *equal time commutator*. In the quantum theory the physical degrees of freedom, $a_{\vec{k}}$, become operators. The complex conjugate $a^*_{\vec{k}}$ becomes a Hermitean conjugate $a^\dagger_{\vec{k}}$. Substituting (2.4) into the commutation relation we arrive at a condition on the

operators:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = 2\omega(\vec{k})(2\pi)^3\delta^3(\vec{k}-\vec{k}') \ .$$

We have now arrived at the same point as in the previous discussion of multi-particle quantum mechanics, if we make the appropriate choice of vacuum.

2.4. Interactions

We have now seen the most trivial example of a field theory. It is specified by a Lagrangian density (usually one calls this just the "Lagrangian")

$$\mathcal{L} = \tfrac{1}{2}\left[\partial_\mu\varphi(\vec{x},t)\partial^\mu\varphi(\vec{x},t) - m^2\varphi(\vec{x},t)^2\right] \ . \tag{2.6}$$

This field theory is called the "free massive real scalar boson".

It is a boson because the quantum properties are described in terms of commutators. Later we will encounter fermionic fields whose quantization involves anti-commutators.

It is a scalar because φ has only one component and there are no indices; therefore it cannot be rotated into anything else and must be a singlet. The particles this theory describes are therefore spin-0 particles or scalars. Later we will encounter fields with additional indices that transform non-trivially under rotations. To describe the standard model we will in fact need only two types of fields: spin-1/2 fields to describe leptons and quarks, and spin-1 fields to describe photons, gluons and W and Z bosons. Scalars occur also in the standard model Lagrangian: the elusive Higgs boson is a scalar. Other examples of scalar particles are the pions, but they are known to be built out of quarks, and do not appear in the standard model Lagrangian. At very low energies, when their internal structure can be ignored, one can however successfully describe them using quantum field theory.

The previous two properties, "statistics" (fermi or bose) and spin are in fact related to each other via the celebrated *spin-statistics theorem*. This states that any half-integer spin field must be fermionic, and any integer spin field bosonic.

Continuing our list of properties, the scalar bosons considered thus far are *real* because the field φ is real. There are also complex scalars, as we will see below.

Finally, the scalar described by (2.6) is "massive" for obvious reasons, and it is called "free" because an N particle initial state evolves directly (via the Klein-Gordon equation) to an N-particle final state. This can be seen from the fact that the Hamiltonian is bilinear in the creation and annihilation operators. There its matrix elements vanish unless the initial states precisely match the final states. Clearly to get interactions we need higher order terms in the creation and annihilation operators, and hence higher order terms in the fields. For example, we could add to the Lagrangian terms like (the numerical factors are for future purposes)[⋆]

$$\mathcal{L}_I = -\frac{1}{3!}\lambda_3\varphi^3 - \frac{1}{4!}\lambda_4\varphi^4 \ . \tag{2.7}$$

2.5. Complex scalars

The Lagrangian of a free massive complex scalar looks like this

$$\mathcal{L} = \left[\partial_\mu\varphi(\vec{x},t)^*\partial^\mu\varphi(\vec{x},t) - m^2\varphi(\vec{x},t)^*\varphi(\vec{x},t)\right] \ .$$

The structure is constrained to be of this form by two constraints: we want the Hamiltonian to be real and bounded from below.[†]

By a simple change of basis

$$\varphi = \frac{1}{\sqrt{2}}[\varphi_1 + i\varphi_2] \tag{2.8}$$

this becomes the Lagrangian of two real scalars with equal mass. So for free fields there isn't really a difference between complex and real scalars. Note that the complex scalar Lagrangian has a $U(1)$-symmetry $\varphi \to e^{i\theta}\varphi$. We will return to that later.

The difference between real and complex scalars will become important if we add electromagnetic interactions. Real scalars are necessarily neutral, whereas complex ones can carry charge. Also for self-interactions there is a difference. In the case of complex scalars, if we

⋆ The signs of the quartic terms are chosen so that the Hamiltonian is positive for large values of the fields.

† Note that there is a second bilinear expression that is real, obtained by replacing $\varphi^*\varphi$ by $\varphi^2 + (\varphi^*)^2$. By writing this in terms of real fields one finds that now the Hamiltonian is not bounded from below.

want to keep the $U(1)$-symmetry we can only add

$$\mathcal{L}_I = -\frac{1}{4}\lambda(\varphi^*\varphi)^2 \ . \tag{2.9}$$

The quantization of the free complex scalar follows straightforwardly from that of two real scalars, using (2.8). If we use operators $c_{\vec{k}}$ to quantize φ_1 and $d_{\vec{k}}$ (commuting with $b_{\vec{k}}$) to quantize φ_2, then we find for a complex scalar φ

$$\varphi(x^\mu) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega(\vec{k})}\left(e^{-ik\cdot x}a_{\vec{k}} + e^{ik\cdot x}b^\dagger_{\vec{k}}\right) \ . \tag{2.10}$$

where $k\cdot x$ denotes the four-vector product $\omega(\vec{k})t - \vec{k}\cdot\vec{x}$ and

$$a_{\vec{k}} = (c_{\vec{k}} + id_{\vec{k}})/\sqrt{2} \ ; \qquad b_{\vec{k}} = (c_{\vec{k}} - id_{\vec{k}})/\sqrt{2} \ .$$

The a and b oscillators satisfy the usual commutation relations and commute with each other. Obviously we also have‡

$$\varphi(x^\mu)^* = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega(\vec{k})}\left(e^{-ik\cdot x}b_{\vec{k}} + e^{ik\cdot x}a^\dagger_{\vec{k}}\right) \ . \tag{2.11}$$

2.6. Summary

The main steps in constructing a quantum field theory describing particles and their interactions are

— Construct a classical field that has the right transformation properties under Lorentz and internal symmetries (*e.g.* charges) to represent the particle of interest.

— Find the proper relativistic wave equation for that field.

— Construct a bilinear Lagrangian density whose Euler-Lagrange equation is the wave equation. This defines the "free" theory.

‡ Since this is an operator we should have replaced the complex conjugation on φ by Hermitean conjugation, writing $\varphi^\dagger$ instead of φ^*. Nevertheless, for convenience we keep using the notation φ^* in the following.

— Now construct all possible higher order terms in the Lagrangian that are allowed by all the symmetries.

Now we have defined our theory. We will consider several other examples later in these lectures. But first we want to discuss how all this perhaps somewhat abstract formalism leads to very concrete predictions for experimentally observable quantities.

3. From Lagrangians to cross sections

Given the Lagrangian of a quantum field theory, we can derive all (differential and total) cross section for processes where two particles (*i.e.* quanta of the fields in the Lagrangian) scatter and produce any number of other particles. The basic steps are as follows.

$$\begin{aligned}&\text{Lagrangian} \to \text{Hamiltonian} \to \text{Time evolution} \to \\ &\text{S-matrix} \to \text{Transition Amplitude} \to \text{Cross Section}\end{aligned}$$

Let us examine these steps one by one. The first one was already discussed.

3.1. From Hamiltonian to time evolution

This is a standard piece of ordinary quantum mechanics. There are two standard ways of dealing with time evolution in quantum mechanics. They are referred to as the *Schrödinger picture* and the *Heisenberg picture*. In the Schrödinger picture the time dependence is put in the wave function ψ. The operators $\mathcal{O}$ are time independent:

$$\begin{aligned} i\frac{d}{dt}\psi_S(t) &= H\psi_S(t) \\ i\frac{d}{dt}\mathcal{O}_S &= 0 \ . \end{aligned} \tag{3.1}$$

These operators can be any quantum operator related to a classical observable, for example momentum.

One may "integrate" this differential equation, *i.e.* solve it for finite time intervals. This will yield an operator $U(t, t_0)$ that maps states the time t to states at time t_0:

$$\psi(t) = U(t, t_0)\psi(t_0) \ .$$

Obviously, because of (3.1) the time evolution operator must satisfy

$$i\frac{d}{dt}U(t, t_0) = HU(t, t_0) \ ,$$

with boundary condition $U(t_0, t_0) = 1$.

In the Heisenberg picture the states are kept constant, and the time dependence is put in the operators. This amounts to making a time-dependent basis transformation

$$\begin{aligned} \psi_H(t) &= \psi_S(t_0) \\ \mathcal{O}_H(t) &= U(t, t_0)^{-1}\mathcal{O}_S U(t, t_0) \ . \end{aligned}$$

The equations of motion are in this picture

$$\begin{aligned} i\frac{d}{dt}\psi_H(t) &= 0 \\ i\frac{d}{dt}\mathcal{O}_H &= [\mathcal{O}_H, H]. \end{aligned} \tag{3.2}$$

In our case the "operators" will be the quantum fields. If we were to adopt the Schrödinger picture we would have to fix them at some time t_0 and treat them as time independent. This introduces an artificial distinction between space and time, which is not nice in a relativistic theory. If we choose the Heisenberg picture that would be evaded.

Unfortunately the Heisenberg picture is not very easy to work with in practice as soon as there are interactions. It turns out that we can have the best of both worlds: we may treat the free part of the Hamiltonian in the Heisenberg picture, and the interactions in the Schrödinger picture. To do this we split the Hamiltonian in a "free" part (in practice this is always the part bilinear in the fields) and an interaction part:

$$H = H_0 + H_I \ .$$

The fields will evolve according to H_0, and since this is the free theory, the fields will simply evolve according to the Klein-Gordon equation. The states will be constant as long as the

interactions are negligible. The interactions will change the initial states into final states. The interaction Hamiltonian is described in terms of the field operators evolving according to H_0, *i.e.* in terms of free fields. This is called the *interaction picture*

The time-evolution operator corresponding to the free Hamiltonian H_0 is simply

$$U_0(t,t_0) = \exp(-iH_0(t-t_0)) \ . \tag{3.3}$$

In the Schrödinger picture Ψ evolves in time according to $U(t,t_0)$. To go to the interaction picture we remove the part of the time evolution that is due to the free part of the Hamiltonian:

$$\psi_I(t) = \exp(iH_0t)\psi_S(t) \ .$$

Now the operators acquire a time-dependence:

$$\mathcal{O}_I(t) = \exp(iH_0t)\mathcal{O}_S \exp(-iH_0t) \ .$$

It is a simple exercise to work out the equations of motion in the interaction picture

$$\begin{aligned} i\frac{d}{dt}\psi_I(t) &= H_I(t)\psi_I(t) \ , \\ i\frac{d}{dt}\mathcal{O}_I &= [\mathcal{O}_I, H_0] \ . \end{aligned} \tag{3.4}$$

Now we wish to write a finite time-difference version of the first equation. We define

$$\psi_I(t) = U(t,t_0)\psi_I(t_0) \ ,$$

so that

$$i\frac{d}{dt}U(t,t_0) = H_I(t)U(t,t_0) \ . \tag{3.5}$$

This is easy enough to solve if the Hamiltonian at a time t commutes with the Hamiltonian at any other time. Then the solution is

$$U(t,t_0) = \exp(-i\int_{t_0}^{t} H_I dt') \ , \tag{3.6}$$

which equals $\exp(-iH_I(t-t_0))$ if the Hamiltonian is time independent. This situation was valid in the derivation of (3.3). One may expect the Hamiltonian to be time-independent

because energy is conserved. Indeed, in the Heisenberg picture the choice $\mathcal{O}_H = H$ in (3.2) leads to a conserved Hamiltonian. However, here we evolve with H_I, and H_I is not separately conserved, only the total energy is conserved. Indeed, we have

$$i\frac{d}{dt}H_I = [H_I, H_0]$$
$$[i\frac{d}{dt}H_I, H_I] = [[H_I, H_0], H_I] \ ,$$

and there is no reason why the right hand side should vanish in either case. Then the proof that (3.6) satisfies (3.5) fails, as one may easily check.

However, there is a small modification that *does* work: we define

$$U(t, t_0) = \mathrm{T}\exp(-i\int_{t_0}^{t} H_I dt') \tag{3.7}$$

where T denotes time-ordering. This simply means that if we expand the exponential, all factors $H_I(t)$ are ordered in such a way that larger times are to the left of smaller ones:

$$\mathrm{T}H_I(t_1)H_I(t_2) = \begin{cases} H_I(t_1)H_I(t_2) & (t_1 > t_2) \\ H_I(t_2)H_I(t_1) & (t_1 < t_2) \end{cases} .$$

Now the operator at the largest time is always on the left, and the proof that (3.7) is satisfied is easy.

3.2. The S-matrix

Having solved this problem, we can write down a transition amplitude between a state with n initial and m final particles:

$$\begin{aligned}\mathcal{A}(\vec{p}_1, \ldots, \vec{p}_m, \text{final}; \vec{k}_1, \ldots, \vec{k}_n, \text{initial}) &\equiv \langle \vec{p}_1, \ldots, \vec{p}_m | U(t_f, t_i) | \vec{k}_1, \ldots, \vec{k}_n \rangle \\ &= \langle \vec{p}_1, \ldots, \vec{p}_m | \mathrm{T}\exp(-i\int_{t_i}^{t_f} H_I) | \vec{k}_1, \ldots, \vec{k}_n \rangle \ .\end{aligned}$$

In field theory the interaction Hamiltonian is an integral over space:

$$H_I = \int d^3x \mathcal{H}_I(x) \ ,$$

where $\mathcal{H}_I(x)$ is the interaction Hamiltonian density, which consists of all terms in the Hamiltonian density that are of cubic and higher order in the fields.

Usually we take the limit $t_i \to -\infty$, $t_f \to \infty$. This is legitimate, since in any case we are assuming that the particles are free and well-separated long before and long after the interaction. In this limit, the matrix U is called the *S-matrix*:

$$S = \lim_{t\to\infty} U(t, -t) \ .$$

Substituting this in the interaction Hamiltonian and taking this limit we get

$$\mathcal{A}(\vec{p}_1, \ldots, \vec{p}_m, \text{final}; \vec{k}_1, \ldots, \vec{k}_n, \text{initial}) = \langle \vec{p}_1, \ldots, \vec{p}_m | \mathrm{T} \exp(-i \int d^4x \mathcal{H}_I) | \vec{k}_1, \ldots, \vec{k}_n \rangle \ . \quad (3.8)$$

3.3. Perturbation Theory

This expression is the starting point for *perturbation theory*. We assume that the effect of interactions is small enough for it to make sense to expand the exponential. Usually the interaction Hamiltonian has a coefficient, and if we are lucky that coefficient is small, and we can use it as an expansion parameter. This is a good approximation for the electro-weak theory. Here the expansion parameter is essentially the fine-structure constant $\alpha = \frac{1}{137.04}$, which indeed is pretty small. It is a much less good approximation in the theory of strong interactions, QCD. Here the expansion parameter α_s is of order .1, which is not extremely small. Perturbation theory may converge better or worse for different energy scales. In the case of QCD it becomes much worse at lower energies, and then perturbation theory becomes useless. Nevertheless, we are lucky enough to be able to apply perturbation theory in many cases, for electro-weak as well as high energy QCD processes. The techniques presented here are useful for scattering and decay processes. They are useless for computing bound state properties.

If we expand (3.8) and consider a given term in the expansion, we are faced with the problem of computing a matrix element of some time-ordered product of interaction Hamiltonians. In the interaction picture these interaction Hamiltonians are functions of the free field operators. Exactly what function depends on the theory one considers, but in any case the general form is some polynomial in the fields and their space-time derivatives. For simplicity we will assume for the moment that $\mathcal{H}_I$ consists of just one term, *e.g.* $\mathcal{H}_I = \frac{1}{6}\lambda_3\varphi^3$. The term of order N in the perturbative expansion looks like this:

$$\frac{1}{N!}\langle \vec{p}_1, \ldots, \vec{p}_m | \ \mathrm{T} \left[-i \int d^4x_1 \mathcal{H}_I(x_1) \right] \ldots \left[-i \int d^4x_N \mathcal{H}_I(x_N) \right] \ | \vec{k}_1, \ldots, \vec{k}_n \rangle \ . \quad (3.9)$$

How can we compute such a term?

