19 | Hidden gauge symmetry

The existence of different realizations of a symmetry, one in which the symmetry is manifestly realized and another one in which it is realized in a spontaneously broken way, naturally raises the question whether a similar phenomenon could exist for theories with local symmetries. It turns out that this is indeed the case. There exist alternative realizations where the theory no longer reflects the original gauge invariance. The most conspicuous aspect of this realization is that gauge fields acquire a mass, while on the other hand the Goldstone particle, whose presence is such a characteristic feature of spontaneous symmetry breaking, will be absent.

What is the reason for this difference? As was already stressed in chapter 11 rigid and local symmetries have entirely different consequences. A rigid symmetry implies that the system contains degenerate degrees which transform systematically among themselves under the symmetry. When a rigid symmetry is realized in a spontaneously broken way, these degrees of freedom no longer exhibit this degeneracy, because the ground state is not invariant. More simply, although the potential is invariant, one is considering the fields in the neighbourhood of a point in configuration space that is itself not invariant. Or in more picturesque language, for a person located on the top of a mountain, there is no preferred direction when descending, assuming that the mountain has a symmetric shape. Under two-dimensional rotations in the horizontal plane (with the vertical axis of rotation going through the top) only the top of the mountain is invariant. Once one moves down from the top, the rotational symmetry is no longer manifest. In directions tangential to the mountain one will stay at the same altitude (as a consequence of the symmetry) and in the other directions one will move up or down the slope.

A local symmetry, on the other hand, implies that certain degrees of freedom are absent. In the case of the mountain, the mountain is then not really a mountain, because all points at the same altitude are related by a rotation and, as the rotational degree of freedom is suppressed, these points should thus be *identified*. This implies that we are not dealing with a mountain area, but with a half-line, where the endpoint of the line is associated with the mountain top. The endpoint is, however, qualitatively different from other, more generic, points. In the context of gauge theories, the endpoint corresponds to the realization with manifest gauge invariance. The latter is required in order to write down a local theory with manifest Lorentz invariance. The gauge fields are massles in that case. Other generic points on the line correspond to the 'spontaneously broken' situation, where, as we shall show in due course, the degrees of freedom re-arrange themselves. Unlike in the usual normal (symmetric) realization, one can rewrite the theory such that the gauge degrees of freedom decouple manifestly from the theory, without losing locality and Lorentz invariance. In that way the theory no longer reflects the gauge symmetry that was manifest in the original version. This explains the term 'hidden' gauge invariance, which is the more appropriate characterization of the spontaneously broken *gauge* symmetry. This is the terminology we adopt below.

To exhibit the consequences of the 'spontaneously broken' realization we first discuss two simple examples, one with an abelian and the other one with a non-abelian gauge group. We then present a more general description and demonstrate how to perform quantum corrections in this hidden symmetry phase.

19.1. The Brout-Englert-Higgs mechanism

To analyze the consequences of the hidden realization of a gauge symmetry we discuss a simple example based on the model of section 18.1. We introduce an abelian gauge field and require invariance under local U(1) transformations. In the next section we will discuss a second example based on a non-abelian gauge group. The transformation rules for the U(1) model, which is based on a complex scalar field ϕ and an abelian gauge field A_μ, read

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = \mathrm{e}^{\mathrm{i}q\xi(x)}\,\phi(x)\,, \\ A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x)\,. \end{aligned} \tag{19.1}$$

Following the prescription of the previous chapters, it is easy to write down a Lagrangian invariant under these transformations. Adding a kinetic term for the gauge field to the Lagrangian (18.1), and replacing the derivatives of ϕ by covariant derivatives, we obtain,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2(A) - |D_\mu\phi|^2 - V(|\phi|)\,, \tag{19.2}$$

where we used the definitions

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad D_\mu\phi = \partial_\mu\phi - \mathrm{i}qA_\mu\phi\,. \tag{19.3}$$

We assume that the potential acquires a minimum for nonvanishing field values of ϕ. Therefore it makes sense to adopt the decomposition (18.4), so that the gauge transformation is expressed by

$$\theta(x) \rightarrow \theta'(x) = \theta(x) + qv\,\xi(x)\,, \tag{19.4}$$

and the covariant derivative takes the form

$$D_\mu\phi = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\theta/v}\left(\partial_\mu\rho - \mathrm{i}q\rho\left(A_\mu - (vq)^{-1}\partial_\mu\theta\right)\right). \tag{19.5}$$

Let us now define a new field B_μ by

$$B_\mu = A_\mu - (vq)^{-1}\partial_\mu\theta\,, \tag{19.6}$$

which is gauge invariant. The covariant derivative can then be written as

$$D_\mu\phi = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\theta/v}\left(\partial_\mu\rho - \mathrm{i}q\rho B_\mu\right). \tag{19.7}$$

Since the relation between B_μ and A_μ takes the form of a (field-dependent) gauge transformation, we can simply replace the field strength $F_{\mu\nu}(A)$ by the corresponding tensor $F_{\mu\nu}(B)$. Therefore the Lagrangian can be expressed entirely in terms of the fields ρ and B_μ, which are both gauge invariant,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2(B) - \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}q^2\rho^2B_\mu^2 - V\big(\rho/\sqrt{2}\big). \tag{19.8}$$

If we now expand the field ρ about its vacuum expectation value v, we find that the Lagrangian (19.8) describes a *massive* spin-1 field B_μ, with a mass given by

$$M_B = |qv|. \tag{19.9}$$

Observe that the massless field corresponding to the Goldstone particle in the model of section 18.1 has now simply disappeared! The massive spinless field remains and has the same mass as before introducing the local symmetry.

At this point we realize that we could have derived (19.8) directly, by exploiting the gauge invariance to set $\theta(x) = 0$ from the beginning. This amounts to choosing a gauge condition $\phi(x) = \rho(x)/\sqrt{2}$. This condition is called the *unitary gauge*. The advantage of the unitary gauge is that the physical content of the model is immediately clear. However, this gauge is extremely inconvenient for calculating quantum corrections, because it tends to lead to more severe ultraviolet divergences than other, so-called renormalizable, gauge conditions. We shall discuss this in more detail in section 19.3.

It is important to realize that the degeneracy of the ground state that was present in the case without local gauge invariance, has disappeared. The degeneracy is related to the fictitious degrees of freedom that are subject to the gauge transformations. These degrees of freedom have no physical content. Hence the phase of ϕ becomes irrelevant and only the radial degree of freedom, which is gauge invariant, is physically significant. The same remark applies to the vacuum-expectation value of ϕ. Actually the term vacuum-expectation value is somewhat misleading in this context. Since the *physical* states are gauge invariant it is not possible to have nonvanishing expectation values for quantities that are themselves not gauge invariant. Strictly speaking the only relevant quantity is the expectation value of $|\phi|$, while the phase of ϕ is not relevant. This represents a crucial difference with the situation described in the previous chapter, where the phase of ϕ does represent a physical degree of freedom. In that case the physical states are not required to be invariant under the symmetry, which leads to the connection between a non-zero expectation value and an infinite degeneracy of the ground state.

When proposing the decomposition (18.4) we assumed that $\rho(x)$ was positive. However, changing ρ to $-\rho$ can be accomplished by a gauge transformation with $\xi = \pi/q$. Therefore we can let the field ρ take values from $-\infty$ to $+\infty$, leaving a two-fold *finite* degeneracy (which generates a discrete subgroup of the infinite-dimensional gauge group), so that we do not have to impose an inconvenient restriction on the field ρ. These finite degeneracies have already been discussed in the previous chapter (in particular, see section 18.4). However, for loop diagrams there may still be complications, which can be resolved by a proper treatment of the gauge-fixing procedure (see the general discussion in chapter 13).

Hence we have discovered that in the hidden symmetry realization, the gauge-dependent degrees of freedom, which effectively reside in the phase θ, decouple from the theory. The reason for this decoupling is obvious, since a gauge invariant theory cannot depend on gauge degrees of freedom. Unlike in the realization where the potential acquires a minimum at $\phi = 0$ and the gauge field remains massless, the decoupling takes place in a purely algebraic manner without the necessity of making nonlocal field redefinitions. The number of field components has not been changed in this way. Previously we had a complex field and a gauge field. A complex field represents two components and a gauge field represents three, one less than the number of components of a four-vector because we subtract one component associated with the gauge transformations. Therefore we count $2+3 = 5$ field components. In the new realization the spinless field has only one component and the vector field, which is no longer subject to gauge transformations, has four. Hence we still count $1 + 4 = 5$ field components. Also the number of physical degrees of freedom has not changed. Originally the Lagrangian described two scalar particles and a massless spin-1 particle; since the latter has two physical degrees of freedom, the total number of physical degrees of freedom for a given value of the momentum equals four. In the spontaneously broken realization we have one scalar and one massive spin-1 particle. Massive spin-1 particles have three polarizations, so that we count again four physical degrees of freedom.

It is possible to understand the above phenomenon in more physical terms. Identifying the gauge field A_μ with the electromagnetic vector potential, it mediates a force between electrically charged particles which is of long range.

However, when this field is generated in a medium, it is not a priori clear that it will still manifest itself as a long-range force. The medium may polarize under the influence of an electromagnetic field, so that the electromagnetic forces will be screened. In this case the force becomes of short range and the field that mediates between charges has a mass. The characteristic screening length is inversely proportional to the mass of the gauge field.

In fact a related phenomenon is known in superconductivity, where the screening gives rise to the so-called Meissner effect and the formation of narrow tubes of quantized magnetic flux. In the context of the present model, the latter are known as Abrikosov-Nielsen-Olesen flux tubes and we refer to problem 19.1 for a discussion of them. Anderson pointed out the existence of the phenomenon of a spontaneously broken realization of gauge invariance in the context of superconductivity. In the context of relativistic field theory this was then demonstrated by Englert and Brout, and by Higgs.

Before closing this section we should draw attention to the extra scalar field ρ, often called the Higgs field. In the above model this field seems to play a rather minor role. If we set it equal to a constant v then we are just dealing with the Proca Lagrangian for a free massive spin-1 particle, which we have already discussed at length in section 4.1. Why should we take into account an extra scalar field and not concentrate directly on the massive vector field? To understand some of the reasons for this, we should point out that the potential $V(\rho/\sqrt{2})$ that will force ρ to a constant value must be infinite everywhere except at $\rho = v$. In other words, we must let the mass of the scalar particle and its interactions move to infinity, keeping the minimum of the potential fixed at $\rho = v$. At the classical level such a procedure is harmless and the field ρ can safely be ignored in the limit. However, at the quantum level taking this limit can affect the possible renormalizability of the model. To see this, the reader may consult section 4.1, where it was shown that the propagator for a massive spin-1 field equals

$$\Delta_{\mu\nu}(p) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{p^2+M^2}\Big(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\Big)\,. \tag{19.10}$$

At large momenta this propagator does not behave as $1/p^2$, which enhances the divergences in the quantum-mechanical corrections corresponding to the Feynman diagrams with closed loops. Thus the quantum theory becomes much more divergent and tends to become unrenormalizable so that one can no longer derive sensible predictions from it. As it turns out, the role of the the scalar field is such that it has a smoothening effect on the quantum divergences. Actually, this problem is most pressing in the nonabelian context, which we discuss in the next section. For the abelian case it turns out that the theory still has a chance of being renormalizable in spite of the bad behaviour of the propagator at asymptotic momenta (see problem 19.2).

