22 | Anomalies

So far we have been evaluating Feynman diagrams in a straightforward manner, using dimensional regularization whenever encountering ultraviolet divergences. There are, however, situations where a regularization scheme is obviously not correct. Take for instance the Lorentz invariant four-dimensional epsilon tensor $\varepsilon_{\mu\nu\rho\sigma}$. In $n \neq 4$ dimensions there is no four-rank antisymmetric tensor that is Lorentz invariant. A related quantity in spinor space, which is also Lorentz invariant in precisely four dimensions, is the matrix $\gamma_5 = \frac{1}{24}\mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, with $\varepsilon_{0123} = -1$. Whenever we encounter these quantities in divergent expressions the application of dimensional regularization becomes problematic.

In itself this does not necessarily pose a problem. One might still be able to adopt some definition of the troublesome quantities when moving away from four dimensions and in that way still be able to perform the calculation without ambiguities. However, whatever the definition, one must then show that certain undesirable aspects, such as a lack of Lorentz invariance, either disappear or can be taken care of, for instance, by making appropriate subtractions corresponding to (finite) renormalizations. Alternatively one might be able to find an alternative regularization method that does not suffer from these problems. When this is the case, the above problems are only an artefact of the dimensional regularization method and can be ignored.

However, assume for a moment that there exists no regularization scheme that is entirely free of these difficulties. In that case it is possible that we are dealing with a so-called *anomaly*. This implies that, whatever definition we give to the ambiguous diagrams, we will always have to sacrifice some property of the classical theory. Sometimes this will not necessarily render the theory inconsistent, but in other cases the underlying theory will become ill-defined. Fortunately a lot is known about anomalies and it is usually rather clear whether or not the theory becomes untenable in their presence. In this chapter, where we give an elementary account of anomalies, we will consider both situations. We should stress that we will primarily be concerned with the so-called *chiral anomaly*, which is related to the γ_5-matrix. There also exist other anomalies of a different nature that we will not discuss.

22.1. The chiral anomaly

Consider a Lagrangian describing massive fermions coupled to external vector and axial-vector fields, V_μ and A_μ, and to a pseudoscalar field ϕ,

$$\mathcal{L} = -\bar\psi(\partial\!\!\!/ + m)\psi + \mathrm{i}g_V V_\mu\,\bar\psi\gamma^\mu\psi + \mathrm{i}g_A A_\mu\,\bar\psi\gamma^\mu\gamma_5\psi + \mathrm{i}g_P\phi\,\bar\psi\gamma_5\psi\,. \tag{22.1}$$

The diagrams of interest consist of a closed fermion loop with a certain number of external lines associated with V_μ, A_μ and ϕ. Diagrams of this type have frequently been discussed in other chapters and their corresponding expressions have been written down and evaluated. In the case of divergent diagrams we usually relied on dimensional regularization. Here we are interested in those diagrams that depend crucially on being in four space-time dimensions. All fermions loops are proportional to a trace over a number of gamma matrices. In four dimensions γ_5 anticommutes with all other gamma matrices and its square equals the unit matrix. Therefore, when the number of γ_5-matrices is even, they can be moved through the other gamma matrices and subsequently removed. Spinor traces that are free of γ_5-matrices do not depend sensitively on the number of space-time dimensions, so that one expects that the resulting expressions can in principle be dealt with by employing dimensional regularization.

Obviously, when the trace contains an odd number of γ_5-matrices, it can be written as the trace over a single γ_5 with a number of other gamma matrices. In $n = 4$ space-time dimensions the relevant spinor trace is $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 4\mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}$. However, when moving away from $n = 4$, as one does in dimensional regularization, the answer will depend sensitively on how γ_5 is defined. Assuming that γ_5 still anticommutes with all n gamma matrices leads, for instance, to (for μ, ν, ρ, σ not equal and summing over all possible indices λ),

$$\begin{aligned} n\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) &= \mathrm{Tr}(\gamma^\lambda\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) \\ &= -\mathrm{Tr}(\gamma^\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_5) \\ &= -(n-4-4)\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5)\,, \end{aligned} \tag{22.2}$$

where in the second line we use the cyclicity of the trace, and in the third line we distinguish between the $n - 4$ indices values of λ that are not equal to μ, ν, ρ, σ and the four remaining ones. This equation shows that $(n-4)\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 0$. Hence, only in $n = 4$ dimensions can the trace of γ_5 with four different gamma matrices be non-vanishing! This result is obvious from the fact that in n dimensions there exists no Lorentz-invariant rank-four antisymmetric tensor. Of course, one may also choose γ_5 equal to the product of the gamma matrices associated with the first four coordinates, so that γ_5 anticommutes with γ_μ for $\mu = 1, 2, 3, 4$ but *commutes* with the extra $n - 4$ gamma matrices. Now the trace is different from zero, but Lorentz invariance in n dimensions is lost as we no longer treat all the gamma matrices on equal footing.

