9 | Dimensional regularization and radiative corrections

Higher-order corrections in perturbation theory involve divergent integrals, and although it may be possible to remove the corresponding infinities in a consistent fashion, it is often necessary to work with infinite expressions at an intermediate stage of the calculation. Therefore one needs to give a more precise meaning to the infinities. This is done in the context of a regularization procedure, which makes all integrals rigorously defined. A particularly convenient method is dimensional regularization, which is based on the observation that the Feynman integrals become finite if the dimension of space-time is sufficiently small. Therefore it is possible to utilize a regularization method based on an analytic continuation of the Feynman integrals to n space-time dimensions. As we will see in the next section the original infinities then arise as poles in n when one approaches the value $n = 4$. The important advantage of this method is that the Feynman rules and almost all symmetries of the theory do not depend on the specific number of space-time dimensions. The symmetries will thus remain manifest during the calculation, which involves in principle only slight modifications of the calculation done directly in 4 space-time dimensions. In this chapter we will explain the method of dimensional regularization and utilize it to calculate all divergent one-loop graphs in quantum electrodynamics. We will determine both finite and infinite terms, perform the renormalization and discuss the evaluation of radiative corrections. The chapter ends with the calculation of the radiative corrections to the "decay" of a virtual photon, where we include the bremsstrahlung of a photon to obtain a result which is free of both ultraviolet and infrared divergences.

9.1. Field theories in n space-time dimensions

To understand the internal consistency of dimensional regularization it is often helpful to consider field theories in space-time dimensions other than four. Since the action is always dimensionless with our choice of units, the Lagrangian must have the dimension $[\text{mass}]^n$. The dimension of the fields follows then from the corresponding kinetic energy terms in the Lagrangian. Since boson fields have kinetic energy terms with two derivatives, their dimension is equal to $d_b = \frac{1}{2}(n-2)$. Fermion kinetic energy terms contain only one derivative, so that the canonical dimension of a fermion field is $d_f = \frac{1}{2}(n-1)$. It is then straightforward to read off the dimension of the coupling constants in the interaction terms by using ordinary dimensional analysis. For instance, the electric charge is the coupling constant of the QED interaction $\mathrm{i}e\bar{\psi}\gamma_\mu\psi A^\mu$, and thus has dimension $d_\mathrm{e} = n - d_\mathrm{A} - 2d_\psi = -\frac{1}{2}(n-4)$ the coupling constant for a scalar field interaction $\lambda\phi^3$ is $d_A = n - 3d_\phi = 1 - \frac{1}{2}(n-4)$; the coupling constant of a Yukawa interaction $G\bar{\psi}\psi\phi$ is $d_G = n - d_\phi - 2d_\psi = -\frac{1}{2}(n-4)$.

Concerning Feynman diagrams in n space-time dimensions we note that Fourier transformations now lead to a factor $(2\pi)^n$ rather than $(2\pi)^4$ and that the δ-function representation is

$$\delta^n(x) = \frac{1}{(2\pi)^n}\int \mathrm{d}^n k \mathrm{e}^{\mathrm{i}k\cdot x}\,.$$

Naturally, the Feynman integrals now have an n-dependent dimension. For instance, the integrand in

$$I = \int \mathrm{d}^n q \frac{1}{(q^2+m^2)^2}$$

has dimension $[\mathrm{mass}]^{-4}$, and the integration volume $\mathrm{d}^n q$ has dimension $[\mathrm{mass}]^n$. Therefore, the integral has dimension $[\mathrm{mass}]^{n-4}$. It is sometimes convenient to scale out a dimensional factor by introducing a reference mass μ, and writing the integral as

$$I = \mu^{n-4}\int \mathrm{d}^n \hat{q}\frac{1}{(\hat{q}^2+m^2/\mu^2)^2}\,,$$

where $\hat{q}$ is now a dimensionless vector. The factor μ^{n-4} can then be combined with the coupling constants to yield functions that are manifestly of the correct dimensionality if $n \neq 4$.

An additional complication concerns the γ-matrices for spinor fields in an n-dimensional space-time. Obviously there have to be n independent γ-matrices, which still satisfy the defining anticommutation relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\,, \qquad \mu\,,\nu = 1,2,\cdots n\,. \tag{9.1}$$

Since $\eta_\mu^{\ \mu} = n$ the maninpulation of γ-matrices induces n-dependent factors, such as in

$$\gamma_\mu\gamma_\rho\gamma^\mu = (2-n)\gamma_\rho\,.$$

In general the γ-matrices must be $2^{n/2}\times 2^{n/2}$ for even n, so that the trace of the unit matrix in spinor space is (see appendix E)

$$\mathrm{Tr}(\mathrm{I}) = 2^{n/2}\,. \tag{9.2}$$

In practice this number is somewhat arbitrary. It measures the number of fermionic components, which in most theories can be chosen at will, so that the analytic continuation around $n = 4$ is arbitrary. Of course, this is not true if there is a symmetry that is sensitive to the number of fermionic degrees of freedom (this is the case in supersymmetric theories, where this aspect requires more care). For our purposes it is more convenient to suppress as many n-dependent terms as possible and maintain the relation

$$\mathrm{Tr}(\mathrm{I}) = 4\,. \tag{9.3}$$

In that case, one may use relations such as

$$\begin{aligned}&\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}\,,\\&\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})\,.\end{aligned} \tag{9.4}$$

Nevertheless, it is obvious that the generalization of the matrix

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

does present a problem, because $\varepsilon_{\mu\nu\rho\sigma}$ is a Lorentz-invanant tensor only in 4 dimensions. Of course, for even values of n one may still define

$$\gamma_{n+1} = -\frac{(-\mathrm{i})^{n/2}}{n!}\varepsilon_{\mu_1\cdots\mu_n}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\,, \tag{9.5}$$

but a problem then arises when taking the trace over strings of γ-matrices that involve γ_5. For instance, the $n = 4$ result

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 4\varepsilon_{\mu\nu\rho\sigma} \tag{9.6}$$

vanishes for $n > 4$, if one uses (9.5). For a discussion of γ-matrices in arbitrary dimensions we refer the reader to appendix E.

9.2. *Evaluation of Feynman integrals in n space-time dimensions*

A systematic analysis of quantum corrections in a relativistic field theory requires the evaluation of Feynman integrals. Those integrals which are finite require no special treatment. However, integrals which are divergent must be regularized, and we will do this by using an analytic continuation in n, the number of space-time dimensions (original references are given at the end of chapter 8). To explain this let us evaluate a typical integral,

$$I(n,\alpha) = \int \frac{\mathrm{d}^n q}{(q^2 + m^2 - \mathrm{i}\varepsilon)^\alpha}\,, \tag{9.7}$$

which is finite for $n < 2\alpha$. Since the n-dimensional vector q has $n-1$ real components and one imaginary component the denominator in (9.7) has zeros near $q_0 = \pm\sqrt{\mathbf{q}^2 + m^2}$, which have been displaced slightly into the upper-left and lower-right quadrants of the complex q_0-plane, as was shown in fig. 2.2. In chapter 2 we have evaluated similar integrals by closing the integration contour in the upper or lower half plane, because we had to ensure that the exponential factor in those integrals remained finite (cf. fig. 2.5). Therefore the integration contour necessarily encloses one of the propagator poles. However, in (9.7) there are no exponential factors so we may now close the contour in a different way by rotating the line of integration counterclockwise to run along the imaginary axis. This so-called Wick *rotation* is shown in fig. 9.1. Since the contour C does not enclose singularities, the integral over C must vanish. When the arcs are moved to infinity their contribution vanishes as well, so that

$$\oint \mathrm{d}q_0 = \int_{-\infty}^{+\infty} \mathrm{d}q_0 + \int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}q_0 = 0\,.$$

Hence, we may replace the variable q_0 by $\mathrm{i}q_0$ and the integral (9.7) becomes equal to

$$I(n,x) = \mathrm{i}\int \frac{\mathrm{d}^n q}{(\mathbf{q}^2 + q_0^2 + m^2)^\alpha}\,. \tag{9.8}$$

The integrand in (9.8) is now positive definite and the integral can be evaluated without difficulty.

Let us digress for a moment to discuss this point further. The effect of the Wick rotation is to replace the Minkowskian "length" $\mathbf{q}^2 - q_0^2$ by a Euclidean "length" $\mathbf{q}^2 + q_0^2$. By this procedure we are thus calculating the Green's functions of the corresponding field theory, but now in a four-dimensional Euclidean space with $x_\mu = (\mathbf{x}, \tau)$. Subsequently we make an analytic continuation of τ to imaginary values, i.e. $\tau = \mathrm{i}t$, in which case we expect to recover the Green's functions of the theory in Minkowski space, where $x_\mu = (\mathbf{x}, \mathrm{i}t)$. The analytic continuation of the time variable t by rotating over $-1/2\pi$ in the complex t plane requires a simultaneous continuation of the energy variable q_0 by rotating over $1/2\pi$ in the complex q_0-plane, when one makes a Fourier transformation. This corresponds to the Wick rotation based on the integration contour of fig. 9.1, which was uniquely prescribed by the iε term in the propagator. As we have seen in (9.8) this leads to a change in the integration volume, $\mathrm{d}^n q = \mathrm{i}\mathrm{d}^n q_\mathrm{E}$, where the subscript E denotes that we are dealing with Euclidean space. Performing an opposite rotation on the time variable it follows that $\mathrm{d}^n x = -\mathrm{i}\mathrm{d}^n x_\mathrm{E}$; consequently δ-functions change according $\delta^n(q) = -\mathrm{i}\delta^n(q_\mathrm{E})$ and $\delta^n(x) = \mathrm{i}\delta^n(x_\mathrm{E})$.

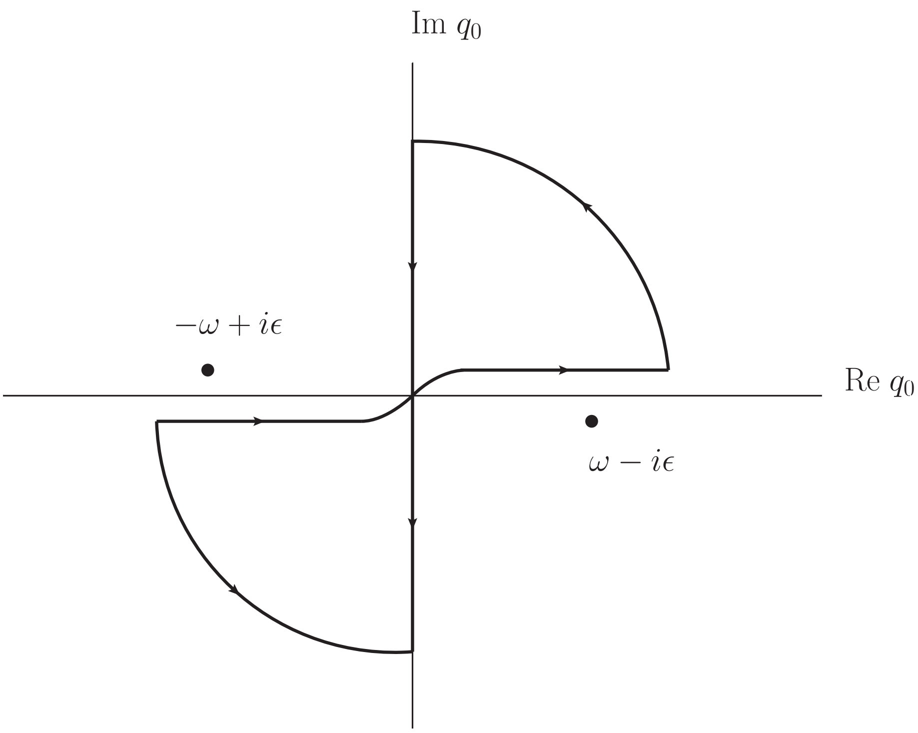

Figure 9.1: Contour used to perform a Wick rotation in the complex q_0-plane.

The Euclidean postulate states that the Green's functions in Minkowski space are to be constructed from Euclidean space by the above analytic continuation. Hence, this postulate prescribes how to deal with the poles that arise in the propagators in Minkowski space, and as we have seen from the Wick rotation, this prescription implies the same $\mathrm{i}\varepsilon$ modification to the propagator that we have adopted in chapter 2 on the basis of more heuristic arguments (we recall that the sign of the $\mathrm{i}\varepsilon$ term has direct physical consequences for causality and the conservation of probability, i.e. unitarity). However, the Euclidean postulate is not always sufficient for defining a consistent treatment of propagator poles. For instance, for massless particles one encounters problems with infrared divergences.

Now we return to (9.8) and consider an n-dimensional Euclidean space, where $q_\mu = (q_1, \cdots, q_n)$ is real. The angular integration can be done in the n-dimensional space where the vector q_μ is characterized by its length q and $n-1$ angles $\theta_1, \cdots, \theta_{n-1}$. It is not difficult to find the expression for the volume element $\mathrm{d}^n q$ in terms of these angles and q (see problem 9.1), so that (9.8) becomes

$$I(n,\alpha) = \mathrm{i}\int_0^\infty \frac{\mathrm{d}q q^{n-1}}{(q^2+m^2)^\alpha}\int_0^{2\pi}\mathrm{d}\theta_1\int_0^\pi \mathrm{d}\theta_2 \sin\theta_2 \cdots \int_0^\pi \mathrm{d}\theta_{n-1}\sin^{n-2}\theta_{n-1}\,. \tag{9.9}$$

The angular integration can be carried out with the help of the formula

$$\int_0^\pi \mathrm{d}\theta \sin^k\theta = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}k)}{\Gamma(1+\frac{1}{2}k)}\,, \tag{9.10}$$

which involves the Euler gamma function, whose definition and some important properties have been collected in table 8.1. Remember that $\Gamma(z)$ is analytic for positive z and has single poles at all nonpositive integers.

Applying (9.10) $n-1$ times we find that the angular integrations yield a factor $2\pi^{1/2n}/\Gamma(\frac{1}{2}n)$. The remaining integral over the length of q_μ, can be evaluated using the integral Table 8.1 The Euler gamma function.

$$\int_0^\infty \frac{x^{2\beta-1}\mathrm{d}x}{(x^2+a^2)^\alpha} = \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\frac{1}{2(a^2)^{\alpha-\beta}}\,, \tag{9.11}$$

valid for $\alpha > \beta > 0$. Hence, collecting all the factors, we find

$$I(n,\alpha) = \mathrm{i}\pi^{n/2}\frac{\Gamma(\alpha-\frac{1}{2}n)}{\Gamma(\alpha)}(m^2)^{\frac{1}{2}n-\alpha}\,, \tag{9.12}$$

with $\alpha > \frac{1}{2}n$. Note that if we go outside this range then $I(n, \alpha)$ has poles at nonpositive values of $\alpha - \frac{1}{2}n$, and zeros for nonpositive values of α. For even values of n some of the poles and the zeros may cancel.

For $\alpha > 2$ one is dealing with a finite integral in 4 space-time dimensions and (9.12) represents the final result after substituting $n = 4$. For divergent integrals where $\alpha \leqslant 2$ the result (9.12) still gives a representation of the integral as long as $\alpha - \frac{1}{2}n$ is not equal to a nonpositive integer. This observation forms the basis for the dimensional regularization method, where one employs an analytic continuation of the integrals to complex values of n. The infinities re-emerge in the limit $n \to 4$ in the form of simple poles. Only at the end of the calculation does one take this limit, and the poles are absorbed by a proper renormalization of the theory, as has been described in the previous chapter.

It is not difficult to understand how the analytic continuation works for more general integrals. Using the chain rule, we write

$$\int \mathrm{d}^n q \frac{\partial}{\partial q_\mu}(q_\mu f(q)) = \int \mathrm{d}^n q\, q_\mu \Big(\frac{\partial}{\partial q_\mu} f(q)\Big) + n \int \mathrm{d}^n q f(q)\,, \qquad (9.13)$$

where we have used $\partial q_\mu / \partial q_\mu = n$. By Gauss' theorem the left-hand side is a surface integral, which may be dropped if $f(q)$ vanishes sufficiently rapidly at infinity. This is certainly the case for finite integrals, but the analytic continuation will be implemented by ignoring the surface term irrespective of the asymptotic behaviour of the integrand. The advantage of this procedure is that it is consistent with standard algebraic manipulations on the integrands. Dropping the surface term we can write

$$n \int \mathrm{d}^n q f(q) = - \int \mathrm{d}^n q\, q_\mu \Big(\frac{\partial}{\partial q_\mu} f(q)\Big)\,, \qquad (9.14)$$

which can be used to express divergent integrals in terms of finite ones for complex values of n. To see how this works, consider an example where $f(q) = (q^2 + m^2)^{-\alpha}$. Then (9.14) becomes

$$n \int \mathrm{d}^n q \frac{1}{(q^2 + m^2)^\alpha} = 2\alpha \int \mathrm{d}^n q \frac{q^2}{(q^2 + m^2)^{\alpha+1}}\,,$$

so that, after writing $q^2 = (q^2 + m^2) - m^2$ and rearranging, we find

$$\int \mathrm{d}^n q \frac{1}{(q^2 + m^2)^\alpha} = \frac{\alpha m^2}{\alpha - \frac{1}{2}n} \int \mathrm{d}^n q \frac{1}{(q^2 + m^2)^{\alpha 1}}\,. \qquad (9.15)$$

The integral on the right-hand side is finite for $n < 2\alpha + 2$, whereas the integral on the left-hand side is only finite for $n < 2\alpha$. Therefore (9.15) defines an analytic continuation of the integral $\int \mathrm{d}^n q (q^2 + m^2)^{-\alpha}$ into the region

$2\alpha < n < 2\alpha + 2$ where it is not convergent. The extension is well defined except for the discrete point $n = 2\alpha$, where the right-hand side has a pole with finite residue. Repeating this procedure allows a further continuation to higher values of n. For instance,

$$\int \mathrm{d}^n q \frac{1}{(q^2+m^2)^\alpha} = \frac{(\alpha+1)\alpha m^4}{(\alpha+1-\frac{1}{2}n)(\alpha-\frac{1}{2}n)} \int \mathrm{d}^n q \frac{1}{(q^2+m^2)^{\alpha+2}} . \quad (9.16)$$

where the right-hand side is finite for $n < 2\alpha + 4$ apart from single poles at $n = 2\alpha$ and $n = 2\alpha + 2$. Proceeding further we can extract a set of poles at discrete points in n until the integral is finite near the number of space-time dimensions in which we are interested.

It is easy to verify that (9.15) and (9.16) are completely consistent with the basic integral (9.12), by using $\Gamma(z+1) = z\Gamma(z)$. However, we should point out that the argument based on (9.14) leads to simple poles only if the function $f(q^2)$ behaves asymptotically as a power of q^2. If this is not the case multiple poles will occur as we shall shortly demonstrate.

