20 | The standard model for electroweak interactions

This chapter deals with the electroweak sector of the standard model. Most elements in its construction have already been introduced in the preceding chapters. Its characteristic features are the $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge group, a single doublet of scalar fields which causes a mixing between the Z-boson and the photon, parametrized by the electroweak mixing angle θ_{W}, and the chiral assignments of the fermions.

The model has turned out to be extremely successful. Present-day experiments confirm predictions derived from it to a very high degree of accuracy. Yet many open questions remain. From a theoretical viewpoint there are many gauge models that one can write down, so naturally there are intriguing questions concerning the possible reasons behind the assumptions that underly this model. A central one regards the generation structure of quarks and leptons, for which the model offers no explanation. The requirement of asymptotic freedom for the strong interactions restricts the number of generations to eight, while there are arguments from astrophysics restricting the number of light neutrinos to three or four. The number of known generations is precisely three, and measurements of Z-boson decays show that there are no additional families, at least none containing fermions lighter than about half the Z-boson mass. Of course, this is all under certain (mild) assumptions, but from a theoretical point of view we know of no argument why different generations, let alone, how many, should exist. Here we should add that an intriguing feature of the generation structure is that it allows for CP violation, provided we have at least three generations.

In this chapter we first discuss a simple prototype model for a pair of fermions, which contains all the essential features of the standard model except for the mixing of different generations. This aspect is dealt with separately in later sections. We briefly discuss the cancellation of anomalies and the constraints this imposes on the standard model. The more intricate aspects of anomalies are discussed at a later point in chapter 22. Finally, in the last section, we formulate the complete model in a renormalizable class of gauges, which will form the starting point for the evaluation of quantum corrections. These and some of the more phenomenological implications of the standard model are further elaborated on in chapter 21.

20.1. The prototype model

In this section we present a prototype model that incorporates all the basic ingredients of the standard electroweak model. It is based on the gauge group SU(2) × U(1) and contains two fermions denoted by p and n. As the gauge group is of dimension 4, there will be four gauge fields; the three gauge fields of SU(2) and the U(1) gauge field, denoted by ${W_\mu}^a$ and B_μ, respectively. However, the Brout-Englert-Higgs mechanism now introduces a novel feature. Initially the three gauge fields of SU(2) and the gauge field of U(1) are massless and have no direct interactions. After the emergence of mass terms, it turns out that precisely one linear combination of these gauge fields remains massless. This field, which will describe the photon, is therefore associated with a nontrivial subgroup of SU(2) × U(1). The underlying reason for the existence of just one massless gauge field is directly linked to the choice of the representation for the scalar field, which transforms as a doublet under SU(2).

However, let us first comment on the way in which the fermions transform under the gauge transformations. Just as before, we decompose the fields in chiral components with the help of the projection operators $\frac{1}{2}(1 \pm \gamma_5)$. Their left-handed components are assigned to a doublet representation of SU(2), and their right-handed counterparts are SU(2) singlets:

$$\psi_L = \begin{pmatrix} p_L \\ n_L \end{pmatrix} \; ; \quad p_R \, ; \quad n_R \, . \tag{20.1}$$

In the original leptonic version of this model p and n correspond to the neutrino and the electron, respectively. The right-handed neutrino was chosen to decouple from the gauge fields in that case, and was a free field (which could be dropped). Noadays, we know that neutrinos have a small mass, as we shall discuss in due course, so that the right-handed neutrino must be kept. For hadrons, p and n may, for instance, correspond to the 'up' and the 'down' quark, respectively.

Now that we have grouped the fields into SU(2) doublets and singlets it remains to specify their transformations under the additional U(1) group. We parametrize these transformations as follows

$$\phi \to \phi' = \mathrm{e}^{\frac{1}{2}\mathrm{i}q\xi} \; \phi \, , \qquad \psi_L \to \psi'_L = \mathrm{e}^{\frac{1}{2}\mathrm{i}q_1\xi} \; \psi_L \, , \tag{20.2}$$

and the singlets as

$$p_R \to p'_R = \mathrm{e}^{\frac{1}{2}\mathrm{i}q_2\xi} \; p_R \, , \qquad n_R \to n'_R = \mathrm{e}^{\frac{1}{2}\mathrm{i}q_3\xi} \; n_R \, , \tag{20.3}$$

where ξ is the parameter of the U(1) transformations and, for the moment, q, q_1, q_2 and q_3 are arbitrary numbers.

We now assume that the potential is such that it acquires a minimum for $\phi \neq 0$. In that case we can decompose ϕ according to (19.15). Setting the matrix Φ to unity, which amounts to choosing the unitary gauge, gives

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \rho(x) \end{pmatrix} , \tag{20.4}$$

with $\rho(x)$ a real scalar field. A crucial observation is that the form of (20.4) remains invariant under a nontrivial U(1) subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$. To identify this subgroup, consider first a somewhat larger subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$ consisting of diagonal matrices. They are parametrized as follows,

$$U(\xi^a, \xi) = \begin{pmatrix} \mathrm{e}^{\frac{1}{2}\mathrm{i}(g\xi^3 + q\xi)} & 0 \\ 0 & \mathrm{e}^{\frac{1}{2}\mathrm{i}(-g\xi^3 + q\xi)} \end{pmatrix} , \tag{20.5}$$

where we have rescaled the SU(2) parameter ξ^3 with the SU(2) gauge coupling constant g according to the procedure outlined for non-abelian gauge fields at the end of section 12.1. The subgroup consisting of the transformations (20.5) involves two independent phase factors and is therefore equal to $\mathrm{U}(1)\times\mathrm{U}(1)$. It is convenient to reparametrize these phases in terms of linear combinations. One combination, denoted by ξ^{EM}, will contribute only to the first phase, $g\xi^3 + q\xi$, and not to the second one, $-g\xi^3 + q\xi$, so that the transformations induced by ξ^{EM} leave the scalar field (20.4) invariant. The latter phase thus coincides with a second linear combination, i.e. $-g\xi^3 + q\xi \propto \xi^{\mathrm{Z}}$. In this way we arrive at the following decomposition of ξ^3 and ξ,

$$\begin{aligned} \xi &= \cos\theta_{\mathrm{W}}\, \xi^{\mathrm{EM}} - \sin\theta_{\mathrm{W}}\, \xi^{\mathrm{Z}} , \\ \xi^3 &= \cos\theta_{\mathrm{W}}\, \xi^{\mathrm{Z}} + \sin\theta_{\mathrm{W}}\, \xi^{\mathrm{EM}} , \end{aligned} \tag{20.6}$$

where the *weak mixing angle* θ_{W} is defined by the condition

$$\tan\theta_{\mathrm{W}} = \frac{q}{g} . \tag{20.7}$$

Indeed, as the reader may verify explicitly, the U(1) group associated with the parameter ξ^{EM} leaves the scalar field (20.4) invariant. Therefore it will remain a manifest local gauge symmetry when the field ρ takes a non-zero value (20.4).[1] Henceforth this group will be denoted by $\mathrm{U}(1)^{\mathrm{EM}}$. Hence ξ^{EM}

[1]Note that this conclusion is independent of the precise form for the value of ϕ; if the field is pointing in a different direction than specified by (20.4), there still is a manifest U(1) invariance, which will involve a different subgroup of SU(2). In the language of section 18.3 the stability subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$ associated with a single doublet field, is equal to either U(1) or to the full $\mathrm{SU}(2) \times \mathrm{U}(1)$ group (when the scalar field vanishes altogether).

parametrizes the electromagnetic gauge transformations in this model, and the weak mixing angle $\theta_{\rm W}$ characterizes the embedding of $\mathrm{U}(1)^{\rm EM}$ into the full gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)$. Although we have not yet considered a Lagrangian, the symmetry structure of the model is already to a large extent determined by the representation content of the scalar fields. The fact that the model of this section has precisely one massless gauge field is a direct consequence of introducing a single doublet field. For other scalar field assignments one obtains a different mass spectrum for the gauge fields. The gauge fields W_μ^3 and B_μ are now redefined in accordance with the decomposition (20.6),

$$\begin{aligned} B_\mu &= \cos\theta_{\rm W}\, A_\mu - \sin\theta_{\rm W}\, Z_\mu \,, \\ W_\mu{}^3 &= \cos\theta_{\rm W}\, Z_\mu + \sin\theta_{\rm W}\, A_\mu \,. \end{aligned} \tag{20.8}$$

Let us now examine how the various gauge fields transform under the two U(1) transformations parametrized by $\xi^{\rm EM}$ and $\xi^{\rm Z}$. Using the infinitesimal gauge transformations of W_μ^a and B_μ in terms of the original parameters of $\mathrm{SU}(2)\times\mathrm{U}(1)$,

$$\delta W_\mu^a = \partial_\mu \xi^a + g\epsilon^a{}_{bc} W_\mu^b\, \xi^c \,, \qquad \delta B_\mu = \partial_\mu \xi \,, \tag{20.9}$$

it follows that the fields A_μ and Z_μ transform according to

$$\delta A_\mu = \partial_\mu \xi^{\rm EM} \,, \qquad \delta Z_\mu = \partial_\mu \xi^{\rm Z} \,, \tag{20.10}$$

which identifies A_μ as the photon field. The other field, denoted by Z_μ, corresponds to a neutral massive vector boson, whose mass will be different from that of the fields $W_\mu^{1,2}$ because of the electroweak mixing. The fields $W_\mu^{1,2}$ are electrically charged since they transform under electromagnetic gauge transformations. It is convenient to decompose them according to

$$W_\mu^\pm = \tfrac{1}{2}\sqrt{2}\,(W_\mu^1 \mp {\rm i} W_\mu^2) \,. \tag{20.11}$$

Under infinitesimal electromagnetic gauge transformations $W_\mu^\pm$ transform as

$$\delta W_\mu^\pm = \pm{\rm i}(g\sin\theta_{\rm W})\,\xi^{\rm EM} W_\mu^\pm , \tag{20.12}$$

which shows that the W-bosons associated with $W_\mu^\pm$ carry an electric charge equal to $\pm(g\sin\theta_{\rm W})$. This charge will be denoted by e, so that we define

$$e = g\sin\theta_{\rm W} = q\cos\theta_{\rm W} \,. \tag{20.13}$$

Henceforth we use a notation in terms of a complex field,

$$\begin{aligned} W_\mu &= W_\mu^+ = \tfrac{1}{2}\sqrt{2}(W_\mu^1 - {\rm i}W_\mu^2) \,, \\ \bar W_\mu &= W_\mu^- = \tfrac{1}{2}\sqrt{2}(W_\mu^1 + {\rm i}W_\mu^2) \,. \end{aligned} \tag{20.14}$$