Hence, for fermion loops involving an odd number of γ_5-matrices the use of dimensional regularization seems problematic. Of course, as we stressed already, it is in principle possible that there exists some other regularization method that will avoid these problems and yields more reasonable answers. Or, one might hope that the result of the integral is free of divergences, in which case one can stay in four dimensions and evaluate the answer without encountering ambiguities. Nevertheless in certain cases it turns out that problems are unavoidable, irrespective of the regularization prescription that is being used.

To examine the situation more closely consider the triangle diagrams consisting of a closed fermion line with one external axial-vector and two external vector fields attached. Obviously the fermion loop leads to a trace over gamma matrices that involves a single γ_5. The invariant amplitude can be written as

$$\mathcal{M}(\mathrm{A,V,V}) = g_A\, g_V^2\, M_{\mu\nu\rho}(p,q)\,, \tag{22.3}$$

where p and q are the *outgoing* momenta associated with the vector fields carrying indices ν and ρ, respectively. The *incoming* momentum associated with the axial-vector field (with index μ) is therefore $Q = p + q$. There are two one-loop diagrams that contribute, which are shown in figure 22.1. Their corresponding expression equals

$$\begin{aligned} M_{\mu\nu\rho}(p,q) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{Tr}\Big(&\gamma_\mu\gamma_5 \frac{1}{\mathrm{i}(\not k - \not p) + m}\gamma_\nu \frac{1}{\mathrm{i}\not k + m}\gamma_\rho \frac{1}{\mathrm{i}(\not k + \not q) + m} \\ &+ \gamma_\mu\gamma_5 \frac{1}{\mathrm{i}(\not k - \not q) + m}\gamma_\rho \frac{1}{\mathrm{i}\not k + m}\gamma_\nu \frac{1}{\mathrm{i}(\not k + \not p) + m}\Big)\,, \end{aligned} \tag{22.4}$$

where we included an extra minus sign for the closed fermion loop.

It turns out that we will also need the diagrams consisting of a fermion loop with two vector fields and a pseudoscalar field attached. Again we define the invariant amplitude by

$$\mathcal{M}(\mathrm{P,V,V}) = g_P\, g_V^2\, S_{\nu\rho}(p,q)\,. \tag{22.5}$$

The corresponding expression is almost the same as (22.4) and given by

$$\begin{aligned} S_{\nu\rho}(p,q) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{Tr}\Big(&\gamma_5 \frac{1}{\mathrm{i}(\not k - \not p) + m}\gamma_\nu \frac{1}{\mathrm{i}\not k + m}\gamma_\rho \frac{1}{\mathrm{i}(\not k + \not q) + m} \\ &+ \gamma_5 \frac{1}{\mathrm{i}(\not k - \not q) + m}\gamma_\rho \frac{1}{\mathrm{i}\not k + m}\gamma_\nu \frac{1}{\mathrm{i}(\not k + \not p) + m}\Big)\,. \end{aligned} \tag{22.6}$$

This amplitude was already studied in section 8.1 in the calculation of the decay amplitude for $\pi^0 \to \gamma\gamma$, where we observed that the two triangle diagrams are in fact equal. From a more general viewpoint this was discussed in problem 8.3.

Figure 22.1: The two triangle diagrams with three external gauge fields

Clearly both expressions (22.4) and (22.6) involve linearly divergent integrals. Nevertheless the resulting expressions are still finite, owing to the fermion trace which suppresses the divergences. For the second expression this result was already explicitly established in section 8.1. It follows from Lorentz invariance and parity reversal. Namely, (22.6) must be a second-rank tensor that is negative under parity. The only tensor that has these properties and that involves at most two independent momenta is proportional to $\varepsilon_{\nu\rho\lambda\tau}\, p^\lambda q^\tau$. However, the contributions of the divergent integral proportional to *two* external momenta is finite. This follows for instance from Taylor expanding the integrals, in the same way as we did in section 14.3. So the expression (22.6) is finite and unambiguous, provided we insist on manifest invariance under Lorentz transformations and parity reversal. The first expression (22.4) turns out to be finite as well, but here the arguments are more involved, because an explicit Lorentz covariant decomposition (which we intend to present later on) reveals that the integral has contributions that are only *linear* in the external momenta. To see that (22.4) is nevertheless finite we may use the formulae (14.20), which give the infinite parts of certain integrals in dimensional regularization. We refer the reader to problem 22.5 for an explicit treatment. From the fact that the expressions are in fact finite, we anticipate no problems and continue our discussion in the context of $n = 4$ space-time dimensions.