At the end of the calculation one must take the limit $n \to 4$ and extract all the poles at $n = 4$. To show how this goes let us choose $\alpha = 2$ and expand in powers of $\varepsilon = n - 4$. We start from

$$I(n,2) = \mathrm{i}\pi^2 \frac{\Gamma(-\frac{1}{2}\varepsilon)}{\Gamma(2)} (\pi m^2)^{\frac{1}{2}\varepsilon} ,$$

and using the result $x^\varepsilon = \exp(\varepsilon \ln x) = 1 + \varepsilon \ln x + O(\varepsilon^2)$ and the properties of the Γ function given in table 8.1 we derive

$$I(n,2) \approx -2i\pi^2 \Big\{ \frac{1}{\varepsilon} + \tfrac{1}{2}\gamma_{\mathrm{E}} + \tfrac{1}{2}\ln(\pi m^2) + O(\varepsilon) \Big\} . \quad (9.17)$$

However, this naive expansion is dimensionally incorrect, since the left-hand side has dimension $[\text{mass}]^\varepsilon$. To correct this error one may introduce an arbitrary reference mass μ, and following the argument in section 9.1, we rewrite

$$I(n,2) = \mathrm{i}\pi^2 \mu^\varepsilon \frac{\Gamma(-\frac{1}{2}\varepsilon)}{\Gamma(2)} \Big(\frac{\pi m^2}{\mu^2}\Big)^{\frac{1}{2}\varepsilon} ,$$

The expansion

$$I(n,2) \approx -2\mathrm{i}\pi^2 \mu^\varepsilon \Big\{ \frac{1}{\varepsilon} + \tfrac{1}{2}\gamma_{\mathrm{E}} + \tfrac{1}{2}\ln\Big(\frac{\pi m^2}{\mu^2}\Big) + O(\varepsilon) \Big\} . \quad (9.18)$$

is now dimensionally correct. As we have already seen in chapter 8 the arbitrary mass determines the precise decomposition of the integral into an infinite term and a finite remainder. All regularization schemes have this feature;

when the result of the calculation is expressed in terms of physical quantities, such as masses and coupling constants, calculated to the same order in perturbation theory, this ambiguity in splitting off the infinite term should no longer play a role. We return to this aspect in chapter 10 when discussing the renormalization group.

We have now shown how single poles arise. Multiple poles are possible for diagrams with more loops. At first sight these seem to arise from the fact that each of the subintegrations may lead to a divergence which is of the single pole type. A subsequent integration would thus lead to a double pole, and so on. However, we recall from the previous chapter that each of the subgraphs has already been rendered finite by the inclusion of an appropriate counterterm which precisely cancels the single pole. Therefore, it is not this feature that gives rise to multiple poles. A more careful analysis reveals that a cancellation of the single poles in the subgraphs is not sufficient to prevent the occurrence of multiple poles, because there remain logarithmic terms after the subtraction of the pole terms; it is the integration over these logarithmic terms that leads to double poles. To see this, apply (9.14) to the function $f(q) = (q^2+m^2)^{-\alpha}\ln(q^2+m^2)$

$$\int \frac{\mathrm{d}^n q \ln(q^2+m^2)}{(q^2+m^2)^\alpha} = \frac{\frac{1}{2}nm^2}{(\alpha-\frac{1}{2}n)^2}\int\frac{\mathrm{d}^n q}{(q^2+m^2)^{\alpha+1}} + \frac{\alpha m^2}{\alpha-\frac{1}{2}n}\int\frac{\mathrm{d}^n q\ln(q^2+m^2)}{(q^2+m^2)^{\alpha+1}}\,. \tag{9.19}$$

The right-hand side now contains a double pole at $n=2\alpha$ with finite residue. By induction we may verify that diagrams with m loops yield poles of order m, $m-1$, .. ., 1 at $n=2\alpha$, after proper renormalization of the subgraphs.

It is not difficult to find generalizations of the basic integral (9.8). For instance

$$\int \mathrm{d}^n q \frac{q_\mu}{(q^2+m^2)^\alpha} = 0\,, \tag{9.20}$$

since the integrand is antisymmetric under $q \to -q$. Another way of coming to the same conclusion is to note that the integral (9.20) should behave as a vector under Lorentz transformations. However, there is no vector available in (9.20) other than the integration variable, so the result must be zero. By similar arguments the integral with $q_\mu q_\nu$ in the numerator must be proportional to a Lorentz-invariant second-rank tensor; the only such tensor is $\eta_{\mu\nu}$, so that

$$\int \mathrm{d}^n q \frac{q_\mu q_\nu}{(Q^2+m^2)^\alpha} = a\eta_{\mu\nu}\,,$$

where a is constant. To find a we contract the indices μ and ν, and use $\eta^{\mu}_{\mu} = n$:

$$a = \frac{1}{n} \int \mathrm{d}^n q \frac{q^2}{(q^2+m^2)^{\alpha}} \,.$$

Hence we find

$$\int \mathrm{d}^n q \frac{q_\mu q_\nu}{(q^2+m^2)^{\alpha}} = \frac{1}{n} \eta_{\mu\nu} [I(n,\alpha-1) - m^2 I(n,\alpha)] \,. \tag{9.21}$$

This technique is often referred to as symmetric integration. Note that further integrals follow by exploiting the substitution $q \to q+k$, where k_μ is a constant n-dimensional vector. A number of useful integrals is listed in appendix F.

We have now derived all the basic results we need. However, one aspect of dimensional regularization which sometime causes confusion is the nonuniformity of limits when discussing massless theories. For example, the limits $\varepsilon \to 0$ and then $m \to 0$ do not give the same result for the integral (9.12) as first taking $m \to 0$ and then $\varepsilon \to 0$. In the latter case the answer depends on whether $\varepsilon \to 0$ is approached from above or below. If we return to (9.15) and set $m^2 = 0$ then $(\alpha - \frac{1}{2}n) \int \mathrm{d}^n q q^{-2\alpha} = 0$, so unless $n = 2\alpha$, $\int \mathrm{d}^n q q^{-2\alpha} = 0$. A consistent rule is therefore to take all integrals of homogeneous functions of q^2 to be identically zero (often it is not crucial to adopt this rule, since such terms cancel against similar terms from other diagrams or they can be absorbed in the renormalization constants). The nonuniformity of the limits is relevant in the context of theories with infrared divergences, where one often introduces a small cut-off mass for the massless particles. This mass must then be kept throughout the calculation, and only after one has performed the renormalization and obtained physical quantities that are infrared finite, may one set the cut-off mass equal to zero. Alternatively, one may use dimensional regularization to regularize both the infrared and the ultraviolet divergences, but in that case the limit $n \to 4$ cannot be taken until after the renormalization procedure has been carried out, and the final infrared finite cross section has been computed.

9.3. Feynman parameters

The integrals corresponding to realistic Feynman diagrams are more complicated than (9.7), because they involve integrands that are products of propagators with different momenta. Therefore the denominators have typically the form $[(p_1+q)^2+m_1^2][(p_2+q)^2+m_2^2]\cdots[(p_\alpha+q)^2+m_\alpha^2]$ rather than $[q^2+m^2]^\alpha$. Also the numerators may be more complicated, but they can often be simplified by taking partial fractions. To bring the denominators into a form that allows us to perform the momentum integrals we may use an identity due to

Feynman. In the simplest case it takes the form

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{(Ax + B(1-x))^2}\,, \tag{9.22}$$

which can be verified straightforwardly by working out the integral. To show how one makes use of this identity let us evaluate the following integral which we will need later on:

$$I(k^2, m_1^2, m_2^2) = \frac{1}{(2\pi)^n} \int \frac{\mathrm{d}^n q}{((q+\frac{1}{2}k)^2 + m_1^2)((q-\frac{1}{2}k)^2 + m_2^2)}\,. \tag{9.23}$$

Identifying $A = (q+\frac{1}{2}k)^2 + m_1^2$ and $B = (q-\frac{1}{2}k)^2 + m_2^2$, we insert the identity (9.22). If $n < 4$ the integral is convergent and the two integrations may be interchanged, leading to

$$I(k^2, m_1^2, m_2^2) = \frac{1}{(2\pi)^n} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n q}{(q^2 + 2q\cdot K + M^2)^2}\,, \tag{9.24}$$

where $K = (x - \frac{1}{2})k$ and $M^2 = \frac{1}{4}k^2 + m^2 + (m_1^2 - m_2^2)x$. Changing the integration variable to $Q = q + K$, and using (9.12), leads to

$$\begin{aligned} I(k^2, m_1^2, m_2^2) &= \frac{1}{(2\pi)^n} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n Q}{(Q^2 + M^2 - K^2)^2} \\ &= \mathrm{i}(4\pi)^{-\frac{1}{2}n}\Gamma(2 - \tfrac{1}{2}n) \int_0^1 \frac{\mathrm{d}x}{(M^2 - K^2)^{2-\frac{1}{2}n}}\,, \end{aligned} \tag{9.25}$$

with $M^2 - K^2 = m_2^2 + (m_1^2 - m_2^2)x + k^2 x(1-x)$. Near $n = 4$ this result takes the form (cf. 9.18)

$$\begin{aligned} I(k^2, m_1^2, m_2^2) = -\frac{\mathrm{i}\mu^\varepsilon}{8\pi^2}\Big\{\frac{1}{\varepsilon} \\ + \tfrac{1}{2}\gamma_{\mathrm{E}} + \tfrac{1}{2}\int_0^1 \mathrm{d}x \ln\frac{m_2^2 + (m_1^2 - m_2^2)x + k^2 x(1-x)}{4\pi\mu^2} + O(\varepsilon)\Big\} \end{aligned} \tag{9.26}$$

with μ an arbitrary reference mass.

Hence we see that the Feynman identity allows an easy evaluation of the momentum integral, while the integral over Feynman parameters is performed at a later stage. As we have shown in section 3.6 the integral (9.26) becomes complex if $-k^2 = s > (m_1 + m_2)^2$. This can be verified directly in (9.26) because the argument of the logarithm is positive definite inside the range of integration, unless $s > (m_1 + m_2)^2$. In that case the argument of the logarithm will have zeros at $x = x_\pm$, where

$$2sx_\pm = s - m_1^2 + m_2^2 \pm \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \qquad \text{and} \qquad 0 < x_\pm < 1\,.$$

Hence $I(-s, m_1^2, m_2^2)$ develops a branch cut in the complex s plane. We have already been dealing with such a situation in section 8.1. Here we note again that the sign of the imaginary part of $I(-s, m_1^2, m_2^2)$ follows from the iε prescription (Euclidean postulate), which implies that we give the masses a small but negative imaginary part. We will further evaluate (9.26) for the case of equal masses in section 9.4.

The integral (9.22) can be extended to the general case. We only quote the result,

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \times \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1} \delta(1 - x_1 - x_2 - \cdots - x_n)}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^{\alpha_1 + \alpha_2 + \cdots \alpha_n}} . \tag{9.27}$$

(in problem 8.2 this formula is proven for positive real $\alpha_1, ..., \alpha_n$). This expression can be used when evaluating more complicated diagrams. For the vertex function that we will calculate in the next section we need an integral over three propagators. Hence we write

$$\begin{aligned} \frac{1}{ABC} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(1-x-y-z)}{(xA+yB+zC)^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{(xA+yB+(1-x-y)C)^3} , \end{aligned} \tag{9.28}$$

where the integration region is restricted by $0 < z = 1 - x - y < 1$. We have already used this integral in section 8.1. Let us use it again to calculate

$$J(t, p^2, p'^2) = \frac{1}{(2\pi)^n} \int \frac{\mathrm{d}^n q}{((p+q)^2 + m^2)((p'+q)^2 + m^2)(q^2 + \kappa^2)} , \tag{9.29}$$

which we will need in section 9.4, where $t = -(p' - p)^2$. Inserting (9.28) and interchanging the order of integrations, we have

$$J(t, p^2, p'^2) = \frac{2}{(2\pi)^n} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^n q}{(q^2 + 2K\cdot q + M^2)^3} , \tag{9.30}$$

where $K = xp + yp'$ and $M^2 = p^2 x + p'^2 y + m^2(x+y) + \kappa^2(1-x-y)$. Changing the integration variable to $Q = q + K$, and performing the momentum integral, the result takes the form

$$J(t, p^2, p'^2) = \frac{\mathrm{i}\Gamma(3 - \frac{1}{2}n)}{(4\pi)^{1/2n}} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{(M^2 - K^2)^{3 - \frac{1}{2}n}} . \tag{9.31}$$

Since the momentum integral is finite we take $n = 4$. Hence

$$J(t, p^2, p'^2) = \frac{\mathrm{i}}{16\pi^2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y [-xyt + p^2 x(1-x-y)$$
$$+ p'^2 y(1-x-y) + m^2(x+y) + \kappa^2(1-x-y)]^{-1} \,. \tag{9.32}$$

Figure 9.2: Integration region of the integral in (8.32).

In a certain range of values for t, p^2, p'^2 this integral is well defined. Let us now take $p^2 = p'^2 = -m^2$. The integrand in (9.32) is then equal to $[-xyt + m^2(x+y)^2 + \kappa^2(1-x-y)]^{-1}$. It is now advantageous to exploit the symmetry of the $x - y$ integration region shown in fig. 9.2 by using different variables. This is most easily done in two steps. One first chooses $s_+ = x + y$ and $s_- = x - y$ with $0 \leqslant s_+ \leqslant 1$ and $-s_+ \leqslant s_- \leqslant s_+$. Hence the integral takes the form

$$\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y = \tfrac{1}{2} \int_0^1 ds_+ \int_{-s_+}^{s_+} ds_- \,. \tag{9.33}$$

The factor 2 arises from the Jacobian of the transformation as can be verified immediately by comparing the surface areas that result from both integrations. To make the boundary of the s_--integration independent of s_+ we scale s_- by a factor $1/s_+$. Combining these changes leads to the variables

$$u = x + y \,, \qquad v = \frac{x-y}{x+y} \,,$$

and $J(t, -m^2, -m^2)$ takes the form

$$J(t, -m^2, -m^2) = \frac{\mathrm{i}}{32\pi^2} \int_0^1 \mathrm{d}u \int_{-1}^1 \mathrm{d}v \frac{u}{u^2[m^2 - \frac{1}{4}t(1-v^2)] + \kappa^2(1-u)} \,. \tag{9.34}$$

This type of integral arises frequently in quantum electrodynamics with extra terms in the numerator. The parameter κ then represents the photon mass, which is actually zero. However, κ must be kept as a cut-off parameter because the integral diverges in the limit $\kappa \to 0$ (if the numerator contains terms of higher order in u one can safely put κ to zero). Hence (9.34) must be evaluated for finite κ. To do this we note the relation

$$\begin{aligned} &\int_0^1 \mathrm{d}u \frac{[m^2 - \frac{1}{4}t(1-v^2)]u - \frac{1}{2}\kappa^2}{[m^2 - \frac{1}{4}t(1-v^2)]u + \kappa^2(1-u)} \\ = \; &\tfrac{1}{2} \int_0^1 \mathrm{d}u \frac{\partial}{\partial u} \ln[u^2(m^2 - \tfrac{1}{2}t(1-v^2)) + \kappa^2(1-u)] \\ = \; &\tfrac{1}{2} \ln \frac{m^2 - \frac{1}{4}t(1-v^2)}{\kappa^2} \,. \end{aligned} \tag{9.35}$$

In the limit $\kappa \to 0$ the κ^2 term in the numerator on the left-hand side of this equation can be dropped, so that we find the following expression for $J(t, -m^2, -m^2)$

$$J(t, -m^2, -m^2) = \frac{\mathrm{i}}{32\pi^2} \int_0^1 dv \frac{1}{m^2 - \frac{1}{4}t(1-v^2)} \ln \frac{m^2 - \frac{1}{4}t(1-v^2)}{\kappa^2} + O(\kappa^2) \,, \tag{9.36}$$

This integral will be further evaluated in appendix D.

9.4. Divergent one-loop diagrams in quantum electrodynamics

Dimensional regularization will now be used to calculate the one-loop diagrams described in section 8.2. In n dimensions expressions (8.27)-(8.29) remain valid once we replace $(2\pi)^{-4}\mathrm{d}^4 q$ by $(2\pi)^{-n}\mathrm{d}^n q$ and switch to n-dimensional momentum vectors and appropriate γ-matrices. The objective of this section is to extract the residue of the pole terms at $n = 4$, and determine the finite remainder. Since the pole term is ultimately removed by the renormalization procedure, it is the finite terms that are physically significant. Unfortunately their determination usually requires laborious calculations.

Before starting the actual calculations we should point out that there are various ways to proceed. One of them is to straightforwardly introduce Feynman parameters to combine the propagators into one common denominator.

This was shown in the previous section. Then one routinely performs the momentum integrals and expands about $n = 4$ to extract the pole term. Finally one evaluates the Feynman parameter integrals. In principle this is all straightforward, but since the divergences in the multi-loop integrals are sometimes transferred into the Feynman parameter integrals, it should be handled with care.