In order to construct a renormalizable theory with massive vector bosons, the prescription is that one starts from a gauge theory of massless vector bosons, coupled to scalar fields, which is renormalizable *by power counting*. The latter means that the theory should not contain any coupling constants of negative mass dimension. Masses are then generated through the Brout-Englert-Higgs mechanism, which respects the renormalizability of the original theory. All consistent nonabelian theories of massive vector bosons can be found via this recipe. We discuss the consistency, which often turns up in systems that describe a combination of massive and massless vector fields, in an example in problem 19.3. These features hinge on the gauge invariance of the original theory, which is itself not of direct physical relevance. This forms the prime motivation for constructing theories according to the Brout-Englert-Higgs procedure.

19.2. Non-abelian examples

The previous section dealt with abelian gauge groups. In this section we shall present two characteristic examples of nonabelian theories, one based on the group SU(2) and the other on the group SO(3). These two groups are closely related. For infinitesimal transformations the multiplication rules of the group elements are in one-to-one correspondence. Therefore the structure constants are the same. Only globally (i.e., for arbitrary finite group elements) there are differences[1]. Because there is no difference in the structure constants, the nonabelian field strengths coincide and so do the gauge field Lagrangians. However, the difference between the two theories will reside in the representation of the matter fields. In the SU(2) example, we introduce scalar fields which comprise a complex two-component field transforming according to the doublet representation of SU(2), while in the SO(3) model the scalars comprise a real three-component field transforming according to the triplet (vector) representation of SO(3). Hence in both cases the Lagrangian decomposes as

$$\mathcal{L} = -\tfrac{1}{4}\big(G_{\mu\nu}{}^{a}\big)^2 + \mathcal{L}_{\rm scalars}\,, \tag{19.11}$$

where the field strengths read $(a, b, \ldots = 1, 2, 3)$,

$$G^a_{\mu\nu} = \partial_\mu W_\nu{}^a - \partial_\nu W_\mu{}^a + g\,\varepsilon_{abc}\,W_\mu{}^b W_\nu{}^c\,, \tag{19.12}$$

and the gauge fields transform infinitesimally according to

$$W_\mu{}^a \to (W_\mu{}^a)' = W_\mu{}^a + g\,\varepsilon_{abc}\,W_\mu{}^b\xi^c + \partial_\mu\xi^a\,, \tag{19.13}$$

[1] In more technical terms, SU(2) is the covering group of SO(3). A more explicit discussion of these global features is presented in appendix C

where ε_{abc} is the completely antisymmetric tensor with $\varepsilon_{123} = 1$.

Let us now turn to the case of SU(2). We introduce a complex doublet of spinless fields denoted by ϕ. The relevant Lagrangians were already given in section 12.2 (cf. 12.65) and the combined Lagrangian equals

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu W_\nu{}^a - \partial_\nu W_\mu{}^a)^2 - g\varepsilon_{abc}W^{\mu\,a}\,W_\mu^{\nu\,b}W_\nu{}^c \\ &-\tfrac{1}{4}g^2\varepsilon_{abc}\varepsilon_{ade}\,W_\mu{}^b\,W^{\mu\,d}\,W_\nu{}^c\,W^{\nu\,e} \\ &-|\partial_\mu\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi|^4 \\ &-\tfrac{1}{2}\mathrm{i}gW_\mu{}^a\left(\phi^*\tau_a\overset{\leftrightarrow}{\partial}_\mu\phi\right) - \tfrac{1}{4}g^2|\phi|^2(W_\mu{}^a)^2\,. \end{aligned} \tag{19.14}$$

Here we introduced an SU(2)-invariant potential of the renormalizable type with two parameters μ^2 and λ. When $\mu^2, \lambda > 0$ the potential acquires a minimum for a nonzero value of the field ϕ. Following the treatment of the previous section we decompose ϕ according to

$$\phi(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = \frac{1}{\sqrt{2}}\,\Phi(x)\begin{pmatrix}0\\ \rho(x)\end{pmatrix}\,, \tag{19.15}$$

where $\Phi(x)$ is an x-dependent SU(2) matrix. This matrix is is a generalization of the phase factor $\exp\mathrm{i}\theta$ (which is a U(1) “matrix”) used in the previous section. Here we make use of the fact that the doublet ϕ can generally be brought into the form $(0, \rho/\sqrt{2})$ by a suitable local SU(2) transformation. The field ρ represents the SU(2) invariant length of the doublet field. As before we find it convenient to let ρ take both positive and negative values, which means that we still have a discrete degeneracy. Positive and negative values of the field ρ are equivalent by a gauge transformation. This feature has, however, very little consequences for most of what we do, which is based on perturbation theory. Under SU(2) the field Φ transforms as

$$\Phi(x) \to U(x)\,\Phi(x)\,, \tag{19.16}$$

where $U(x)$ is space-time dependent SU(2) matrix. This relation is the analogue of (19.4).

It is not so difficult to substitute the parametrization (19.15) into the Lagrangian, provided we redefine the gauge fields in close analogy with (19.6). Hence we define new gauge fields which are related to the old ones through a gauge transformation given by the matrix Φ^{-1},

$$\hat{W}_\mu(x) = \Phi^{-1}(x)W_\mu(x)\Phi(x) + g^{-1}[\partial_\mu\Phi^{-1}(x)]\,\Phi(x)\,, \tag{19.17}$$

where $W_\mu = \frac{1}{2}W_\mu{}^a\tau_a$. The quantity $\hat{W}_\mu$ is the analogue of the field B_μ of the abelian theory in the previous section. As the reader may check by combining the transformation rules (12.11) and (19.16), the new gauge fields $\hat{W}_\mu^a$ are now gauge invariant. Also the following result is straightforward to derive,

$$D_\mu\phi(x) = \frac{1}{\sqrt{2}}\Phi(x)\left[\partial_\mu - \tfrac{1}{2}\mathrm{i}g\hat{W}_\mu{}^a(x)\,\tau_a\right]\begin{pmatrix}0\\ \rho(x)\end{pmatrix}.$$

Using the above results it readily follows that the generalized phase factor Φ disappears entirely from the Lagrangian. We are thus left with the three gauge fields and one single scalar field ρ. Again this is a direct implication of the gauge invariance of the original Lagrangian. By redefining the fields into gauge invariant combinations, Φ remains the only field that varies under the gauge transformations. Gauge invariance therefore implies that this field should disappear from the Lagrangian. As before, an easier way to obtain the same result makes use of the unitary gauge. Namely, one notes that by an appropriate local gauge transformation, the field Φ, which itself parametrizes elements of the SU(2) gauge group, can be set equal to the identity matrix. In this way one replaces the doublet field ϕ by $(0,\rho/\sqrt{2})$. The Lagrangian then takes the form (we now drop the carrot on W_μ^a)

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu W_\nu{}^a - \partial_\nu W_\mu{}^a)^2 - g\,\varepsilon_{abc}W^{\mu\,a}\,W^{\nu\,b}\,\partial_\mu W_\nu{}^c\\ &-\tfrac{1}{4}g^2\varepsilon_{abc}\varepsilon_{ade}\,W^\mu\;W_\mu{}^d\,W^{\nu\,c}\,W_\nu{}^e\\ &-\tfrac{1}{2}(\partial_\mu\rho)^2 + \tfrac{1}{2}\mu^2\rho^2 - \tfrac{1}{4}\lambda\rho^4 - \tfrac{1}{8}g^2\,\rho^2\,(W_\mu{}^a)^2\,.\end{aligned} \tag{19.18}$$

To appreciate the consequences of the above result, let us now determine the values $\rho = \pm v$ for which the potential $V(\rho) = -\frac{1}{2}\mu^2\rho^2 + \frac{1}{4}\lambda\rho^4$ acquires a minimum, assuming μ^2 and λ positive. The derivative of $V(\rho)$ vanishes whenever $\rho = 0$ or $-\mu^2 + \lambda\rho^2 = 0$. At $\rho = 0$ we have a local maximum, while the minima are at $\rho = \pm v$ with

$$v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{19.19}$$

The mass of the so-called Higgs particle, associated with the field ρ, equals

$$m_\rho^2 = 2\lambda v^2, \tag{19.20}$$

while the gauge fields acquire equal masses, whose value follows from substituting $\rho^2 = v^2$ into the last term in the Lagrangian (19.18),

$$M_W^2 = \tfrac{1}{4}g^2v^2. \tag{19.21}$$

Just as in the abelian model discussed in section 19.1, the field ρ seems to play an ancillary role. It only interacts with the gauge fields through the $\rho^2\,W^2$ interaction. In the limit $\lambda\to\infty$, keeping v fixed, the degrees of freedom associated with ρ are suppressed, and one is left with the standard Lagrangian for SU(2) gauge fields with an extra mass term. At the *classical* level this procedure is harmless and there is no reason why one cannot drop the Higgs field. However, for the *quantum* theory, the situation is quite different, at least in the nonabelian case. With an explicit mass term the theory is not renormalizable, so that there is no way to obtain sensible predictions. As it turns out the presence of the extra scalar field has a smoothening effect on the the quantum corrections which is vital for making the theory renormalizable. This aspect, and not so much the gauge invariance of the original theory which is no longer manifest in (19.18) anyway, requires us to incorporate the Higgs fields according to the Brout-Englert-Higgs procedure.

