7 | Chirality, helicity and discrete symmetries

In earlier chapters we have discussed how fields associated with particles of spin 0, $\frac{1}{2}$ and 1 transform under Lorentz transformations. These transformations depend on six continuous parameters, namely three angles and three boost parameters. In addition there are also discrete symmetries which have some relation with space and time, such as parity reversal, time reversal and charge conjugation. In this chapter we shall introduce some of these symmetries. In a separate section we also demonstrate how, for certain theories, these symmetries can be exploited to facilitate the actual calculations of cross sections and decay rates. Here we are alluding to the so-called *spinor-helicity methods*. In principle these methods are designed to deal with massless particles, but they can also be used in processes where the relevant energies are so high that particle masses can be neglected.

We conclude this chapter with a brief exposition of Monte Carlo methods, which allow the efficient computation of the cross sections and decay rates by computer and are flexible enough to incorporate a large variety of experimental acceptance cuts. Although this topic is not directly related to quantum field theory, it is such an important tool in present-day experimental and theoretical particle physics that we believe it deserves some attention. Readers who are less inclined to get involved in such explicit computations may safely skip the later part of this chapter.

7.1. Discrete symmetries

From the Lagrangian (5.41) for quantum-electrodynamics one can easily conclude that the photon field A_μ transforms under parity reversal as a *vector* field. The reason is that A_μ couples to matter through the "minimal coupling substitution" $\partial_\mu \to \partial_\mu - \mathrm{i}A_\mu$. Because the derivative transforms as a vector under parity reversal, so must the vector field A_μ. Of course, this presupposes that the theory as a whole is invariant under parity reversal. It is important to realize that one can only unambiguously assign symmetry transformations to the fields in the context of a symmetric theory.

However, the above argument is not conclusive. To see this let us consider the following Lagrangian of a massless spin-1 field (5.41), but now with a $\gamma_\mu\gamma_5$ term in its interaction,

$$\mathcal{L} = -\bar\psi\,\gamma^\mu\big(\partial_\mu + \mathrm{i}(q + q'\,\gamma_5)\,A_\mu\big)\psi - m\,\bar\psi\psi\,. \tag{7.1}$$

The interaction with the field A_μ suggests that the spinor fields must transform under gauge transformations according to (cf. 5.42)

$$\psi \to \psi' = \mathrm{e}^{\mathrm{i}(q+q'\gamma_5)\xi}\,\psi\,, \qquad \bar\psi \to \bar\psi' = \bar\psi\,\mathrm{e}^{-\mathrm{i}(q-q'\gamma_5)\xi}\,, \tag{7.2}$$

where the second transformation follows directly from the definition of the adjoint spinor field $\bar\psi$. Note that the exponential factor is now a matrix and can be written as

$$\mathrm{e}^{\mathrm{i}(q+q'\,\gamma_5)\xi} = \mathrm{e}^{\mathrm{i}q\xi}\left[\mathrm{I}\,\cos q'\xi + \mathrm{i}\gamma_5\,\sin q'\xi\right].$$

This was the reason why we wrote the exponential factor in (7.2) on the right-hand side of the row vector $\bar\psi$. The above transformation rules together with the standard transformation rule for the gauge field leave the Lagrangian (7.2) gauge invariant provided $m\,q' = 0$. As long as q and q' are both nonzero (so that m must vanish), it is not possible to assign a definite parity to the gauge field A_μ, as it couples to a fermionic current that is a linear combination of a vector and an axial-vector current. Clearly, the Lagrangian is only invariant under parity reversal provided that either q' or q vanishes, in which case the gauge field tranforms as a vector or as an axial-vector, respectively.

Let us explore the symmetries of this Lagrangian a bit further. The behaviour of a spinor field under parity reversal was already discussed in problem 5.3, where we found the transformation rule

$$\psi(\mathbf{x},x^0) \xrightarrow{P} \eta_p\,\psi^p(\mathbf{x},x^0)\,, \qquad \bar\psi(\mathbf{x},x^0) \xrightarrow{P} \eta_p^*\,\bar\psi^p(\mathbf{x},x^0)\,. \tag{7.3}$$

Here η_p is some phase factor, which must be assigned to ψ such that the theory becomes invariant under parity reversal, and

$$\psi^p(\mathbf{x},x^0) \equiv \mathrm{i}\gamma_4\,\psi(-\mathbf{x},x^0)\,, \qquad \bar\psi^p(\mathbf{x},x^0) \equiv -\mathrm{i}\,\bar\psi(-\mathbf{x},x^0)\,\gamma_4\,. \tag{7.4}$$