The computation is in fact quite simple. We write $\mathcal{H}_I$ in terms of the fields, and then we write the fields in terms of creation and annihilation operators, as in (2.2).

Using the commutation relations for the creation/annihilation operators (2.1) we systematically commute all creation operators to the left, and all annihilation operators to the right, keeping track of all commutator terms (this is called *normal ordering*). Note that the commutator (2.1) yields a number, not an operator, so we can take it outside the expectation value. Then we apply the same ordering procedure to the remaining creation/annihilation operators in the commutator terms.

After a lot of straightforward work, one gets a sum of many terms. A typical term looks like this

$$[\text{Commutators}]\ \langle \vec{p}_1, \ldots, \vec{p}_m | a^\dagger_{\vec{p}_1} \ldots a^\dagger_{\vec{p}_{m'}} a_{\vec{k}'_1} \ldots a_{\vec{k}'_{n'}} | \vec{k}_1, \ldots, \vec{k}_n \rangle \ , \tag{3.10}$$

where "[Commutators]" denotes the product of all commutators one has picked up. This expectation value clearly vanishes if there are more annihilation (creation) operators than initial (final) states. If there are fewer, it means that one or more initial state is mapped directly to a final state without participating in the scattering process. This possibility is part of the S-matrix, but not the part we are interested in. For example, if there are no creation/annihilation operators at all, the result is non-vanishing if and only if the initial state has the same number of particles and with the same momenta as the final state. Then there is no scattering at all.

Hence the only case of interest is when the creation/annihilation operators exactly match the final/initial states, in multiplicity as well as momentum. Then each operator destroys precisely one particle: the annihilation operators destroy one particle in the initial state by acting to the right, the creation operators destroy one final state particle acting to the left. We are left with the vacuum-to-vacuum amplitude $\langle 0|0\rangle = 1$ times the commutators.

Any annihilation operator was part of a field, so instead of working with single operators, it is more convenient to split each field in an annihilation and a creation part,

$$\varphi = \varphi^- + \varphi^+ \ ,$$

as in (2.2). When the annihilation part φ^- acts on an incoming state we get

$$\begin{aligned}
\varphi^-(x)|\vec{k}\rangle &= \frac{1}{(2\pi)^3}\int \frac{d^3k'}{2\omega(\vec{k}')} e^{i\vec{k}'\cdot\vec{x}-i\omega(\vec{k}')t}\ a_{\vec{k}'}|\vec{k}\rangle \\
&= \frac{1}{(2\pi)^3}\int \frac{d^3k'}{2\omega(\vec{k}')} e^{i\vec{k}'\cdot\vec{x}-i\omega(\vec{k}')t}\ a_{\vec{k}'}a^\dagger_{\vec{k}}|0\rangle \\
&= \frac{1}{(2\pi)^3}\int \frac{d^3k'}{2\omega(\vec{k}')} e^{i\vec{k}'\cdot\vec{x}-i\omega(\vec{k}')t}\ [a_{\vec{k}'}, a^\dagger_{\vec{k}}]\,|0\rangle \\
&= \frac{1}{(2\pi)^3}\int \frac{d^3k'}{2\omega(\vec{k}')} (2\pi)^3 2\omega(\vec{k})\delta^3(\vec{k}-\vec{k}') e^{i\vec{k}'\cdot\vec{x}-i\omega(\vec{k}')t}\ |0\rangle \\
&= e^{-ik_\mu x^\mu}\,|0\rangle\ .
\end{aligned} \tag{3.11}$$

Here $x^\mu = (t, \vec{x})$ is the argument of the field from which this term originated. This is therefore an integration variable of one of the interaction Hamiltonians in (3.9). The four-momentum is $k^\mu = (\omega(k), \vec{k})$, and satisfies $k^2 = m^2$. This corresponds to an on-shell initial state.

Similarly, for a final state we get a contribution

$$\langle\vec{p}|\varphi^+(x) = \langle 0|e^{ip_\mu y^\mu}\ ,$$

where y is the integration variable of another interaction Hamiltonian. It can happen that two states (initial or final) are related to the same factor in (3.9), in which case the variables x and y are of course the same.

The generalization to the case of multiple operators acting on a multi-particle state can also be analyzed, but we will omit the details. The result is just a product of phases $e^{-ik_\mu x^\mu}$, one for each incoming particle, and similarly for outgoing ones. Furthermore one gets a sum over all permutations of combining the particle momenta with the fields φ^- or φ^+.

Finally we need to consider the commutator terms. They always come from the commutation of one annihilation and one creation operator. Since the right hand side of (2.1) is not an operator but a number, there are no multiple commutators.

The result of "normal ordering" two fields can be written as follows

$$\mathrm{T}\varphi(x)\varphi(y) = \mathrm{N}\varphi(x)\varphi(y) + \Delta(x,y)\ .$$

Here "N" denotes the normal ordered combination, and Δ is the commutator. It can be computed by expressing the left-hand side in terms of creation/annihilation operators and

normal-ordering them explicitly. A simpler way is to take the vacuum expectation value of both sides. The normal-ordered terms vanish, because either the left, or the right, or both vacua are annihilated. Hence we get

$$\Delta(x,y) = \langle 0|\,\mathrm{T}\varphi(x)\varphi(y)\,|0\rangle \ .$$

This is called the *propagator*, and we will say more about it in a moment. Let us first return to the analysis of a term of the form (3.10).

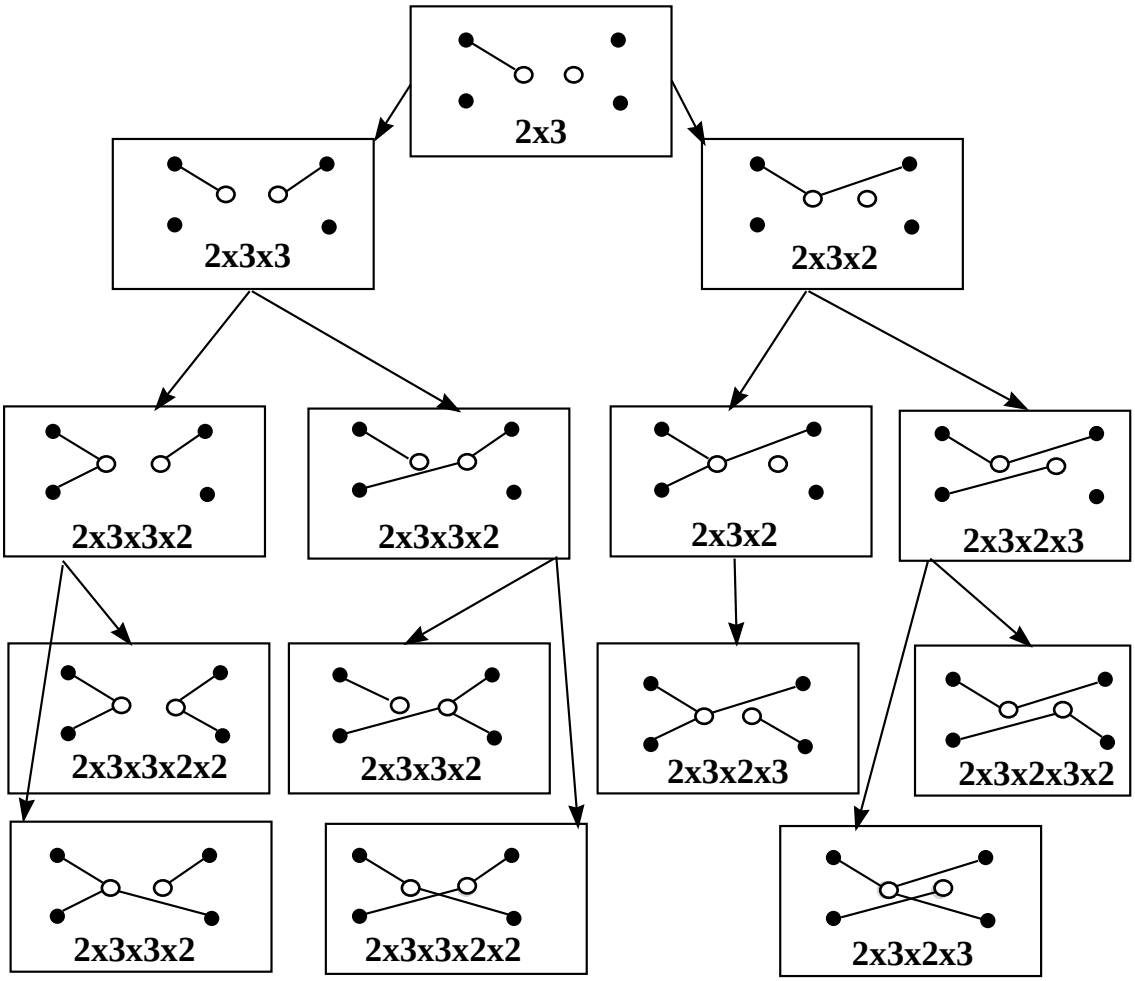
More precisely, such a terms looks like this:

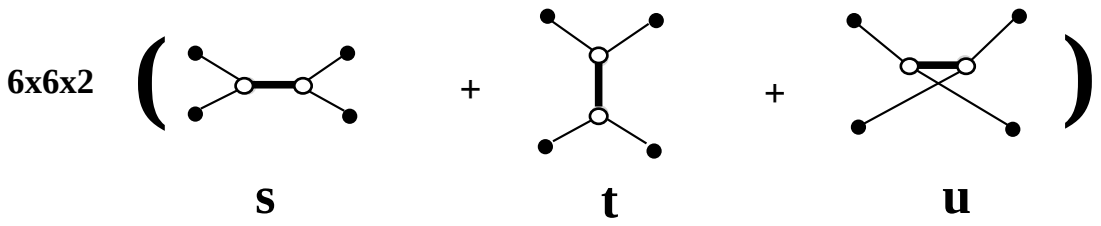
$$\begin{aligned}&\langle 0|\,\mathrm{T}\varphi(x_1)\varphi(y_1)\,|0\rangle \ldots \langle 0|\,\mathrm{T}\varphi(x_l)\varphi(y_l)\,|0\rangle \\ &\qquad \times \langle \vec{p}_1,\ldots,\vec{p}_m|\varphi^+(z_1)\ldots\varphi^+(z_m)\varphi^-(w_1)\ldots\varphi^-(w_n)|\vec{k}_1,\ldots,\vec{k}_n\rangle \ .\end{aligned} \tag{3.12}$$

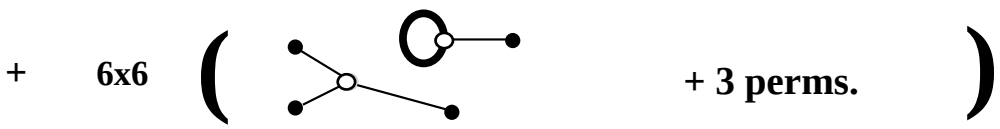
The arguments x_i, y_i, z_i and w_i correspond in some way to the integration variables of the various factors in (3.9). It is clear that the fields appearing in each such term (3.12), either in propagators or in the last factor, must precisely match the fields appearing in (3.9), with each argument $x_1 \ldots x_N$ appearing exactly as many times as in (3.9). In fact the complete expansion of (3.9) is a sum of all possible terms of the form (3.12). This is most easily represented diagrammatically. For each term (3.12) we draw a diagram, which is called a *Feynman diagram*.

First one represents each external particle (final or initial) by a point. One does the same with each interaction Hamiltonian. The number of interaction Hamiltonians depends on the order of the expansion of the S-matrix one is considering. The points corresponding to the interaction Hamiltonians are called "vertices".

A term of the form (3.10) is now obtained by connecting each external particle point to a vertex. This connection is called an "external line". It associates with the initial or final particle a field from one of the interaction Hamiltonians, which then acts as explained in (3.11). Now each interaction Hamiltonian will in general still have some unused fields. These are linked to each other in some way, by lines connecting one vertex to another. This must be done in such a way that from every vertex as many lines are emerging as there are fields in the interaction Hamiltonian. The external lines correspond to the fields φ^+ and φ^- in (3.12), whereas the lines that connect vertices to each other correspond to the propagators.







This is illustrated in the figure. It shows the computation of a process with two incoming and two outgoing scalars. The interaction Hamiltonian is $\frac{1}{6}\varphi^3$ and we expand to second order

($N = 2$ in (3.9)). The external particles are represented here by black dots, the vertices by open ones. We are not indicating which ones are incoming and which ones are outgoing, but we will worry about that later. The connections are made step-by-step, and in each step we keep track of the number of different ways we could have made them. For example, for the first line we choose a connection from particle 1, which has a definite momentum and hence is unique, to one of the vertices. Since there are two vertices each with three fields, we have six choices for the field to connect to particle 1.

When we are finished we end up with a total of seven contributions. Three of them are the so-called s-channel, t-channel and u-channel contributions to the scattering amplitude. The other four look rather strange: One of the external lines ends on vertex that is connected only to itself. Such self-connections are undesirable (for reasons that will not be discussed here), but can be explicitly avoided by demanding that the interaction vertex itself is already normal ordered. After all, when we write $\mathcal{H}_I = \frac{1}{6}\varphi^3$ we write down a classical expression. The correct quantum version could be just the cube of the quantum field, but it could also be another ordering of the operators. The preferable choice turns out to define the quantum version of φ^3 as $\mathrm{N}(\varphi^3)$ *i.e.* to normal order the interaction Hamiltonian itself *before* using it in perturbation theory. Then self-connection "tadpole" diagrams are absent.

The s,t,u channel scattering contributions come with a factor $6^2 \times 2$. This is nicely cancelled by the factor $\frac{1}{6}$ we had introduced in the interaction Hamiltonian, and the factor $\frac{1}{N!}$ in (3.9). To write down the answer we need to know the propagator.

3.4. The propagator

The expression

$$\Delta(x, y) = \langle 0|\, \mathrm{T}\varphi(x)\varphi(y)\, |0\rangle$$

can be interpreted as the amplitude for the creation of a particle at time y^0, followed by its annihilation at the later time x^0 (or vice-versa, if y^0 is later). In the interaction picture the fields appearing in this expression are just the free fields, and then the propagator can be computed quite easily using (2.2). The result is

$$\Delta(x - y) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(\vec{k})} e^{i\vec{k}\cdot(\vec{x}-\vec{y}) - i\omega(\vec{k})|x^0 - y^0|} \,. \tag{3.13}$$

Note a slight change of notation: since the propagator depends only on the *difference* of the

positions (as expected in view of translation invariance, we give it a single argument $x - y$. The propagator can be re-written as follows:

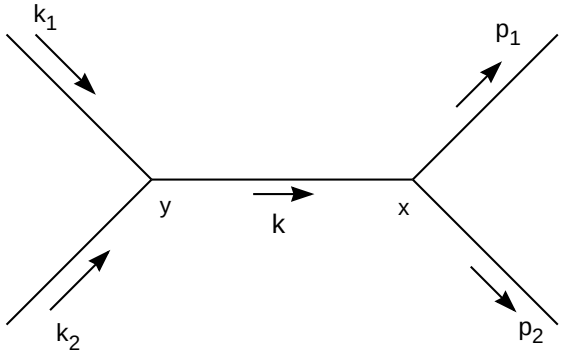
$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik_\mu x^\mu} \Delta(k) \ ,$$
$$\Delta(k) = \frac{i}{k^2 - m^2 + i\epsilon} \ .$$

Eqn. (3.13) can be derived from this by performing the k^0 integration using complex contour deformation and the residue theorem (k^0 is continued into the complex plane). The $-i\epsilon$ term gives an infinitesimal shift of the poles into the complex plane, in precisely the right way. Although we will usually ignore these $-i\epsilon$ terms, they are very important for the analytic properties of Feynman diagrams and the unitarity of the theory. But that is beyond the scope of these lectures.

This is the propagator of a massive scalar boson. For other particles there will be other formulas later in these notes.

