17

Perturbative quantum chromodynamics

In the preceding chapter we discussed certain non-perturbative aspects of QCD such as baryon and meson spectroscopy, and quark confinement. Our analysis of the QCD beta function showed that a perturbative approach to making predictions with QCD was possible at large energy scales, where the running coupling has become small enough to use it as an expansion parameter, and we demonstrated this for the R ratio in e^+e^- collisions. In the present chapter we shall show why and how one may use perturbative QCD in collisions involving hadrons, as indeed occur in present-day colliders. To this end we shall extend the ideas of the parton model in section 6.6 to higher orders in perturbation theory, and apply them to well-known reactions such as deep-inelastic scattering, the Drell-Yan process, and jet production.

17.1. Factorization

In section 15.1 we saw that the effective scale-dependent coupling of a non-abelian gauge theory such as QCD has the property that at high scales it becomes small enough to use perturbation theory. This property, together with the infrared safety provided by the KLN theorem discussed in the previous chapte,r suffices to allow a perturbative calculation of the inclusive cross section for e^+e^- collisions. What we have not yet understood is how perturbative QCD could describe high-energy scattering with hadrons in the initial state. The problem is in fact two-fold. To appreciate the first, recall that in chapter 3 we defined cross sections in terms of particles prepared at times long before the collision, whereas the particles exiting the collision are detected long after. This requires participating particles (here: quarks and gluons) to exist as free particles for very long times, at least relative to the time interval during which the collision takes place. Quarks and gluons, however, are confined by a mechanism not yet fully understood. The second problem involves collinear and infrared diverges one encounters In perturbation theory with massless partons. The KLN theorem again ensures their cancellation if one sums over degenerate final states, but cannot help when they are associated with the initial, where we do not sum over degenerate states. We shall see however that these vices becomes virtues, and can help establish QCD predictive power for hadronic collisions.

The property of QCD that ensures this is called factorization. Before discussing details, let us review the logic behind the argument. The key is that these initial state "collinear" divergences are process independent. These divergences show up at every order in perturbation theory beyond the lowest order, and their name indicates that they are associated with collinear branchings of partons into more partons long before they collide with other particles. These collinear divergences can be collectively written as a scalar "transition" function multiplying a divergence-free partonic cross section. These transition functions are therefore very similar to the Z-factors, comprising UV divergences, multiplying renormalized quantities. Predictive power is then obtained as follows. First, one uses these transition functions to renormalize the parton distributions (see section 6.5). Because the collinear divergences are process independent, this procedure will indeed remove these divergences for any process. Next, one devotes a certain number of experiments to fit these finite, renormalized parton distributions. For any other process, then, one can then use them, by their universality, to predict the (differential) cross section.

While the validity of the procedure just described should properly be proven to all orders in perturbation theory, we shall take a more heuristic approach, as is our habit, and consider a one-loop case in detail. Our approach is illustrated in Fig. 17.1, which shows schematically the scattering amplitude for a process involving an incoming proton, from which a quark participates in a hard scattering process after emission of a number of gluons. In this section we shall compute the pre-scattering part of the process, which will lead to us to explicit expressions for the transition functions. In the next sections we show that these indeed remove the initial state collinear divergences for specific processes. Because we will deal with divergences, we work in n dimensions.

We define the transition function Φ_{fi} by the statement that, if the initial particle i has momentum $\mathbf{p} = (0, 0, P)$, the probability of, after radiation, emerging as a particle f with momentum between $(\mathbf{k_T}, zP,)$ and $(\mathbf{k_T} + \mathbf{dk_T}, (z + dz)P)$ is given by

$$\Phi_{fi}(z, \mathbf{k_T}) d\mathbf{k_T} dz \,. \tag{17.1}$$

Here $\mathbf{k_T}$ is the $n - 2$-component vector transverse to z direction. For our present purpose the azimuthal orientation of this transverse vector is not relevant, so we define in n dimensions

$$\int d\mathbf{k_T} \Phi_{fi}(z, k_T) = \pi^{n/2-1} \int dk_T^2 (k_T^2)^{(n-4)/2} \Phi_{fi}(z, k_T) = \Phi(z) \tag{17.2}$$