The factor i in the above definition is such that it also applies to a real spinor (a so-called Majorana spinor, which we will discuss in more detail shortly). Note that we have $(\psi^p)^p = -\psi$, as was also discussed in problem 5.3. [1] The phase factor in (7.4) can only be complex for a Dirac field. For the standard Dirac action which is invariant under rigid phase transformations, one has the freedom of choosing an arbitrary phase factor in the definition of P. In an interacting theory, this phase factor may need adjusting in order that the interactions with other fields remain invariant under parity reversal (see e.g. problem 7.x). Observe that the phase factors are constrained by the fact that repeated application of the P transformation should should not lead to

[1] The reader may also consult G. Racah, Nuovo Cim. **14** (1937) 322 and C.N. Yang and J. Tiomno, Phys. Rev. **79** (1950) 495, where this phase convention is motivated from various viewpoints.

a proliferation of new symmetries that no nontrivial theory will be able to satisfy. This is clearly no problem when P^2 is proportional to the identity, up to sign factor for the fermion fields, as the theory is invariant under a uniform change of sign of the fermionic fields.

For fermionic bilinears $\bar\psi_1\Gamma\psi_2$, where $\Gamma_{\alpha\beta}$ is some spinor matrix, we find

$$\bar\psi_1^p\Gamma\psi_2^p = \bar\psi_1\,\gamma_4\Gamma\gamma_4\,\psi_2\,. \tag{7.5}$$

For vector and axial-vector bilinears, this result gives

$$\bar\psi_1\gamma_\mu\psi_2 \quad \stackrel{P}{\longrightarrow} \quad -\eta_{p1}^*\,\eta_{p2}\,(1-2\delta_{\mu 0})\,\bar\psi_1\gamma_\mu\psi_2, \tag{7.6}$$

$$\bar\psi_1\gamma_\mu\gamma_5\psi_2 \quad \stackrel{P}{\longrightarrow} \quad \eta_{p1}^*\,\eta_{p2}\,(1-2\delta_{\mu 0})\,\bar\psi_1\gamma_\mu\gamma_5\psi_2\,. \tag{7.7}$$

We remind the reader that the coordinates of the fields are simultaneously changed according to $(\mathbf{x},x^0)\to(-\mathbf{x},x^0)$. With these results it is straightforward to show that the Dirac action is invariant under parity reversal (see problem 7.x).

There is another important transformation acting on the spinor fields, which is induced by complex conjugation. This transformation is called *charge conjugation*, because complex fields are usually employed in the description of electrically charged particles and the charge assignment changes sign when one interchanges a field with its complex conjugate. For a spinor field this interchange is somewhat subtle, as ψ and $\psi^\dagger$ (or $\bar\psi$) do not transform in the same way under Lorentz transformations. Therefore in order to define charge conjugation in such a way that it commutes with Lorentz transformations, one must introduce a charge-conjugation matrix $C_{\alpha\beta}$ (see problem 7.1) such that

$$\psi_\alpha \stackrel{C}{\longrightarrow} \eta_c\,\psi^c_\alpha\,, \qquad \bar\psi_\alpha \stackrel{C}{\longrightarrow} \eta_c^*\,\bar\psi^c_\alpha\,, \tag{7.8}$$

where we assigned a phase factor η_c and

$$\psi^c_\alpha \equiv C_{\alpha\beta}\,\bar\psi_\beta = \big(C\,\bar\psi^{\rm T}\big)_\alpha\,, \qquad \bar\psi^c_\alpha \equiv \psi_\beta\,(\gamma_4^{\rm T}\,C^\dagger\,\gamma_4)_{\beta\alpha}\,, \tag{7.9}$$

Here the superscript T indicates the transpose of a matrix, or the fact that we are writing a column (row) vector such as ψ ($\bar\psi$) as a row (column) vector.

The charge-conjugation matrix C was already introduced in section 5.3 and problem 7.1. In the representation for the gamma matrices chosen in in chapter 5, it takes the form $C=\gamma_4\,\gamma_2$. For general representations of gamma matrices, the properties of the charge conjugation matrix are discussed extensively in appendix E.6. For our purpose the defining condition for $C_{\alpha\beta}$ is

$$C^{-1}\gamma_\mu C = -\gamma_\mu^{\rm T}\,, \tag{7.10}$$

but we stress that this condition cannot be satisfied for any number of space-time dimensions. One can prove that C can be determined uniquely from (7.10) modulo a multiplicative constant, which can be chosen such that C is unitary. Furthermore C is antisymmetric in four space-time dimensions, so that

$$C^\dagger = C^{-1}\,, \qquad C^{\mathrm{T}} = -C\,. \tag{7.11}$$

This result implies that the second formula of (7.9) can be written as

$$\bar\psi^c_\alpha = -\psi_\beta\, C^{-1}_{\beta\alpha} = -\big(\psi^{\mathrm{T}}\, C^{-1}\big)_\alpha\,. \tag{7.12}$$