3.5. Momenta

Consider now one of the diagrams constructed in the figure. Let us put some more information



We have indicated the momenta of the particles. In the following we will always let the incoming particles come from the left, and the outgoing ones move out to the right, whenever we draw a Feynman diagram. (unfortunately quantum mechanical amplitudes are always written in just the opposite way, as $\langle \text{out}|S|\text{in}\rangle$). The complete amplitude is

$$\tfrac{1}{2}\,[-i\int d^4x]\,[-i\int d^4y]\,(\frac{1}{3!}\lambda_3)^2(6^2\times 2)e^{-ik_1 y}e^{-ik_2 y}e^{ip_1 x}e^{ip_2 x}\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\Delta(k) \ .$$

We may now integrate over x and y, using

$$\int d^4x e^{ikx} = (2\pi)^4\delta^4(k) \ .$$

The result is

$$(-i\lambda_3)^2 \int \frac{d^4k}{(2\pi)^4}[(2\pi)^4\delta^4(k-k_1-k_2)][(2\pi)^4\delta^4(p_1+p_2-k)]\Delta(k) \ .$$

Note that the result of the space-time integrations is to impose four-momentum conservation at all the vertices. Finally we may also perform the integral over k; this removes one of the δ-functions The final result for this diagram is

$$(-i\lambda_3)^2 \ [(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)] \ \frac{i}{s-m^2} \ ,$$

where $s = (k_1+k_2)^2$. The other diagrams yield exactly the same expression, but with s replaced by $t = (k_1-p_1)^2$ and $u = (k_1-p_2)^2$ respectively. The final result at this order is the sum of these three contributions.

3.6. Feynman rules

The foregoing discussion can be summarized by a very simple set of rules, so that we do not have to go through all these steps again. Suppose we wish to compute the scattering amplitude at order N of a given field theory, and with a given set of ingoing and outcoming momenta. Consider for example the free boson theory with an interaction Lagrangian $-\frac{1}{M!}\lambda_M\varphi^M$. This gives rise to vertices with M lines. Then we go through the following steps.

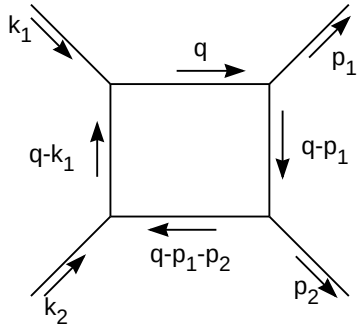
— Draw all distinct diagrams with N vertices. Two diagrams are "distinct" if they cannot be deformed into each-other without breaking any lines and without moving the external lines.

— Assign a momentum to each internal line, so that momentum conservation at the vertices is respected.

— The contribution to the amplitude due to each separate diagram is obtained by multiplying the following factors

1. A factor $(-i\lambda_M)$ for each vertex.
2. A factor $i\Delta(k)$ for each propagator, where k is the momentum that flows through the propagator.
3. A combinatorial factor (see below).
4. A momentum conservation δ-function $(2\pi)^4\delta^4(\sum p_i - \sum k_i)$.
5. An integration $\int \frac{d^4k}{(2\pi)^4}$ for each closed loop.

The last point is a new feature. To illustrate it, consider the following diagram, which occurs at order 4 (together with other diagrams not drawn here).

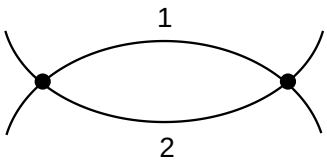


In this diagram there are four vertices and four propagators, hence four momentum integrals and four δ-functions. However, one of those four is the overall momentum conservation δ-function. The other three kill three integrations, but then there is still one integral left. Hence we end up with a single four-dimensional integral; the result is

$$(-i\lambda_3)^4\ [(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)]\int \frac{d^4q}{(2\pi)^4}(\frac{i}{q^2-m^2})(\frac{i}{(q-p_1)^2-m^2})$$
$$(\frac{i}{(q-p_1-p_2)^2-m^2})(\frac{i}{(q-k_1)^2-m^2})\ .$$

The computation of such integrals is beyond the scope of these lectures.

A combinatorial factor appears if the number of ways of building a diagram is not exactly cancelled by the factorials in the interaction Hamiltonian and the expansion of the S-matrix. This happens if a diagram has symmetries under the exchange of vertices and/or propagators (without moving the external lines). In the example of fig 1. the s, t, u channel diagrams have no such symmetries. A diagram with symmetries is



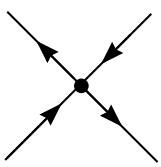
The symmetry is the interchange of propagators 1 and 2. In general such diagrams have an extra factor $\frac{1}{L}$, where L is the number of symmetries. In the case shown here $L = 2$.

The preferable choice for the numerical factor for terms in the Lagrangian is $\frac{1}{n!}$ for each field appearing n times, with complex conjugates counted as different. Then the combinatorial factors are easiest.

3.7. Generalizations

The foregoing discussion can be easily generalized to more complicated situations. First of all there can be more than one term in the interaction Lagrangian. Then one simply introduces a vertex for each term, and builds Feynman diagrams out of any possible combination of vertices. There can also be more than one field in the theory. In that case each field has its own propagator, and to draw diagrams one has to use different lines (*e.g.* dashed) to distinguish different fields. Each vertex has a definite number of lines of each types, and lines of type "a" can only be connected by propagators of type a.

Finally there is the possibility of having complex scalars. It is not hard to show that they have the same propagator as real scalars, but this propagator can only connect fields to their complex conjugate. Hence we need a way to indicate that a line on a vertex belongs to φ or φ^*. This is usually done by adding arrows to the lines. By convention, the arrow always points from φ to φ^*. If the arrow points towards a vertex, it refers to a field φ^* in the interaction Lagrangian, if it points away then it refers to φ. For example the interaction $-\frac{1}{4}\lambda(\varphi^*\varphi)^2$ leads to a vertex



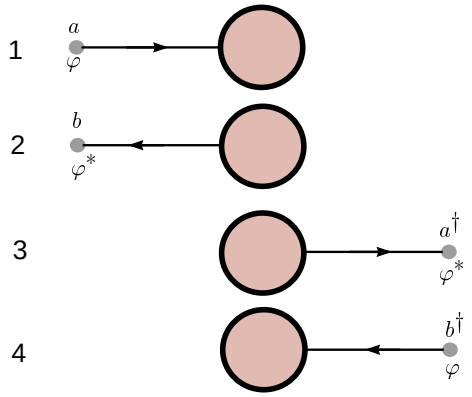
$-i\lambda$

where we have indicated the resulting Feynman rule.

3.8. Anti-particles

Lines with arrows can appear as external lines in two ways, with the arrow pointing inward or outward. In one case the incoming or outgoing state is created by the field φ, in the other by φ^*. These states are rather similar: they are both scalars and have the same mass. Later, when we introduce electric charges, we will see that the particles created by φ and φ^* have opposite charges. These particles are each other's anti-particle. Quantum field theory predicts the existence of an anti-particle for any particle. The quanta of a real scalar field are their own anti-particle; for complex field the anti-particle corresponds to the conjugate.

Since external lines can correspond to incoming states as well as outgoing states, there are actually four situations to be distinguished.



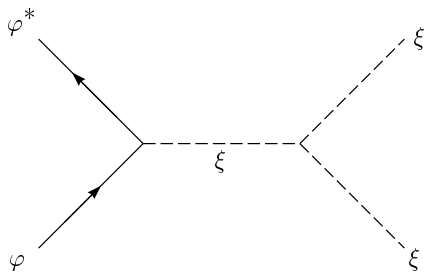
Here the small grey dot represents the in or out state, and the big blob can be any Feynman diagram, which may have other external lines. In case 1 we have an incoming particle (by definition of what we mean by "particle"), corresponding to a field φ on the in-state. Since only the destruction part of φ acts on the in-state due to normal ordering, this assigns the operator a in (2.10) to particle creation and destruction. Then case 2 corresponds to an incoming anti-particle, destroyed by the operator b in (2.11). Case 3 represents φ^* acting on the out state. On the out state only the creation part of the fields acts, which in this case is $a^\dagger$. So this is an outgoing particle. Finally, case 4 is an outgoing anti-particle.

Particles and anti-particles have opposite quantum numbers and can in principle annihilate, if their are interactions allowing that to happen. For example, if we couple the complex

scalar theory to a real scalar ξ in the following way*

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \tfrac{1}{2}\partial_\mu \xi \partial^\mu \xi - \frac{1}{3!}\xi^3 - \varphi^* \varphi \xi \ .$$

then the following process is possible, and corresponds to the annihilation of a φ particle and anti-particle into two ξ scalars



This summarizes essentially all that can happen with scalars. Now we have to extend the discussion to fermions and vector bosons. The general idea will be the same, except for different expressions for propagators and vertices, and a new rule having to do with fermi statistics.

3.9. From amplitudes to cross sections

Before closing this chapter we still have to discuss how to compute measurable quantities from these amplitudes. First one rewrites the S-matrix element as follows

$$\begin{aligned}\langle \vec{p}_1, \ldots, \vec{p}_m | S | \vec{k}_1, \ldots, \vec{k}_n \rangle &= \langle \vec{p}_1, \ldots, \vec{p}_m | \vec{k}_1, \ldots, \vec{k}_n \rangle \\ &\quad - i(2\pi)^4 \delta^4 \Big(\sum_i p_i - \sum_j k_j \Big) T(\vec{k}_1, \ldots, \vec{k}_n; \vec{p}_1, \ldots, \vec{p}_m),\end{aligned}$$

which defines the transition amplitude T (not to be confused with the time ordering symbol used earlier). This is what we are really interested in. The S matrix includes contributions without any scattering at all, *i.e.* with final states identical to the initial states. They are given by the first term, and correspond in terms of Feynman diagrams to graphs without any vertices at all. This kind of process is not what one looks for in an experiment, and is therefore not taken into account.

$\star$ Extra terms would be needed to make the energy bounded from below.

There are two kinds of processes one encounters: with one incoming particle or with two incoming particles. The former correspond to decay processes, the latter to cross sections. To compute the decay width or cross section is a straightforward exercise which has not much to do with quantum field theory; this can be found in any textbook on non-relativistic quantum mechanics. The corresponding relativistic expressions are

$$d\Gamma = \frac{1}{2E}|T(\vec{k};\vec{p}_1,\ldots,\vec{p}_m)|^2\Phi_m \ .$$

Here $d\Gamma$ is the differential decay width of a particle with energy E and Φ_M is the M-particle phase space

$$\Phi_m = (2\pi)^4\delta^4\Big(\sum_i p_i - \sum_j k_j\Big)\prod_{i=1}^{M}\frac{d^3p_i}{(2\pi)^3 2E_i} \ . \tag{3.14}$$

To get the total decay width one must integrate over all outgoing momenta, and sum over all possible final states. In the rest frame of the particle the correct answer is obtained by replacing E by the mass m.

For a relativistic differential cross section one gets

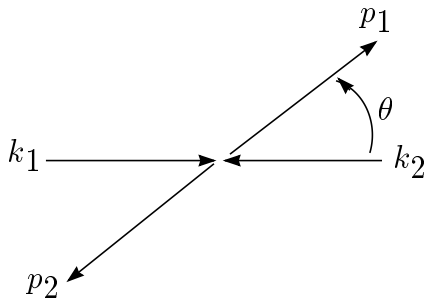
$$d\sigma = \frac{|T(\vec{k}_1,\vec{k}_2;\vec{p}_1,\ldots,\vec{p}_m)|^2}{4\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2}}\ \Phi_m \tag{3.15}$$

with the same phase space element. Here $k_1\cdot k_2 \equiv k_1^\mu(k_2)_\mu$, showing that the result is indeed relativistic. Note that all external momenta are "on-shell", *i.e.* they satisfy $k_j^2 - m_j^2 = p_i^2 - m_i^2 = 0$. Hence from the knowledge of the 3-vector $\vec{k}_j$ and the mass m_j we know the fourth component of the four-vector k_j.

Expressions for the phase space elements for two or three particles can be found, for example, in the particle data book. A useful expression for the two-particle phase space is

$$\Phi_2 = \frac{|\vec{p}|d\Omega}{16\pi^2\sqrt{s}} \tag{3.16}$$

where $\vec{p} = \vec{p}_1 = -\vec{p}_2$ is the center-of-mass momentum of one of the particles and $d\Omega = d\cos\theta d\phi$, where θ is the center-of-mass scattering angle (the angle between $\vec{p}_1$ and $\vec{k}_1$), and ϕ the azimuthal angle.



The phase space elements include all the kinematics of a process, and are completely determined by the outgoing particles and their momenta, independent of the process. The interesting physics is in the transition amplitudes T.

4. Fermions

4.1. Spinors in Non-relativistic Quantum Mechanics

It is well-know how to deal with spin-$\frac{1}{2}$ particles in non-relativistic quantum mechanics. One introduces two-component spinors χ whose components can be interpreted as the amplitude for the particle having spin up or down along a certain direction:

$$\chi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} \ .$$

To define the spin one introduces the Pauli matrices σ_i. Under rotations R the spinors transform with some unitary two-by-two matrix $U(R)$:

$$\chi' = U(R)\chi \ ,$$

(in mathematical terms they form a "two-dimensional representation of the rotation group"). This transformation is such that the combination $\chi^\dagger \vec{\sigma} \chi$ transforms as a vector:

$$\chi'^\dagger \sigma_i \chi' = \chi^\dagger U(R)^\dagger \sigma_i U(R) \chi = R_{ij} \chi^\dagger \vec{\sigma}_j \chi \ , \tag{4.1}$$

where R_{ij} is the three-by-three rotation matrix that acts on vectors.

This sort of relation is usually checked in infinitesimal form, using small rotations. Then one has

$$U(R(\vec{\theta})) \approx 1 - i\vec{J}\cdot\vec{\theta}\ .$$

and

$$R_{ij}(\vec{\theta}) \approx \delta_{ij} - \epsilon_{ijk}\theta_k\ ,$$

where $\vec{J}$ is the generator of rotations and $\vec{\theta}$ is a set of three angles that parametrize rotations R. In our case the generator of rotations is the spin operator $\vec{S} = \frac{1}{2}\vec{\sigma}$. Substituting this into (4.1), and expanding to first order in θ one gets the well-known relation

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\ .$$

In addition to the vector $\chi^\dagger\sigma_i\chi$, another quantity of interest is the rotation invariant

$$\chi^\dagger\chi\ .$$

It is invariant because $U(R)$ is unitary (or, equivalently, $\vec{J}$ is Hermitean).

4.2. Relativistic spinors

To make this description relativistic all one has to do is extend the rotations by boosts. Group-theoretically this amounts to extending the rotation group $SO(3)$ to the Lorentz-group $SO(3,1)$. Instead of matrices $U(R)$ depending on three parameters we will have matrices $U(L)$ depending on the six parameters of the Lorentz-transformations L.

It turns out that – for massive particles – one cannot do this with just two-component spinors. A minimum of four is required (and the minimum is also enough for elementary particles). Intuitively the difference between massive and massless particles can be understood as follows. Instead of using a fixed axis, one can project the spin on the velocity direction of the particle itself. This is called the helicity:

$$\lambda = \frac{\vec{S}\cdot\vec{v}}{|\vec{v}|}\ .$$

If a particle is massive we can always boost it from velocity $\vec{v}$ to $-\vec{v}$. This flips the velocity, but does not affect the spin, and hence flips the helicity. A massless particle moves with the

velocity of light, which cannot be flipped (one cannot "overtake" the particle). Hence it is at least consistent for a massless particle to have just one helicity state. It must then be in an eigenstate of the helicity operator, *i.e.* either $\lambda = +\frac{1}{2}$ or $\lambda = -\frac{1}{2}$. Since it can still spin in any direction, a two component spinor is minimally required to describe a massless spin-$\frac{1}{2}$ particle. Then for a *massive* particle, which can have either helicity depending on the Lorentz frame, a four component spinor is required. Note that also parity flips the helicity (it flips velocity, but not spin) so that if we want to have a parity invariant theory four-component spinors are also required for that reason. Indeed, the only known spin-$\frac{1}{2}$ fermions that might be massless are the neutrinos. In the standard model they have just one helicity, and their interactions break parity.