The definitions in (17.1) and (17.2) imply that the Φ_{fi} must be normalized such that

$$\sum_f \int_0^1 dz \Phi_{fi}(z) = 1 \tag{17.3}$$

to any order in perturbation theory. It expresses the fact that the sum of all transition probabilities for particle i is one. To be more specific, let us compute the fermion-to-fermion transition function Φ_{ff} to next-to-leading order in QCD perturbation theory. At lowest order, a fermion of momentum p does not emit, so we must consider the amplitude $A^{(0)}_\alpha(p)$ in Fig.17.2, which is simply given by the polarization spinor $u_\alpha(p)$. The conjugate amplitude is

$$\bar{A}^{(0)}_\beta(p) = \bar{u}_\beta(p), \qquad . \tag{17.4}$$

We now define the transition probability as

$$\Phi^{(0)}_{ff}(z, \mathbf{k_T}) d\mathbf{k_T} dz = N_p \sum_{\alpha\beta} \bar{A}^{(0)}_\beta(p) K^{\beta\alpha} A^{(0)}_\alpha(p) \delta(\mathbf{k_T}) \delta(1-z)\, d\mathbf{k_T} dz\,, \tag{17.5}$$

with N_p a normalization factor. The absence of any emission is also reflected in the presence of the δ functions. Notice that we have contracted the spinor indices by $K^{\beta\alpha}$, to obtain a scalar function. Its purpose is to project out the largest components, and to decouple the spin degrees of freedom between the initial state collinear branching process and the actual scattering process. Summing over the fermion spin we find

$$\Phi^{(0)}_{ff}(z, \mathbf{k_T}) = N_p \sum_{\text{spin}} \bar{u}(p) K u(p) \delta(\mathbf{k_T}) \delta(1-z), \tag{17.6}$$

leading to

$$\Phi^{(0)}_{ff}(z, \mathbf{k_T}) = N_p \mathrm{Tr}\,(-\mathrm{i}\not{p} K)\ \delta(\mathbf{k_T}) \delta(1-z) \tag{17.7}$$

From this lowest order calculation we can infer what K must be, as it should not depend on the order of perturbation theory. Out of a complete set of γ matrices (see Appendix E), only the choice $K = -i\gamma^+$ gives a positive, non-zero answer. Choosing $N_p = 1/2P$ we have the intuitive result

$$\Phi^{(0)}_{ff}(z, \mathbf{k_T}) = \delta(1-z) \delta(\mathbf{k_T})\,. \tag{17.8}$$

At this order the condition (17.3) is clearly satisfied. At next order, we must compute the radiative contribution in Fig.17.3. To regularize divergences, we work in $n = 4 + \epsilon$ dimensions. The real emission correction is given by

$$\Phi^{(1),r}_{ff}(z, \mathbf{k_T}) = e^2 \sum_{s,\lambda} \int \frac{d^{n-1}k}{(2\pi)^{n-1} 2E} \times \bar{u}(p,s) \not{\epsilon}^*(k,\lambda) \frac{\not{p} - \not{k}}{(p-k)^2} (-\mathrm{i}\gamma^+) \frac{\not{p} - \not{k}}{(p-k)^2} \not{\epsilon}(k,\lambda') u(p,s) \tag{17.9}$$