Consider now the gauge group SO(3) and introduce three real scalar fields $\phi^a(x)$ transforming as a vector. Under infinitesimal transformations the scalar fields thus transform as

$$\phi^a \to (\phi^a)' = \phi^a + g\,\varepsilon_{abc}\,\phi^b\xi^c\,. \tag{19.22}$$

The gauge invariant Lagrangian for this theory takes the following form

$$\mathcal{L} = \tfrac{1}{2}\big(D_\mu\phi^a\big)^2 + \tfrac{1}{2}\mu^2(\phi^a)^2 - \lambda\big((\phi^a)^2\big)^2 - \tfrac{1}{4}\big(G_{\mu\nu}{}^a\big)^2\,, \tag{19.23}$$

where the the field strengths are still defined by (19.12) and the covariant derivative of the scalar field equals

$$D_\mu\phi^a = \partial_\mu\phi^a - g\varepsilon_{abc}\,\phi^b W_\mu^c\,. \tag{19.24}$$

We assume that the potential acquires its minimum at some nonvanishing values of ϕ^a. Just as in the abelian case, we can then choose a decomposition of the fields ϕ^a in terms of the radius $\rho(x)$ of the three-component field and a rotation matrix, which directs ϕ^a along a certain direction,

$$\begin{pmatrix}\phi^1(x)\\ \phi^2(x)\\ \phi^3(x)\end{pmatrix} = \begin{pmatrix}O_{11}(x) & O_{12}(x) & O_{13}(x)\\ O_{21}(x) & O_{22}(x) & O_{23}(x)\\ O_{31}(x) & O_{32}(x) & O_{33}(x)\end{pmatrix}\begin{pmatrix}0\\ 0\\ \rho(x)\end{pmatrix}\,. \tag{19.25}$$

Here $O(x)$ is an SO(3) matrix, which in principle depends on three x-dependent angles. However, one of these angular parameters is always left undetermined, namely the one associated with rotations around the vector $(0,0,\rho)$. Therefore the matrix depends effectively on only two fields, say $\theta_1(x)$ and $\theta_2(x)$. Together with $\rho(x)$, they describe the three degrees of freedom corresponding to the original fields $\phi^a(x)$.

Rather than substituting the above expression into the Lagrangian we may exploit the local SO(3) invariance to direct the vector ϕ^a along the third axis without loss of generality,

$$\phi^a(x) = \big(0,0,\rho(x)\big)\,. \tag{19.26}$$

The covariant derivative then reads

$$D_\mu\phi^a = \big(g\rho\, W_\mu{}^2,\, -g\rho\, W_\mu{}^1,\, \partial_\mu\rho\big)\,, \tag{19.27}$$

so that the Lagrangian reads

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\rho)^2 - V(\rho) - \tfrac{1}{4}(G^a_{\mu\nu})^2 - \tfrac{1}{2}g^2\,\rho^2\big((W^1_\mu)^2 + (W^2_\mu)^2\big)\,. \tag{19.28}$$

Expanding about its vacuum expectation value v, we find that two of the vector bosons, W^1 and W^2, acquire a mass equal to $M = gv$, whereas the third one, W^3, remains massless. Hence the phenomenon of spontaneous symmetry breaking also occurs in nonabelian theories. In this case not all of the gauge fields have acquired a mass. One of the gauge fields remains massless. This fact is easy to understand as a consequence of the fact that a vector in a three-dimensional space remains always invariant under rotations around that vector. Consequently, for any finite value of $\phi^a(x)$ there must be an SO(2) subgroup that leaves $\phi^a(x)$ invariant.

The SO(2) transformations correspond to rotations of $\phi^1(x)$ and $\phi^2(x)$, the two fields that were put to zero by (11.13), and $W_\mu{}^1$ and $W_\mu{}^2$. Under these transformations ρ is inert, so that the gauge field W^3 associated with these rotations is not affected by the fact that ρ has a non-zero vacuum-expectation value. Therefore W^3 remains massless. Hence the massless gauge fields are always associated with the residual gauge invariance of the theory that remains manifest when vacuum expectations are present. The nature of the residual invariance group is crucially related to the representations to which the scalar fields have been assigned. This will be discussed more systematically in the next chapter.

The physical relevance of the residual gauge invariances is now clear. Apart from the gravitational field the massless gauge fields in nature are those associated with the spin-1 quanta of QED and QCD, i.e. the photon and the SU(3) vector gluons that mediate the electromagnetic and strong interactions, respectively. Hence the residual gauge group in nature is therefore $\mathrm{U}(1)\times\mathrm{SU}(3)$. In order to construct realistic models one must have scalar fields in representations that permit such a residual invariance.

19.3. Choosing a gauge

As we stressed in the preceding sections, a gauge invariant theory does not depend on certain gauge degrees of freedom that are usually associated with (generalized) phase factors of the matter fields. In the "normal" realization it is not possible to consistently remove these degrees of freedom, without sacrificing manifest Lorentz invariance or locality of the underlying field theory. This was discussed already in chapter 13. The special feature of a hidden symmetry realization is that these degrees of freedom decouple algebraically. We are thus left with an action that contains the standard Proca Lagrangian for a massive vector field. However, just as in the massless realization, one could choose to ignore that the theory does not depend on certain degrees of freedom and straightforwardly adopt a gauge condition. In chapter 13 we discussed how to do this for massless gauge fields. In the massive realization there are some new features and they are discussed below. This topic is of some importance in view of the fact that the renormalization properties of the field theory will depend rather sensitively on the type of gauge condition that one adopts.

For our purpose it suffices to discuss all of this in the abelian case. So we consider a gauge field A_μ and a complex scalar field ϕ transforming as in (19.1) and described by the Lagrangian (19.2). One obvious gauge condition is to just suppress the phase of ϕ by restricting the field to be real,

$$\phi(x) = \phi^*(x)\,. \tag{19.29}$$

In this way one recovers directly the Lagrangian (19.8) upon identifying the real part of ϕ with the field $\rho/\sqrt{2}$. The identification is exact provided we allow ρ to take both positive and negative values, something that we did discuss before. Indeed the condition (19.29) does not fix the gauge freedom completely as we can still perform gauge transformations with $\exp(\mathrm{i}q\,\xi) = -1$, which is achieved when ξ is equal to an odd multiple of π/q. Therefore, there remains a residual gauge invariance acting on a real field ϕ according to

$$\phi(x) \to -\phi(x)\,. \tag{19.30}$$

This discrete symmetry does not prevent us from setting up perturbation theory, so we can allow ρ to take both positive and negative values.

The gauge condition (19.29) is called the *unitary gauge*, because in this gauge the physical content of the theory is most manifest in the sense that all fields are directly associated with physical particles. However, in such a gauge the quantum corrections to the theory tend to lead to severe ultraviolet divergences. The reason is that the propagator of the vector field has the form (19.10) and, as stressed before, this propagator does not vanish at large momenta. This means that the divergences will be of a type that is known to occur in non-renormalizable field theories. However, the question of quantum divergences can be studied in alternative gauges, where the propagators do vanish for large momenta, thus giving rise to divergences that are of the renormalizable type. Since the *physical* results of the theory should not depend on the choice of the gauge condition, one may exploit the possibility of evaluating the quantum corrections in a so-called *renormalizable gauge*.

Renormalizable gauges have one drawback. In such gauges there are extra fields which do not correspond to physical particles. This implies that one has to deal with many more diagrams. However, the short-distance behaviour of these diagrams is similar to the diagrams that arise in any renormalizable theory. Therefore their evaluation is standard. In renormalizable gauges, one should be careful to isolate the terms that are physically relevant. For instance, in the evaluation of physical quantities, such as cross sections and decay rates, one should restrict oneself to amplitudes that refer exclusively to fields that are associated with physical particles. We will discuss some of these features below.

The starting point is still the original action (19.2), but now we decompose the field ϕ into its real and imaginary parts according to

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(\sigma(x) + \mathrm{i}\chi(x)\Big)\,. \tag{19.31}$$

By an appropriate field redefinition we may always assume that ϕ acquires a real vacuum-expectation value (strictly speaking, only the expectation value of $|\phi|$ is relevant), whereas the vacuum-expectation value of χ remains zero. The covariant derivative now takes the form

$$D_\mu\phi = \frac{1}{\sqrt{2}}\Big(\partial_\mu\sigma + qA_\mu\chi\Big) + \frac{\mathrm{i}}{\sqrt{2}}\Big(\partial_\mu\chi - qA_\mu\sigma\Big)\,. \tag{19.32}$$

Expanding σ about its vacuum expectation value

$$\sigma(x) = v + \sigma'(x), \tag{19.33}$$

we find

$$D_\mu\phi = \frac{1}{\sqrt{2}}\Big(\partial_\mu\sigma' + qA_\mu\chi\Big) + \frac{\mathrm{i}}{\sqrt{2}}\Big(\partial_\mu\chi - MA_\mu - qA_\mu\,\sigma'\Big)\,, \tag{19.34}$$

where M has been defined in (19.9). Therefore the part of the Lagrangian quadratic in the fields σ', χ and A_μ takes the form

$$\begin{aligned}\mathcal{L}_{\rm quadr} = &-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}M^2A_\mu^2 - \tfrac{1}{2}(\partial_\mu\chi)^2 + MA^\mu\partial_\mu\chi \\ &-\tfrac{1}{2}(\partial_\mu\sigma')^2 - \tfrac{1}{2}m_\sigma^2\sigma'^{\,2}\,,\end{aligned} \tag{19.35}$$

where m_σ^2 is defined by $m_\sigma^2 = \partial^2V(\sigma,\chi)/\partial\sigma^2$ taken at the minimum of the potential at $\sigma = v$ and $\chi = 0$. As the potential is function of $\sigma^2 + \chi^2$, one can show that there is no mass term for the field χ in accord with the Goldstone theorem discussed in section 18.1.

The Lagrangian (19.35) contains a mass term for the vector field, but now the Goldstone field χ is still present and seems not to decouple from the theory. However, further inspection shows that it does decouple. As it turns out the mass term and the χ-dependent terms can be written in the form

$$-\tfrac{1}{2}(\partial_\mu\chi)^2 - \tfrac{1}{2}M^2A_\mu^2 + MA_\mu\partial_\mu\chi = -\tfrac{1}{2}M^2(A_\mu - M^{-1}\partial_\mu\chi)^2\,, \tag{19.36}$$

so that χ can be removed from the theory by absorbing it into the definition of A_μ. Hence we are in a similar situation as in the previous sections.

It is also instructive to analyze the physical content of the theory on the basis of the fields A_μ and χ. The field equations for these fields can be derived from (19.35) and read

$$\begin{aligned} \partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu - M^2 A_\mu + M\,\partial_\mu \chi &= 0\,, \\ \partial^2 \chi - M\,\partial^\mu A_\mu &= 0\,. \end{aligned} \tag{19.37}$$

Note that these equations are not independent; the second equation follows by taking the divergence of the first one. This is feature related to the gauge invariance of the theory.

Let us first adopt the unitary gauge $\chi = 0$. According to (19.37) this implies that $\partial^\mu A_\mu = 0$, which leads to the solution

$$A_\mu(x) = \varepsilon_\mu(k)\,\mathrm{e}^{\mathrm{i}k\cdot x}\,, \qquad \chi(x) = 0\,, \tag{19.38}$$

with $k^\mu \varepsilon_\mu(k) = 0$ and $k^2 + M^2 = 0$. Hence we obtain three physical polarizations as is appropriate for a spin-1 massive particle.