Another method, which will be followed in this section, is to first manipulate the momentum integrals in order to exhibit their tensorial structure in terms of external momenta. Usually there are important cancellations which will become manifest at this point. Unfortunately this approach tends to become rather unwieldy if we are dealing with diagrams with several external momenta, as we shall see shortly. Now let us get down to evaluating the expressions (8.27)-(8.29):

(a) *Vacuum polarization diagram*

One starts by making the denominator in (8.27) symmetric under $q \to -q$ by replacing q by $q - \frac{1}{2}k$. Subsequently odd powers of q can be dropped in the numerator, which yields

$$\Pi_{\mu\nu}(k) = \frac{\mathrm{i}e^2}{(2\pi)^n} \int \mathrm{d}^n q \frac{\mathrm{Tr}(-\gamma_\mu \not{q} \gamma_\nu \not{q} + \frac{1}{4}\gamma_\mu \not{k} \gamma_\nu \not{k} + m^2 \gamma_\mu \gamma_\nu)}{((q + \frac{1}{2}k)^2 + m^2)((q - \frac{1}{2}k)^2 + m^2)} \,. \tag{9.37}$$

Now use (8.3)-(8.4) to find

$$\Pi_{\mu\nu}(k) = \frac{4\mathrm{i}e^2}{(2\pi)^n} \int \mathrm{d}^n q \frac{q^2 \eta_{\mu\nu} - 2q_\mu q_\nu + (m^2 - \frac{1}{4}k^2)\eta_{\mu\nu} + \frac{1}{2}k_\mu k_\nu}{((q + \frac{1}{2}k)^2 + m^2)((q - \frac{1}{2}k)^2 + m^2)} \,. \tag{9.38}$$

Before proceeding we remark that one could use the anticommutation relations for terms like $\gamma_\mu \not{q} \gamma_\nu \not{q}$ to reduce the number of γ-matrices. It then suffices to only use $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}$

Since the answer must be a second-rank Lorentz-covariant tensor it must be expressible in terms of $\eta_{\mu\nu}$ and $k_\mu k_\nu$. For a Lorentz-invariant function $f(q, k)$ this observation implies

$$\int \mathrm{d}^n q f(q,k) q_\mu q_\nu = a\eta_{\mu\nu} + bk_\mu k_\nu \,,$$

where we can find a and b by solving the two algebraic equations formed by contracting with $\eta_{\mu\nu}$ (remember $\eta_{\mu\mu} = n$) and $k_\mu k_\nu$. This means that we can make the following replacement

$$\begin{aligned}\int \mathrm{d}^n q f(q,k) q_\mu q_\nu \; &= \\ &\int \mathrm{d}^n q f(q,k) \frac{1}{n-1} \Big\{ \Big(q^2 - \frac{(q\cdot k)^2}{k^2}\Big)\eta_{\mu\nu} - \Big(q^2 - n\frac{(q\cdot k)^2}{k^2}\Big)\frac{k_\mu k_\nu}{k^2} \Big\} \,.\end{aligned} \tag{9.39}$$

The drawback of this method is that it introduces spurious poles at $k^2 = 0$ which only cancel later in the calculation. Straightforward application of (9.39) to (9.38) can be followed by a partial fractioning of the $(q.k)^2$ term, using

$$\frac{(q\cdot k)^2}{((q+\frac{1}{2}k)^2+m^2)((q-\frac{1}{2}k)^2+m^2)} = \\ -1 + \tfrac{1}{2}\frac{q^2+\frac{1}{4}k^2+m^2}{(q+\frac{1}{2}k)^2+m^2} + \tfrac{1}{2}\frac{q^2+\frac{1}{4}k^2+m^2}{(q-\frac{1}{2}k)^2+m^2}\,.$$

This leads to new integrals with single denominators, which, after changing the integration variable q to $q \pm \frac{1}{2}k$, give integrals with a simple denominator $q^2 + m^2$. Collecting together the pieces we see that (9.33) is proportional to $k^2\eta_{\mu\nu} - k_\mu k_\nu$. This can be understood as a consequence of the gauge invariance of quantum electrodynamics, which implies $k_\mu \Pi_{\mu\nu}(k) = 0$ (see problem 8.3). Therefore we define

$$\Pi_{\mu\nu}(k^2) = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) \tag{9.40}$$

where

$$\Pi(k^2) = \\ \frac{\mathrm{i}e^2}{(2\pi)^n}\Big(\frac{2}{n-1}\Big(1+\frac{4m^2}{k^2}\Big)-2\Big)\int \mathrm{d}^n q\frac{1}{((q+\frac{1}{2}k)^2+m^2)((q-\frac{1}{2}k)^2+m^2)} \\ + \frac{4\mathrm{i}e^2}{(2\pi)^n}\frac{n-2}{n-1}\frac{1}{k^2}\int \mathrm{d}^n q\frac{1}{q^2+m^2}\,. \tag{9.41}$$

The second term in $\Pi(k^2)$ looks more singular than the first. However, the presence of a factor $(n-2)$ reminds us of (9.15). Indeed, substituting (9.15) with $\alpha = 1$, we can rewrite (9.41) in the convenient form

$$\Pi(k^2) = 4\mathrm{i}e^2\Big\{-\tfrac{1}{2}\frac{n-2}{n-1}I(k^2,m^2,m^2)+\frac{2m^2}{n-1}\frac{I(k^2,m^2,m^2)-I(0,m^2,m^2)}{k^2}\Big\} \tag{9.42}$$

where the function $I(k^2, m^2, m^2)$ has been calculated in the previous section (cf. 9.23). Note that expression (9.42) is regular as $k^2 \to 0$. Substituting (9.26) for $I(k^2, m^2, m^2)$ we find $\Pi(k^2)$ expanded about $\varepsilon = n - 4 = 0$,

$$\Pi(k^2) = \frac{-e^2\mu^\varepsilon}{6\pi^2}\Big[\frac{1}{\varepsilon}+\tfrac{1}{2}\gamma_{\mathrm{E}}-\tfrac{1}{2}\ln 4\pi\Big] + \Pi^f(k^2) + O(\varepsilon)\,,$$

which is the result quoted in (**??**), where we have now explicitly determined the finite momentum-dependent remainder (suppressing the factor μ^ε)

$$\Pi^f(k^2) = \frac{-e^2}{12\pi^2}\Big\{\ln\frac{m^2}{\mu^2}+\tfrac{1}{3}+\Big(1-\frac{2m^2}{k^2}\Big)\int_0^1 \mathrm{d}x\ln\Big(1+\frac{k^2}{m^2}x(1-x)\Big)\Big\}\,.$$

$$(9.43)$$

Note that the expansion of (9.42) about $\varepsilon = 0$ involves a term proportional to ε/ε, which must be retained! By explicit integration of (9.43) one finds that

$$\begin{aligned}
\Pi^f(k^2) &= \frac{-e^2}{6\pi^2 k^2}\Big\{2m^2 + 2k^2\Big(\ln\frac{m^2}{\mu^2} - \tfrac{5}{3}\Big) \\
&+ (k^2 - 2m^2)\sqrt{\frac{4m^2 + k^2}{-k^2}}\arctan\sqrt{\frac{-k^2}{4m^2+k^2}}\Big\}\,, \qquad -4m^2 < k^2 < 0 \\
&= \frac{-e^2}{6\pi^2 k^2}\Big\{2m^2 + \tfrac{1}{2}k^2\Big(\ln\frac{m^2}{\mu^2} - \tfrac{5}{3}\Big) \\
&+ \tfrac{1}{2}(k^2 - 2m^2)\sqrt{1 + \frac{4m^2}{k^2}}\Big[\ln\Big|\frac{1+\sqrt{1+4m^2/k^2}}{1-\sqrt{1+4m^2/k^2}}\Big| - i\pi\theta(-k^2 - 4m^2)\Big]\Big\} \\
&\qquad\qquad k^2 < -4m^2, \quad k^2 > 0\,. \qquad (9.44)
\end{aligned}$$

For $k^2 \to 0$ and $k^2 \to \pm\infty$ this result takes the form

$$\Pi^f(k^2) - \Pi^f(0) = -\frac{e^2}{60\pi^2}\frac{k^2}{m^2} + O\Big(\Big(\frac{k^2}{m^2}\Big)^2\Big)\,, \tag{9.45}$$

$$\Pi^f(k^2) - \Pi^f(0) = \frac{e^2}{12\pi^2}\Big\{-\ln\Big(\frac{|k|^2}{m^2}\Big) + \frac{5}{3} + i\pi\theta(-k^2 - 4m^2)\Big\} + O\Big(\frac{m^2}{k^2}\Big)\,. \tag{9.46}$$

The term linear in k^2 in (9.45) was first found by Uehling. Figure 9.3 shows the behaviour of the real and imaginary part of $\Pi^f(k^2)$.

(b) *Electron self-energy diagram*

The expression for the electron self-energy diagram is given by (8.28). In the first term proportional to $\eta_{\mu\nu}$ one multiplies the numerator and denominator by $-\mathrm{i}(\not p + \not q) + m$, and uses the relation $\gamma_\mu\gamma_\rho\gamma_\mu = (2-n)\gamma_\rho$ in n dimensions to simplify the result. In the second term one writes one of the q factors as $-\mathrm{i}(\mathrm{i}(\not p + \not q) + m) + \mathrm{i}(\mathrm{i}\not p + m)$. The first term cancels against the fermion propagator, whereas the second term is evaluated in the standard way. The result is

$$\begin{aligned}
\Sigma(p) \;&=\; \frac{-\mathrm{i}e^2}{(2\pi)^n}\int \mathrm{d}^n q\Big\{\frac{\mathrm{i}(n-2)(\not p + \not q) + nm}{((q+p)^2 + m^2)q^2} \\
&\quad -\mathrm{i}(1-\lambda^{-2})(\mathrm{i}\not p + m)\frac{(-\mathrm{i}(\not p + \not q) + m)q}{((q+p)^2 + m^2)q^4}\Big\}\,,
\end{aligned} \tag{9.47}$$

where in the last term we have discarded the term containing $\not q/q^4$ because it is antisymmetric under $q \to -q$.

Figure 9.3: The vacuum polarization function $\Pi^f(s/m2) - \Pi^f(0)$ expressed as a function of $s/m^2 = -k^2/m^2$. The solid line denotes its real part. The imaginary parts of the order-α and order-α^2 contributions are indicated by dot-dashed lines. The latter, exhibited at a scale a factor 200 larger, will be calculated in section 9.6. When giving the order-α^2 terms, care should be taken to specify the definition of e and m in the order-α terms, because those may differ from the physical charge and mass of the electron by higher order terms. This is explained in section 9.5.

Just as in (9.39) we use Lorentz invariance to deduce the relation

$$\int \mathrm{d}^n q q_\mu f(q,p) = a p_\mu \,,$$

where $f(q,p)$ is a Lorentz-invariant function of two momentum vectors. The coefficient a is found by contracting with p_μ, i.e.

$$\int \mathrm{d}^n q q_\mu f(q,p) = \frac{p_\mu}{p^2} \int \mathrm{d}^n q p \cdot q f(q,p) \,. \tag{9.48}$$

This is sufficient to generally decompose $\Sigma(p)$ in terms of two scalar functions according to (8.33)

$$\Sigma(p) = mA(p^2) + \mathrm{i}\not{p}B(p^2) \,,$$

where

$$A(p^2) = \frac{-\mathrm{i}e^2}{(2\pi)^n} \int \mathrm{d}^n q \frac{n + \lambda^{-2} - 1}{((q+p)^2 + m^2)q^2} \,, \tag{9.49}$$

$$\begin{aligned} B(p^2) \quad = \quad & \frac{-\mathrm{i}e^2}{(2\pi)^n} \int \mathrm{d}^n q \Big\{ \frac{(n-2)(1 + p \cdot q p^{-2})}{((q+p)^2 + m^2)q^2} \\ & -(1-\lambda^{-2}) \frac{q^2 + (p^2 + m^2) p \cdot q p^{-2}}{((q+p)^2 + m^2)q^4} \Big\} \,. \end{aligned} \tag{9.50}$$

Using (8.14) we find the identity

$$\begin{aligned} &(n-2) \int \mathrm{d}^n q \frac{p \cdot q}{((q+p)^2 + m^2)q^2} = \\ &\quad \int \mathrm{d}^n q \frac{p \cdot q}{((q+p)^2 + m^2)^2} - (p^2 + m^2) \int \mathrm{d}^n q \frac{p \cdot q}{((q+p)^2 + m^2)^2 q^2} \,. \end{aligned} \tag{9.51}$$

The first integral on the right-hand side can be simplified by shifting the integration variable from q to $q + p$ and dropping the term odd in q. This integral then reduces to an integral over $p^2(q^2 + m^2)^{-2}$. Both $A(p^2)$ and $B(p^2)$ can now be expressed in terms of the integral I that was defined in (9.23) and a function $\bar{B}$ which is free of ultraviolet divergences. The results take the form

$$A(p^2) = -\mathrm{i}e^2(n - 1 + \lambda^{-2}) I(p^2, m^2, 0) \,, \tag{9.52}$$

$$B(p^2) = -\mathrm{i}e^2 \Big\{ (n - 3 + \lambda^{-2}) I(p^2, m^2, 0) - I(0, m^2, m^2) - \frac{p^2 + m^2}{p^2} \bar{B}(p^2) \Big\} \,, \tag{9.53}$$

where

$$\bar{B}(p^2) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n q \frac{p \cdot q}{((q+p)^2 + m^2)q^2} \left(\frac{1}{(q+p)^2 + m^2} + \frac{1 - \lambda^{-2}}{q^2} \right). \tag{9.54}$$

The finite integral (9.54) can be done in 4 dimensions. First one uses (9.27) after which the momentum integration is straightforward (cf. 9.12). For later use we quote a slightly more general result than what is needed for (9.54), namely

$$\begin{aligned} &\int d^4 q \frac{p \cdot q}{((q+p)^2 + m^2)^2 (q^2 + \mu^2)} = \\ &-\mathrm{i}\pi^2 \Big\{ -1 + \int_0^1 \mathrm{d}x \frac{(p^2 + m^2 - \mu^2)x + \mu^2}{p^2 x(1-x) + m^2 x + \mu^2(1-x)} \Big\}, \\ &\int d^4 q \frac{p \cdot q}{((q+p)^2 + m^2)(q^2 + \mu_1^2)(q^2 + \mu_2^2)} = \\ &\quad -\mathrm{i}\pi^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{p^2 x}{p^2 x(1-x) + (m^2 - \mu_2^2)x + (\mu_1^2 - \mu_2^2)y + \mu_2^2}, \end{aligned} \tag{9.55}$$

For $\mu^2 = \mu_1^2 = \mu_2^2 = 0$ these results simplify, and one easily derives

$$\bar{B}(p^2) = \frac{\mathrm{i}}{16\pi^2} \Big\{ \lambda^{-2} - \Big(1 + \lambda^{-2} \frac{m^2}{p^2}\Big) \ln \frac{p^2 + m^2}{m^2} \Big\}, \tag{9.56}$$

which has a logarithmic singularity at $p^2 = -m^2$.

The function $I(p^2, m^2, 0)$ and $I(0, m^2, m^2)$ can be directly obtained from (9.26)

$$\begin{aligned} I(p^2, m^2, 0) &= -\frac{\mathrm{i}\mu^\varepsilon}{8\pi^2} \Big\{ \frac{1}{\varepsilon} + \tfrac{1}{2}\gamma_{\mathrm{E}} + \tfrac{1}{2} \ln \Big(\frac{m^2}{4\pi\mu^2} \Big) \\ &\quad + \tfrac{1}{2} \int_0^1 \mathrm{d}x \ln \frac{m^2 x + p^2 x(1-x)}{m^2} \Big\}, \\ I(0, m^2, m^2) &= -\frac{\mathrm{i}\mu^\varepsilon}{8\pi^2} \Big\{ \frac{1}{\varepsilon} + \tfrac{1}{2}\gamma_{\mathrm{E}} + \tfrac{1}{2} \ln \Big(\frac{m^2}{4\pi\mu^2} \Big) \Big\}. \end{aligned} \tag{9.57}$$

The integral over the logarithm is straightforward

$$\int_0^1 \mathrm{d}x \ln \frac{m^2 x + p^2 x(1-x)}{m^2} = -2 + \frac{p^2 + m^2}{p^2} \ln \frac{p^2 + m^2}{m^2}.$$

Combining all the various terms leads to the results quoted in (8.33, 8.34,8.35)

$$
\begin{aligned}
A(p^2) &= -\frac{e^2\mu^\varepsilon}{8\pi^2}(3+\lambda^{-2})\Big[\frac{1}{\varepsilon}+\tfrac{1}{2}\gamma_{\rm E}-\tfrac{1}{2}\ln 4\pi\Big]+A^f(p^2)\,,\\
B(p^2) &= -\frac{e^2\mu^\varepsilon}{8\pi^2}(\lambda^{-2})\Big[\frac{1}{\varepsilon}+\tfrac{1}{2}\gamma_{\rm E}-\tfrac{1}{2}\ln 4\pi\Big]+B^f(p^2)\,,
\end{aligned}
$$

where (carefully retaining ε/ε terms and suppressing the factor μ^ε)

$$
\begin{aligned}
A^f(p^2) &= \frac{e^2}{8\pi^2}\Big\{2+\lambda^{-2}-\tfrac{1}{2}(3+\lambda^{-2})\ln\frac{m^2}{\mu^2}\\
&\qquad -\tfrac{1}{2}(\lambda^{-2}+3)\frac{p^2+m^2}{p^2}\ln\frac{p^2+m^2}{m^2}\Big\}\,. \qquad (9.58)\\
B^f(p^2) &= \frac{e^2\lambda^{-2}}{8\pi^2}\Big\{1-\tfrac{1}{2}\ln\frac{m^2}{\mu^2}-\frac{p^2+m^2}{p^2}\Big(\tfrac{1}{2}+\ln\frac{p^2+m^2}{m^2}\Big)\\
&\qquad +\tfrac{1}{2}\Big(\frac{p^2+m^2}{p^2}\Big)^2\ln\frac{p^2+m^2}{m^2}\Big\}\,. \qquad (9.59)
\end{aligned}
$$

Because all these results do not yet refer to physical quantities they still contain the arbitrary gauge-fixing parameter λ. Furthermore we note that although (9.58) and (9.59) are finite when $p^2=-m^2$, these functions are not analytic at that point. Already the first derivative diverges at $p^2=-m^2$. Therefore, the proper definition of the probability amplitude according to the prescription presented in chapter 3 runs into difficulty, because the propagator will not exhibit a simple pole structure at the electron mass but a branch cut. The origin of this phenomenon is the masslessness of the photon. Let us therefore digress for a moment and discuss the regularization of these infrared divergences.

The standard way of dealing with infrared divergences is to introduce a finite photon mass in the propagator (there are alternative ways of treating infrared divergences; for instance it is possible to use dimensional regularization but this will require the calculation of cross sections and decay rates in n dimensions). This procedure is rather ad hoc, and in particular for propagators with arbitrary λ (cf. 4.42) which contain two different tensors, there seems a variety of ways to do this. For example, one could replace (4.42) by

$$
\varDelta_{\mu\nu}(k)\to\varDelta^{\rm reg}_{\mu\nu}(k)=\frac{1}{{\rm i}(2\pi)^4}\Big\{\frac{\eta_{\mu\nu}}{k^2+\kappa_1^2}-\frac{(1-\lambda^{-2})k_\mu k_\nu}{(k^2+\kappa_2^2)(k^2+\kappa_3^2)}\Big\}\,. \qquad (9.60)
$$

One particular form of (9.60) follows from straightforwardly introducing a mass term in the photon Lagrangian, i.e.

$$
\mathcal{L}=-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2-\tfrac{1}{2}(\lambda\partial_\mu A_\mu)^2-\tfrac{1}{2}\kappa^2A_\mu^2\,.
$$

This leads to the propagator (9.60) with

$$
\kappa_1^2=\kappa_2^2=\kappa^2\,,\qquad \kappa_3^2=\kappa^2\lambda^{-2}\,. \qquad (9.61)
$$

Because a photon mass term violates the gauge invariance of the original quantum-electrodynamics Lagrangian it is not immediately obvious that physical quantities remain independent of λ. However, detailed analysis based on the Ward-Takahashi identity shows that the λ-independence holds irrespective of the choice of the mass parameters κ_1, κ_2 and κ_3. We will not prove this result, but calculate the derivative of $A^f(p^2) - B^f(p^2)$ at $p^2 = -m^2$ using the regularization (9.60). Since we need this expression later on in the calculation, this will allow us to verify the λ-independence when putting all results together.