We sum over the spin degrees of freedom of the fermion s and the photon λ, using (4.38) to perform the sum over the photon helicities. The energy of the emitted photon is

$$E = \sqrt{k_T^2 + (1-z)^2 P^2} \simeq (1-z)P + \frac{k_T^2}{2(1-z)P} \tag{17.10}$$

where we use the fact P is very large. As a result,

$$p \cdot k \simeq -\frac{k_T^2}{2(1-z)}, \qquad p \cdot k^* \simeq 2(1-z)^2 P^2 \tag{17.11}$$

where again we have only kept the dominant terms. Rewriting the photon phase space measure as

$$\int \frac{d^{n-1}k}{(2\pi)^{n-1} 2E} = \frac{1}{16\pi^2} \frac{1}{(4\pi)^{\epsilon/2}} \int \frac{dk_T^2 (k_T^2)^{\epsilon/2} dz}{(1-z)} \tag{17.12}$$

we arrive at the following expression for the radiative part of the first order correction to the transition function

$$\Phi_{ff}^{(1),r}(z) = \delta(1-z) + \frac{\alpha(\mu)}{\pi}(4\pi)^{-\epsilon/2}\mu^{-\epsilon} \int \frac{dk_T^2}{(k_T^2)^{1-\epsilon/2}} \frac{1+z^2}{1-z} + \mathcal{O}\left(\frac{1}{P^2}\right), \tag{17.13}$$

where we have also introduced the dimensional regularization scale μ. Because P, the momentum of the incoming parton, is very large, we can safely ignore the last correction.

The virtual correction, in Fig.17.3, is proportional to $\delta(1-z)\delta(\mathbf{k_T})$. Rather than compute it explicitly, we shall determine it by the condition (17.3). Because the Born contribution already saturates the bound, the full integral over the next-order correction should vanish. To help express this property we introduce the notion of the plus-distribution, which is defined in integrals with smooth testfunctions

$$\int_0^1 dz f(z)[g(z)]_+ \equiv \int_0^1 dz (f(z) - f(1)) \, [g(z)] \tag{17.14}$$

Note that if $f(z)$ is a constant, the integral is zero. We can therefore write

$$\Phi_{ff}^{(1)}(z) = \delta(1-z) + \frac{\alpha(\mu)}{\pi}(4\pi)^{-\epsilon/2}\mu^{-\epsilon} \int \frac{dk_T^2}{(k_T^2)^{1-\epsilon/2}} \left[\frac{1+z^2}{1-z}\right]_+ \tag{17.15}$$

We have not yet specified the integral over the recoil momentum k_T. We let this integral range from 0, the collinear limit, to a new scale μ_F, the factorization scale. This scale signifies the separation between the part of the photon emission spectrum that is included into the transition function ($k_T < \mu_F$) and the part that is assigned to the actual scattering ($k_T > \mu_F$)[1] The result then reads

$$\Phi_{ff}^{(1)}(z) = \delta(1-z) + \frac{\alpha(\mu)}{\pi}\frac{-2}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2}\left[\frac{1+z^2}{1-z}\right]_+ \tag{17.16}$$

where the collinear divergence is made explicit.

So far we considered photon emission from a charged fermion. By the same methods (see problems), we can compute the complete set of transition functions Φ_{ij}, $i,j = q,\bar{q},g$ from emission of partons by partons. In addition to the normalization (17.3) we now have sum rules that express the fact that the total momentum of all partons in a proton is not changed when higher order corrections are included

$$\begin{aligned}
&\int_0^1 dzz\left\{\Phi_{qq}^{(1)}(z) + \Phi_{qg}^{(1)}(z)\right\} = 0\,,\\
&\int_0^1 dzz\left\{2n_f\Phi_{gq}^{(1)}(z) + \Phi_{gg}^{(1)}(z)\right\} = 0\,,
\end{aligned} \tag{17.17}$$

where n_f is the number of massless fermions flavors allowed in the initial state.

