18 | Spontaneously broken symmetry

Can a theory that is exactly invariant under a continuous symmetry, have a non-invariant ground state? In general a symmetric theory gives rise to both invariant and non-invariant states. Take, for instance, the hydrogen atom, which is described by a rotationally invariant Hamiltonian. Its energy eigenstates of nonvanishing angular momentum l are not invariant under rotations as there exist $2l + 1$ independent states of angular momentum l transforming into each other. However, the actual hydrogen ground state has zero angular momentum and is therefore a singlet state that remains invariant under rotations.

In contrast, the ferromagnet is an example of a rotationally invariant system that is realized in such a way that the ground state is *not* rotationally invariant. Non-ferromagnetic materials have a rotationally invariant ground state in which the atomic spins are randomly oriented, so that the gross magnetization vanishes. However, in a ferromagnet the spin-spin interactions are such that all spins align in the state of lowest energy. Therefore the ground state gives rise to a finite magnetization which breaks rotational symmetry (we concentrate here on the rotations of the electron spins and ignore that, in a realistic ferromagnet, also the positions of the atoms are affected by a rotation). Although the underlying Hamiltonian is rotationally invariant, the groundstate is not in a singlet state and carries angular momentum. Hence one is dealing with a situation where the rotational symmetry is realized in a *spontaneously broken* way. This does *not* imply that rotational symmetry has no consequences anymore, but only that the most obvious implication of the symmetry is now absent. An important feature of a spontaneously broken realization is that the ground state must be *infinitely degenerate.* This is obvious: by applying (a continuous set of) symmetry transformations on a non-symmetric ground state, one constructs a continuous variety of different ground states. All these different states must have the same energy as the original one, because the Hamiltonian commutes with all the symmetry transformations of the theory. Indeed for a ferromagnet, an infinite set of ground states can be obtained by applying rotations to the magnet. These ground states can be characterized by the spatial orientation of the magnetization.

It is important to point out here that classical degeneracies do not necessarily survive in a quantum-mechanical context. Consider, for instance, a simple quantum-mechanical model where the classical potential has two absolute minima, corresponding to classical configurations of lowest energy. Due to quantum tunneling, the true ground state is not given by either one of these classical ground states, but, in a certain approximation, by a linear combination of them. When there is a (discrete) symmetry under which the two classical groundstates are interchanged, then the unique quantum-mechanical groundstate will be an eigenstate of this symmetry. Because of quantum tunneling, the unique ground state is a superposition of the two classical ground states.

Another, even simpler, example of this phenomenon, where one is dealing with a continuous symmetry, is the free particle. Classically a free particle has an infinite number of ground states, namely those where the particle is at rest at some given, but arbitrary, position. The theory is obviously invariant under translations, under which these ground states are transformed into one another. Quantum-mechanically, the free-particle states can be represented by wave packets of positive energy. In the limiting case of zero energy (corresponding to the ground state) the wave function tends to a constant. Therefore there exists only one single ground-state wave function which, because it is constant, is invariant under translations. (We ignore here that this wave function is not normalizable. See problem 18.1). Clearly, there are situations where there is a degeneracy of the classical ground state, whereas quantum-mechanically there exists only one single ground state. The reason for the latter is quantum tunneling. The ground-state wave function will be a superposition of all the classical ground states.

In quantum field theory the situation is different. Unlike in simple quantum-mechanical models, one is now dealing with an infinite number of degrees of freedom. Because of this, the tunneling amplitudes vanish in most cases. Again, consider the ferromagnet, whose ground state is characterized by the direction of its magnetization. Two different groundstates, with their magnetization pointing in different directions, are obviously related by a rotation. However, when the ferromagnet is infinite in size, it will require an infinite amount of energy to rotate the magnet. This infinite amount of energy is also responsible for a suppression of the tunneling amplitude between the various classical ground states. In other words, whereas it is, in principle, possible for some isolated spins to tunnel to another vacuum, it is impossible for *all* the spins to tunnel to the other ground-state configuration. Consequently the classical degeneracies are *not* removed by quantum tunneling.

In particle physics spontaneously broken realizations of symmetries play an important role. In this chapter we explain this phenomenon and its consequences. We start in a simple (classical) field-theoretic model, bearing in mind the remarks made above. Subsequently we discuss the implications for chiral symmetry in hadrons. In later sections we discuss some more technical notions that play a role in describing spontaneously broken symmetries.

18.1. Spontaneously broken realizations of a symmetry

Let us demonstrate the phenomenon of a spontaneously broken realization of a symmetry in the context of a simple field-theoretic model. Consider the Lagrangian based on a complex spinless field ϕ,

$$\mathcal{L} = -|\partial_\mu \phi|^2 - V(|\phi|)\,, \tag{18.1}$$

which is invariant under constant phase transformations,

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\xi}\,\phi(x)\,. \tag{18.2}$$

Such U(1) transformations can also be represented as two-dimensional rotations of the real and imaginary parts of ϕ. Hence the groups U(1) and SO(2) are equivalent.

In theories such as (18.1) the fields are usually expanded about some constant value for which the potential (and thus the energy) has an absolute minimum. This value characterizes the ground state of the system, in the same way as the magnetization characterizes the ground state of a ferromagnet. Of course, the actual value for the field at the minimum of the potential may change when quantum corrections are included, but such effects do not concern us here. The constant field value which represents the field configuration with minimal energy is often called the vacuum-expectation value, because in the quantum theory it represents the expectation value of the field operator ϕ for the ground state. The actual value of the field ϕ may fluctuate about this vacuum-expectation value and these fluctuations reflect the dynamical degrees of freedom described by the field. Such degrees of freedom can be associated with particles. The relevant information is encoded in the Lagrangian. Expanding the field about its vacuum-expectation value v, we obtain a Lagrangian of the conventional (Klein-Gordon) type for the two field components contained in ϕ, which describe particles with a mass determined by the second derivative of $V(|\phi|)$ at $\phi = v$. However, it turns out that, depending on whether the potential exhibits a minimum at the origin of the field configuration space ($\phi = 0$) or not, the model leads to qualitatively different situations.

As a first possibility assume that the potential acquires its minimum at $\phi = 0$. Expanding the potential about $\phi = 0$ and retaining only the terms quadratic in ϕ then gives rise to

$$\mathcal{L} = -|\partial_\mu \phi|^2 - \mu_0^2 |\phi|^2 + \cdots\,, \tag{18.3}$$

where μ_0^2 parametrizes the second derivative of V at $\phi = 0$; note that μ_0^2 must be positive as we have assumed that we are expanding the potential about a minimum. The above Lagrangian shows that the excitations described by ϕ

correspond to particles with mass μ_0. We may thus distinguish two kinds of particles, associated with the real and imaginary part of ϕ. They have equal mass, which can be understood on the basis of the symmetry (18.2) that rotates the real and imaginary parts of the field. Such a symmetric realization of the theory is sometimes called the Wigner-Weyl realization.

Now consider the case where the minimum is acquired for a non-zero value of the field. In that case one immediately realizes that there must be an infinite set of minima because of the symmetry (18.2). In the plane of real and imaginary components of ϕ these minima are located on a circle (see fig. 18.1) and each of them represents a possible ground state. This situation describes a spontaneously broken realization of the symmetry (18.2). Because we are forced to consider the theory for a nonvanishing vacuum-expectation value the symmetry is no longer manifest. Further inspection shows that the symmetry still has an important implication. Because the potential stays constant in the "angular" direction, i.e., the direction that is swept out by applying a rotation, the second derivative of the potential along that direction equals zero. Therefore, expanding about the minimum, the component of ϕ associated with rotations corresponds to a massless field. This is a specific result implied by the so-called Goldstone theorem, according to which there exists a massless particle corresponding to every generator of the symmetry group that is broken by the vacuum-expectation value of the field. This particle is called the Goldstone particle, and the spontaneously broken realization is called the Goldstone realization. One recognizes that the massless degrees of freedom are related to the symmetry that is broken, as it is this symmetry that is responsible for the degeneracy of the ground state. In the case at hand, the symmetry is generated by a scalar parameter that affects the phase of ϕ, so the Goldstone particle is a scalar particle. But more complicated examples of spontaneously broken symmetries are possible, as we will exhibit in due course.

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Figure 18.1: The potential $V(|\phi|)$, which has a local maximum at $\phi = 0$ and a locus of minima away from the origin.

Let us consider the Goldstone realization in a more quantitative way. Since we are expanding the field about some nonvanishing value, it is convenient to make a decomposition

$$\phi(x) = \frac{1}{\sqrt{2}}\,\rho(x)\;\mathrm{e}^{\mathrm{i}\theta(x)/v}\ . \tag{18.4}$$

Here v is the value for ρ where the potential has a minimum. We introduced this quantity into the exponential in order to give the field $\theta(x)$ the canonical dimension for a scalar field. The above decomposition leads to

$$\partial_\mu\phi = \frac{1}{\sqrt{2}}\, \mathrm{e}^{\mathrm{i}\theta(x)/v}\, \Big(\partial_\mu\rho + \mathrm{i}\frac{\rho\,\partial_\mu\theta}{v}\Big)\,, \tag{18.5}$$

which, when inserted into the Lagrangian (18.1), yields

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}(\rho^2/v^2)(\partial_\mu\theta)^2 - V(\rho/\sqrt{2})\,. \tag{18.6}$$

Clearly the radial degrees of freedom corresponding to ρ describe a particle with a mass given by

$$m_\rho^2 = \frac{\partial^2}{\partial\rho^2}V(\rho/\sqrt{2})\Big|_{\rho=v} = \tfrac{1}{2}V''(v/\sqrt{2})\,. \tag{18.7}$$

But the angular degrees of freedom corresponding to θ do not acquire a mass and we find a standard kinetic term for a massless scalar field with some additional derivative interactions.[1] This confirms the result of our qualitative considerations and is in agreement with Goldstone's theorem.