The calculation of $A^f - B^f$ starts from (9.49) and (9.50) where we now introduce infrared regulator masses according to (9.60)

$$A(p^2) - B(p^2) = \frac{-\mathrm{i}e^2}{(2\pi)^n} \int \mathrm{d}^n q \Big\{ \frac{2-(n-2)p\cdot q p^{-2}}{((q+p)^2+m^2)(q^2+\kappa_1^2)} + \frac{p^2+m^2}{p^2} \frac{(1-\lambda^{-2})p\cdot q}{((q+p)^2+m^2)(q^2+\kappa_2^2)(q^2+\kappa_3^2)} \Big\} . \qquad (9.62)$$

Subsequently one uses a slightly modified version of (9.51) and obtains a linear combination of $I(p^2, m^2, \kappa_1^2)$, $I(0, m^2, m^2)$ and the integrals given in (9.55). Using (9.26) and (9.55) leads to integrals over Feynman parameters; this result can easily be differentiated with respect to p^2. Afterwards one puts $p^2 = -m^2$ and extracts all contributions that do not vanish when the regulator masses tend to zero. Rather than explicitly presenting the calculation we give one intermediate result for the second integral of (9.55):

$$\int \mathrm{d}^4 q \frac{p\cdot q}{((q+p)^2+m^2)(q^2+\kappa_1^2)(q^2+\kappa_2^2)}\Big|_{p^2=-m^2} = -\tfrac{1}{2}\mathrm{i}\pi^2 \frac{1}{\kappa_1^2-\kappa_2^2} \Big\{ \kappa_1^2 - \kappa_2^2 + \kappa_1^2 \ln\frac{\kappa_1^2}{m^2} - \kappa_2^2 \ln\frac{\kappa_2^2}{m^2} + O(\kappa_1^3) + O(\kappa_2^3) \Big\} . \qquad (9.63)$$

Collecting all contributions from (9.62) the result that we need is

$$\frac{\mathrm{d}}{\mathrm{d}p^2}(A^f(p^2) - B^f(p^2))|_{p^2=-m^2} = \frac{e^2 m^{-2}}{32\pi^2} \Big\{ 3 - \lambda^{-2} + 2\ln\frac{\kappa_1^2}{m^2} + \frac{1-\lambda^{-2}}{\kappa_2^2-\kappa_3^2} \Big(\kappa_2^2 \ln\frac{\kappa_2^2}{m^2} - \kappa_3^2 \ln\frac{\kappa_3^2}{m^2} \Big) \Big\} \qquad (9.64)$$

where we have replaced A and B by their finite parts, because the infinite terms are independent of p^2.

(c) *Vertex diagram*

The relevant expression for the vertex diagram has been given in (8.29). To perform the manipulations with the γ matrices in n dimensions should by now be standard. For the first term in (8.29) we write

$$\begin{aligned}
& \gamma_\sigma[-\mathrm{i}(\not{q}+\not{p}')+m]\gamma_\mu[-\mathrm{i}(\not{q}+\not{p})+m]\gamma_\sigma \\
= \quad & \gamma_\mu[(2-n)q^2-4p\cdot p'-4p\cdot q-4p'\cdot q]-4\mathrm{i}mq_\mu \\
+ \quad & 4(p+p')_\mu\not{q}-2(2-n)q_\mu\not{q} \\
+ \quad & (\mathrm{i}\not{p}'+m)[-2m\gamma_\mu-4\mathrm{i}p_\mu-4\mathrm{i}q_\mu+\mathrm{i}(6-n)\gamma_\mu\not{q}] \\
+ \quad & [-2m\gamma_\mu-4\mathrm{i}p'_\mu-4\mathrm{i}q_\mu+\mathrm{i}(6-n)\not{q}\gamma_\mu](\mathrm{i}\not{p}+m) \\
+ \quad & (6-n)(\mathrm{i}\not{p}'+m)\gamma_\mu(\mathrm{i}\not{p}+m)\,. \qquad (9.65)
\end{aligned}$$

In the second term we again make substitutions such as $i\not{q}=(\mathrm{i}(\not{p}'+\not{q})+m)-(\mathrm{i}\not{p}'+m)$ and cancel as many propagators as possible. In this way one finds

$$\begin{aligned}
& q\frac{1}{1(p'+q)+m}\gamma_\mu\frac{1}{\mathrm{i}(\not{p}+\not{q})+m}\not{q}= \\
& -\gamma_\mu+\gamma_\mu\frac{1}{\mathrm{i}(\not{p}+\not{q})+m}(\mathrm{i}\not{p}+m) \\
& +(\mathrm{i}\not{p}'+m)\frac{1}{\mathrm{i}(\not{p}+\not{q})+m}\gamma_\mu \\
& \qquad -\mathrm{i}(\not{p}'+m)\frac{1}{\mathrm{i}(\not{p}'+\not{q})+m}\gamma_\mu\frac{1}{\mathrm{i}(\not{p}+\not{q})+m}(\mathrm{i}\not{p}+m)\,. \qquad (9.66)
\end{aligned}$$

One advantage of using (9.65) and (9.66) is that it is immediately obvious which terms contribute if the diagram is contracted with Dirac spinors for the external electron lines, because terms multiplied from the right by $\mathrm{i}\not{p}+m$ or from the left by $\mathrm{i}\not{p}'+m$ will vanish irrespective of whether we are dealing with electrons or positrons (note that these terms contain no ultraviolet divergences). In this section we intend to calculate the on-shell terms. To that order let us first consider the following integrals which are free of infrared divergences:

$$S_\mu(P,Q)=\frac{1}{(2\pi)^n}\int \mathrm{d}^nq\frac{q_\mu}{(q^2+2p'\cdot q)(q^2+2p\cdot q)q^2}\,, \qquad (9.67)$$

$$T_{\mu\nu}(P,Q)=\frac{1}{(2\pi)^n}\int \mathrm{d}^nq\frac{q_\mu q_\nu}{(q^2+2p'\cdot q)(q^2+2p.q)q^2}\,, \qquad (9.68)$$

where $p^2=p'^2=-m^2$ and

$$Q=p'-p\,, \qquad P=p'+p\,. \qquad (9.69)$$

Note that $P\cdot Q=0$ and $P^2+Q^2=-4m^2$. Integrals (9.67) and (9.68) have a decomposition in terms of P_μ and Q_μ according to

$$S_\mu(P,Q)=S(Q^2)P_\mu\,, \qquad (9.70)$$

$$T_{\mu\nu}(P,Q) = T_1(Q^2)\eta_{\mu\nu} + T_2(Q^2)P_\mu P_\nu + T_3(Q^2)Q_\mu Q_\nu \,, \tag{9.71}$$

where we have used that (9.67) and (9.68) are symmetric under $p' \leftrightarrow p$, so that there can be no term in (9.70) or (9.71) that is odd in Q.

Let us start by evaluating $T_{\mu\nu}(P,Q)$. Contraction with P_μ or Q_μ leads to

$$\begin{aligned} P_\mu T_{\mu\nu} = \frac{1}{(2\pi)^n}\int \mathrm{d}^n q \Big\{ \tfrac{1}{2}\frac{q_\nu}{(q^2+2p\cdot q)q^2} + \tfrac{1}{2}\frac{q_\nu}{(q^2+2p'\cdot q)q^2} \\ -\frac{q_\nu}{(q^2+2p'\cdot q)(q^2+2p\cdot q)}\Big\} \end{aligned} \tag{9.72}$$

$$Q_\mu T_{\mu\nu} = \frac{1}{(2\pi)^n}\int \mathrm{d}^n q \Big\{ \tfrac{1}{2}\frac{q_\nu}{(q^2+2p\cdot q)q^2} - \tfrac{1}{2}\frac{q_\nu}{(q^2+2p'\cdot q)q^2}\Big\} . \tag{9.73}$$

Each of these integrals is proportional to p_μ and/or p'_μ. This follows from using (9.48), so that:

$$\begin{aligned} \int \mathrm{d}^n q \frac{q_\mu}{(q^2+2p\cdot q)q^2} &= p_\nu p^{-2}\int \mathrm{d}^n q \frac{p\cdot q}{(q^2+2p\cdot q)q^2} \\ &= \tfrac{1}{2}p_\nu p^{-2}\int \mathrm{d}^n q \Big\{\frac{1}{q^2} - \frac{1}{q^2+2p\cdot q}\Big\} \\ &= \tfrac{1}{2}p_\nu p^{-2}\int \mathrm{d}^n q \Big\{\frac{1}{q^2} - \frac{1}{q^2+m^2}\Big\} , \end{aligned} \tag{9.74}$$

where we have shifted the integration variable in the second term. Likewise

$$\begin{aligned} &\int \mathrm{d}^n q \frac{q_\mu}{(q^2+2p'\cdot q)(q^2+2p\cdot q)} = \\ &\int \mathrm{d}^n q \frac{q_\nu - p'_\nu}{(q^2+m^2)((q-Q)^2+m^2)} = \\ &\int \mathrm{d}^n q \Big\{ \tfrac{1}{2}\frac{Q_\nu}{Q^2}\Big(-\frac{1}{q^2+m^2} + \frac{1}{(q-Q)^2+m^2} \\ &\quad + \frac{Q^2}{(q^2+m^2)((q-Q)^2+m^2)} - \frac{p'_\nu}{(q^2+m^2)((q-Q)^2+m^2)}\Big\} = \\ &\qquad\qquad -\tfrac{1}{2}P_\nu \int \mathrm{d}^n q \frac{1}{(q^2+m^2)((q-Q)^2+m^2)} . \end{aligned} \tag{9.75}$$

From (9.72)-(9.75) we conclude that

$$\begin{aligned} &T_1(Q^2) + P^2 T_2(Q^2) \\ = \;& \frac{1}{4m^2}\frac{1}{(2\pi)^n}\int \mathrm{d}^n q \Big\{ -\frac{1}{q^2} + \frac{1}{q^2+m^2} \\ &\qquad + \frac{2m^2}{(q^2+m^2)((q-Q)^2+m^2)}\Big\} , \end{aligned} \tag{9.76}$$

$$T_1(Q^2) + Q^2 T^3(Q^2) = \frac{1}{4m^2}\frac{1}{(2\pi)^n}\int \mathrm{d}^n q \Big\{ \frac{1}{q^2} - \frac{1}{q^2+m^2} \Big\}. \qquad (9.77)$$

Contracting $T_{\mu\nu}$ with $\eta_{\mu\nu}$ yields a third identity

$$nT_1(Q^2) + P^2 T_2(Q^2) + Q^2 T_3(Q^2) = \frac{1}{(2\pi)^n}\int \mathrm{d}^n q \frac{1}{(q^2+m^2)((q-Q)^2+m^2)}\,, \qquad (9.78)$$

so that we can now solve for T_1, T_2 and T_3.

The evaluation of S_μ goes in a similar way, and one finds

$$P^2 S(Q^2) = \frac{1}{(2\pi)^n}\int \mathrm{d}^n q \Big\{ \frac{1}{(q^2+2p\cdot q)q^2} - \frac{1}{(q^2+m^2)((q-Q)^2+m^2)} \Big\}. \qquad (9.79)$$

All results can now be expressed in terms of the function $I(k^2, m_1^2, m_2^2)$ which we have defined in (9.23), and can be summarized as follows

$$\begin{aligned} T_1(Q^2) &= \tfrac{1}{2}\frac{1}{n-2} I(Q^2, m^2, m^2)\,, \\ T_2(Q^2) &= \frac{1}{Q^2+4m^2}\Big\{ \tfrac{1}{4} I(0, m^2, 0) - \tfrac{1}{2}\frac{n-3}{n-2} I(Q^2, m^2, m^2) \Big\}\,, \\ T_3(Q^2) &= \frac{1}{Q^2}\Big\{ \tfrac{1}{4} I(0, m^2, 0) - \tfrac{1}{2}\frac{1}{n-2} I(Q^2, m^2, m^2) \Big\}\,, \\ S(Q^2) &= \frac{1}{Q^2+4m^2}\Big\{ - I(-m^2, m^2, 0) + I(Q^2, m^2, m^2) \Big\}\,. \qquad (9.80) \end{aligned}$$

Since $I(Q^2, m^2, m^2)$ satisfies the equations

$$\begin{aligned} & I(0, m^2, 0) = \frac{2}{n-2} I(0, m^2, m^2)\,, \\ & I(-m^2, m^2, 0) = \frac{1}{n-3} I(0, m^2, m^2)\,, \\ & I(-4m^2, m^2, m^2) = \frac{1}{n-3} I(0, m^2, m^2)\,, \qquad (9.81) \end{aligned}$$

one can deduce that the expressions (8.81) do not exhibit poles at $Q^2 = 0$ or $Q^2 = -4m^2$. Furthermore it follows that the divergences for $n \to 4$ are entirely contained in T_1.

Collecting the various terms leads to

$$\begin{aligned}
&\Lambda_\mu(p',p) = \\
&e^3\Big(\gamma_\mu\Big\{-(n-2)[(n-2)T_1(Q^2)-(Q^2+4m^2)T_2(Q^2)+Q^2T_3(Q^2)] \\
&+2(Q^2+2m^2)J(-Q^2,-m^2,-m^2) \\
&+4(Q^2+4m^2)S(Q^2)+\frac{1-\lambda^{-2}}{(2\pi)^n}\int\frac{\mathrm{d}^nq}{q^4}\Big\} \\
&\qquad -4(\mathrm{i}mP_\mu-P_\mu\not{P})S(Q^2)+2(n-2)[T_2(Q^2)P_\mu\not{P}+T_3(Q^2)Q_\mu\not{Q}]\Big)\,,
\end{aligned} \tag{9.82}$$

where we have ignored all terms proportional to $(i\not{p}+m)$ or $(i\not{p}'+m)$ since they cancel when we contract $\Lambda_\mu(p',p)$ with Dirac spinors. Both the function $J(-Q^2,-m^2,-m^2)$, defined in (9.29), and the integral $\int \mathrm{d}^nq{-4}$ are infrared divergent. Assuming the regularization (9.60) we must use a photon mass κ_1 in $J(-Q^2,-m^2,-m^2)$; the integral over q^{-4} is replaced by an integral over $((q^2+\kappa_2)(q^2+\kappa_3))^{-1}$.

We now add (9.82) to the lowest-order vertex, and contract the full vertex function with Dirac spinors. Using the Dirac equation for the spinors one easily verifies that

$$\begin{aligned}
&\bar u(\mathbf{p}')\not{P}u(\mathbf{p}) = 2\mathrm{i}m\bar u(\mathbf{p}')u(\mathbf{p})\,, \qquad \bar u(\mathbf{p}')\not{Q}u(\mathbf{p})=0\,, \\
&P_\mu\bar u(\mathbf{p}')u(\mathbf{p}) = u(\mathbf{p}')[2im\gamma_\mu+\tfrac12(\gamma_\mu\not{Q}-\not{Q}\gamma_\mu)]u(\mathbf{p})\,,
\end{aligned}$$

These identities also hold for v spinors. Hence the full vertex function can be written as

$$\bar u(\mathbf{p}')[-\mathrm{i}e\gamma_\mu+\Lambda_\mu(p',p)]u(\mathbf{p}) = -\mathrm{i}e\bar u(\mathbf{p}')\Big\{F_1(Q^2)\gamma_\mu-2mF_2(Q^2)\sigma_{\mu\nu}Q_\nu\Big\}u(\mathbf{p}) \tag{9.83}$$

with

$$\begin{aligned}
&F_1(Q^2)+F_2(Q^2) \\
&=1+\mathrm{i}e^2\Big\{-(n-2)[(n-2)T_1(Q^2)-(Q^2+4m^2)T_2(Q^2)+Q^2T_3(Q^2)] \\
&\qquad +2(Q^2+2m^2)J(-Q^2,-m^2,-m^2) \\
&\qquad +4(Q^2+4m^2)S(Q^2)+(1-\lambda^{-2})I(0,\kappa_2^2,\kappa_3^2)\Big\} \\
&=1+\mathrm{i}e^2\Big\{(7-n)I(Q^2,m^2,m^2)-\frac{4}{n-3}I(0,m^2,m^2) \\
&\qquad +2(Q^2+2m^2)J(-Q^2,-m^2,-m^2)+(1-\lambda^{-2})I(0,\kappa_2^2,\kappa_3^2)\Big\}\,,
\end{aligned} \tag{9.84}$$

$$\begin{aligned} F_2(Q^2) &= 8\mathrm{i}m^2e^2[S(Q^2)+(n-2)T_2(Q^2)] \\ &= \frac{\mathrm{i}m^2e^2(5-n)}{m^2+\frac{1}{4}Q^2}\Big(I(Q^2,m^2,m^2)-\frac{1}{n-3}I(0,m^2,m^2)\Big)\,, \end{aligned} \tag{9.85}$$

where we have used (9.80)-(9.81). Inserting (9.36) for the function J and the following result for $I(Q^2,m^2,m^2)$ (cf. 9.26)

$$\begin{aligned} I(Q^2,m^2,m^2) &= -\frac{\mathrm{i}\mu^\varepsilon}{8\pi^2}\Big\{\frac{1}{\varepsilon}+\tfrac{1}{2}\gamma_{\mathrm{E}}+\tfrac{1}{2}\ln\Big(\frac{m^2}{4\pi\mu^2}\Big) \\ &\qquad +\tfrac{1}{2}\int_0^1 \mathrm{d}v\ln\frac{m^2+\frac{1}{4}Q^2(1-v^2)}{m^2}\Big\} \end{aligned} \tag{9.86}$$

(we changed the integration variable to $v=2x-1$), one finds the result quoted in (8.36)

$$\begin{aligned} F_1(Q^2) &= -\frac{e^2\mu^\varepsilon\lambda^{-2}}{8\pi^2}\Big[\frac{1}{\varepsilon}+\tfrac{1}{2}\gamma_{\mathrm{E}}-\tfrac{1}{2}\ln 4\pi\Big]+F_1^f(Q^2)\,, \\ F_2(Q^2) &= F_2^f(Q^2)\,, \end{aligned} \tag{9.87}$$

F_1^f and F_2^f are equal to

$$\begin{aligned} F_1^f(Q^2) &= 1+\frac{e^2}{8\pi^2}\Big\{\tfrac{1}{2}\lambda^{-2}\Big(1-\ln\frac{m^2}{\mu^2}\Big)+\tfrac{3}{2}+\tfrac{1}{2}\frac{1-\lambda^{-2}}{\kappa_2^2-\kappa_3^2}\Big(\kappa_2^2\ln\frac{\kappa_2^2}{m^2}-\kappa_3^2\ln\frac{\kappa_3^2}{m^2}\Big) \\ &+ \tfrac{1}{4}\frac{Q^2}{m^2+\frac{1}{4}Q^2}+\frac{m^2+\frac{3}{8}Q^2}{m^2+\frac{1}{4}Q^2}\int_0^1\mathrm{d}v\ln\frac{m^2+\frac{1}{4}Q^2(1-v^2)}{m^2} \\ &- \int_0^1\mathrm{d}v\frac{m^2+\frac{1}{2}Q^2}{m^2+\frac{1}{4}Q^2(1-v^2)}\ln\frac{m^2+\frac{1}{4}Q^2(1-v^2)}{\kappa_1^2}\Big\}\,, \end{aligned} \tag{9.88}$$

$$F_2^f(Q^2)=\frac{e^2}{8\pi^2}\frac{m^2}{m^2+\frac{1}{4}Q^2}\Big\{1+\tfrac{1}{2}\int_0^1\mathrm{d}v\ln\frac{m^2+\frac{1}{4}Q^2(1-v^2)}{m^2}\Big\}\,, \tag{9.89}$$

where we have suppressed a factor μ^ε and were careful not to drop terms of order ε. This completes the calculation of all divergent one-loop diagrams in quantum electrodynamics (see problem 9.2). The infinities found above agree with the results quoted in (8.31)-(8.36). There we have discussed in detail how to render the theory finite by absorbing the infinities into the parameters of the original Lagrangian. The renormalization procedure therefore yields finite answers; using the counterterms given in (8.50) the resulting finite expressions are given in (9.43), (9.58), (9.59), (9.88) and (9.89). In this connection it is important to observe that (9.85) is free of ultraviolet divergences. This result is crucial for the renormalizability of the theory, because the initial quantum-electrodynamics Lagrangian does not contain an interaction term corresponding to (9.85). Such a term would be of the form $F_{\mu\nu}\bar\psi\sigma_{\mu\nu}\psi$ with a coupling constant of dimension $[\text{mass}]^{-1}$. If (9.85) had been infinite then such a coupling should have been introduced in the Lagrangian in order to absorb the infinity. However, this coupling would in fact worsen the situation in higher orders where it leads to new divergences requiring more and more new terms in the Lagrangian, so that all predictive power of the theory is lost.