The one-loop transition functions for splittings involving quarks and gluons may be computed in a similar way (see Problem 16.x). It is easy to see that for the quark-to-quark splitting the previous calculation only requires two trivial modifications. The first involves the fact that quarks and gluons have colour, leading to factor $C_2(R)$ in Φ_{qq}, e.g. The second follows from the dimension regularization of the coupling constant α_s which introduces a factor $\mu^{-\epsilon}$. The other splittings require calculations of the type just performed. The results

[1]Beyond first order it is no longer straigtforward to interpret μ_F so directly. It is then best to interpret it similarly to the renormalization scale in chapter 10.

are

$$\Phi_{qq}(z,\mu_F) = \delta(1-z) + \frac{\alpha_s(\mu)}{2\pi}\frac{1}{\epsilon}C_2(R)\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2}\left[\frac{1+z^2}{1-z}\right]_+ \quad (17.18)$$

$$\Phi_{gg}(z,\mu_F) = \delta(1-z) + \frac{\alpha_s(\mu)}{2\pi}\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2}\left(2C_2(G)\Big[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\Big] + \tfrac{1}{2}\left(\frac{11C_2(G)-4n_fT_F}{3}\right)\delta(1-z)\right) \quad (17.19)$$

$$\Phi_{qg}(z,\mu_F) = \frac{\alpha_s(\mu)}{2\pi}\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2}C_2(R)\left(\frac{1+(1-z)^2}{z}\right) \quad (17.20)$$

$$\Phi_{gq}(z,\mu_F) = \frac{\alpha_s(\mu)}{2\pi}\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2}T_F\left(z^2+(1-z)^2\right) \quad (17.21)$$

It may be checked that they satisfy the conditions 17.3 and 17.17. In the next sections we show that the initial state collinear divergences for various different processes can be removed by renormalizing the parton distributions with the Φ factors above, in complete analogy with the renormalization by Z factors in section 8.3. The transition functions are therefore universal, can be determined in a few dedicated processes, and then subsequently used to predict many other ones.

Besides the infinite terms in the transition functions, which are necessary to cancel the collinear divergences in cross sections with initial state partons, we may include convenient finite terms in their definition. This corresponds to a choice of factorization scheme. The expessions above refer to the *minimal substraction scheme*, as they contain pure poles in ϵ. The most common scheme is to replace

$$\frac{1}{\epsilon} \to \frac{1}{\epsilon} + \tfrac{1}{2}\gamma_E - \tfrac{1}{2}\ln 4\pi \quad (17.22)$$

the *modified minimal subtraction* ($\overline{\text{MS}}$) *scheme*. Of course, to be consistent one should stick with a chosen scheme, and not use parton distribution functions determined in one scheme in a calculation carried out in another.

17.2. Partonic cross section for Drell-Yan scattering

An important ingredient in the evaluation of QCD corrections to high-energy reactions are the cross sections for the partons. Although partons are not observable as physical particles one can envisage doing standard field-theoretic calculations in perturbation theory to study their interaction with photons or W or Z bosons. The results of these calculations can then serve as input for the calculation of deep-inelastic lepton production or for Drell-Yan reactions. As we pointed out already, the use of the quark-parton model can be avoided for deep-inelastic lepton production, where operator-product expansions can be used. However, for Drell-Yan reactions, where one deals with two incoming hadrons, there exists no purely field-theoretic approach and one has to make use of the less rigorous framework offered by the parton model.

Unfortunately, even at the partonic level, there are a number of subtleties in the perturbative calculations. Obviously we encounter ultraviolet divergences, which can be absorbed into appropriate renormalizations in the standard way. But in addition, the masslessness of the partons leads to infrared divergences which have to be carefully separated from the hard scattering reactions of the partons and can then be absorbed into quantities whose determination will be left to the experiments.

We will now examine the complications involved with higher-order partonic calculations by focussing one specific process, namely the production of a lepton pair in proton-antiproton collisions corresponding to the rection $p+\bar{p} \to l+\bar{l}+X$ (commonly called the Drell-Yan reaction). In (6.121) we gave a formula for the triple differential cross section in $\hat{s}$, the invariant mass of the lepton-antilepton pair (or the timelike mass of the virtual photon with momentum Q, so $\hat{s} = -Q^2 > 0$), in y, the longitudinal momentum of the virtual photon and in $\cos\hat{\theta}$, where $\hat{\theta}$ is the angle of the outgoing lepton momentum in the dilepton center-of-mass frame. For simplicity we will now only examine the QCD corrections to the single differential cross section in the invariant mass $\hat{s}$. Therefore the process is inclusive in all the hadron final states which should allow us to apply the Kinoshita-Lee-Nauenberg (KLN) theorem. We begin by considering the reaction at the partonic level, where the Born approximation only involves quark-antiquark annihilation into a virtual photon, which couples to the lepton-antilepton pair.