Consider now the following algebraic identities,

$$\begin{aligned}
\frac{1}{\mathrm{i}(\not k + \not q) + m}(\not p + \not q)\gamma_5 \frac{1}{\mathrm{i}(\not k - \not p) + m} = \\
&- \mathrm{i}\gamma_5\, \frac{1}{\mathrm{i}(\not k - \not p) + m} - \frac{1}{\mathrm{i}(\not k + \not q) + m}\, \mathrm{i}\gamma_5 \\
&+ 2\mathrm{i}m\, \frac{1}{\mathrm{i}(\not k + \not q) + m}\gamma_5 \frac{1}{\mathrm{i}(\not k - \not p) + m}\,,
\end{aligned} \tag{22.7}$$

$$\frac{1}{\mathrm{i}(\not k - \not p) + m}(\not p + \not q)\frac{1}{\mathrm{i}\not k + m} = -\mathrm{i}\, \frac{1}{\mathrm{i}(\not k - \not p) + m} + \mathrm{i}\, \frac{1}{\mathrm{i}\not k + m}. \tag{22.8}$$

These identities can be used to find expressions for $(p+q)^\mu M_{\mu\nu\rho}$, $p^\nu M_{\mu\nu\rho}$, $q^\rho M_{\mu\nu\rho}$, $p^\nu S_{\nu\rho}$ and $q^\rho S_{\nu\rho}$. It turns out that, with the exception of the last term on the right-hand side of (22.7), all contributions vanish. To show a typical example, consider for instance the first term in (22.4) corresponding to the first diagram in 22.1, contract with $(p+q)^\mu$ and make use of (22.7) and the cyclicity of the trace,

$$(p+q)^\mu \big[M_{\mu\nu\rho}(p,q)\big]_{\text{first diagram}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}$$
$$\times \Big\{ \mathrm{Tr}\Big(-\mathrm{i}\gamma_5 \frac{1}{\mathrm{i}(\not{k}-\not{p})+m}\gamma_\nu \frac{1}{\mathrm{i}\not{k}+m}\gamma_\rho - \mathrm{i}\gamma_5\gamma_\nu \frac{1}{\mathrm{i}\not{k}+m}\gamma_\rho \frac{1}{\mathrm{i}(\not{k}+\not{q})+m}\Big)$$
$$+ 2\mathrm{i}m\, \mathrm{Tr}\Big(\gamma_5 \frac{1}{\mathrm{i}(\not{k}-\not{p})+m}\gamma_\nu \frac{1}{\mathrm{i}\not{k}+m}\gamma_\rho \frac{1}{\mathrm{i}(\not{k}+\not{q})+m}\Big)\Big\}\,. \tag{22.9}$$

To obtain the corresponding contribution from the second diagram one simply interchanges ν with ρ and p with q. With this contribution we have two more terms that contain two propagators, just as the terms exhibited in the first spinor trace in (22.9). As it turns out, these resulting terms cancel pairwise upon using the cyclicity of the trace and shifting the integration momentum k.[1]

In this way one proves the following relations,

$$(p+q)^\mu M_{\mu\nu\rho}(p,q) = 2\mathrm{i}m\, S_{\nu\rho}(p,q)\,, \tag{22.10}$$
$$p^\nu M_{\mu\nu\rho}(p,q) = 0 = q^\rho M_{\mu\nu\rho}(p,q)\,, \tag{22.11}$$
$$p^\nu S_{\nu\rho}(p,q) = 0 = q^\rho S_{\nu\rho}(p,q)\,. \tag{22.12}$$

As we just recalled the tensor $S_{\nu\rho}$ was evaluated in section 8.1, where we derived the result

$$S_{\nu\rho}(p,q) = -\varepsilon_{\nu\rho\lambda\tau}\, p^\lambda q^\tau \, \frac{1}{m}\, S(Q^2,p^2,q^2)\,, \tag{22.13}$$

where the dimensionless function $S(Q^2,p^2,q^2)$ takes the form

$$S(Q^2,p^2,q^2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \frac{8\mathrm{i}\, m^2}{\big((k-p)^2+m^2\big)\big(k^2+m^2\big)\big((k+q)^2+m^2\big)}\,. \tag{22.14}$$