Having argued that we should have four-component spinors ψ we now introduce a four-by-four generalization of the Pauli-matrices. They are called the Dirac matrices γ^μ. It turns out that the only condition they should satisfy is the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ . \tag{4.2}$$

This is called the *Clifford algebra*. There are many ways to satisfy it, but in any case it seems convenient to choose some of the matrices Hermitean. However, we cannot do that with all of them: with our choice for the metric we have

$$(\gamma^i)^2 = -1, \quad (\gamma^0)^2 = 1 \ ,$$

so that we may choose γ^i anti-Hermitean and γ^0 Hermitean.

The action of Lorentz-transformations L on a spinor are defined in terms of an operator $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, the generalization of $\vec{S}$ above. An infinitesimal Lorentz-transform has the form

$$U(\omega) = 1 - \frac{i}{2}\Sigma^{\mu\nu}\omega_{\mu\nu} \tag{4.3}$$

for some set of parameters $\omega_{\mu\nu}$. Note that because of the anti-symmetry of $\Sigma^{\mu\nu}$ there are six relevant parameters $\omega_{\mu\nu}$, precisely the number of independent rotations and boosts.

Just as above we constructed a scalar and a three vector out of the non-relativistic spinors, we now construct a scalar and a four-vector. The scalar is not, as one might perhaps have

expected, $\psi^\dagger\psi$ but

$$\bar{\psi}\psi \equiv \psi^\dagger\gamma^0\psi \ ,$$

The reason for the difference is that Lorentz-transformations are not unitary, due to the relative sign in the metric between space and time. Indeed, from (1.4) we see that the transformation acting on the coordinates themselves is not an orthogonal transformation, and (since it is real) it is therefore also not unitary. The same is true for the action on spinors; indeed, one may easily verify that not all matrices $\Sigma^{\mu\nu}$ are Hermitean, and hence the matrix $U(\omega)$ is not unitary for arbitrary ω. But if $U^\dagger(\omega)U(\omega) \neq 1$, then $\psi^\dagger\psi$ is not Lorentz-invariant. The extra factor γ^0 corrects that. If $\psi' = U(\omega)\psi$ then $\psi'^\dagger = \bar{\psi}U(\omega)^\dagger$ but $\bar{\psi}' = \bar{\psi}U(\omega)^{-1}$, as one may easily verify.

It is now easy to check that the following quantity transforms as a four-vector:

$$\bar{\psi}\gamma^\mu\psi \ .$$

To check this, one first determines the infinitesimal form of a Lorentz transformation from (1.4), and then one compares this using (4.3) and (4.2) with the transformation of $\bar{\psi}\gamma^\mu\psi$. The only input one needs is that $\bar{\psi}$ transforms with U^{-1} plus the Clifford algebra.

4.3. Free fermion Lagrangian

Now that we know how to construct Lorentz-invariant quantities we can start building Lagrangians. The free theory consists of bilinear terms. We have already seen one candidate, $\bar{\psi}\psi$, but it doesn't contain derivatives, which leads to trivial equations of motion. One might also consider $\bar{\psi}\partial_\mu\partial^\mu\psi$, but that leads to four decoupled Klein-Gordon equations for the four components, and that is also not what we are looking for (the spin then plays a trivial rôle). The correct guess turns out to be

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \ .$$

4.4. The Dirac Equation

Using the Euler-Lagrange equations one can derive an equation of motion from this Lagrangian. In doing so one has to treat ψ and ψ^* as independent variables, just as for complex scalars. In fact, in its present form the Lagrangian depends on $\bar{\psi}$, but not on its derivative, so the Euler-Lagrange equations are rather simple:

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi \ .$$

One usually defines $\gamma^\mu \partial_\mu = \not{\partial}$ (and in general $\gamma^\mu v_\mu = \not{v}$ for any vector v), and writes the equation as

$$(i\not{\partial} - m)\psi = 0 \ .$$

This is the famous Dirac equation. Note that by "m" here we mean m times the unit matrix. It is instructive to act on the Dirac equation with the operator $(i\not{\partial} + m)$. Using $\not{\partial}^2 = \partial_\mu \partial^\mu$ one then obtains the Klein-Gordon equation. Obviously any solution to the Dirac equation satisfies the Klein-Gordon equation, which is as it should be because the latter is a general relativistic wave equation.

4.5. Plane wave solutions to the Dirac equation

Let us now try to find plane wave solutions to this equation. Write

$$\psi = u e^{-ik\cdot x} \ ,$$

where k and x are four-vectors and u a constant spinor. Then one finds

$$(\not{k} - m)u = 0 \ . \tag{4.4}$$

This is a set of linear equations, which one can solve by choosing an explicit set of Dirac γ matrices. A convenient choice is

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad ; \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Writing the Dirac equation in the the rest frame ($k^\mu = (m, 0, 0, 0)$; note that k must satisfy

the on-shell condition $k^2 = m^2$) we get

$$\not{k} - m = \gamma^0 k^0 - m = 2m \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} .$$

The matrix equation $(\not{k} - m)u = 0$ obviously has two independent solutions, which may be taken as

$$u^1 = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$u^2 = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with the standard normalization used in the literature. However, there are two more solutions to the Dirac equation. A second solution to $k^2 = m^2$ is $k^\mu = (-m, 0, 0, 0)$. Now we get

$$\not{k} - m = -2m \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} ,$$

and the solution are

$$v^1 = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$v^2 = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

Instead of saying that these are solutions to $(\not{k} - m)v = 0$ with $k = (-m, 0, 0, 0)$ one usually

views them as solutions of $(\not{k} + m)v = 0$ with $k = (m, 0, 0, 0)$.

It is a simple matter to generalize these solutions to an arbitrary Lorentz frame, for example by boosting them. Alternatively one may check that the following solutions do indeed satisfy the Dirac equations $(\not{k} - m)u = 0$ and $(\not{k} + m)v = 0$

$$u^i(\vec{k}) = \sqrt{m + \omega(\vec{k})} \begin{pmatrix} \chi^i \\ \frac{\vec{\sigma}\cdot\vec{k}}{m+\omega(\vec{k})}\ \chi^i \end{pmatrix} \ ; \quad v^i(\vec{k}) = \sqrt{m + \omega(\vec{k})} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{m+\omega(\vec{k})}\ \chi^i \\ \chi^i \end{pmatrix} \ , \tag{4.5}$$

where χ^i are orthonormal two-component spinors.

Now we have found the complete set of solutions in momentum space. We can combine them to obtain the most general solution in space-time:

$$\psi(x) = \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int \frac{d^3k}{2\omega(\vec{k})} [u^i(\vec{k}) e^{-ik\cdot x}\ a^i_{\vec{k}} + v^i(\vec{k}) e^{ik\cdot x}\ (b^i_{\vec{k}})^\dagger] \ . \tag{4.6}$$

This may be compared to the expression for the complex scalar field, (2.10). Note that a Dirac spinor is necessarily complex, because the Dirac matrices acting on them are complex (and there doesn't exist any other choice so that they are real). Hence there is no relation between a_i and b_i. With some foresight we have used in (4.6) the complex conjugate of b_i. For the conjugate spinor we find

$$\bar{\psi}(x) = \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int \frac{d^3k}{2\omega(\vec{k})} [\bar{v}^i(\vec{k}) e^{-ik\cdot x}\ b^i_{\vec{k}} + \bar{u}^i(\vec{k}) e^{ik\cdot x}\ (a^i_{\vec{k}})^\dagger] \ . \tag{4.7}$$

4.6. Quantization

The quantization procedure of the free fermion differs in just one respect from that of the free boson: instead of commutation relations one must impose anti-commutation relations. Otherwise the theory is not consistent. This follows from a very general theorem, the *spin-statistics theorem* which states that particles with integer spin have bose statistic, and those with half-integer spin fermi-statistics.

Going through the standard procedure one finds

$$\{a^i_{\vec{k}}, (a^j_{\vec{k}'})^\dagger\} = 2\omega(\vec{k})(2\pi)^3 \delta^3(\vec{k} - \vec{k}')\delta^{ij} \ .$$
$$\{b^i_{\vec{k}}, (b^j_{\vec{k}'})^\dagger\} = 2\omega(\vec{k})(2\pi)^3 \delta^3(\vec{k} - \vec{k}')\delta^{ij} \ .$$

while a and b anti-commute with each other; furthermore all other anti-commutators also vanish.

Now we just replace $a, a^\dagger, b$ and $b^\dagger$ by operators and we have quantized the theory. Admittedly the fact that the quantum variables anti-commute rather than commute makes the classical description of the theory a bit dubious. However, this can all be justified, provided one is careful not to interchange two classical fields without adding a $-$ sign.

4.7. Chirality

The fermionic action consists of two terms, a kinetic term and a mass term. We have argued above, in a hand-waving way, that massless fermions are in some respects essentially different from massive ones. We can make that explicit in the following way. Define the matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ .$$

It is easy to check that γ_5 is Hermitean and anti-commutes with γ^μ. Furthermore one may check that $\gamma_5^2 = 1$, and hence γ_5 has eigenvalues 1 or -1. Now define the operators

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_5) \ .$$

These are projection operators: $P_+P_+ = P_+$ and the same for P_-; furthermore $P_+P_- = 0$. Under the action of $P_\pm$ the spinor splits into two γ_5 eigenspaces:

$$\psi_L \equiv P_+\psi \ , \qquad \psi_R \equiv P_-\psi \ .$$

The γ_5-eigenvalue is called *chirality*. We have denoted these subspaces as "L" and "R" respectively because the spinors are also eigenvectors of the helicity operators, they are respectively left-handed and right-handed. (The helicity operator is $\frac{1}{2}\epsilon_{ijk}\Sigma_{ij}p^k/|\vec{p}|$).

It is easy to write the action in terms of ψ_L and ψ_R:

$$\mathcal{L} = i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R - m\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L \ .$$

We observe that the mass term mixes left- and right-handed fermions, but the kinetic terms do not. Hence if $m = 0$ ψ_L and ψ_R decouple from each other, and we may remove either one of them, if we wish. The result, a purely left- or right-handed fermion is often called a Weyl-fermion. The neutrino (if it is massless) is an example. If $m \neq 0$ we must have both chiralities.

This can all be made still more explicit. Instead of the basis used earlier, we may also choose a basis so that γ_5 is explicitly diagonal:

$$\gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad ; \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

This is called the Weyl basis. In this basis we have, obviously

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} .$$

4.8. Feynman rules

The Feynman rules for fermions are in some respects quite similar to those for complex scalars. Since fermion fields are complex we draw lines with arrows. The propagator of a fermion is not much harder to compute than that of a scalar. One must evaluate

$$\Delta^F_{\alpha\beta}(x-y) = \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$$

Here one should note the following

— Since the fermions are four-component spinor, the propagator becomes a matrix in spinor space.

— One could also try to compute a propagator for two ψ's or two $\bar{\psi}$'s, but the result would vanish. This is equivalent to saying that one must respect the direction of the arrows when connecting fermion vertices. (Exactly the same remarks apply to complex scalars.)

— We are treating ψ and $\psi^\dagger$ as independent variables. This allows us to make a change of variables for $\psi^\dagger$ by including a γ^0. Hence we may work directly with $\bar{\psi}$, and since this is what appears in the Lagrangian, it is much more convenient.

— We anticipated already that the propagator will only depend on the difference of the positions.

The main difference with the computation for scalars is that we get an extra factor due to the polarization vectors of the fermions. To evaluate it we need the following *polarization sums*

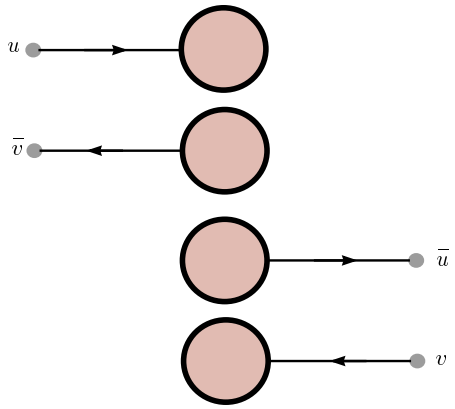
$$\begin{aligned}\sum_{i=1}^{2} u^i_\alpha(\vec{k})\bar{u}^i_\beta(\vec{k}) &= (\not{k}+m)_{\alpha\beta} \\ \sum_{i=1}^{2} v^i_\alpha(\vec{k})\bar{v}^i_\beta(\vec{k}) &= (\not{k}-m)_{\alpha\beta}\ .\end{aligned} \tag{4.8}$$

The result is

$$\begin{aligned}\Delta^F_{\alpha\beta}(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik_\mu x^\mu} \Delta^F_{\alpha\beta}(k)\ , \\ \Delta^F_{\alpha\beta}(k) &= \frac{i(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\epsilon}\ .\end{aligned}$$

where k is the momentum in the direction of the arrow on the fermion line, *i.e.* from ψ to $\bar{\psi}$.

The discussion of external lines is nearly identical as for complex scalars. The main difference is the presence of the spinors $u, v, \bar{u}, \bar{v}$ in the fermion field. Depending on whether the operator that acts on the initial or final state is $a, b, a^\dagger$ or $b^\dagger$ one gets for an external line a factor $u^i(\vec{k}), v^i(\vec{k}), \bar{u}^i(\vec{k}), \bar{v}^i(\vec{k})$ respectively. This is shown in the following picture

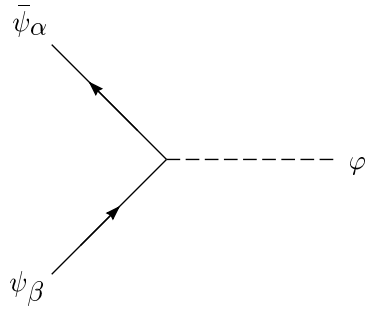


The argument of the spinors is of course the momentum of the external state. This momentum points towards the "blob" (which represents the rest of the Feynman diagram) in the first two cases (incoming particles) and away from the blob in the last two cases (outgoing particles). The index $i = 1, 2$ depends on the spin of the incoming state along some fixed axis. In addition the spinors u^i etc. also carry a spinor index $\alpha = 1, \ldots, 4$ labelling their four components.

For the purpose of showing some more features of fermion Feynman rules, let us introduce an interaction. There are several possibilities, using fields already discussed. For example, one might consider a four-fermion interaction $(\bar{\psi}\psi)^2$, but we will restrict ourselves to the simple case of an additional real scalar coupling to the fermion as

$$\mathcal{L}_{\rm int} = g\bar{\psi}M\psi\varphi \equiv \sum_{\alpha\beta} g\bar{\psi}_\alpha M_{\alpha\beta}\psi_\beta\varphi \ ,$$

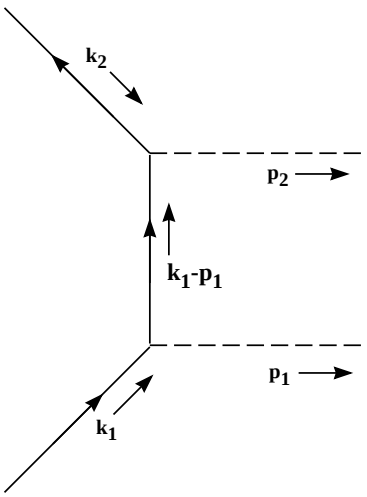
where g is the coupling constant. This gives rise to a vertex plus Feynman rule



$$igM_{\alpha\beta}$$

The choices for M are severely restricted by Lorentz invariance, and in this case there are only two: $M = 1$ and $M = \gamma_5$ (the γ_5 coupling occurs for example in the pion-nucleon vertex). In general, this kind of fermion-scalar coupling term is called a Yukawa coupling. This is how the Higgs scalar couples to the quarks and leptons.