It is straightforward to determine how quantities bilinear in fermion fields transform under charge conjugation. Using (7.9) and (7.12) we find

$$\bar\psi^c_1\,\Gamma\,\psi^c_2 = -\psi^{\mathrm{T}}_1\, C^{-1}\Gamma C\,\bar\psi^{\mathrm{T}}_2 = \bar\psi_2\, C\Gamma^{\mathrm{T}} C^{-1}\,\psi_1\,. \tag{7.13}$$

Observe that we have added a minus sign when interchanging the order of the spinor fields ψ_1 and ψ_2 in the last line in order to respect Fermi-Dirac statistics; therefore fermion fields should be treated as anticommuting quantities.

Again it is instructive to compare the behaviour of vector and axial-vector bilinears $\bar\psi_1\gamma_\mu\psi_2$ and $\bar\psi_1\gamma_\mu\gamma_5\psi_2$ under C conjugation. From (7.13) we find

$$\bar\psi_1\gamma_\mu\psi_2 \quad \stackrel{C}{\longrightarrow} \quad -\eta^*_{c1}\,\eta_{c2}\,\bar\psi_2\gamma_\mu\psi_1\,, \tag{7.14}$$

$$\bar\psi_1\gamma_\mu\gamma_5\psi_2 \quad \stackrel{C}{\longrightarrow} \quad \eta^*_{c1}\,\eta_{c2}\,\bar\psi_2\gamma_\mu\gamma_5\psi_1\,. \tag{7.15}$$

Hence vector and axial vector bilinears transform with opposite signs under both parity reversal and charge conjugation. Under the combined transformation, denoted by CP, they therefore transform with the same phase, so that the Lagrangian (7.1), which is not separately invariant under P and C for general values of q and q', is still invariant under CP. Under CP the gauge field transforms as

$$A_\mu(\mathbf{x},x^0) \;\stackrel{CP}{\longrightarrow}\; (1-2\delta_{\mu0})\,A_\mu(-\mathbf{x},x^0)\,, \tag{7.16}$$

with no possibility of assigning an arbitrary phase factor.

It is convenient to include γ_5 via chiral projection operators, defined by

$$P_\pm = \tfrac12(1\pm\gamma_5)\,. \tag{7.17}$$

They satisfy

$$P_+^2 = P_+, \qquad P_-^2 = P_-\,, \qquad P_+P_- = P_-P_+ = 0\,. \tag{7.18}$$

Using these projection operators one can decompose every fermion field into chiral components

$$\psi = \psi_L + \psi_R\,, \tag{7.19}$$

with

$$\psi_L \equiv P_+\psi\,, \qquad \psi_R \equiv P_-\psi\,. \tag{7.20}$$

It is obvious from (**to be fixed**) (7.d17) that ψ_L and ψ_R can now independently be assigned to representations of the gauge group. For the abelian example, we thus find (cf. 7.2)

$$\psi_L \to \psi'_L = \mathrm{e}^{\mathrm{i}q_L\,\xi}\psi_L\,, \qquad \psi_R \to \psi'_R = \mathrm{e}^{\mathrm{i}q_R\,\xi}\,\psi_R\,, \tag{7.21}$$

with $q_L = q + q'$ and $q_R = q - q'$. Observe that the gauge transformations take the form of phase transformations when acting on chiral components.

The operators $P_\pm$ are called *chiral* projection operators because in certain limits they project out particles of a certain helicity (chirality = "handedness"). In our conventions a left-handed (right-handed) polarization corresponds to positive (negative) helicity. In section 5.3 we introduced the helicity operator, which measures the spin (in units of $\frac{1}{2}\hbar$) along the direction of motion of the particle. The helicity operator takes the form (cf. 5.72)

$$h(\mathbf{P}) = \frac{-\mathrm{i}\gamma_5\gamma_4\gamma^i P_i}{|\mathbf{P}|}\,,$$

where $\mathbf{P}$ is the momentum of the particle. For vanishing mass or high momenta, the helicity operator tends to minus γ_5 (cf. problem 5.5),

$$h(\mathbf{P})\,u(\mathbf{P}) = -\gamma_5\,u(\mathbf{P}) + \mathrm{O}\big(m/|\mathbf{P}|\big)\,,$$

when acting on a spinor $u(\mathbf{P})$ that satisfies the Dirac equation. Hence for massless fermions or at momenta large compared to the mass, the eigenstates of $P_\pm$ coincide with the helicity eigenstates.

