Anomalies, Characters & Strings

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ANOMALIES, CHARACTERS AND STRINGS

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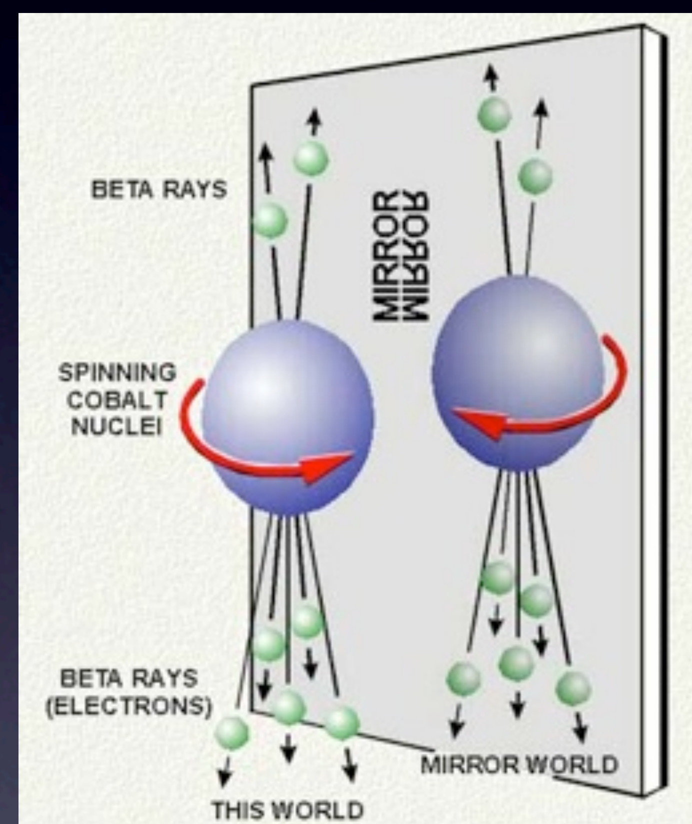
Parity violation (1957)

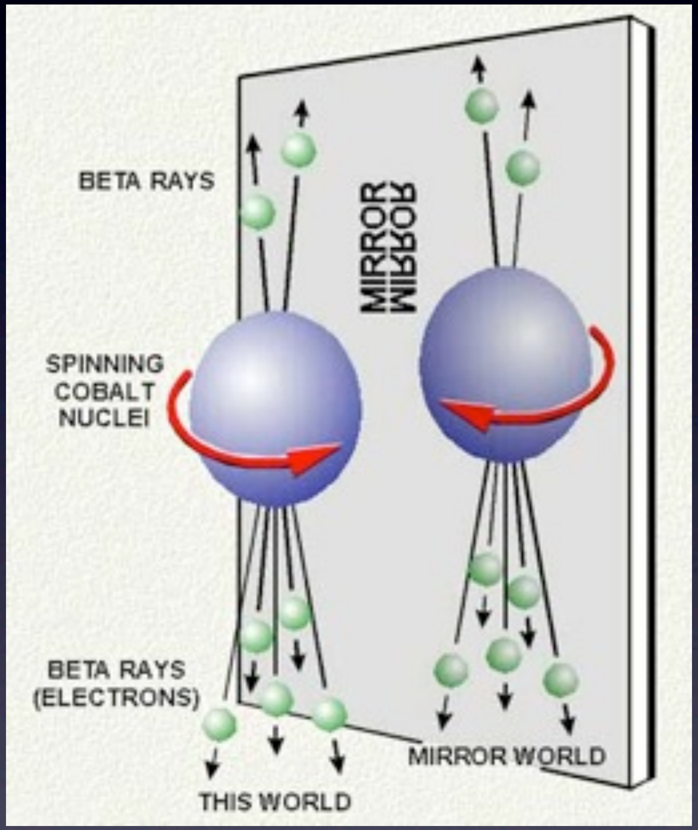
T.D. Lee and C.N. Yang



C.S. Wu

Parity violation (1957)

T.D. Lee and C.N. Yang



C.S. Wu

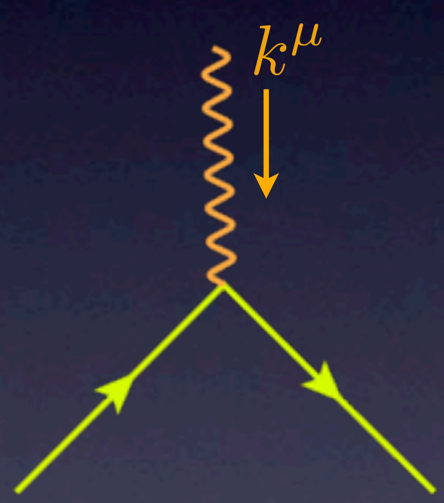
Pauli:

Now after the first shock is over, I begin to collect myself

Current Conservation

Electromagnetic current $\partial_\mu J^\mu(x) = 0$

In momentum space



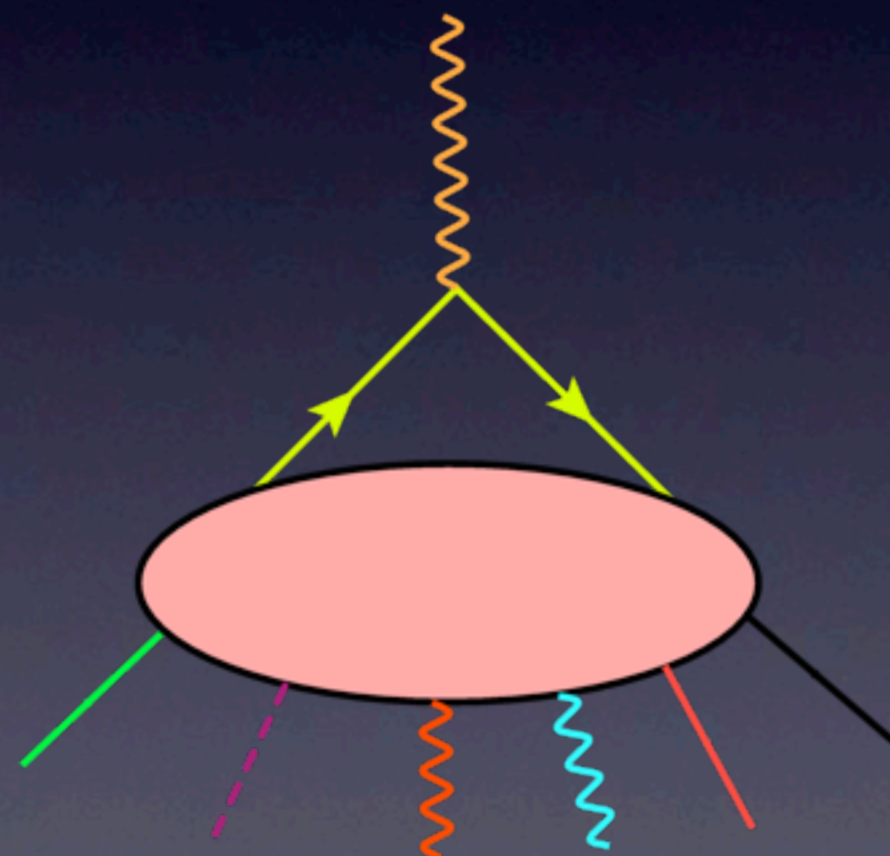
$$k^\mu \overline{\psi} \gamma_\mu \psi = 0$$

Essential for unitarity and renormalizability of gauge theories

Anomalies

Classical symmetries that are not symmetries of the quantum theory

$k^\mu \Bigg($



$\Bigg) = 0$?

Anomalies

S. Adler (1969), J. Bell and R. Jackiw (1969)

$$J^{\alpha} = -\delta\mathfrak{L}/\delta(\partial_{\alpha}\Lambda) = \bar{\psi}\gamma^{\alpha}\gamma_5\psi\,,$$

$$\partial_{\alpha}J^{\alpha} = (\alpha_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\,\epsilon_{\xi\sigma\tau\rho}\,.$$

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Anomalies

- Breaking of global symmetries: implications for

$$\pi^0 \rightarrow \gamma\gamma$$

η' mass ("*U(1)*-problem")

- Breaking of local symmetries: Must be avoided.

$$\mathrm{STr}\ T^aT^bT^c = \frac{1}{2}\mathrm{Tr}\ \{T^a, T^b\}T^c = 0$$

$\mathrm{Tr}\ T^a = 0$ *(U(1) x (Graviton)^2 anomaly)*

Anomaly Cancellation in the Standard Model

		SU(3)	SU(2)	SU(2)×U(1)	SU(3)×U(1)	$U(1)^3$	Grav.
Q	$(3, 2, \frac{1}{6})$	2	0	½	⅓	$\frac{1}{36}$	1
U^c	$(\bar{3}, 1, -\frac{2}{3})$	-1	0	0	-⅔	$-\frac{8}{9}$	-2
D^c	$(\bar{3}, 1, \frac{1}{3})$	-1	0	0	⅓	$\frac{1}{9}$	1
L	$(1, 2, -\frac{1}{2})$	0	0	-½	0	$-\frac{1}{4}$	-1
E^c	$(1, 1, 1)$	0	0	0	0	1	1
		✓	✓	✓	✓	✓	✓

Generalizations

(many authors, 1970 - 1984)

- Adler-Bardeen theorem: Only one loop diagrams are relevant.
- In four dimensions also box and pentagon diagrams may contribute. Complicates structure of anomalies.
- In 2N dimensions the leading contribution is an $(N+1)$-gon.
- Diagrams with external gravitons can be anomalous. Purely gravitational anomalies in $4N+2$ dimensions. Mixed gauge-gravitational in all dimensions
- Anomalies arise not only from fermion loops, but also from loops of (anti)-selfdual anti-symmetric tensors (in $4N+2$ dimensions)
- All of this is summarized beautifully by relating anomalies to the Atiyah-Singer index theorem.

Anomalies from the index theorem

$$\mathrm{Index}(\gamma^a D_a) = \int_M \left[\hat{A}(R)\mathrm{Ch}(F)\right]_{\mathrm{Vol}}$$

$\hat{A}(R)$ Dirac Genus

$\mathrm{Ch}(F)$ Chern Character

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

Field strength two-form

$F \equiv F^a T^a$

$$R^\alpha{}_\beta \equiv \frac{1}{2} R^\alpha{}_{\beta\delta\gamma} dx^\delta \wedge dx^\gamma$$

Curvature two-form

($SO(N)$-valued)

$$\mathrm{Ch(F) = Tr\ e^{iF/2\pi}}$$

$$\hat{A}(R) = \prod_a \frac{x_a/2}{\sinh(x_a/2)}$$

x_a : Skew eigenvalues of R

$$\begin{aligned}\hat{A}(R) = 1 &+ \frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{Tr}R^2 \\ &+ \frac{1}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{Tr}R^2)^2 + \frac{1}{360}\mathrm{Tr}R^4\right] \\ &+ \frac{1}{(4\pi)^6}\left[\frac{1}{10368}(\mathrm{Tr}R^2)^3 + \frac{1}{4320}\mathrm{Tr}R^2\mathrm{Tr}R^4 + \frac{1}{5670}\mathrm{Tr}R^6\right] \\ &\ldots\end{aligned}$$

Anomalies from the index theorem

To get the contribution to the anomaly of a Weyl fermion in *2N* dimensions, take the *2N+2* volume-form in the expansion of $\hat{A}(R)\mathrm{Ch}(F)$. (Order *N+1* in *F* and *R*)

The apply the "method of descent" to the resulting polynomial in *F* and *R*. This gives the precise expression for the right-hand side of $D_\mu J^\mu$.

To check anomaly cancellation the precise form of the anomaly is not needed. It is sufficient to check that the polynomial vanishes.

Anomalies due to other fields

- Spin 3/2

$$\mathrm{Index}(D_{3/2}) = \int_M \left[\hat{A}(R)\mathrm{Ch}(F)\{\mathrm{Ch}(R) - 1\}\right]_{\mathrm{Vol}}$$

- Anti-symmetric tensor (rank *N-1*, *N* odd)

$$\mathrm{Index}(D_A) = \frac{1}{4}\int_M [L(R)]_{\mathrm{Vol}}$$

$$L(R) = 2^N \prod_a \frac{x_a/2}{\tanh(x_a/2)}$$

(Hirzebruch signature)

Anomalies due to other fields

- Spin 3/2

$$\text{Index}(D_{3/2}) = \int_M \left[\hat{A}(R)\text{Ch}(F)\boxed{\{\text{Ch}(R) - 1\}}\right]_{\text{Vol}}$$

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Cancellation of Gravitational Anomalies

Alvarez-Gaumé and Witten (1983)

Ten-dimensional field theory with Majorana-Weyl spinors, gravitino's and (anti)self-dual anti-symmetric tensors

$$\hat{I}_{1/2} = \tfrac{1}{967\,680}(-31p_1^3 + 44p_1 p_2 - 16p_3)\,,$$

$$\hat{I}_{3/2} = \tfrac{1}{967\,680}(225p_1^3 - 1620p_1 p_2 + 7920p_3)\,,$$

$$\hat{I}_{A} = \tfrac{1}{967\,680}(-256p_1^3 + 1664p_1 p_2 - 7936p_3)\,.$$

$$\hat{I}_{1/2} - \hat{I}_{3/2} - \hat{I}_A = 0$$

Green-Schwarz anomaly cancellation (1984)

Chiral gravitino, anti-chiral Weyl spinor plus a chiral gaugino.

$$\text{Anomaly} = I_{3/2}(R) - I_{1/2}(R) + I_{1/2}(R, F)$$

$$I_{3/2}(R) = -\frac{11}{8064}\text{Tr}R^6 + \dots$$

$$I_{1/2}(R) = \frac{1}{362880}\text{Tr}R^6 + \dots$$

$$I_{1/2}(R, F) = \dim(\mathcal{G})\frac{1}{362880}\text{Tr}R^6 + \dots$$

If there are precisely *496* gauge bosons, the leading trace cancels.

Green-Schwarz anomaly cancellation (1984)

Chiral gravitino, anti-chiral Weyl spinor plus a chiral gaugino.

$$
\begin{aligned}
\text{Anomaly} &= I_{3/2}(R) - I_{1/2}(R) + I_{1/2}(R,F) \\
&\propto -\frac{1}{15}\mathrm{Tr}F^6 + \frac{1}{24}\mathrm{Tr}R^2\mathrm{Tr}F^4 + \frac{1}{8}\mathrm{Tr}R^2\mathrm{Tr}R^4 \\
&\quad + \frac{1}{32}(\mathrm{Tr}R^2)^3 - \frac{1}{240}\mathrm{Tr}F^2\mathrm{Tr}R^4 - \frac{1}{192}\mathrm{Tr}F^2(\mathrm{Tr}R^2)^2
\end{aligned}
$$

Green-Schwarz anomaly cancellation (1984)

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If a group can be found that satisfies:

$$\text{Tr}F^6 = \frac{1}{48}\text{Tr}F^2\text{Tr}F^4 - \frac{1}{14400}(\text{Tr}F^2)^3$$

Then:

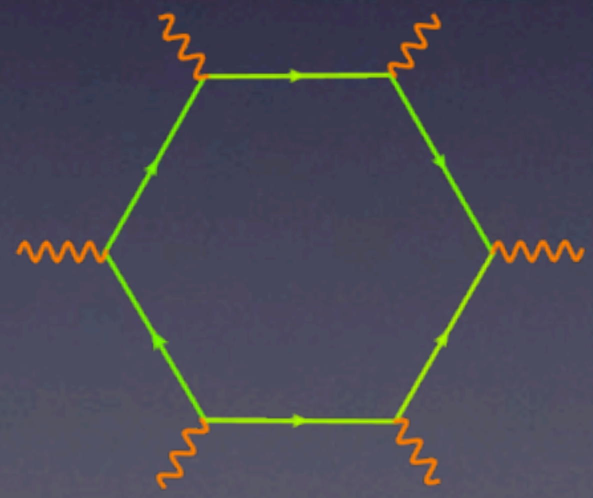
$$\text{Anomaly} \propto \left(\text{Tr}R^2 - \frac{1}{30}\text{Tr}R^2\right) \times X_8(R,F)$$

There are two (non-abelian) solutions two these conditions:

SO(32) (Green and Schwarz, 1984)
$E_8 \times E_8$ (Thierry-Mieg, 1984)

With fermions in the adjoint representation

The anomalies still don't cancel, but now they can be cancelled by adding extra terms to the action.