Although well-defined, the above results are not yet of direct physical relevance, because they pertain to Green's functions rather than to physical amplitudes. This explains why most of the expressions still depend on the arbitrary gauge parameter λ. The next section will show how to normalize the vertex function and obtain physically meaningful results. The troublesome infrared divergences will be dealt with in section 9.6.

9.5. The eeγ amplitude

In this section we describe how to extract some physically meaningful results from the previous calculations. In particular we will determine the eeγ amplitude expressed in terms of the fine-structure constant and the electron mass. We recall from section 8.3 that the parameters e, m and λ that occur in the (finite) expressions that we have just been calculating, no longer refer to the parameters of the original quantum-electrodynamics Lagrangian. We have been absorbing the ultraviolet infinities in the latter parameters according to the renormalization prescription given in (8.48) and (8.50). The parameters e and m are implicitly defined by this procedure, and do not coincide with the electric charge and the mass of the electron. The precise relation will be derived shortly.

Already in section 2.6 we have discussed how to sum self-energy diagrams and to include them into the propagator. The result is the Dyson equation (2.64) which can easily be generalized to spin-1 and spin-$\frac{1}{2}$ fields; one finds that the inverse photon propagator in one-loop order is given by

$$(\Delta_{\mu\nu}(k))^{-1} = \mathrm{i}(2\pi)^4\{(k^2\eta_{\mu\nu} - k_\mu k_\nu)[1 + \Pi^f(k^2)] + \lambda^2 k_\mu k_\nu\}\,, \tag{9.90}$$

where the sign of $\Pi^f(k^2)$ follows from the normalization of the vacuum polarization adopted in (8.27). Similarly for the inverse electron propagator we find

$$(\Delta(p))^{-1} = \mathrm{i}(2\pi)^4\{m[1 + A^f(k^2)] + \mathrm{i}\not{p}[1 + B^f(p^2)]\}\,. \tag{9.91}$$

Note that the normalization adopted for the vacuum polarization and the self-energy functions, Π^f, A^f and B^f, is such that these terms can be added directly to the lowest-order results.

The photon propagator that follows from (9.90)

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\Big\{\Big(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\frac{1}{1+\Pi^f(k^2)} + \lambda^{-2}\frac{k_\mu k_\nu}{k^2}\Big\}\,, \tag{9.92}$$

still exhibits a pole at $k^2 = 0$. Consequently the photon remains massless.

In order to normalize physical amplitudes with photons correctly one needs the residue of the propagator at $k^2 = 0$. However, in this case, the residue factor is

$$z_A = (1 + \Pi^f(0))^{-1} = 1 + \frac{e^2}{12\pi^2} \ln \frac{m^2}{\mu^2} + O(e^4) \,. \tag{9.93}$$

Note that z_A is a finite quantity in contradistinction with the renormalization constant Z_A defined in (8.50) which contains an ultraviolet divergent term.

The electron propagator corresponding to (9.91)

$$\Delta(p) = \frac{1}{\mathrm{i}(2\pi)^4} \frac{1}{\mathrm{i}\not{p}[1 + B^f(p^2)] + m[1 + A^f(p^2)]} \,, \tag{9.94}$$

no longer has a pole at $p^2 = -m^2$, but at $p^2 = -M^2$, where M is defined by

$$M[1 + B^f(-M^2)] = m[1 + A^f(-M^2)] \,. \tag{9.95}$$

In perturbation theory it is easy to solve this equation and one finds that the physical electron mass equals

$$\begin{aligned} M \ &= m[1 + A^f(-m^2) - B^f(-m^2)] + O(e^4) \\ &= m\Big[1 + \tfrac{e^2}{8\pi^2}\Big(2 - \tfrac{3}{2}\ln\tfrac{m^2}{\mu^2}\Big) + O(e^4)\Big] \,. \end{aligned} \tag{9.96}$$

Since the electron mass M is a true physical quantity it should not depend on the gauge parameter λ, and indeed the λ-dependent terms contained in A^f and B^f cancel precisely. This is not the case for the residue factor at the pole. To determine this factor we write

$$\begin{aligned} \Delta(p) = \frac{1}{\mathrm{i}(2\pi)^4} \frac{1}{\mathrm{i}\not{p} + M} \times \\ \frac{1}{1 + B^f(-M^2) + [ipB^{f\prime}(-M^2) + mA^{f\prime}(-M^2)](-ip + M) + O(p^2 + M^2)} \,, \end{aligned} \tag{9.97}$$

where $A^{f'}$ and $B^{f'}$ are the derivatives of A^f and B^f with respect to p^2. At the pole the residue factor is thus equal to

$$\begin{aligned} z_\psi &= [1 + B^f(-M^2) - 2M(MB^{f'}(-M^2) - mA^{f'}(-M^2))]^{-1} \,, \\ &= \ 1 - \frac{e^2}{16\pi^2}\Big\{3 + \lambda^{-2}\Big(1 - \ln\frac{m^2}{\mu^2}\Big) + 2\ln\frac{\kappa_1^2}{m^2} \\ &\quad + \frac{1-\lambda^{-2}}{\kappa_2^2 - \kappa_3^2}\Big(\kappa_2^2 \ln\frac{\kappa_2^2}{m^2} - \kappa_3^2 \ln\frac{\kappa_3^2}{m^2}\Big)\Big\} + O(e^4) \,, \end{aligned} \tag{9.98}$$

where we have used (9.59), (9.64) and (9.96). For cut-off masses κ_1, κ_2 and κ_3 that approach zero uniformly (9.98) remains finite if $\lambda^2 = \frac{1}{3}$ (Fried-Yennie gauge). Note again that z_ψ is finite in contradistinction with Z_ψ, defined in (8.50).

In order to determine the correct normalization for the eeγ amplitude we recall from chapter 3 that all the external lines must be normalized by the square root of the residue factor of the corresponding propagator. Let us briefly summarize how this criterion arises. First one multiplies the vertex function by the full propagators, i.e.

$$\varDelta_{\mu\nu}(Q)\varDelta(p')[-ie\gamma_\nu + \Lambda_\nu(p',p)]\varDelta(p)\,. \tag{9.99}$$

In the approximation that we are working in this means that we sum the diagrams shown in fig. 9.4. When putting external lines on-shell, after extracting propagator poles and sandwiching between fermion spinors, one has

$$z_A z_\psi^2 \bar{u}(\mathbf{p}')[-\mathrm{i}e\gamma_\mu + \Lambda_\mu(p',p)]u(\mathbf{p})\,.$$

Figure 9.4: Feynman diagrams contributing to (9.99) to order e^3.

However, in order to have propagators with the residue factor equal to unity, we must multiply all fields, and therefore divide the corresponding external lines, by $\sqrt{z_A}$ or $\sqrt{z_\psi}$. Hence we are left with the amplitude

$$\mathcal{M}(\mathrm{ee}\gamma) = -\mathrm{i}e\sqrt{z_A}z_\psi\bar{u}(\mathbf{p}')\Big(F_1(Q^2)\gamma_\mu - \frac{1}{2m}F_2(Q^2)\sigma_{\mu\nu}Q_\nu\Big)u(\mathbf{p})\,. \tag{9.100}$$

This implies that the contribution of the self-energy graphs in fig. 9.4 is reduced by a factor $\frac{1}{2}$.

The electric charge of the electron (or positron) is defined as the amplitude for emitting or absorbing a zero-frequency photon. Therefore the charge is given by

$$\begin{aligned} e_P &= e\sqrt{z_A}z_\psi F_1^f(0) \\ &= e\Big[1 + \frac{e^2}{24\pi^2}\ln\frac{m^2}{\mu^2} + O(e^4)\Big]\,. \end{aligned} \tag{9.101}$$

Note that the vertex and propagator corrections cancel each other in (9.101) and that the same cancellation occurs between the infinite parts; according to (8.49) the infinite part of the vertex function is given by $(Z_e Z_\psi Z_A^{1/2} - 1)$, which is indeed equal to $(Z_\psi - 1)$ as shown in (8.50). This result can be generalized to all orders based on the celebrated Ward-Takahashi identity for the vertex function (see problem 7.3).

Obviously, both e and m are just parameters, which have been defined implicitly by our choice of the infinite renormalization constants specified in (8.50). Because they lack an intrinsic meaning it is more sensible to express the above results in terms of the actual physical charge and mass parameters. This is trivial to do in perturbation theory. Decomposing the amplitude (9.100) according to

$$\mathcal{M}(\mathrm{ee}\gamma) = -\mathrm{i}e_P \bar{u}(\mathbf{p}')\Big(f_1(Q^2)\gamma_\mu - \frac{1}{2M} f_2(Q^2)\sigma_{\mu\nu}Q_\nu\Big)u(\mathbf{p}) \,, \tag{9.102}$$

a combination of the previous results yields the charge form factor, which measures the charge density of the electron in units of the electron charge,

$$\begin{aligned} f_1(Q^2) = \\ 1 + \frac{\alpha}{8\pi}\int_0^1 \frac{\mathrm{d}v}{M^2 + \frac{1}{4}Q^2(1-v^2)}\Big(Q^2(1+2v^2) \\ + Q^2(1+v^2)\ln\frac{\kappa_1^2}{M^2} \\ - 4(M^2 + \tfrac{1}{2}Q^2)\ln\frac{M^2 + \frac{1}{4}Q^2(1-v^2)}{M^2}\Big) + O(\alpha^2)\,, \end{aligned} \tag{9.103}$$

where we have integrated by parts on one term and have introduced the fine-structure constant $\alpha = e_\mathrm{P}^2/4\pi$. This result no longer depends on the gauge parameter λ. Furthermore the κ_2 and κ_3 dependent terms have cancelled so the infrared divergence is associated only with the cut-off mass κ_1. This can be understood from the fact that κ_2 and κ_3 regularize the longitudinal part of the photon propagator (cf. 9.60), which is related to unphysical photon degrees of freedom. Therefore these terms should not contribute to a physical quantity such as (9.103). The cut-off mass κ_1 on the other hand corresponds to the physical degrees of freedom of the photon, and can still be present at this point; the corresponding infrared divergence will be dealt with in section 8.6.

The anomalous magnetic moment form factor is already of order α. It measures the anomalous magnetic moment in Bohr magnetons $\frac{1}{2}e_\mathrm{P}/M$. After a partial integration of (9.89) we find

$$f_2(Q^2) = \frac{\alpha}{2\pi}\int_0^1 \frac{\mathrm{d}v}{1 + \frac{1}{4}(1-v^2)Q^2/M^2} + O(\alpha^2)\,, \tag{9.104}$$

(note that this integral and also 9.103 becomes complex for $Q^2 < -4M^2$). The anomalous magnetic moment is given by (9.104) taken at $Q^2 = 0$, The corresponding integral is trivial and we find

$$a_e^{(2)} = \frac{\alpha}{2\pi}\,. \tag{9.105}$$

By the same token we have also determined the anomalous magnetic moment of the muon, because (9.104) does not depend on the fermion mass (hence $a_e^{(2)} = a_\mu^{(2)}$. This feature does not persist in higher orders.

Figure 9.5: Diagrams contributing to the anomalous magnetic moment of the electron and muon in order α^2.

The program for carrying out the calculation of higher order terms in the power series expansion for the anomalous magnetic moment of the electron and muon has occupied many physicists over the past twenty years. In fourth order there are seven new graphs which contribute to a_e namely those shown in fig. 9.5. An analytical result is known for a_e in this order, namely

$$a_e^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 [\tfrac{197}{144} + \tfrac{1}{12}\pi^2 + \tfrac{3}{4}\zeta(3) - \tfrac{1}{2}\pi^2 \ln 2] \tag{9.106}$$

where $\zeta(3) = \sum_n n^{-3} = 1.20206$ is the Riemann zeta function. Numerically the fourth-order term yields

$$a_e^{(4)} = -0.328478966\left(\frac{\alpha}{\pi}\right)^2 . \tag{9.107}$$

The sixth-order contribution to a_e for the electron involves the evaluation of seventy-two Feynman graphs. Analytic expressions are known for some of them, but for the others it has not been possible to find analytical results and the integrals are computed numerically. At present the best result seems to be

$$a_e^{(6)} = 1.1765(13)\left(\frac{\alpha}{\pi}\right)^3 , \tag{9.108}$$

where the number enclosed in the brackets represents the uncertainty in the final two digits. If one uses the currently accepted value of the fine-structure constant as measured by the ac Josephson effect, $\alpha^{-1} = 137.035963(15)$, then the prediction for $a_e = a_e^{(2)} + a_e^{(4)} + a_e^{(6)}$ is

$$a_e = 1\,159\,652\,459(34) \times 10^{-12} . \tag{9.109}$$

Extremely accurate experiments have been performed to measure a_e. The latest results are

$$\begin{aligned} a_{e^-} &= 1\,159\,652\,200(40) \times 10^{-12} , \\ a_{e^+} &= 1\,159\,652\,222(50) \times 10^{-12} , \end{aligned} \tag{9.110}$$

which are not in complete agreement with theory. Kinoshita and Lindquist, in a heroic effort, have now calculated numerically the 891 Feynman diagrams which contribute in eighth order of quantum electrodynamics. They find that

$$a_e^{(8)} = (-0.8 \pm 2.5)\left(\frac{\alpha}{\pi}\right)^4 .$$

Adding very small contributions from muons (2.8×10^{-12}), tau leptons (0.1×10^{-12}) and hadrons ($1.6(2) \times 10^{-12}$) which appear in some of the closed loops inside the higher-order graphs, the theoretical result is

$$a_e = 1\,159\,652\,459(43) \times 10^{-12} ,$$

which is now consistent with the experimental result at the two standard deviation level.

One can turn the argument around and use theory to redetermine the value for α. This leads to $\alpha^{-1} = 137.035993(10)$ which is in excellent agreement with α^{-1} determined by measuring the hyperfine splitting in muonium (the same quantity is also measured using the quantum Hall effect).

9.6. Radiative corrections

In the previous section we have seen that it is possible to extract meaningful results from a renormalizable field theory in spite of the fact that quantum corrections contain infinities associated with large momenta, and thus short distances. However, the charge form factor is still plagued by infrared divergences, caused by the masslessness of the photon. These divergences, associated with large distances, are related to the fact that a charged particle can continuously radiate or absorb zero-frequency photons. In this process both the charged particle and the photon are physical, i.e. they remain on their respective mass shells.

In chapter 4 we already discussed the inherent difficulty in defining experimental quantities caused by the photon having zero mass. We noted that the bremsstrahlung graphs diverge for zero photon momenta. Experimentally the bremsstrahlung events are counted as contributions to a nonradiative process as soon as the photon momenta are below an experimental threshold. Therefore they must be included in a theoretical determination of the nonradiative rate. When we calculate cross sections and decay rates in this way soft-photon divergences cancel with the infrared divergences in the loop graphs, and we are left with finite terms which are called radiative corrections. These corrections often depend sensitively on the details of the experimental set-up.

We consider here the radiative corrections to the reaction $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$, since precision tests of quantum electrodynamics have been made on this process; these corrections are also relevant for determining the electroweak asymmetry, which was discussed in section 6.7. To keep matters as simple as possible, we concentrate on the corrections to the total cross section; in order to discuss the differential cross section one also has to deal with the detection efficiency for the muon momenta, since the inability to detect the emission of soft photons is directly related to the inability to determine the muon momenta with infinite precision.

The lowest-order diagrams for $e^+e^- \to \mu^+\mu^-$ have been discussed in section 6.7. In next order there are the diagrams shown in fig. 9.6, which we have subdivided into three sets: (a) the single-photon exchange diagrams with self-energy and vertex insertions, (b) the external radiation diagrams, and (c) the two-photon exchange graphs and the single-photon exchange graph with vacuum polarization insertion. since we want to concentrate on the infrared divergent terms we drop the diagrams of class (c). Although the contribution of the vacuum polarization graphs could be simply included in the calculation of the graphs of class (a) we refrain from doing so in view of certain applications later on.

The graphs in (a) and (b) decompose into corrections to the electron lines and corrections to the muon lines, which are clearly separate. Therefore it suffices to consider only the one-loop corrections to the "decay" of a (virtual) time-like photon γ^*, i.e. the diagrams of fig. 9.4. The relevant amplitude was already given in (9.101), but the spinors change because we are dealing with an outgoing particle and an outgoing antiparticle:

$$\mathcal{M}_\rho(\gamma^* \to \mu^+\mu^-) = \mathrm{i}e\bar{u}(\mathbf{p}')\Big(\gamma_\rho[f_1(Q^2) + f_2(Q^2)] + \frac{\mathrm{i}}{2m} f_2(Q^2) P_\rho\Big) v(\mathbf{p})\,, \tag{9.111}$$

where $Q = p + p'$, $P = p - p'$, and m, e are the muon mass and electric charge, respectively (i.e. the *physical* mass and charge containing the effects of quantum corrections). Call

$$\begin{aligned} &f_1(-s) + f_2(-s) = 1 + \frac{\alpha}{\pi} C(s)\,, \\ &f_2(-s) = \frac{\alpha}{\pi} D(s)\,, \qquad s = -Q^2 > 4m^2\,, \end{aligned} \tag{9.112}$$

then it is easy to determine the square of (9.111), summed over the muon spin polarizations, from the results of section 6.2,

$$\begin{aligned} \sum_{\text{spins}} \mathcal{M}_\rho \overline{\mathcal{M}}_\sigma = 4e^2 \Big\{ &[p_\rho p'_\sigma + p'_\rho p_\sigma + (m^2 - p\cdot p')\eta_{\rho\sigma}]\Big(1 + \frac{2\alpha}{\pi}\mathrm{Re}\, C\Big) \\ &+ \frac{\alpha}{\pi} P_\rho P_\sigma \mathrm{Re}\, D + O(\alpha^2)\Big\}\,. \end{aligned} \tag{9.113}$$

In order to obtain the total cross section we must integrate this result over the fermion momenta. To proceed we observe that

$$\mathit{\Gamma}^{(a)}_{\rho\sigma} = \frac{1}{(2\pi)^2} \int \frac{\mathrm{d}^3 p}{2\omega(\mathbf{p})} \frac{\mathrm{d}^3 p'}{2\omega(\mathbf{p}')} \delta^4(p + p' - Q) \sum_{\text{spins}} \mathcal{M}_\rho \bar{\mathcal{M}}_\sigma \tag{9.114}$$

must be a real Lorentz-covariant tensor depending only on the remaining momentum vector Q_μ. Therefore there are two independent structures, namely ($\eta_{\rho\sigma}$ and $Q_\rho Q_\sigma$ which can be projected out of (9.114) by using the methods of section 9.4. The resulting integrals can then be calculated by going to the centre-of-mass frame of the "decaying" photon, where $Q = \sqrt{s}(0,0,0,\mathrm{i})$, $p = \frac{1}{2}\sqrt{s}(\beta_0, 0, 0, \mathrm{i})$, $p' = \frac{1}{2}\sqrt{s}(0,0,0,\mathrm{i})$. Here β_0 is the fermion velocity in the centre-of-mass frame, $\beta_0 = \sqrt{1 - 4m^2/s}$. Subsequently one follows the same approach as for the phase space integrals in section 3.4. However, in this particular case there is a short-cut, because we know that the original amplitude is conserved: $Q_\rho \mathcal{M}_\rho = 0$. Therefore (9.114) must be proportional to the tensor $\eta_{\rho\sigma} Q^2 - Q_\rho Q_\sigma$. Hence we write

$$\mathit{\Gamma}^{(a)}_{\rho\sigma} = \tfrac{1}{3}\Big(\eta_{\rho\sigma} - \frac{Q_\rho Q_\sigma}{Q^2}\Big) \mathit{\Gamma}^{(a)}_{\tau\tau}\,, \tag{9.115}$$