Alternatively one may also choose the Coulomb gauge,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0\,, \tag{19.39}$$

so that the first equation (19.37) decomposes into

$$\begin{aligned} (\boldsymbol{\nabla}^2 - \partial_t^2)\mathbf{A} - M^2\mathbf{A} &= -\,\boldsymbol{\nabla}(\partial_t A_0 + M\chi)\,, \\ (\boldsymbol{\nabla}^2 - M^2)A_0 + M\,\partial_t\chi &= 0\,. \end{aligned} \tag{19.40}$$

Taking the divergence of the first equation and using the gauge condition (19.39) leads to $\boldsymbol{\nabla}^2(\chi + M^{-1}\partial_t A_0) = 0$, from which one infers that $\chi = -M^{-1}\partial_t A_0$. Using this result the two equations show that both $\mathbf{A}$ and A_0 satisfy the massive Klein-Gordon equation, wherea the second equation (19.37) is now identically satisfied.

Combining the resulting equations with the Coulomb gauge condition shows that the solutions take the following form

$$A_\mu(x) = \varepsilon_\mu(k)\,\mathrm{e}^{\mathrm{i}k\cdot x}\,, \qquad \chi(x) = \frac{\mathrm{i}k_0}{M}\varepsilon_0(k)\,\mathrm{e}^{\mathrm{i}k\cdot x}, \tag{19.41}$$

with $k^2 + M^2 = 0$ and $\mathbf{k}\cdot\varepsilon(\mathbf{k}) = 0$. Hence two degrees of freedom reside in the transverse polarizations of $\mathbf{A}$ and the third degree of freedom is related to χ and A_0. These three degrees of freedom can be rearranged into a single vector field $B_\mu = A_\mu - M^{-1}\partial_\mu\chi$, subject to the massless Klein-Gordon equation and the condition $\partial^\mu A_\mu = 0$. The analysis of the degrees of freedom thus leads to the same result, irrespective of the gauge condition that one adopts.

We now discuss the propagators corresponding to the theory (19.8) for a more general class of gauges. As we have discussed before, the propagators follow from the terms in the Lagrangian quadratic in the fields. These have been given in (19.35). Ignoring the field σ', whose propagator is simple to determine, we note that the fields A_μ and χ cannot be considered separately because of the presence of the term $MA_\mu\,\partial^\mu\chi$ in (19.35). Therefore the quadratic terms for A_μ and χ are characterized by a 5×5 matrix $[\Delta^{-1}]^{\hat\mu\hat\nu}$ acting on the fields $A_{\hat\mu} = (A_\mu, \chi)$, so that the indices $\hat\mu, \hat\nu$ run over five values associated with the four space-time indices μ, ν and the field χ. To make this explicit, consider the action associated with the Lagrangian (19.35) in the momentum representation, retaining only the terms depending on A_μ and χ,

$$\int \mathrm{d}^4x\ \mathcal{L}_{\rm quadr} = -\tfrac{1}{2}(2\pi)^4 \int \mathrm{d}^4k\ A_{\hat\mu}(-k)\ \Big[\Delta^{-1}(k)\Big]^{\hat\mu\hat\nu} A_{\hat\nu}(k)\ , \tag{19.42}$$

with

$$\Big[\Delta^{-1}(k)\Big]^{\hat\mu\hat\nu} = \begin{pmatrix} (k^2+M^2)\eta^{\mu\nu} - k^\mu k^\nu & \mathrm{i}Mk^\mu \\ & \\ -\mathrm{i}Mk^\nu & k^2 \end{pmatrix} . \tag{19.43}$$

The propagator is the inverse of this 5×5 matrix, but because of local gauge invariance this matrix has one zero eigenvalue, so that its inverse will not exist. This is the same situation that arose with massless gauge fields and we adopt the same strategy for constructing the propagator. Namely, one introduces a quadratic gauge-fixing term to the original Lagrangian. A convenient choice is

$$\mathcal{L}^{\rm g.f.} = -\tfrac{1}{2}(\lambda\,\partial_\mu A_\mu + \lambda' M\,\chi)^2\,, \tag{19.44}$$

with λ and λ' arbitrary parameters. The term in parentheses must be such that, when fixed to a certain value, it removes the freedom of making gauge transformations. In other words, the gauge-fixing term $\lambda\,\partial_\mu A_\mu + \lambda' M\chi$ should not be inert under gauge transformations. One easily verifies that this is the case by applying an infinitesimal gauge transformation,

$$\delta(\lambda\,\partial^\mu A_\mu + \lambda' M\chi) = \lambda\,\partial^2\xi - \lambda' M^2\xi - \lambda' qM\sigma'\xi\,. \tag{19.45}$$

We now add (19.44) to the original gauge-invariant Lagrangian and recalculate the matrix occurring in (19.42). This time the matrix should have an inverse, since the presence of the gauge-fixing term breaks the gauge invariance. Therefore we find the inverse propagator, which is still a 5×5 matrix,

$$\Big[\Delta^{-1}(k)\Big]^{\hat\mu\hat\nu} = \begin{pmatrix} (k^2+M^2)\eta^{\mu\nu} + (\lambda^2-1)k^\mu k^\nu & \mathrm{i}M(1-\lambda\lambda')k^\mu \\ \\ -\mathrm{i}M(1-\lambda\lambda')k^\nu & k^2+\lambda'^2M^2 \end{pmatrix} . \tag{19.46}$$

In principle this matrix can be inverted to derive the propagators. But to keep matters simple, it is advantageous to first make a convenient choice for the gauge parameters λ and λ', such that the off-diagonal terms in (19.46) will vanish. This happens for $\lambda\lambda' = 1$. In that case the mixing between the fields χ and A_μ disappears, so that we have a 4×4 propagator for A_μ, and a simple propagator for χ. The result of inverting (19.46) then gives

$$\begin{aligned} \Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\Big[\frac{1}{k^2+M^2}\Big(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big) + \frac{1}{\lambda^2}\frac{1}{k^2+(M/\lambda)^2}\frac{k_\mu k_\nu}{k^2}\Big] , \\ \Delta_\chi(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2+(M/\lambda)^2} . \end{aligned} \tag{19.47}$$

The most conspicuous feature of these propagators is that they exhibit poles at $k^2 = -M^2$, corresponding to transverse polarizations, and at $k^2 = -(M/\lambda)^2$. The latter poles must clearly correspond to unphysical degrees of freedom, since their location is gauge dependent. Physical results should not depend on the choice of gauge and therefore should not depend on the arbitrary parameter λ . We will come back to this in the next section.

The gauge choice that we have introduced is called renormalizable because the propagators (19.47) are inversely proportional to k^2 at large (Euclidean) values of the momenta. It is instructive to consider some special values for λ. Choosing $\lambda = 1$ defines the so-called Feynman-'t Hooft gauge,

$$\begin{aligned} \Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{\delta_{\mu\nu}}{k^2+M^2} , \\ \Delta_\chi(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2+M^2} . \end{aligned} \tag{19.48}$$

In the limit $\lambda \to \infty$ one has the Landau gauge, characterized by the fact that the vector-boson propagator is transversal, i.e., $k^\mu\Delta_{\mu\nu}(k) = 0$, as can be verified from the explicit result,

$$\begin{aligned} \Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2+M^2}\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big) , \\ \Delta_\chi(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2} . \end{aligned} \tag{19.49}$$

Finally, when λ approaches zero, we recover the unitary gauge,

$$\begin{aligned} \Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2+M^2}\Big(\delta_{\mu\nu}+\frac{k_\mu k_\nu}{M^2}\Big)\,, \\ \Delta_\chi(k) &= 0\,. \end{aligned} \tag{19.50}$$

The gauge-dependent mass becomes infinite in this case, which suppresses the contribution of the field χ.

With these propagators we can now evaluate arbitrary tree diagrams. However, when evaluating diagrams with closed loops we also have to include the ghost fields introduced in section 13.3. The ghost Lagrangian associated with the gauge-fixing term (19.44) the ghost field Lagrangian follows directly from (19.45) and reads,

$$\mathcal{L}_{\text{ghost}} = \mathrm{i}\lambda\,\big[b\,\partial^2 c - \lambda'\lambda^{-1}M^2 b\,c\big] - \mathrm{i}q\lambda' M\,\sigma\,b\,c\,. \tag{19.51}$$

where c denotes the ghost field and b the anti-ghost field. For $\lambda\lambda' = 1$ the ghost propagator takes the form,

$$\Delta_{bc}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{\mathrm{i}\lambda[k^2+(M/\lambda)^2]}\,, \tag{19.52}$$

and exhibits the same unphysical pole as the gauge field propagator shown in (19.47). We recall that the ghosts are anti-commuting fields so that closed ghost loops will acquire an extra minus sign, just as closed fermion loops.

The important lesson learned in this section is that gauge theories can be quantized by adopting different gauge conditions. In the unitary gauge the physical content of the theory is immediately obvious, but the structure of the quantum divergencies is very complicated. On the other hand, renormalizable gauges are much more convenient for calculating quantum corrections. The divergencies that one encounters have a simple structure, which facilitates the renormalization procedure. However, the physics is more subtle to extract in a renormalizable gauge, and in particular the proof that the theory is unitary at the quantum level is more involved. The fact that physical quantities do not depend on the gauge condition, however, allows one to take advantage of both kinds of gauges.

19.4. Gauge independence; some examples

The general proof that physical quantities are independent of the gauge condition is rather complicated and its description is outside the scope of this book. Instead we shall perform a few one-loop calculations in this section to demonstrate the gauge independence in a simple U(1) gauge theory, this time coupled to both a complex scalar and a fermion field. The model that we consider is a straightforward extension of the model of the previous section by including a spin-$\frac{1}{2}$ field ψ, which transforms under gauge transformations according to

$$\psi(x) \to \psi'(x) = \mathrm{e}^{\frac{1}{2}\mathrm{i}q\,\xi(x)\,\gamma_5}\,\psi(x)\,. \tag{19.53}$$

Note that this transformation contains a spinor matrix γ_5; such transformations are called chiral transformations and they were already introduced in chapter **??**. As the reader can easily check, the conjugate field $\bar{\psi}(x)$ transforms under chiral transformations as

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}q\,\xi(x)\,\gamma_5}\,. \tag{19.54}$$

The covariant derivative on ψ has the form

$$D_\mu\psi(x) = \partial_\mu\psi(x) - \tfrac{1}{2}\mathrm{i}qA_\mu(x)\gamma_5\psi(x)\,, \tag{19.55}$$

and the complete gauge invariant Lagrangian of the model reads

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F_{\mu\nu}^2 - |D_\mu\phi|^2 + \mu^2\,|\phi|^2 - \lambda\,|\phi|^4 \\ &- \bar{\psi}\not{D}\psi - \tfrac{1}{2}\sqrt{2}\,G\,[\phi\,\bar{\psi}(1-\gamma_5)\psi + \bar{\phi}\,\bar{\psi}(1+\gamma_5)\psi]\,.\end{aligned} \tag{19.56}$$

Note that gauge invariance excludes the presence of a mass term for ψ, but it does allow a Yukawa coupling between fermions and the scalar field, owing to the fact that the spinor transformation rule involves a factor $\frac{1}{2}q$ and the scalar transformation rule a factor q in the exponentials (cf. problem 19.8).