The coupling of the gauge fields A_μ, Z_μ and W_μ can be conveniently summarized in terms of the generators of the group SU(2) × U(1). Denoting the generator of the U(1) group by $\frac{1}{2}\mathrm{i}q\,Y$, so that Y takes the values 1, q_1/q, q_2/q and q_3/q, for the fields ϕ, ψ_L, p_R and n_R, respectively, the coupling appears in the combination $\mathrm{i}(\frac{1}{2}q\,B_\mu\,Y + gW_\mu{}^a T_a)$, where the hermitean matrices T_a are defined by $t_a = \mathrm{i}T_a$. The quantity Y is often called "weak hypercharge" and measures the U(1) coupling in units of the coupling constant q associated with the scalar field (cf. (20.2)). Hence for the field ϕ the weak hypercharge equals $Y = 1$, while its complex conjugate ϕ^* its value has $Y = -1$. The fermions carry weak hypercharges equal to q_1/q, q_2/q and q_3/q. Similarly the SU(2) is sometimes called "weak isospin". Dropping the overall factor i the four gauge-field couplings can be decomposed as

$$\begin{aligned}\tfrac{1}{2}qB_\mu\,Y + gW_\mu^a\,T_a =&\, eA_\mu\,Q^{\mathrm{EM}} + \frac{g}{\cos\theta_{\mathrm{W}}}Z_\mu\,Q^{\mathrm{Z}}\\ &+ \frac{g}{\sqrt{2}}\Big[W_\mu\,T_+ + \bar{W}_\mu\,T_-\Big]\,,\end{aligned} \tag{20.15}$$

where $T_\pm = T_1 \pm \mathrm{i}T_2$. The electric charge Q^{EM}, measured in units of e, appears straightforwardly upon the substitution of (20.8) into the left-hand side of (20.15). One then finds

$$e\,Q^{\mathrm{EM}} = g\sin\theta_{\mathrm{W}}T_3 + \tfrac{1}{2}q\,\cos\theta_{\mathrm{W}}\,Y = e\left(T_3 + \tfrac{1}{2}Y\right)\,, \tag{20.16}$$

where we made use of (20.13). From group theory we know that the eigenvalues of the matrices T_a are equal to $-m, -m+1, \ldots, m$, where m is a positive integer or half-integer, and furthermore, the sum of the T_3 eigenvalues vanishes, i.e. $\mathrm{Tr}\,(T_a) = 0$. Therefore charge *differences* within SU(2) multiplets are necessarily multiples of e and the average charge within a multiplet is given by one-half times the value of the weak hypercharge. Likewise we derive for the coupling Q^{Z} to the Z-boson (in units of $g/\cos\theta_{\mathrm{W}}$),

$$\begin{aligned}\frac{g}{\cos\theta_{\mathrm{W}}}\,Q^{\mathrm{Z}} =&\, g\cos\theta_{\mathrm{W}}\,T_3 - \tfrac{1}{2}q\sin\theta_{\mathrm{W}}\,Y\\ =&\, \frac{g}{\cos\theta_{\mathrm{W}}}\Big(T_3 - \sin^2\theta_{\mathrm{W}}\,Q^{\mathrm{EM}}\Big).\end{aligned} \tag{20.17}$$

Observe that for the fermions the above formula should be applied to their chiral components separately.

The kinetic terms for the gauge fields can now be presented. We note the

Lie-algebra valued form,

$$\begin{aligned} gW_\mu &= g\,W^a_\mu(\tfrac{1}{2}\mathrm{i}\tau_a) \\ &= \tfrac{1}{2}\mathrm{i}\begin{pmatrix} g\cos\theta_\mathrm{W} Z_\mu + eA_\mu & g\sqrt{2}\,W_\mu \\ & \\ g\sqrt{2}\,\bar{W}_\mu & -g\cos\theta_\mathrm{W} Z_\mu - eA_\mu \end{pmatrix}, \end{aligned} \tag{20.18}$$

and its corresponding field strength tensor,

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g[W_\mu, W_\nu]\,, \tag{20.19}$$

or, in components,

$$G_{\mu\nu}{}^a = \partial_\mu W_\nu{}^a - \partial_\nu W_\mu{}^a + g\,\epsilon_{abc}\,W_\mu{}^b W_\nu{}^c\,. \tag{20.20}$$

It is convenient to decompose the SU(2) field strengths according to

$$\begin{aligned} G_{\mu\nu} &\equiv \tfrac{1}{2}\sqrt{2}\,(G^1_{\mu\nu} - \mathrm{i}G^2_{\mu\nu}) \\ &= \partial^\mathrm{EM}_\mu W_\nu - \partial^\mathrm{EM}_\nu W_\mu + \mathrm{i}g\cos\theta_\mathrm{W}(W_\mu Z_\nu - W_\nu Z_\mu)\,, \\ G^3_{\mu\nu} &= \cos\theta_\mathrm{W}(\partial_\mu Z_\nu - \partial_\nu Z_\mu) + \sin\theta_\mathrm{W}(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &\quad - \mathrm{i}g(W_\mu\bar{W}_\nu - W_\nu\bar{W}_\mu)\,, \end{aligned} \tag{20.21}$$

where $\partial^\mathrm{EM}_\mu W_\nu = (\partial_\mu - \mathrm{i}eA_\mu)W_\nu$. The U(1) field strength becomes

$$\begin{aligned} G^0_{\mu\nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu \\ &= \cos\theta_\mathrm{W}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \sin\theta_\mathrm{W}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\,. \end{aligned} \tag{20.22}$$

In terms of the above field strengths the gauge field Lagrangian equals,

$$\mathcal{L}_G = -\tfrac{1}{2}G_{\mu\nu}\,\bar{G}^{\mu\nu} - \tfrac{1}{4}G^3_{\mu\nu}\,G^{3\mu\nu} - \tfrac{1}{4}G^0_{\mu\nu}\,G^{0\mu\nu}\,, \tag{20.23}$$

and gives rise to the following kinetic terms

$$\begin{aligned} \mathcal{L}_0 = &-\tfrac{1}{2}|\partial^\mathrm{EM}_\mu W_\nu - \partial^\mathrm{EM}_\nu W_\mu|^2 \\ &-\tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\,. \end{aligned} \tag{20.24}$$

Note that the normalization factor for the charged field W_μ is different from the factor of the neutral fields, because it is complex. In addition to the contributions from the electromagnetic derivatives there are cubic and quartic gauge field interactions. These have all be collected in section 20.4 in a renormalizable gauge and in appendix G in the unitary gauge. Here we only note one of these terms which involves the photon field and gives rise to a magnetic moment coupling of the W-boson,

$$\mathcal{L}^{\rm mag} = {\rm i}e(\partial_\mu A_\nu - \partial_\nu A_\mu)W^\mu\,\bar{W}^\nu. \tag{20.25}$$

To derive the expressions for the masses of the W_μ and Z_μ bosons we examine the Lagrangian for the scalar field ϕ,

$$\mathcal{L}_\phi = -|D_\mu\phi|^2, \tag{20.26}$$

where the explicit form for the covariant derivative on ϕ is equal to

$$\begin{aligned} D_\mu\phi = &\begin{pmatrix} \partial_\mu^{\rm EM}\phi_1 \\ \\ \partial_\mu\phi_2 \end{pmatrix} \\ &+ \begin{pmatrix} -\tfrac{1}{2}{\rm i}\dfrac{g}{\cos\theta_{\rm W}}(1-2\sin^2\theta_{\rm W})Z_\mu & -\tfrac{1}{2}\sqrt{2}\,{\rm i}gW_\mu \\ \\ -\tfrac{1}{2}\sqrt{2}\,{\rm i}g\bar{W}_\mu & \tfrac{1}{2}{\rm i}\dfrac{g}{\cos\theta_{\rm W}}Z_\mu \end{pmatrix} \begin{pmatrix} \phi_1 \\ \\ \phi_2 \end{pmatrix} \\ = &\begin{pmatrix} -\tfrac{1}{2}{\rm i}gW_\mu\rho \\ \\ \tfrac{1}{2}\sqrt{2}\left(\partial_\mu\rho + \tfrac{1}{2}{\rm i}\dfrac{g}{\cos\theta_{\rm W}}Z_\mu\rho\right) \end{pmatrix}, \end{aligned} \tag{20.27}$$

where $\partial_\mu^{\rm EM}\phi_1 = (\partial_\mu - {\rm i}e\,A_\mu)\phi_1$. The first equation may be compared to (20.15), (20.16), (20.17); in the second equation we used (20.4). Substituting this result into (20.26) we find,

$$\mathcal{L}_\phi = -\tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{8}\frac{g^2}{\cos^2\theta_{\rm W}}\rho^2 Z_\mu^2 - \tfrac{1}{4}g^2\rho^2|W_\mu|^2\,. \tag{20.28}$$

We can now read off the masses of the W- and Z-bosons after combining (20.24) with (20.28) and substituting $\rho = v$,

$$M_{\rm W} = \tfrac{1}{2}gv, \qquad M_{\rm Z} = \tfrac{1}{2}\frac{gv}{\cos\theta_{\rm W}}. \tag{20.29}$$

This leads to the well-known relation

$$\frac{M_{\rm W}}{M_{\rm Z}} = \cos\theta_{\rm W}\,, \tag{20.30}$$

so that $M_{\rm Z} \geqslant M_{\rm W}$. The contribution to the Lagrangian of the potential for the field ρ does not involve new features and will be recorded in section 20.4 and appendix G.

The gauge boson couplings to the fermions follow directly from substituting covariant derivatives into the free massless Dirac Lagrangians,

$$
\begin{aligned}
\mathcal{L}_{\text{fermion}} &= -\,\bar{\psi}_L \not{D}\psi_L - \bar{p}_R \not{D} p_R - \bar{n}_R \not{D} n_R \\
&= -\,\bar{\psi}_L \hat{\not{\partial}}\psi_L - \bar{p}_R \hat{\not{\partial}} p_R - \bar{n}_R \hat{\not{\partial}} n_R \\
&\quad + \tfrac{1}{2}\sqrt{2}\,\mathrm{i}g\Big(\bar{W}_\mu\,\bar{n}_L\gamma^\mu p_L + W_\mu\,\bar{p}_L\gamma^\mu n_L\Big) \\
&\quad + \frac{\mathrm{i}g}{2\,cos\theta_{\mathrm{W}}} Z_\mu\Big(\bar{p}_L\gamma^\mu p_L - \bar{n}_L\gamma^\mu n_L\Big),
\end{aligned}
\tag{20.31}
$$

where we have used (20.17) and the definition

$$
\hat{\partial}_\mu = \partial_\mu - \mathrm{i}\Big(eA_\mu - gZ_\mu\frac{\sin^2\theta_{\mathrm{W}}}{\cos\theta_{\mathrm{W}}}\Big)Q^{\mathrm{EM}}. \tag{20.32}
$$

Some of these couplings are summarized in appendix G and in table 5.5.

The Fermi coupling constant G_{F} is defined by the strength of the four-fermion coupling (as, for instance, appearing in neutron decay, $n \to p + e^- + \bar{\nu}_e$, caused by the exchange of a charged intermediate W-boson. Since the momentum trasfer in these processes is much smaller than the W-boson mass, we can drop the momentum transfer in the propagator. In that way we find an expression of the G_{F} in this model. given by[2]

$$
\begin{aligned}
\frac{G_{\mathrm{F}}}{\sqrt{2}} &= \frac{1}{4}\,\frac{(\frac{1}{2}\sqrt{2}g)^2}{{M_{\mathrm{W}}}^2} \\
&= \frac{1}{2v^2} \\
&= \frac{e^2}{8{M_{\mathrm{W}}}^2\sin^2\theta_{\mathrm{W}}}\,.
\end{aligned}
\tag{20.33}
$$

At this point let us consider some typical parameter values. From the value of the Fermi coupling constant, $G_{\mathrm{F}} = 1.166 \times 10^{-5}$ GeV^{-2}, we conclude from (20.33) that[3] $v \approx 250$ GeV. Using (20.29) and the value of the W-mass, $M_{\mathrm{W}} = 80.22$ GeV$/c^2$, we deduce that $g^2/4\pi \approx 0.0025$. The value for $\sin^2\theta_{\mathrm{W}}$ follows from $1 - {M_{\mathrm{W}}}^2/{M_{\mathrm{Z}}}^2$ according to (20.30). Substituting the value for the Z mass, $M_{\mathrm{Z}} = 91.187$ GeV$/c^2$, one obtains $\sin^2\theta_{\mathrm{W}} \approx 0.22$, which in turn gives

[2]See problem 20.1.

[3]We are a little cavalier here with units, because we have set $\hbar = c = 1$ thoughout. Therefore the proper dimension of a field (and thus its vacuum-expectation value) is somewhat ambiguous, as we have been introducing them within these restricted conventions. Here we assign a field the dimension of an energy, while for masses we keep the obvious GeV$/c^2$ dimension. The actual dimension of the Fermi coupling constant times $(\hbar c)^3$ is (GeV)$^{-2}$.