Figure 9.6: Diagrams contributing to the reaction $e^+e^- \to \mu^+\mu^-$ in order α^2: (a) the self-energy and vertex insertions, (b) the external radiation graphs and (c) the vacuum polarization and two-photon exchange graphs.

with

$$\Gamma^{(a)}_{\tau\tau} = \frac{4e^2}{(2\pi)^2}\int \frac{\mathrm{d}^3p}{2\omega(\mathbf{p})}\frac{\mathrm{d}^3p'}{2\omega(\mathbf{p}')}\delta^4(p+p'-Q) \\ \times\Big\{(2m^2-Q^2)\Big(1+\frac{2\alpha}{\pi}\mathrm{Re}\,C\Big)-\frac{\alpha}{\pi}(4m^2+Q^2)\mathrm{Re}\,D+O(\alpha^2)\Big\}, \tag{9.116}$$

where we made use of (9.113), $2p\cdot p' = Q^2+2m^2$ and $P^2=(p-p')^2 = -Q^2-4m^2$. Substituting expression (3.49) for the two-particle phase space integral, and expressing the result in terms of β_0, one arrives at

$$\Gamma^{(a)}_{\tau\tau} = 2\alpha s\Big\{\tfrac{1}{2}\beta_0(3-\beta_0^2)\Big(1+\frac{2\alpha}{\pi}\mathrm{Re}\,C\Big)+\frac{\alpha}{\pi}\beta_0^3\mathrm{Re}\,D\Big\}+O(\alpha^3)\,. \tag{9.117}$$

Using definitions (9.102), (9.104) and (9.112) and the evaluation of the relevant integrals over Feynman parameters given in appendix D, one finds the following expressions

$$\begin{aligned}\mathrm{Re}\,C = -1+\tfrac{1}{2}\Big(\tfrac{1+\beta_0^2}{\beta_0}\ln\tfrac{1+\beta_0}{1-\beta_0}-2\Big)\ln\tfrac{\kappa_1}{m}-\tfrac{3}{4}\beta_0\ln\tfrac{1-\beta_0}{1+\beta_0} \\ +\tfrac{1+\beta_0^2}{2\beta_0}\Big[\ln\tfrac{2\beta_0}{1+\beta_0}\ln\tfrac{1-\beta_0}{1+\beta_0}-\tfrac{1}{4}\ln^2\tfrac{1-\beta_0}{1+\beta_0}+\tfrac{\pi^2}{3}+\mathrm{Li}_2\Big(\tfrac{1-\beta_0}{1+\beta_0}\Big)\Big]\,, \\ \mathrm{Re}\,D = \tfrac{1-\beta_0^2}{4\beta_0}\ln\tfrac{1-\beta_0}{1+\beta_0}\,,\end{aligned} \tag{9.118}$$

which can be substituted straightforwardly into (9.117).

The rate for the decay of the virtual photon into fermions follows from (9.114), contracted with polarization vectors and divided by $2\omega(\mathbf{Q})$. On the other hand, the decay rate is directly proportional to the imaginary part of the photon self-energy diagram. Comparing the relevant expressions with (3.73) and (3.76) one concludes that (9.114) must be related to the imaginary part of the vacuum polarization graphs according to

$$\mathrm{Im}(\Pi_{\rho\sigma}(Q)) = \tfrac{1}{2}\Gamma^{(a)}_{\rho\sigma}(Q)+\cdots\,, \tag{9.119}$$

where the dots indicate contributions from other decay reactions, such as $\gamma^*\to\mu^+\mu^-\gamma$. To order α the result (9.119) can be verified by comparing (9.117) to (9.44).

In order to cancel the infrared divergence in (9.118) we must consider the bremsstrahlung reaction $\gamma^*\to\mu^+\mu^-\gamma$ described by the diagrams shown in fig. 9.7. For any experimental set-up there is a range of photon energies and angles where the detection efficiency becomes negligibly small (this includes photons with an infinitesimally small energy). Such events are then counted as $\gamma^*\to\mu^+\mu^-$ events, which should thus be included in a theoretical determination of the $\gamma^*\to\mu^+\mu^-$ event rate. As stated in section 4.6, the infrared divergence of the soft photon bremsstrahlung then cancels against the infrared

Figure 9.7: The diagrams contributing to the reaction $\gamma^* \to \mu^+\mu^-\gamma$.

divergence contained in (9.118). For the moment we will keep the photon energy and angle fixed, and evaluate the rate for $\gamma^* \to \mu^+\mu^- + \gamma$. We denote the momentum and polarization index of the extra photon by k and λ, respectively. Since the diagrams of fig. 9.7 diverge for small k, we assign a small mass to the photon equal to the photon mass κ_1 used in the virtual corrections. The corresponding amplitude for this process is

$$\mathcal{M}_{\rho\lambda}(\gamma^* \to \mu^+\mu^-\gamma) = \mathrm{i}e^2\bar{u}(\mathbf{p}')\Big(\frac{2p'_\lambda + \gamma_\lambda \not{k}}{2k\cdot p' + k^2}\gamma_\rho - \gamma_\sigma\frac{2p_\lambda + \not{k}\gamma_\lambda}{2k\cdot p + k^2}\Big)v(\mathbf{p})\,, \tag{9.120}$$

where we have used the Dirac equation for the spinors to remove terms that explicitly depend on the fermion mass. In this form it is straightforward to see that the amplitude is conserved with respect to both photons separately, i.e.

$$k_\lambda \mathcal{M}_{\rho\lambda} = 0\,, \qquad Q_\rho\mathcal{M}_{\rho\lambda} = 0\,, \qquad Q = p + p' + k\,. \tag{9.121}$$

To verify the second identity requires some manipulations involving again the Dirac equation (note that with different assignments of spinors and external momenta, (9.120) is just the (off-shell) amplitude for Compton scattering). Subsequently we square the amplitude and sum over the fermion spins

$$\begin{aligned}\sum_{\text{spins}} \mathcal{M}_{\rho\lambda}\bar{\mathcal{M}}_{\sigma\tau} = e^4\mathrm{Tr}\Big\{&\Big(\frac{2p'_\lambda + \gamma_\lambda\not{k}}{2k\cdot p' + k^2}\gamma_\rho - \gamma_\rho\frac{2p_\lambda + \not{k}\gamma_\lambda}{2k\cdot p + k^2}\Big)(-\mathrm{i}\not{p} - m)\\ &\times\Big(-\gamma_\sigma\frac{\not{k}\gamma_\tau + 2p'_\tau}{2k\cdot p' + k^2} + \frac{2p_\lambda + \not{k}\gamma_\lambda}{2k\cdot p + k^2}\gamma_\sigma\Big)(-\mathrm{i}\not{p}' - m)\Big\}\,.\end{aligned} \tag{9.122}$$

This can be reduced to a trace over at most four γ-matrices, by the identity (cf. appendix E)

$$\gamma_\rho \not{k} \gamma_\lambda = k_\rho \gamma_\lambda + k_\lambda \gamma_\rho - \eta_{\rho\lambda} \not{k} - \varepsilon_{\rho\mu\lambda\tau} k_\mu \gamma_5 \gamma_\tau \,.$$

In principle, it is now straightforward to evaluate (9.122), but the result will contain a few hundred terms as there are many independent fourth-rank Lorentz covariant tensors depending on three independent momenta. Fortunately the result (9.122) is too general for what we need, since we are not interested in the polarization of the bremsstrahlung photon. At this point the reader may wonder what the polarization states are for this photon, because we are not yet considering an on-shell photon and we know from chapter 4 that the polarizations for massive and massless spin-1 particles are not the same. However, if we contract (9.122) with the photon polarization vectors and sum over them, then the result is independent of whether we choose massive or massless polarizations. This is so because the two polarization sums differ only by terms proportional to the photon momentum, and since the amplitude is conserved (cf. 9.121) these terms vanish. Hence we may simply contract (9.122) over the indices λ and τ as is appropriate for massless photons, and obtain a second-rank tensor containing terms proportional to the 9 independent Lorentz covariant tensors constructed from products of p_μ, p'_μ and k_μ, or to the invariant tensor $\eta_{\mu\nu}$. If we define $N = 2k \cdot p + k^2$ and $N' = 2k \cdot p' + k^2$, then a straightforward but somewhat tedious calculation yields

$$\begin{aligned}
&\sum_{\text{spins}} \mathcal{M}_{\rho\lambda}\bar{\mathcal{M}}_{\sigma\lambda} \\
= \; & e^4\Big\{ -16[\eta_{\rho\sigma}(m^2 - p\cdot p') + p_\rho p'_\sigma]\Big(\frac{2p\cdot p'}{NN'} + \frac{m^2}{N^2} + \frac{m^2}{N'^2}\Big) \\
+ \; & 4\eta_{\rho\sigma}\Big[(2m^2 + 4p\cdot p' - k^2)\Big(\frac{1}{N} + \frac{1}{N'}\Big) + 2m^2\Big(\frac{N}{N'^2} + \frac{N'}{N^2}\Big) + \frac{N}{N'} + \frac{N'}{N} \\
- \; & k^2\Big(\frac{N}{N'^2} + \frac{N'}{N^2} + \frac{2p\cdot p'}{N^2} + \frac{4p\cdot p'}{NN'} + \frac{2p\cdot p'}{N'^2}\Big) + k^2\Big(\frac{1}{N^2} + \frac{1}{N'^2}\Big)\Big] \\
+ \; & 16p_\rho p_\sigma\Big(\frac{1}{N} - \frac{k^2}{NN'}\Big) + 16p'_\rho p'_\sigma\Big(\frac{1}{N'} - \frac{k^2}{NN'}\Big) \\
- \; & 8(p_\rho p'_\sigma + p'_\rho p_\sigma)\Big(\frac{1}{N} + \frac{1}{N'} - \frac{k^2}{N^2} - \frac{k^2}{N'^2} - \frac{2k^2}{NN'}\Big) \\
- \; & 8(k_\rho p_\sigma + k_\rho p'_\sigma + p_\rho k_\sigma + p'_\rho k_\sigma)\Big(\frac{2p\cdot p'}{NN'} + \frac{m^2}{N^2} + \frac{m^2}{N'^2}\Big) \\
- \; & 8(k_\rho p_\sigma + p_\rho k_\sigma)\Big(-\frac{m^2}{N^2} + \frac{m^2}{N'^2} + \frac{1}{N'} - \frac{k^2}{N'^2}\Big) \\
- \; & 8(k_\rho p'_\sigma + p'_\rho k_\sigma)\Big(\frac{m^2}{N^2} - \frac{m^2}{N'^2} + \frac{1}{N} - \frac{k^2}{N^2}\Big) \\
- \; & 32k_\rho k_\sigma \frac{m^2}{NN'}\Big\}\,.
\end{aligned} \tag{9.123}$$

As a check one may verify that this expression still vanishes when contracted with Q_ρ or Q_σ.

Subsequently we integrate over the fermion momenta. The momentum k of the bremsstrahlung photon will be kept fixed at this stage. Hence we consider (only integrating over $\mathbf{p}$ and $\mathbf{p}'$)

$$\mathrm{d}\Gamma^{(b)}_{\rho\sigma} = \frac{1}{(2\pi)^5}\int \frac{\mathrm{d}^3k}{2\omega(\mathbf{k})}\int \frac{\mathrm{d}^3p}{2\omega(\mathbf{p})}\frac{\mathrm{d}^3p'}{2\omega(\mathbf{p}')}\delta^4(p + p' - Q)\sum_{\text{spins}} \mathcal{M}_{\rho\lambda}\bar{\mathcal{M}}_{\sigma\lambda}\,, \tag{9.124}$$

which is a second-rank Lorentz covariant tensor depending on the momenta k_μ and Q_μ. However, the tensor (9.123) must vanish when contracted with Q_ρ or Q_σ, so it must be proportional to $\eta_{\rho\sigma}Q^2 - Q_\rho Q_\sigma$ i.e.

$$\mathrm{d}\Gamma^{(b)}_{\rho\sigma} = \tfrac{1}{3}\Big(\eta_{\rho\sigma} - \frac{Q_\rho Q_\sigma}{Q^2}\Big)\mathrm{d}\Gamma^{(b)}_{\tau\tau}\,, \tag{9.125}$$

with

$$\begin{aligned}
\mathrm{d}\Gamma^{(b)}_{\tau\tau} &= \frac{16e^4}{(2\pi)^5}\frac{\mathrm{d}^3k}{2\omega(\mathbf{k})}\int\frac{\mathrm{d}^3p}{2\omega(\mathbf{p})}\frac{\mathrm{d}^3p'}{2\omega(\mathbf{p}')}\delta^4(p+p'+k-Q) \\
&\quad\times\Big\{2(p\cdot p'-2m^2)\Big(\frac{2p\cdot p'}{NN'}+\frac{m^2}{N^2}+\frac{m^2}{N'^2}\Big) \\
&+ (2p\cdot p'+M^2)\Big(\frac{1}{N}+\frac{1}{N'}\Big) \\
&+ \tfrac{1}{2}\Big(\frac{N}{N'}+\frac{N'}{N}\Big)+m^2\Big(\frac{N'}{N^2}+\frac{N}{N'^2}\Big) \\
&+ k^2\Big[m^2\Big(\frac{1}{N^2}+\frac{1}{N'^2}\Big)-p\cdot p'\Big(\frac{1}{N^2}+\frac{1}{N'^2}\Big) \\
&- \tfrac{1}{2}\Big(\frac{1}{N}+\frac{1}{N'}\Big)-\tfrac{1}{2}\Big(\frac{N'}{N^2}+\frac{N}{N'^2}\Big)\Big] \\
&+ \tfrac{1}{2}k^4\Big(\frac{1}{N^2}+\frac{1}{N'^2}\Big)\Big\}
\end{aligned} \tag{9.126}$$

An alternative way to obtain this result is by contracting the indices ρ and σ, and λ and τ directly in the spinor trace in (9.122). After some algebra one is then left with a trace that contains products of at most two γ-matrices, which is easy to perform. Of course, the final result should again coincide with (9.126).

We now switch to the phase-space variables of section 3.7 in the rest frame of γ^* with $M=\sqrt{s}$, $m_1=\kappa$ and $m_2=m_3=m$ (henceforth we denote the photon mass κ_1 by κ). The parametrization (3.98) then reads

$$\omega(\mathbf{k})=\omega\,,\qquad \omega(\mathbf{p})=x+\tfrac{1}{2}(M-\omega)\,,\qquad \omega(p')=-x+\tfrac{1}{2}(M-\omega)\,. \tag{9.127}$$

The three-particle phase space is now expressed in terms of the photon energy and angle, ω and Ω, the variable x and an angle ϕ ($0\leqslant\phi\leqslant 2\pi$) which corresponds to rotations around the photon momentum $\mathbf{k}$. The boundary values of ω and x follow from (3.99) and (3.101).

$$\begin{aligned}
&\kappa\leqslant\omega\leqslant\omega_m=\tfrac{1}{2}M^{-1}(M^2-4m^2+\kappa^2)\,, \\
&x_\pm=\pm\tfrac{1}{2}\sqrt{\frac{(\omega_m-\omega)(\omega^2-\kappa_2)}{\omega_m-\omega+2m^2/M}}\,, \\
&\qquad=\pm\tfrac{1}{2}\beta\sqrt{\omega^2-\kappa^2}\,,
\end{aligned} \tag{9.128}$$

where β is the velocity of the muons in the dimuon rest frame.

$$\beta=\sqrt{\frac{M^2-2M\omega-4m^2+\kappa^2}{M^2-2M\omega+\kappa^2}}\,. \tag{9.129}$$

It is also straightforward to verify that

$$\begin{aligned}
N &\equiv 2k\cdot p + k^2 = -M(\omega - 2x)\,,\\
N' &\equiv 2k\cdot p' + k^2 = -M(\omega + 2x)\,,\\
2p\cdot p' &= -M^2 + 2M\omega + 2m^2 - \kappa^2\,.
\end{aligned} \tag{9.130}$$

Finally we use the result (3.102) which implies that

$$\frac{1}{(2\pi)^5}\frac{\mathrm{d}^3p}{2\omega\mathbf{p})}\frac{\mathrm{d}^3p'}{2\omega\mathbf{p}')}\frac{\mathrm{d}^3k}{2\omega\mathbf{k})}\delta^4(p+p'+k-Q) = \frac{1}{256\pi^5}\mathrm{d}\omega\mathrm{d}\Omega\mathrm{d}\phi\mathrm{d}x\,, \tag{9.131}$$

and substitute (9.130) and (9.131) into (9.126). The necessary x-integrals take the form

$$\begin{aligned}
&\int_{x_-}^{x_+}\mathrm{d}x\frac{1}{N} = \int_{x_-}^{x_+}\mathrm{d}x\frac{1}{N'} = -\frac{1}{2M}\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}}\,,\\
&\int_{x_-}^{x_+}\mathrm{d}x\frac{N}{N'} = \int_{x_-}^{x_+}\mathrm{d}x\frac{N'}{N} = -\beta\sqrt{\omega^2-\kappa^2} + \omega\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}}\,,\\
&\int_{x_-}^{x_+}\mathrm{d}x\frac{N}{N'^2} = \int_{x_-}^{x_+}\mathrm{d}x\frac{N'}{N^2}\\
&\qquad = \frac{1}{M}\Big(\tfrac{1}{2}\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}} - \frac{2\beta\omega\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta^2)+\beta^2\kappa^2}\Big)\,,\\
&\int_{x_-}^{x_+}\mathrm{d}x\frac{1}{NN'} = \frac{1}{2M^2\omega}\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}}\,,\\
&\int_{x_-}^{x_+}\mathrm{d}x\frac{1}{N^2} = \int_{x_-}^{x_+}\mathrm{d}x\frac{1}{N'^2} = \frac{1}{M^2}\frac{\beta\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta^2)+\beta^2\kappa^2}\,.
\end{aligned} \tag{9.132}$$

Combining all results and integrating over the angle θ gives (using $k^2 = -\kappa^2$)

$$\begin{aligned}
\frac{\mathrm{d}^2\Gamma^{(b)}_{\tau\tau}}{\mathrm{d}\omega\mathrm{d}\Omega} &= \frac{2\alpha^2}{\pi^2}\Big\{\Big(-1-\frac{2m^2(M^2+2m^2)+2m^2\kappa^2+M^2\kappa^2}{M^2(\omega^2(1-\beta^2)+\beta^2\kappa^2)}\Big)\beta\sqrt{\omega^2-\kappa^2}\\
&+ \Big(\omega^2 - \frac{\omega(M^2+2m^2+\kappa^2)}{M} + \frac{M^4-4m^4}{2M^2} + \kappa^2\Big\}\\
&\qquad\times\frac{1}{\omega}\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}}\Big\}\,.
\end{aligned} \tag{9.133}$$