Let us now compute the Feynman diagram

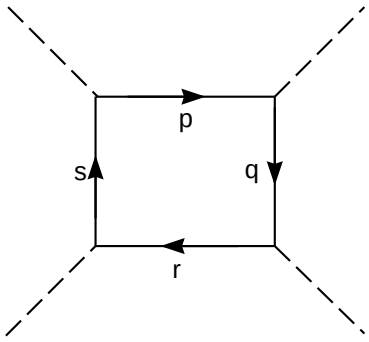


This is annihilation of a fermion and anti-fermion into two scalars. The result is

$$\sum_{\alpha,\beta,\gamma,\delta} \bar{v}^i_\alpha(k_2)(ig)M_{\alpha\beta}\frac{i(\not{k}_1 - \not{p}_1 + m)_{\beta\gamma}}{(k_1-p_1)^2 - m^2 + i\epsilon}(ig)M_{\gamma\delta}u^j_\delta(k_1) \ .$$

We will leave out the explicit summations from now on. The spinor structure is in general a string of gamma-matrices terminated by spinors, *i.e.* something of the form $\bar{u}M_1 \ldots M_n u$ (or v instead of u at either end). Each such string corresponds to a fermion line that can be followed throughout the diagram, starting at the outgoing end of the line.

Fermion lines may also form loops, for example



In this case the fermion line gives a trace

$$\text{Tr } M\not{p}M\not{s}M\not{r}M\not{q} \ ,$$

where for simplicity we've omitted the mass and all further details.

The most important difference between bosons and fermions is their statistics. In the manipulations leading from the product of time-ordered interaction Hamiltonians to the amplitude this gives rise to some signs one has to keep track of. The net result is easy to state:

— There is an overall $-$ sign for every closed fermion loop a diagram contains.

— If two diagrams differ only by the interchange of two external fermion lines, they have a relative minus sign.

5. Gauge theories

5.1. Classical electrodynamics

Classical electrodynamics can be derived from the following simple Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \ , \tag{5.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

To verify this statement we simply derive the Euler-Lagrange equations that follow from this Lagrangian

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} = \frac{\partial \mathcal{L}}{\partial A_\sigma} \ .$$

This yields

$$\partial^\nu F_{\mu\nu} = J_\mu \ . \tag{5.2}$$

Now define electric and magnetic fields

$$E_i = F_{0i} \ , \quad B_k = \tfrac{1}{2} \epsilon_{ijk} F_{jk} \ ,$$

and the equation takes the form

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= \vec{J} \\ \vec{\nabla} \cdot \vec{E} &= J_0 \end{aligned}$$

These are two of the four Maxwell equations (the other two,

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \ . \end{aligned}$$

are trivially satisfied if we express the electric and magnetic fields in terms of a vector

potential A_μ). Consistency of (5.2) clearly requires

$$\partial^\mu J_\mu = \partial^\mu \partial^\nu F_{\mu\nu} = 0 \ .$$

because of the antisymmetry of $F_{\mu\nu}$. This implies that J must be a conserved current. For such a current one can define a charge

$$Q = \int d^3x J_0 \ .$$

where the integral is over some volume V. This charge is conserved if the flux of the current $\vec{J}$ into the volume vanishes.

5.2. Gauge invariance

Consider the bilinear terms in the Lagrangian (5.1). If we quantize it in the way discussed before, it seems that we will end up with particles having 4 degrees of freedom, since A_μ has four components. However, this is incorrect for two reasons. First of all, one degree of freedom is not dynamical, *i.e.* does not appear with a time derivative, namely A_0. This means that the corresponding canonical momentum does not exist, and one will not obtain creation/annihilation operators for this degree of freedom. In addition to this there is one degree of freedom that does not really appear in the action at all. Suppose we replace A_μ by $A_\mu + \partial_\mu \Lambda(x)$, where $\Lambda(x)$ is some function. It is easy to see that $F_{\mu\nu}$ does not change at all under this transformation, and therefore the action is also invariant. This is called *gauge invariance*. Hence the action does not depend on Λ, which removes another degree of freedom. We conclude that there are just two degrees of freedom instead of 4. These two degrees of freedom correspond to the two polarizations of light. The quanta of A_μ are called photons.

If we add a mass term $m^2 A_\mu A^\mu$ to the Lagrangian it is still true that A_0 is not dynamical, but gauge invariance is broken. Therefore now we have three degrees of freedom. Just as fermions, massless and massive vector fields have very different properties.

Now consider the coupling $A_\mu J^\mu$. This is *not* invariant under gauge transformations, but

observe what happens if instead of the Lagrangian density we consider the action,

$$S_J = \int d^4x A_\mu J^\mu \ .$$

This transforms into itself plus a correction

$$\delta S_J = \int d^4x \partial_\mu \Lambda J^\mu \ .$$

Integrating by parts, and making the assumption that all physical quantities fall off sufficiently rapidly at spatial and temporal infinity, we get

$$\delta S_J = -\int d^4x \Lambda \partial_\mu J^\mu \ ,$$

which vanishes if the current is conserved, as we've seen it should be.

Gauge invariance (or current conservation) are our main guiding principles in constructing an action coupling the electromagnetic field to other fields. Consider for example the free fermion. It is not difficult to write down a Lorentz-invariant coupling:

$$\mathcal{L}_{\rm int} = eqA_\mu \bar{\psi}\gamma^\mu \psi \ ,$$

which is to be added to the kinetic terms

$$\mathcal{L}_{\rm kin} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} \ .$$

Note that we have introduced two new variables here: the coupling constant e and the charge q. The latter quantity depends on the particle one considers; for example for the electron $q = -1$ and for quarks $q = \frac{2}{3}$ or $q = -\frac{1}{3}$. The coupling constant determines the strength of the interaction. This quantity is the same for all particles. It turns out that the combination $\alpha = \frac{e^2}{4\pi}$ is small, $\approx 1/137.04$. This is the expansion parameter of QED, and its smallness explains why perturbation theory is successful for this theory. Although only the product eq is observable, it is convenient to make this separation.

With this choice for the interaction, the current is

$$J^\mu = eq\bar{\psi}\gamma^\mu\psi \ . \tag{5.3}$$

Using the equations of motion (*i.e.* the Dirac equation) one may verify that this current is indeed conserved, so that the theory is gauge invariant. But there is a nicer way of seeing that. Notice that the fermion kinetic terms as well as the interaction are invariant under the transformation

$$\psi \to e^{ieq\Lambda}\psi \ ; \quad \bar{\psi} \to e^{-ieq\Lambda}\bar{\psi} \ , \tag{5.4}$$

if Λ is independent of x. Because of the derivative this is not true if Λ does depend on x. However, the complete Lagrangian $\mathcal{L}_{\rm kin}+\mathcal{L}_{\rm int}$ is invariant under the following transformation

$$\begin{aligned} &\psi \to e^{ieq\Lambda(x)}\psi \ ; \quad \bar{\psi} \to e^{-ieq\Lambda(x)}\bar{\psi} \\ &A_\mu + \partial_\mu\Lambda(x) \ . \end{aligned}$$

This is the gauge transformation, extended to act also on the fermions. This is sufficient for our purposes: it shows that also in the presence of a coupling to fermions one degree of freedom decouples from the Lagrangian, so that the photon has only two degrees of freedom.

5.3. Noether's theorem

Actually all these facts are related, and the missing link is *Noether's theorem*. Simply stated, this works as follows. Suppose an action is invariant under a global (x-independent) transformation of the fields. Suppose it is not invariant under the corresponding local (x-dependent) transformation. Then the variation must be proportional to the derivative of the parameter $\Lambda(x)$ of the transformation (for simplicity we assume here that only first derivatives appear, but this can be generalized). Hence the variation of the action must have the form

$$\delta S = \int d^4x \partial^\mu\Lambda(x) J_\mu[\text{Fields}] \tag{5.5}$$

where $J_\mu[\text{Fields}]$ is some expression in terms of the fields of the theory. The precise form of J_μ depends on the action under consideration, and follows in a straightforward way from the symmetry.

The equations of motion are derived by requiring that the action is a stationary point of the action (see (1.10)), which means that terms linear in the variation, such as (5.5) must vanish. Integrating by parts we get then

$$\int d^4x\Lambda(x)\partial^\mu J_\mu[\text{Fields}] = 0 \ .$$

Since $\Lambda(x)$ is an arbitrary function, it follows that the it Noether current $J_\mu[\text{Fields}]$ is conserved. It is an easy exercise to show that the symmetry (5.4) of the free fermion action does indeed yield the current (5.3).

5.4. Covariant derivatives

Checking gauge invariance can be made easier by introducing the *covariant derivative*

$$D_\mu = \partial_\mu - ieqA_\mu \tag{5.6}$$

This has the property that under a gauge transformation

$$D_\mu \to e^{ieq\Lambda} D_\mu e^{-ieq\Lambda}$$

If we now write the Lagrangian as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi$$

checking gauge invariance is essentially trivial. One can simply pull the phases through D_μ, even if they are x-dependent!

Replacing normal derivatives by covariant ones is called *minimal substitution*, and the resulting interaction terms *minimal coupling*. It is a general principle: an action can be made gauge invariant by replacing all derivatives by covariant derivatives. For example the coupling of a photon to a complex scalar is given by the Lagrangian

$$\mathcal{L} = (D_\mu(q)\varphi)^*(D^\mu(q)\varphi) \ ,$$

where q is the charge of φ. Note that φ must be a complex field since the gauge transformation multiplies it by a phase. Note also that the field $\phi = \varphi^*$ has opposite charge.

The Lagrangian of the vector bosons can also be written down in terms of covariant derivatives. We have (for any $q \neq 0$)

$$-ieqF_{\mu\nu} = [D_\mu(q), D_\nu(q)] \ ,$$

from which gauge invariance of the action follows trivially. Here q has no special significance, and any non-zero value can be used.

5.5. Plane waves

Just as we did for the free fermion we can try to solve the equation of motion for the free spin-1 boson in momentum space. Choose plane waves

$$A_\mu(x) = A_\mu(k)e^{-ik\cdot x} \ .$$

Then we get

$$k^2 A_\mu(k) - k_\mu k^\nu A_\nu = 0 \ .$$

To solve this we choose a suitable basis. Clearly k^μ plays a distinguished rôle and will be one of the basis vectors. We may rotate and boost it k^μ to the form $(k^0, 0, 0, k^3)$ (we may assume $k^3 \neq 0$ since any particle can be boosted to non-zero momentum; note that $k^3 = 0$ would not be equally general, since it would not allow lightlike vectors). To complete the basis we may choose two vectors

$$\epsilon_1^\mu(k) = (0, 1, 0, 0) \ \text{ and } \ \epsilon_2^\mu = (0, 0, 1, 0) \ ,$$

plus a vector $n^\mu = (1, 0, 0, 0)$. Expanding plane waves in this basis we get

$$A^\mu(k) = a_i\epsilon_i^\mu(k) + b(k)n^\mu + c(k)k^\mu \ .$$

Substituting this into the equation of motion we get (note that $k_\mu\epsilon_i^\mu(k) = 0$)

$$k^2 a_i\epsilon_i^\mu(k) + b(k)(k^2 n^\mu - k^\mu n\cdot k) = 0 \ .$$

This equation can be analysed easily by considering the four components $\mu = 0, 1, 2, 3$

separately. We find then

$$b(k)(k^3)^2 = 0; \quad a_i(k)k^2 = 0,\ i = 1, 2\ .$$

The first equation implies $b(k) = 0$ since $k^3 \neq 0$. The second and third imply that $a_1(k)$ vanishes unless $k^2 = 0$. There are no constraints on $c(k)$. This was to be expected, since it is simply the momentum space analog of gauge invariance:

$$A_\mu(k) \to A_\mu(k) + k_\mu \Lambda(k)$$
$$c(k) \to c(k) + \Lambda(k)\ .$$

5.6. The Quantum Field

If we expand the free field in terms of these plane waves we get in the by now familiar way

$$A_\mu(x) = \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int \frac{d^3k}{2\omega(\vec{k})} [\epsilon_\mu^i(\vec{k}) e^{-ik\cdot x}\ a^i(\vec{k}) + k_\mu e^{-ik\cdot x}\ c(\vec{k}) + \text{c.c}]\ , \tag{5.7}$$

where "c.c" means "complex conjugate". The correct quantization procedure is more complicated than for scalars and spin-$\frac{1}{2}$ particles due to the absence of a canonical momentum for A_0 and due to gauge invariance. However, one does find the expected creation/annihilation operators for a_i:

$$[a^i(\vec{k}), a^j(\vec{k}')^\dagger] = 2\omega(\vec{k})(2\pi)^3 \delta^3(\vec{k} - \vec{k}')\delta^{ij}\ ,$$

but there is no such relation for c, because there is no canonical momentum corresponding to c: the Lagrangian doesn't depend on c at all. Hence c remains a numerical parameter, a "c-number".

5.7. Gauge fixing

The presence of an arbitrary, unquantized parameter in the expression for the gauge field implies also that the propagator will have such a parameter. It will turn out that these gauge parameters will drop out in the final result for any physical quantity. For this reason it is often convenient to fix the gauge. A standard choice in electrodynamics is the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. In momentum space this yields $\vec{k} \cdot \vec{A} = 0$, and in the special basis chosen above it implies that $c(k) = 0$.

There are other gauges which are more convenient. Usually one adds a "gauge-fixing term" to the Lagrangian of the form

$$-\tfrac{1}{2}\lambda(\partial^\mu A_\mu)^2 \ .$$

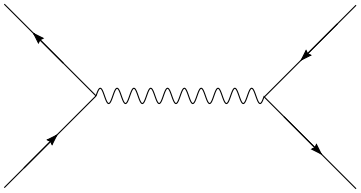
This is covariant, so that Lorentz invariance is manifest. The extra term breaks gauge invariance, and in this sense it "fixes the gauge". Of course one cannot simply add terms to the Lagrangian. It must be shown that this does not affect the results. This can be done most easily in the path integral approach to quantum field theory, which is beyond the scope of these lectures.

If one works out the propagator in the presence of the gauge-fixing term one gets, in momentum space

$$\Delta_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} - (1-\frac{1}{\lambda})\frac{k_\mu k_\nu}{k^2}\right) \ .$$

If indeed the gauge-fixing term is irrelevant, then physical amplitudes must be independent of λ. The reason why this is indeed true is that these terms are proportional to $k_\mu k_\nu$. The propagator is on both sides coupled to a current, as $J_1^\mu(k)\Delta_{\mu\nu}(k)J_2^\nu(k)$. Current conservation in momentum space implies $k^\mu J_\mu(k) = 0$, and therefore the extra terms vanish.

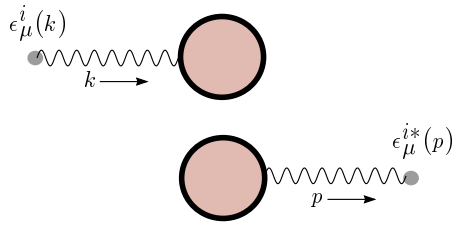
Unfortunately this argument is a bit too simplistic. It works for simple diagrams like



where the currents couple directly to external lines (the wavy line denoted the photon, the straight ones fermions or complex scalars). This implies that the fields out of which the current is built satisfy the equations of motion (*i.e.* are on-shell), and only in that case can we directly use current conservation. If the photon propagator ends on another internal line of the diagram, the argument is more subtle, but still true. The cancellation of the extra terms may require adding several Feynman diagrams. Since they cancel anyway, one may simplify the computation by choosing a convenient value for λ. Popular values are $\lambda \to \infty$ (Landau gauge) and $\lambda = 1$ (Feynman gauge). However, many people prefer to keep λ as a parameter, because its cancellation in the final result is a useful check on the computations.

5.8. External photons and polarization sums

Analogous to fermions, the Feynman rule for external fermions involves the factor ϵ^μ that appears in common with the creation/annihilation operators. The rule is



Note that for the outgoing particle we have written ϵ^* to allow complex polarization vectors. For example, a photon can have circular polarization with polarization tensors $\epsilon^1 \pm i\epsilon^2$.

Suppose we are considering a process with external photons, but we do not detect the polarization of those photons. Then the cross section is related to the square of the amplitude, summed over the polarizations.