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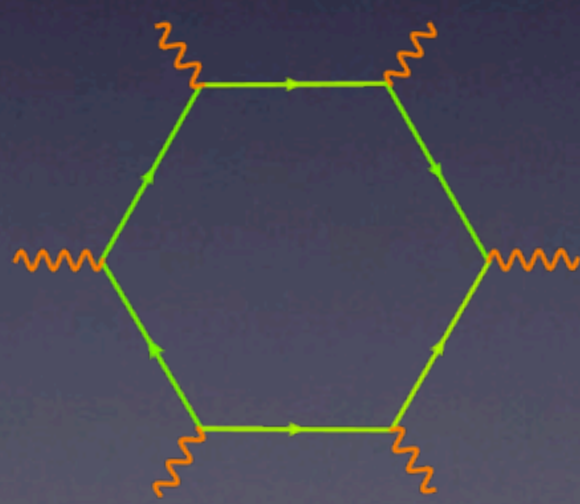

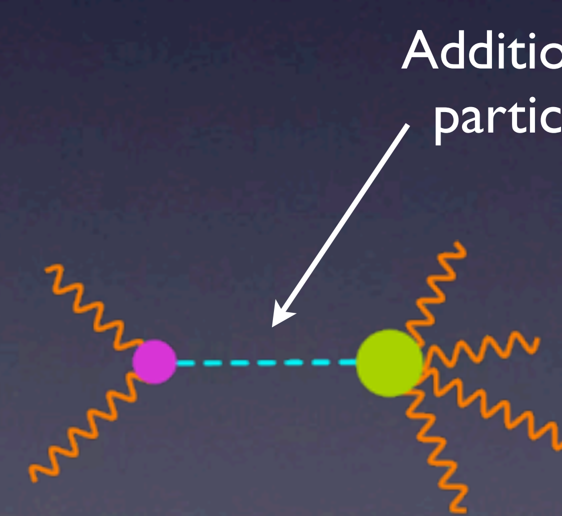

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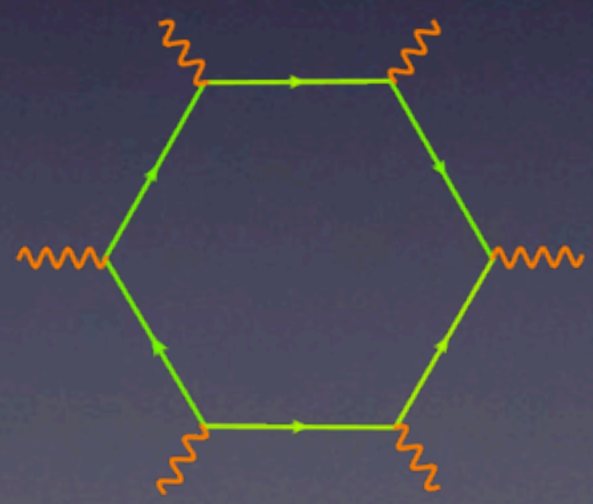
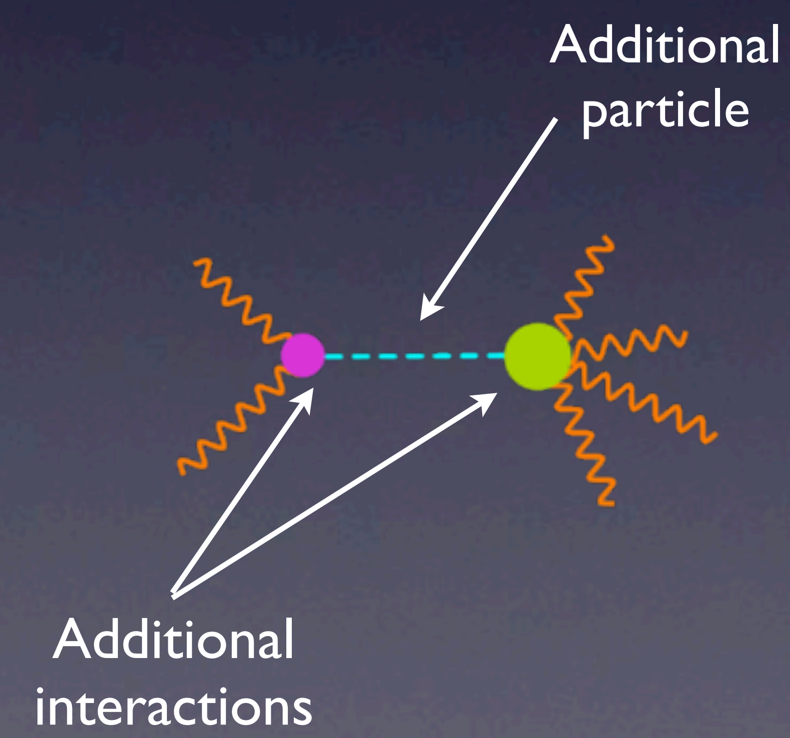
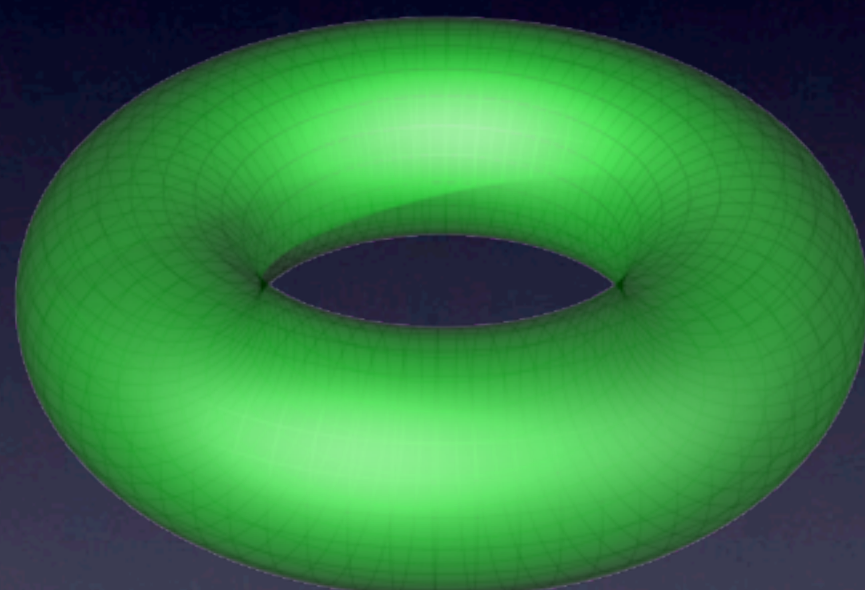




Why do these miracles occur?

All these field theories originated from string theories.
All can be described as field theory limits of closed string theories(*).

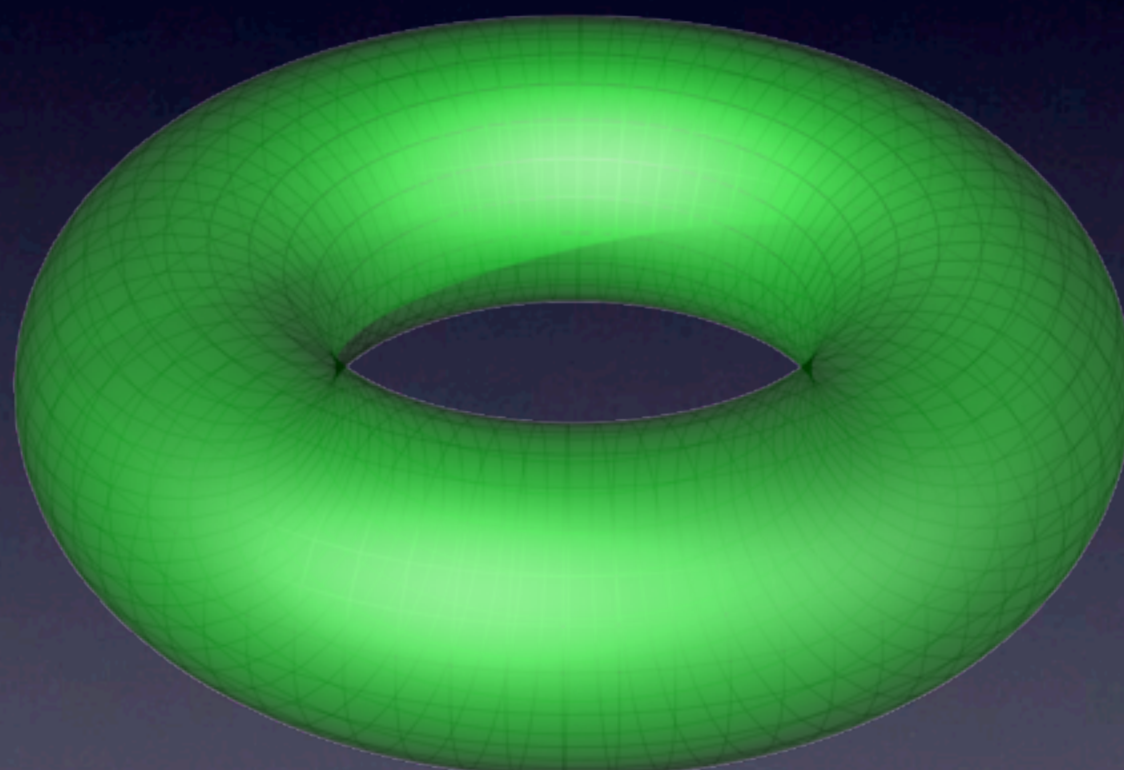


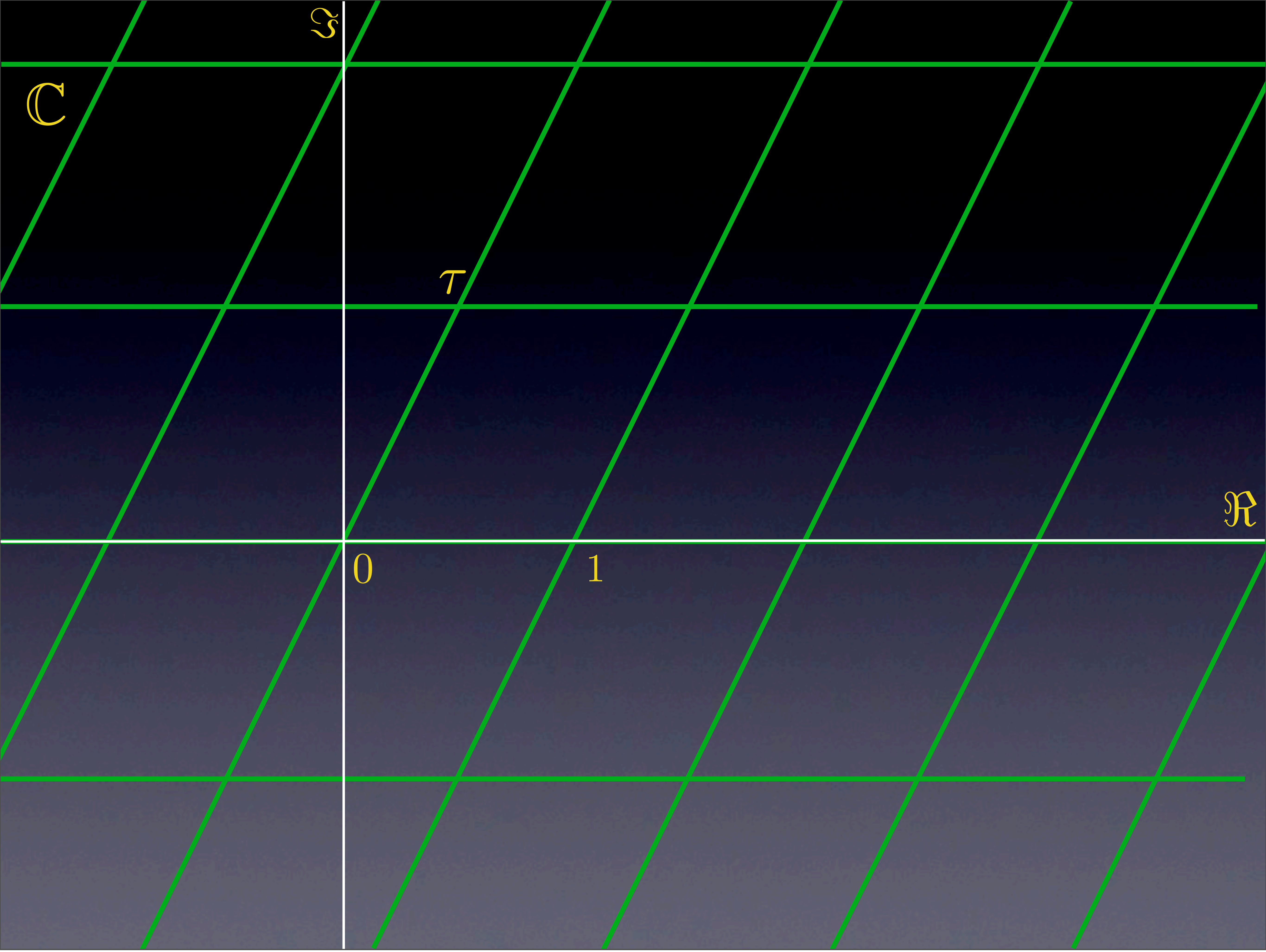
Loop graphs of closed string theories have a remarkable property:
Modular Invariance.

This singles out the gauge groups $SO(32)$ and $E_8 \times E_8$ in 10 dimensions.