The function $S(Q^2,p^2,q^2)$ was explicitly calculated for $p^2=q^2=0$ and found equal to

$$S(Q^2,0,0) = \frac{J(\xi)}{2\pi^2\,\xi}\,, \quad \text{with} \quad \xi = -\frac{Q^2}{m^2}\,, \tag{22.15}$$

where $J(\xi)$ is the dimensionless function listed in (8.22). Obviously $S_{\nu\rho}(p,q)$ satisfies the conditions (22.12). However, as we will explain shortly, it is not possible to satisfy also the two preceding identities (22.10) and (22.11)!

[1] Actually in this case one can argue that each of the four terms vanishes separately. Each integral depends on only one linear combination of external momenta and must be proportional to the epsilon tensor. It is not possible to write down rank-two Lorentz tensors that satisfy these criteria. However, this case is special and we prefer the more general arguments presented in the text.

To see this we decompose $M_{\mu\nu\rho}$ into a number of scalar functions. First, we observe that this tensor must be odd under parity reversal (meaning that the tensor should change sign when all spatial vectors change sign, which requires that $M_{\mu\nu\rho}$ must be proportional to the epsilon tensor. Furthermore we use the so-called Schouten identity, cf. (A.17)

$$\varepsilon_{\mu\nu\rho\sigma}\, P_\tau + \varepsilon_{\nu\rho\sigma\tau}\, P_\mu + \varepsilon_{\rho\sigma\tau\mu}\, P_\nu + \varepsilon_{\sigma\tau\mu\nu}\, P_\rho + \varepsilon_{\tau\mu\nu\rho}\, P_\sigma = 0\,, \tag{22.16}$$

which follows from the observation that the left-hand side is totally antisymmetric in five indices. In four dimensions this expression must therefore vanish for any four-vector P! By means of this identity we can ensure that the index μ is always attached to the epsilon tensor without loss of generality. Insisting on Bose symmetry (so that the tensor is invariant under the simultaneous interchange of ν, ρ and p, q, the most general decomposition of $M_{\mu\nu\rho}(p,q)$ takes the form,

$$\begin{aligned} M_{\mu\nu\rho}(p,q) &= A(Q^2,p^2,q^2)\,\varepsilon_{\mu\nu\rho\lambda}\,p^\lambda - A(Q^2,q^2,p^2)\,\varepsilon_{\mu\nu\rho\lambda}\,q^\lambda \\ &\quad + B(Q^2,p^2,q^2)\,p_\nu\,\varepsilon_{\mu\rho\lambda\tau}\,p^\lambda q^\tau - B(Q^2,q^2,p^2)\,q_\rho\,\varepsilon_{\mu\nu\lambda\tau}\,p^\lambda\,q^\tau \\ &\quad + C(Q^2,p^2,q^2)\,p_\rho\,\varepsilon_{\mu\nu\lambda\tau}\,p^\lambda\,q^\tau - C(Q^2,q^2,p^2)\,q_\nu\,\varepsilon_{\mu\rho\lambda\tau}\,p^\lambda\,q^\tau\,, \end{aligned} \tag{22.17}$$

where A, B and C are Lorentz-invariant functions (so that they depend only on p^2, q^2 and Q^2).

With the help of these decompositions we obtain the following results

$$\begin{aligned} (p+q)^\mu M_{\mu\nu\rho}(p,q) &= -\big[A(Q^2,p^2,q^2) + A(Q^2,q^2,p^2)\big]\varepsilon_{\nu\rho\lambda\tau}\,p^\lambda q^\tau\,, \\ p^\nu M_{\mu\nu\rho}(p,q) &= \big[A(Q^2,q^2,p^2) + p^2\,B(Q^2,p^2,q^2) \\ &\qquad - \tfrac{1}{2}(Q^2-p^2-q^2)\,C(Q^2,q^2,p^2)\big]\varepsilon_{\mu\rho\lambda\tau}\,p^\lambda q^\tau\,, \\ q^\rho M_{\mu\nu\rho}(p,q) &= -\big[A(Q^2,p^2,q^2) + q^2\,B(Q^2,q^2,p^2) \\ &\qquad - \tfrac{1}{2}(Q^2-p^2-q^2)\,C(Q^2,p^2,q^2)\big]\varepsilon_{\mu\nu\lambda\tau}\,p^\lambda q^\tau\,. \end{aligned} \tag{22.18}$$