To integrate this expression over ω is extremely complicated for finite κ (note that the logarithm contains the square root of a fourth-order polynomial!). However, we only need (9.133) in the limit that $\kappa^2 \ll m^2, M^2$. Therefore we can safely drop the κ-dependence in β, so that from now on we write

$$\beta = \sqrt{\frac{\beta_0^2 - 2\omega/M}{1-2\omega/M}}\,, \tag{9.134}$$

with

$$\beta_0 = \sqrt{1 - 4m^2/M^2} \tag{9.135}$$

the maximal velocity of the muons in the dimuon rest frame. Then we isolate the terms that diverge when $\kappa \to 0$, by writing

$$\begin{aligned}
\frac{\beta\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta^2)+\beta^2\kappa^2} &= \frac{\beta_0\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta_0^2)+\beta_0^2\kappa^2} \\
&\quad + \sqrt{\omega^2-\kappa^2}\Big(\frac{\beta}{\omega^2(1-\beta^2)+\beta^2\kappa^2} - \frac{\beta_0}{\omega^2(1-\beta_0^2)+\beta_0^2\kappa^2}\Big), \\
\frac{1}{\omega}\ln\frac{\omega+\beta\sqrt{\omega^2-\kappa^2}}{\omega-\beta\sqrt{\omega^2-\kappa^2}} &= \frac{1}{\omega}\ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}} \\
&\quad + \frac{1}{\omega}\Big(\ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}} - \ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}}\Big).
\end{aligned} \tag{9.136}$$

In the second terms we can now safely let κ go to zero. Therefore (9.133) takes the form

$$\begin{aligned}
\frac{\mathrm{d}^2\Gamma^{(b)}_{\tau\tau}}{\mathrm{d}\omega\mathrm{d}\Omega} &= \frac{\alpha^2M^2}{2\pi^2}(3-\beta_0^2) \\
&\times\Big\{-\beta_0(1-\beta_0^2)\frac{\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta_0^2)+\beta_0^2\kappa^2} + \frac{1+\beta_0^2}{\omega}\ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}} \\
&-\frac{1-\beta_0^2}{\omega}\Big(\frac{\beta}{1-\beta^2}-\frac{\beta_0}{1-\beta_0^2}\Big) + \frac{1+\beta_0^2}{\omega}\Big(\ln\frac{1+\beta}{1-\beta}-\ln\frac{1+\beta_0}{1-\beta_0}\Big)\Big\} \\
&\qquad + \frac{\alpha^2}{\pi^2}\Big\{-2\beta\omega + [2\omega - M(3-\beta_0^2)]\ln\frac{1+\beta}{1-\beta}\Big\}.
\end{aligned} \tag{9.137}$$

The above expression must now be integrated over the photon momentum, where we should make the following distinction. If the photon momentum is such that the photon is not observable in a given experiment then the contribution of (9.137) must be included into the expression for the nonradiative rate. This is the range of photon energies $\kappa < \omega \leqslant \Delta(\Omega)$, where $\Delta(\Omega)$ is the energy (which may depend on the direction in which the photon is emitted) below which the photon detection efficiency of the actual experimental set-up is negligible. For photon energies $\Delta(\Omega) \leqslant \omega \leqslant \omega_m = \frac{1}{2}M\beta_0^2$ the contribution of (9.137) is counted as a radiative decay. Hence we need the integral of (9.137) over ω with integration boundaries that depend on the experiment in question. Therefore it is useful to know the primitive of (9.137); we quote the results for the various terms in (9.137), which can easily be verified by explicit differentiation. The first two terms in (9.137), which still depend on the regulator mass κ, are given by

$$\frac{\sqrt{\omega^2-\kappa^2}}{\omega^2(1-\beta_0^2)+\beta_0^2\kappa^2} = \tfrac{1}{2}\frac{1}{1-\beta_0^2}\frac{\mathrm{d}}{\mathrm{d}\omega}\Big(\ln\frac{1+t}{1-t}-\frac{1}{\beta_0}\ln\frac{1+\beta_0 t}{1-\beta_0 t}-\ln\frac{M}{\kappa}\Big)\,, \tag{9.138}$$

$$\begin{aligned}
&\frac{1}{\omega}\ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}}\\
= \;& \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\Big(\ln(1-t)\ln\frac{1+\beta_0}{1-\beta_0}-\ln(\beta_0(1+t))\ln\frac{1+\beta_0 t}{1-\beta_0 t}\\
&\qquad -\ln(1-\beta_0 t)\ln(1-\beta_0^2)\\
+\;& \tfrac{1}{2}\ln^2(1-\beta_0 t)+\tfrac{1}{2}\ln^2(1+\beta_0 t)+\mathrm{Li}_2\Big(\frac{\beta_0(1-t)}{1+\beta_0}\Big)+\mathrm{Li}_2\Big(\frac{\beta_0(1-t)}{1-\beta_0 t}\Big)\\
+\;& \mathrm{Li}_2\Big(\frac{1-\beta_0)}{1+\beta_0 t}\Big)+\mathrm{Li}_2\Big(\frac{1-\beta_0 t)}{1+\beta_0}\Big)-\ln\frac{1+\beta_0}{1-\beta_0}\ln\frac{M}{\kappa}\Big)\,,
\end{aligned} \tag{9.139}$$

where $t=\omega^{-1}\sqrt{\omega^2-\kappa^2}$. Note that we have included certain ω-independent terms, proportional to $\ln M/\kappa$, so that the results remain finite when $\kappa\to 0$ for fixed values of $\omega>\kappa$. The other integrals do not depend on κ and follow from

$$\begin{aligned}
\frac{1}{\omega}\Big(\frac{\beta}{1-\beta^2}-\frac{\beta_0}{1-\beta_0^2}\Big) &= \frac{\mathrm{d}}{\mathrm{d}\omega}\Big(-\frac{2\beta_0}{1-\beta_0^2}\ln(\beta_0+\beta)+\frac{1}{1-\beta_0^2}\ln\frac{1+\beta}{1-\beta}\\
&\quad +\frac{\beta_0}{1-\beta_0^2}\ln(1-\beta^2)+\frac{\beta}{1-\beta^2}-\tfrac{1}{2}\ln\frac{1+\beta}{1-\beta}\Big)\,,
\end{aligned} \tag{9.140}$$

$$\begin{aligned}
\frac{1}{\omega}\Big(\ln\frac{1+\beta}{1-\beta}-\frac{1+\beta_0}{1-\beta_0}\Big) &= \frac{\mathrm{d}}{\mathrm{d}\omega}\Big(-\ln^2(1+\beta)+\ln(1-\beta_0^2)\ln(1-\beta)\\
&\quad -2\ln 2\ln(1-\beta)\\
&\quad +\ln(1-\beta)\ln(1+\beta)+\ln(1-\beta^2)\ln\frac{1+\beta_0}{1-\beta_0}\\
&\quad -\ln(\beta_0+\beta)\ln\frac{1+\beta_0}{1-\beta_0}+\ln(\beta_0+\beta)\ln\frac{1+\beta}{1-\beta}\\
&\quad -\mathrm{Li}_2\Big(\frac{\beta_0-\beta}{1+\beta_0}\Big)-\mathrm{Li}_2\Big(\frac{1-\beta}{1+\beta_0}\Big)\\
&\quad -\mathrm{Li}_2\Big(\frac{1-\beta_0}{1+\beta}\Big)-\mathrm{Li}_2\Big(\frac{\beta_0-\beta}{1-\beta}\Big)+2\mathrm{Li}_2\Big(\frac{1-\beta}{2}\Big)\Big)\,,
\end{aligned} \tag{9.141}$$

$$\begin{aligned}
\beta\omega &= \tfrac{1}{8}M^2\frac{\mathrm{d}}{\mathrm{d}\omega}\Big(\frac{1-\beta_0^2}{1-\beta^2}\Big(-2\beta+(1-\beta^2)\ln\frac{1+\beta}{1-\beta}\Big)\\
&\quad +\tfrac{1}{2}\Big(\frac{1-\beta_0^2}{1-\beta^2}\Big)^2\Big(2\beta-\beta(1-\beta^2)-\tfrac{1}{2}(1-\beta^2)^2\ln\frac{1+\beta}{1-\beta}\Big)\Big)\,,
\end{aligned} \tag{9.142}$$

$$\ln\frac{1+\beta}{1-\beta} = -\tfrac{1}{4}M(1-\beta_0^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\Big(-\frac{1+\beta^2}{1-\beta^2}\ln\frac{1+\beta}{1-\beta} - \frac{2\beta}{1-\beta^2}\Big)\,, \tag{9.143}$$

$$\begin{aligned}\beta\omega\ln\frac{1+\beta}{1-\beta} &= \tfrac{1}{8}M^2\frac{\mathrm{d}}{\mathrm{d}\omega}\Big(-\frac{1-\beta_0^2}{1-\beta^2}\Big(-(1+\beta^2)\ln\frac{1+\beta}{1-\beta}+2\beta\Big)\\ &+\tfrac{1}{2}\Big(\frac{1-\beta_0^2}{1-\beta^2}\Big)^2\Big([2-\tfrac{3}{4}(1-\beta^2)^2]\ln\frac{1+\beta}{1-\beta}-\tfrac{5}{2}\beta+\tfrac{3}{2}\beta^3\Big)\Big)\,,\end{aligned} \tag{9.144}$$

It is now straightforward to use (9.138-9.144) and calculate the ω-integral of (9.137) over any desired range of photon energies. As an example we consider the situation where no photons can be detected, so that we integrate (9.137) over the entire ω-range from $\omega=\kappa$ to $\omega=\omega_m=\frac{1}{2}M\beta_0^2$. In that case (9.138-9.144) give the relevant integrals

$$\int_\kappa^{\omega_m}\mathrm{d}\omega\frac{\sqrt{\omega^2-\kappa^2}}{\omega^2-\beta_0^2(\omega^2-\kappa^2)} = \frac{1}{1-\beta_0^2}\Big(\ln\frac{M\beta_0^2}{\kappa}-\frac{1}{2\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}\Big)+O(\kappa)\,, \tag{9.145}$$

$$\begin{aligned}\int_\kappa^{\omega_m}\frac{\mathrm{d}\omega}{\omega}\ln\frac{\omega+\beta_0\sqrt{\omega^2-\kappa^2}}{\omega-\beta_0\sqrt{\omega^2-\kappa^2}} &= \ln\frac{1+\beta_0}{1-\beta_0}\ln\frac{M\beta_0^2}{\kappa}+\mathrm{Li}_2\Big(\frac{1-\beta_0}{1+\beta_0}\Big)-\tfrac{1}{6}\pi^2\\ &+ \tfrac{1}{4}\ln\frac{1+\beta_0}{1-\beta_0}\ln(1-\beta_0^2)\\ &+ \tfrac{1}{2}\ln\frac{1+\beta_0}{1-\beta_0}\ln\frac{1+\beta_0}{4\beta_0^2}+O(\kappa)\,,\end{aligned} \tag{9.146}$$

$$\begin{aligned}\int_\kappa^{\omega_m}\frac{\mathrm{d}\omega}{\omega}\Big(\frac{\beta}{1-\beta^2}-\frac{\beta_0}{1-\beta_0^2}\Big) &= \tfrac{1}{2}\frac{1}{1-\beta^2}\Big(-\beta_0\ln\frac{1-\beta_0^2}{4}-\ln\frac{1+\beta_0}{1-\beta_0}-\beta_0\Big)\\ &+\tfrac{1}{2}\ln\frac{1+\beta_0}{1-\beta_0}\,,\end{aligned} \tag{9.147}$$

$$\begin{aligned}\int_\kappa^{\omega_m}\frac{\mathrm{d}\omega}{\omega}\Big(\ln\frac{1+\beta}{1-\beta}-\frac{1+\beta_0}{1-\beta_0}\Big) &= \mathrm{Li}_2\Big(\Big(\frac{1-\beta_0}{1+\beta_0}\Big)^2\Big)-\tfrac{1}{6}\pi^2\\ &+ 2\ln\frac{1+\beta_0}{1-\beta_0}\ln\frac{1+\beta_0}{2\beta_0}\,,\end{aligned} \tag{9.148}$$

$$\int_\kappa^{\omega_m}\mathrm{d}\omega\beta\omega = \tfrac{1}{8}M^2\Big(\beta_0+\frac{1}{2}\beta_0(1-\beta_0^2)-(1-\beta_0^2)[1-\tfrac{1}{4}(1-\beta_0^2)]\ln\frac{1+\beta_0}{1-\beta_0}\Big)\,, \tag{9.149}$$

$$\int_\kappa^{\omega_m}d\omega\ln\frac{1+\beta}{1-\beta} = \tfrac{1}{4}M\Big((1+\beta_0^2)\ln\frac{1+\beta_0}{1-\beta_0}-2\beta_0\Big)\,, \tag{9.150}$$

$$\int_{\kappa}^{\omega_m} d\omega\omega \ln\frac{1+\beta}{1-\beta} = \tfrac{1}{8}M^2\Big(-\tfrac{3}{4}\beta_0(1+\beta_0^2)+[\beta_0^2+\tfrac{3}{8}(1-\beta_0^2)^2]\ln\frac{1+\beta_0}{1-\beta_0}\Big)\,. \tag{9.151}$$

In deriving (9.146) and (9.148) we have used relations among the dilogarithmic functions given in appendix D. Some of the above results can be checked by expansion of both sides for small β_0 or by differentiation with respect to β_0 (noting that ω_m is a function of β_0).

Performing the angular integration of (9.137) yields a factor 4π, so that

$$\begin{aligned}
&\Gamma_{\tau\tau}^{(b)} = \int_{\kappa}^{\omega_m}\mathrm{d}\omega\int\mathrm{d}\Omega\frac{\mathrm{d}^2\Gamma_{\tau\tau}^{(b)}}{\mathrm{d}\omega\mathrm{d}\Omega}\\
=\;& \frac{\alpha^2M^2}{\pi}(3-\beta_0^2)\beta_0\\
\times\;& \Big\{\Big(\frac{1+\beta_0^2}{\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}-2\Big)\ln\frac{m}{\kappa}+2+\frac{2+\beta_0^2}{\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}+3\ln\frac{1-\beta_0^2}{4}\\
-\;& 4\ln\beta_0+\frac{1+\beta_0^2}{\beta_0}\Big[\mathrm{Li}_2\Big(\frac{1-\beta_0}{1+\beta_0}\Big)+\mathrm{Li}_2\Big(\Big(\frac{1-\beta_0}{1+\beta_0}\Big)^2\Big)\\
-\;& \tfrac{1}{3}\pi^2+\tfrac{1}{4}\ln^2\frac{1+\beta_0}{1-\beta_0}+\ln\Big(\frac{1+\beta_0}{1-\beta_0}\Big)\ln\frac{(1+\beta_0)^2}{4\beta_0}\Big]\Big\}\\
+\;& \frac{3\alpha^2M^2}{8\pi}\Big\{10\beta_0-6\beta_0^2-(5+6\beta_0^2-3\beta_0^4)\ln\frac{1+\beta_0}{1-\beta_0}\Big\}\,,
\end{aligned} \tag{9.152}$$

where we have grouped the terms in direct correspondence with (9.137). As the detector is entirely blind to photons in this example, the above result should be added to (9.117-9.118), and the infrared divergences should cancel. Indeed, using $s = M^2$ and $\kappa_1 = \kappa$ the desired cancellation takes place, and we are left with a finite result

$$\begin{aligned}
&\Gamma_{\tau\tau}^{(a)}+\Gamma_{\tau\tau}^{(b)} = \alpha M^2\beta_0(3-\beta_0^2)\\
+\;& \frac{\alpha^2M^2}{\pi}\Big\{(1+\beta_0^2)(3-\beta_0^2)\Big[3\ln\frac{1+\beta_0}{2}\ln\frac{1+\beta_0}{1-\beta_0}-2\ln\beta_0\ln\frac{1+\beta_0}{1-\beta_0}\\
&\quad+\mathrm{Li}_2\Big(\frac{1-\beta_0}{1+\beta_0}\Big)+\mathrm{Li}_2\Big(\Big(\frac{1-\beta_0}{1+\beta_0}\Big)^2\Big)\Big]\\
&\quad+[\frac{11}{8}(1+\beta_0^2)(3-\beta_0^2)+\tfrac{1}{2}\beta_0^4-3\beta_0(3-\beta_0^2)]\ln\frac{1+\beta_0}{1-\beta_0}\\
&\quad+6\beta_0(3-\beta_0^2)\ln\frac{1+\beta_0}{2}-4\beta_0(3-\beta_0^2)\ln\beta_0+\tfrac{3}{4}\beta_0(5-3\beta_0^2)\Big\}
\end{aligned} \tag{9.153}$$

The result (9.153) is now free of infrared and ultraviolet divergences and can be compared to experiment. In fact (9.153) also remains finite in the limit that the muon mass approaches zero (the absence of mass divergences will be discussed in section 9.5). The result in that limit equals

$$\lim_{m\to 0}(\mathit{\Gamma}^{(a)}_{\tau\tau}+\mathit{\Gamma}^{(b)}_{\tau\tau}) = 2M^2\Big\{\alpha+\frac{3\alpha^2}{4\pi}+O(\alpha^3)\Big\}\,. \tag{9.154}$$

The calculations in this chapter illustrate the laborious procedure by which physically relevant results can be extracted from quantum corrections. Needless to say powerful numerical techniques are required if one considers more complicated processes or if one wishes to take into account certain features of an actual detector. Finally we note that (9.153) extends (9.119) to second order in α

$$\mathrm{Im}(\Pi_{\rho\sigma}(Q)) = \tfrac{1}{2}(\mathit{\Gamma}^{(a)}_{\rho\sigma}(Q)+\mathit{\Gamma}^{(b)}_{\rho\sigma}(Q)) = \tfrac{1}{6}\Big(\eta_{\rho\sigma}-\frac{Q_\rho Q_\sigma}{Q^2})(\mathit{\Gamma}^{(a)}_{\tau\tau}+\mathit{\Gamma}^{(b)}_{\tau\tau})\,, \tag{9.155}$$

so that we have obtained the order α and the order-α^2 contributions to the imaginary part of the vacuum polarization function $\Pi^f(s/m^2)$, which are nonvanishing in the region $s/m^2 > 4$ [remember that $s = -Q^2 = M^2 = 4m^2(1-\beta_0^2)^{-1}$]; the result is

$$\mathrm{Im}\Pi^f(s/m^2) = \frac{\mathit{\Gamma}^{(a)}_{\tau\tau}+\mathit{\Gamma}^{(b)}_{\tau\tau}}{6M^2}+O(\alpha^3)\,. \tag{9.156}$$

Both the order-α and order-α^2 contributions have been indicated in fig. 9.3 (with different scales) and the corresponding diagrams are shown in fig. 9.8. Note that in the calculation in this section e and m denote the physical muon charge and mass, in contradistinction with the parameters used in section 9.5. From (9.156) one can also determine the real part of $\Pi^f(s/m^2)$ to the same order in α by using a dispersion relation.