One obtains the following polarization sum:

$$P_{\mu\nu} = \sum_{i=1}^{2} \epsilon^i_\mu \epsilon^{i*}_\nu \ .$$

In the basis we have chosen this is obviously equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

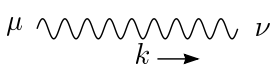
but we need to know the result also when $\vec{k}$ is pointing in an arbitrary direction. The generalization is easy: we simply define $n^\mu \epsilon^i_\mu = \vec{k} \cdot \vec{\epsilon}^{\,i} = 0$ and choose the two ϵ's orthonormal. Then we get

$$P_{\mu\nu} = \begin{cases} -g_{\mu\nu} - \frac{k_\mu k_\nu}{|\vec{k}|^2} & \text{if} \quad \mu \neq 0 \quad \text{and} \quad \nu \neq 0 \\ 0 & \text{if} \quad \mu = 0 \quad \text{or} \quad \nu = 0 \end{cases} .$$

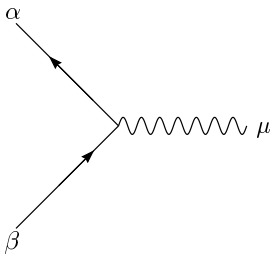
This is not a covariant expression, because we did not work in a covariant gauge. However, again gauge invariance simplifies our life. It turns out that the correct result is obtained if we simply set $P_{\mu\nu} = -g_{\mu\nu}$ since all other terms do not contribute.

5.9. Feynman rules

The Feynman rules are, in Feynman gauge



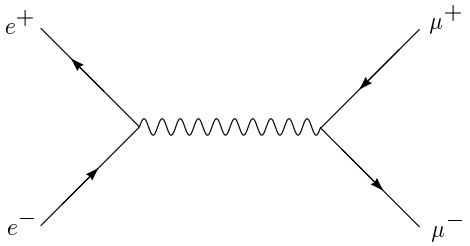
$$-\frac{i}{k^2}g_{\mu\nu}$$



$$ieq\gamma^{\mu}_{\alpha\beta}$$

5.10. An example

Consider the process $e^+e^- \to \mu^+\mu^-$ at tree level. There is just one Feynman diagram, namely



We choose momenta as follows: $e^+(k_1)e^-(k_2) \to \mu^+(p_1)\mu^-(p_2)$. Applying all the Feynman rules we easily get the amplitude:

$$T(k_1,k_2,p_1,p_2,i,j,m,n) = \bar{v}^i(\vec{k}_1)ieq_e\gamma^\rho u^j(\vec{k}_2)\frac{-ig_{\rho\sigma}}{(k_1+k_2)^2}\bar{u}^m(\vec{p}_2)ieq_\mu\gamma^\sigma v^n(\vec{p}_1)\ .$$

Here q_e is the electron charge and q_μ the muon charge. By definition, q is the charge of the particle (as opposed to the anti-particle) and also by definition the particles in this case are e^- and μ^-, whereas e^+ and μ^+ are the anti-particles. Hence $q_e = q_\mu = -1$.

Let us suppose that the incoming electron and positron beams are not polarized, and that we cannot observe the polarization of the outgoing muons. Then we should *average* over the polarizations of the incoming particles, and *sum* over the polarizations of the outgoing ones. The difference between summing and averaging is just a normalization factor. Hence we get

$$d\sigma = \frac{1}{4} \sum_{i,j,m,n} |T(k_1, k_2, p_1, p_2, i, j, m, n)|^2 \frac{\Phi_2}{4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2)}}$$

using (3.15). The amplitude squared yields

$$\frac{1}{4} \sum_{i,j,m,n} |T(k_1, k_2, p_1, p_2, i, j, m, n)|^2 = \frac{e^4}{4s^2} L_{\rho\sigma}(e, k_1, k_2) L^{\rho\sigma}(\mu, p_1, p_2) \ ,$$

where $s = (k_1 + k_2)^2$ and

$$L^{\rho\sigma}(e, k_1, k_2) = \sum_{i,j} \bar{v}^i(\vec{k}_1) \gamma^\rho u^j(\vec{k}_2) \left[\bar{v}^i(\vec{k}_1) \gamma^\sigma u^j(\vec{k}_2) \right]^* \ ,$$

and something analogous for the muon part. Now we need a few standard tricks. First of all, since $\bar{v}Mu$ is a number, for any matrix M, one may write

$$(\bar{v}Mu)^* = (\bar{v}Mu)^\dagger = u^\dagger M^\dagger (\bar{v})^\dagger \ .$$

In our case $M = \gamma^\rho$. This matrix is not hermitean, but

$$(\gamma^\rho)^\dagger = \gamma^0 \gamma^\rho \gamma^0 \ .$$

The factors γ^0 can be absorbed into the spinors, so that we get

$$(\bar{v}\gamma^\rho u)^* = \bar{u}\gamma^\rho v \ .$$

Similar manipulations hold for arbitrary matrices M built out of γ-matrices, but the details (especially signs) may differ slightly.

Using this we get

$$L_{\rho\sigma} = \sum_{i,j} \bar{v}^i_\alpha(\vec{k}_1)\gamma^\rho_{\alpha\beta}u^j_\beta(\vec{k}_2)\bar{u}^j_\eta(\vec{k}_2)\gamma^\sigma_{\eta\zeta}v^i(\vec{k}_1)_\zeta \ ,$$

where we have explicitly written the spinor indices, which are to be summed. With the indices written explicitly we may also write this as

$$L_{\rho\sigma} = \sum_{i,j} v^i(\vec{k}_1)_\zeta\bar{v}^i_\alpha(\vec{k}_1)\gamma^\rho_{\alpha\beta}u^j_\beta(\vec{k}_2)\bar{u}^j_\eta(\vec{k}_2)\gamma^\sigma_{\eta\zeta} \ .$$

Now we may perform the sums over i and j using (4.8):

$$L_{\rho\sigma} = (\not{k}_1 - m)_{\zeta\alpha}\gamma^\rho_{\alpha\beta}(\not{k}_2 + m)_{\beta\eta}\gamma^\sigma_{\eta\zeta} = \ \mathrm{Tr}\ (\not{k}_1 - m)\gamma^\rho(\not{k}_2 + m)\gamma^\sigma \ .$$

To computing this trace we may use the formulas

$$\begin{aligned}\mathrm{Tr}\ \gamma^\mu\gamma^\nu &= 4g^{\mu\nu} \\ \mathrm{Tr}\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\rho\nu}\right] \ .\end{aligned}$$

Then we get

$$L^{\rho\sigma} = 4(k_1^\rho k_2^\sigma + k_2^\rho k_1^\sigma - g^{\rho\sigma}(k_1\cdot k_2 + m^2)) \ .$$

Combining this with the muon contribution and ignoring the electron and muon mass (which is a good approximation at high energy) we get

$$L^{\rho\sigma}(e,k_1,k_2)L_{\rho\sigma}(\mu,p_1,p_2) = 32((k_1\cdot p_1)(k_2\cdot p_2) + (k_1\cdot p_2)(k_2\cdot p_1)) = 8(t^2+u^2) \ ,$$

expressed in terms of the Mandelstam variables

$$\begin{aligned}s &= (k_1+k_2)^2 \approx 2k_1\cdot k_2 \approx 2p_1\cdot p_2 \\ t &= (k_1-p_1)^2 \approx -2k_1\cdot p_1 \approx -2k_2\cdot p_2 \\ u &= (k_1-p_2)^2 \approx -2k_1\cdot p_2 \approx -2k_2\cdot p_2 \ .\end{aligned}$$

In the massless limit there are some more simplifications. We can take the momenta as

follows

$$\begin{aligned} k_1 &= (k, k, 0, 0) \\ k_2 &= (k, -k, 0, 0) \\ p_1 &= (k, k\cos\theta, k\sin\theta, 0) \\ p_2 &= (k, -k\cos\theta, -k\sin\theta, 0) \end{aligned}$$

Then $s = 4k^2, t = 2k^2(1-\cos\theta)$ and $u = 2k^2(1+\cos\theta)$, and

$$\tfrac{1}{4}\sum |A|^2 = e^2(1+\cos^2\theta)$$

The phase space factor is

$$\frac{\Phi_2}{4\sqrt{(k_1\cdot k_2)^2 - m_1^2 m_2^2}} \approx \frac{1}{64\pi^2 s} d\Omega$$

Putting all this together we find

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{64\pi^2 s}(1+\cos^2\theta)\ . \\ &= \frac{\alpha^2}{4s}(1+\cos^2\theta) \end{aligned}$$

Note that if we consider $e^+e^- \to e^+e^-$ the result will be different because there are extra diagrams!

5.11. Non-abelian gauge theories

The field transformations we used to construct QED

$$\psi \to e^{ieq\Lambda(x)}\psi$$

are local (x-dependent) elements of the group $U(1)$, the group of phases $e^{i\theta}$. Since all elements of this group commute it is called *abelian*.

The whole formalism can be extended in a rather straightforward way to non-abelian groups. For simplicity we restrict ourselves here to the group $SU(2)$. This group is well-know as the rotation group for spinors, but here it will play a totally different rôle.

Suppose we have a field ψ^i with an extra index i. For definiteness we will assume that the field is a fermion (it could also be a complex scalar) and that i just takes two values, 1,2.

The kinetic terms are

$$\mathcal{L}_{\text{kin}} = i \sum_{i=1}^{2} \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i \ .$$

This Lagrangian is invariant under transformations

$$\psi^i \to U^{ij} \psi^j \ ,$$

where U is a unitary two-by-two matrix (note that $\bar{\psi}^i \to (\bar{\psi} U^\dagger)^i$). Under multiplication these matrices form a group, $U(2)$, and for simplicity we restrict ourselves to the subgroup $SU(2)$ of matrices with determinant 1 (the overall phase is just a $U(1)$ transformation).

Now suppose we consider a space-time dependent transformation $U(x)$. Because of the derivative this is no longer an invariance of the Lagrangian. Imitating QED, we can try to cure that by introducing a covariant derivative D_μ that must transform as

$$D_\mu \to U D_\mu U^{-1} \ . \tag{5.8}$$

Without loss of generality we may assume that D_μ has the form $\partial_\mu + A_\mu$, so that it reduces to the ordinary derivative for $A_\mu = 0$. Since A_μ must act on the indices i it must be a two-by-two matrix.

Therefore we can expand it into a complete basis of two-by-two matrices. Any such matrix can be written as $a + \vec{b} \cdot \vec{\sigma}$, where a and $\vec{b}$ are four complex constants and σ are the Pauli matrices. In this case we want A_μ to be anti-hermitean (just as ∂_μ) so the constants must be purely imaginary. Furthermore we will set $a = 0$. This is not necessary, but the constant component of A_μ corresponds to an abelian gauge field that belongs to the overall phase in $U(2)$ in comparison with $SU(2)$. Since we are only considering $SU(2)$ here, only the components proportional to $\vec{\sigma}$ are interesting for us. Instead of the Pauli matrices we will use the matrices

$$T^a = \tfrac{1}{2} \sigma^a \ .$$

This avoids several factors $\frac{1}{2}$ in formulas, and also prevents confusion with the Pauli-matrices

used for spin. Then we write the gauge fields in the following way

$$A_\mu = -ig \sum_a A_\mu^a T^a \ ,$$

where we have introduced a factor $-ig$ for future purposes. The component fields A_μ^a are real. The factor g will play the rôle of the coupling constant, just as e in QED. Note that there are three gauge fields, for $a = 1, 2, 3$.

To see how A_μ should transform, it is instructive to consider infinitesimal transformations

$$U(\vec{\theta}) = 1 + i\vec{\theta} \cdot \vec{T} \ .$$

Expanding (5.8) to first order in θ we get

$$A_\mu \to A_\mu - i\partial_\mu \vec{\theta} \cdot \vec{T} + ig\vec{\theta} \left[\vec{T}, A_\mu\right] \ .$$

In terms of the components we get

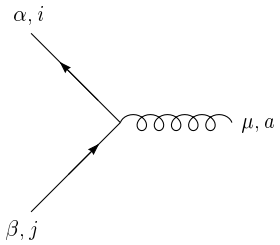
$$A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu \theta^a - \theta^b \epsilon^{abc} A_\mu^c \ .$$

5.12. Coupling to fermions

Replacing in the fermion action ∂_μ by D_μ we have coupled a three-component gauge field to the fermion. The action is explicitly

$$\begin{aligned}
\mathcal{L} &= i \sum_{i,j=1}^{2} \bar{\psi}^i \gamma^\mu D_\mu^{ij} \psi^j \\
&= i \sum_{i,j=1}^{2} \bar{\psi}^i \gamma^\mu \left[\partial_\mu \delta_{ij} - ig A_\mu^a T_{ij}^a\right] \psi^j \\
&= i\bar{\psi}^i \gamma^\mu \partial_\mu \psi^i + g A_\mu^a \bar{\psi}^i \gamma^\mu T_{ij}^a \psi^j \\
&= \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} \ .
\end{aligned}$$

In perturbation theory the first term gives rise to the fermion propagator, which in comparison to the one of QED has an extra factor δ_{ij}. The second term is a perturbation, which yields the Feynman rule (the curly line represents a non-abelian gauge boson, see below)



$$ig\gamma^{\mu}_{\alpha\beta}T^{a}_{ij}$$

The fermion spinors $u, v, \bar{u}, \bar{v}$ now get extra indices $i, j, \ldots$ in addition to their spinor indices. The matrices T^a are multiplied together along a fermion line, starting at an outgoing arrow and following the line against the arrow direction. If there is a closed fermion loop, one obtains a trace of a product of matrices T. Combinatorically this works exactly as for the γ matrices.

5.13. Gauge kinetic terms

We can also write down a kinetic term for the gauge fields. First define

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ . \tag{5.9}$$

Just like A_μ, the *field strength tensor* $F_{\mu\nu}$ is a two-by-two matrix, and it can be expanded in terms of Pauli matrices as

$$F_{\mu\nu} = -ig \sum_a F^a_{\mu\nu} T^a \ ,$$

Now we can express the components of $F_{\mu\nu}$ in terms of those of A_μ:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu \ .$$

The reason for writing $F_{\mu\nu}$ as in (5.9) is that it has a nice transformation law under gauge transformations

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1} \ .$$

Note that in contrast to the field strength of QED, the field strength tensor of non-abelian gauge theories is not gauge invariant. However, we can make a gauge invariant combination,

$$\mathcal{L}_{\text{gauge}} = \frac{1}{2g^2} \ \text{Tr} \ F_{\mu\nu} F^{\mu\nu} \ . \tag{5.10}$$

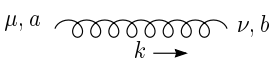
where the trace is over the two-by-two matrices. Because of the cyclic property of the trace this quantity is gauge invariant. It is also manifestly Lorentz invariant, and hence it is a

good candidate for the Lagrangian of the non-abelian gauge fields. If we write it out in components we get

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{\mu\nu,a} \ .$$

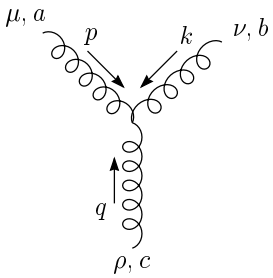
5.14. Feynman rules

Note that the linear terms in $F^a_{\mu\nu}$ are just like those for QED. If that was all there was we just had three copies of QED, for $a = 1, 2, 3$. The quadratic terms in $F^a_{\mu\nu}$ give rise to cubic terms in the Lagrangian that are proportional to g, and quartic terms proportional to g^2. These are interactions. Just as in QED, we use the bilinear terms in the action to define a propagator, which in fact is identical to the one of QED except for a factor δ^{ab}. To distinguish non-abelian gauge bosons from photons we use another kind of line:

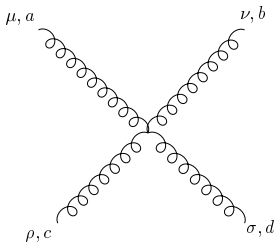


$$-\frac{i}{k^2}g_{\mu\nu}\delta^{ab}$$

The cubic and quartic term give rise to interactions, whose Feynman rules are



$$g\epsilon^{abc}\left[(q-p)_\nu g_{\mu\rho} + (k-q)_\mu g_{\nu\rho} + (p-k)_\rho g_{\mu\nu}\right]$$



$$\begin{aligned} -ig^2[&\epsilon_{eab}\epsilon_{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \\ &+\epsilon_{eac}\epsilon_{edb}(g_{\nu\rho}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma}) \\ &+\epsilon_{ead}\epsilon_{ebc}(g_{\sigma\rho}g_{\mu\nu} - g_{\mu\rho}g_{\nu\sigma})] \end{aligned}$$

Just like photons, non-abelian gauge bosons A^a_μ have two components (two for each value of the index a of course), and when they appear as external lines they are represented by

polarization tensors ϵ^a_μ. Gauge fixing is required also in the non-abelian case, and there is an extra complication. Extra fields have to be introduced that only appear in loop diagrams, the *Fadeev-Popov ghosts*. They do not correspond to physical particles, but are required in order to get gauge-invariant answers.