The substitution (18.4) is very convenient for our purposes, but a somewhat disturbing feature is that the Lagrangian (18.6) is not of the renormalizable type. Nevertheless the original Lagrangian is renormalizable provided the potential consists of only two terms, $|\phi|^2$ and $|\phi|^4$. An alternative parametrization that leads to a Lagrangian that is renormalizable by power counting, is based on a decomposition of ϕ into real and imaginary parts. Expanding about one of the minima, we may write

$$\phi = \frac{1}{\sqrt{2}}\,\big(v + \phi_1 + \mathrm{i}\phi_2\big)\,, \tag{18.8}$$

so that $|\phi|^2 = \frac{1}{2}(v^2 + 2v\,\phi_1 + \phi_1^2 + \phi_2^2)$. The Lagrangian then remains of the renormalizable type. Note that ϕ_2 is now associated with the Goldstone boson, as explicit evaluation of the potential will show. The Lagrangian remains invariant under the reflection symmetry $\phi_2 \to -\phi_2$. This type of parametrization is suitable when considering quantum corrections.

Hence a symmetry can generally be realized in two different ways. One is the symmetric realization, in which the ground state is symmetric and all physical quantities are subject to the obvious implications of the symmetry, such as the fact that two particles that are related by the symmetry have

[1]This statement is somewhat misleading, as the field $\theta(x)$ is restricted to take values between $0 \leqslant \theta(x) < 2\pi v$. In perturbation theory, which pertains to local fluctuations of the fields, this restriction has no consequences. In a non-perturbative treatment of the theory the restriction may, however, be important. The same comment applies to the field $\rho(x)$, which is restricted to taking positive values. We can, however, avoid restrictions on $\rho(x)$, by imposing a stronger bound on θ, namely $0 \leqslant \theta(x) < \pi v$

equal masses. The other is the spontaneously broken realization, in which the implications of the symmetry are more subtle. Here the ground state is degenerate and there is a massless particle associated with each spontaneously broken symmetry.[2]We should stress here that we have been considering continuous symmetries, such as rotations or phase transformations. Also discrete symmetries can be spontaneously broken. However, in that case the ground state is only finitely degenerate. In other words there is a finite number of ground states, which are not connected by continuous transformations but only by discrete ones. In that case there is *no* Goldstone particle.

When, for a given theory, one can change some of its parameters by external means, such as by the temperature, it may be possible to actually observe transitions between the two realizations. When the observable quantities change *continuously* from one phase to another then the transition is called second-order. On the other hand, when there are discontinuities one is dealing with a first-order transition. In the analogous case of the ferromagnet the phase transition is usually observed by changing the temperature. Above the so-called Curie temperature the magnet is in the symmetric phase. Below that temperature this realization becomes unstable, and the system is realized in the spontaneously broken mode in which it exhibits a spontaneous magnetization (the direct analogue of a nonzero vacuum-expectation value). For examples, see problems 18.2 and 18.3.

Indeed we observe the same phenomenon for the theory described by (18.1), assuming the potential $V(|\phi|) = \mu^2|\phi|^2 + \lambda|\phi|^4$, if we determine the vacuum expectation value v and other observables such as the masses of the particles described by ϕ, as functions of μ^2. This result is depicted in fig. 18.2. Obviously at $\mu^2 = 0$ the transition is of second order. For positive μ^2 the quantity v vanishes irrespective of the precise value for μ^2. The masses of the two particles are equal and given by μ. For negative μ^2 we have $v^2 = |\mu^2/\lambda|$, while the mass associated with the field θ vanishes and the mass associated with ρ is equal to $\sqrt{-2\mu^2}$.

We close this section by considering the Noether current associated with the symmetry (18.2). It takes the form

$$J_\mu = -\mathrm{i}q\,\phi^* \overleftrightarrow{\partial}_\mu\,\phi\,. \tag{18.9}$$

[2]Note that we will be discussing Goldstone bosons in the text, and not Goldstone fermions. The reason is that we are discussing symmetries whose parameters are 'bosonic' in the sense that they are ordinary commuting numbers. In principle, the nature of the Goldstone particle follows from the nature of the corresponding symmetry transformations. For instance, to have Goldstone particles with spin, it is necessary to have symmetry transformations whose parameters carry spin. In a relativistic context that usually means that they have to transform nontrivially under the Lorentz transformations. To have Goldstone fermions, one must have symmetry transformations with anticomuting parameters that transform as spinors under the Lorentz group. Such a symmetry exists and it is known as *supersymmetry.* Under supersymmetry bosons transform into fermions, and vice versa.

V V V

$\mu^2 < 0$ $\mu^2 = 0$ $\mu^2 > 0$

v^2 m^2

Figure 18.2: The typical profiles of the potential $V(\rho/\sqrt{2}) = \frac{1}{2}\mu^2\,\rho^2 + \frac{1}{4}\lambda\,\rho^4$ for $\mu^2 < 0$, $\mu^2 = 0$ and $\mu^2 > 0$, with the square of vacuum-expectation value v of ϕ and the masses squared of the two corresponding two spinless particles as a function μ^2.

As the reader may verify, this current is conserved by virtue of the equations of motion, provided that the potential depends only on the norm of ϕ (as is required by the invariance (18.2)). One may wonder what the relevance of current conservation is in a spontaneously broken realization of the corresponding symmetry. In the parametrization (18.4) the current reads

$$J_\mu = \frac{q\,\rho^2}{v}\,\partial_\mu\theta\,. \tag{18.10}$$

The equation $\partial_\mu J^\mu = 0$ precisely coincides with the equation of motion for the field θ. In section 11.4 we described how current conservation gives rise to charge conservation. However, in the spontaneously broken realization this connection no longer exists. To see this, consider again the form factor decomposition (11.50), which gives the amplitude of a current with two on-shell scalar particles,

$$\mathcal{M}_\mu(p',p) = F_1(k^2)\,(p'+p)_\mu + F_2(k^2)\,k_\mu\,, \tag{18.11}$$

with p and p' the momentum of the incoming and outgoing particle, respectively, and $k = p' - p$. Current conservation leads to $(p'^2 - p^2)\,F_1(k^2) + k^2\,F_2(k^2) = 0$. In section 11.3 we concluded from this that $F_2(k^2) = 0$ for the case of equal-mass particles, while for unequal masses $F_1(0)$ must vanish. Furthermore we derived that $F_1(0)$, the charge form factor at zero momentum transfer, can be identified as the charge of the particle, as measured in Thomson scattering (assuming that the photon couples to the current in question). This charge is additively conserved in elementary particle processes, as derived in section 11.4.

The above conclusions are, however, invalidated in the spontaneously broken case, because of the presence of the Goldstone boson. Due to this massless particle the form factors may exhibit poles at zero momentum transfer. Therefore, when considering the amplitude of the current with an incoming on-shell θ and an outgoing on-shell ρ particle, we find that

$$k^2\,F_2(k^2) = m_\rho^2\,F_1(k^2)\,. \tag{18.12}$$

This identity seems to indicate that $F_1(0) = 0$. This conclusion is, however, false, because F_2 is not regular at $k^2 = 0$ and exhibits a pole due to the fact that the current has a nonvanishing coupling to a virtual Goldstone boson, which subsequently couples to the on-shell particles. To see this explicitly let us simply evaluate the form factors in tree approximation. It turns out that two diagrams contribute, shown in fig. 18.3. In the first diagram the current is directly connected to the external particles. This vertex follows from expanding the current (18.10) by writing $\rho = v + \tilde\rho$, which leads to a term $2\,q\,\tilde\rho\,\partial_\mu\theta$. In the second diagram the coupling acts through a virtual θ

particle. The coupling of θ with the current equals $qv\,\partial_\mu\theta$, whereas the $\tilde\rho\theta\theta$ coupling follows from expanding the second term in the Lagrangian (18.6), which yields $-\frac{1}{2}((\tilde\rho/v)^2+2(\tilde\rho/v)+1)(\partial_\mu\theta)^2$. The two diagrams thus lead to the following expression,

$$\mathcal{M}_\mu(p',p) = 2\mathrm{i}q\,p_\mu - 2\mathrm{i}q\,p\cdot k\,\frac{1}{k^2}\,k_\mu = 2\mathrm{i}q\Big(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)p^\nu\,, \tag{18.13}$$

where we also included the θ-propagator in the second term. This expression is indeed conserved; it vanishes when contracted with k^μ by virtue of the transverse projection operator. On shell we use $p\cdot k = -\frac{1}{2}(m_\rho^2+k^2)$, so that

$$\mathcal{M}_\mu(p',p) = \mathrm{i}q\,(p'+p)_\mu + \mathrm{i}\frac{q\,m_\rho^2}{k^2}\,k_\mu\,. \tag{18.14}$$

Indeed F_1 is nonvanishing while F_2 exhibits a pole at $k^2=0$.[3]

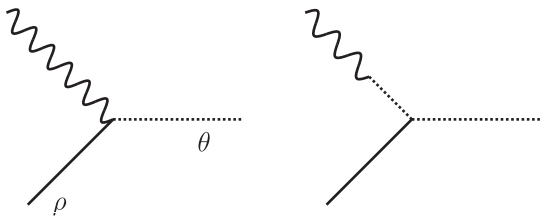

Figure 18.3: Tree diagrams contributing to the form factors of the Noether current with an incoming θ and an outgoing ρ particle.