We now summarize some relations for chiral spinor components. First of all we have

$$\bar\psi_L = \tfrac{1}{2}[(1+\gamma_5)\psi_L]^\dagger\gamma_4 = \tfrac{1}{2}\psi_L^\dagger(1+\gamma_5)\gamma_4 = \tfrac{1}{2}\bar\psi_L(1-\gamma_5)\,. \tag{7.22}$$

Likewise

$$\bar\psi_R = \tfrac{1}{2}\bar\psi_R(1+\gamma_5)\,. \tag{7.23}$$

The relations (7.22,7.23) lead directly to the following decompositions for fermionic bilinears,

$$\bar\psi\gamma_\mu\chi = \bar\psi_L\gamma_\mu\chi_L + \bar\psi_R\gamma_\mu\chi_R, \tag{7.24}$$

$$\bar\psi\gamma_\mu\gamma_5\chi = \psi_L\gamma_\mu\chi_L - \bar\psi_R\gamma_\mu\chi_R, \tag{7.25}$$

$$\bar\psi\chi = \bar\psi_R\chi_L + \bar\psi_L\chi_R,$$

$$\bar{\psi}\gamma_5\chi \quad = \bar{\psi}_R\chi_L - \bar{\psi}_L\chi_R. \tag{7.26}$$

Note that the standard fermionic mass terms are formed from products of positive and negative chirality components.

The discussion of P and C can also be given for chiral spinors by projecting the chirality components of the defining equations (7.4) and (7.9). This leads to

$$\psi_L^p \equiv \tfrac{1}{2}(1+\gamma_5)\,\psi^p = \mathrm{i}\gamma_4\,\psi_R\,, \qquad \psi_R^p \equiv \tfrac{1}{2}(1-\gamma_5)\,\psi^p = \mathrm{i}\gamma_4\,\psi_L\,, \tag{7.27}$$
$$\psi_L^c \equiv \tfrac{1}{2}(1+\gamma_5)\,\psi^c = C\bar{\psi}_R^{\mathrm{T}}\,, \qquad \psi_R^c \equiv \tfrac{1}{2}(1-\gamma_5)\,\psi^c = C\bar{\psi}_L^{\mathrm{T}}\,. \tag{7.28}$$

where we used $\gamma_5\gamma_4 = -\gamma_4\gamma_5$ and $\gamma_5 C = C\gamma_5^{\mathrm{T}}$. Hence, under P and C a left-handed spinor is converted into a right-handed one, and vice versa. The above result also shows that it is possible to express a general spin-$\frac{1}{2}$ field in terms of left- and right-handed spinors ψ_L and ψ_R, or, alternatively, in terms of two left-handed spinors, ψ_L and ψ_L^c, or two right-handed spinors ψ_R and ψ_R^c.

Obviously under the combined CP transformation left- and right-handed spinors transform among themselves. One easily verifies that

$$\psi_L^{cp} \equiv \tfrac{1}{2}(1+\gamma_5)\,\psi^{cp} = \mathrm{i}\gamma_4 C\bar{\psi}_L^{\mathrm{T}}T\,, \qquad \psi_R^{cp} \equiv \tfrac{1}{2}(1-\gamma_5)\,\psi^{cp} = \mathrm{i}\gamma_4 C\bar{\psi}_R^{\mathrm{T}}\,. \tag{7.29}$$

Using the definions above we give some additional material about spinors. The charge conjugation matrix C enables one to write down Lorentz-invariant expressions that contain both the original and the C-conjugated fields. An example of this is the so-called Majorana mass term, which reads

$$\mathcal{L}_{\mathrm{mass}} = -\tfrac{1}{2}m\big(\bar{\psi}\psi^c + \bar{\psi}^c\psi\big) = -\tfrac{1}{2}m\big(\bar{\psi}C\bar{\psi}^{\mathrm{T}} - \psi^{\mathrm{T}}C^{-1}\psi\big)\,. \tag{7.30}$$

Obviously this expression is *not* invariant under phase transformations since $\bar{\psi}$ and ψ^c transform in exactly the same way as follows from the definition of ψ^c (7.9).