(*) Heterotic Strings. Gross, Harvey, Martinec, Rohm (1984)

Modular Invariance





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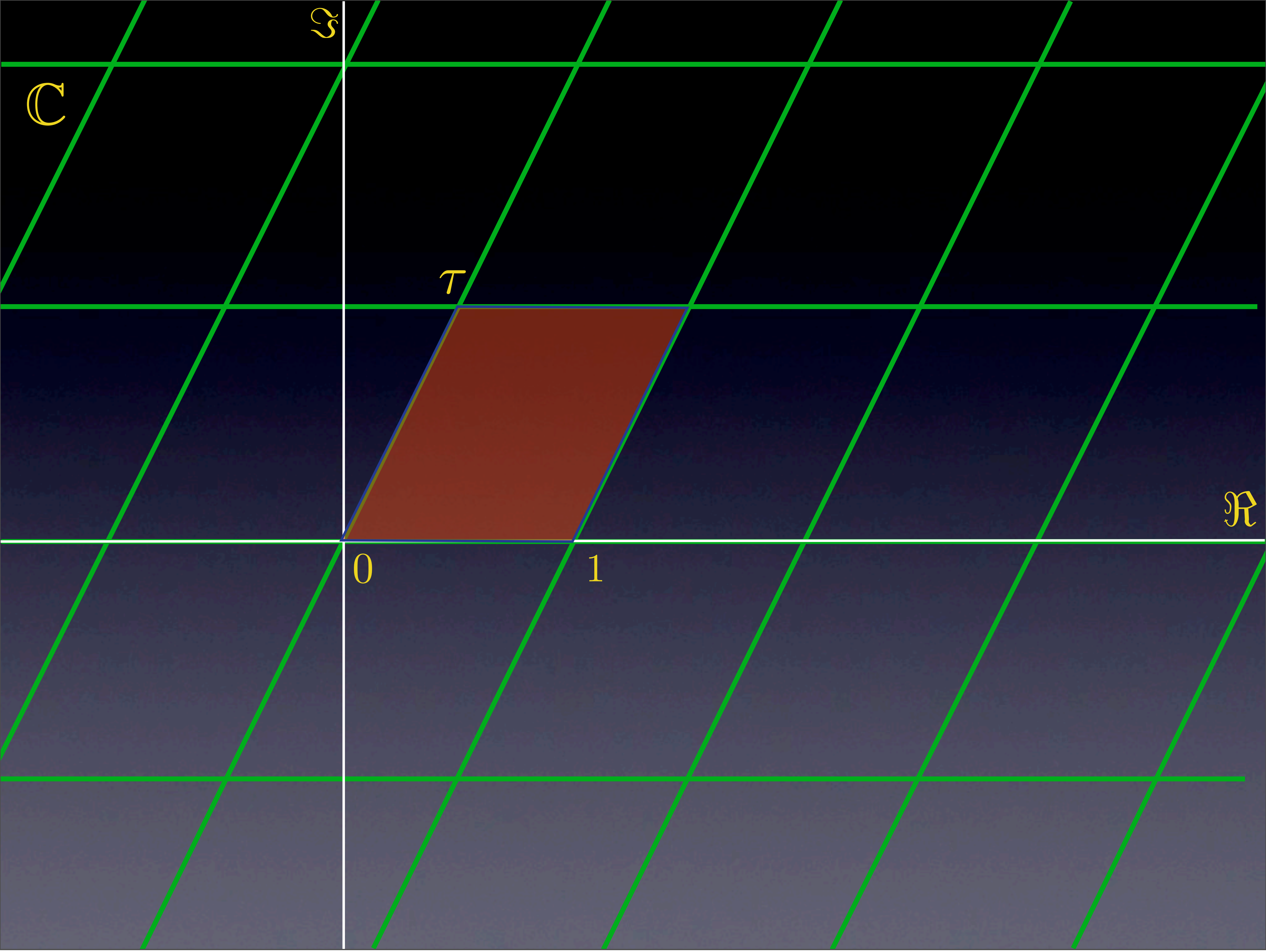

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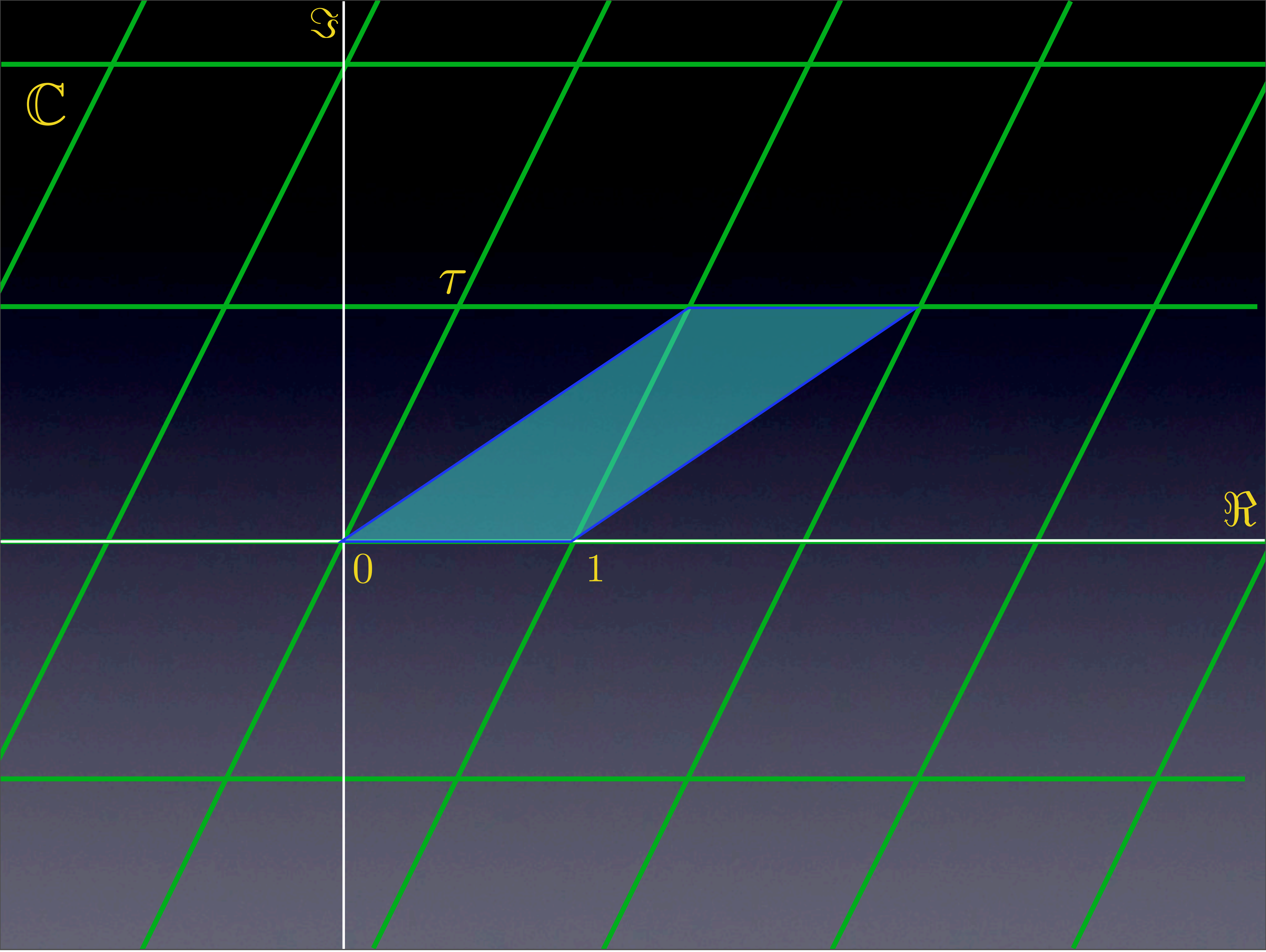
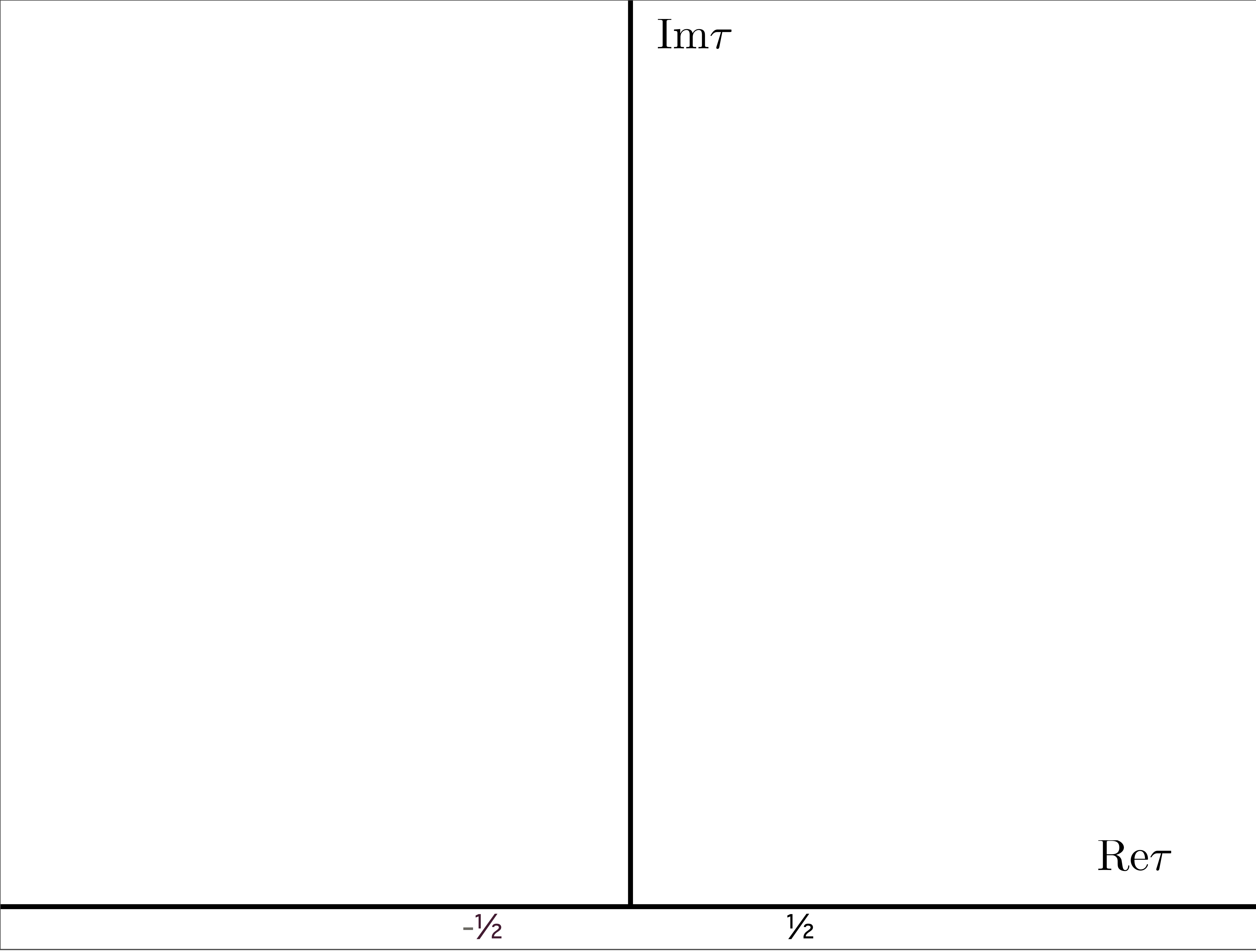
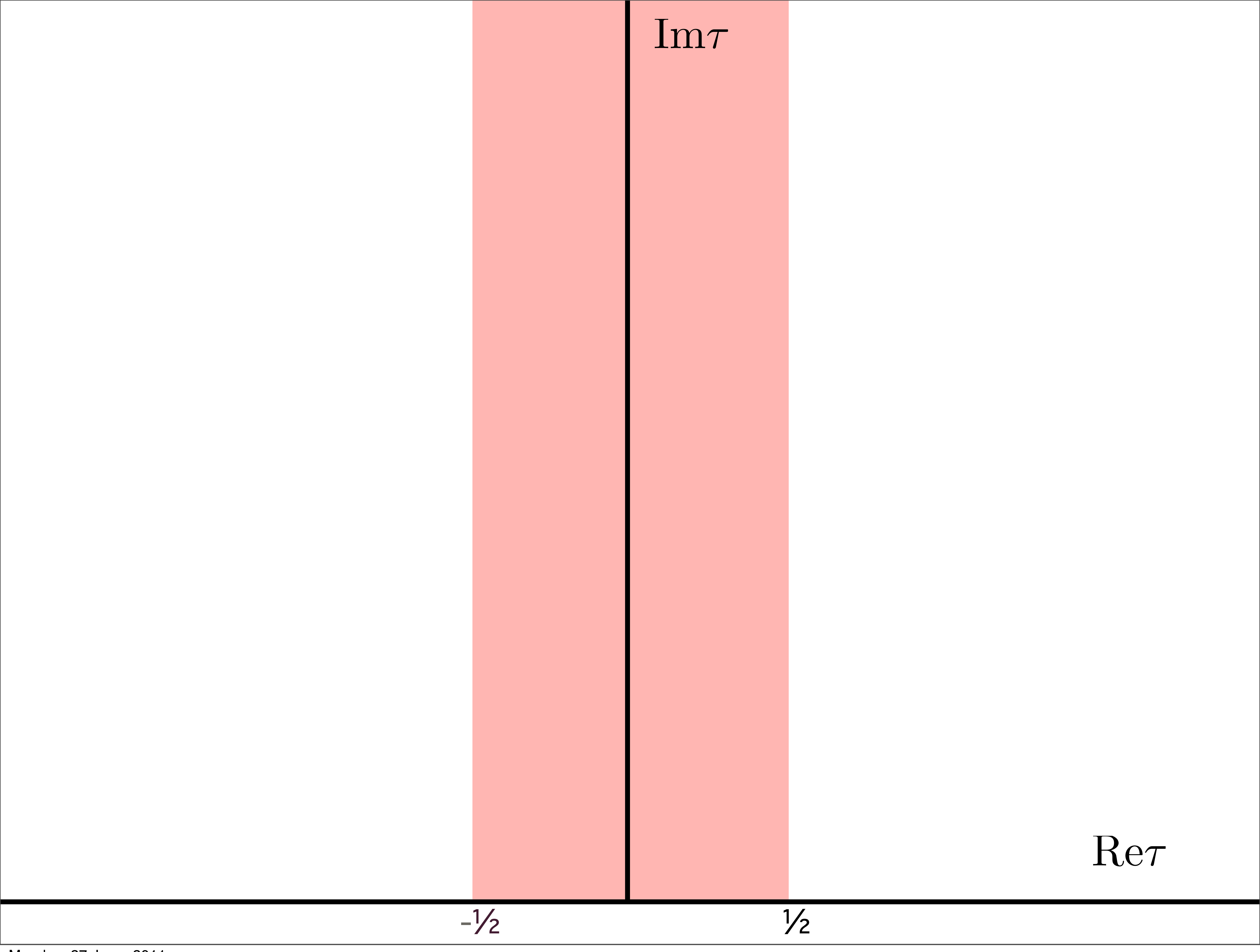