Hence we identified two characteristic features of spontaneous symmetry breaking. First the presence of the massless Goldstone bosons, which lead to poles in various form factors that are crucial for preserving current conservation. At the same time the charge form factor at zero momentum transfer will no longer correspond to a quantity that is additively conserved in particle reactions, in contrast with the situation described in section 11.4. Second, the current has a transition to a single particle. In the model of this section the amplitude for the current and a single θ particle with incoming momentum k is equal to $\mathrm{i}qv\,k_\mu$. In the general case the amplitude for a current and a single scalar (off-shell) particle P can be written in terms of a single function,

$$\mathcal{M}_\mu(k) = \mathrm{i}k_\mu\,f_{\mathrm{P}}(k^2)\,. \tag{18.15}$$

[3]Note that it is not necessary here to retrict the ρ-momentum to its mass-shell. This is related to the fact that the field ρ is invariant under the U(1) symmetry.

As the current is conserved when the particle is on its mass shell, we derive the equation $m_{\rm P}^2 f_{\rm P}(-m_{\rm P}^2) = 0$. In the symmetric realization all particles are usually massive, so that there can be no transition between the current and a single particle (i.e., $f_{\rm P}(-m_{\rm P}^2) = 0$). In the spontaneously broken realization there is always the massless Goldstone boson, which allows the corresponding $f_{\rm P}(0)$ to differ from zero. It turns out that this quantity is always proportional to the vacuum-expectation value of the field that is responsible for the spontaneous symmetry breaking.

18.2. Spontaneously broken chiral symmetry

There exist many relevant extensions of the model of section 18.1. In this section we couple it to a single fermion field by combining the phase transformations (18.2) with chiral transformations on the fermions. Subsequently we consider an extension of these phase transformations to a non-abelian group of transformations in order to obtain a more realistic model.

The phase transformations (18.2) can be implemented on the fermion in a variety of ways. Here we assume that the fermion field transforms under the symmetry group by means of a chiral transformation. Following section 7.1 we decompose the fermion field into chiral components, $\psi_L = \frac{1}{2}(1+\gamma_5)\psi$ and $\psi_R = \frac{1}{2}(1-\gamma_5)\psi$, and define the combined transformations as follows,

$$\begin{aligned} \phi(x) &\to \phi'(x) = \mathrm{e}^{\mathrm{i}q\xi}\,\phi(x)\,, \\ \psi_L(x) &\to \psi'_L(x) = \mathrm{e}^{\frac{1}{2}\mathrm{i}q\xi}\,\psi_L(x)\,, \\ \psi_R(x) &\to \psi'_R(x) = \mathrm{e}^{-\frac{1}{2}\mathrm{i}q\xi}\,\psi_R(x)\,. \end{aligned} \tag{18.16}$$

A Lagrangian invariant under the above transformations reads

$$\begin{aligned} \mathcal{L} = &- \bar\psi_L \partial\!\!\!/ \psi_L - \bar\psi_R \partial\!\!\!/ \psi_R - \sqrt{2}\,G\big(\phi\,\bar\psi_L\psi_R + \phi^*\,\bar\psi_R\psi_L\big) \\ &- |\partial_\mu\phi|^2 - V(|\phi|)\,, \end{aligned} \tag{18.17}$$

where G is a real coupling constant (a possible phase of G can be absorbed into the definition of the fields. See problem 18.5 for a discussion). Note that the symmetry does not allow a mass term for the fermion field.

When the symmetry is manifestly realized, one may expand the field ϕ about $\phi = 0$. The Lagrangian then describes the coupling of a massless fermion to a massive scalar and pseudoscalar field. This is obvious when decomposing ϕ according to $\phi = (S + \mathrm{i}P)/\sqrt{2}$ and combining the chiral components into a single Dirac field,

$$\begin{aligned} \mathcal{L} = &- \bar\psi \partial\!\!\!/ \psi - G\big(S\,\bar\psi\psi - \mathrm{i}P\,\bar\psi\gamma_5\psi\big) \\ &- \tfrac{1}{2}\big((\partial_\mu S)^2 + (\partial_\mu P)^2\big) - V(S,P)\,. \end{aligned} \tag{18.18}$$

When the potential exhibits a minimum for $\phi \neq 0$, the symmetry is realized in a spontaneously broken way. This leads to a massless Goldstone boson, as was discussed before. However, the non-zero vacuum-expection value now also leads to the emergence of a mass term for the fermion field, proportional to G times the vacuum-expectation value of ϕ. To see this we use again the decomposition (18.4), so that the Lagrangian acquires the form

$$\mathcal{L} = -\bar{\psi}_L \partial\!\!\!/ \psi_L - \bar{\psi}_R \partial\!\!\!/ \psi_R - G\,\rho\big(\mathrm{e}^{\mathrm{i}\theta/v}\,\bar{\psi}_L\psi_R + \mathrm{e}^{-\mathrm{i}\theta/v}\,\bar{\psi}_R\psi_L\big) \\ - \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}(\rho^2/v^2)(\partial_\mu\theta)^2 - V(\rho/\sqrt{2})\,. \qquad (18.19)$$

Expanding ρ about $\rho = v$ gives the fermion a mass $M = G\,v$. Hence particles that are initially massless because of a symmetry, may acquire a non-zero mass when the symmetry is realized in a spontaneously broken way. This phenomenon plays an important role in the generation of masses for elementary particles.

Let us again consider the Noether current associated with the symmetry transformations (18.16), which, in this case, contains also contributions from the fermions,

$$J_\mu = \frac{q\,\rho^2}{v}\,\partial_\mu\theta + \tfrac{1}{2}\mathrm{i}q\,\bar{\psi}\gamma_\mu\gamma_5\psi\,. \qquad (18.20)$$

Assigning negative intrinsic parity to the field θ, the above current transforms as an axial-vector current under parity reversal.

Just as in section 18.1 we may verify current conservation for the amplitude with two fermions. Using the Dirac equation and Lorentz invariance, one can show that the amplitude can be decomposed in terms of three form factors. One of them is usually equal to zero by virtue of some discrete symmetry (in this case charge conjugation), so that we are left with two form factors, which appear as follows,

$$\mathcal{M}_\mu(p',p) = \bar{u}(\mathbf{p}')\big[\mathrm{i}g_A(k^2)\,\gamma_\mu\gamma_5 + k_\mu g_P(k^2)\,\gamma_5\big]u(\mathbf{p})\,, \qquad (18.21)$$

where g_A and g_P are functions of k^2 with $k = p' - p$. Imposing current conservation and using the Dirac equation on the spinors we derive

$$2M\,g_A(k^2) + k^2\,g_P(k^2) = 0\,. \qquad (18.22)$$

It is important that one does not directly consider the limit $k \to 0$, as the above expression is multiplied with $\bar{u}\gamma_5 u$, which itself vanishes in the limit of equal fermion momenta. Furthermore the range of k^2 is restricted by kinematics. However, by analytic continuation one may use the above equation also for $k^2 = 0$. As the mass and the axial-vector form factor g_A are not zero, the second form factor should contain a $1/k^2$ pole, again due to the exchange of the Goldstone boson between the current and the fermion vertex. Using the parametrization (18.15) of the current amplitude with the Goldstone boson, it follows that

$$g_P(k^2) = \frac{f_\theta\, G_{\theta FF}}{k^2} + \cdots \tag{18.23}$$

where $G_{\theta FF}$ is the coupling constant of the Goldstone boson with the fermions, f_θ parametrizes the transition between the current and a single Goldstone particle and the ellipses denote terms regular at $k^2 = 0$. In the model at hand, we have $g_A(k^2) = \frac{1}{2}q$, $f_\theta = qv$ and $G_{\theta FF} = -G$ in tree approximation.

Substituting (18.23) into (18.22) one obtains the so-called Goldberger-Treiman formula,

$$2M\, g_A(0) + f_\theta\, G_{\theta FF} = 0\,. \tag{18.24}$$

The correctness of this result is easily verified in the tree approximation by virtue of the equation for the fermion mass, $M = G\,v$, derived above. The relevant diagrams are shown in figure 18.4.

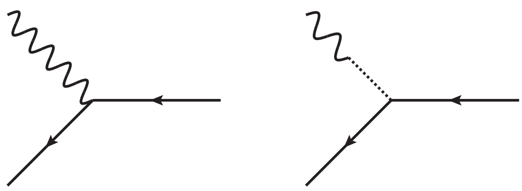

Figure 18.4: Diagrams contributing to the form factors for the amplitude of the Noether current (18.21) with two fermions.