$q^2/4\pi \approx 0.072$ after use of (20.13). The results for these parameters can then be confronted with high-precision data for the various electroweak coupling constants for the fermions and the value for the fine-structure constant. All these results lead to excellent agreement, even more so after including one-loop quantum corrections. Some of the one-loop corrections will be discussed in chapter 21.

A specific prediction of the model is the existence of the so-called neutral-current interactions, induced by he exchange of the neutral vector boson Z between the fermions. Historically the discovery of these neutral-current interactions provided the first experimental evidence that the standard model was correct.

We now discuss the generation of fermion masses in this model. Mass terms are constructed from the product of a right-handed and a left-handed fermion field. However, right- and left-handed fermions belong to different SU(2) representations, so that a direct construction of an invariant mass term is excluded. The only way for the fermions to acquire masses is via a Yukawa coupling of the scalar doublet ϕ to products of a right- and a left-handed fermion field. Expanding ϕ about v will then lead to fermionic mass terms. In order to construct the necessary Yukawa couplings we first form two left-handed SU(2) singlets, ψ_1 and ψ_2, by taking the SU(2) invariant products of ψ_L with ϕ,

$$\begin{aligned} \psi_1 &= \sqrt{2}\,\phi^{*a}\psi_{L_a} = \sqrt{2}\,(\phi_1^*\,p_L + \phi_2^*\,n_L)\,, \\ \psi_2 &= -\sqrt{2}\,\epsilon^{ab}\phi_a\psi_{L_b} = \sqrt{2}\,(-\phi_1\,n_L + \phi_2\,p_L)\,. \end{aligned} \tag{20.34}$$

Using the parametrization (20.4) these singlets take the following form

$$\psi_1(x) = \rho(x)\,n_L(x)\,, \qquad \psi_2(x) = \rho(x)\,p_L(x)\,. \tag{20.35}$$

Under U(1) ψ_1 and ψ_2 transform as

$$\begin{aligned} \psi_1 \rightarrow \psi_1' &= \mathrm{e}^{\frac{1}{2}\mathrm{i}(q_1-q)\xi}\,\psi_1\,, \\ \psi_2 \rightarrow \psi_2' &= \mathrm{e}^{\frac{1}{2}\mathrm{i}(q_1+q)\xi}\,\psi_2\,. \end{aligned} \tag{20.36}$$

We can now construct two invariant Yukawa couplings, if we assume the following relations among the U(1) coupling constants,

$$\begin{aligned} q_2 &= q_1 + q\,, \\ q_3 &= q_1 - q\,. \end{aligned} \tag{20.37}$$

The corresponding invariants are, respectively,

$$\begin{aligned} \mathcal{L}_p &= -\,G_p\,\overline{p}_R\psi_2 + \text{h.c.}\,, \\ \mathcal{L}_n &= -\,G_n\,\overline{n}_R\psi_1 + \text{h.c.}\,, \end{aligned} \tag{20.38}$$

which, using (20.35), can be written as

$$\begin{aligned}\mathcal{L}_p &= -G_p\,\rho\,\overline{p}_R p_L + \text{h.c.}\,,\\ \mathcal{L}_n &= -G_n\,\rho\,\overline{n}_R n_L + \text{h.c.}\,.\end{aligned} \tag{20.39}$$

The coupling constants G_p and G_n can be chosen real by absorbing possible phases into the definitions of p_R and n_R. Expanding $\rho(x)$ about the constant v then gives rise to the following expressions for the masses

$$m_p = G_p v\,, \qquad m_n = G_n v\,, \tag{20.40}$$

which shows that the Higgs field ρ tends to couple weakly because it is proportional to the fermion masses divided by v. Indeed for the light quarks, for instance, the coupling to the Higgs is very small indeed (even for a bottom quark it is still about 0.02), but for the top quark, whose mass is about 175 GeV/c^2, the coupling constant is of order unity. Needless to say, such a large coupling raises questions about the validity of perturbation theory.

Let us now briefly outline the two versions of this prototype model. In its *hadronic* version the model requires the presence of two Yukawa couplings in order to generate masses for both the quarks corresponding to p and n. Because of the two restrictions (20.37) we have three independent gauge coupling constants, say, q_1, $\theta_{\rm W}$, and g. Since the Yukawa couplings (20.39) will be invariant under electromagnetic gauge transformations, the coupling of the photon must be purely vectorlike. This can indeed be verified explicitly by considering (20.16) for the right- and left-handed quarks. According to (20.38), the right-handed quarks have hypercharges equal to $y \pm 1$, where the hypercharge of the left-handed quarks is denoted by y. The right-haded quarks are SU(2) singlets, so that their electric charges are proportional to their hypercharge and equal to $\frac{1}{2}e(y \pm 1)$, according to (20.16). Because the left-handed quarks are SU(2) doublets, the same relation (20.16) shows that the electric charges of the left-handed quarks also equal to $\frac{1}{2}e(y \pm 1)$. The choice $y = \frac{1}{3}$ leads to the conventional electric charges for the quarks, namely $\frac{2}{3}e$ and $-\frac{1}{3}e$, respectively. In that case the left-handed quarks have $Y = \frac{1}{3}$, and the right-handed quarks have $Y = \frac{4}{3}$ and $Y = -\frac{2}{3}$.

Incidently, one may also consider the value $y = 1$, leading to the electic charges e and 0 for the fields p and n respectively. These are the values that correspond to the identification of p and n with the proton and the neutron, respectively.

In the *leptonic* version of the model, p and n correspond to the neutrino and its corresponding lepton, respectively. When including a right-handed neutrino, this model will coincide with the hadronic version, and one has to make an appropriate choice for the weak hypercharge. In its original incarnation the right-handed neutrino did not seem to exist experimentally atht the time, so that the field p_R was supresssed. Consequently, the corresponding

Yukawa coupling $\mathcal{L}_p$ and the first condition (20.38) were not present. In that case the model still depends on three independent gauge coupling constants, namely q_1, θ_W, and g. Denoting the hypercharge of the left-handed doublet again by y, then the second condition (20.37) implies that the right-handed lepton must have $Y = y - 1$. The lepton has then a vectorlike coupling to the photon with charge $\frac{1}{2}e(y-1)$. Also the eletric charge of the neutrino remains arbitrary and is equal to $\frac{1}{2}e(y+1)$, as follows again from (20.16). Requiring the (left-handed) neutrino to be electrically neutral requires the choice $y = -1$. In this case the left-handed lepton has $Y = -1$ and the right-handed lepton has $Y = -2$. Hence there is a relative factor 3 between the hypercharges of the left-handed quarks and the left-handed leptons. However, this original model without right-handed neutrino is no longer supported by the data. We will return to this in section 20.3.

20.2. Generations and anomaly cancellation

One of the intriguing features of the standard model is that fermions appear in so-called *generations*. Each generation consists of four pairs of fermions: one lepton doublet and three quark doublets comprising a colour triplet. One generation is thus based on the prototype model of the previous section with four fermion doublets. At present we know that three generations exist in Nature. The charged leptons corresponding to these generations are the electron, the muon and the tau, each with a corresponding neutrino and accompanied by the ‘up’ and ‘down’, the ‘charm’ and ‘strange’, and the ‘top’ and ‘bottom’ quarks (see table 14.1). The neutrinos are almost massless and the mass of the positively charged quark is always bigger than that of the negatively charged quark. All fermion species corresponding to these three generations have been detected. Their masses extend over a large range from the electron mass of 0.5 MeV/c^2 to the mass of the top quark at approximately 175 GeV/c^2.

To combine the quarks and leptons into one generation is straightforward. Leptons and quarks cannot mix, because the SU(3) gauge symmetry acts only on the quarks and not on the leptons. The quarks change *colour* through their interaction with the gluons, while they change their *flavour* through the weak interactions. Likewise the two lepton species in a generation can change into each other via the weak interactions. There are three independent fermion mass parameters per generation, one for the lepton and two for the quarks. Models with several generations are more complicated and quarks coming from different generations can mix through the weak interactions. We return to the phenomenon of generation mixing in the next section.

We already emphasized that the generation structure is one of the most intriguing features of the standard model. One cannot help but wonder why the fermions are organized in multiples of four fermion pairs. There is at least a partial answer to this question. While the prototype model of the previous section is of the renormalizable type, it turns out that there is nevertheless a subtlety at the quantum level, which is related to the fact that the weak vector bosons have both vector and axial-vector interactions with the fermions. It is known that axial-vector couplings can lead to so-called anomalies. So far we have not discussed these anomalies and we present a more detailed account of them in chapter 22. Here it suffices to say that anomalies associated with vector bosons destroy the renormalizability of the theory. Fortunately, anomalies can be avoided by choosing the fermions in certain balanced decompositions of gauge-group representations. Suprisingly enough, the decomposition of the fermions in a single generation is precisely such that the effects of the anomalies cancel!

In fact we can uniquely determine the standard weak hypercharge assignments within a generation by requiring the absence of anomalies. The basic anomaly diagram consists of a triangle diagram in which a closed fermion loop couples to three gauge fields. Such diagrams are linearly divergent, at least generically, but in practice they are finite but ambiguous. When there is an anomaly we cannot assign a value to the diagram (consistent with Lorentz invariance, Bose symmetry and locality) such that all gauge symmetries remain unaffected. The fermion masses are not important in this respect, because their contribution to the triangle diagram is generically less divergent. These issues are explained in chapter 22. To verify whether there are any anomalies one computes the numeric factor for the three-point gauge couplings generated by the triangle graphs, which is proportional to the symmetric tensor

$$D_{abc} = \mathrm{Tr}\,(t_a t_b t_c + t_a t_c t_b)\,. \tag{20.41}$$

The sum in the trace extends over both colour and flavour. The symmetrization is due to Bose symmetry and follows from the fact that there are two ways to construct the triangle diagram for given external gauge fields with generators t_a, t_b and t_c. To see whether or not there is an anomaly for a certain fermion representation, one must construct D_{abc} for both the right- and the left-handed fermions. If they are not equal then one finds an anomaly. For gauge fields with pure vector couplings, such as QCD, the right- and left-handed fermions transform identically under the gauge group, so there is no anomaly for pure QCD. However, for the electroweak $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge group, the right- and the left-handed fermions transform differently so there is a potential anomaly. Fortunately, for certain groups the tensor D_{abc} vanishes identically. An example of this is SU(2). For instance, in the doublet representation one has $t_b t_c + t_c t_b = -\frac{1}{2}\delta_{bc}\,\mathbf{I}$. Taking the trace with a third generator t_a gives zero because the t_a are traceless. So a pure SU(2) theory has no anomalies. Similarly one can show that this happens for all groups SO(N) (see problem 20.2). Furthermore D_{abc} vanishes whenever one of the indices refers to SU(3) and another one to SU(2). Combining all this information one concludes that potential anomaly diagrams fall in three classes. One, denoted by $\langle Bgg \rangle$, corresponds to one U(1) and two SU(3) gauge fields, another one, denoted by $\langle BWW \rangle$, corresponds one U(1) and two SU(2) gauge fields, and the third denoted by $\langle BBB \rangle$, corresponds to three U(1) gauge fields.

Let us consider the case $\langle Bgg \rangle$, where the tensor D_{abc} carries two SU(3) indices and the remaining index refers to the generator Y. Since SU(3) acts identically on all quarks we are left with Y summed over all quark fields. Hence the absence of anomalies requires

$$\sum_{\substack{\text{left-handed}\\ \text{quarks}}} Y = \sum_{\substack{\text{right-handed}\\ \text{quarks}}} Y \,. \tag{20.42}$$

It turns out that this condition is automatically satisfied already in the prototype model. Denoting the hypercharge of the left-handed quark doublet by y, the two right-handed quarks will have $Y = y \pm 1$. Therefore the two components of the left-handed doublet each contributes y to the left-hand side, while the right-handed ones yield $(y+1)+(y-1) = 2y$ to the right-hand side of the equation. Hence the mixed $\langle Bgg \rangle$ anomaly contribution vanishes.