ℑ
ℂ
τ
ℜ
0
1


Imτ
Reτ
−½
½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Reτ
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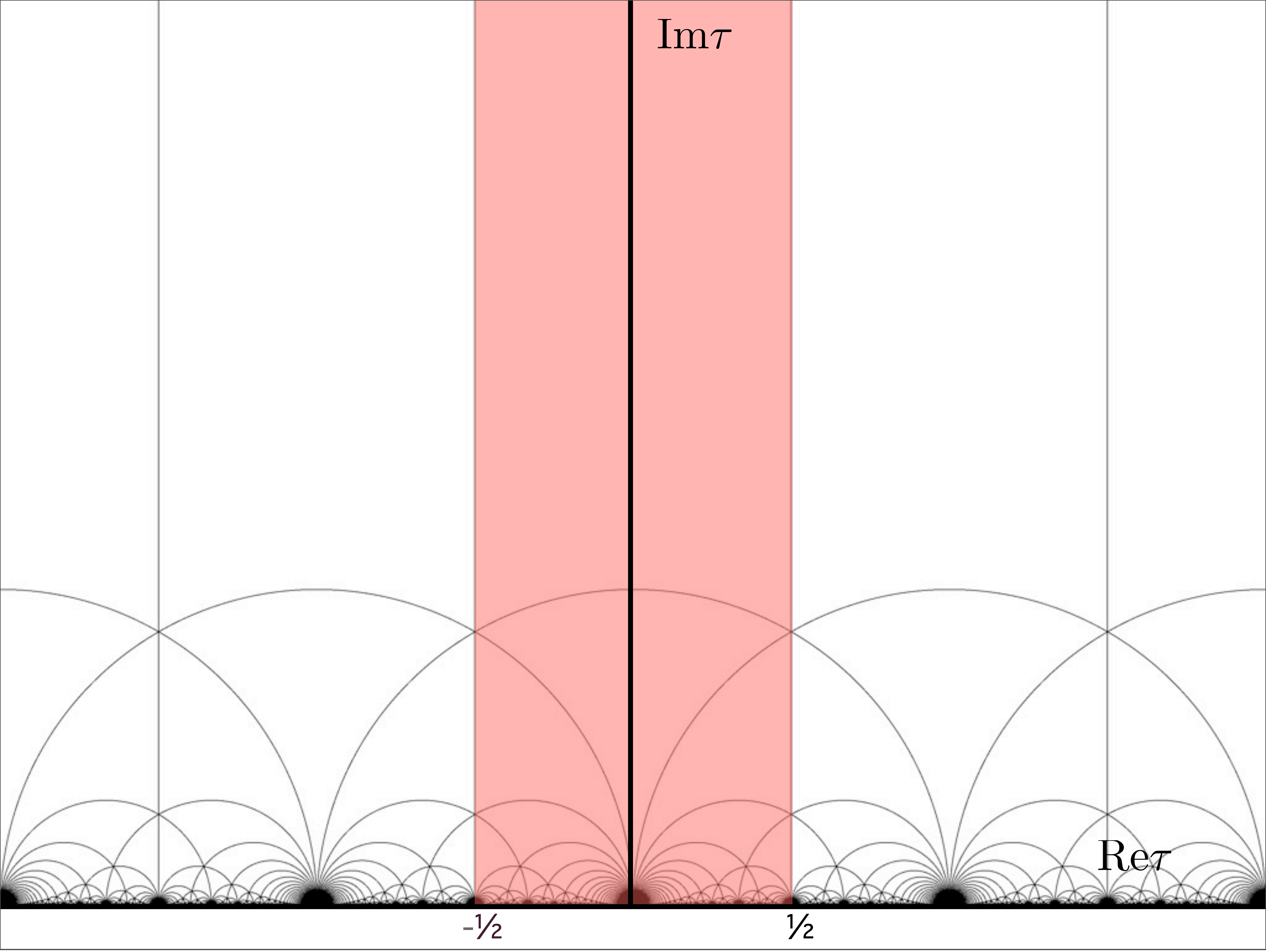
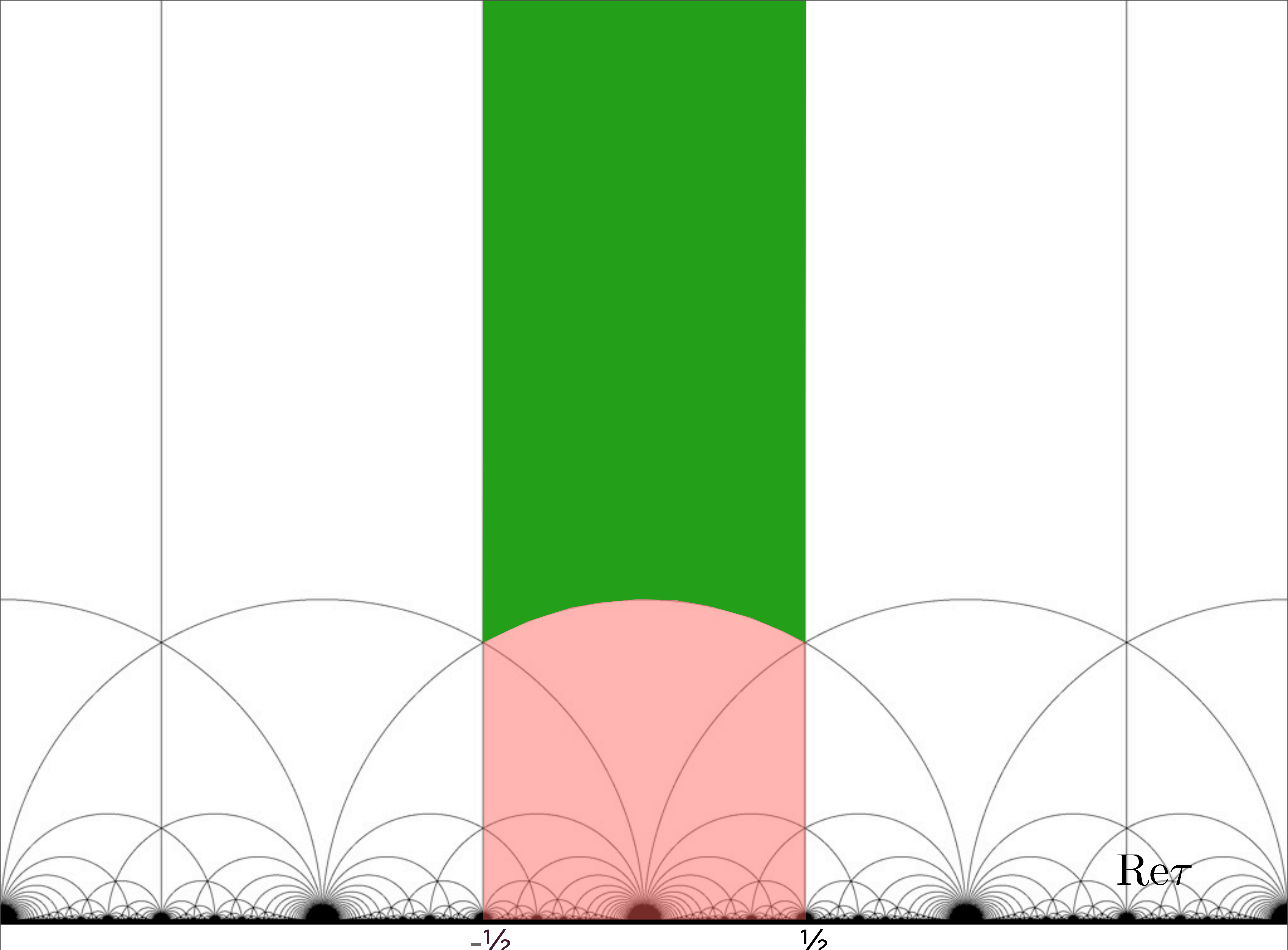
½



Imτ
Reτ
-½
½


Reτ
−½
½

Modular Invariance

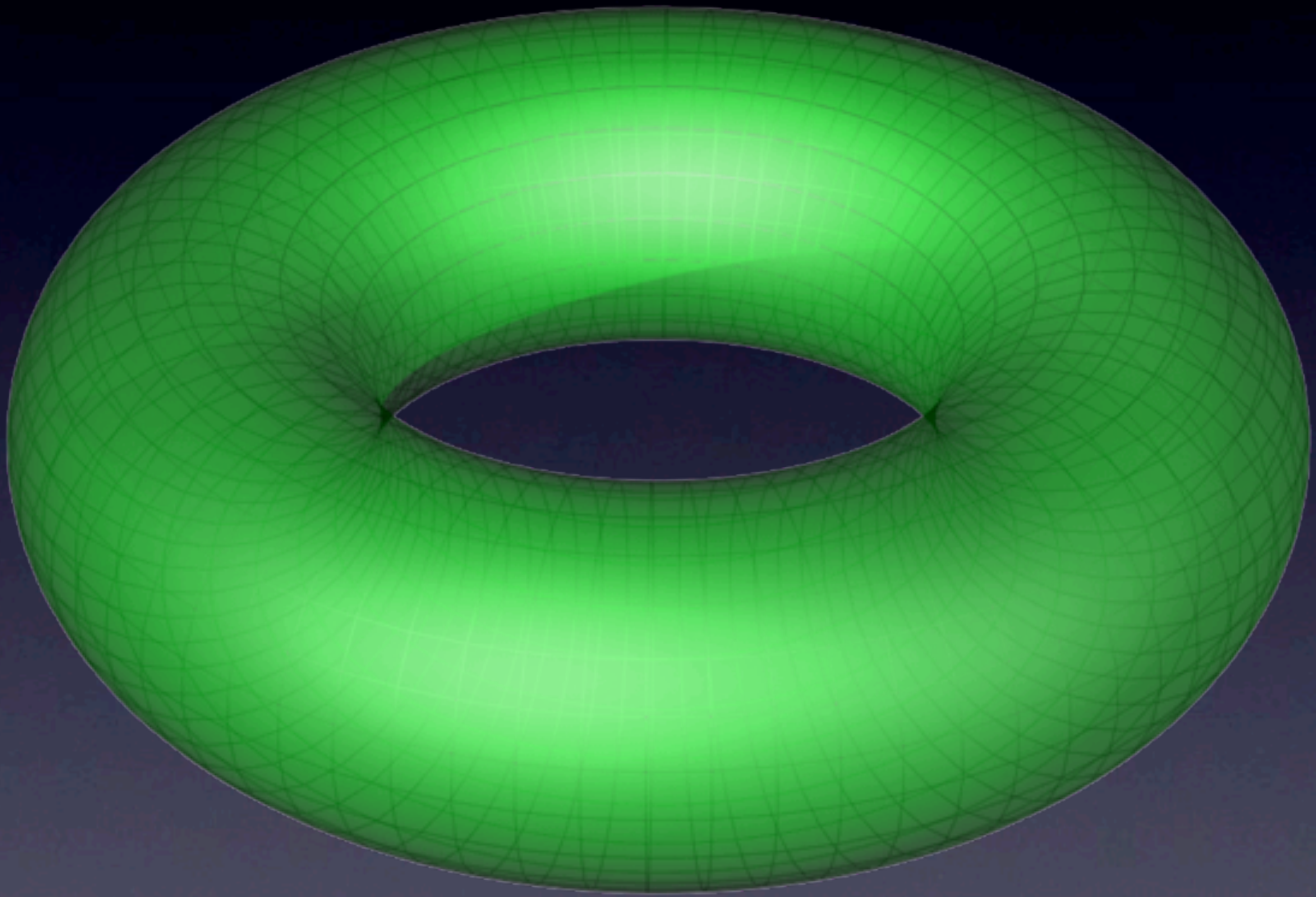
$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \mathrm{Tr}\; e^{-\mathrm{Im}\tau H}$$

Integrand must be invariant under $SL_2(\mathbb{Z})/\mathbb{Z}_2$

$$\left.\begin{array}{l} \tau \to \tau + 1 \\ \\ \tau \to -\dfrac{1}{\tau} \end{array}\right\} \Rightarrow \quad \tau \to \frac{a\tau + b}{c\tau + d}$$

$$ad - bc = 1;\; a, b, c, d \in \mathbb{Z}$$

Strings vs. Particles



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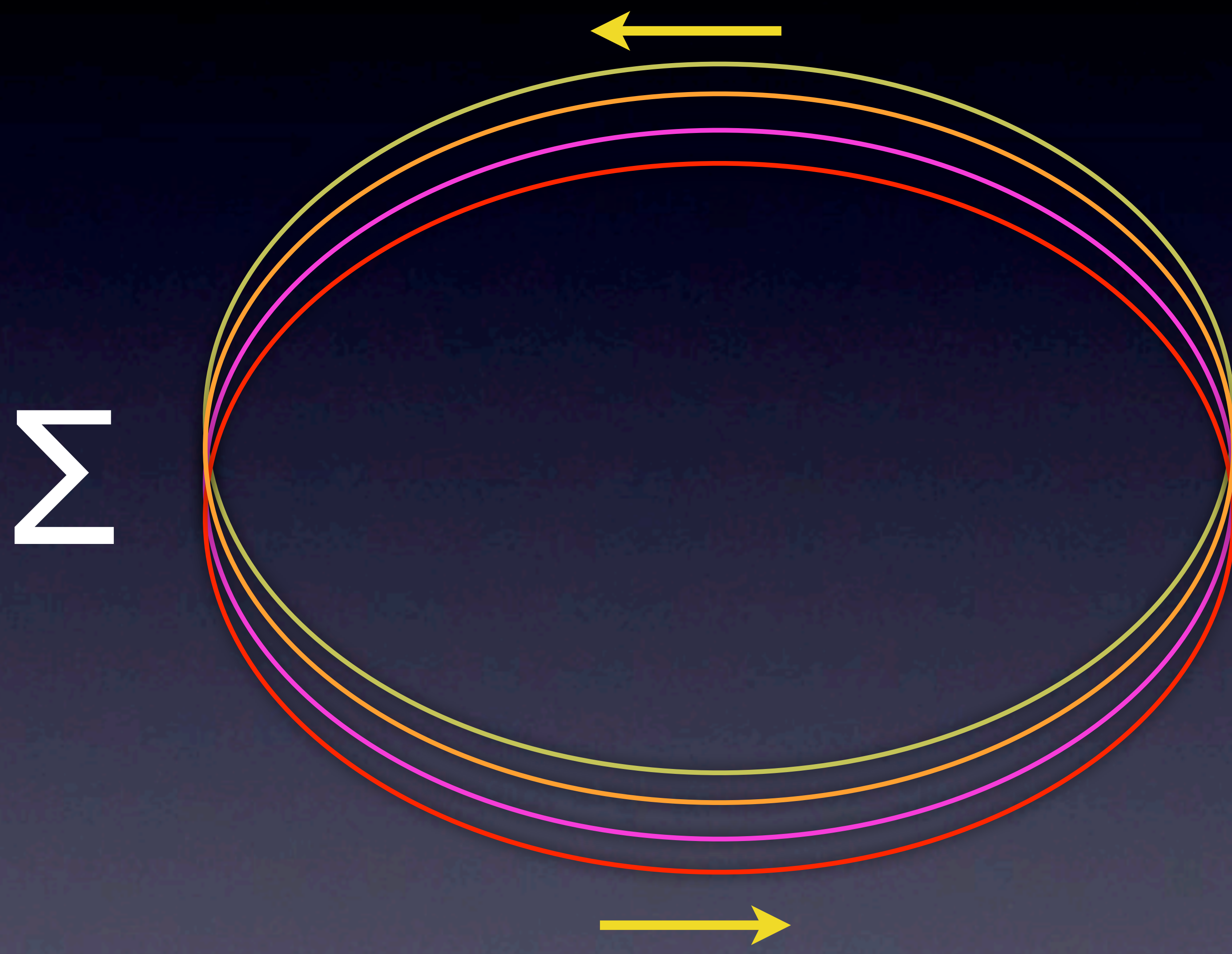