Observe that the functions B and C are supposed to be unabiguously defined, while A is logarithmically divergent by powercounting, and therefore ambiguous.

from here there may be slight discrepancies due to inconsistent conventions of the $\varepsilon_{\mu\nu\rho\sigma}$ tensor! Comparing the right-hand sides of (22.18) to the equations (22.10)-(22.12), yields an unambiguous definition for the function A, as well as to the result

$$\begin{aligned} S(Q^2,p^2,q^2) &= -\left[A(Q^2,p^2,q^2)+A(Q^2,q^2,p^2)\right] \\ &= -\left[p^2\,B(Q^2,p^2,q^2)+q^2\,B(Q^2,q^2,p^2)\right] \\ &\qquad + \tfrac{1}{2}(Q^2-p^2-q^2)\left[C(Q^2,q^2,p^2)+C(Q^2,p^2,q^2)\right] . \end{aligned} \tag{22.19}$$

At this point we discover a discrepancy. According to the above equation $S(Q^2,p^2,q^2)$ tends to zero when $Q^2,p^2,q^2 \to 0$. On the other hand we can explicitly evaluate the function $S(Q^2,p^2,q^2)$ in this limit and we find (cf. 8.23),

$$S(0,0,0) = -\frac{1}{4\pi^2} . \tag{22.20}$$

Hence we see that the explicit results, although finite, disagree with the equations that we derived formally before performing the integral, shifting the integration momentum when necessary. Obviously when the integrals are well defined, no such inconsistency should arise and the formal derivations should be in accord with explicit evaluations.

The difficulty resides in the contributions to $M_{\mu\nu\rho}$ that are *linear* in the external momenta. Owing to the spinor trace, the actual expressions take the form of differences of linearly divergent integrals. Although the divergencies cancel at the end, the finite remainders depend on the way in which the integrals are evaluated. For instance, the definition of the trace may depend on the regularization, so that terms which tend to zero when the regularization is removed, still yield finite terms when multiplied by integrals that tend to infinity when the regularization is removed. This ambiguity is associated with all terms that are *generically* infinite. The terms in (22.4) that are linearly proportional to the external momenta are generically logarithmically divergent. Although their infinite part cancels by virtue of the spinor trace, their finite part is still ambiguous. It thus follows that the function $A(Q^2,p^2,q^2)$ is only determined up to an additive constant! Assuming Lorentz invariance and Bose symmetry $M_{\mu\nu\rho}$ is therefore determined up to

$$M_{\mu\nu\rho} \to M_{\mu\nu\rho} + a\,\varepsilon_{\mu\nu\rho\sigma}\,(p-q)^\sigma , \tag{22.21}$$

where a is an arbitrary constant. In terms of the vector and axial-vector fields the extra term corresponds to the following addition to the Lagrangian

$$\mathcal{L}_{\text{ambiguity}} = -\mathrm{i}a\,g_A\,g_V^2\,\varepsilon^{\mu\nu\rho\sigma}\,A_\mu V_\nu\,\partial_\rho V_\sigma . \tag{22.22}$$

The presence of this ambiguity leads to corresponding ambiguities in the expressions for $(p+q)^\mu M_{\mu\nu\rho}$, $p^\nu M_{\mu\nu\rho}$ and $q^\rho M_{\mu\nu\rho}$. Therefore we should expect the identities (22.10)-(22.12) to hold, up to such ambiguities.

The ambiguous term is implicitly fixed by *assuming* one or two of the identities (22.10)-(22.12) to be valid. According to the previous analysis, it is then not possible to statisfy the remaining identities either. For instance, assuming that (22.11) is valid, we conclude that the function A must tend to zero for vanishing external momenta (cf. (22.18). In that case we must allow for an extra term so as to ensure that the right-hand side exhibits the required behaviour for small external momenta. This means that (22.10) must be replaced by

$$(p+q)^\mu M_{\mu\nu\rho}(p,q) = 2\mathrm{i}m\, S_{\nu\rho}(p,q) + \frac{1}{2\pi^2}\varepsilon_{\nu\rho\lambda\tau}\, p^\lambda q^\tau\, . \tag{22.23}$$

With this extra term there is no longer any inconsistency. Another way in which this result can be formulated is in terms of the following identity,

$$\partial_\mu\big(\mathrm{i}\,\bar\psi\gamma^\mu\gamma_5\psi\big) = 2\mathrm{i}m\,\bar\psi\gamma_5\psi - \frac{\mathrm{i}g_V^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial_\rho V_\sigma - \partial_\sigma V_\rho)\, . \tag{22.24}$$

In the limit $m \to 0$ the axial-vector current is conserved, but the anomalous term violates this conservation law. Classically, the anomalous term is not present, so that we are dealing with a situation where it is not possible to preserve all symmetries of a classical theory at the quantum level.