There is another phenomenon that is characteristic for spontaneously broken symmetries, namely that the interaction strength for *soft* Goldstone particles tends to vanish. By a soft (massless) particle, we mean that the particle momentum is small with respect to the typical mass scale of the particle reaction. One way to see how this comes about is by redefining the fermion fields,

$$\begin{aligned}\psi_L(x) &\rightarrow \mathrm{e}^{\frac{1}{2}\mathrm{i}\theta(x)/v}\,\psi_L(x)\,,\\ \psi_R(x) &\rightarrow \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta(x)/v}\,\psi_R(x)\,.\end{aligned} \tag{18.25}$$

After this redefinition the Lagrangian takes a simple form

$$\begin{aligned}\mathcal{L} = & - \bar{\psi}\partial\!\!\!/\psi - \frac{\mathrm{i}}{2v}\,\partial_\mu\theta\,\bar{\psi}\gamma^\mu\gamma_5\psi - G\,\rho\,\bar{\psi}\psi \\ & - \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}(\rho^2/v^2)\,(\partial_\mu\theta)^2 - V(\rho/\sqrt{2})\,,\end{aligned} \tag{18.26}$$

in which only derivatives of the field θ appear. This means that vertices that involve the Goldstone particle are proportional to the momentum of this particle and thus tend to vanish when the particle becomes soft. However, from this fact one cannot conclude that all processes involving soft Goldstone bosons must vanish, as there can be diagrams that contain kinematical singularities, such as for instance the Born-approximation graphs, discussed in chapter 11. However, in that case one can specify the amplitude for soft Goldstone bosons in terms of other physical quantities. Such results are called low-energy theorems. Earlier, in section 2.5 and in problems 2.6 and 5.4, we already considered a few applications of low-energy theorems, in anticipation of a more complete treatment of spontaneous symmetry breaking. For the model at hand, problem 18.6 discusses an example.

Spontaneous breaking of chiral symmetry is realized in particle physics, albeit only in an approximate sense. Pions can be regarded as the (approximate) Goldstone bosons of an underlying chiral $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry. This naturally explains why the pion mass is substantially smaller than the mass of the other hadrons. The nucleons have thus acquired a mass through the mechanism of spontaneous symmetry breaking. The approximate chiral symmetry is present in the underlying theory of quarks and gluons, as we discussed in chapter 16. In quantum chromodynamics the chiral symmetry is not exact because of the presence of the quark masses and of the electroweak interactions.

We previously discussed various features of a realistic model containing nucleons and pions that exhibits chiral symmetry (cf. sections 2.5, 5.2 and problems 2.6, 5.4). It is called the sigma model[4] and it was written down long ago by Gell-Mann and Lévy. The model, which is a generalization of the model introduced at the beginning of this section, contains pion fields, written as an isospin vector $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ and a scalar field σ, assembled into a 2×2 matrix Φ,

$$\Phi = \sigma\mathbf{I} + \mathrm{i}\vec{\pi}\cdot\vec{\tau} = \begin{pmatrix} \sigma + \mathrm{i}\pi_3 & \pi_2 + \mathrm{i}\pi_1 \\ & \\ -\pi_2 + \mathrm{i}\pi_1 & \sigma - \mathrm{i}\pi_3 \end{pmatrix} . \tag{18.27}$$

[4]This model is sometimes called the *linear* sigma model to distinguish it from so-called nonlinear sigma models in which the symmetry transformations act nonlinearly. An example of the latter is discussed in problem 2.7.

This matrix has the following properties

$$\Phi^* = \tau_2\,\Phi\tau_2\,, \qquad \Phi^\dagger = \sigma\,\mathbf{I} - i\vec{\pi}\cdot\vec{\tau}\,, \qquad \Phi\,\Phi^\dagger = (\sigma^2+\vec{\pi}^2)\,\mathbf{I}\,. \tag{18.28}$$

The matrix Φ is subject to two independent SU(2) transformations according to

$$\Phi \to \Phi' = U\,\Phi\,V^\dagger\,, \tag{18.29}$$

where U and V are independent SU(2) matrices, i.e., they are 2×2 unitary matrices with unit determinant. In passing we mention that we could also write σ and $\vec{\pi}$ in terms of a four-dimensional vector. The (real) rotations of this vector coincide with the SU(2)$\times$SU(2) transformations. Indeed, according to (18.28) the invariant matrix $\Phi\,\Phi^\dagger$ is just proportional to the length squared of the four-component vector $(\sigma,\pi_1,\pi_2,\pi_3)$. The matrix representation makes it convenient to include the nucleons. The nucleon fields are combined into a two-component isospinor (cf. (5.46)),

$$N(x) = \begin{pmatrix}\psi_{\rm p}(x)\\ \psi_{\rm n}(x)\end{pmatrix}\,,$$

from which we can construct two chiral doublet fields,

$$N_L = \tfrac{1}{2}(1+\gamma_5)N\,, \qquad N_R = \tfrac{1}{2}(1-\gamma_5)N\,. \tag{18.30}$$

Under SU(2) $\times$ SU(2) these fermion doublets transform as

$$N_L \to N_L' = U\,N_L\,, \qquad N_R \to N_R' = V\,N_R\,. \tag{18.31}$$

With these result it is straightforward to write down an invariant Lagrangian,

$$\begin{aligned}\mathcal{L} = &-\bar{N}_L\partial\!\!\!/ N_L - \bar{N}_R\partial\!\!\!/ N_R - G\bar{N}_L\Phi N_R - G\bar{N}_R\Phi^\dagger N_L\\ &-\tfrac{1}{4}\mathrm{Tr}[\partial^\mu\Phi\,\partial_\mu\Phi^\dagger] - \tfrac{1}{4}\mu^2\,\mathrm{Tr}[\Phi\,\Phi^\dagger] - \tfrac{1}{16}\lambda\left(\mathrm{Tr}[\Phi\,\Phi^\dagger]\right)^2.\end{aligned} \tag{18.32}$$

In addition, this Lagrangian turns out to be invariant under phase transformations of the fermion fields, $N_{L,R}\to \mathrm{e}^{\mathrm{i}\alpha}N_{L,R}$, so that the actual symmetry group is equal to SU(2)$\times$SU(2)$\times$U(1). Note that $\mathrm{Tr}[\Phi\,\Phi^\dagger\,\Phi\,\Phi^\dagger] = (\mathrm{Tr}[\Phi\,\Phi^\dagger])^2$ by virtue of (18.28), so that at this stage we have included all possible invariant couplings of the renormalizable type. The Lagrangian may also be written as

$$\begin{aligned}\mathcal{L} = &-\bar{N}\partial\!\!\!/ N - G\,\sigma\,\bar{N}N + \mathrm{i}G\,\vec{\pi}\cdot(\bar{N}\gamma_5\vec{\tau}N)\\ &-\tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 - \tfrac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2 - \tfrac{1}{2}\mu^2\left(\sigma^2+\vec{\pi}^2\right) - \tfrac{1}{4}\lambda\left(\sigma^2+\vec{\pi}^2\right)^2.\end{aligned} \tag{18.33}$$

For negative μ^2, the potential acquires its minima at $\sigma^2+\vec{\pi}^2 = v^2 = -\mu^2/\lambda$ and $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is realized in a spontaneously broken way. These minima parametrize a three-dimensional sphere in the four-dimensional configuration space. Because the degeneracy of the vacuum configuration is three-dimensional, there will be three Goldstone bosons. According to the arguments presented earlier, we may just choose one of these minima, say at $\sigma = v$, $\vec{\pi} = 0$, which leaves the so-called diagonal subgroup, defined by $U = V$, invariant as follows from inspection of (18.29). For this choice, the Goldstone bosons correspond to the fields $\vec{\pi}$. The matrices $U = V$ define an SU(2) invariance group which is 'vectorlike', meaning that it acts identically on both chiral components of the fermion fields. The fields $\vec{\pi}$ transform as an iso-vector under this SU(2). The manifest vectorlike SU(2) group corresponds to the group of isospin transformations. The two nuclons acquire a mass equal to $M = G\,v$. The fact that the proton and the neutron have equal mass is a consequence of the isospin invariance.

Expanding about $\sigma = v$ by substituting $\sigma = v + \tilde{\sigma}$, the Lagrangian takes the form,

$$
\begin{aligned}
\mathcal{L} =& -\bar{\psi}_{\rm p}(\partial\!\!\!/ + M)\psi_{\rm p} - \bar{\psi}_{\rm n}(\partial\!\!\!/ + M)\psi_{\rm n} - \tfrac{1}{2}\big[\big(\partial_\mu\tilde{\sigma}\big)^2 + m^2\,\tilde{\sigma}^2\big] - \tfrac{1}{2}\big(\partial_\mu\vec{\pi}\big)^2 \\
& - G\,\tilde{\sigma}\,\big(\bar{\psi}_{\rm p}\psi_{\rm p} + \bar{\psi}_{\rm n}\psi_{\rm n}\big) + {\rm i}G\,\pi_3\big(\bar{\psi}_{\rm p}\gamma_5\psi_{\rm p} - \bar{\psi}_{\rm n}\gamma_5\psi_{\rm n}\big) \\
& + {\rm i}\,G\,(\pi_1 - {\rm i}\pi_2)\,\bar{\psi}_{\rm p}\gamma_5\psi_{\rm n} + {\rm i}\,G\,(\pi_1 + {\rm i}\pi_2)\,\bar{\psi}_{\rm n}\gamma_5\psi_{\rm p} \\
& - \lambda v\,\tilde{\sigma}\big(\tilde{\sigma}^2 + \vec{\pi}^2\big) - \tfrac{1}{4}\lambda\,\big(\tilde{\sigma}^2 + \vec{\pi}^2\big)^2 . \qquad (18.34)
\end{aligned}
$$

with $m^2 = 2\,\lambda\,v^2 = -2\,\mu^2$ and $M^2 = G^2\,v^2 = (G^2/2\,\lambda)\,m^2$. This pion-nucleon sigma model has played a central role in the study of chiral symmetry breaking for hadrons. Of course, pions are not massless in reality, and for that reason one can introduce a small external symmetry-breaking term to the Lagrangian, linearly proportional to the sigma field. Such a term is discussed in a simpler model in problem 18.3 and leads to a nonzero pion mass, while the isospin symmetry remains preserved. An alternative way to break the symmetry is by assigning mass terms for the fermions in the original symmetric Lagrangian (18.32).