The second case is $\langle BWW \rangle$. Because the SU(2) gauge fields act only on the (left-handed) doublets we obtain

$$\sum_{\substack{\text{fermion}\\ \text{doublets}}} Y = 0 \,. \tag{20.43}$$

Denoting the hypercharge of the left-handed quark doublet by y and of the left-handed lepton doublet by y', this restriction gives

$$3y + y' = 0 \,, \tag{20.44}$$

where the factor 3 originates from the three quark colours. Hence the left-handed lepton and quark hypercharges should differ by a factor -3.

Finally consider the case $\langle BBB \rangle$. Since the gauge field B_μ couples to the weak hypercharge Y the tensor D_{abc} is proportional to Y^3, summed over all right- or left-handed representations. Hence anomaly cancellation requires

$$\sum_{\substack{\text{left-handed}\\ \text{fermions}}} Y^3 = \sum_{\substack{\text{right-handed}\\ \text{fermions}}} Y^3 \,. \tag{20.45}$$

The left-hand side of this equation yields $6y^3 + 2y'^3$, where we counted three quark and one lepton doublets. For the right-hand side it makes a difference whether or not we have right-handed neutrinos. The six right-handed quarks contribute $3(y+1)^3 + 3(y-1)^3 = 6y(y^2+3)$. Without a right-handed neutrino, the right-handed lepton contributes $(y'-1)^3$ and with a right-handed neutrino, we obtain $(y'+1)^3 + (y'-1)^3 = 2y'(y'^2+3)$. Hence (20.45) takes the form

$$6y^3 + 2y'^2 = \begin{cases} 6y(y^2+3) + 2y'(y'^2+3), & \text{(with right-handed neutrino)} \\ 6y(y^2+3) + (y'-1)^3. & \text{(without right-handed neutrino)} \end{cases} \tag{20.46}$$

Combining this result with (20.44) shows that there is no restriction in the presence of a right-handed neutrino, other than (20.44). In case one wishes to suppress the right-handed neutrino, it is necessary to fix the hypercharge assignment of the left-handed leptons to $y' = -1$. This choice leads to the same assignment discussed in the previous section.

This is a striking result! Without a right-handed neutrino the quantum-mechanical consistency of the one-generation model uniquely predicts the electric charges of all the fermions, precisely in agreement with what is observed in Nature. The neutrino is necessarily massless in this case as well as electrically neutral. With a right-handed neutrino such constraints do not exist. The neutrino can be massive and it can also carry an electric charge. Assuming that it is electrically neutral leads to $y' = -1$ and thus again to the standard electric charges for all the fermions. Then the right-handed neutrino is non-interacting; its only role is to enable the neutrino to acquire a mass.

Of course there is no a priori reason why the anomaly cancellation should be realized within one generation. On the other hand, without such a cancellation it would be natural to expect that the mass scales in different generations would be much more comparable. It is also interesting that the cancellation requires a hypercharge relation between quarks and leptons, particles which hitherto could be viewed as totally independent. Of course, the existence of precisely three different generations remains a puzzle.

As we alluded to at the beginning of this section, quarks and leptons cannot mix in the single-generation model, for the simple reason that quarks carry colour while leptons do not. Therefore lepton and baryon number are separately conserved, meaning that the number of leptons minus the number of antileptons, as well as the number of quarks minus the number of antiquarks, should remain separately invariant in reactions between elementary particles. More precisely one assigns one unit of lepton number L to each lepton and minus one unit of lepton number to each antilepton. Lepton number conservation is then associated with the fact that the Lagrangian is invariant under the certain (constant) phase transformations of the lepton fields. Similarly, baryon number B is defined such that one assigns one-third unit of baryon number to each quark and minus one-third unit of baryon number to each antiquark. The factor of one-third is chosen such that physical baryons (consisting of three quarks) carry baryon number one.

The decomposition of fermions into leptons and quarks seems to point at some underlying structure, which cannot be addressed in the framework of the standard model. One might hope to explain the generation structure by assuming that the higher-mass generations can be thought of as excited states of the lowest-mass generation. This would mean that the quarks and leptons are build from even smaller constituents, whose binding forces would somehow give rise to bound states that exhibit the pattern of generations. However, it is then difficult to understand why the excited states should carry the same spin as well as other quantum numbers, while the conservation of lepton types and quark flavours (in the absence of weak interactions) seems equally mysterious in such a framework.

After combining the three generations, leptons and quarks remain very distinctive, but with more generations there is a priori the possibility that the various generations mix in some nontrivial way. For leptons no such mixing has been observed and electron-, muon- and tau-type lepton numbers are separately conserved to a high degree of accuracy. For quarks the situation is different. All the higher-generation quarks are unstable and decay into lower-mass quarks, so that none of the quark flavours, or linear combinations thereof, are conserved. The only exception is baryon number. In the next section we explain how these mixing phenomena are incorporated into the standard model.

20.3. Generation mixing and CP violation

When considering N generations, one introduces $3N$ pairs of quarks, so that all quarks appear in three colours, and N pairs of leptons. However, because quarks and leptons do not mix and the colour degeneracy is trivial, the essence is fully captured by studying N copies of the prototype model. Generations will be labeled by indices $A, B, \ldots = 1, 2, \ldots, N$, so that we consider N doublet fields ψ_{LA} and $2N$ right-handed fields p_{RA} and n_{RA}. All these fields transform identically under the action of the $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge group; in particular they carry the same weak hypercharge. In what follows we concentrate exclusively on the fermions. As far as the Lagrangian is concerned, the kinetic terms for the gauge and scalar fields and the scalar field potential remain as before.

If the generations do not mix, then we simply write down the sum of the fermion Lagrangians, consisting of the properly covariantized kinetic terms (20.31) and the masslike terms (20.38)-(20.39). For the latter we need the obvious generalization of (20.34), namely,

$$\psi_{1A} \equiv \sqrt{2}\,\phi^{*a}\psi_{LaA}\,, \qquad \psi_{2A} \equiv -\sqrt{2}\,\epsilon^{ab}\phi_a\psi_{LbA}\,. \tag{20.47}$$

However, as there is no difference whatsoever between corresponding fields of these generations, there is no reason why these terms should not mix. For instance, the general gauge-invariant kinetic term for the left-handed spinors takes the form,

$$\mathcal{L} = -\sum_{A,B} \mathcal{H}_{AB}\, \bar{\psi}_{LA} \not{D} \psi_{LB}\,, \tag{20.48}$$

Here $\mathcal{H}$ is an N-by-N matrix, which must be hermitean (so that the Lagrangian is real) and positive definite (so that its eigenvalues are positive and the kinetic terms have the correct sign). However, one can always diagonalize an hermitean matrix and absorb the (positive) eigenvalues into the fermion fields. Hence without any loss of generality, we may assume

$$\mathcal{L} = -\sum_{A} \bar{\psi}_{LA} \not{D} \psi_{LA}\,. \tag{20.49}$$

The same argument applies for the kinetic terms for the right-handed fields. Therefore the kinetic terms for the fermion fields are just (20.31), summed over all generations,

$$\mathcal{L}_f = -\sum_{A} \Big[\bar{\psi}_{LA} \not{D} \psi_{LA} + \bar{p}_{RA} \not{D} p_{RA} + \bar{n}_{RA} \not{D} n_{RA}\Big]\,. \tag{20.50}$$

The normalizations of the fields are now determined, but we can still change the left- and right-handed fields by independent unitary rotations. More precisely, under redefinitions

$$\psi_{LA} \to (V_\psi)_{AB}\, \psi_{LB}\,, \qquad \psi_{RA} \to (U_p)_{AB}\, p_{RB}\,, \qquad n_{RA} \to (U_n)_{AB}\, n_{RB}\,, \tag{20.51}$$

where V_ψ, U_p and U_n are arbitrary unitary matrices, the kinetic terms in (20.50) remain unaffected.

After these preparatory remarks we turn to the fermion mass terms, where we encounter the same problem. A priori the various generations can mix in the mass terms, without affecting gauge invariance. Hence, the two Yukawa couplings related to the mass terms can be written as

$$\begin{aligned} \mathcal{L}_p &= -\sum_{A,B} (G_p)_{AB}\, \bar{p}_{RA} \psi_{2B} + h.c.\,, \\ \mathcal{L}_n &= -\sum_{A,B} (G_n)_{AB} \bar{n}_{RA} \psi_{1B} + h.c.\,, \end{aligned} \tag{20.52}$$

where G_p and G_n are now arbitrary complex N-by-N matrices.

Now we make use of a mathematical result (which we will prove in problem 20.3) acccording to which any complex matrix M can be written as a product of two unitary matrices U and V and a positive diagonal matrix D, according to

$$M = U\,D\,V^\dagger\,. \tag{20.53}$$

Using this result, we can decompose the matrices G_p and G_n as (here and henceforth we suppress summation signs)

$$\begin{aligned}(G_p)_{AB} &= (U_p)_{AC}(G_p)_C(V_p^\dagger)_{CB}\,,\\ (G_n)_{AB} &= (U_n)_{AC}(G_n)_C(V_n^\dagger)_{CB}\,,\end{aligned} \tag{20.54}$$

with G_{pA} and G_{nA} positive numbers and V_p, V_n, U_p and U_n unitary $N \times N$ matrices. Without loss of generality we can absorb the matrices U_p and U_n into the definitions of the right-handed fermion fields by substituing,

$$p_{RA} \to (U_p)_{AB}\,p_{RB}\,, \qquad n_{RA} \to (U_n)_{AB}\,n_{RB}\,. \tag{20.55}$$

This redefinition does not affect the kinetic terms for p_R and n_R, as we observed already below (20.50). In the same way we can also absorb V_p or V_n into the doublet fields. However, it is not possible to get rid of both V_p and V_n, because ψ_1 and ψ_2 are both proportional to the same doublet field ψ_L. For instance, making the replacement

$$\psi_{LA} \to (V_p)_{AB}\psi_{LB}\,, \tag{20.56}$$

which again leaves the kinetic terms (20.50) unaffected, yields the same redefinition for SU(2) singlets ψ_1 and ψ_2,

$$(\psi_{1,2})_A \to (V_p)_{AB}(\psi_{1,2})_B\,. \tag{20.57}$$

Hence we are not able to redefine ψ_1 and ψ_2 independently.

If we drop the second right-handed fermions n_R, then all mixing affects can consistently be absorbed into the definition of the fields. This conclusion is directly relevant for the leptons. In the absence of right-handed neutrinos, we have no mixing between different lepton types. Therefore there is a natural conservation of electron- muon- and tau-lepton numbers.

In the presence of two right-handed fermion singlets per generation it is thus no longer possible to diagonalize both Yukawa couplings simultaneously. This mixing phenomenon is known to take place for quarks. After the various definitions derived above (20.53) takes the form

$$(G_p)_{AB} = (G_p)_A\delta_{AB}\,, \qquad (G_n)_{AB} = (G_n)_A V^\dagger_{AB}\,, \tag{20.58}$$

Choosing the form (20.4) for the scalar field the Yukawa interactions have the following form

$$\mathcal{L}_p + \mathcal{L}_n = \rho\left[G_{pA}\,\bar p_{RA}p_{LA} + G_{nA}\,V^\dagger_{AB}\,\bar n_{RA}n_{LB} + h.c.\right], \tag{20.59}$$

with G_{pA} and G_{nA} positive numbers and V a unitary matrix related to the previous matrices by

$$V = V_p^\dagger \, V_n \, . \tag{20.60}$$

It is usually more convenient to work with a diagonal mass matrix. The fermion mass matrix, which follows from setting $\rho = v$ in (20.59), is not diagonal due to the presence of the mixing matrix V. However, the mass matrix can be diagonalized by making another unitary redefinition of the fields n_L, but not of p_L. Redefining the two components in the doublet fields differently interferes, of course, with the structure of the SU(2) gauge invariance. This will reflect itself in a modification of the gauge field interactions that depends on the mixing matrix V. Nevertheless, a unitary redefinition of the fields n_{LA} will not interfere with the interactions of the *neutral* gauge fields, A_μ and Z_μ, because the $\bar{n}_L\gamma^\mu n_L$ terms couple with uniform strength to the neutral gauge fields. This phenomenon is often called the GIM mechanism, after Glashow, Iliopoulos and Maiani who discovered it in the case of two generations (or, equivalently, four quark flavours).