In view of the fact that μ^2 and g are taken positive, the potential acquires minima for $|\phi| = \frac{1}{2}\sqrt{2}\,v$ with $v = \sqrt{\mu^2/\lambda}$. Choosing the minimum at a positive real value of ϕ, we introduce the decomposition

$$\phi = \frac{1}{\sqrt{2}}\Big(v + \sigma + \mathrm{i}\chi\Big)\,, \tag{19.57}$$

similar to (19.31) and (19.33), where we have dropped the prime on σ. The fermionic part of the Lagrangian then takes the form

$$\begin{aligned}\mathcal{L}_{\text{fermionic}} = &-\bar{\psi}(\not{\partial} + m_\psi)\psi + \tfrac{1}{2}\mathrm{i}qA_\mu\,\bar{\psi}\gamma^\mu\gamma_5\psi \\ &- G\,\sigma\,\bar{\psi}\psi - \mathrm{i}G\,\chi\,\bar{\psi}\gamma_5\psi\,,\end{aligned} \tag{19.58}$$

where the vacuum expectation value of ϕ has generated a mass term for ψ with the mass m_ψ given by

$$m_\psi = G\,v\,. \tag{19.59}$$

We also need the bosonic terms of the Lagrangian. The quadratic terms were already given in (19.35), where the mass m_σ is now given by

$$m_\sigma = \sqrt{2\lambda}\, v\,. \tag{19.60}$$

To these quadratic terms we will add the gauge-fixing term (19.44) with $\lambda\lambda' = 1$ so that we can make use of the propagators (19.47). Furthermore we find the following interaction terms cubic and quartic in the bosonic fields,

$$\begin{aligned}\mathcal{L}_{\text{int}} =& qA_\mu(\sigma\,\partial^\mu\chi - \partial^\mu\sigma\,\chi) - qM\sigma\,{A_\mu}^2 - \tfrac{1}{2}q^2{A_\mu}^2(\sigma^2+\chi^2)\\ &- g\lambda\,\sigma(\sigma^2+\chi^2) - \tfrac{1}{4}g(\sigma^2+\chi^2)^2\,.\end{aligned} \tag{19.61}$$

To confirm that physical quantities are independent of the choice of gauge we present some calculations below. The first one concerns fermion-fermion scattering in tree approximation.The second and third calculation deal with one-loop diagrams, where one must include the ghost fields.

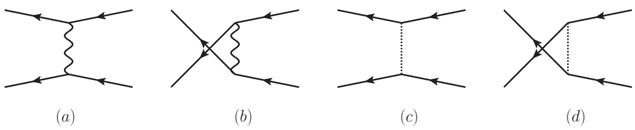

Figure 19.1: Diagrams contributing to fermon-fermion scattering in tree approximation

(i) Fermion-fermion scattering: We calculate the scattering of two fermions in this model in tree approximation. The incoming fermions carry momenta p_1 and p_2 whereas the outgoing ones are assigned the momenta p_3 and p_4. The corresponding spinors are denoted by u_1-u_4. The relevant diagrams are given in fig. 19.1. The diagrams (a) and (b) describe the exchange of the massive vector field and the diagrams (c) and (d) the exchange of the field ϕ; the latter decompose into two contributions corresponding to σ and χ exchange. Note that the diagrams (a) and (b) are related by an interchange of the indices 3 and 4, where an relative minus sign must be included in order to be consistent with Fermi-Dirac statistics. The same comment applies to the diagrams (c) and (d).

The amplitude in tree approximation takes the form

$$\begin{aligned}\mathcal{M}(1+2\to 3+4) = {}& \mathrm{i}(2\pi)^4\left(\tfrac{1}{2}\mathrm{i}q\,\bar u_3\gamma^\mu\gamma_5 u_1\right)\left(\tfrac{1}{2}\mathrm{i}q\,\bar u_4\gamma^\nu\gamma_5 u_2\right)\Delta_{\mu\nu}(p_1-p_3)\\ &+\mathrm{i}(2\pi)^4\left(G\,\bar u_3 u_1\right)\left(G\,\bar u_4 u_2\right)\Delta_\sigma(p_1-p_3)\\ &+\mathrm{i}(2\pi)^4\left(\mathrm{i}G\,\bar u_3\gamma_5 u_1\right)\left(\mathrm{i}G\,\bar u_4\gamma_5 u_2\right)\Delta_\chi(p_1-p_3)\\ &-\mathrm{i}(2\pi)^4\left\{3\longleftrightarrow 4\right\}\,.\end{aligned} \tag{19.62}$$

The first three terms correspond to the diagrams (a) and (c); the last term generates the contributions from the diagrams (b) and (d). For our purposes it is obviously sufficient to restrict the attention to the first three lines of (19.62). Using the propagator in the renormalizable gauge, specified in (19.47), the gauge dependent part of the amplitude follows from the λ-dependent part of the gauge field and the χ propagator. This yields

$$\begin{aligned}\mathcal{M}_{\rm g.d}(1+2\to 3+4) &= \frac{1}{(p_1-p_3)^2+(M/\lambda)^2}\\ &\times\Big[\frac{q^2}{4\,\lambda^2\,(p_1-p_3)^2}\Big(\bar u_3(\not p_3-\not p_1)\gamma_5 u_1\Big)\Big(\bar u_4(\not p_4-\not p_2)\gamma_5 u_2\Big)\\ &\qquad -G^2\Big(\bar u_3\gamma_5 u_1\Big)\Big(\bar u_4\gamma_5 u_2\Big)\Big]\\ &-\Big\{3\longleftrightarrow 4\Big\}\,. \end{aligned}\qquad(19.63)$$

Note that we have used energy-momentum conservation to write $p_1-p_3 = -(p_2-p_4)$. Obviously the above expression is λ-dependent for arbitrary momenta p_1-p_4, but we will demonstrate that for a physical scattering amplitude where the external fermions are on the mass shell, (19.63) will no longer depend on the gauge parameter λ. To see this we use the Dirac equation on the spinors, $(\not p_j - \mathrm{i}m_\psi)u_j = 0$, and on the conjugate spinors, $\bar u_j(\not p_j-\mathrm{i}m_\psi)=0$, which allows us to write

$$\bar u_3(\not p_3-\not p_1)\gamma_5 u_1 = \bar u_3(\not p_3\gamma_5+\gamma_5\not p_1)u_1 = 2\mathrm{i}m_\psi\,\bar u_3\gamma_5 u_1\,. \qquad(19.64)$$

A similar identity holds for the expression involving the momenta and spinors of the fermions 3 and 4. Furthermore, we use (19.9) and (19.59) to write

$$q^2\,m_\psi{}^2 = q^2\,G^2\,v^2 = G^2\,M^2\,. \qquad(19.65)$$

This enbles us to show straightforwardly that (19.63) becomes equal to

$$\mathcal{M}_{\rm g.d}(1+2\to 3+4) = -\frac{G^2\,(\bar u_3\gamma_5 u_1)(\bar u_4\gamma_5 u_2)}{(p_1-p_3)^2} - \Big\{3\longleftrightarrow 4\Big\}\,. \qquad(19.66)$$

which no longer depends on the gauge parameter λ.

(ii) One-loop diagrams with one external line: As an intermediate result we determine the expression for one-loop diagrams with one external line, often called tadpole diagrams. The result of this calculation will be needed for the determination of the one-loop correction to the fermion mass, which we will discuss shortly. Because of the symmetries involved the tadpole diagrams exist only with external σ-lines. The field χ is a pseudoscalar field, as one can verify from the Lagrangian. This excludes diagrams with one external χ-line. Diagrams with one external A_μ-line are excluded because they

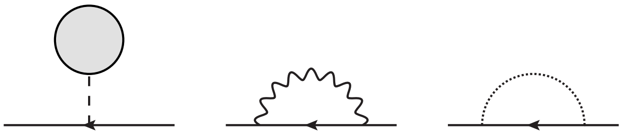

Figure 19.2: Diagrams contributing to the fermion selfenergy in one-loop order.

should transform as a four-vector. This requires them to be proportional to the four-momentum of the external line, which, however, vanishes by energy-momentum conservation.

Hence we restrict ourselves to external σ-lines. There are five different diagrams, distinguished by the internal line. This line is associated with either the field ψ, σ, χ, A_μ, and with the ghost field. The total result can thus be decomposed into

$$T_\sigma = T(\psi) + T(\sigma) + T(\chi) + T(A) + T(bc)\,, \tag{19.67}$$

where

$$\begin{aligned} T(\psi) &= \mathrm{i}(2\pi)^4\, G \int \mathrm{d}^4k\ \mathrm{Tr}[\Delta_\psi(k)]\ , \\ T(\sigma) &= -3\mathrm{i}(2\pi)^4\, gv \int \mathrm{d}^4k\ \Delta_\sigma(k)\ , \\ T(\chi) &= -\mathrm{i}(2\pi)^4\, gv \int \mathrm{d}^4k\ \Delta_\chi(k)\ , \\ T(A) &= -\mathrm{i}(2\pi)^4\, qM \int \mathrm{d}^4k\ \Delta_\mu{}^\mu(k)\ , \\ T(bc) &= -(2\pi)^4\, qM\lambda^{-1} \int \mathrm{d}^4k\ \Delta_{bc}(k)\ . \end{aligned} \tag{19.68}$$

We included an extra minus sign for the closed fermion and ghost loops. A quick perusal shows that the gauge-dependent part of $T(A)$ cancels against the ghost diagram $T(bc)$. Using the simple identity,

$$g\,v = \frac{q\,m_\sigma{}^2}{2\,M} = \frac{G\,m_\sigma{}^2}{2\,m_\psi}\,, \tag{19.69}$$

the full result for T_σ can be expressed as follows,

$$\begin{aligned} T_\sigma = \frac{1}{M}\int \mathrm{d}^4k\ &\left\{\frac{4\,GMm_\psi}{k^2+m_\psi{}^2} - \frac{3\,qm_\sigma{}^2}{2[k^2+m_\sigma{}^2]} - \frac{3\,qM^2}{k^2+M^2}\right\} \\ &- \frac{G\,m_\sigma{}^2}{2m_\psi}\int \frac{\mathrm{d}^4k}{k^2+(M/\lambda)^2}\,. \end{aligned} \tag{19.70}$$

Observe that the gauge-dependent part is proportional to ${m_\sigma}^2$, a feature that will be important in the next calculation.