18.3. Counting Goldstone bosons

To gain more insight in the structure of spontaneously broken realizations of a symmetry we consider a more general class of models, described by a potential $V(\phi)$ that is invariant under a group G of continuous transformations. Here ϕ generically denotes a number of scalar fields ϕ_i. We will be interested in constant fields ϕ for which the action is stationary. These may be associated with possible Lorentz and translationally invariant ground states. Therefore, terms in the action that depend on derivatives of the fields, can be ignored in the analysis and it suffices to consider only the potential. The restriction to constant fields is, in principle, not essential for the analysis (although certainly convenient). Many conceptual aspects will apply equally well to more general field configurations.

Consider a generic constant field configuration ϕ and apply all possible symmetry transformations on it. In general this configuration will not be invariant under all transformations belonging to the group G, although it could remain invariant under some subgroup $\mathrm{H} \subset \mathrm{G}$. When the field configuration corresponds to a stationary point of the potential, then the physics associated with the corresponding ground state will depend crucially on this subgroup, which is called the *stability subgroup*, or the *isotropy group*.[5] When a point ϕ is invariant under G then $\mathrm{H} = \mathrm{G}$. Here it is not important whether or not ϕ is an isolated point in the field configuration space in the sense that it is not continuously connected to points that remain invariant under the full group G. When ϕ is G-invariant, the symmetry is unaffected and there is no spontanous symmetry breaking associated with ϕ. When, on the other hand, ϕ is not invariant under G, then ϕ is transformed to some other point in the field configuration space. The potential will necessarily take the same value at this new point, just because the potential is G-invariant. Obviously, ϕ is never an isolated point in that case, because the G-transformations are continuous and thus sweep out an orbit in the field configuration space for which the potential has the same value. The *orbit* associated with a point ϕ, which we denote by $G(\phi)$, consists of all the points that are generated by applying all possible G-transformations on ϕ. Obviously these orbits form submanifolds in the field configuration space and for all points on a given orbit the potential takes the same value, so that an orbit corresponds to an *equipotential submanifold*. Moreover two different points belonging to the same orbit must be connected by a symmetry transformation. Hence the orbit swept out from these two points must be identical,

$$G(\phi) = G(g\,\phi)\,. \tag{18.35}$$

Generically points belonging to the same orbit $G(\phi)$ have an equivalent isotropy group. Namely, when ϕ_2 is obtained by applying a G-transformation $g \in \mathrm{G}$ on ϕ_1, then the isotropy group H_{ϕ_1} and the isotropy group H_{ϕ_2} of ϕ_2 are related by conjugation, $\mathrm{H}_{\phi_2} = g\,\mathrm{H}_{\phi_1}\,g^{-1}$. The nature of the orbit is thus to a large extent characterized by the associated isotropy group. For instance, consider the dimension of an orbit $G(\phi)$, which can be determined by applying infinitesimal G-transformations on ϕ. Under some of these transformations, namely those associated to the subgroup H, ϕ will remain unchanged. Only

[5] It is not difficult to see that the transformations that leave a certain quantity invariant, must form a group. See, for instance, problem 18.7.

under those infinitesimal transformations that do *not* belong to H will the point ϕ change to a neighbouring point on the orbit $G(\phi)$. The number of independent infinitesimal transformations of a group is called the dimension of the group. Of all possible infinitesimal transformations belonging to G only those lead to a neighbouring point that are *not* associated with the subgroup H. Hence the dimension of $G(\phi)$ is given by the difference of the dimensions of G and H,[6]

$$\dim[G(\phi)] = \dim[\mathrm{G}] - \dim[\mathrm{H}_\phi]\,. \tag{18.36}$$

This result allows us to determine the number of Goldstone bosons. Along the direction of the orbit the potential is flat so that the corresponding fields that parametrize the orbit will be massless. This is precisely in agreement with Goldstone's theorem, which states that the number of Goldstone particles is equal to the number of broken generators of the group, i.e. to those infinitesimal G-transformations that do not leave the stationary point of the potential invariant but move it along the orbit. Hence the number of Goldstone particles is equal to the dimension of the orbit which contains the stationary point at $\phi = v$ and is thus also given by

$$\#\ \text{Goldstone bosons} = \dim[G(v)] = \dim[\mathrm{G}] - \dim[\mathrm{H}_v]\,, \tag{18.37}$$

where v denotes the stationary point. Each point on the orbit $G(v)$ defines an equivalent ground state. Because the ground state is invariant under the isotropy group H, this group will remain a manifest symmetry group. This symmetry will, for instance, be reflected by the mass spectrum of the realization. The full symmetry group G, on the other hand, will not be manifest and its presence can only be deduced from the presence of Goldstone particles and from phenomena related to them.

We can be a little more specific here and present a simple derivation of the Goldstone theorem. Assume that the symmetry group G leads to infinitesimal transformations $\delta_{\mathrm{G}}\phi$, so that the invariance of the potential implies that

$$\delta V(\phi) = \frac{\partial V(\phi)}{\partial \phi_i}\,\delta_{\mathrm{G}}\phi_i = 0\,. \tag{18.38}$$

Now consider a second, arbitrary, variation of the fields by differentiating with respect to ϕ_j,

$$\frac{\partial^2 V(\phi)}{\partial \phi_j\,\partial \phi_i}\,\delta_{\mathrm{G}}\phi_i + \frac{\partial V(\phi)}{\partial \phi_i}\,\frac{\partial \delta_{\mathrm{G}}\phi_i}{\partial \phi_j} = 0\,. \tag{18.39}$$

[6] In mathematics lterature these orbits are called *coset spaces* and they are denoted by G/H.

Evaluate this result at a stationary point $\phi = v$ of the potential, where $\partial V/\partial\phi_i = 0$. The second derivative of the potential is then proportional to the square of the mass matrix,

$$[\mathcal{M}^2]^{ij} = \tfrac{1}{2}\frac{\partial^2 V(\phi)}{\partial\phi_i\,\partial\phi_j}\Big|_{\phi=v}\,. \tag{18.40}$$

Using this relation we see that $\mathcal{M}^2$ has null eigenvectors, defined by the nonvanishing $\delta_{\rm G}\phi_i|_{\phi=v}$,

$$[\mathcal{M}^2]^{ij}\,\delta_{\rm G}\phi_i|_{\phi=v} = 0\,. \tag{18.41}$$

These null eigenvectors are precisely the tangent vectors along the orbit $G(v)$ and the number of these independent vectors defines the dimension of this orbit.

Perhaps it is best to introduce a simple example at this point. Other examples will follow in due course. Consider an N-dimensional array of fields ϕ transforming under continuous real rotations, so that $\rm G = SO(N)$. There exists a single point, namely the origin $\phi = 0$, that remains invariant under SO(N), so that $\rm H = SO(N)$. The dimension of an isolated point is equal to zero, and this is consistent with (18.36) as the dimension of the corresponding orbit vanishes. Any other point ϕ, however, will be affected by the action of SO(N), but there always exists an $\rm SO(N-1)$ subgroup, consisting of all possible rotations around ϕ that leave ϕ invariant. Hence, for generic points ϕ the isotropy group equals $\rm H = SO(N-1)$. Therefore it follows that the orbit associated with ϕ is either the isolated point at the origin, which has zero dimension, or an $(N-1)$-dimensional sphere S^{N-1} swept out by rotations applied on a nonvanishing N-dimensional vector. And indeed, this result is in accord with (18.36), as $\dim[\rm SO(N)] = \tfrac{1}{2}N(N-1)$ and $\dim[\rm SO(N-1)] = \tfrac{1}{2}(N-1)(N-2)$, so that their difference equals $N-1$, which is indeed the dimension of S^{N-1}. The additional field residing in ϕ corresponds to the radius. So the whole field configuration space can be described in terms of the radius and the corresponding orbit. The orbit is trivial only at the origin. Everywhere else it is isomorphic with the unit sphere S^{N-1}.

The above example makes it clear that the approach of this section amounts to setting up some generalized angular decomposition of the fields, separating invariant components (a generalization of the radius) from generalized orbital variables. Therefore, let us now start from the other end and discuss the complete set of independent invariant variables, which we denote by $\{\rho\}$. Depending on the representation content of the fields there may be a large variety of these invariants. For a single vector transforming under real rotations, the only invariant is its length. But when there are, for instance, two independent vectors, then there are three invariants under rotations, namely the length of each of the vectors and the relative angle between them. Generically, the isotropy group is just the identity, and we have three independent invariants and three orbital parameters. These orbital parameters parametrize the group SO(3). Together the three invariants and the three orbital variables comprise the six components associated with two independent vector fields. In the special situation where the two vectors are parallel or anti-parallel, there are only two unconstrained invariant variables (namely the lengths of the two vectors) and the orbit is isomorphic to the two-dimensional sphere S^2 (the isotropy goup is the one-dimensional subgroup of rotations around the two (anit-)parallel vectors). Hence, we are dealing with a two-dimensional variety of orbits that are isomorphic to S^2, which constitute a four-dimensional subspace of the full six-dimensional field configuration space. Finally, there is the isolated point where both vectors vanish with the isotropy group equal to the full group of three-dimensional rotations.