Let us consider the effect of the mixing in more detail. To obtain a diagonal interaction (20.59) we replace n_{LA} according to

$$n_{LA} \to V_{AB} \, n_{LB} . \tag{20.61}$$

Then (20.59)(17.c13) takes the form

$$\mathcal{L}_p + \mathcal{L}_n = \rho \, (G_{pA} \, \bar{p}_A p_A + G_{nA} \, \bar{n}_A n_A). \tag{20.62}$$

When $\rho(x)$ is expanded about the constant value v this leads to masses for the p and n fermions equal to

$$m_{pA} = G_{pA} v, \qquad m_{nA} = G_{nA} v. \tag{20.63}$$

The interactions with the gauge fields can be written down in their standard form, but now on the basis of left-handed doublets $(p_{LA}, V_{AB} n_{LB})$. Indeed, the effect of the mixing matrix V cancels for the neutral currents, whose left-handed terms are linear combinations of $\bar{p}_{LA}\gamma_\mu p_{LA}$ and $\bar{n}_{LA}\gamma_\mu n_{LA}$. But the mixing does manifest itself for the charged current interactions, which take the form

$$\mathcal{L}_{\text{charged}} = \tfrac{1}{2}\sqrt{2}\mathrm{i}g \Big[W_\mu^+ \, \bar{p}_{LA} \gamma^\mu U_{AB} \, n_{LB} + W_\mu^- \, \bar{n}_{LA} \gamma^\mu V_{AB}^\dagger \, p_{LB} \Big] . \tag{20.64}$$

In principle the unitary mixing matrix V depends on N^2 free parameters, which can be decomposed into $\frac{1}{2}N(N-1)$ angles and $\frac{1}{2}N(N+1)$ phases. However it is still possible to absorb some of the phases into the definitions of the fermion fields, without affecting any of the other terms in the Lagrangian. Obviously $2N$ independent phase factors can be absorbed into the fields p_{LA} and n_{LA}, without modifying the mass terms and the neutral gauge interactions. But in the charged gauge interaction, these phase factors will multiply the matrix elements of the mixing matrix V. Note, however, that a *uniform* phase factor for all fields p_R and n_R would cancel in all terms of the Lagrangian, so that in this way we can only modify $2N-1$ phase parameters of the mixing matrix V. Without loss of generality it is thus clear that V can be parametrized in terms of $\frac{1}{2}N(N-1)$ angles and $\frac{1}{2}N(N+1)-(2N-1) = \frac{1}{2}(N-1)(N-2)$ phases. This is under the assumption that the flavours are non-degenerate. If for instance some of the Yukawa coupling constants are equal then the number of independent parameters in U can be further reduced. However, this situation is not realized in Nature.

Let us illustrate this for the case of $N = 2$ and 3 generations, corresponding to four and six quark flavours, respectively. For $N = 2$ the mixing matrix can be parametrized in terms of a single angle, the Cabibbo angle θ,

$$V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{20.65}$$

The charged current interactions then have the form

$$\begin{aligned} \mathcal{L}^{N=2}_{\text{charged}} \;=\; & \tfrac{1}{4}\sqrt{2}\mathrm{i}gW^+_\mu \Big[\cos\theta\,[\bar{u}\gamma^\mu(1+\gamma_5)d + \bar{c}\gamma^\mu(1+\gamma_5)s] \\ & \qquad + \sin\theta\,[\bar{u}\gamma^\mu(1+\gamma_5)s - \bar{c}\gamma^\mu(1+\gamma_5)d] \Big] \\ & + \tfrac{1}{4}\sqrt{2}\mathrm{i}gW^-_\mu \Big[\cos\theta\,[\bar{d}\gamma^\mu(1+\gamma_5)u + \bar{s}\gamma^\mu(1+\gamma_5)c] \\ & \qquad + \sin\theta\,[\bar{s}\gamma^\mu(1+\gamma_5)u - \bar{d}\gamma^\mu(1+\gamma_5)c] \Big] \end{aligned} \tag{20.66}$$

Experimentally the Cabibbo angle is small; $\cos\theta$ can be extracted from nucleon or nuclear beta decay after taking into account radiative corrections. Presently accepted values are $\cos\theta = 0.974 \pm 0.002$ or $\theta = (13.1 \pm 0.5)^\circ$, which is consistent with $\sin\theta = 0.24$ (from strange-particle decays). Thus the strangeness conserving weak processes will dominate at low energy. Notice that there are no phase factors in (20.65)- (20.66), which means that the two-generation model cannot accommodate CP violation in the weak interaction sector.

This is different in the case of three generations, corresponding to six quark flavours. The 3×3 mixing matrix V can now be parametrized in terms of three angles and one phase factor. The latter gives rise to CP-violating interactions.

The standard parametrization of V is

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \tag{20.67}$$

It can be expressed in terms of three angles θ_{12}, θ_{23} and θ_{13}, and one phase δ, in the following way,

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-im\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \tag{20.68}$$

with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. This matrix iis called the the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Note that there are other ways to parametrize this matrix but (20.93) is the one that is most commonly used. All the angles can be chosen between 0-degrees and 90-degrees. The first angle θ_{12} corresponds to the Cabibbo angle and in the case of exact Cabibbo universality θ_{23} and θ_{13} must vanish, which would have the consequence that particles with bottom flavour are stable under weak interactions. The presence of the extra angles introduces a violation of Cabibbo universality corresponding to the situation described by (20.65). Since such violations are small experimentally the extra angles are small. Therefore it is safe to assume that $s_{13} \leqslant s_{23} \leqslant s_{12}$. In the Wolfenstein parametrization one defines

$$\begin{aligned} s_{12} &= \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}\,, s_{23} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right| \\ s_{13}\mathrm{e}^{\mathrm{i}\delta} &= V_{ub}^* = A\lambda^3(\rho + \mathrm{i}\eta) = \frac{A\lambda^3(\bar\rho + \mathrm{i}\bar\eta)\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}[1 - A^2\lambda^4(\bar\rho + \mathrm{i}\bar\eta)]} \end{aligned} \tag{20.69}$$

and one can check that $\bar\rho + \mathrm{i}\bar\eta = -(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)$ is phase independent. If we expand in terms of λ and choose $\bar\rho = \rho(1 - \lambda^2/2 + \cdots)$ then we can write the CKM matrix as

$$V = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - \eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - \mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{20.70}$$

with corrections of order Λ^4. The unitarity of the CKM matrix requires that $\Sigma_i V_{ij} V^*_{ik} = \delta_{jk}$ and $\Sigma_j V_{ij} V^*_{kj} = \delta_{ik}$. These relations define triangles in the $(\bar{\rho}, \bar{\eta})$ plane. Data are usually shown by using the relation

$$V_{ud} V^*_{ub} + V_{cd} V^*_{cb} + V_{td} V^*_{tb} = 0\,, \tag{20.71}$$

and dividing each side by the value of $V_{cd} V^*_{cb}$. Then the vertices are at $(0, 0)$, $(1, 0)$ and $(\bar{\rho}, \bar{\eta})$ and the area is given by one-half of the Jarlskog invariant J defined by

$$\mathrm{Im}[V_{ij} V_{kl} V^*_{il} V^*_{kj}] = J \Sigma_{m,n} \epsilon_{ikm} \epsilon_{jln}\,, \tag{20.72}$$

Many experiments have yielded data for the above parameters. A recent fit for the Wolfenstein parameters gives

$$\begin{aligned} \lambda &= 0.2257^{+0.0009}_{-0.0010}\,, A = 0.814^{+0.021}_{-0.022} \\ \bar{\rho} &= 0.135^{+0.031}_{-0.016}\,, \bar{\eta} = 0.349^{+0.015}_{-0.017} \end{aligned} \tag{20.73}$$

The fit results for the magnitudes of all nine CKM elements are

$$V = \begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415^{+0.0010}_{-0.0011} \\ 0.00874^{+0.00026}_{-0.00037} & 0.0407 \pm 0.0010 & 0.999133^{+0.000044}_{-0.000043} \end{pmatrix}. \tag{20.74}$$

The experimental information on the angles in the triangle are as follows. One defines the angles

$$\begin{aligned} \beta &= \phi_1 = arg\Big(- \frac{V_{cd} V^*_{cb}}{V_{td} V^*_{tb}} \Big) \\ \alpha &= \phi_2 = arg\Big(- \frac{V_{td} V^*_{tb}}{V_{ud} V^*_{ub}} \Big) \\ \beta &= \phi_1 = arg\Big(- \frac{V_{ud} V^*_{ub}}{V_{cd} V^*_{cb}} \Big) \end{aligned} \tag{20.75}$$

The notation is such that the angle ϕ_1 is between the two lines which meet at the origin, the angle ϕ_2 is between the two lines which meet at the point (1,0) and the angle ϕ_3 is the angle between the two lines which meet at the point $(\bar{\rho}, \bar{\eta})$. The world average of measurements yields

$$\sin 2\beta = 0.681 \pm 0.025 \tag{20.76}$$

$$\alpha = (88^{+6}_{-5})^o \tag{20.77}$$

$$\gamma = (77^{+30}_{-32})^o \tag{20.78}$$

Therefore the sum of the three angles of the unitarity triangle is

$$\alpha + \beta + \gamma = (186^{+31}_{-32})^o \tag{20.79}$$

which is consistent with the expectations of the standard model. The Jarlskog invariant is $J = (3.05^{+0.19}_{-0.20}) \times 10^{-5}$. These measurements leave very little room for possible extensions of the standard model. Model independent conclusions on new physics are very unreliable because new flavor physics contain so many new parameters. The minimal supersymmetric model (MSSM) is one such extension but it contains 69 new CP-conserving parameters and 41 CP-violating phases. However one conclusion is that the value of the Jarlskog invariant (the magnitude of CP-violating effects in the Standard Model) is too small to explain the present domination of matter over anti-matter in the observable Universe.

In the Standard Model it is assumed that there are three massless neutrinos ν_i, where $i = 1, 2, 3$ refers to ν_e, ν_μ, and ν_τ respectively. They are all left-handed and lepton number is conserved. Evidence for nonzero neutrino masses is now overwhelming and requires an extension of the model. Finite neutrino masses thus imply the existence of right-handed neutrino components. Within the context of the previous SU(2) $\times$ U(1) models it is easy to accommodate massive neutrinos by repeating the same construction as for quarks. Hence we introduce a unitary leptonic mixing matrix U analogously to (17.c11) which leads to a mixing of neutrino generations. To be explicit assume that there are three neutrino mass eigenstates ν_α where $\alpha = 1, 2, 3$ which are associated with the charged lepton mass eigenstates e, μ and τ which we call l_i, $i = 1, 2, 3$. Since the leptons can mix the weak interaction coupling of a W-boson to leptons is more complicated. If we assume the vertex for the W^+-decay into a specific combination of leptons $l_\alpha + \nu_i$ is $U^*_{\alpha i}$ then the neutrino state created in the decay $W^+ \to l^+_\alpha + \nu$ is the state

$$|\nu_\alpha >= \Sigma_i U^*_{\alpha i} |\nu_i > \ . \tag{20.80}$$

Thus when we refer to a neutrino of flavour α which is produced in association with a charged lepton of flavor α we mean this superposition of mass eigenstates. The unitarity of U, which is implied by invariance of the theory under the CPT transformation allows us to express a specific mass eigenstate ν_i as a superposition of neutrinos with definite flavour, namely

$$|\nu_i >= \Sigma_\beta U_{\beta i} |\nu_\beta > \ . \tag{20.81}$$

When a neutrino with flavour α propagates (in vacuum or matter) it can change into one of flavour β. This quantum mechanical process is called neutrino oscillations.