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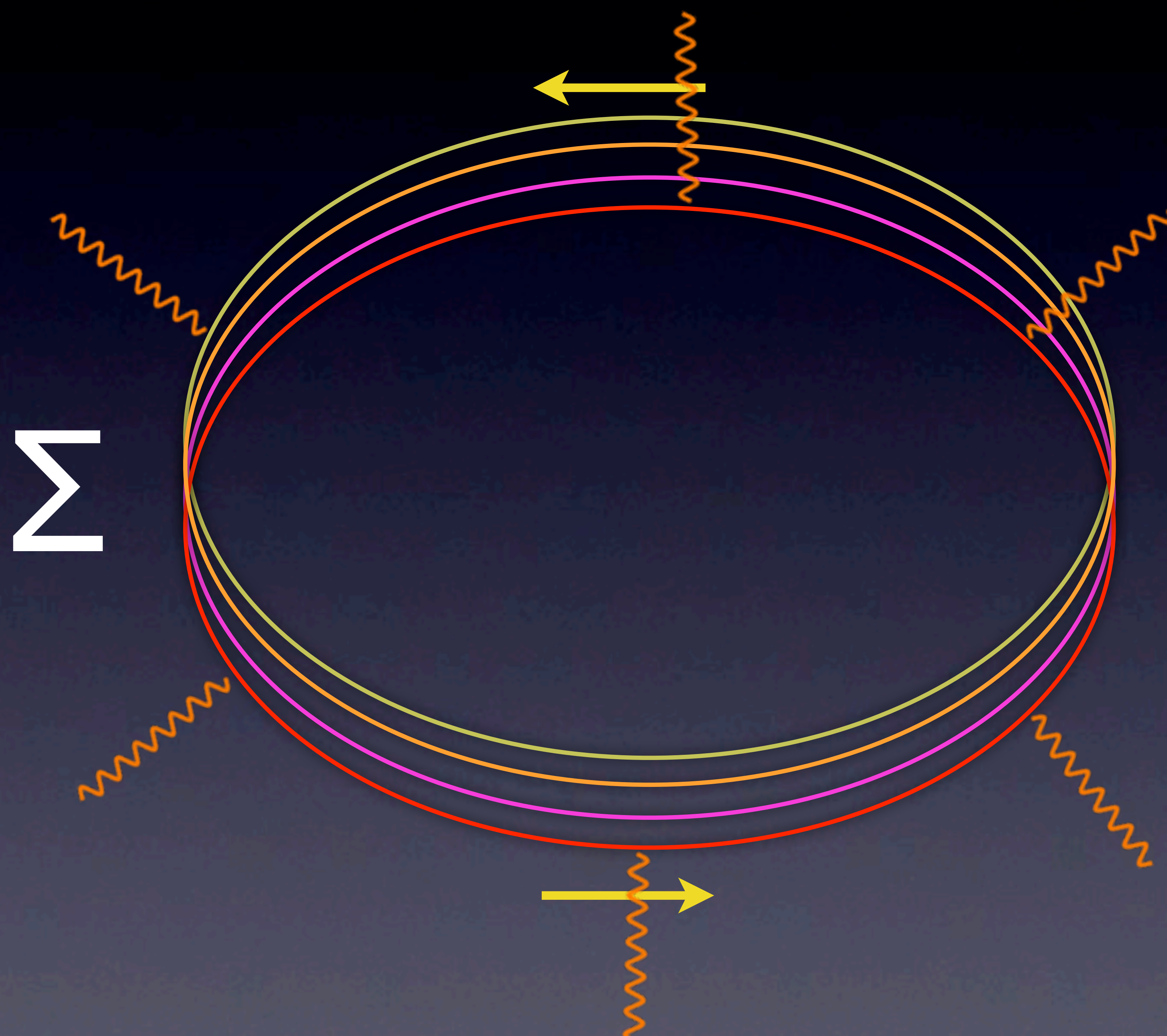


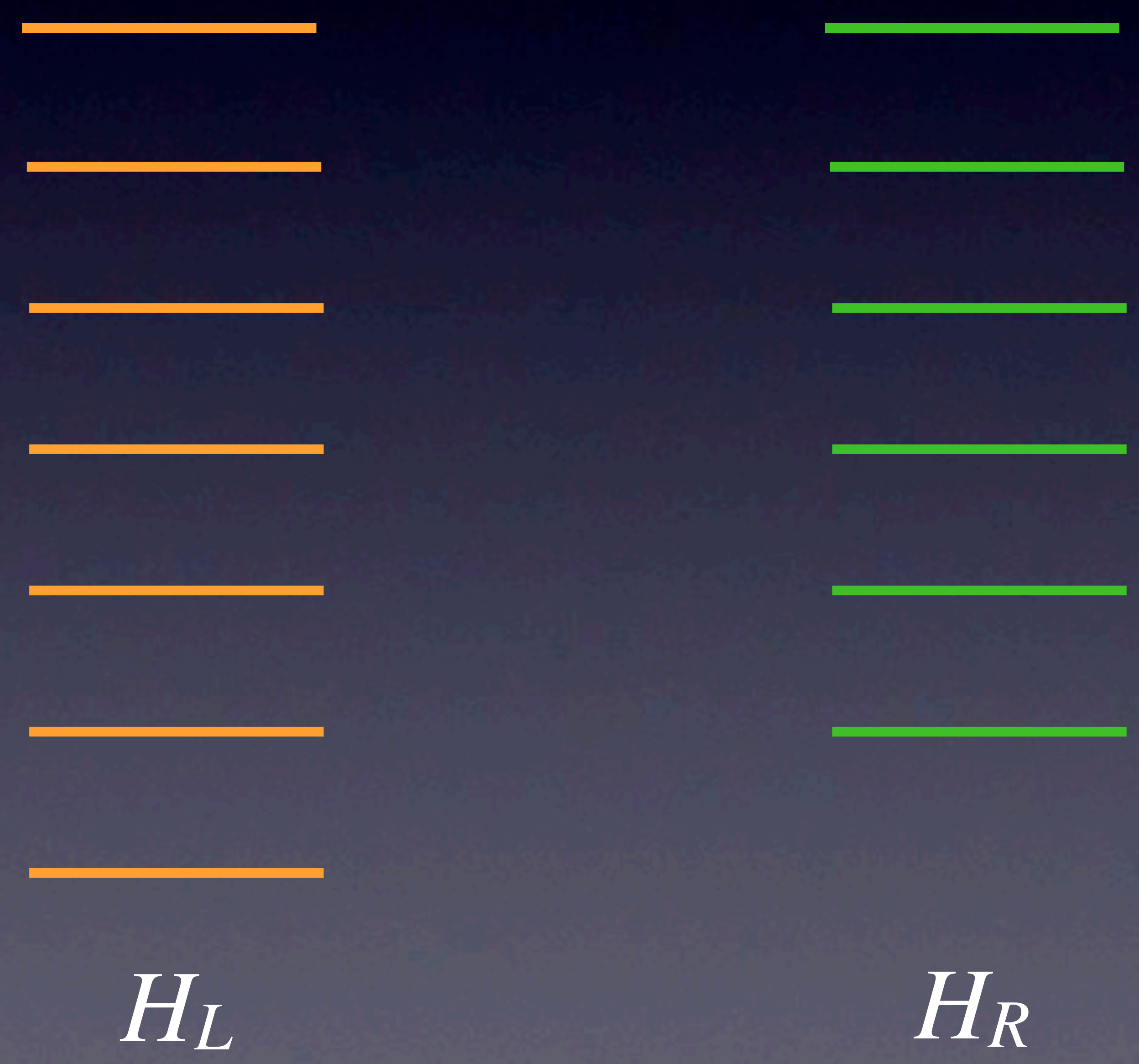
Heterotic Strings

Partition function: $P(\tau,\bar{\tau}) = \mathrm{Tr}\ e^{2\pi i\tau H_L - 2\pi i\bar{\tau} H_R}$

- H_L: Bosonic 2D-CFT giving rise to gauge groups and gauge representations.

- H_R: Fermionic 2D-CFT giving rise to Lorentz reps.

Level Matching



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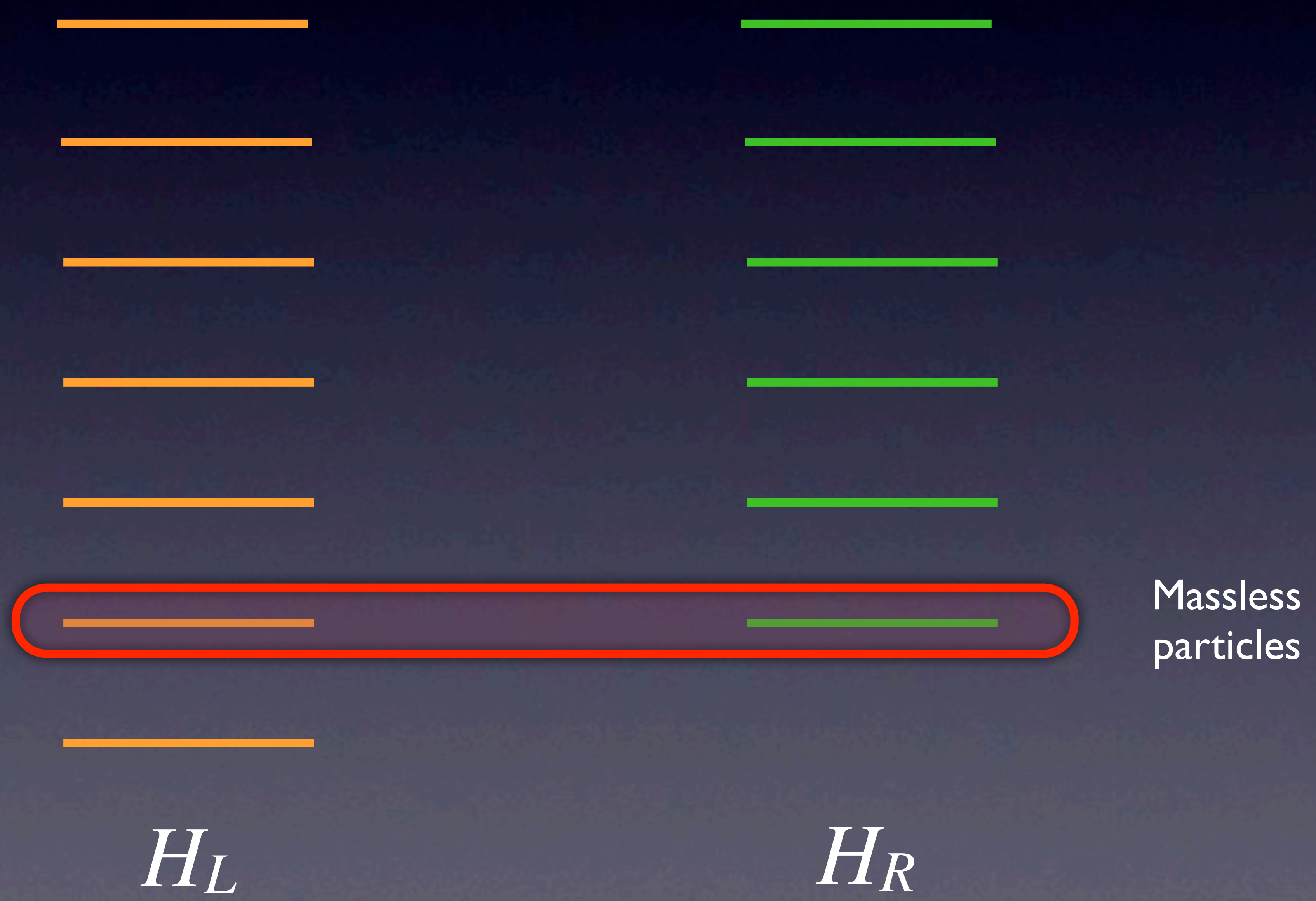

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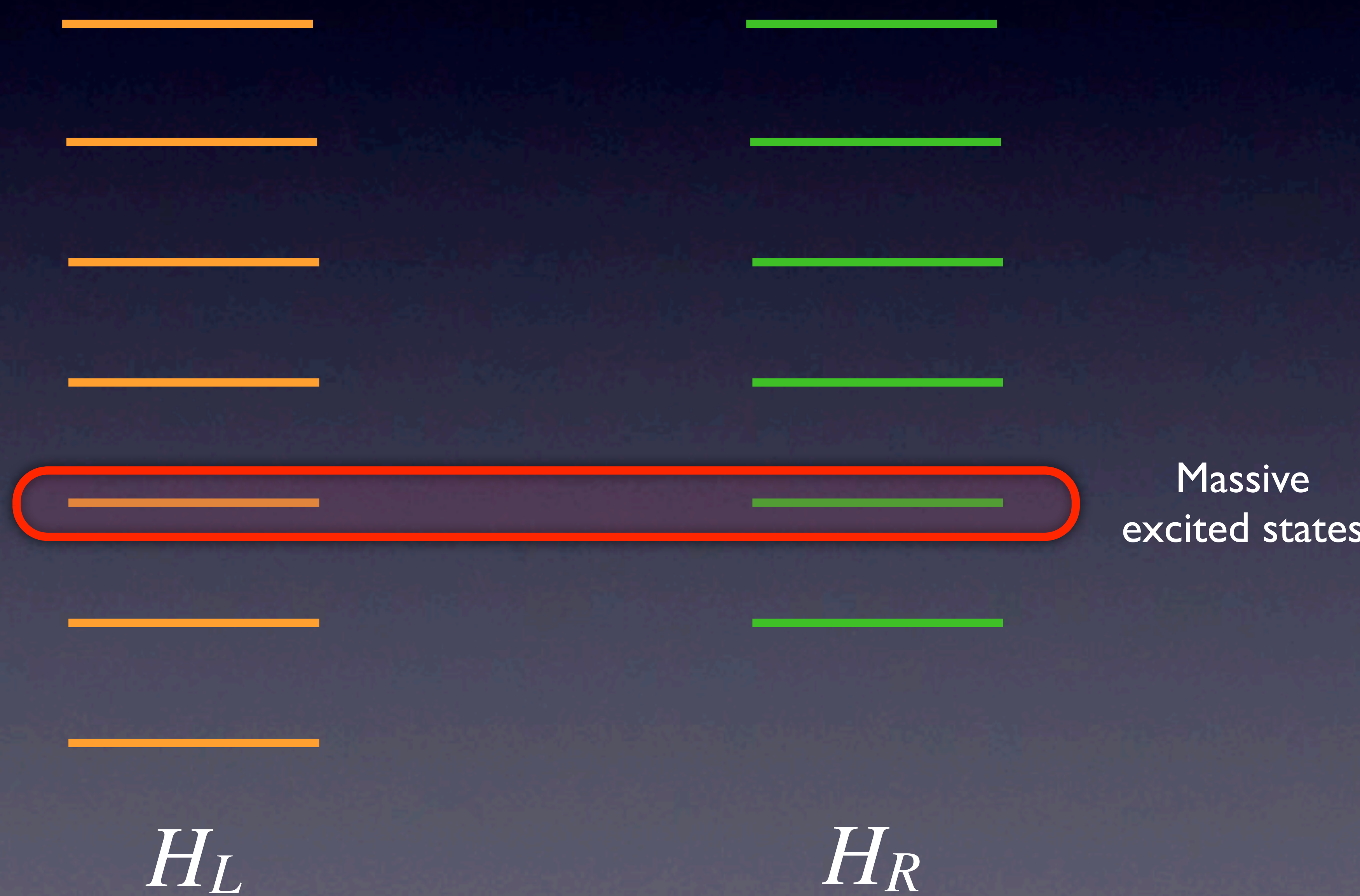