An important observation is that the potential is invariant and therefore will only depend on the invariant combinations of the fields. Therefore, the potential is only a function of the full set of independent invariant variables $\{\rho\}$ and those are the variables that will be fixed at stationary points of the potential. The stationary points have a certain (continuous) degeneracy, and this degeneracy is reflected in the orbital structure at the particular value of the $\{\rho\}$. Depending on the precise form of the potential and the values for the coupling constants, the theory will be realized in a certain phase which can be characterized by the isotropy group. The latter is the manifest symmetry group associated with that phase. In the example above, this group is equal to either the identity, SO(2) or SO(3), and the number of associated Goldstone bosons equals 3, 2 or 0, respectively.[7]

Let us briefly demonstrate the above for the case of two independent vector fields, $\vec{\phi}_1$ and $\vec{\phi}_2$. In that case we define the following SO(3) invariants,

$$\rho_1 = (\vec{\phi}_1)^2\,, \qquad \rho_2 = (\vec{\phi}_2)^2\,, \qquad \rho_3 = \vec{\phi}_1 \cdot \vec{\phi}_2\,. \tag{18.42}$$

Any SO(3) invariant potential $\mathcal{V}(\rho)$ is a function of these invariants. Stationary points are obtained by putting the the derivatives of the potential with respect to $\vec{\phi}_1$ and $\vec{\phi}_2$ to zero. This leads to two vector equations,

$$\begin{aligned} &2\,\vec{\phi}_1\,\mathcal{V}_1 + \vec{\phi}_2\,\mathcal{V}_3 = 0\,,\\ &2\,\vec{\phi}_2\,\mathcal{V}_2 + \vec{\phi}_1\,\mathcal{V}_3 = 0\,, \end{aligned} \tag{18.43}$$

where $\mathcal{V}_{1,2,3}$ denote the derivatives of $\mathcal{V}$ with respect to $\rho_{1,2,3}$. If we assume that both $\vec{\phi}_1$ and $\vec{\phi}_2$ are finite, then they should be parallel or anti-parallel. In that case one finds the condition

$${\mathcal{V}_3}^2 - 4\,\mathcal{V}_1\,\mathcal{V}_2 = 0\,. \tag{18.44}$$

[7]In these discussions it is important that one considers the full symmetry group of the potential and a complete set of independent invariants.

The two (anti-)parallel vectors leave an SO(2) subgroup invariant, which leads to ${\rho_3}^2 - \rho_1\rho_2 =$. This result follows also from contracting (18.43) with $\vec{\phi}_1$ and $\vec{\phi}_2$, which leads to four degenerate equations expressed entirely in the invariants ρ_1, ρ_2 and ρ_3,

$$\begin{aligned} 2\,\rho_1\,\mathcal{V}_1 + \rho_3\,\mathcal{V}_3 &= 0\,, & 2\,\rho_2\,\mathcal{V}_2 + \rho_3\,\mathcal{V}_3 &= 0\,,\\ 2\,\rho_3\,\mathcal{V}_1 + \rho_2\,\mathcal{V}_3 &= 0\,, & 2\,\rho_3\,\mathcal{V}_2 + \rho_1\,\mathcal{V}_3 &= 0\,. \end{aligned} \tag{18.45}$$

From these equations it is ea sy to derive that (18.44) implies the same condition on the variables $\rho_{1,2,3}$ as noted above.

To elucidate this even further, let us move to another class of examples, based on a field transforming in the adjoint representation of SU(N). Such a field can be written as an $N \times N$ hermitean, traceless, matrix Φ, transforming according to $\Phi \to U\Phi U^{-1}$. Invariants can be constructed by taking traces of integer powers Φ^n. One can show that these traces are not all independent and for $n > N$ can be expressed in terms of (products of) traces of lower powers.[8] Hence one irreducible adjoint representation of SU(N) has $N-1$ invariant components, $\rho_n = \mathrm{Tr}[\Phi^n]$ with $n = 2, \ldots, N$. Observe that $\rho_1 = \mathrm{Tr}[\Phi] = 0$, and has therefore been suppressed. Generically the number of orbital components is equal to the number of independent fields contained in Φ (which equals $N^2 - 1$) minus the number of invariants. Thus the dimension of the orbits is generically equal to $(N^2-1) - (N-1) = N(N-1)$. This means that the dimension of the orbit is generically equal to $N(N-1)$, so that the dimension of the isotropy group equals $N-1$. Indeed, as we shall see, the isotropy group for a generic point of the configuration space is equal to $[\mathrm{U}(1)]^{N-1}$. Of course, depending on the values taken by the invariants, there may be symmetry enhancement implying that the isotropy group H becomes larger than in the generic case. In that case, the number of orbital components is decreased (according to (18.36)), and so is the effective number of invariant components (simply because they are restricted to those points where the symmetry enhancement takes place).

To be more specific let us consider a field Φ transforming in the adjoint representation of SU(3). Hence Φ can be written as a 3×3, hermitean, traceless, matrix, transforming under SU(3) according to

$$\Phi \to U\Phi U^{-1}\,. \tag{18.46}$$

Obviously Φ comprises eight independent fields. By means of an SU(3) transformation Φ can be always be written in diagonal form, which we may

[8]It turns out that for a general $N \times N$ matrix A, $\mathrm{Tr}[A^n]$ with $n > N$, and $\det[A]$ can be expressed in terms of products of traces of A^k with $k \leqslant N$.See, exercise 18.9 for an outline of the proof and further discussion.

parametrize as follows,

$$\Phi = \begin{pmatrix} \dfrac{a}{\sqrt{3}} + b & 0 & 0 \\ 0 & \dfrac{a}{\sqrt{3}} - b & 0 \\ 0 & 0 & -\dfrac{2\,a}{\sqrt{3}} \end{pmatrix} . \tag{18.47}$$

The invariants associated with Φ are

$$\rho_2 = \mathrm{Tr}[\Phi^2] = 2(a^2 + b^2)\,, \qquad \rho_3 = \mathrm{Tr}[\Phi^3] = -2\sqrt{3}\,a(\tfrac{1}{3}a^2 - b^2)\,. \tag{18.48}$$

There are no other independent invariants. The reader may, for instance, verify by explicit calculation that $\det[\Phi]$ and $\mathrm{Tr}[\Phi^4]$, which are both invariant, can be expressed in terms of the two invariants, ρ_2 and ρ_3.

For generic values of a and b, Φ is invariant under the diagonal subgroup of SU(3), whose elements take the following form,

$$U = \begin{pmatrix} \exp[\mathrm{i}\alpha] & 0 & 0 \\ 0 & \exp[\mathrm{i}\beta] & 0 \\ 0 & 0 & \exp[-\mathrm{i}(\alpha+\beta)] \end{pmatrix} , \tag{18.49}$$

where α and β are real variables which can be restricted to the interval $(0, 2\pi)$. Hence the isotropy group equals $\mathrm{H} = \mathrm{U}(1) \times \mathrm{U}(1)$, which has dimension 2. Therefore the eight independent fields contained in Φ can be decomposed into two invariant variables and $8 - 2 = 6$ orbital ones (cf. 18.36).

For special values of a and b, namely, $b = 0$, or $b = \pm\sqrt{3}\,a$, where Φ has two equal eigenvalues, the symmetry is enhanced from $\mathrm{U}(1) \times \mathrm{U}(1)$ to $\mathrm{H} = \mathrm{U}(1) \times \mathrm{SU}(2)$, which is of dimension 4. In that case the invariants are restricted by ${\rho_2}^3 - 6\,{\rho_3}^2 = 0$. Because of this restriction there is only one relevant invariant variable left and there are $8 - 4 = 4$ orbital components.

Finally, maximal symmetry enhancement takes place at $\Phi = 0$, which leaves the full SU(3) group invariant. In this case there are no orbital components, confirming that we are dealing with an isolated point in the configuration space.

Summarizing, we are generically dealing with a two-dimensional variety of ground states, left invariant by $\mathrm{H} = \mathrm{U}(1) \times \mathrm{U}(1)$. Each of these ground states has a six-dimensional degeneracy corresponding to six Goldstone bosons. There exists a one-dimensional subvariety of ground states, characterized by ${\rho_2}^3 - 6\,{\rho_3}^2 = 0$, which are invariant under $\mathrm{H} = \mathrm{U}(1) \times \mathrm{SU}(2)$. Each of these ground states has a four-dimensional degeneracy correponding to four Goldstone bosons. Finally there is an isolated ground state with SU(3) symmetry and not further degeneracy, and therefore without Goldstone bosons.

Hence models based on Φ can in principle exhibit three different types of realizations (configurations with the same symmetry group). One has a manifest $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry group with six singlet Goldstone bosons. A second one has a manifest $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry and has four Goldstone bosons, transforming in the $(\mathbf{2},\mathbf{1})+(\bar{\mathbf{2}},\bar{\mathbf{1}})$ representation of this group (see, 18.11, where this phase is discussed). Finally there is a phase with manifest SU(3) symmetry with no Goldstone bosons. Which one of these phases will be realized in the framework of a given theory will depend on the details of the Lagrangian. Here we refer to problem 18.10 where we work out an example for $\mathrm{SU}(N)$.