Of course, there is a certain arbitrariness in our prescription that the equations (22.11) must be preserved. This arbitrariness can be made more explicit by uniformly modifying $M_{\mu\nu\rho}$ according to (22.21), so that (22.10) and (22.11) will take the form

$$\begin{aligned}
(p+q)^\mu M_{\mu\nu\rho}(p,q) &= 2\mathrm{i}m\, S_{\nu\rho}(p,q) + \frac{1-2a}{2\pi^2}\varepsilon_{\nu\rho\lambda\tau}\, p^\lambda q^\tau\, ,\\
p^\nu M_{\mu\nu\rho}(p,q) &= \frac{a}{2\pi^2}\varepsilon_{\mu\rho\lambda\tau}\, p^\lambda q^\tau\, ,\\
q^\rho M_{\mu\nu\rho}(p,q) &= -\,\frac{a}{2\pi^2}\varepsilon_{\mu\nu\lambda\tau}\, p^\lambda q^\tau\, .
\end{aligned} \tag{22.25}$$

By choosing suitable values for the parameter a we can thus shift the effect of the anomaly to certain identities. This phenomenon is typical when dealing with anomalies. An anomaly forces one to modify at least one of the equations, because it is the combined set of equations that are incompatible. However, one still has a choice in deciding which equation should be sacrificed.

22.2. Triangle anomaly

Subsequent sections:
- the consequences of the anomalies; pion decay
- coupling the vector and axial vector currents to gauge fields
- AAA anomaly; nonabelian anomaly; anomaly cancellation

22.3. Once more pion decay

In this section we consider all one-loop diagrams with fermions that may give rise to ultraviolet divergences. Subsequently, in the next section, we turn to the diagrams without fermions, which are more difficult to evaluate. The first set of graphs that we consider here are those with a closed fermion loop. They are the only divergent one-loop diagrams that are proportional to N_f, as one can easily verify. Therefore they form an independent

22.4. Gauge fields

In this section we consider all one-loop diagrams with fermions that may give rise to ultraviolet divergences. Subsequently, in the next section, we turn to the diagrams without fermions, which are more difficult to evaluate. The first set of graphs that we consider here are those with a closed fermion loop. They are the only divergent one-loop diagrams that are proportional to N_f, as one can easily verify. Therefore they form an independent

Problems

22.1. Consider the expression (22.4) for the two triangle diagrams. Let us first concentrate on the momentum integrals, and leave the evaluation of the spinor trace until later. There are two divergent momentum integrals, namely,

$$\begin{aligned} I^{(1)}_{\sigma\lambda\tau}(p,q) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{(k-p)_\sigma k_\lambda (k+q)_\tau}{((k-p)^2+m^2)(k^2-m^2)((k-q)^2-m^2)} \,, \\ I^{(2)}_{\sigma\lambda}(p,q) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{m\, k_\sigma k_\lambda}{((k-p)^2+m^2)(k^2-m^2)((k-q)^2-m^2)} \,. \end{aligned} \tag{1}$$

These integrals can be evaluated with dimensional regularization and we can colloect the infinite terms, represented by $1/\epsilon$ poles, from the equations (14.20). Subsequently contract the result for the $1/\epsilon$ terms with the spinor trace (taken in $n=4$ dimensions and prove that the residue of the pole vanishes for both integrals. This shows that the result for the triangle diagrams is actually finite. Argue now why the finite terms are ambiguous.