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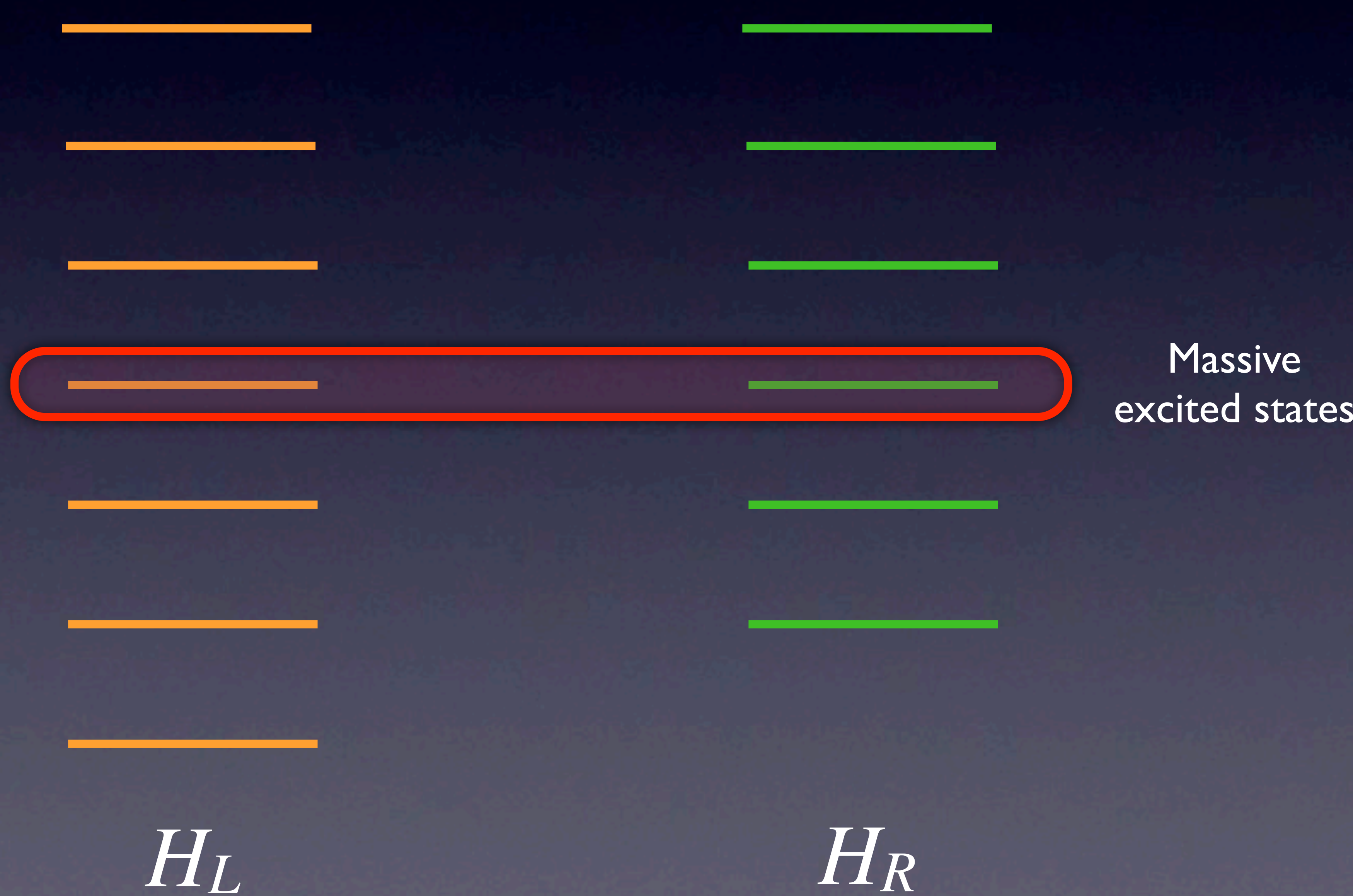

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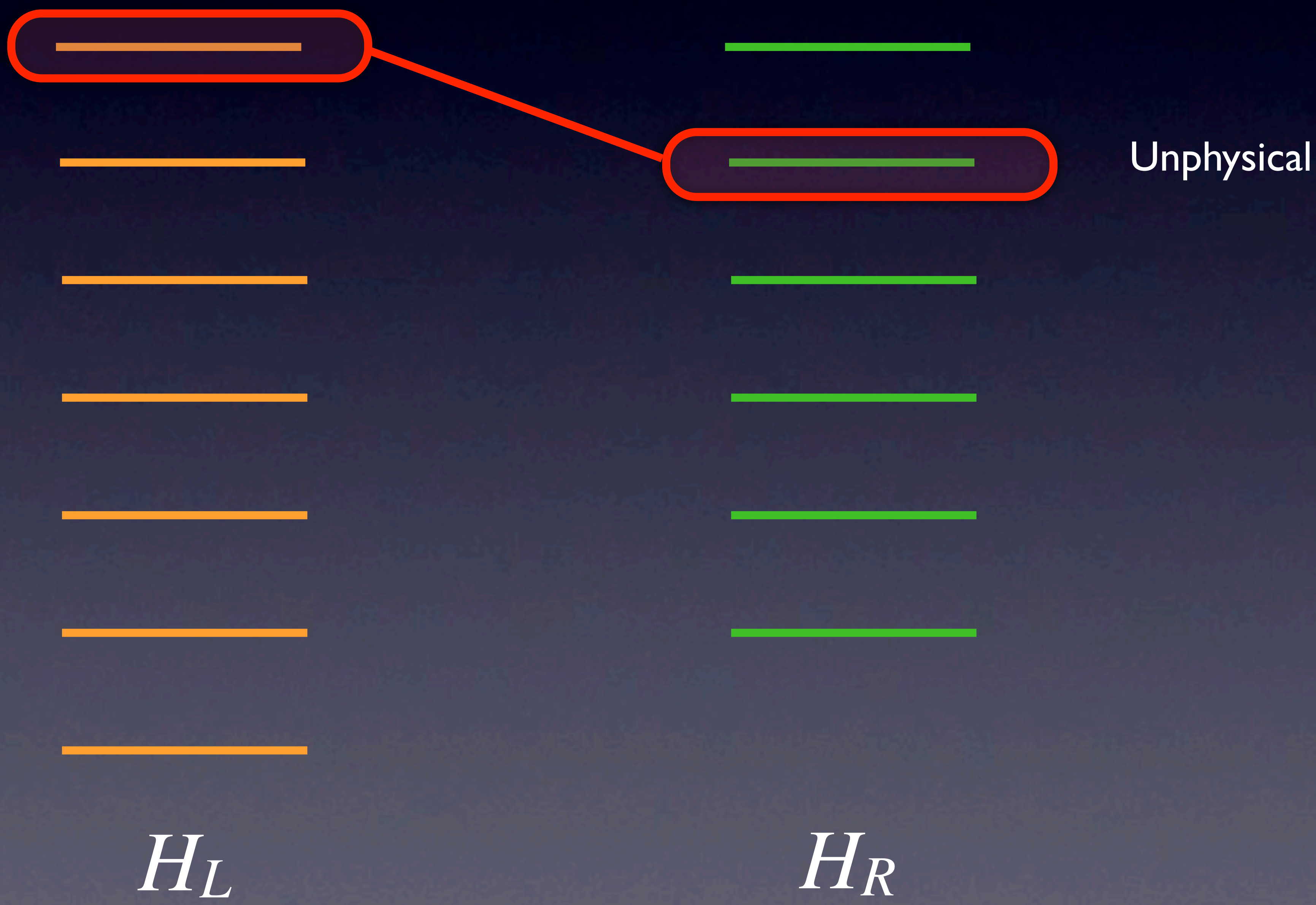

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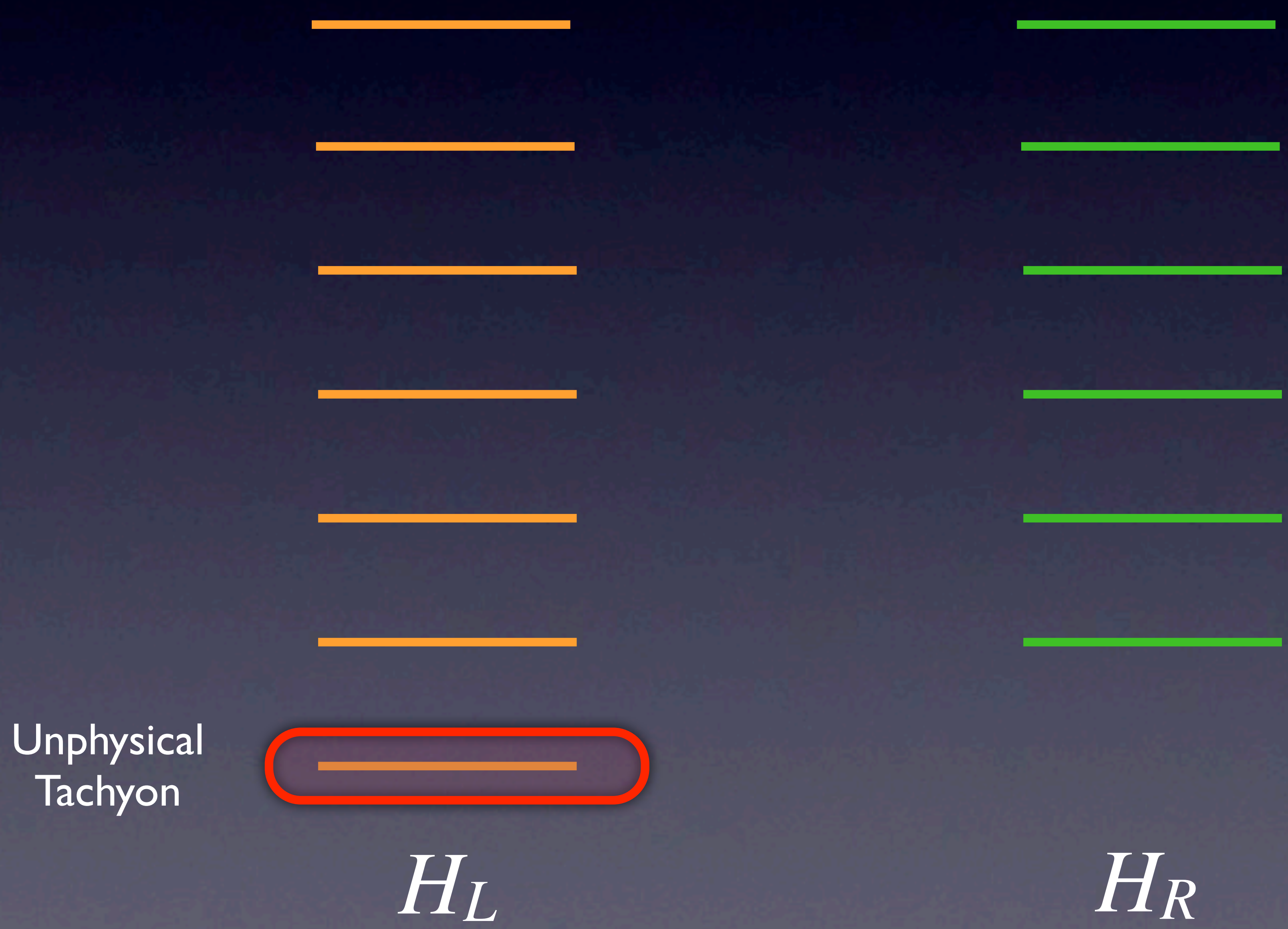
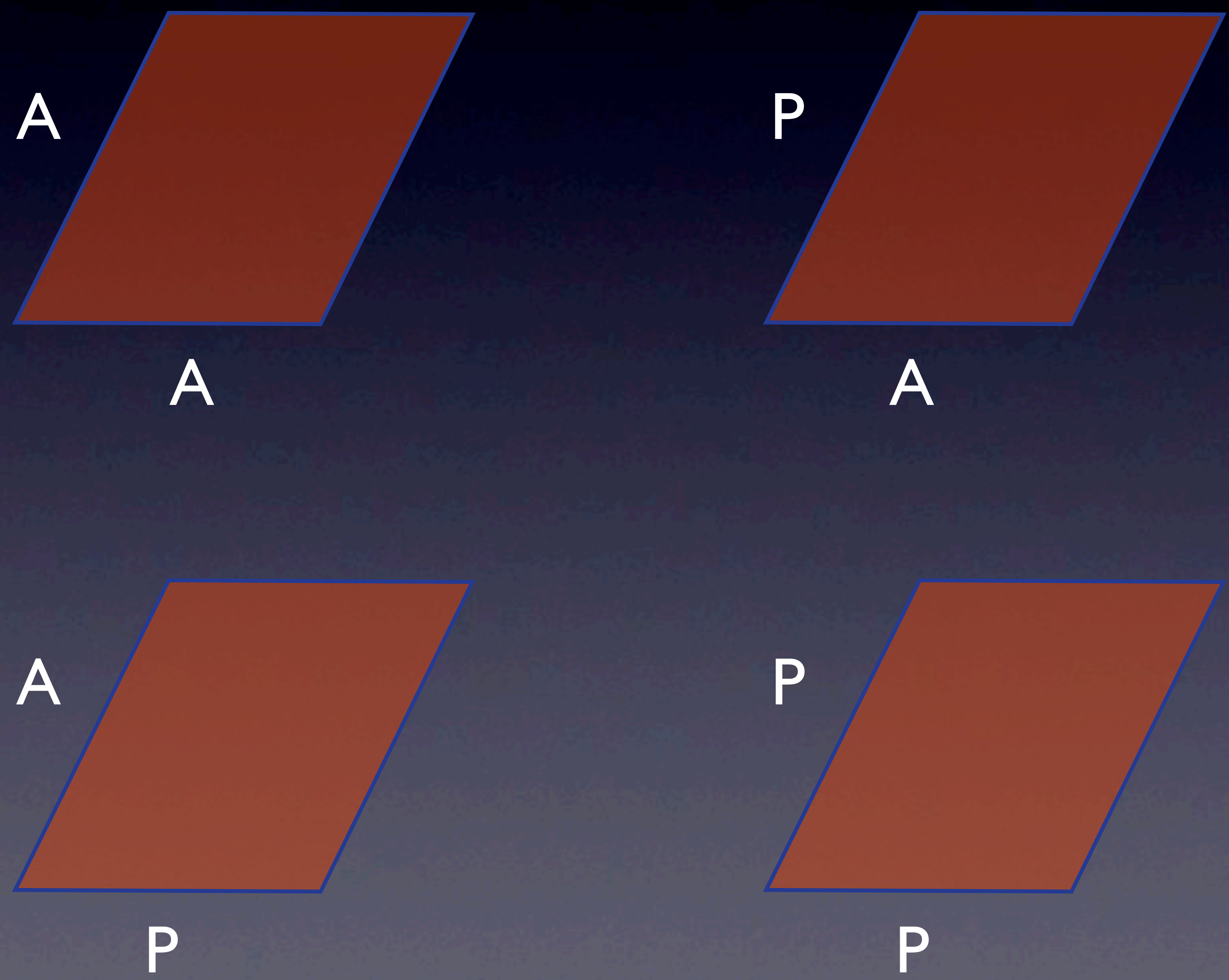
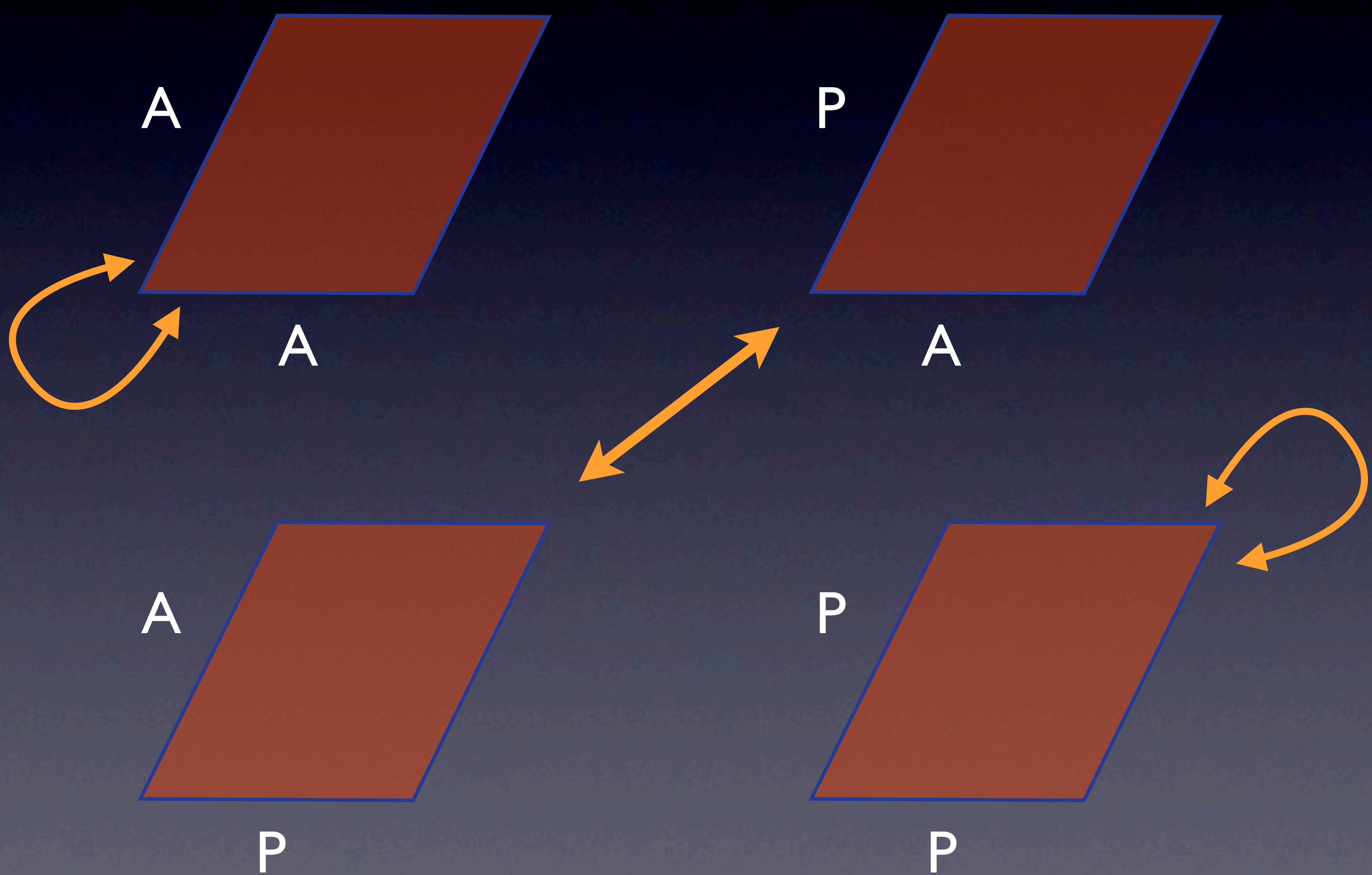