There is one aspect that has been ignored so far. The two invariants $\rho_{2,3}$ were expressed in terms of the parameters a and b, which parametrize the eigenvalues of the matrix field Φ according to (18.47). Obviously, a and b are invariant under the generic isotropy group $\mathrm{U}(1)\times\mathrm{U}(1)$, so that it seems that different values of (a,b) will correspond to inequivalent $\mathrm{U}(1)\times\mathrm{U}(1)$ invariant ground states. However, this is not necessarily the case. The parameters a and b determine the eigenvalues of Φ, which yield the original matrix Φ up to an SU(3) transformation. As it turns out the parametrization of the diagonal matrix (18.47) in terms of a and b is redundant, because there exist SU(3) transformations that will interchange these eigenvalues. Therefore, matrices Φ that are related by an interchange of eigenvalues define equivalent ground states.

Oviously there exists a finite group of elements of SU(3) that implement the possible interchanges of the three eigenvalues, which also includes the identity matrix. These elements parametrize the permutation group S_3 which has dimension six (corresponding to the number of independent permutation of three numbers). On the parameters (a,b) these elements are represented by two-by-two matrices which define a number of finite rotations and reflections in the (a,b)-plane. Correspondingly they divide the (a,b)-plane into six regions which should be regarded as equivalent.

To see this in more detail, let us consider the two-dimensional plane, parametrized by a and b as shown in fig. 9.5, where we choose a along the horizontal, and b along the vertical axis. Each point on this plane now corresponds to a certain diagonal matrix (18.47). In this plot we distinguish three different straight lines running through the origin at relative angles of 60°. For points located on any one of these lines, two of the three eigenvalues of Φ are equal. These lines separate six different domains (also known as Weyl chambers), which generically correspond to matrices (18.47) that are equivalent up to a permutation of their eigenvalues. For instance, the line with $b=0$ characterizes the diagonal matrices Φ whose first two eigenvalues are equal. Away from the line $b=0$ the first two eigenvalues are different and they can be interchanged by applying an SU(3) transformation according to

$$\Phi \to U_{(2,1,3)}\, \Phi\, U^{-1}_{(2,1,3)}\,. \tag{18.50}$$

where the label $(2, 1, 3)$ indicates the permutation, and

$$U_{(2,1,3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{18.51}$$

Observe that the matrix $U_{(2,1,3)}$ is only defined up to certain phase factors, but we chose it such that its square equals the identity matrix. In the (a, b)-plane, the interchange of eigenvalues is effected by a two-by-two matrix,

$$T_{(2,1,3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{18.52}$$

which represents a reflection $b \to -b$. Obviously its square is also equal tot the identity. We refer to problem 18.12 for a more elaborate discussion along these lines.

To demonstrate the equivalence of the six different domains in the (a, b)-plane, consider fig. 18.5. The six domains constitute a wedge with an opening angle of 60°. At their boundaries two of the three eigenvalues of Φ are equal. From the dotted, the dashed and the dashed-dotted curves presented, which denote fixed values for the gauge invariant quantities ρ_2 and ρ_3, it readily follows that the six domains cover all possible values of ρ_2 and ρ_3 and are therefore equivalent.

Problems

18.1. In conventional quantum mechanics the phenomenon of spontaneous symmetry breaking does not exist, as was explained at the beginning of this chapter. This was elucidated by considering free particle states, which has the technical disadvantage that the ground-state wave function is not square integrable. The following example does not have this drawback. We consider a particle of mass m in two dimensions in a spherically symmetric potential $V(r)$. The relevant classical Hamiltonian equals,

$$H = \frac{1}{2\,m}\left({p_r}^2 + \frac{{p_\varphi}^2}{r^2}\right) + V(r)\,, \tag{1}$$

where $r \geqslant 0$ and $0 \leqslant \varphi < 2\pi$ denote the radius and the angle of the two-dimensional plane, and p_r and p_φ are the associated momenta. We assume that the potential has a minimum at $r = r_0$, so that classically we are dealing with a degenerate ground

Figure 18.5: Two-dimensional plot of the gauge-invariant components of a field Φ transforming in the adjoint representation of SU(3). The horizontal axis ($b = 0$) and the two solid lines under an angle of 60° contain points where Φ has two equal eigenvalues. For these points, the invariants ρ_2 and ρ_3 are subject to ${\rho_2}^3 - 6\,{\rho_3}^2 = 0$, and the isotropy group equals $\mathrm{SU}(2) \times \mathrm{U}(1)$. These lines form the boundaries of six gauge-equivalent domains. This equivalence is confirmed by the curves along which ρ_2 and ρ_3 take fixed values. The dotted curves connect points of equal value of ρ_2, the dashed lines indicate points of fixed positve values of ρ_3 and the dashed-dotted curves represent points of fixed and negative values of ρ_3.

state. Subsequently, consider the Schrdinger equation and argue that the state of lowest energy has a wave function $\psi(r)$ with no angular dependance, subject to

$$-\frac{\hbar^2}{2\,m}\frac{\partial^2\psi(r)}{\partial r^2} + V(r)\psi(r) = E\,\psi(r)\,. \tag{2}$$

Argue that, for a suitable potential $V(r)$, this equation has a unique lowest-energy solution with a normalizable wave function.

18.2. Consider the Lagrangian $\mathcal{L} = -|\partial_\mu\phi|^2 - V(\phi)$, with the potential $V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4$, where ϕ is a complex field and λ is a fixed, positive, coupling constant. The Lagrangian is invariant under phase transformations of ϕ. Distinguish the three cases where $\mu^2 > 0$, $\mu^2 = 0$ and $\mu^2 < 0$. Sketch the potential in these three cases and calculate the value of ϕ at stationary stable points of the potential as a function of μ^2. Present the physical masses related to the second derivative of $V(\phi)$ at the local minimum of the potential and plot them as a function of μ^2. Compare the result with fig. 18.2. Argue that the theory has two different phases and that there exists a second-order phase transition at $\mu^2 = 0$.

18.3. Rewrite the potential of problem 18.2 in terms of the real and imaginary parts of $\phi = \frac{1}{2}\sqrt{2}(\phi_1 + \mathrm{i}\phi_2)$. After adding a deformation proportional to the coupling constant c, the potential reads, $V(\phi_1,\phi_2) = \frac{1}{2}\mu^2({\phi_1}^2 + {\phi_2}^2) + \frac{1}{4}\lambda({\phi_1}^2 + {\phi_2}^2)^2 - c\,\phi_1$. Throughout this problem we keep $\lambda > 0$ fixed. First let us assume that $\mu^2 > 0$. Consider the values of ϕ_1 and ϕ_2 at stationary points of the potential and describe the masses as a function of c. (Hint: This involves a cubic equation which can be analyzed graphically.) Consider the realization as a function of the coupling constant c and argue that the theory exhibits a second-order phase transition at $c = 0$. Subsequently, assume $\mu^2 < 0$. Again consider the values of ϕ_1 and ϕ_2 where the potential has an extremum as a function of c, by using graphical analysis. Then consider the values of the masses as a function of c and argue that the theory has a first-order phase transition at $c = 0$. Prove that the product of c and the value of ϕ_1 at the stationary point must always be positive.

18.4. Consider the model discussed in problem 18.3 and show that the Noether current (18.9) can be written as $J_\mu = q(\phi_1\,\partial_\mu\phi_2 - \phi_2\,\partial_\mu\phi_1)$. Because of the symmetry breaking term in the potential, the current is only 'partially conserved', implying that $\partial_\mu J^\mu$ does not vanish but is equal to a term of low canonical dimension. Prove that indeed we have $\partial_\mu J^\mu = qc\,\phi_2$, where the dimension of the operator on the right-hand side equals unity. In analogy with (18.15) write the amplitude for the current with an outgoing ϕ_2 line as $\mathcal{M}_\mu(k) = \mathrm{i}k_\mu\,f(k^2)$, where k_μ is the incoming momentum of the ϕ_2-line. Denote the mass associated with ϕ_2 by m and show that relation for the current implies the identity $m^2 f(-m^2) = qc$. This relation can be proven to all orders in perturbation theory. In lowest order of perturbation theory, we know that $f(k^2) = qv$. Show that this identity implies that vc must be positive, in line with what we derived at the end of the previous problem.

18.5. Reconsider the Lagrangian (18.17) and write the interaction term as the real part of $-\sqrt{2}G\,\phi\,\bar\psi_L\psi_R$, where the coupling constant G is now *complex*. Demonstrate explicitly how G can be made real by absorbing the phase of G into the definition of the fields, without affecting any of the remaining interaction terms.

18.6. Consider the Lagrangian (18.26) and assume that the mass m_ρ of the ρ particle is larger than twice the fermion mass M, so that this particle can decay into two fermions: $\rho \to F + \bar{F}$. Because there is a massless particle associated with the field θ, another possible decay is then $\rho(\mathbf{p}) \to F(\mathbf{P}_2) + \bar{F}(\mathbf{P}_1) + \theta(\mathbf{q})$. Write down the Feynman diagrams in tree approximation, contributing to these two decays.
Give the corresponding decay amplitudes. Show that the amplitude for the second process, which receives contributions from three diagrams, is equal to

$$\mathcal{M} = \frac{G^2}{M}\left\{\frac{m_\rho{}^2 + 3\,p\cdot q}{m_\rho{}^2 + 2\,p\cdot q}\,\mathrm{i}\,\bar{u}_2\gamma_5 v_1 + \Big(\frac{M}{2\,P_1\cdot q} + \frac{M}{2\,P_2\cdot q}\Big)\,\bar{u}_2\not{q}\gamma_5 v_1\right\}, \qquad (1)$$

and does not tend to zero for vanishing θ momentum q_μ. Explain the result.