22.2. To understand chiral anomalies in a somewhat simpler context we consider quantum electrodynamics in two space-time dimensions. This model is also known as the Schwinger model. It contains a massive fermions coupled to a photon field, described by the following Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \bar\psi(\not\partial + m)\psi + \mathrm{i}e\,\bar\psi \not{V}\psi \,, \tag{1}$$

where e denotes the electric charge of the fermion and V_μ the photon field. Observe that the charge e has the dimension of mass. For the moment we disregard possible

couplings to axial-vector and pseudoscalar fields. They will be considered in the next two problems. In two space-time dimensions, the only divergent diagrams are those that contribute to the vacuum polarization. These diagrams are logarithmically divergent.
Argue that the one-loop vacuum polarization diagram takes the form,

$$\Pi_{\mu\nu}(q) = \Big(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big)\,\Pi(q^2)\,. \tag{2}$$

Find an expression for the function $\Pi(q^2)$, by evaluating the contraction $\eta^{\mu\nu}\,\Pi_{\mu\nu}$. This can be done in two different ways, namely by assuming n dimensions, or by assuming exactly two dimensions. To see the difference evaluate first the expression for $\gamma^\mu \not{k}\gamma_\mu$ in n and in 2 space-time dimensions. For arbitrary n the integral for $\eta^{\mu\nu}\,\Pi_{\mu\nu}$ reads,

$$\eta^{\mu\nu}\,\Pi_{\mu\nu}(q) = \mathrm{Tr}(\mathbf{1})\int \frac{\mathrm{d}^n k}{\mathrm{i}(2\pi)^2}\,\frac{(n-2)(k^2+k\cdot q)+n\,m^2}{(k^2+m^2)((k+q)^2+m^2)}\,, \tag{3}$$

where $\mathrm{Tr}(\mathbf{1})$ equals the trace of the unit matrix in the spinor space (which can be taken two-dimensional). It is is easy to see that this integral is finite. However, it makes a difference whether or not the limit $n\to 2$ is taken before or after this evaluation. In the latter case one rewrites the numerator as $\frac{1}{2}(n-2)[(k^2+m^2)+((k+q)^2+m^2)-q^2]+2\,m^2$, and subsequently make use of the n-dimensional formula (2) of problem 9.2. Show that the results for n and for 2 dimensions lead to (we suppress coupling constants and normalize as in (22.4))

$$\Pi(q^2) = 2\,\mathrm{Tr}(\mathbf{1})\begin{cases} I(q^2/m^2) - I(0) & (n \text{ dimensions}) \\ \\ I(q^2/m^2) & (2 \text{ dimensions}) \end{cases} \tag{4}$$

with the finite, dimensionless, function $I(x)$ defined by

$$I(q^2/m^2) = \frac{1}{\mathrm{i}(2\pi)^2}\int \mathrm{d}^2 k\,\frac{m^2}{(k^2+m^2)((k+q)^2+m^2)}\,. \tag{5}$$

The evaluation in n dimensions is claimed to be consistent with gauge invariance. Can you give an argument based on the explicit result of your calculation why this claim is indeed correct, while the evaluation performed directly in two dimensions must be incorrect?
Use the results from sections 9.2 and 9.3 to determine $I(x)$ (with $x = k^2/m^2$). Analyze the behaviour of $I(x)$ for $|x|\to\infty$, $x\to 0$ and $|x+4|\to 0$. Draw both the real and imaginary parts of $I(x)$. With these results, give $\Pi_{\mu\nu}(q^2)$ for the case of massless fermions.
Finally, determine the full propagator for the vector field in the one-loop approximation for $m=0$, by solving the Dyson equation. What is your conclusion with regard to the photon mass?

22.3. We return to the Lagrangian (1) of problem 22.2 and consider first the definition of the gamma matrices. Verify that they can be defined by $\gamma_1 = \sigma_1$ and

$\gamma_0 = -\mathrm{i}\sigma_3$, where σ_1, σ_3 are the two real Pauli spin matrices; the analogue of γ_5 in two space-time dimensions is given by $\gamma_5 = \gamma_0\gamma_1$, which is hermitean and whose square equals unity. Show that $\gamma_\mu\gamma_5 = \varepsilon_{\mu\nu}\,\gamma^\nu$, where $\varepsilon_{10} = 1$.
Assuming that the fermion mass vanishes, the Lagrangian is invariant under chiral transformations, which generate the product of two U(1) groups. The explicit transformations take the following form on the spinor fields,