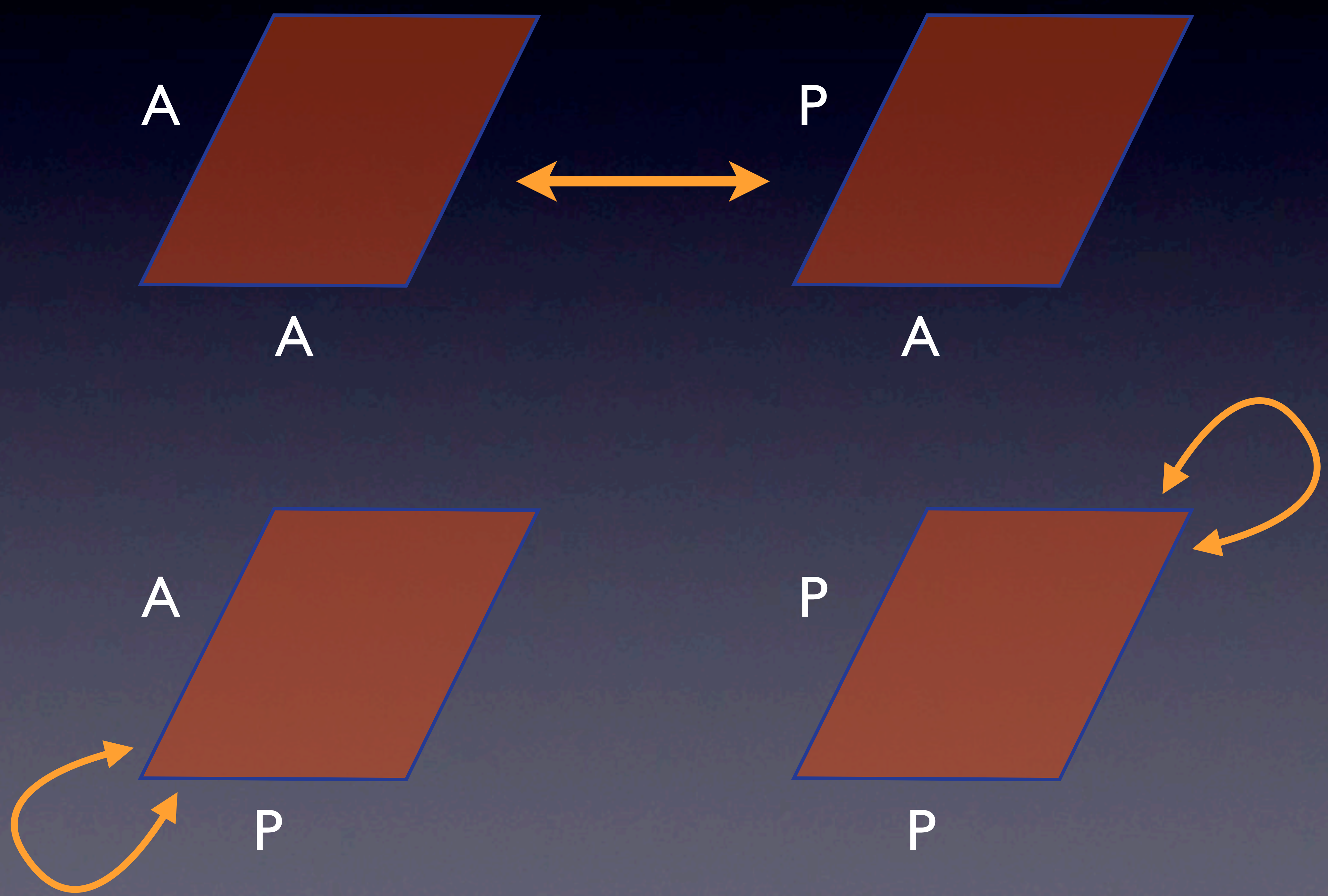
Fermion Boundary Conditions



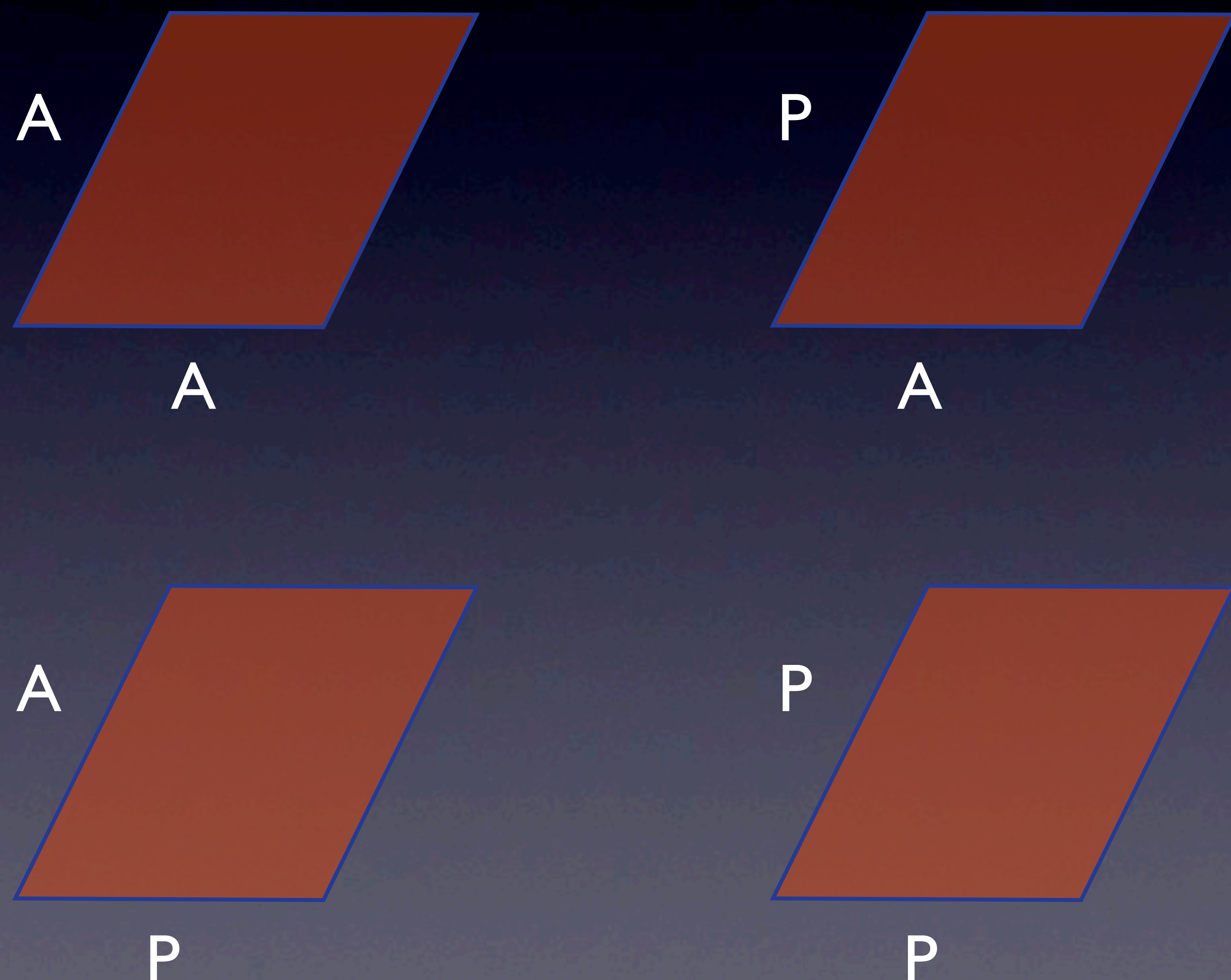
$$\tau \longrightarrow -\frac{1}{\tau}$$



$$\tau \to \tau + 1$$



Partition functions



Partition functions

$$\mathrm{Tr}\ e^{-2\pi i \bar{\tau} H_A} = \left[\frac{\theta_3(0|\bar{\tau})}{\eta(\bar{\tau})} \right]^{(D-2)/2}$$

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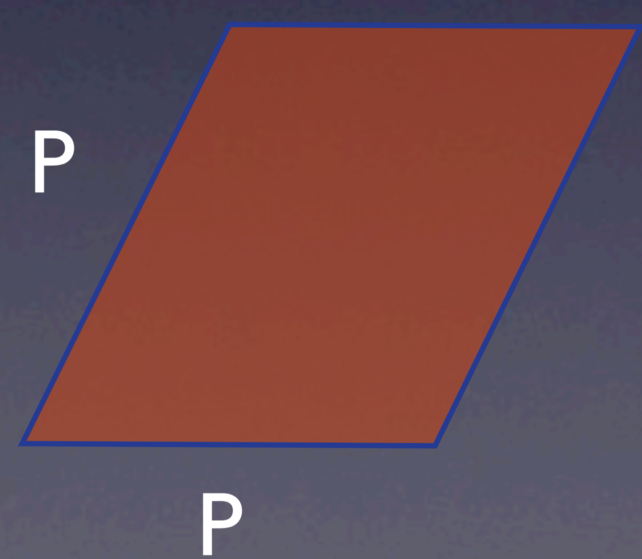

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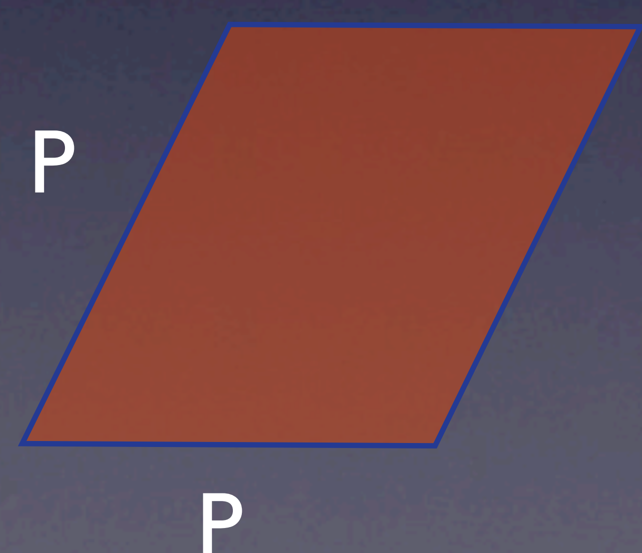
$$\text{Tr}\ e^{-2\pi i\bar{\tau}H_P} = \left[\frac{\theta_2(0|\bar{\tau})}{\eta(\bar{\tau})}\right]^{(D-2)/2}$$



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$\theta_i(0|\tau)$ Jacobi θ functions

$\eta(\tau)$ Dedekind η function

Partition functions

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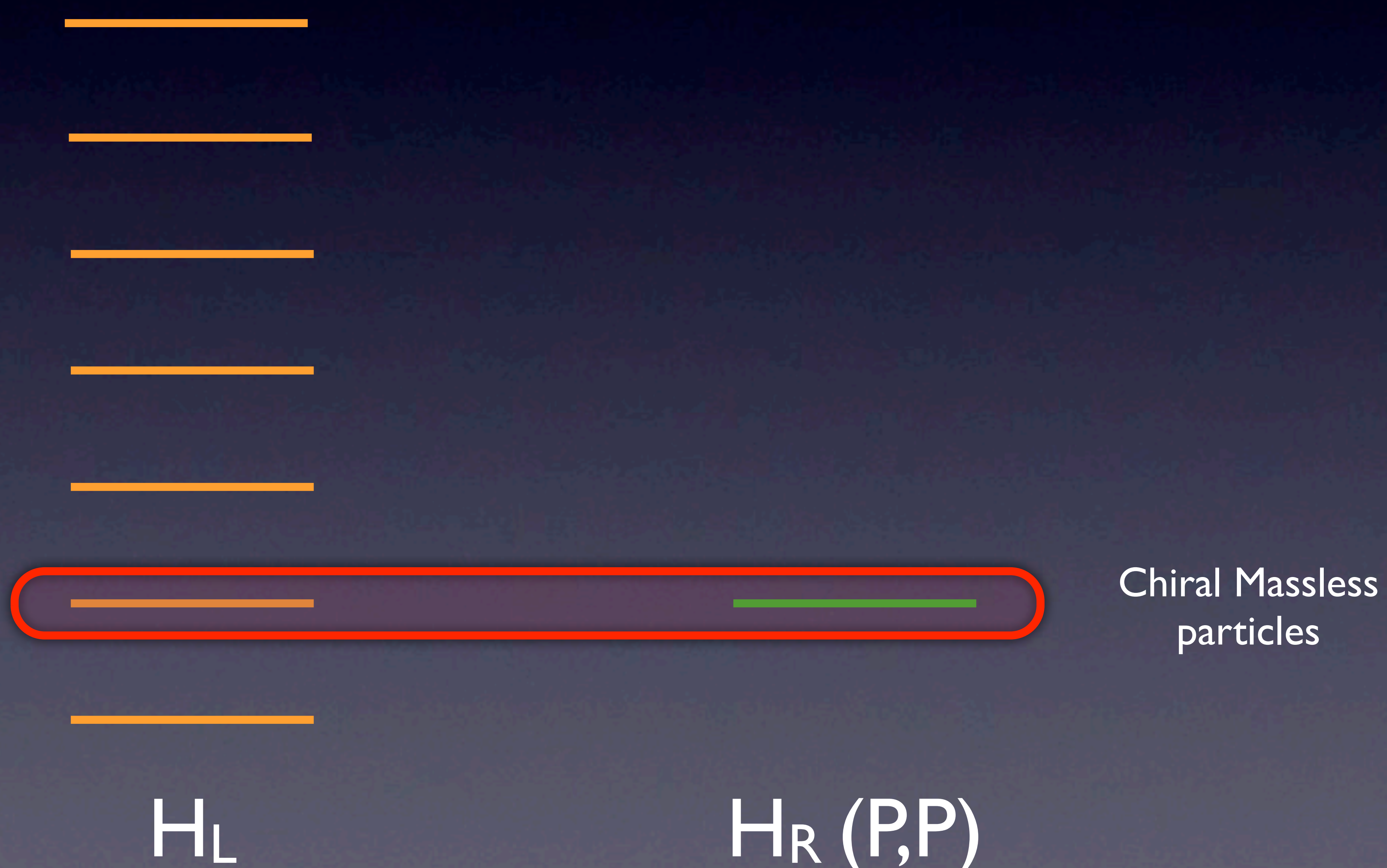
Bosons

$$\mathrm{Tr}\ e^{-2\pi i\bar{\tau}H_P} = \left[\frac{\theta_2(0|\bar{\tau})}{\eta(\bar{\tau})}\right]^{(D-2)/2} \qquad \mathrm{Tr}\ (-1)^F e^{-2\pi i\bar{\tau}H_P} = \left[\frac{\theta_1(0|\bar{\tau})}{\eta(\bar{\tau})}\right]^{(D-2)/2}$$