Note that we evaluated the expressions for four space-time dimensions, while the integrals have ultraviolet divergencies. The calculations of this chapter can easily be done in n space-time dimensions with a definition for γ_5 that anti-commutes with the gamma matrices γ_μ. For the class of diagrams considered here, this defines a consistent form of dimensional regularization. However, the reader should bear in mind that the theory that we discuss is anomalous, so that there are certain sectors where subtleties will arise. However, in this chapter we will not encounter anomalous graphs. We return to a more thorough discussion of anomalies in chapter 22.

(iii) Fermion self-energy corrections: We now consider the one-loop corrections to the fermion propagator. One correction we have already determined above, involves the tadpole diagram which yields a contribution

$$\Sigma_{(\mathrm{a})} = -\frac{G\,T_\sigma}{{m_\sigma}^2}\,, \tag{19.71}$$

where the factor $1/{m_\sigma}^2$ arises from the σ propagator at zero momentum, as is shown in diagram (a) in fig. 19.2.

The remaining diagrams are due to the exchange of the vector bosons and of the scalars, and correspond to the diagrams (b) and (c) depicted in fig. 19.2 Their corresponding expression reads

$$\begin{aligned}\Sigma_{(\mathrm{b})+(\mathrm{c})}(p) = \mathrm{i}(2\pi)^4 \int & \frac{\mathrm{d}^4k}{(p-k)^2+m_\psi^2} \\ & \times \Big\{ -\tfrac{1}{4}q^2[\gamma^\mu\gamma_5(-\mathrm{i}\not{p}+\mathrm{i}\not{k}+m_\psi)\gamma^\nu\gamma_5]\,\Delta_{\mu\nu}(k) \\ & \quad + G^2\,[-\mathrm{i}\not{p}+\mathrm{i}\not{k}+m_\psi]\,\Delta_\sigma(k) \\ & \quad - G^2\,[\gamma_5(-\mathrm{i}\not{p}+\mathrm{i}\not{k}+m_\psi)\gamma_5]\,\Delta_\chi(k)\Big\}\,,\end{aligned} \tag{19.72}$$

After some gamma-matrix algebra this result decomposes into two parts,

$$\begin{aligned}
\Sigma_{(\mathrm{b})+(\mathrm{c})}(p) = \\
&\int \frac{\mathrm{d}^4 k}{(p-k)^2+m_\psi^2}\left\{-\frac{q^2(\mathrm{i}\not{p}-\mathrm{i}\not{k}-2\,m_\psi)}{2[k^2+M^2]}+\frac{G^2\,(-\mathrm{i}\not{p}+\mathrm{i}\not{k}+m_\psi)}{k^2+{m_\sigma}^2}\right\} \\
&+\int \frac{\mathrm{d}^4 k}{(p-k)^2+m_\psi^2}\left\{\frac{q^2[\mathrm{i}\not{k}(p^2-(p-k)^2)+k^2(\mathrm{i}\not{p}-m_\psi)]}{4k^2}\right. \\
&\qquad\qquad \times\left[\frac{1}{k^2+M^2}-\frac{1}{\lambda^2[k^2+(M/\lambda)^2]}\right] \\
&\qquad\qquad \left.+\frac{G^2\,(-\mathrm{i}\not{p}+\mathrm{i}\not{k}-m_\psi)}{k^2+(M/\lambda)^2}\right\}\,.
\end{aligned} \tag{19.73}$$

We expect the self-energy corrections to be gauge independent only on the mass shell, so we now assume $p^2+{m_\sigma}^2=0$ and apply the above result on a Dirac spinor satisfying $(\not{p}-\mathrm{i}m_\psi)u(p)=0$. Concentrating on the gauge dependent terms, contained in the second integral in both (19.70) and (19.73), we can then drop one integral because its integrand is antisymmetric in k_μ, and we obtain for the combined on-shell, gauge-dependent terms in (19.71) and (19.73),

$$\begin{aligned}
\Sigma_{\mathrm{g.d.}}(p) = &\int \frac{\mathrm{d}^4 k}{[(p-k)^2+m_\psi^2]\,[k^2+(M/\lambda)^2]} \\
&\times\left\{\frac{G^2\,M^2}{2\,\lambda^2\,m_\psi}+\mathrm{i}G^2\,\not{k}+\frac{G^2}{2m_\psi}[(p-k)^2+m_\psi^2]\right\}\,,
\end{aligned} \tag{19.74}$$

where in the first term we made use of (19.65). The third term represents the contribution from the tadpole diagrams.

to be completed

19.5. Phase structure of gauge theories

In section 18.3 we discussed the structure of spontanously broken realizations of a symmetry. In this discussion the so-called *stability subgroup* (also called the *isotropy group*) played a central role. This group, denoted by H, is a subgroup of the full symmetry group G, so that H ⊂ G. It is defined as the residual symmetry group of a a point ϕ in the (scalar) field configuration space. We assume that only Lorentz scalars have non-zero values, as we are interested in Lorentz invariant ground states. The scalar fields do not necessarily correspond to elementary fields, but they can take the form of composite operators. Therefore one often speaks generically of *order parameters*. In perturbation theory, one is usually restricted to elementary fields, but it is not difficult to construct composite order parameters, such as $\bar{\psi}\psi$, where ψ is a fermion field, which is an order parameter whose finite value leads to chiral symmetry breaking. Another one, $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$, where $F_{\mu\nu}$ is the non-abelian field strength, is a (Lorentz and gauge invariant) order parameter that is expected to be relevant in non-perturbative QCD.

At each point ϕ in the configuration space the group G sweeps out a certain orbit $\mathrm{G}(\phi)$ whose dimension is equal to the difference of the dimensions of the two groups G and H, as is specified by (18.36). In the previous chapter, where we were discussing rigid symmetries, this implied that the orbit characterizes a physical degeneracy parametrized by the fields corresponding to the massless Goldstone bosons. However, when the symmetry group (or a subgroup thereof) parametrizes a *local* invariance, the situation is fundamentally different. The orbit swept out by a local invariance group has no physical significance as all points on the orbit are connected by local gauge transformations. This implies that all points on the orbit should be counted as *identical*! Physically what happens is that the orbital parameters are absorbed in corresponding gauge fields which will become massive, in the same way as we have been discussing earlier for simple models. In section 18.3 we already noted that the orbits can be fully characterized in terms of invariants. The values of these invariants determine the stability subgroup, the dimension of the orbit and the number of gauge fields that will acquire a mass. For elementary scalar fields we have been discussing this phenomenon in a number of simple examples in section 18.3.

To appreciate the implications of removing the orbital gauge degrees of freedom, we return again to scalar fields transforming in the adjoint representation of SU(3), which was discussed at the end of subsection 18.3. These fields can be described as a 3×3 matrix Φ, which was parametrized in terms of two real parameters a and b, up to an SU(3) transformation used to bring the matrix in diagonal form as shown in (18.47). In fig. 19.3 the two-dimensional plane spanned by these parameters is shown, with three different lines, characterized by $a = 0$, and $\pm a + b\sqrt{3} = 0$, which devide the plot into six different areas, known as Weyl chambers. A generic point on this graph has a six-dimensional orbit associated to it, as $\mathrm{H} = \mathrm{U}(1) \times \mathrm{U}(1)$. On the three lines, two of the eigenvalues of the matrix Φ are equal, so that $\mathrm{H} = \mathrm{SU}(2)$ and the orbit is five-dimensional. At the origin of the diagram, $\mathrm{H} = \mathrm{SU}(3)$, and the associated orbit is then eight-dimensional.

Just as before the orbital structure may change depending on the values taken by the order parameters. When they change, one is dealing with a different phase (realization) of the theory, which obviously has a different physical spectrum as the number of massive and massless gauge fields will

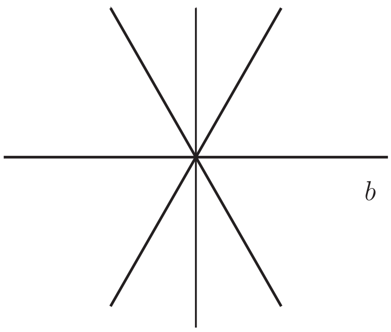

Figure 19.3: Two-dimensional plot of the gauge-invariant components of a field Φ transforming in the adjoint representation of SU(3).

change. The relevance of the stability subgroup is clear. It is the subgroup of gauge transformations that is left invariant and therefore it still realized as a local symmetry group of that particular realization. The massless vector fields will therefore constitute a gauge theory associated with the stability subgroup H. Changes in the values of the order parameters may, in principle, be effected by external means, for instance, by changing the temperature or by applying external fields. The question whether a particular phase is stable, is determined by the underlying dynamics of the theory.

Let us now describe a gauge theory based on a group G, which is realized in a phase characterized by a stability subgroup isomorphic to a group H. In other words, the order parameters ϕ take classical values in one of the possible orbits with a stability subgroup equivalent to H. To decompose ϕ according to this orbital structure we choose the parametrization

$$\phi(x) = U(x)\phi(x), \tag{19.75}$$

where $U(x)$ is an element of G which contains the dependence on the orbital parameters, and $\phi(x)$ describes the remaining degrees of freedom. Hence $U(x)$ corresponds to a certain parametrization of the coset space G/H. The restricted fields ϕ do not contain any of the orbital degrees of freedom; their classical value defines an element of the orbit in question. If this orbit has the largest possible stability subgroup then ϕ can be expressed entirely in terms of the gauge-invariant radial variables; otherwise ϕ contains non-invariant components as well.