To illustrate this work with neutrino mass eigenstates in vacuum. Then the amplitude for the oscillation $\nu_\alpha \to \nu_\beta$ is a coherent sum over all contributions from the ν_i, so

$$Amp(\nu_\alpha \to \nu_\beta) = \Sigma_i U^*_{\alpha i} Prop(\nu_i) U_{\beta i} \,. \tag{20.82}$$

Quantum mechanics says that the propagator factor is $\exp(-i m_i \tau_i)$, where m_i is the mass of the ν_i and τ_i is the proper time of the propagation in the ν_i rest frame. If we assume the neutrinos are very relativistic with E the laboratory frame energy and L the laboratory frame distance then (see problem..) the propagator factor is $\exp(-i m_i^2 L/(2E))$. When we square the amplitude the quantities which enter are $\Delta m^2_{ij} = m_i^2 - m_j^2$, L and E which are expressed in eV^2, in km and GeV units respectively. Inserting all the appropriate factors of $\bar{h}$ and c we find

$$\Delta m^2_{ij}(L/4E) \approx 1.27 \Delta m^2_{ij}(\mathrm{eV}^2) L(\mathrm{km})/E(\mathrm{GeV}). \tag{20.83}$$

Now we can square the amplitude and use the unitarity if the U-matrix to find

$$\begin{aligned} &P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha,\beta} \\ &-4\Sigma_{i>j}|\mathrm{RE}(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(1.27 \Delta m^2_{ij}(L/E)] \\ &+2\Sigma_{i>j}|\mathrm{Im}(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(2.54 \Delta m^2_{ij}(L/E)] \,. \end{aligned} \tag{20.84}$$

The probability for an antineutrino oscillation into another antineutrino is equal to that for a neutrino into another neutrino by CPT invariance. There is another relation which follows from the above equation, namely

$$P(\nu_\beta \to \nu_\alpha; U) = P(\nu_\alpha \to \nu_\beta; U^*) \,, \tag{20.85}$$

so from CPT

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; U^*) = P(\nu_\alpha \to \nu_\beta; U^*) \,. \tag{20.86}$$

If U is not real then the neutrino and anti-neutrino oscillation probabilities are different because the last term in the above equation changes sign. It may be the case that only two mass eigenstates are relevant because the third one does not couple significantly to the lepton in question. In that case there is only one Δm^2 and the two-by-two unitary mixing matrix is real and parametrized by one angle θ (like the Cabibbo for quarks). In this case the complicated formula above reduces to

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \sin^2 2\theta \sin^2(1.27 \Delta m^2 (L/E)) \,, \tag{20.87}$$

with $\beta \neq \alpha$ and

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\alpha) = 1 - \sin^2 2\theta \sin^2(1.27\Delta m^2(L/E)) \,. \tag{20.88}$$

The actual experimental situation is more complicated because the neutrinos usually travel through matter such as the Sun or the Earth. In this case they can coherently scatter in the forward direction from their interactions with the matter and this modifies their propagation characteristics. However this scattering does not change their flavour and it can be analysed as follows. There is a Schroedinger equation which describes the evolution of a neutrino state vector, which has one component for each flavour. The Hamiltonian in the equation is a matrix which includes the interaction energies which appear from the coherent forward scattering. The diagonal elements of the Hamiltonian arise from the virtual W exchange with the electrons in the matter. Hence it is proportional to N_e, the number of electrons per unit volume and to the Fermi constant G_F. One usually includes the interaction energy

$$V = \sqrt{2} G_F N_e \,, \tag{20.89}$$

for the $\nu_e - \nu_e$ element of the Hamiltonian. There is also Z-boson exchange but this has the same form for the ν_e, ν_μ and ν_τ elements of the Hamiltonian so it adds a common constant and therefore has no effect.

For example consider what happens to solar neutrinos in the Sun. They are born as ν_e in nuclear reactions and turn into a combination of $\nu_m u$ and ν_τ. Let us call this combination ν_x. The Hamiltonian is a two-by-two matrix in $\nu_e - \nu_x$ space. As a function of the distance r from the center of the Sun, the Hamiltonian has two parts

$$V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{20.90}$$

and

$$H_1 = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \,, \tag{20.91}$$

and

$$H_2 = \begin{pmatrix} V(r) & 0 \\ 0 & 0 \end{pmatrix} \,. \tag{20.92}$$

The first term contains the vacuum Hamiltonian and the second term the matter Hamiltonian. The angleθ is the solar mixing angle and the splitting $\Delta m^2 = m_2^2 - m_1^2$. Here we simply define $\nu_2 = \nu_x$ to be the heavier state so

Δm^2 is positive. $V(r)$ is the interaction energy with $N_e(r)$ the electron density at distance r from the center of the Sun. Note that the $\nu_e - \nu_e$ element of the solar Hamiltonian contains both terms in $\cos 2\theta$ and $V(r)$ so it does distinguish between the two possibilities $\theta < \pi/4$, where the lighter mass eigenstate is more ν_e than ν_x and $\theta' = \pi/2 - \theta$, where the lighter mass eigenstate is more ν_x than ν_e.

lepton type	charged lepton mass [MeVc^{-2}]	neutrino mass [MeVc^{-2}]
electron	$m(e) = 0.5110034 \pm .0000014$	$m_{\nu_e} \leqslant 0.05$
muon	$m(\mu) = 105.65946 \pm .00024$	$m_{\nu_\mu} \leqslant 0.57$
tau	$m(\tau) = 1784 \pm 4$	$m_{\nu_\tau} \leqslant 250$

Table 20.1: Mass values in MeVc^{-2} of the three lepton generations.

In conventional weak interactions it has generally been assumed that neutrinos are massless. Evidence for nonzero neutrino masses have been reported from nuclear reactor experiments but refuted by other experiments. The situation regarding neutrino flux measurements from the Sun is confusing. The experimental situation remains unclear, but has prompted many people to examine the effects of nonzero neutrino masses. One obvious theoretical motivation is that the quarks exhibit flavour mixing, so probably the leptons should mix in a similar way.

Within the context of the previous $SU(2) \times U(1)$ models it is easy to accommodate massive neutrinos by repeating the same construction as for quarks. Hence we introduce a mixing matrix V analogously to (20.57) which leads to a mixing of neutrino generations. Finite neutrino masses thus imply the existence of right-handed neutrino components. For Majorana neutrinos the above schemes have to be modified considerably. We prefer to let the reader delve into the literature by himself and give a few references as a starting point.

20.4. The standard model in renormalizable gauge

In the previous sections we derived and discussed the electroweak interactions in the standard model. However, the form in which the model was presented is rather unsuitable for evaluating quantum corrections. The calculation of these corrections will inevitably lead to terms which diverge more than quadratically and cannot be absorbed into the fields and the coupling constants of the original Lagrangian. Although such divergent terms must cancel amongst themselves when considering *physical* quantities, they make the calculations very cumbersome. As we have pointed out previously the Lagrangian in this formulation coincides with what is conventionally called the *unitary gauge*. However, we have also shown that it is possible to formulate the theory in alternative gauges in which the divergences are less severe and can all be absorbed in the fields and coupling constants of the Lagrangian. Such gauges are called *renormalizable gauges*. It is obvious that it much more convenient to work in such a renormalizable gauge. We already discussed renormalizable gauges in previous chapters so the ideas are not new. The purpose of section is to formulate the standard model in a renormalizable gauge in order to pave the way for the one-loop corrections that we evaluate in the next section.

In the renormalizable gauge the scalar doublet field $\phi = (\phi_1\,,\,\phi_2)$ is no longer set equal to the field ρ, as in (20.4). Instead we keep it general and parametrize its components according to

$$\phi_1 = \mathrm{i}\phi_{\mathrm{W}}\,, \qquad \phi_2 = \frac{\sqrt{2}M_{\mathrm{W}}}{g} + \tfrac{1}{2}\sqrt{2}(H - \mathrm{i}\phi_{\mathrm{Z}})\,, \tag{20.93}$$

which involves the complex scalar field ϕ_{W} (its complex conjugate will be denoted by $\bar{\phi}_{\mathrm{W}}$), and neutral scalar fields H and ϕ_{Z}. The field ϕ_{W} carries the same charge as $W_\mu \equiv W_\mu^+ = (W_\mu^1 - \mathrm{i}W_\mu^2)/\sqrt{2}$. Observe that we directed the vacuum-expectation of ϕ in the same direction as before, such that $\rho = v + H$ with H the field associated with the physical Higgs particle. The other fields ϕ_{W}, $\bar{\phi}_{\mathrm{W}}$ and ϕ_{Z} do not correspond to physical particles. In this section, we express the vacuum-expectation value v and the mass and ϕ^4 coupling constant in terms of g, M_{W} and m_{H}, which represent the SU(2) coupling constant and the masses of W and H in tree approximation. This implies that H will develop a vacuum-expectation value in higher orders of perturbation theory, which, as we shall show in the next chapter, is crucial in order to get sensible results.

For convenience we first list the infinitesimal transformation rules for the gauge fields and the scalars in the conventions that we will use here and in the next chapter. They take the form

$$\begin{aligned}
\delta A_\mu &= \partial_\mu \xi^{\rm EM}\,,\\
\delta W_\mu &= \partial_\mu^{\rm EM}\xi^{\rm W} + {\rm i}\Big(e\,\xi^{\rm EM} + g\cos\theta_{\rm W}\,\xi^{\rm Z}\Big)\,W_\mu - {\rm i}g\cos\theta_{\rm W}\,\xi^{\rm W}\,Z_\mu\,,\\
\delta Z_\mu &= \partial_\mu\xi^{\rm Z} + {\rm i}g\cos\theta_{\rm W}\Big(\bar\xi^{\rm W}\,W_\mu - \xi^{\rm W}\,\bar W_\mu\Big)\,,\\
\delta\phi_{\rm W} &= M_{\rm W}\,\xi^{\rm W} + \tfrac12 g\,\xi^{\rm W}(H - {\rm i}\phi_{\rm Z}) + {\rm i}\Big(e\,\xi^{\rm EM} + g\,\frac{1-2\sin^2\theta_{\rm W}}{2\cos\theta_{\rm W}}\,\xi^{\rm Z}\Big)\phi_{\rm W}\,,\\
\delta\phi_{\rm Z} &= M_{\rm Z}\,\xi^{\rm Z} + \tfrac12\frac{g}{\cos\theta_{\rm W}}\,\xi^{\rm Z}\,H - \tfrac12{\rm i}g\Big(\bar\xi^{\rm W}\,\phi_{\rm W} - \xi^{\rm W}\,\bar\phi_{\rm W}\Big)\,,\\
\delta H &= -\tfrac12\frac{g}{\cos\theta_{\rm W}}\xi^{\rm Z}\,\phi_{\rm Z} - \tfrac12 g\Big(\bar\xi^{\rm W}\,\phi_{\rm W} + \xi^{\rm W}\,\bar\phi_{\rm W}\Big)\,,
\end{aligned}\qquad(20.94)$$

where $\xi^{\rm W} = (\xi^1 - {\rm i}\xi^2)/\sqrt{2}$.