Non-chiral Fermions

Chiral Fermions

Chiral Sector



One-loop integral

$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^2}(\mathrm{Im}\tau)^{(D-2)/2}\sum_{i=1}^{4}\left(\frac{\theta_i(0|\bar{\tau})}{\eta(\bar{\tau})}\right)^{(D-2)/2} P_i(\tau,\bar{\tau})$$

Chiral fermion contributions

$$P_1(\tau,\bar{\tau}) = P_1(\tau) = \sum_{k=-1}^{\infty} d_k q^k$$

$$q = e^{2i\pi\tau}$$

The anomaly generating function

Note:

$$\mathrm{Ch}(F) = \mathrm{Tr}\ e^{iF/2\pi}$$

$$\mathrm{Ch}(0)_k = d_k$$

Now we can write down the anomaly generating function for the entire chiral sector

$$A(q, F, R) = \hat{A}(R) \sum_k q^k \mathrm{Ch}(F)_k \mathrm{Ch}(R)_k$$

Spin contributions from bosonic sector

Modular transformation

The anomaly generating function must be modular invariant for $F=R=0$.

On the other hand, it can be written in terms of characters of affine Lie algebras.
It is known how such characters transform for $F\neq 0$ or $R\neq 0$.
Consider, for example, the θ-functions (related to $SO(N)$ characters)

$$\theta_i\left(\frac{F}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\right)=\sum_j S_{ij}\sqrt{c\tau+d}\;\; e^{i\pi F^2 c/(c\tau+d)}\;\;\theta_j(F|\tau)$$

Phase — Modular Weight

The phases S_{ij} cancel in the final assembly, because the result is modular invariant.
The overall weight is also determined by modular invariance.

Modular transformation

$$A(\frac{a\tau+b}{c\tau+d},\frac{F}{c\tau+d},\frac{R}{c\tau+d})$$
$$=\exp\left[\frac{ic}{32\pi^3(c\tau+d)}(\mathrm{Tr}F^2-\mathrm{Tr}R^2)\right](c\tau+d)^{-(D-2)/2}A(\tau,F,R)$$

Note: F normalized as in *SO(N)* vector (not adjoint)

The anomaly

1. Expand *A(q, F, R)* to order *(D+2)/2* in *F* and *R*
2. Take only the coefficients of q^0

The result of 1. is a coefficient function $f(\tau)$ for each combination of traces of *F* and *R*.
From the transformation of *A(q, F, R)* we infer,
if we ignore the phases involving Tr F^2 - Tr R^2

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2 f(\tau)$$

This is a meromorphic modular function of weight 2

Theorem:

Any meromorphic function of weight 2 can be written as

$$f(\tau) = \frac{d}{d\tau} P(\tau)$$

Therefore there is no term q^0.

Hence there is no anomaly if Tr F^2 = Tr R^2.

Therefore the anomaly is proportional to Tr F^2 - Tr R^2

The type-II miracle

Type-II may be viewed as heterotic with the left-moving light-cone Lorentz group interpreted as an *SO(8)* gauge group.

Hirzebruch signature:

$$\hat{A}(R)\mathrm{Ch}(R)_{SO(8),\mathrm{spinor}}$$

The anomaly factorizes as

$$\left[\mathrm{Tr}F^2 - \mathrm{Tr}R^2\right] \times X_8(F,R) = \left[\mathrm{Tr}R^2 - \mathrm{Tr}R^2\right] \times X_8(R,R) = 0$$

Path integral derivation

With K. Pilch, N. Warner (october 1986)

A derivation of $A(q,F,R)$ from the string path integral in gauge and gravitational backgrounds.

(1)

THE WORK I WILL BE TALKING ABOUT WAS MOTIVATED BY PAPERS OF

LANDWEBER & STONG
OCHANINE
D. & G. CHUDNOVSKY
LANDWEBER - STONG - RAVENEL & HOPKINS.
also - ZAGIER

AND IS ALSO CLOSELY RELATED TO WORK ON ANOMALIES BY

SCHELLEKENS - WARNER
and recent paper with PILCH.

E. Witten, december 1986

$$ {}^{1/16}\sum_{k} q^{k/2}\, \mathrm{INDEX}_k\, Q $$

$$ = q^{-\frac{d}{16}}\, \hat{A}\, ch \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T $$

$$ \bigotimes_{m=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots} \Lambda_{q^m} T $$

(ELLIPTIC GENUS, OR
GENERATING F'NAL OF ANOMALIES

THIS HAS A PATH INTEGRAL REPRESENTATION, WHICH REVEALS ITS MODULAR PROPERTIES.

Meromorphic CFT’s

2D conformal field theories that have only left-moving modes and are modular invariant.

Partition function: $P(\tau, \bar{\tau}) = \mathrm{Tr}\ e^{2\pi i \tau H_L - 2\pi i \bar{\tau} H_R}$

Examples exist with free bosons with left-moving momenta on even self-dual Euclidean lattices.

These only exist if the dimension is a multiple of 8

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Free Boson Theories

$N=8$	E_8 (root lattice)
$N=16$	$E_8 \times E_8$ $D_{16}(0+S)$
$N=24$	24 "Niemeier lattices"
$N=32$	$> 10^7$

The 23 Niemeier lattices that are Lie algebra lattices. Square brackets indicate cyclic permutation

Lie algebra	conjugacy class generators
D_{24}	(s)
$D_{16}E_8$	$(s, 0)$
E_8^3	$(0, 0, 0)$
A_{24}	(5)
D_{12}^2	$(s, v), (v, s)$
$A_{17}E_7$	$(3, 1)$
$D_{10}E_7^2$	$(s, 1, 0), (c, 0, 1)$
$A_{15}D_9$	$(2, s)$
D_8^3	$([s, v, v])$
A_{12}^2	$(1, 5)$
$A_{11}D_7E_6$	$(1, s, 1)$
E_6^4	$(1, [0, 1, 2])$
$A_9^2D_6$	$(2, 4, 0), (5, 0, s), (0, 5, c)$
D_6^4	even permutations of $(0, s, v, c)$
A_8^3	$([1, 1, 4])$
$A_7^2D_5^2$	$(1, 1, s, v), (1, 7, v, s)$
A_6^4	$(1, [2, 1, 6])$
$A_5^4D_4$	$(2, [0, 2, 4], 0), (3, 3, 0, 0, s), (3, 0, 3, 0, v), (3, 0, 0, 3, c)$
D_4^6	$(s, s, s, s, s, s), (0, [0, v, c, c, v])$
A_4^6	$(1, [0, 1, 4, 4, 1])$
A_3^8	$(3, [2, 0, 0, 1, 0, 1, 1])$
A_2^{12}	$(2, [1, 1, 2, 1, 1, 1, 2, 2, 2, 1, 2])$
A_1^{24}	$(1, [0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1])$

... but these are just free bosons.

One may also consider conformally invariant interacting theories (CFTs).

They must have a Virasoro central charge that is a multiple of 8.

Can these also be classified?

For $c=8$ and $c=16$: nothing new.
First (and last) challenge: $c=24$

The partition function of such a CFT is a meromorphic function of q with a single pole at $q=0$.

This function must be fully modular invariant.

Then it is determined up to a constant.

The absolute modular invariant

$$P(q) = j(q) + \text{constant}$$

$$j(q) = \frac{1}{q} + 744 + 196884\ q + 21493760\ q^2 + \ldots$$

Its higher coefficients are equal to sums of dimensions of the monster group. This is a the largest "sporadic" group, a discrete group with

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elements.

"String" interpretation (in two dimensions)

$$P(q) = \frac{1}{q} + N + 196884\, q + \ldots$$

$1/q$	Vacuum
N	Massless spin 1 excitations
$196884\, q$	Massive excitations
$\ldots$	Massive excitations

The N spin-1 excitations must form an "affine Lie algebra" or Kac-Moody algebra:

$$\left[J^a_m, J^b_n\right] = i f^{abc} J^c_{m+n} + k m \delta^{ab} \delta_{m+n,0}$$

If $N > 0$ there are "gauge symmetries", and $P(q)$ can be generalized to $P(q,F)$, a character-valued partition function.

We know how it transforms under modular transformations

$$P(\frac{a\tau+b}{c\tau+d}, \frac{F}{c\tau+d}) = \exp\left(\frac{-ic}{8\pi(c\tau+d)}\frac{k}{g}\mathrm{Tr}_{\mathrm{Adj}}\ F^2\right)P(\tau,F)$$

g: Dual coxeter number (depends on algebra)

This function can be expressed in terms of a few basic modular functions

Eisenstein functions

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}$$

$$E_6(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}$$

Modular transformation

$$E_n\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n E_n(\tau) - \frac{6i}{\pi}c(c\tau+d)\delta_{2,n}$$

Weight *n*

Modular anomaly
($n=2$ only)

Any holomorphic (no poles) weight N modular function can be written as a polynomial in E_4 and E_6 of total weight N

In general, the function must have the form

$$P(q, F_1, \ldots, F_L) = e^{\frac{1}{48} E_2(q) \mathcal{F}^2} (\eta(q))^{-24} \sum_{m=0}^{\infty} \sum_i \mathcal{E}_{12+m}(i) \mathcal{T}_i^m$$

$$\mathcal{F}^2 = \sum_{\ell=1}^{L} \frac{k_\ell}{g_\ell} \mathrm{Tr}_{\mathrm{Adj}} F_\ell^2$$

$\mathcal{T}_i^m$: Some trace of a combination of F_j of total order m

$\eta(q)$: The Dedekind η-function

$\mathcal{E}_{12+m}(q)$: A combination of the Eisenstein functions E_4 and E_6

This leaves just a few parameters

$\mathcal{E}_{12} = \alpha(E_4)^2 + \beta(E_6)^2$	2
$\mathcal{E}_{14} = \alpha(E_4)^2 E_6$	1
$\mathcal{E}_{16} = \alpha(E_4)^4 + \beta(E_6)^2 E_4$	2
$\mathcal{E}_{18} = \alpha(E_4)^3 E_6 + \beta(E_6)^3$	2
$\mathcal{E}_{20} = \alpha(E_4)^5 + \beta(E_2)^3 (E_4)^2$	2
$\mathcal{E}_{22} = \alpha(E_4)^4 E_6 + \beta(E_6)^3 E_4$	2
$\mathcal{E}_{24} = \alpha(E_4)^6 + \beta(E_6)^4 + \gamma(E_3)^2 (E_6)^2$	3
$\mathcal{E}_{26} = \alpha(E_4)^5 E_6 + \beta(E_6)^3 (E_4)^2$	2
$\ldots$	3

Strategy

One parameter is fixed by the vacuum (singlet).
Then the quadratic traces are fully fixed at all levels.

At the first excited level we encounter only adjoints, whose traces are now known.

This determines the possible combinations of groups.

Given the vacuum and first excited level, all traces of order 4, 6, 8, 10 and 14 are known at all levels.

This is sufficient information to determine all solutions.

The Lie groups

From the quadratic traces:

$$\frac{g_\ell}{k_\ell} = \frac{1}{24}N - 1$$

This determines the total Virasoro central charge of the "gauge" part: *c=24*.

Hence either there are no gauge symmetries at all, or the saturate the full central charge.

In the latter case, there 222 solutions.

Higher excitations

Now find representations that satisfy the trace identities.
(up to order six if necessary).
This is possible in only 69 of the 222 cases.

Finally, check modular invariance for those 69 candidates.
This guarantees that the trace identities are satisfied to *any* order.

The list

- One CFT without any Lie-algebra. ("The Monster Module")
- One $U(1)^{24}$ lattice ("the Leech Lattice")
- 23 Niemeier lattices
- 14 Z2 Orbifolds of Niemeier lattices. (Goddard, Olive, Montague, 1990)
- 2 already known cases. (Schellekens and Yankielowicz, 1989)
- 30 new cases

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
0	0	—			[10]
1	24	$U(1)^{24}$		(0)	[41]
2	36	$(A_{1,4})^{12}$	$1[1;(0;)^{10}]$	see text	[12]
3	36	$D_{4,12}A_{2,6}$	$(0,1)+(s,0)+(v,0)$	$(0000+0006+0060+0066+0400+3033,00)$ $+(0204+0240+0300+0244+1411+2122,03)$ $+(0044+0600+1213+1231+1233+2022,11)$ $+(0004+0040+0048+0320+0302+0324$ $+1033+1035+1053+3\times 2222,22)$	
4	36	$C_{4,10}$	1	$0000+0024+0040+0044+00{,}10{,}0+0260+0321$ $+0323+0500+0800+1051+1430+1431+2\times 2222$ $+2242+3031+4140$	
5	48	$(A_{1,2})^{16}$	$11[11;(00;)^{6}]$ $+1010[1010;(0000;)^{2}]$ $+(1000)^{4}$	see text	[12]
6	48	$(A_{2,3})^{6}$	$1[1;(0;)^{4}]$	$(00)^6+\{(11;)^4(00;)^2\}+(01)^5(12)+(10)^5(21)+6\times(1,1)^6$	
7	48	$(A_{3,4})^{3}A_{1,2}$	$[1;0;0]1$	$((000)^3+(012)^3,0)+(\{002;010;111\},1)+4\times((111)^3,1)$ $+([000;020;020],2)+([012;020;020],2)$	
8	48	$A_{5,6}C_{2,3}A_{1,2}$	$(1,0,1)+(0,1,1)$	$(00000+02020,00,0)+(00003+00211,30,1)$ $+(00200+02020,20,2)+(00130+03100,11,1)$ $+(00022,01,0)+(00030,00,2)+(01102,10,1)$	

The number 71

In total there are 71 (candidate) CFT's.

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=

$2^{46} . 3^{20} . 5^{9} . 7^{6} . 11^{2} . 13^{3} . 17 . 19 . 23 . 29 . 31 . 41 . 47 . 59 . 71$

71 is the largest prime factor in the order of the monster group...