5.15. Other gauge groups

All the foregoing can easily be generalized to other symmetries. Instead of $SU(2)$ we may use other groups like $SU(N)$ or $SO(N)$. In general, one has instead of the Pauli matrices some other set of hermitean matrices T^a. These matrices satisfy a generalized set of commutation relations,

$$\left[T^a, T^b\right] = if^{abc}T^c \ .$$

where f^{abc} is a set of real numbers that are called the *structure constants* of the group. They are fully anti-symmetric in all three indices. In addition to the commutation relations, the only other property one needs to know about these matrices is their normalization. Often one uses[⋆]

$$\text{Tr } T^aT^b = \tfrac{1}{2}\delta^{ab} \ ,$$

which is indeed satisfied by the $SU(2)$ matrices we used. In (5.10) this normalization is implicitly assumed.

To write down Lagrangians, transformations and Feynman rules for another group, simply make everywhere the replacement

$$\epsilon^{abc} \to f^{abc} \ .$$

In interesting special case is the group $SU(3)$, with fermions in triplet representations. There are eight traceless hermitean three-by-three matrices T^a. This yields QCD (quantum chromodynamics). Corresponding to the eight matrices there are eight gauge bosons, called gluons, while the fermions are called quarks. It is now completely straightforward to write down the QCD Lagrangian.

⋆ In some theories different fields may transform according to different *representations* of the gauge group. Then the normalization can be fixed for only one representation, and will in general be different for other fields. This does not occur in the standard model: although quarks and anti-quarks are in different $SU(3)$ representations, the traces have the same normalization.

Note that the entire discussion of non-abelian gauge theories is completely analogous to that of QED. This is in fact a special case, obtained by replacing

$$\begin{aligned} T^a &\to q \\ g &\to e \\ \epsilon_{abc} &\to 0 \\ \vec{\theta}(x) &\to e\Lambda(x) \ . \end{aligned}$$

6. The Higgs mechanism

6.1. Vacuum expectation values

The classical value of a field is the value it has when all quantum fluctuations are in their ground state. Up to now we have always implicitly assumed that the classical value of any field, $\langle 0|\phi|0\rangle$, vanishes. Indeed, this follows immediately if one computes the vacuum expectation value (v.e.v.) of any quantum field we have written down so far, since the v.e.v. of any creation/annihilation operator vanishes. But this is not necessarily true. In general one can have $\phi = \phi_{\rm cl} + \phi_{\rm qu}$, with all quantum fluctuations in the second term, and $\phi_{\rm cl} \neq 0$. The classical field must be a solution to the equations of motion, or in other words a stationary point of the action. Quantum mechanics gives rise to fluctuations around this classical solution. In all examples we have seen so far $\phi = 0$ was a solution to the equations of motion, but in some cases there may be other solutions.

The classical value, $\phi_{\rm cl}$, serves as a new, non-trivial ground state of the theory. One defines the vacuum in such a way that $\langle 0|\phi_{\rm qu}|0\rangle = 0$. The properties of the vacuum state are determined by $\phi_{\rm cl}$. The possible values of $\phi_{\rm cl}$ are restricted by the symmetries the theory should have. In general, with $\phi_{\rm cl} \neq 0$ there will be fewer symmetries than with $\phi_{\rm cl} = 0$.

In any case we want our vacuum to be translation invariant and Lorentz-invariant. This restricts $\phi_{\rm cl}$ to be a constant over all of space-time, and it restricts ϕ to be a scalar field. But ϕ may also transform non-trivially under some internal gauge symmetry. In general such a symmetry will then not be a symmetry of the new vacuum.

6.2. Higgs mechanism for abelian gauge symmetry

As an example, consider a complex scalar, coupled to an abelian gauge field. The Lagrangian is

$$\mathcal{L}_{\rm scalar} = (D_\mu\phi)^* D^\mu\phi \ . \tag{6.1}$$

where $D_\mu = \partial_\mu - ieA_\mu$. The gauge symmetry of this Lagrangian is $\phi \to e^{ie\Lambda}\phi$. Let us assume that ϕ has a v.e.v. equal to v, which we will take to be real. If we expand around $\phi = v$, the fluctuations will not have the gauge symmetry anymore, since v is fixed and doesn't transform. This is puzzling at first sight, because we had argued before that the gauge symmetry was essential for having a massless photon with two physical polarizations.

To see what happens we rewrite the Lagrangian in a peculiar way, by choosing

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\xi(x)} \ , \tag{6.2}$$

so that η are the real fluctuations and ξ the imaginary ones. In the quantum theory the quanta of η and ξ will yield the fluctuations, and they will have the usual expansion in terms of oscillators, as in (2.2). Substituting (6.2) into the Lagrangian we get

$$\tfrac{1}{2}|(\partial_\mu\eta + i(v+\eta)(\partial_\mu\xi - eA_\mu)|^2 \ .$$

Now we replace everywhere A_μ by $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\xi$. Now the Lagrangian becomes

$$\tfrac{1}{2}|(\partial_\mu\eta - i(v+\eta)eB_\mu)|^2 \ .$$

Expanding this yields

$$\tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta + \tfrac{1}{2}e^2v^2B_\mu B^\mu + \tfrac{1}{2}e^2B_\mu B^\mu\eta(2v+\eta) \ .$$

Now suppose that there are other terms in the Lagrangian in addition to (6.1). This includes in particular the kinetic terms. All the additional terms must be gauge invariant. The replacement $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\xi$ can be realized on all other terms as a gauge transformation, which may include ξ-dependent transformations of other fields. This implies that the other terms in the Lagrangian remain unchanged, except that A_μ is replaced everywhere by B_μ.

To summarize, suppose we started with a Lagrangian

$$-\tfrac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{rest}}(A) \ .$$

Then after shifting the vacuum and some changes of variables we end up with

$$-\tfrac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) + \tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta + \tfrac{1}{2}e^2v^2B_\mu B^\mu + \tfrac{1}{2}e^2B_\mu B^\mu\eta(2v+\eta) + \mathcal{L}_{\text{rest}}(B) \ .$$

Two observations can now be made:

— The field ϕ had two real degrees of freedom, η and ξ, but the latter has disappeared completely.

— The quadratic term in B_μ gives a mass ev to the vector field.

This magic is called the *Higgs mechanism*, after one of its inventors. It allows us to give a mass to the gauge boson, simultaneously breaking the gauge symmetry. The field ξ is not really gone. As we have seen, a massive gauge boson has three degrees of freedom, a massless one only two. When we made the transformation $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\xi$ we have absorbed ξ into the gauge field to provide the extra degree of freedom needed to make it massive. One often says that ξ was "eaten" by the gauge field.

Massive vector bosons occur in the theory of weak interactions, the $W^\pm$ and Z bosons. You may wonder why we couldn't simply have added the mass term by hand. The reason is that such a procedure makes the theory inconsistent. It explicitly breaks gauge invariance, and gauge invariance is essential for consistency of theories with spin-1 particles. In the procedure explained above gauge invariance is not manifest anymore in the shifted ground state, but it is still present in a less obvious form.

6.3. The potential

The theory we have obtained after shifting the vacuum has a massive vector boson plus a massless scalar. The presence of the scalar is a prediction of the Higgs mechanism, but if it could only be massless it would already contradict experiment. If one tries to add a mass term for the scalar ϕ one finds that now only the trivial vacuum $\phi = 0$ is a solution to the

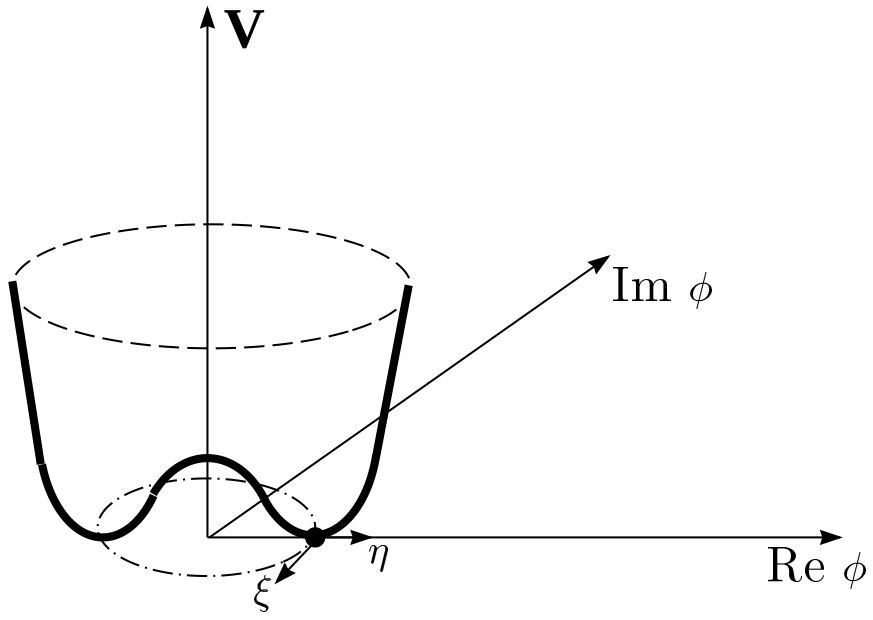
equation of motion (remember that we only consider constant solutions). But by adding interaction terms one can circumvent this easily. We consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi\phi^* - \tfrac{1}{4}\lambda(\phi\phi^*)^2 \ .$$

For constant fields the kinetic terms may be ignored, and the equations of motion read

$$m^2\phi + \frac{1}{2}\lambda\phi(\phi\phi^*) = 0 \ .$$

If $m^2 < 0$ his has two solutions, $\phi = 0$ and the non-trivial ground state $\phi\phi^* = \frac{-2m^2}{\lambda}$. Note that only the norm of ϕ is fixed, not its phase. We have a continuous set of ground states to choose from, and above we have chosen just one of them, $v = \sqrt{\frac{-2m^2}{\lambda}}$ real. The continuum is illustrated in the following picture of the potential as a function of ϕ



The black dot indicates our choice of the ground state, but any choice on the bottom of the "Mexican hat" would have been fine as well. By making a choice we break the gauge symmetry, *i.e.* the phase rotations of ϕ. We have also indicated the directions of the small perturbations η and ξ.

If one shifts the value of ϕ one finds that η now gets a mass, and ξ disappears, as before. The mass of η, the Higgs boson, is a free parameter, and is in principle unrelated to the mass of the vector boson, ev.

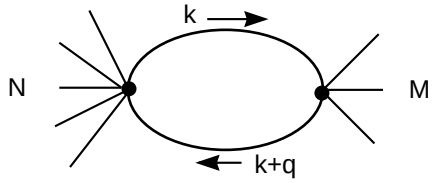
Observe that in order to find a non-trivial ground state we had to take $m^2 < 0$. If one expands around the trivial ground state $\phi = 0$ this negative value of m^2 leads to trouble: the theory now contains particles with imaginary mass. This means that the theory with this vacuum choice is sick. The only correct vacuum is the non-trivial one, and all fluctuations around it have positive or zero mass. If on the other hand $m^2 > 0$ the only vacuum is $\phi = 0$. By continuously changing m^2 we can go from the symmetric vacuum $\phi = 0$ to the vacuum with broken gauge symmetry. This phenomenon is called *spontaneous symmetry breaking*.

All the above can be generalized to non-abelian gauge theories, but we will not discuss that in any detail. The main features are the same. Some symmetries within a symmetry group are spontaneously broken, and the corresponding vector bosons acquire a mass by each "eating" a scalar. The resulting spectrum always contains (at least) one Higgs scalar, whose mass is a free parameter and hence cannot be predicted.

7. Loops

7.1. Ultraviolet Divergences

Consider a simple loop diagram in a scalar theory, such as



To be rather general we have left the number of external lines as a free parameter, and we have used an N+2-point vertex with coupling constant λ_{N+2} and an L+2-point vertex with coupling constant λ_{L+2}. The interaction Lagrangians are thus $\frac{1}{(N+2)!}\lambda_{N+2}\phi^{N+2}$ and the same with L instead of N. The loop integral is

$$(i\lambda_{N+2})(i\lambda_{L+2}) \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{k^2 - m^2}\right] \left[\frac{i}{(k+q)^2 - m^2}\right] \ .$$

Here the integral is over all of momentum space.

For large k this integral behaves as

$$V = \int d^4k \frac{1}{k^2(q+k)^2} \ .$$

This integral diverges for the same reason why $\int dx(1/x)$ diverges. One calls this a logarithmic divergence. Since it occurs for large momenta it is also called an *ultraviolet divergence*.

7.2. Regularization

One can make the ultraviolet divergence explicit by cutting off the integral. Instead of integrating over all of momentum space, one integrates over a finite sphere of radius Λ^2, so that $k^2 < \Lambda^2$.[⋆] After introducing the cutoff the integral is finite, but now it depends on the cutoff,

$$V \propto \lambda_{N+2}\lambda_{L+2} \log(\frac{\Lambda^2}{q^2})$$

and we cannot take the cutoff to infinity.

The process of making the integral finite is called *regularization*. There are other ways of achieving this, and since it has no obvious physical meaning, all physical quantities one finally obtains should be independent of the regularization procedure. But first we have to get rid of the divergences.

7.3. The origin of ultraviolet divergences

What is the reason for the infinity? Note that when we integrate over all of momentum space we are doing something that is physically ridiculous. Large momentum corresponds to large energies, and to short distances. Experimentally we have been able to explore nature up to several hundred GeV, and without doing further experiments we cannot pretend to know what happens at larger energies or shorter distances. Suppose that at shorter distances space-time has a crystalline structure. Then the inverse of the cell size would provide a maximum momentum, since wavelengths smaller than the cell size make no sense. In this situation the momentum cut off introduced above *would* have a physical meaning.

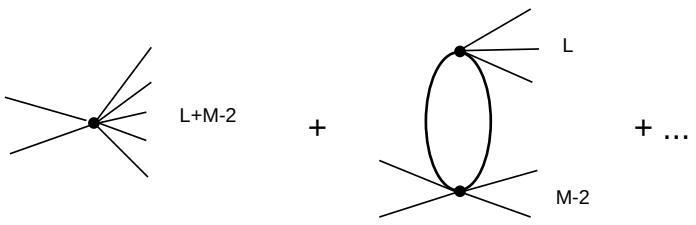
⋆ Since we are in Minkowski space this requires a bit more discussion, since it is not obvious what a "sphere" is. In fact all these manipulations are always done after one has analytically continued the integrand to Euclidean space, and then the cutoff make sense. We leave out all technical details here.

One may also envisage changes to the vertices that are small at low energies, but cut off the integral at large energies. For example, suppose the Feynman rule for a vertex was not λ_{L+2} but something like $\lambda_{L+2}\ [\Lambda^2/(P^2+\Lambda^2)]$, where P is the sum of the incoming momenta and Λ a large mass (larger than 1 TeV, say). A low energy observer would experimentally detect the existence of the $\lambda_{L+2}\phi^{L+2}$ vertex by scattering two ϕ particles, and measuring the probability that L such scalars come out. At low energies $P^2 \ll \Lambda^2$, and the correction factor is almost one. If Λ is large enough, it would be impossible to observe it. However, if we insert the same vertex in a loop diagram we integrate over all momenta, and we are sensitive to any such factor. Factors of this kind do indeed occur, for example if our ϕ particle were not elementary, but is in fact a bound state of two other particles. Then the interaction vertices are corrected by "form-factors". If the binding scale is sufficiently high, a low energy observer cannot resolve the sub-structure, and for all practical purposes sees the particle as elementary.