18.7. To prove that transformations that leave a quantity invariant, must constitute a group, one may follow the following steps. First, argue that the identity transformation belongs to it, as well as products of invariance transformations. Subsequently, argue that the quantity must also be invariant under the inverse transformation.

18.8. Consider a field configuration consisting of two real three-dimensional vector fields, $\vec{V}_1$ and $\vec{V}_2$, transforming under SO(3). As explained in section 18.3 the stability subgroup H $\subset$ G, where G = SO(3) in this case, can be non-trivial for special values of the fields. Show that this happens whenever $(\vec{V}_1\cdot\vec{V}_2)^2 - V_1{}^2 V_2{}^2 = 0$, and give the possible groups H.
Consider also a field in the adjoint representation of SU(3) and show that non-trivial stability subgroups H $\subset$ G, where G = SU(3), exist whenever $(\mathrm{Tr}[\Phi^2])^3 = 6\,(\mathrm{Tr}[\Phi^3])^2$. *Hint*: Write the field as in (18.47) and solve the equation for the eigenvalues parametrized in a and b. Show that the solutions correspond to the case of two equal eigenvalues and argue that H is then equal to either SU(2) or SU(3).

18.9. Consider an $N\times N$ matrix A, and the following product of matrix elements $A^i{}_j$,

$$A^{i_1}{}_{[j_1}\,A^{i_2}{}_{j_2}\cdots A^{i_n}{}_{j_n]}\ , \qquad (1)$$

fully antisymmetrized over all indices $j_1,\ldots j_n$. Argue that this expression must vanish for $n > N$, while, for $n = N$ it is proportional to the determinant of A. Subsequently contract with the n-fold product of Kronecker deltas, $\delta^{j_1}{}_{i_1}\,\delta^{j_2}{}_{i_2}\cdots\delta^{j_n}{}_{i_n}$. Show that this leads to a polynomial in $\mathrm{Tr}\,[A^k]$, with $k \leqslant n$. Prove that the determinant $\det[A]$ can be written as a polynomial in $\mathrm{Tr}\,[A^k]$ with $k \leqslant N$. Discuss the situation for a symmetric matrix, which can be diagonalized, in terms of its eigenvalues for $N = 3$.

18.10. A field transforming in the adjoint representation of SU(N) can be written in the form of a traceless, hermitean, $N\times N$ matrix Φ. Consider the most general invariant potential,

$$V(\Phi) = \tfrac{1}{2}\mu^2\,\mathrm{Tr}[\Phi^2] + \tfrac{1}{3}\lambda_3\,\mathrm{Tr}[\Phi^3] + \tfrac{1}{4}\lambda_4\,\mathrm{Tr}[\Phi^4] + \tfrac{1}{4}\lambda_5\,\left(\mathrm{Tr}[\Phi^2]\right)^2\,, \tag{1}$$

with only couplings of the renormalizable type. Argue that, for $N = 2$ or 3, one can restrict the number of terms in the potential to two or three, respectively. Derive the following matrix equation for stationary points of the potential,

$$(\mu^2 + \lambda_5\,\mathrm{Tr}[\Phi^2])\,\Phi + \lambda_3\,\Phi^2 + \lambda_4\,\Phi^3 = \frac{1}{N}\mathbf{I}\Big[\lambda_3\,\mathrm{Tr}[\Phi^2] + \lambda_4\,\mathrm{Tr}[\Phi^3]\Big]\,, \tag{2}$$

where we included a term on the right-hand side so that the trace of the above equation is trivially satisfied. This is necessary because the equation is generated by varying the potential with respect to the traceless field Φ. By taking products with powers of Φ and taking the trace subsequently, one finds a set of equations involving only invariant traces. In this way one derives, for instance,

$$(\mu^2 + \lambda_5\,\mathrm{Tr}[\Phi^2]\Big)\,\mathrm{Tr}[\Phi^2] + \lambda_3\,\mathrm{Tr}[\Phi^3] + \lambda_4\,\mathrm{Tr}[\Phi^4] = 0\,. \tag{3}$$

Obviously the Lagrangian has a stationary point for $\Phi = 0$. Discuss the symmetry properties at this point and give the values for the masses of the $(N^2 - 1)$ fields. Subsequently, consider the possiblity that there exists a stationary point for which Φ has $N - n$ equal eigenvalues λ and n equal eigenvalues λ'. Show that

$$\lambda' = \frac{n-N}{n}\,\lambda\,. \tag{4}$$

Argue that the residual symmetry group is equal to $\mathrm{S(U}(n)\times\mathrm{U}(N-n)) \cong \mathrm{SU}(n)\times\mathrm{SU}(N-n)\times\mathrm{U}(1)$.
Subsequently prove the following identities,

$$\begin{aligned}\mathrm{Tr}[\Phi^2] &= \frac{N(N-n)\,\lambda^2}{n}\,,\\ \mathrm{Tr}[\Phi^3] &= -\frac{N(N-n)(N-2n)\,\lambda^3}{n^2}\,.\end{aligned} \tag{5}$$

Substitute these expressions into (2), and derive two cubic euations for λ. Provided that $\lambda \neq 0$, show that one obtains the following quadratic equation for λ,

$$\mu^2 - \lambda_3\,\frac{N-2n}{n}\,\lambda + \left[\lambda_5\frac{N(N-n)}{n} + \lambda_4\frac{N^2-3nN+3n^2}{n^2}\right]\lambda^2 = 0\,, \tag{6}$$

Can you understand that there is just one quadratic equation?

18.11. Consider the SU(3) matrix Φ specified in (18.47) with $b = 0$, so that the first two eigenvalues are equal and the corresponding symmetry group equals $\mathrm{U}(1)\times\mathrm{SU}(2)$. To sweep out an orbit by acting with an SU(3) transformation on Φ according to (18.46), we consider infinitesimal SU(3) transformation. Argue that, because of the U)(1) $\times$ SU(2) invariance, it will sufice to consider the infinitesimal transformation,

$$U = \mathbf{1} + \begin{pmatrix} 0 & 0 & \varphi_1 \\ 0 & 0 & \varphi_2 \\ -\bar{\varphi}_1 & -\bar{\varphi}_2 & 0 \end{pmatrix} + \mathcal{O}(\varphi^2) \ , \tag{1}$$

where (φ_1, φ_2) is a complex doublet. Verify that this is an infinitesimal transformation by showing that $U^\dagger = U^{-1}$ to first order in φ_i or $\bar{\varphi}_i$. Show that this transformation changeds Φ into

$$\Phi = \frac{a}{\sqrt{3}} \begin{pmatrix} 1 & 0 & -3\,\varphi_1 \\ 0 & 1 & -3\,\varphi_2 \\ 3\,\bar{\varphi}_1 & 3\,\bar{\varphi}_2 & -2 \end{pmatrix} + \mathcal{O}(\varphi^2) \ , \tag{2}$$

Argue that the fields φ_i correspond to the Goldstone bosons in this phase, and, by considering U)(1) $\times$ SU(2) transformations on (**??**), show that these Goldstone bosons transform in the $(\mathbf{2}_2 + \bar{\mathbf{2}}_{-2})$ of this group. Here the subscripts denote the U(1) eigenvalues. this result

18.12. Let us consider the elements of the permutation group S_3 that permute the eigenvalues of the matrix Φ given in (18.47). As explained in the main text, these permutations are induced by finite SU(3) transformations denoted by $U_{(i,j,k)}$, where the label $(i, j, k,)$ denotes the permutation, according to

$$\Phi \to U_{(i,j,k)}\, \Phi\, U^{-1}_{(i,j,k)} \ . \tag{1}$$

In the (a, b)-plane the permutations are described by the two-dimensional matrices $T_{(i,j,k)}$.
Let us first consider the three odd permutions $(2, 1, 3)$, $(1, 3, 2)$ and $(3, 2, 1)$ that interchange two possible eigenvalues. All other permutation can be obtained by successively applying a number of these three special permutations, so that they can be regarded as the generators of the finite group. Demonstrate that these permutations can be described by the following matrices $T_{(i,j,k)}$ and $U_{(i,j,k)}$,

$$T_{(2,1,3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad U_{(2,1,3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{2}$$

$$T_{(1,3,2)} =, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ , \qquad U_{(1,3,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \tag{3}$$

$$T_{(3,2,1)} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ , \qquad U_{(3,2,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ . \tag{4}$$

Show that these matrices square to the identity and have determinant -1. Demonstrate that each of the above generators $T_{(i,j,k)}$ leaves one of the boundary lines depicted in Fig. 18.4 invariant. Prove als that the generators do not commute with each other, so that the permutation group is non-abelian.
Even permutations are described by products of an even number of matrices. Argue that, besides the identity, the only other independent permuations are the even ones, $(2,3,1)$ and $(3,1,2)$. Write down the corresponding matrices $T_{(i,j,k)}$ and $U_{(i,j,k)}$.