To define a gauge we introduce a gauge-fixing term for each of the gauge fields. Here we assume three independent gauge parameters, denoted by $\lambda_{\rm A}$, $\lambda_{\rm W}$ and $\lambda_{\rm Z}$. The sum of the squares of the gauge-fixing terms leads to the Lagrangian,

$$\begin{aligned}
\mathcal{L}^{\rm g.f.} &= -\tfrac12\Big(\lambda_{\rm A}\,\partial_\mu A^\mu\Big)^2 - \Big|\lambda_{\rm W}\,\partial_\mu^{\rm EM}W^\mu - \frac{M_{\rm W}}{\lambda_{\rm W}}\phi_{\rm W}\Big|^2\\
&\quad -\tfrac12\Big(\lambda_{\rm Z}\,\partial_\mu Z^\mu - \frac{M_{\rm Z}}{\lambda_{\rm Z}}\phi_{\rm Z}\Big)^2\,.
\end{aligned}\qquad(20.95)$$

We can now use the general construction explained in section 13.4 to find the corresponding ghost Lagrangian. We introduce ghost fields $c^{\rm A}$, $c^{\rm W}$ and $c^{\rm Z}$ and corresponding antighost fields $b_{\rm A}$, $b_{\rm W}$ and $b_{\rm Z}$. Observe that $c^{\rm W}$ and $b_{\rm W}$ are complex and carry the same charge as the gauge field W_μ, while the other (anti)ghost fields are real and electrically neutral [4]. The ghost Lagrangian is determined from the variation of the gauge-fixing terms under the infinitesimal gauge transformations, given in (20.94). All of this was explained previously (cf. sections 13.3 and 13.4); the only new feature is the use of a complex basis for the charged fields. However, by decomposing complex into real fields, we can make direct contact with the prescription presented in section 13.4.

It is convenient to decompose the ghost Lagrangian into three terms. The first one comprises the kinetic terms, covariantized with respect to electro-

[4] Note that there are also ghost fields associated with the gluons of quantum chromodynamics. Because we have already introduced them in chapters 13 and 14 and they do not couple to the weak and the electromagnetic sector, there is no reason to include them here.

magnetic gauge transformations,

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm ghosts} = \quad & -\mathrm{i}\lambda_{\rm A}\,(\partial_\mu b_{\rm A})\,(\partial^\mu c^{\rm A}) \\ & -\mathrm{i}\lambda_{\rm W}\Big[(\partial^{\rm EM}_\mu \bar b_{\rm W})\,(\partial^{{\rm EM}\,\mu} c^{\rm W}) + (M_{\rm W}/\lambda_{\rm W})^2\,\bar b_{\rm W}\,c^{\rm W} \\ & \qquad +(\partial^{\rm EM}_\mu b_{\rm W})\,(\partial^{{\rm EM}\,\mu}\bar c^{\rm W}) + (M_{\rm W}/\lambda_{\rm W})^2\,b_{\rm W}\,\bar c^{\rm W}\Big] \\ & -\mathrm{i}\lambda_{\rm Z}\Big[(\partial_\mu b_{\rm Z})\,(\partial^\mu c^{\rm Z}) + (M_{\rm Z}/\lambda_{\rm Z})^2\,b_{\rm Z}\,c^{\rm Z}\Big]\,, \end{aligned} \tag{20.96}$$

which shows that the ghost associated with the photon is massless, whereas the other ghosts have gauge-dependent masses equal to $M_{\rm W}/\lambda_{\rm W}$ and $M_{\rm Z}/\lambda_{\rm Z}$. Then we distinguish two types of interaction terms. The first one is diagonal in the sense that it is proportional to a ghost and an antighost field of the same type. These terms read as follows,

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm ghosts} \;=\; & -\lambda_{\rm W}\,g\cos\theta_{\rm W}\,Z_\mu\Big[(\partial^\mu \bar b_{\rm W})\,c^{\rm W} - (\partial^\mu \bar b_{\rm W})\,c^{\rm W}\Big] \\ & -\tfrac12\mathrm{i}\lambda_{\rm W}^{-1}\,gM_{\rm W}\Big[\bar b_{\rm W}\,c^{\rm W}\,(H-\mathrm{i}\phi_{\rm Z}) + b_{\rm W}\,\bar c^{\rm W}\,(H+\mathrm{i}\phi_{\rm Z})\Big] \\ & -\tfrac12\mathrm{i}\lambda_{\rm Z}^{-1}gM_{\rm Z}\,b_{\rm Z}\,c^{\rm Z}\,H\,. \end{aligned} \tag{20.97}$$

The remaining terms are not diagonal in the ghost and antighost fields. They take the form

$$\begin{aligned}\mathcal{L}^{(3)}_{\rm ghosts} = \quad & +\mathrm{i}g\cos\theta_{\rm W}\lambda_{\rm W}W_\mu\Big[(\partial_\mu\bar b_{\rm Z})c^{\rm W} - (\partial_\mu\bar b_{\rm W})c^{\rm Z}\Big] \\ & +\mathrm{i}e\lambda_{\rm W}W_\mu\Big[(\partial_\mu\bar b_{\rm A})c^{\rm W} - (\partial_\mu\bar b_{\rm W})c^{\rm A}\Big] \\ & +\mathrm{i}g\cos\theta_{\rm W}\lambda_{\rm W}\bar W_\mu\Big[(\partial_\mu\bar b_{\rm W})c^{\rm Z} - (\partial_\mu\bar b_{\rm Z})c^{\rm W}\Big] \\ & +\mathrm{i}e\lambda_{\rm W}W_\mu\Big[(\partial_\mu\bar b_{\rm W})c^{\rm A} - (\partial_\mu\bar b_{\rm A})c^{\rm W}\Big] \\ & +\mathrm{i}e\lambda_{\rm A}A_\mu\Big[(\partial_\mu\bar b_{\rm W})c^{\rm W} - (\partial_\mu\bar c_{\rm W})b^{\rm W}\Big] \\ & -\tfrac12 gM_{\rm W}\lambda_{\rm W}^{-1}\Big[\bar b_{\rm W}c^{\rm W}H + \bar b_{\rm W}c^{\rm W}H + \frac{1}{\cos^2\theta_{\rm W}}\bar b_{\rm Z}c^{\rm Z}H\Big] \\ & -\tfrac12\mathrm{i}g(1-2\sin^2\theta_{\rm W})M_{\rm Z}\lambda_{\rm W}^{-1}\Big[\bar b_{\rm W}c^{\rm Z}\phi_{\rm W} - b_{\rm Z}\bar c^{\rm W}\bar\phi_{\rm W}\Big] \\ & +\tfrac12\mathrm{i}g\lambda_{\rm Z}^{-1}M_{\rm Z}\Big[\bar b_{\rm Z}c^{\rm W}\phi_{\rm W} - b_{\rm W}\bar c^{\rm Z}\bar\phi_{\rm W}\Big] \\ & +\mathrm{i}eM_{\rm W}\lambda_{\rm W}^{-1}\Big[\bar b_{\rm W}c^{\rm A}\phi_{\rm W} - b_{\rm A}\bar c^{\rm A}\bar\phi_{\rm W}\Big] \\ & +\tfrac12\mathrm{i}g\lambda_{\rm W}^{-1}M_{\rm W}\Big[\bar b_{\rm W}c^{\rm W}\phi_{\rm Z} - b_{\rm W}\bar c^{\rm W}\bar\phi_{\rm Z}\Big] \end{aligned} \tag{20.98}$$

Let us now turn to the scalar-field Lagrangian,

$$\mathcal{L}_{\rm scalars} = -|D_\mu\phi_1|^2 - |D_\mu\phi_2|^2 + \mu^2(|\phi_1|^2+|\phi_2|^2) - \lambda(|\phi_1|^2+|\phi_2|^2)^2\,, \tag{20.99}$$

The covariant derivatives of the separate fields follow directly from (20.94),

$$
\begin{aligned}
D_\mu\phi_{\rm W} &= \partial_\mu^{\rm EM}\phi_{\rm W} - M_{\rm W}W_\mu - \tfrac{1}{2}gW_\mu(H-{\rm i}\phi_{\rm Z}) - \tfrac{1}{2}{\rm i}gZ_\mu\frac{1-2\sin^2\theta_{\rm W}}{\cos\theta_{\rm W}}\,\phi_{\rm W}\,,\\
D_\mu\phi_{\rm Z} &= \partial_\mu\phi_{\rm Z} - M_{\rm Z}Z_\mu - \tfrac{1}{2}\frac{g}{\cos\theta_{\rm W}}\,Z_\mu\,H + \tfrac{1}{2}{\rm i}g(\bar W_\mu\phi_{\rm W} - W_\mu\bar\phi_{\rm W})\,,\\
D_\mu H &= \partial_\mu H + \tfrac{1}{2}\frac{g}{\cos\theta_{\rm W}}Z_\mu\,\phi_{\rm Z} + \tfrac{1}{2}g(\bar W_\mu\phi_{\rm W} + W_\mu\bar\phi_{\rm W})\,. \qquad (20.100)
\end{aligned}
$$

Following the treatment in (20.27) through (20.31), (20.96)(17.d4) yields

$$
\begin{aligned}
\mathcal{L}^{(1)}_{\rm scalars} = \; & -\tfrac{1}{2}(\partial_\mu H)^2 - \tfrac{1}{2}m_{\rm H}^2H^2\\
& -|\partial_\mu^{\rm EM}\phi_{\rm W}|^2 - M_{\rm W}^2|\phi_{\rm W}|^2 - \tfrac{1}{2}(\partial_\mu\phi_{\rm Z})^2 - \tfrac{1}{2}M_{\rm Z}^2(\phi_{\rm Z})^2 \qquad (20.101)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm scalars} = \; & -gM_{\rm W}H|W_\mu|^2 - \tfrac{1}{2}g\frac{1}{\cos\theta_{\rm W}}M_{\rm Z}HZ_\mu^2\\
& -\tfrac{1}{4}g\frac{m_{\rm H}^2}{M_{\rm W}}\Big[H^3 + H\phi_{\rm Z}^2 + 2H|\phi_{\rm W}|^2\Big]\\
& +\tfrac{1}{2}{\rm i}g\phi_{\rm Z}\Big[\,\bar W_\mu\,\partial_\mu^{\rm EM}\phi_{\rm W} + \partial_\mu^{\rm EM}\,\bar\phi_{\rm W}\,W_\mu\,\Big]\\
& +\tfrac{1}{2}gH\Big[\,\bar W_\mu\,\partial_\mu^{\rm EM}\phi_{\rm W} + \partial_\mu^{\rm EM}\,\bar\phi_{\rm W}W_\mu\,\Big]\\
& +{\rm i}g\frac{1-2\sin\theta_{\rm W}^2}{2\cos\theta_{\rm W}}Z_\mu\Big[\bar\phi_{\rm W}\overset{\leftrightarrow}{\partial}{}^{\rm EM}_\mu\,\phi_{\rm W}\Big] \qquad (20.102)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(3)}_{\rm scalars} = \; & -\tfrac{1}{32}g^2\frac{m_{\rm H}^2}{M_{\rm W}^2}\Big[H^4 + \phi_{\rm Z}^4 + 4(|\phi_{\rm W}|^2)^2 + 4\phi_{\rm Z}^2|\phi_{\rm W}|^2\\
& +4|\phi_{\rm W}|^2H^2 + 2\phi_{\rm Z}^2H^2\Big]\\
& -\tfrac{1}{4}g^2|W_\mu|^2\Big[H^2 + \phi_{\rm Z}^2 + 2|\phi_{\rm W}|^2\Big]\\
& -\tfrac{1}{8}g^2\frac{1}{\cos\theta_{\rm W}^2}Z_\mu^2\Big[H^2 + \phi_{\rm Z}^2 + 2(1-2\sin\theta_{\rm W}^2)^2|\phi_{\rm W}|^2\Big]\\
& -\tfrac{1}{2}g^2\frac{\sin\theta_{\rm W}^2}{\cos\theta_{\rm W}}Z_\mu\phi_{\rm Z}\Big[W_\mu\bar\phi_{\rm W} + \bar W_\mu\phi_{\rm W}\Big]\\
& -\tfrac{1}{2}{\rm i}g^2\frac{\sin\theta_{\rm W}^2}{\cos\theta_{\rm W}}Z_\mu H\Big[W_\mu\bar\phi_{\rm W} - \bar W_\mu\phi_{\rm W}\Big]\,, \qquad (20.103)
\end{aligned}
$$

where we have used relations (20.13), (20.28) and (20.33)(17.34).