$$\psi \to \exp(\tfrac{1}{2}\Lambda_\pm(1 \pm \gamma_5))\,\psi\,. \tag{1}$$

The symmetry implies that we have conserved vector and axial-vector currents, given by $\mathrm{i}\bar\psi\gamma_\mu\psi$ and $\mathrm{i}\bar\psi\gamma_\mu\gamma_\mu\psi$, respectively. Just as in the Lagrangian (22.1) we may couple a vector and an axial vector field to these currents. The vector field couples then to the photon field as in (1). In this Lagrangian the photon field is dynamic and it is again denoted by V_μ. The axial-vector field, denoted by A_μ is treated as an external field
In principle we can now compute the vacuum polarization diagrams for both the vector and axial vector fields. In the previous problem we calculated $\Pi^{\mathrm{VV}}_{\mu\nu}(q) = \Pi_{\mu\nu}(q)$ for the one-loop diagram with two external vector fields. In a similar way we define $\Pi^{\mathrm{AV}}_{\mu\nu}(q)$ as the amplitude of the one-loop diagram with an external axial-vector and a vector field, and $\Pi^{\mathrm{AA}}_{\mu\nu}(q)$ for the diagram with two external axial-vector fields. Write down the expressions for $\Pi^{\mathrm{AV}}_{\mu\nu}(q)$ and $\Pi^{\mathrm{AA}}_{\mu\nu}(q)$ in terms of corresponding integrals. Using formal manipulations (such as given in (22.7) and (22.8)) show that $\Pi^{\mathrm{AV}}_{\mu\nu}$ and $\Pi^{\mathrm{AA}}_{\mu\nu}$ indeed satisfy current conservation.
Subseqently use the relation between the gamma matrices to express $\Pi^{\mathrm{AV}}_{\mu\nu}$ and $\Pi^{\mathrm{AA}}_{\mu\nu}$ algebraically in terms of $\Pi^{VV}_{\mu\nu}$. Current conservation of the three amplitudes, now leads to a set of identities for $\Pi^{\mathrm{VV}}_{\mu\nu}$. Use these to show that $\Pi^{\mathrm{VV}}_{\mu\nu}$ must vanish. Argue, by referring to our earlier evaluation of $\Pi^{\mathrm{VV}}_{\mu\nu}$, that we are dealing with an anomaly because there is no non-trivial solution that is consistent with current conservation of both the vector and the axial-vector current.
Similar arguments can be used to demonstrate that all the one-loop diagrams with an arbitrary number of external vector and axial-vector fields will vanish. A more rigorous diagrammatic proof of this fact can be given by using arguments similar to those used in problem 8.3.

22.4. Consider the same model as in the previous two problems, but now with massive fermions (i.e., $m \neq 0$). The mass term will break the chiral invariance, so that the axial-vector field no longer couples to a conserved current.
First consider $\Pi^{\mathrm{AV}}_{\mu\nu}(q)$. Show that, by Lorentz invariance and parity reversal, it can be expressed in the form.

$$\Pi^{\mathrm{AV}}_{\mu\nu}(q) = \varepsilon_{\mu\nu}\Pi^{\mathrm{AV}}_1(q^2) + \varepsilon_{\mu\rho}q^\rho q_\nu\,\Pi^{\mathrm{AV}}_2(q^2) + \varepsilon_{\nu\rho}q^\rho q_\mu\,\Pi^{\mathrm{AV}}_3(q^2)\,. \tag{1}$$

Argue, based on the structure of the one-loop integral, that Π_2 and Π_3 are both finite functions of q^2.
Prove that $q^\mu\Pi^{AV}_{\mu\nu}(q^2) \propto \Pi^{PV}_\nu(q^2)$, where Π^{PV}_ν refers to the one-loop diagram with one external photon and one pseudoscalar field ϕ (c.f. 22.1). Express this new diagram in terms of the function $I(x)$ defined in (4) of problem 22.2, and argue using

the same arguments as before that $\Pi^{\rm PV}_\nu$ can be wrtitten as,

$$\Pi^{\rm PV}_\nu(q) = {\rm i}\varepsilon_{\nu\rho} q^\rho \, \Pi^{\rm PV}(q^2)\,, \tag{2}$$

with $\Pi^{\rm PV}(q^2)$ a finite function of q^2. Likewise,

$$\Pi^{\rm VV}_{\mu\nu}(q) = \eta_{\mu\nu}\, \Pi^{\rm VV}_1(q^2) + q_\mu q_\nu \, \Pi^{\rm VV}_2(q^2)\,, \tag{3}$$

with $\Pi^{\rm VV}_2(q^2)$ finite. The same decomposition applies to $\Pi^{\rm AA}_{\mu\nu}(q)$
Noting that all functions are finite and unambiguously defined, with the exception of $\Pi^{\rm AV}_1$, $\Pi^{\rm AV}_1$, and $\Pi^{\rm AV}_1$,
Use these results to discuss the apparent discrepancy between the results from the previous two problems.
Consider again the case $m = 0$. Show that by adding a constant term to Π^{AV}_ν one can impose current conservation on either the vector current or the axial-vector current, but not on both.

22.5. XXX