In other words, if we claim that Feynman diagrams are divergent for large momenta, we are simply making a completely unfounded extrapolation of known physics to extremely short distances. But that leaves us with the question what to do about these integrals.

7.4. Renormalization

Let us return to experiment. Clearly the loop diagram contributes to processes with $N+L$ external lines. Suppose our theory has an additional vertex $\frac{1}{(N+L)!}\lambda_{N+L}\phi^{N+L}$. Suppose we do a scattering experiment to measure this vertex for example 2 ϕ particles to $N+L-2$ such particles. The amplitude, expanded to one-loop level has now schematically the following contributions



(this is just intended schematically, and in particular we didn't draw all diagrams here; there are others with one or both incoming lines attached to the other vertex). An experimentalist

can only measure the sum of these diagrams.† The sum gives an expression like

$$\lambda_{L+N} + C\lambda_{N+2}\lambda_{L+2}\log(\frac{\Lambda^2}{q^2}) + \dots \ ,$$

where C is some numerical coefficient and q is some combination of the external momenta. The explicit form of both follows from the details of the computation, but is not relevant for our purpose. The dots indicate terms that are finite for $\Lambda \to \infty$ plus contributions of higher order diagrams.

The coupling constant λ_{L+N} is a physical parameter of the theory, that is not predicted by the theory itself, but must be measured. To measure it we must specify a physical process. In the present case, that physical process could be ϕ-ϕ scattering to $N + L - 2$ ϕ-particles with precisely specified external momenta. Let us call the value of q for those fixed momenta q_0. Then the physical value of the coupling constant is related in the following way to the parameters in the Lagrangian

$$\lambda^{\text{physical}}_{L+N} = \lambda_{L+N} + C\lambda_{N+2}\lambda_{L+2}\log(\frac{\Lambda^2}{q_0^2}) + \dots \ . \tag{7.1}$$

Experimentalists can only measure finite numbers, so clearly $\lambda^{\text{physical}}_{L+N}$ is finite, and independent of Λ. The terms in the linear combination appearing on the right hand side are physically irrelevant, because we can never measure them separately. If we now re-express the physical process for arbitrary momenta in terms of the physical, measured coupling constant, we get of course

$$\lambda^{\text{physical}}_{L+N} + C\lambda_{N+2}\lambda_{L+2}\log(\frac{q_0^2}{q^2}) + \dots \ ,$$

which is finite.

This process of absorbing short distance singularities into physical quantities is called *renormalization*.‡ The quantity $\lambda^{\text{physical}}_{L+N}$ is usually called the renormalized coupling constant, and the quantities that appeared in the Lagrangian are called *bare* coupling constant. They cannot be measured.

† In fact only the infinite sum of all diagrams is a measurable quantity. Here we work to second order in the coupling constants λ_K, which are assumed to be small. This may seem strange since the loop correction diverges as $\Lambda \to \infty$. But note that for any *finite* choice of Λ we can make the coupling constants small enough so that the next order can be ignored. After computing a physical cross section for small coupling, we continue the coupling to its physical value.

‡ In addition to *physical* quantities, some singularities are absorbed in the normalization of the fields, which is not a physical quantity.

The crucial point is now the following. We can only give one definition of $\lambda_{L+N}^{\text{physical}}$, but of course this coupling constant appears in many different processes. Whenever λ_{L+N} (the bare coupling) appears, we replace it by $\lambda_{L+N}^{\text{physical}}$, using (7.1). If all goes well, this should remove all $\log \Lambda$ terms at the next order. For this to work, it should be true that λ_{L+N} *always* receives at the next order exactly the same loop corrections. To some extent one can see that intuitively, but actually proving it is quite hard.

The foregoing can be summarized by the following prescription:

1. Calculate some process to a given loop order in perturbation theory.
2. Introduce a prescription to cut off all the "divergent" integrals. (regularization).
3. For each physical parameter, choose one specific physical process to define and measure it.
4. Then use this definition in all other processes to substitute the bare parameters by the renormalized ones. If all goes well, one now obtains for each process one computes a perturbative expansion in terms of physical, renormalized parameters, and all dependence on the regulator scale Λ has disappeared.

Note that it doesn't matter whether the momentum integrals are actually infinite or are cut off by some unknown short distance physics. All the unknown physics is absorbed in the renormalized parameters. These parameters depend on unknown physics and are therefore not determined theoretically.

However, in general the number of parameters one needs in this procedure is infinite. We can only absorb a $\log \Lambda$ in a physical parameter if that parameter actually exists. For a scalar theory the procedure outlined above will generate a vertex with $N + L$ lines if there exists a vertex with $N + 2$ and one with $L + 2$ lines. Suppose $N = L = 3$ and $L = 2$, *i.e.* we consider two five point vertices. Then $N + L = 6$, and to absorb the corresponding divergence we need a six-point vertex. Combining that with a five-point vertex gives a seven-point vertex, and clearly this never stops. Then the theory has an infinite number of parameters. To determine it *completely* one needs to do an *infinite* number of experiments.

7.5. Renormalizability

A theory is called *renormalizable* if all divergences can be absorbed into a *finite* number of parameters. This is a very strong restriction, but it makes the theory enormously more powerful. After the determination of a handful of physical parameters, one can make detailed predictions of all physical cross sections and decay rates! In our scalar theory example this allows only two vertices, ϕ^3 and ϕ^4. If there is just one scalar, the only parameters are the couplings λ_3 and λ_4 and the mass of the scalar. The mass is treated in a quite similar way: it also must be determined experimentally, and it is also renormalized.

Other examples of renormalizable theories are QED and QCD. Both have just one parameter, the coupling constants e and g respectively (if one ignores the fermion masses). The coupling constant of QED can be defined by means of the electron-photon coupling at zero photon momentum. For the QCD coupling constant the equivalent procedure cannot work, because we do not have free quarks and gluons, and furthermore because QCD perturbation theory doesn't work at zero gluon momentum. So one necessarily has to define g rather more indirectly, and at a non-vanishing momentum scale.

7.6. Dimensional analysis

An important constraint on the allowed vertices comes from dimensional analysis. Since we have set $\hbar = 1$ and $c = 1$ there is just one physical dimension left, that of a mass. The number of powers of "mass" a physical quantity contains is called its *dimension*. Hence mass has dimension 1 and length has dimension -1; derivatives then have dimension 1.

Actions are dimensionless, and therefore Lagrangian densities must have dimension 4. From the kinetic terms we can then determine the dimensions of all fields. For example

$$\mathcal{L} = \tfrac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi\right]$$

tells us that φ has dimension 1.* Similarly, fermion kinetic terms tell us that fermion fields have dimension $\frac{3}{2}$, and gauge kinetic terms require that gauge fields have dimension 1.

$\star$ Note that we could have allowed for a coefficient in front of the kinetic term, which could have its own dimension. However, we can always absorb such a coefficient by redefining φ. Any other term in the Lagrangian will always have a coefficient.

7.7. Renormalizable theories

Now consider interactions. Since the dimensions of the fields are fixed, dimensional analysis now fixes the dimensions of the coupling constants. Take for example $\lambda_N \varphi^N$. Obviously the dimension of λ_N is $4 - N$. If $N > 4$ the coupling constant has a negative dimension. This turns out to be the origin of non-renormalizability. Feynman diagrams with combinations of such coupling constants can have coefficients with arbitrarily negative dimensions, whose divergences correspond to terms with an arbitrarily large number of fields.

A necessary condition for renormalizability is absence of negative dimensional coupling constants. This leaves only very few possibilities, namely

— φ^3 (with a coupling of dimension 1)

— φ^4

— $\bar{\psi}\psi\varphi$ or $\bar{\psi}\gamma_5\psi\varphi$

— $\partial A A^2$ or A^4

— $\bar{\psi}\gamma^\mu\psi A_\mu$ or $\bar{\psi}\gamma^\mu\gamma_5\psi A_\mu$

— $A^2\phi^2$ or $A\partial\phi\phi$

where φ, ψ and A denote generic scalars, fermions and spin-1 fields. In some cases details of the index structure are suppressed. In all cases except φ^3 the coupling constant is dimensionless. In addition to these interactions also mass terms are allowed.

For spin-1 fields more severe constraints apply, which will not be explained here. Their interactions must not only have the structure listed above, but they must have the precise form we saw in the chapter on gauge theories. This is due to the requirement of gauge invariance. Mass terms for spin-1 bosons are only allowed if they are due to the Higgs mechanism.

7.8. The meaning of renormalizability

There is another way of looking at the requirement of renormalizability. Mass scales in physics usually have a deeper meaning. If a coupling constant has a non-zero dimension, the corresponding mass scale must have a physical interpretation in terms of "new physics". Take for example a fermion four-point vertex $(\bar{\psi}\psi)^2$. This has dimension six, so the coupling

constant has dimension -2. Fermi wrote down an interaction vertex of this type to understand the weak interactions, and this gives a very accurate description of weak interactions at low energies. However, now we know a more fundamental explanation for this four-fermi interaction. In the standard model, the interaction is attributed to exchange of a heavy W-boson, and takes the form[†]

$$\bar{\psi}\gamma^{\mu}\psi\frac{1}{k^2-M_W^2}\bar{\psi}\gamma_{\mu}\psi \ .$$

For low values of k this looks like a four-fermi vertex with a coupling constant $\frac{1}{M_W^2}$, but at higher energies the effect of the propagator momentum k_μ becomes noticeable.

The four-fermion theory is not renormalizable. If one imagines a time before the weak interactions were discovered, but QED was known, then the physicists of that time could live happily with the knowledge that their theory was renormalizable. The discovery of the four-fermion interaction changed that. Its presence hints at new physics. That physics is described by the standard model, which again looks to us as a renormalizable theory. But future experiments may change that again. If evidence for new interactions with negative dimensional coupling constants is found, we may again expect new physics. Nobody knows where that may happen, but the point is that it does not matter as long as the scale of the new physics is large enough. Then the extra interactions are anyway invisible to us. In other words, renormalizability is not some deep property of nature, but rather an inevitable consequence of doing physics well below the next scale where interesting new phenomena occur.

In our description of nature both renormalizable and non-renormalizable theories play a role. For example the standard model of weak, electromagnetic and strong interactions is renormalizable, but the theory of pion-nucleon interactions is not. In the former case that means that we can predict scattering amplitudes of quarks and leptons with – in principle – unlimited accuracy in terms of only a few (about 27) parameters that must be determined experimentally. In the latter case we may be able to describe low-energy pion-nucleon interactions, but if we attempt to go to higher energies more and more parameters are required and finally the description becomes completely inadequate. At sufficiently short distances

† This is only schematic, and important details such as the left-handed nature of the currents are suppressed.

we have to take the quark substructure of pions and nucleons into account, and we cannot pretend that they are fundamental fields.

At some time in the future we may find ourselves in the same situation with the standard model, but only experiment can tell us if and when that happens.

7.9. Running coupling constants

Let us return to the definition of the coupling constant, (7.1). We will consider renormalizable theories, so we choose $L = N = 2$. Then all vertices involved are four-point vertices (we ignore λ_3). To define the coupling constant, it was necessary to fix an energy scale q_0. Suppose we consider the same process at some other energy scale Q. Then the amplitude is

$$\begin{aligned} A &= \lambda_4 + C\lambda_4^2 \log(\frac{\Lambda^2}{Q^2}) + \dots \ . \\ &= \lambda_4 + C\lambda_4^2 \log(\frac{\Lambda^2}{q_0^2}) + C\lambda_4^2 \log(\frac{q_0^2}{Q^2}) + \dots \\ &= \lambda_4^{\text{physical}} + C\lambda_4^2 \log(\frac{q_0^2}{Q^2}) + \dots \\ &= \lambda_4^{\text{physical}} + C(\lambda_4^{\text{physical}})^2 \log(\frac{q_0^2}{Q^2}) + \dots \ . \end{aligned}$$

In the last step we have substituted λ_4 by $\lambda_4^{\text{physical}}$. This is allowed because the difference is of higher order in λ_4. It is in fact precisely what the renormalization prescription tells us to do, in order to remove divergences at the next order of perturbation theory. If we consider scales Q very far from our reference scale q_0 perturbation theory doesn't work very well anymore: there are large logarithmic corrections. They are finite remnants of the $\log \Lambda$ divergence. This situation can be improved by using as the expansion parameter not the constant $\lambda^{\text{physical}}$ but the *running coupling constant*

$$\lambda_4(Q^2) = \lambda_4^{\text{physical}} + C(\lambda_4^{\text{physical}})^2 \log(\frac{q_0^2}{Q^2}) + \dots \ , \tag{7.2}$$

where, as before, $\lambda_4^{\text{physical}}$ is the coupling constant defined at the reference scale q_0. The name "running coupling constant" is of course an internal contradiction: obviously it is not a constant at all.

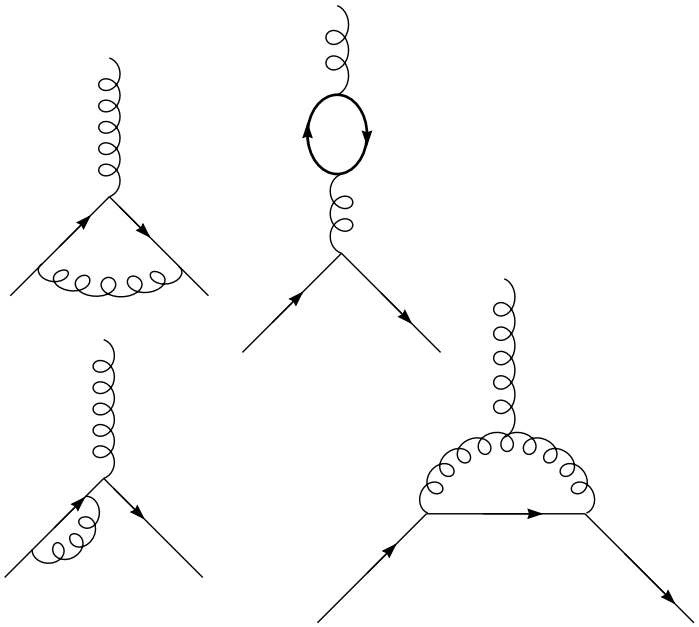
7.10. Beyond one loop

To keep the discussion simple we have only discussed one-loop corrections, but everything goes through in the same spirit at higher loops. The definition of the running coupling constant can be extended in such a way that even the largest contributions of higher loop diagrams (the *leading logs*) are included in the definition of the running coupling constant. In this way the use of running coupling constants effectively sums up part of the infinite perturbation series.

We have been rather vague about the definition of the scale Q to be used, and indeed this is a rather subtle issue. In general, it should be the energy scale of the process one considers, but it is not always clear cut. Bad choices lead to poor convergence of the series, but one will only see that if higher loop orders are computed.

7.11. Asymptotic Freedom

All couplings of the standard model run, but the best known example is the couping constant of QCD. The sign of the one loop correction is rather important, and determines if the coupling constant decreases or increases with increasing energy. If it decreases, perturbation theory effectively gets better and better with increasing energy, and the theory is called *asymptotically free*, *i.e.* it approaches a free theory at small distances. This is the case for QCD. Some of the diagrams responsible for the running of the QCD coupling constant are shown below



Here we consider the gauge-boson/fermion coupling. If instead one uses the three gauge-boson coupling or the four gauge-boson vertex one gets an entirely different set of diagrams, but by the miracle of gauge invariance the result, the renormalization and running of the coupling constant, is exactly the same.

If the coupling constant increases with increasing energy it will eventually become too large for perturbation theory to hold. This is what happens for QED. The relevant diagrams are almost the same as for QCD, but of course the one with the three gluon vertex is absent* The effect of the running of α is quite small though: it is $\frac{1}{137.04}$ at zero energy, and about $\frac{1}{129}$ at M_W. One has to go to extremely large energies before the coupling constant diverges.

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$\star$ In addition the QCD calculation has additional diagrams with ghost loops.