The Lagrangian which contains the couplings between the fermion fields and the scalar fields follows from substituting (20.93) into (20.34)(17.35) and using (20.40)(17.40). This yields

$$
\begin{aligned}
\mathcal{L}_{\text{fermions}} = \quad & -\bar{e}_l\not{\partial}e_l - m_l\bar{e}_l\gamma_\mu e_l - \bar{\nu}_l\not{\partial}\nu_l - \bar{p}_A(\not{\partial}+m_{pA})p_A - \bar{n}_A(\not{\partial}+m_{nA})n_A \\
& +\tfrac{1}{4}\sqrt{2}\frac{m_l}{M_{\rm W}}\Big[-\phi_{\rm W}\bar{\nu}_l(1-\gamma_5)e_l + \bar{\phi}_{\rm W}\bar{e}_l(1+\gamma_5)\nu_l\Big] \\
& -\tfrac{1}{2}g\frac{m_l}{M_{\rm W}}\Big[H\bar{e}_le_l + {\rm i}\phi_{\rm Z}\bar{e}_l\gamma_5e_l\Big] \\
& +\tfrac{1}{4}\frac{{\rm i}g}{M_{\rm W}}\phi_{\rm W}\Big[-m_{nB}(\bar{p}_AV_{AB}(1-\gamma_5)n_B) + m_{pA}(\bar{p}_AV_{AB}(1+\gamma_5)n_B)\Big] \\
& +\tfrac{1}{4}\frac{{\rm i}g}{M_{\rm W}}\bar{\phi}_{\rm W}\Big[m_{nA}(\bar{n}_AV^\dagger_{BA}(1+\gamma_5)p_B) - m_{pA}(\bar{n}_AV^\dagger_{AB}(1-\gamma_5)p_B)\Big] \\
& -\tfrac{1}{2}g\frac{m_{pA}}{M_{\rm W}}\Big[H\bar{p}_Ap_A + {\rm i}\phi_{\rm Z}\bar{p}_A\gamma_5p_A\Big] - \tfrac{1}{2}g\frac{m_{nA}}{M_{\rm W}}\Big[H\bar{n}_An_A + {\rm i}\phi_{\rm Z}\bar{n}_A\gamma_5n_A\Big] .
\end{aligned}
\tag{20.104}
$$

where l runs over e, μ and τ. The subscript on the quark fields p_A runs over u, c and t while the corresponding subscript on the fields n_A runs over d, s and b.

Problems

20.1. Consider the prototype model of section 20.1, but now with two doublet fields, one correspondng to the electron and its neutrino ν_e, and the other one to the proton and the neutron. The hypercharges for the left-handed doublets are equal to $Y = -1, +1$, respectively, as discussed in the text at the end of section 20.1. Derive the electric charges for all the four fermions. Subsequently, write down the amplitude for the neutron decay: $n \to p + e + \bar{\nu}_e$. This process is mediated by a W-boson. Due to the fact that the neutron-proton mass difference is very small as compared to the W-boson mass, the momentum dependence in the W-propagator can be suppressed. Compare the result for the decay amplitude and compare its effective coupling constant with (20.33).
Subsequently consider the scattering reaction $\nu_e + p \to \nu_e + p$, which is mediated by the Z-boson. Write down the corresponding amplitude in the same approximation of small momentum transfer.

20.2. Consider a group with generators t_a and form the following trace $D_{abc} = \mathrm{Tr}(t_at_bt_c + t_at_ct_b)$. For the group SU(2), the generators are expressed into the Pauli spin matrices according to $t_a = \frac{1}{2}\tau_a$. Prove that $D_{abc} = 0$. Subsequently consider the group SO(N). In the defining representation, the $\frac{1}{2}N(N-1)$ generators take the form of real anti-symmetric N-by$N$ matrices t_a. Show again that $D_{abc} = 0$.

20.3. In this problem we consider the decay of a neutral scalar particle, such as the standard-model Higgs boson, into two photons,

$$H(Q) \to \gamma(p) + \gamma(q)\,, \tag{1}$$

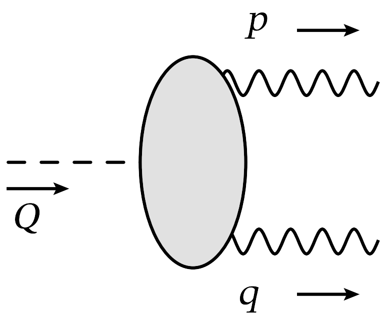

Figure 20.1: Higgs decay into two photons

as is described by the diagram in fig. 20.1. The relevant Lagrangian for describing this process is

$$\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi + ie\,\bar{\psi}\not{A}\psi + g\,H\bar{\psi}\psi\,, \tag{2}$$

where ψ, A_μ, and H are the fermion, photon and Higgs fields, respectively, e is the fermionic electric charge and g is the coupling constant of the Higgs boson.
We start by decomposing the general H-γ-γ amplitude $S_{\mu\nu}(p,q)(p,q)$, where μ,ν and p,q refer to the polarization indices and the momenta of the two photons, such that the on-shell invariant amplitude will take the form

$$\mathcal{M}(H\to\gamma\gamma) = ge^2\,S_{\mu\nu}(p,q)\,\varepsilon^\mu(p)\,\varepsilon^\nu(q)\,, \tag{3}$$

with $p^2 = q^2 = 0$. Note that the Higgs boson momentum equals $Q_\mu = (p+q)_\mu$. Argue that this process cannot be caused by a direct coupling in the standard model Lagrangian, because of the requirement of renormalizability. Therefore this amplitude must originate from closed-loop corrections.
Make a Lorent-covariant decomposition of $S_{\mu\nu}(p,q)$ in terms of the momenta and a number of coefficient functions. Since the photons and the Higgs boson are electrically neutral, this amplitude should satisfy current conservation, even for off-shell momenta. Use this requirement to show that $S_{\mu\nu}(p,q)$ must decompose as follows,

$$\begin{aligned} S_{\mu\nu}(p,q) &= \big(p\cdot q\,\eta_{\mu\nu} - p_\nu q_\mu\big)\,S^{(1)}(p^2,q^2,Q^2) \\ &\qquad + \big(p\cdot q\,p_\mu q_\nu + p^2q^2\eta_{\mu\nu} - q^2p_\mu p_\nu - p^2q_\mu q_\nu\big)S^{(2)}(p^2,q^2,Q^2)\,. \end{aligned} \tag{4}$$

Show that the physical decay amplitude will depend on only one constant, namely $ge^2\,S^{(1)}$ taken at $p^2 = q^2 = 0$ and $-Q^2$ equal to the Higgs boson mass square.
Let us consider an alternative way of constructing the previous decomposition. Namely, one can impose current conservation by extracting two transverse projectors, i.e., $S_{\mu\nu} = \big(\eta_{\mu\rho}p^2 - p_\mu p_\rho\big)\big(\eta_{\nu\sigma}q^2 - q_\nu p_\sigma\big)\hat{S}^{\rho\sigma}(p,q)$. Subsequently one decomposes $\hat{S}^{\rho\sigma}$ in a Lorentz covariant way. Argue now that $\hat{S}^{\rho\sigma}$ contains only two terms, namely one proportional to $\eta^{\rho\sigma}$ and another one proportional to $q^\rho p^\sigma$. Show that this construction leads to the same result (in spite of the fact that this construction seems

to give rise to terms that are at least of fourth order in the momenta). Compare this situation with a similar situation described in section 11.5.
Derive the following expression for the function $S^{(1)}(p^2, q^2, Q^2)$ in terms of the full off-shell amplitude $S_{\mu\nu}(p,q)$, where we set the number of space-time dimensions equal to n,

$$S^{(1)} = \frac{1}{n-2}\,\frac{p\cdot q}{(p\cdot q)^2 - p^2q^2}\left[S_\mu{}^\mu - \frac{(p\cdot q)^2 + (n-2)p^2q^2}{p\cdot q\,\big((p\cdot q)^2 - p^2q^2\big)}\,q^\mu p^\nu S_{\mu\nu}\right]. \tag{5}$$

Given the result from the calculation of a closed-loop diagram, this result enables us to extract the physically relevant function $S^{(1)}$. Note that the above expression simplifies considerably upon substituting the mass-shell condition $p^2 = q^2 = 0$ and $p\cdot q = \frac{1}{2}Q^2$.

20.4. Remaining:

ii) We will work in $n = 4+\epsilon$ dimensions. Argue that the tensor $S_{\mu\nu}(p,q)$ must have the following structure, and prove that

$$S^{(1)} = \frac{1}{Q^2}\Big(S_\mu{}^\mu - \frac{2}{Q^2}q^\mu S_{\mu\nu}p^\nu\Big)\,, \tag{1}$$

$$S^{(2)} = \frac{4}{(n-2)Q^2}\Big(-S_\mu{}^\mu + \frac{2(n-1)}{Q^2}\,q^\mu S_{\mu\nu}p^\nu\Big)\,. \tag{2}$$

iii) Draw the relevant one-loop diagrams for $S_{\mu\nu}(p,q)$ and show that they lead to

$$\begin{aligned} S_{\mu\nu}(p,q) = \frac{1}{\mathrm{i}(2\pi)^n}\int \mathrm{d}^n k\,\mathrm{Tr}\Big(&\frac{1}{\mathrm{i}(\not k - \not p) + m}\gamma_\mu\frac{1}{\mathrm{i}\not k + m}\gamma_\nu\frac{1}{\mathrm{i}(\not k + \not q) + m}\\ &+\frac{1}{\mathrm{i}(\not k - \not q) + m}\gamma_\nu\frac{1}{\mathrm{i}\not k + m}\gamma_\mu\frac{1}{\mathrm{i}(\not k + \not p) + m}\Big)\,. \end{aligned} \tag{3}$$

iv) Prove that the above expression Ward identity $p^\mu S_{\mu\nu} = 0$, and discuss its meaning.

Let us examine the numerator of $S_{\mu\nu}$ as a polynomial in the mass m by working out the spinor trace.

v) Argue that the terms proportional to m^2 and m^0 are zero. Show that terms proportional to m and m^3 term read,

$$\begin{aligned} &\int \frac{d^n k}{\mathrm{i}(2\pi)^n}\,\frac{8\,m}{D_1D_2D_3}\\ &\times\Big\{(m^2 - p\cdot q - k^2)\eta_{\mu\nu} + 4\,k_\mu k_\nu - 2\,p_\mu k_\nu + 2\,k_\mu q_\nu - p_\mu q_\nu + q_\mu p_\nu\Big\}\,, \end{aligned} \tag{4}$$

where $D_1 = k^2 + m^2$, $D_2 = (k+q)^2 + m^2$ and $D_3 = (k-p)^2 + m^2$.

vi) Show that

$$\begin{aligned} S_\mu{}^\mu = 8\mathrm{i}m\Big(&2I(0,m^2,m^2)\\ &- (n-2)I(Q^2,m^2,m^2) - (n\,p\cdot q + 4m^2)K(Q^2,m^2)\Big)\,. \end{aligned} \tag{5}$$

vii) Argue that this expression is finite in the limit $\epsilon \to 0$. Discuss why this must be the case, based on general arguments. Conisder also the finiteness of the functions $S^{(1)}$ and $S^{(2)}$.

Some useful relations (preliminary):

$$\mathrm{Tr}(\gamma_{\mu_1} \dots \gamma_{\mu_i}) = 0 \qquad i \text{ odd} \tag{6}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu} \tag{7}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4\eta_{\mu\nu}\eta_{\rho\sigma} - 4\eta_{\mu\rho}\eta_{\nu\sigma} + 4\eta_{\mu\sigma}\eta_{\rho\nu} \tag{8}$$

$$I(Q^2, m^2, m^2) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k + \frac{1}{2}Q)^2 + m^2]\,[(k - \frac{1}{2}Q)^2 + m^2]} \tag{9}$$

$$= \frac{-i\mu^\epsilon}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) \tag{10}$$

$$K(Q^2, m^2) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{D_1 D_2 D_3} = \text{finite} \tag{11}$$

20.5. Consider the model