To elucidate parametrization such as (17.20) we first discuss the situation described previously in the examples 1 and 2. In the first example we have a field ϕ transforming in the vector representation of $SO(3)$. In the trivial case

(a) where ϕ is expanded about its origin there are no orbital variables; the stability subgroup is the full $SO(3)$ group, and we have $U(x) = 1$ and leave ϕ unrestricted: $\phi = \phi$. In the second case (b), where ϕ is expanded about some constant vector the stability group is SO(2). The orbital variables therefore parametrize the coset space SO(3)/SO(2). If we choose $\phi(x)$ along the positive z-axis,

$$\phi = (0, 0, \rho), \tag{19.76}$$

then $U(x)$ is a 3×3 matrix which can be expressed in the orbital variables α_1 and α_2 according to (17.16). In this way we have achieved a decomposition of ϕ of type (3.20) in terms of one radial variable ρ and two orbital variables α_1, α_2, which renders

$$\phi = \rho(\alpha_1/\alpha \sin\alpha, \alpha_2/\alpha \sin\alpha, \cos\alpha), \tag{19.77}$$

with

$$\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}. \tag{19.78}$$

A more complicated structure arises in the second example where we have two fields ϕ_1 and ϕ_2 both transforming according to the vector representation of $SO(3)$. When the two fields are expanded about zero (a) there are again no orbital variables, and so that $U(x) = 1$ and ϕ_1 and ϕ_2 take general values $\phi_{1,2} = \phi_{1,2}$. In case (b) and (c) one of the fields is about zero, and the other one about some fixed vector. Choosing this field in the positive z-direction, we find one radial variable defined by

$$\phi_2 = (0, 0, \rho). \tag{19.79}$$

The other field is unrestricted

$$\phi_1 = (\sigma_1, \sigma_2, \tau) \tag{19.80}$$

where we have assumed $< \rho >> 0$ and $< \sigma_1 >=< \sigma_2 >=< \tau >= 0$. The two orbital variables α_1 and α_2 which parametrize the $SO(3)/SO(2)$ coset space are again of the form (17.16) and we write according to (17.20)

$$\begin{aligned} \phi_1 &= U(\alpha_1, \alpha_2)\phi_2, \\ \phi_2 &= U(\alpha_1, \alpha_2)\phi_1. \end{aligned} \tag{19.81}$$

Note that the two-dimensional vector (σ_1, σ_2) transforms as a vector under the $SO(2)$ stability subgroup. In case (d) the two vectors are exp expanded about parallel vectors, so that the orbital variables still parametrize $SO(3)/SO(2)$.

Choosing the direction of the fields such that the $SO(2)$ stability group again rotates the 1- and 2-directions, we may choose for the restricted fields

$$\begin{aligned} \phi_1 &= (0, 0, \rho_1), \\ \phi_2 &= (\sigma_1 . \sigma_2, \rho_2). \end{aligned} \tag{19.82}$$

with $< \rho_{1,2} >\neq 0$ and $< \sigma_{1,2} >= 0$. The two-vector $(\sigma_1 \sigma_2)$ rotates again under the elements of the $SO(2)$ stability group.

The case of maximal symmetry breaking arises when the fields $\phi_{1,2}$ are expanded about two non-parallel vectors. The stability group is now the identity, and the orbital variable now parametrizes the full $SO(3)$ group. Such a parametrization has been given in the group-theoretical introduction in this first volume. Hence we have three orbital parameters α_1, α_2, and α_3. The restricted field components can be parametrized as follows

$$\begin{aligned} \phi &= (0, \rho_3/\rho_1, \rho_1) \\ \phi_2 &= (0, -\rho_3/\rho_2, \rho_2) \end{aligned} \tag{19.83}$$

with $< \rho_{1,2,3} >\neq 0$.

We can now perform a gauge transformation characterized by $U(x)$, which parametrizes the G/H coset space, on all fields of the theory in question. However, since that theory was assumed to be locally gauge-invariant, this gauge transformation has no effect; in other words, we have established that the orbital variables decouple from the theory, and that we are thus free to restrict the order parameters ϕ to the subset ϕ. This does not mean that the theory is losing degrees of freedom, because in the absence of the orbital degrees of freedom, we can no longer use the corresponding gauge transformations. Therefore the gauge-invariance is only manifest with respect to the stability subgroup H. At this point the reader should recognize that we have followed the same procedure as in the previous chapter, where we discussed a gauge theory based on $U(1)$. In that case the theory was expressed entirely in gauge-invariant quantities; this corresponds to imposing the unitary gauge. In the more general case that we are discussing here we have chosen the unitary gauge with respect to the symmetries of G/H only. The fact that the gauge freedom with respect g/g_ν has disappeared implies that the corresponding gauge fields contain more than the transversal degrees of freedom. This is also clear from the form of the covariant derivatives on the restricted order parameters.

$$\Delta_\mu \phi(x) = (\partial_\mu - W_\mu^a(x) t_a) \phi(x). \tag{19.84}$$

Because we expand ϕ about its classical value

$$\phi(x) = v \tag{19.85}$$

the derivative contains a term linear in W_μ^a, but only for those values of a that correspond to g/g_ν .

$$D_\mu\phi(x) = \partial_\mu\phi(x) - W_\mu^a(x)(t_a v) + \text{ non-linear terms} . \tag{19.86}$$

When we now consider a standard kinetic term for the order parameters ϕ , they will lead to a mass term for the gauge fields corresponding to g/g_ν, whereas the gauge fields corresponding to the stability subgroup remain massless.

Problems

19.1. Let us study the Lagrangian (19.2) in two spatial dimensions and analyze finite-energy solutions. The solution that we will discover is rotationally invariant and can be embedded into a similar field theory in 3 + 1 dimensions, where they look like a static stringlike configuration. Because the solution carries magnetic fluss, these solutions are known as flux tubes or vortices. They were first studied in **abrikosov, nielsen, olesen**.
In the case of two spatial dimensions the energy equals minus the Lagrangian, so that

$$E = \int \mathrm{d}^2x \left\{ \tfrac{1}{4}F_{ij} + |\boldsymbol{\nabla}_i\phi|^2 - \mu^2|\phi|^2 + \lambda|\phi|^4 \right\} . \tag{1}$$

Note that $\mu^2 > 0$ so that the mass term has the 'wrong' sign.
let us consider static solutions of the corresponding field equations. ETC vortices

19.2. Consider a vector field A_μ coupled to a real scalar field ϕ and a spinor field ψ in four space-time dimensions, described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{M^2}{2\,q^2}\left|(\partial_\mu - \mathrm{i}qA_\mu)\mathrm{e}^{\mathrm{i}q\phi/M}\right|^2 - \bar\psi(\partial\!\!\!/ - \mathrm{i}gA\!\!\!/ + m)\psi\,. \tag{1}$$

Show that the Lagrangian is invariant under the combined gauge transformations,

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\xi\,,\\ \phi &\to \phi + M\,\xi\,,\\ \psi &\to \exp[ig\xi]\,\psi\,, \end{aligned} \tag{2}$$

where $\xi(x)$ is an arbitrary function of space and time.
Collect all terms quadratic in the fields A_μ and ϕ. Argue that the inverse propagator takes the form of a 5×5 matrix and determine this matrix. Does the propagator exist? (Try to motivate the answer in two different ways: both on the basis of the explicit matrix and on the basis of a more general argument.)
Argue that $\phi = 0$ is an admissable gauge condition. Determine the propagator for A_μ in this gauge. What are the physical bosonic states of given momentum described by

the resulting Lagrangian? Is the resulting theory renormalizable by power counting and why (not)?
We now choose another gauge condition by adding the following term to the Lagrangian,

$$\mathcal{L}_{\text{gauge-fixing}} = -\tfrac{1}{2}(\lambda\,\partial_\mu A^\mu - M\lambda^{-1}\phi)^2\,, \tag{3}$$

with λ an arbitrary parameter. Calculate again the propagators for A_μ and ϕ. In this case, specify the physical bosonic states for given momentum described by the corresponding Lagrangian? Compare your result with your previous answer.
In the gauge based on (3), is the theory renormalizable by power counting and why (not)? Write down the expression for the fermion self-energy diagram in the one-loop approximation and determine the degree of divergence of the integral. Compare the degree of divergence to what it would have been with the previous gauge condition $\phi = 0$.
Determine the difference between the expressions for the fermion self-energy diagram in the two gauges on the mass shell, i.e. sandwiched between spinors that satisfy the Dirac equation $(\mathrm{i}\not{p} + m)u = 0$ (this implies $p^2 + m^2 = 0$). Did you expect this result and why (not)?

19.3. Consistency: em coupling of Proca field. $D_\mu V_\nu - D_\nu V_\mu$ etc. Relate to (19.28). use complex W with em covariant derivatives etc.

19.4. In this problem we consider the group $\mathrm{SU}(N)$, acting on a variety of fields belonging to a certain representation. When such a field takes a non-zero value, it will leave a subgroup of the full group invariant, the so-called *stability subgroup*. When assuming that the group is realized locally, one can determine the mass spectrum of the $\mathrm{SU}(N)$ gauge fields. We recall that $\mathrm{SU}(N)$ is the group of unitary $N \times N$ matrices with unit determinant. The dimension of $\mathrm{SU}(N)$ equals $N^2 - 1$, so that the number of independent gauge fields in this problem is equal to $N^2 - 1$.
We start with the N-dimensional defining representation consisting of an N-component array Φ of complex scalar fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}\,, \tag{1}$$

transforming under local $SU(N)$ transformations according to

$$\Phi'(x) = U(x)\,\Phi(x)\,, \qquad U(x) \in \mathrm{SU}(N)\,. \tag{2}$$

Gauge invariance requires that Φ is coupled to $\mathrm{SU}(N)$ gauge fields. The Lagrangian reads,

$$\mathcal{L} = \tfrac{1}{4}\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu}\big] - |D_\mu\Phi|^2 - V(\Phi)\,, \tag{3}$$

where we used the definitions

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi - gW_\mu{}^a t_a\Phi\,,\\ G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu - g\,[W_\mu W_\nu]\,. \end{aligned} \tag{4}$$

Here $W_\mu = W_\mu{}^a t_a$ with t_a the SU(N) generators in the N-dimensional representations, which are anti-hermitean, traceless $N\times N$ matrices normalized according to $\mathrm{Tr}[t_a t_b] = -\delta_{ab}$. Besides being invariant under local SU(N), this theory is also invariant under a rigid U(1), corresponding to constant phase transformations of the field Φ.

Argue that the most general renormalizable (in four space-time dimensions) potential invariant under SU(N) must be of the form

$$V(\Phi) = m^2|\Phi|^2 + \lambda|\Phi|^4\,. \tag{5}$$

Show that the stationary points of this potential are given by

$$m^2\,\Phi + 2\,\lambda\,|\Phi|^2\,\Phi = 0\,. \tag{6}$$

Depending on the values of the constants m^2 and λ there are two possible solutions. One solution (for $m^2, \lambda > 0$) is $\Phi = 0$. Argue that the (rigid) symmetry group of this realization equals SU(N) $\times$ U(1) and that the number of massless gauge fields equals N^2-1, transforming in the adjoint representation of SU(N).

Another solution (for $m^2 < 0$ and $\lambda > 0$ is characterized by $|\Phi|^2 = -m^2/(2\lambda)$. To analyze these symmetries and the mass spectrum, it is convenient to bring the field Φ in a special form by exploiting a suitable local SU(N) transformation. In this way one can achieve that $\phi_i = 0$ for $i = 2,\ldots,N$, while ϕ_1 is real. The vacuum expectation value can then be chosen equal to $\sqrt{-m^2/(2\lambda)}$. Derive that the rigid invariance group of this realization equals SU($N-1$) $\times$ U(1).