For electrically neutral fields there is no a priori reason for a description in terms of complex fields and the discussion can be given in terms of two real fields. Complex spinor fields are called *Dirac* fields, whereas real spinor fields are called *Majorana* fields. The reality condition on the latter is simply

$$\psi^c = \psi, \qquad \text{or} \qquad \psi = C\,\bar{\psi}^{\mathrm{T}}. \tag{7.31}$$

The Dirac field can be decomposed into two Majorana fields ψ_1 and ψ_2 according to (to be fixed with "cases")

$$\psi = \psi_1 + \mathrm{i}\psi_2\,, \qquad \text{where} \tag{7.32}$$

The Dirac Lagrangian with a Majorana mass term or an ordinary mass term (i.e. $m\,\bar\psi\psi$) are then almost the same (here we suppress a total divergence),

$$\mathcal{L} = -\bar\psi_1 \partial\!\!\!/ \psi_1 - \bar\psi_2 \partial\!\!\!/ \psi_2 - m(\bar\psi_1\psi_1 \mp \bar\psi_2\psi_2)\,, \tag{7.33}$$

where the upper sign refers to the Majorana mass term. However, this sign can be changed by making a subsequent field redefinition (cf. problem 7.x)

$$\psi_2 = \mathrm{i}\gamma_5\psi_2'\,, \tag{7.34}$$

so that the two expressions coincide. Note that ψ_2' is still a Majorana field, as it satisfies the Majorana condition.

In general one should be careful with constrained fields when introducing possible (continuous) symmetry transformations. Obviously, real scalar fields cannot transform under complex transformations. Majorana fields can still avoid this obstacle, as they can transform under chiral transformations $\psi \to \psi' = \exp(\mathrm{i}\xi\gamma_5)\,\psi$, because ψ' satisfies the Majorana condition. This can be extended to the general case by basing oneself on chiral components. In terms of chiral components the Majorana condition reads

$$\psi_L = C\bar\psi_R^{\mathrm{T}}, \qquad \psi_R = C\bar\psi_L^{\mathrm{T}}. \tag{7.35}$$

This shows that one cannot have spinors in four space-time dimensions that are both Majorana and chiral (so-called Majorana-Weyl spinors). Furthermore, general (nonabelian) transformations on arrays of Majorana spinors take the form

$$\psi_L \to \psi_L' = U\psi_L, \qquad \psi_R \to \psi_R' = U^*\psi_R. \tag{7.36}$$

In order that the standard kinetic term is invariant U must be a unitary transformation, whereas an invariant mass term requires U to be real and orthogonal.

As shown by (7.33) a Dirac field corresponds to two Majorana fields. As a Majorana field contains four independent real components, whereas a chiral field depends on two complex components, one expects the two to be related somehow. It is easy to see that this is indeed the case. A Majorana spinor describes a real spin-$\frac{1}{2}$ particle (so its antiparticle is just the same), whereas a chiral spinor describes a particle and an antiparticle. In the massless case, these correspond to two different helicity states, whereas in the massive case they together comprise the two states of a spin-$\frac{1}{2}$ particle. All this can be verified by rewriting the Majorana Lagrangian in terms of chiral components,

$$\begin{aligned}
\mathcal{L}_{\mathrm{Majorana}} &= -\tfrac{1}{2}\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}m\,\bar\psi\psi \\
&= -\tfrac{1}{2}\big\{\bar\psi_L\partial\!\!\!/\psi_L + \bar\psi_R\partial\!\!\!/\psi_R\big\} - \tfrac{1}{2}m\big\{\bar\psi_L\psi_R + \bar\psi_R\psi_L\big\} \\
&= -\tfrac{1}{2}\big\{\bar\psi_L\partial\!\!\!/\psi_L - \psi_L^{\mathrm{T}}C^{-1}\partial\!\!\!/ C\bar\psi_L^{\mathrm{T}}\big\} - \tfrac{1}{2}m\big\{\bar\psi_L C\bar\psi_L^{\mathrm{T}} - \psi_L^{\mathrm{T}}C^{-1}\psi_L\big\} \\
&= -\bar\psi_L\partial\!\!\!/\psi_L - \tfrac{1}{2}m\big\{\bar\psi_L C\bar\psi_L^{\mathrm{T}} - \psi_L^{\mathrm{T}}C^{-1}\psi_L\big\}\,,
\end{aligned} \tag{7.37}$$

where we dropped a total derivative and used (7.35) and (7.11). The resulting Lagrangian is the Dirac Lagrangian with a Majorana mass term for a left-handed field. Of course, the Lagrangian can also be expressed exclusively in terms of right-handed fields.

7.2. Spinor-helicity methods

7.3. Monte Carlo techniques