Figure 9.8: Diagrams contributing to the vacuum polarization function in order α and α^2. Their imaginary part is given in (9.156).

Problems

9.1. In (9.9) we use the result for n-dimensional spherical coordinates, that

$$\int \mathrm{d}^n r = \int \mathrm{d}r r^{n-1} \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{\pi} \mathrm{d}\theta_2 \sin\theta_2 \cdots \int_0^{\pi} \mathrm{d}\theta_{n-1} \sin\theta_{n-1} \,. \tag{1}$$

This result is well-known for $n = 1$, 2 and 3, and can be proved by induction. Assume it to be true in n dimensions, then in $n + 1$ dimensional space

$$\int \mathrm{d}^{n+1} r = \int \mathrm{d}x_{n+1} \int \mathrm{d}^n r \,. \tag{2}$$

We now insert (1) into the right-hand side of (2). Denoting the n-dimensional radius by r' we introduce an angle θ_n by $r' = r\sin\theta_n$ and $x_{n+1} = r\cos\theta_n$, where r is the radius in $n+1$ dimensions, i.e., $r = \sqrt{r'^2 + x_{n+1}^2}$. It is then straightforward to write (2) in the form (1), thus proving that (1) holds for $n + 1$ dimensions.

9.2. Show that in n dimensions the vacuum polarization diagram in quantum electrodynamics with zero photon momentum is proportional to

$$\frac{1}{n}\eta_{\mu\nu} \int \mathrm{d}^n q \frac{(n-2)q^2 + nm^2}{(q^2+m^2)^2} \tag{1}$$

by using symmetric integration. Verify that the integrand is a total divergence, so that according to dimensional regularization the vacuum polarization diagram is zero (cf. 9.13). Give an argument based on gauge invariance to explain this result. Now generalize this problem to the box diagrams for photon-photon scattering with zero photon momentum. Exploit the fact that the amplitude for the sum of the box diagrams must be proportional to

$$(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$$

(μ, ν, ρ and σ are the photon indices). Show that according to dimensional regularization

$$\int \mathrm{d}^n q \frac{(n-2\alpha)q^2 + nm^2}{(q^2+m^2)^{\alpha+1}} = 0 \tag{2}$$

and use this to prove that the amplitude for the box diagrams vanishes. Why is this result important for the renormalizability of quantum electrodynamics? Give arguments why any closed fermion loop with an arbitrary number of zero-momentum photon lines should vanish.

9.3. Assuming a $\mathrm{K}_S^0\pi^+\pi^-$ coupling constant equal to g, and using the Lagrangian (4.47), show that the amplitude for the decay $\mathrm{K}_S^0(P) \to \gamma(k_1)+\gamma(k_2)$, when mediated by a virtual pion loop, is

$$\mathcal{M} = \frac{2ge^2}{\mathrm{i}(2\pi)^n} \int \mathrm{d}^n q \frac{4q\cdot\varepsilon_1 q\cdot\varepsilon_2 - (q^2+m^2)\varepsilon_1\cdot\varepsilon_2}{((q+k_1)^2+m^2)((q-k_2)^2+m^2)(q^2+m^2)} \,,$$

where ε_1 and ε_2 are the photon polarizations and $k_1^2 = k_2^2 = 0$. Evaluate the first term in the numerator using the techniques described following (9.67). To do this it is essential to perform the manipulations at $k_1^2 = k_2^2 = -\mu^2$ and set $\mu^2 = 0$ at the end of the calculation.

9.4. Consider the Lagrangian (4.47) for scalar electrodynamics. Specify the two one-loop diagrams contributing to the vacuum polarization. Show that

$$\Pi_{\mu\nu}(k) = (\eta_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2)\,, \tag{1}$$

where, including a factor $[-\mathrm{i}(2\pi)^n]^{-1}$ as in (9.37),

$$\Pi(k^2) = \frac{\mathrm{i}e^2}{(2\pi)^n}\frac{2}{n-1}\frac{1}{k^2}\int \mathrm{d}^n q\Big\{\frac{2q^2}{((q+\frac{1}{2}k)^2+m^2)((q-\frac{1}{2}k)^2+m^2)} - \frac{n}{q^2+m^2}\Big\}\,. \tag{2}$$

Evaluate this expression for $n = 4 + \varepsilon$ and check that for $\varepsilon \approx 0$,

$$\Pi(k^2) = -\frac{e^2\mu^\varepsilon}{24\pi^2}\frac{1}{\varepsilon} + \Pi^f(k^2)\,, \tag{3}$$

where

$$\Pi^f(0) = -\frac{e^2}{48\pi^2}\Big[\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2}\Big]\,, \tag{4}$$

and

$$\Pi^f(k^2) - \Pi^f(0) = \frac{e^2}{12\pi^2}\Big\{\tfrac{1}{3} + \int_0^1 \mathrm{d}x\Big[\tfrac{1}{4} - \frac{2m^2}{k^2} - 3x(1-x)\Big]\ln\frac{m^2 + k^2x(1-x)}{m^2}\Big\}\,. \tag{5}$$

Verify that the $1/\varepsilon$ term is cancelled by the contribution from the counterterm Lagrangian

$$\Delta\mathcal{L}_1 = -\frac{e^2\mu^\varepsilon}{96\pi^2}\frac{1}{\varepsilon}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\,. \tag{6}$$

9.5. Determine the two one-loop diagrams contributing to the scalar propagator in scalar quantum electrodynamics. Including a factor $[-\mathrm{i}(2\pi)^n]^{-1}$ as in (9.47), show that

$$\begin{aligned}\Sigma(p) = \frac{\mathrm{i}e^2}{(2\pi)^n}\int \mathrm{d}^n q\Big\{ &-\frac{1}{q^2+m^2} - \frac{4m^2}{((q+p)^2+m^2)(q+\kappa_1^2)} + \frac{2-n}{q^2+\kappa_1^2} \\ &+ \frac{p^2+m^2}{(p+q)^2+m^2}\Big[\frac{2}{q^2+\kappa_1^2} + (1-\lambda^{-2})\frac{q^2+2p\cdot q}{(q^2+\kappa_2^2)(q^2+\kappa_3^2)}\Big]\Big\}\,,\end{aligned} \tag{1}$$

where we use the regulated photon propagator (9.60). Expand this result as a Taylor series about the point $p^2 = -m^2$, i.e.

$$\Sigma(p) = m^2 A + (p^2+m^2)B + O((p^2+m^2)^2) \tag{2}$$

and check that near $\varepsilon \approx 0$

$$A = -\frac{3e^2\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon} + A^f\,, \tag{3}$$

where

$$A^f = -\frac{e^2}{8\pi^2}\Big(-\tfrac{7}{2} + \tfrac{3}{2}\gamma_{\mathrm{E}} + \tfrac{3}{2}\ln\frac{m^2}{4\pi\mu^2}\Big)\,. \tag{4}$$

Explain why A does not depend on λ. Furthermore derive

$$B = \frac{e^2\mu^\varepsilon}{8\pi^2}(3-\lambda^{-2})\frac{1}{\varepsilon} + B^f\,, \tag{5}$$

where

$$\begin{aligned} B^f = \frac{e^2}{8\pi^2}\Big\{&\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2} - \ln\frac{m^2}{\kappa_1^2} \\ &+\tfrac{1}{2}(1-\lambda^{-2})\Big[-1+\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\kappa_2^2-\kappa_3^2}\Big(\kappa_2^2\ln\frac{\kappa_2^2}{m^2} - \kappa_3^2\ln\frac{\kappa_3^2}{m^2}\Big)\Big]\Big\} \end{aligned} \tag{6}$$

Verify that the $1/\varepsilon$ terms are cancelled by the contributions from the counterterm Lagrangians

$$\Delta\mathcal{L}_2 = -\frac{e^2\mu^\varepsilon}{8\pi^2}m^2\lambda^{-2}\frac{1}{\varepsilon}|\phi|^2\,, \tag{7}$$

$$\Delta\mathcal{L}_3 = \frac{e^2\mu^\varepsilon}{8\pi^2}m^2(3-\lambda^{-2})\frac{1}{\varepsilon}|\partial_\mu\phi|^2\,, \tag{8}$$

9.6. Consider the three one-loop diagrams contributing to the photon scalar vertex, with on-shell scalars, i.e. $p^2 = p'^2 = -m^2$. Extracting a factor of $\mathrm{i}(2n)^n$ as in (9.82), show that $\Lambda_\mu(p,p') = \Lambda_\mu(Q^2)P_\mu$ with

$$\begin{aligned} \Lambda(Q^2) = \mathrm{i}e^3\{&2(Q^2+2m^2)[J(Q^2,-m^2,-m^2) + 2S(Q^2)] \\ &+2I(-m^2,m^2,0) + (1-\lambda^{-2})I(0,\kappa_2^2,\kappa_3^2)\}\,, \end{aligned} \tag{1}$$

where $P = p' + p$, $Q = p' - p$, and $J(Q^2,-m^2,-m^2)$, $S(Q^2)$, $I(-m^2,m^2,0)$ and $I(0,\kappa_2^2,\kappa_3^2)$ are defined in (9.36), (9.80), (9.81) and (9.86), respectively. Evaluate (1) near $\varepsilon \approx 0$ and check that

$$\Lambda(Q^2) = \frac{e^3\mu^\varepsilon}{8\pi^2}(3-\lambda^{-2}) + \Lambda^f(Q^2)\,, \tag{2}$$

where

$$\begin{aligned} \Lambda^f(0) = \frac{e^3}{8\pi^2}\Big\{&\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2} - \ln\frac{m^2}{\kappa_1^2} \\ &+\tfrac{1}{2}(1-\lambda^{-2})\Big[-1+\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\kappa_2^2-\kappa_3^2}\Big(\kappa_2^2\ln\frac{\kappa_2^2}{m^2} - \kappa_3^2\ln\frac{\kappa_3^2}{m^2}\Big)\Big]\Big\} \end{aligned} \tag{3}$$

and

$$\Lambda^f(Q^2) - \Lambda^f(0) = \frac{e^3}{8\pi^2}\Big\{ -2 + \ln\frac{m^2}{\kappa_1^2} +$$
$$+\tfrac{1}{2}(Q^2+2m^2)\int_0^1 dv \frac{1}{m^2+\frac{1}{4}Q^2(1-v^2)}\Big(2 - \ln\frac{m^2+\frac{1}{2}Q^2(1-v^2)}{\kappa_1^2}\Big)\Big\} . \quad (4)$$

Verify that the $1/\varepsilon$ term is cancelled by the contribution from the counterterm Lagrangian

$$\Delta\mathcal{L}_4 = \mathrm{i}\frac{e^2\mu^\varepsilon}{8\pi^2} m^2(3-\lambda^{-2})\frac{1}{\varepsilon} A_\mu \phi^* \overleftrightarrow{\partial}_\mu \phi . \quad (5)$$

9.7. Consider the one-loop graphs with four external lines contributing to the $\gamma\gamma\phi^*\phi$ amplitude. Distinguish the four classes of graphs indicated below. Note that the external lines have to be attached to the graphs in all possible ways and that we normalize with respect to the $A_\mu^2|\phi|^2$ coupling by removing one factor of $\mathrm{i}(2\pi)^n$. For the extraction of the ultraviolet divergences it is sufficient to consider the case when all the external momenta are zero, as higher-order terms in a Taylor expansion about this point are finite. Show that the graphs of each class have an ultraviolet piece equal to

$$\frac{e^4\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}\eta_{\rho\sigma} D$$

with

$$D^{(a)} = 2[1-(1-\lambda^{-2})] , \qquad D^{(b)} = -2[1-(1-\lambda^{-2})] ,$$
$$D^{(c)} = 4[1-(1-\lambda^{-2})] , \qquad D^{(d)} = -2[4-(1-\lambda^{-2})] ,$$

for graphs of the class (a), (b), (c) and (d), respectively, where ρ and σ are the indices of the external photon lines.
Verify that the $1/\varepsilon$ term is cancelled by the contribution from the counterterm Lagrangian

$$\Delta\mathcal{L}_5 = \frac{e^4\mu^\varepsilon}{8\pi^2} m^2(3-\lambda^{-2})\frac{1}{\varepsilon} A_\mu^2|\phi|^2 .$$

9.8. In the previous problems we have considered all the one-loop graphs in scalar quantum electrodynamics apart from graphs with four external scalars. The latter graphs fall into the three classes shown below. Note that the external lines have to be attached to the diagrams in all possible ways. As in the previous problem put external momenta equal to zero to extract the ultraviolet divergent parts and normalize by removing one factor of $\mathrm{i}(2\pi)^n$. Check that the graphs have an ultraviolet piece equal to

$$\frac{e^4\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon} D$$

with

$$D^{(a)} = -4[4 - 2(1-\lambda^{-2}) + (1-\lambda^{-2})^2]\,,$$
$$D^{(b)} = 12[1 - 2(1-\lambda^{-2}) + (1-\lambda^{-2})^2]\,,$$
$$D^{(c)} = -8[1 - 2(1-\lambda^{-2}) + (1-\lambda^{-2})^2]\,,$$

for the graphs of the class (a), (b) and (c), respectively.
Verify that the $1/\varepsilon$ terms are cancelled by the contribution from the counterterm Lagrangian.

$$\Delta\mathcal{L}_6 = \frac{3e^4\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}|\phi|^4\,.$$

We thus have to introduce a $|\phi|^4$ coupling in higher order, so there is no reason to exclude it from the original Lagrangian. Therefore we assume the classical Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 - |\partial_\mu\phi|^2 - m^2|\phi|^2 - \mathrm{i}eA_\mu^2\phi^*\overleftrightarrow{\partial}_\mu\phi - e^2A_\mu^2|\phi|^2 - g|\phi|^4\,. \tag{1}$$

The presence of the $|\phi|^4$ interaction leads to additional one-loop graphs which contribute to the amplitudes calculated previously, namely where the external lines must be attached to the graphs in all possible ways.
Extracting a factor $-\mathrm{i}(2\pi)^n$, show that the self-energy graph (a) yields

$$\frac{gm^2\mu^\varepsilon}{2\pi^2}\Big(\frac{1}{\varepsilon} + \tfrac{1}{2}\gamma_\mathrm{E} - \tfrac{1}{2} + \tfrac{1}{2}\ln\frac{m^2}{4\pi\mu^2}\Big)\,.$$

Its infinite part can be cancelled by the addition of the counterterm Lagrangian

$$\Delta\mathcal{L}_7 = \frac{gm^2\mu^\varepsilon}{2\pi^2}\frac{1}{\varepsilon}|\phi|^2\,.$$

The contribution from graph (b) is identically zero. Give an argument why this is so.
The combined contribution from the graphs (c) and (d) is ultraviolet finite.
The ultraviolet divergent contributions from graphs (e) and (f) are cancelled by the addition of the counterterm Lagrangian

$$\Delta\mathcal{L}_8 = \frac{\mu^\varepsilon}{4\pi^2}(6g^2 - ge^2\lambda^{-2})\frac{1}{\varepsilon}|\phi|^4\,.$$

To derive the last two results it is most convenient to set the external momenta to zero. However pay particular attention to the cancellations among the graphs in (f).

9.9. Consider the Lagrangian (1) of problem 9.8 written in terms of the original fields A_μ^0,ϕ^0 and parameters λ_0, m_0, e_0 and g_0. Show that the counterterms calculated in problems 9.4- 9.8 are generated by the substitutions

$$A^0_\mu = \sqrt{Z_A} A_\mu \qquad \text{with} \qquad Z_A = 1 + \frac{e^2\mu^\varepsilon}{24\pi^2}\frac{1}{\varepsilon},$$
$$\phi^0 = \sqrt{Z_\phi}\phi \qquad \text{with} \qquad Z_\phi = 1 - \frac{e^2\mu^\varepsilon}{8\pi^2}(3-\lambda^{-2})\frac{1}{\varepsilon},$$
$$m_0^2 = Z_m m^2 \qquad \text{with} \qquad Z_m = 1 + \Big(\frac{3e^2\mu^\varepsilon}{8\pi^2} - \frac{g\mu^\varepsilon}{2\pi^2}\Big)\frac{1}{\varepsilon},$$
$$e_0 = Z_e e \qquad \text{with} \qquad Z_e = 1 - \frac{e^2\mu^\varepsilon}{48\pi^2}\frac{1}{\varepsilon},$$
$$g_0 = g + \Big(\frac{3ge^2\mu^\varepsilon}{4\pi^2} - \frac{3e^4\mu^\varepsilon}{8\pi^2} - \frac{3g^2\mu^\varepsilon}{2\pi^2}\Big)\frac{1}{\varepsilon},$$
$$\lambda^0 = Z_\lambda \lambda \qquad \text{with} \qquad Z_\lambda = 1 - \frac{e^2\mu^\varepsilon}{48\pi^2}\frac{1}{\varepsilon}.$$

Note the following consequences of this result. First that the $eA_\mu(\phi^*\overleftrightarrow{\partial}_\mu\phi)$ and $e^2A_\mu^2|\phi|^2$ counterterms are generated by the same substitutions for ϕ_0, A_0 and e_0 as given above. This demonstrates that the relationship between the unrenormalized couplings is preserved at the one-loop level. Note also that $Z_\lambda Z_A^{1/2} = 1$, $Z_e Z_A^{1/2} = 1$ and that the renormalization of g is not multiplicative. Finally explain why Z_m, Z_e and g_0 are independent of λ.

9.10. Express, in the one-loop approximation, the physical mass M and the physical charge e_{P} of the scalar particle in scalar electrodynamics in terms of the parameters e, g and m defined in the previous problems and verify that

$$M^2 = m^2\Big[1 + \frac{e^2}{16\pi^2}\Big(7 - 3\gamma_{\mathrm{E}} - 3\ln\frac{m^2}{4\pi\mu^2}\Big) + \frac{g}{4\pi^2}\Big(\gamma_{\mathrm{E}} - 1 + \ln\frac{m^2}{4\pi\mu^2}\Big)\Big], \tag{1}$$

$$e_{\mathrm{P}} = e\Big[1 + \frac{e^2}{96\pi^2}\Big(\gamma_{\mathrm{E}} + \ln\frac{m^2}{4\pi\mu^2}\Big)\Big]. \tag{2}$$

Show that the charge form factor of the scalar particle is then given by (cf. problem 9.6)

$$F(Q^2) = 1 + \frac{\alpha}{2\pi}\Big\{-2 + \ln\frac{M^2}{\kappa_1^2}$$
$$+\tfrac{1}{2}(Q^2+2M^2)\int_0^1 \mathrm{d}v\frac{1}{M^2+\frac{1}{4}Q^2(1-v^2)}\Big[2 - \ln\frac{M^2+\frac{1}{4}Q^2(1-v^2)}{\kappa_1^2}\Big]\Big\}, \tag{3}$$

where $\alpha = e_{\mathrm{P}}^2/4\pi$ is the fine-structure constant. Note that the infrared divergent term will be cancelled by bremsstrahlung contributions when we calculate physical cross sections or decay rates.

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