Decompose the matrix W_μ accordingly as,

$$W_\mu = \begin{pmatrix} -\mathrm{i}\sqrt{\frac{N-1}{N}}W_\mu{}^0 & -\bar{W}_\mu \\ W_\mu & \sqrt{\frac{1}{N(N-1)}}\mathrm{i}W_\mu{}^0\,\mathbf{1} + \hat{W}_\mu \end{pmatrix}, \tag{7}$$

where $W_\mu{}^0$ is real singlet, W_μ is a complex $(N-1)$-dimensional vector, and $\bar{W}_\mu$ is its complex conjugate written as a row vector. Furhtermore, the matrix $\hat{W}_\mu$ takes values in the Lie algebra of SU($N-1$). Hence $\hat{W}_\mu = \hat{W}_\mu{}^a\hat{t}_a$, with the $\hat{t}_a$ the generators of SU($N-1$): traceless, anti-hermitean $(N-1)\times(N-1)$ matrices normalized according to $\mathrm{Tr}[\hat{t}_a\hat{t}_b] = -\delta_{ab}$, where the indices $a, b, \ldots$ now run over $(N-1)^2-1$ values.

Using these decompositions, derive the mass spectrum for the gauge fields and for the remaining physical scalar. Indicate how the residual U(1) symmetry acts on the fields. In particular prove that the ratio of the mass of the $N-1$ complex fields W_μ and the mass of $W_\mu{}^0$ is given by,

$$\frac{M_{W^0}}{M_W} = \sqrt{\frac{2(N-1)}{N}}\,. \tag{8}$$

19.5. Next, we consider N such N-component fields Φ which we arrange as an $N \times N$ complex scalar matrix field that we denote by $\hat{\Phi}$. Each column of this matrix transforms under local SU(N) transformations, as before. In addition the N columns can also be rotated by rigid U(N) transformations which will act on the $N\times N$ matrix $\hat{\Phi}$ with unitary matrices $V^\dagger$ acting from the right. Hence,

$$\hat{\Phi}'(x) = U(x)\,\hat{\Phi}(x)\,V^\dagger\,, \qquad U(x) \in \mathrm{SU}(N)\,, \qquad V \in \mathrm{U}(N)\,. \tag{1}$$

When coupled to $SU(N)$ gauge fields, the Lagrangian is similar to (3), except that the matrix field $\hat{\Phi}$ requires the presence of traces, such as, for instance, in the kinetic term,

$$|D_\mu \Phi|^2 \longrightarrow \mathrm{Tr}(D^\mu \hat{\Phi}^\dagger D_\mu \hat{\Phi})\,. \tag{2}$$

iii) Argue that the most general renormalizable potential invariant under both the rigid and the local symmetry equals,

$$V(\hat{\Phi}) = m^2\,\mathrm{Tr}(\hat{\Phi}^\dagger \hat{\Phi}) + \lambda_1\,[\mathrm{Tr}(\hat{\Phi}^\dagger \hat{\Phi})]^2 + \lambda_2\,\mathrm{Tr}(\hat{\Phi}^\dagger \hat{\Phi} \hat{\Phi}^\dagger \hat{\Phi})\,. \tag{3}$$

Derive the following matrix equation for stationary points of the potential,

$$m^2\,\hat{\Phi} + 2\,\lambda_1\,\mathrm{Tr}(\hat{\Phi}^\dagger \hat{\Phi})\,\hat{\Phi} + 2\,\lambda_2\,\hat{\Phi}\hat{\Phi}^\dagger\hat{\Phi} = 0\,. \tag{4}$$

We now make use the fact that any complex matrix can be written as a product ADB, where D is a real diagonal matrix with non-negative eigenvalues, and A and B are unitary matrices. Without loss of generality one can restrict A to SU(N) and V to U(N), because the extra U(1) factor commutes with A, D and B. Since the theory is invariant under (1), we can then restrict the discussion of (4) to a diagonal matrix for $\hat{\Phi}$ with non-negative eigenvalues. Obviously zero eigenvalues are always a solution of (4).

iv) Show that the general solutions can therefore be classified in terms of an integer $0 \leqslant n \leqslant N$, which specifies the number of positive eigenvalues. The number of zero eigenvalues is then equal to $N - n$. Prove that the positive eigenvalues are all equal to the same (positive) quantity v, and determine the value of v for given n.

v) For given n, find the stability subgroup of the SU(N) gauge group and specify the number of massless and of massive gauge fields.

vi) The residual rigid symmetry group is given by the stability subgroup of SU(N) $\times$ U(N). This is the symmetry group that is the manifestly realized after some of the gauge fields have acquired masses. For given value of n specify this group. As an example, assume that $n = N$ so that the matrix $\hat{\Phi}$ is proportional to the identity matrix. In that case specify the number of massive vector fields and specify the symmetry group after the Brout-Englert-Higgs mechanism has been effected.

19.6. Consider a non-abelian gauge theory based on the group SU(N) with anti-hermitean, traceless, generators t_a, coupled to N fermion fields ψ transforming in the fundamental representation of SU(N), and to $N^2 - 1$ scalar fields described by an hermitean, traceless, matrix Φ, transforming in the adjoint representation of the gauge group. We write the gauge fields in Lie-algebra valued form, $W_\mu(x) = W_\mu{}^a(x)\, t_a$. Also the scalar fields can be decomposed in this way, $\Phi(x) = \mathrm{i}\phi^a(x)\, t_a$. Hence the local SU($N$) transformation rules are

$$W_\mu \to U W_\mu U^{-1} + \frac{1}{g}(\partial_\mu U)U^{-1}, \qquad \psi \to U\psi, \qquad \Phi \to U\Phi U^{-1} \tag{1}$$

Consider the Lagrangian

$$\begin{aligned}\mathcal{L} = {}& \tfrac{1}{4}\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu}\big] - \bar\psi(\partial\!\!\!/ - gW\!\!\!\!/\,)\psi \\ & - \tfrac{1}{2}\mathrm{Tr}\big[D_\mu\Phi\, D^\mu\Phi\big] - \tfrac{1}{2}\mu^2\mathrm{Tr}\big[\Phi^2\big] - \lambda_1\big(\mathrm{Tr}\big[\Phi^2\big]\big)^2 - \lambda_2\mathrm{Tr}\big[\Phi^4\big]\,.\end{aligned} \tag{2}$$

Note that the canonical normalization for the kinetic terms is obtained for $\mathrm{Tr}[t_a t_b] = -\delta_{ab}$.

i) Write down the expression for $G_{\mu\nu}$, and for $D_\mu\Phi$. Note that the two expressions share a certain similarity, but there are also differences. Can you comment on these features?

ii) Assume that the field Φ has a non-zero expectation value. Substitute $\Phi \to \Phi + \delta\Phi$ in the potential in Eq. (2). Collect all terms linearly proportional to $\delta\Phi$, and give the resulting expression. When this expression vanishes for all $\delta\Phi$, then Φ is a stationary point of the potential. Show that $\Phi = 0$ is such a stationary point. Do any of the SU(N) gauge fields acquire a mass in this case?

iii) Show that any stationary point of the potential must satisfy the equation,

$$\tfrac{1}{4}(\mu^2 + 2\,\lambda_1\mathrm{Tr}[\Phi^2])\Phi + \lambda_2\Phi^3 = \mathbf{1}\,\lambda_2 N^{-1}\mathrm{Tr}[\Phi^3]\,. \tag{3}$$

Argue that Φ can be diagonalized so that this equation gives the condition for possible eigenvalues of Φ.

iv) Assume that N is even and that $\Phi \neq 0$ has $\frac{1}{2}N$ eigenvalues a and $\frac{1}{2}N$ eigenvalues $-a$. Determine the value of a.

v) Give the number of massless vector bosons in this situation.

vi) *Bonus question*: the remaining vector bosons acquire a mass. Can you say anything about their mass differences?

19.7. Consider the following Lagrangian for SU(N) gauge fields coupled to scalar fields H,

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \frac{M^2}{2g^2}\mathrm{Tr}\left(D_\mu H\, D^\mu H^\dagger\right)\,. \tag{1}$$

Here $G_{\mu\nu} = G_{\mu\nu}{}^a\, t_a$ is the covariant field strength tensor, expressed in Lie-algebra valued form. We take the generators t_a in the fundamental representation, so that they are N-by-N traceless, antihermitian, matrices. Consequently $\mathrm{Tr}[t_a t_b]$ is a negative-definite matrix, where $a, b = 1, 2, \ldots, N^2 - 1$. The field H also takes the form of an N-by-N matrix, whose precise definition will be given later. Under SU(N), H transforms as $H(x) \to U(x)\, H(x)$.

i) Give the definitions of $G_{\mu\nu}$, $D_\mu H$ and $D_\mu H^\dagger$ in matrix form, and write out the second term in the Lagrangian (1) into a kinetic term for H, a term linear in W_μ and a term quadratic in W_μ. In particular, note that the last term is proportional to $\mathrm{Tr}\,\big(W_\mu W^\mu H\, H^\dagger\big)$.

We now assume that H can be written in the following form,

$$H(x) = \exp[\Phi(x)/M]\,, \qquad (2)$$

where Φ is a Lie-algebra valued field, so that it can be expressed in terms of N^2-1 real fields $\phi^a(x)$ according to $\Phi(x) = \phi^a(x)\, t_a$.

ii) Show that $H(x)$ is a unitary N-by-N matrix with unit determinant, so that H is an element of the group SU(N). Subsequently argue that by a suitable gauge choice one can choose $H(x) = \mathbf{1}$. Consider the Lagrangian in this gauge and determine the relevant propagator(s). In view of the large-momentum behaviour of the propagator(s), explain whether the theory is renormalizable or not (by power counting).

iii) Upon expanding the Lagrangian for H, as given in (1), in powers of Φ, one obtains terms of order M^2, M, and M^0, as well as terms proportional to negative powers in M. Write down all the terms of (1) quadratic in the fields W_μ and Φ. From these terms we must construct the propagators. Because of gauge invariance, a gauge-fixing term must be added and we make the following choice,

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2g^2}\mathrm{Tr}\,\big((\lambda\,\partial_\mu W^\mu + \lambda^{-1} M\,\Phi)^2\big)\ . \qquad (3)$$

Subsequently write down the kinetic terms for the fields and give the propagators (up to a proportionality factor). Argue that the propagators are of the renormalizable type.

iv) Now determine all the interaction terms that are *not* proportional to negative powers of M.

v) Determine also interaction terms linear proportional to M^{-1}. Note that the remaining terms yield interactions proportional to $M^{2-p}\,\Phi^p$, with $p>3$. Specify how many derivatives they can contain and how many fields W_μ. Argue that all these terms should vanish in case the gauge group had been abelian.

vi) Is the theory renormalizable by power counting in the gauge (3)?

19.8. Fermion gauge invariant Yukawa etc, see (19.56)