The total cross section for quark-antiquark annihilation in the reaction $q(p_1) + \bar{q}(p_2) \to l(q_1) + \bar{l}(q_2)$ is

$$\sigma^{(0)}_{q\bar{q}}(\hat{s}) = \frac{1}{N_c^2 2^2 2\hat{s}} \int \frac{d^3q_1}{(2\pi)^3 2\omega_1} \int \frac{d^3q_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta(p_1+p_2-q_1-q_2) \sum |\mathcal{M}|^2 \,, \tag{17.23}$$

where the sum is over all initial and final spin and colour indices. The only relevant Feynman diagram is the one photon exchange diagram for which the square of the amplitude yields

$$\sum |\mathcal{M}|^2 = e^4 q_f^2 \mathrm{Tr}(\gamma_\mu p_1 \gamma_\nu p_2) \mathrm{Tr}(\gamma_\mu q_2 \gamma_\nu q_1) \frac{1}{\hat{s}^2} \,, \tag{17.24}$$

where the charge of the quark is $q_f e$, and the final trace is over the unit N_c-dimensional matrix labelled by the colour indices. It is trivial to work out (17.24) using the results from section 6.6. The integration over the lepton trace follows from application of Lenard's identity in problem 6.4. The final result

$$\sigma^{(0)}_{q\bar{q}}(\hat{s}) = \frac{4\pi\alpha^2}{3N_c\hat{s}}\,, \tag{17.25}$$

is only a function of $\hat{s}$. Now introduce the cross section differential with respect to $Q^2 = (q_1 + q_2)^2$ by the formula

$$\frac{d\sigma^{(0)}_{q\bar{q}}(Q^2)}{dQ^2} = \left[\frac{4\pi\alpha^2}{3N_c(Q^2)^2}\right]\delta\left(1 - \frac{Q^2}{\hat{s}}\right)\,, \tag{17.26}$$

which satisfies

$$\sigma^{(0)}_{q\bar{q}}(\hat{s}) = \int \frac{d\sigma^{(0)}_{q\bar{q}}(Q^2)}{dQ^2}dQ^2 = \frac{4\pi\alpha^2}{3N_c}\int \frac{dQ^2}{(Q^2)^2}\delta\left(1 - \frac{Q^2}{\hat{s}}\right) = \frac{4\pi\alpha^2}{3N_c\hat{s}} \tag{17.27}$$

Note that (17.26) is no longer a function but a distribution as it only exists at the point where $Q^2 = \hat{s}$. Therefore as far as the lowest order formula is concerned we can write $d\sigma^{(0)}_{q\bar{q}}/dQ^2$ or $d\sigma^{(0)}_{q\bar{q}}/d\hat{s}$. The expression in square brackets in (17.26) we will later call $\sigma^{(0)}_\gamma$ for brevity.

For the calculations in what follows we are not very interested in the part of the diagram in which the photon decays into leptons. Therefore, let us compute the amplitude squared for the process $q(p_1) + \bar{q}(p_2) \to \gamma(Q)$ and then work out the factor K defined by

$$\sigma(l_1 l_2) = K\sigma(\gamma) \tag{17.28}$$

It is important to calculate both factors in dimensional regularization, as we will use K in higher order calculations., To be finished..

We remind the reader that the hadronic cross section follows by convoluting this result with partonic densities in the incoming hadrons as in section 6.7. We follow (6.116) and write $\hat{s} = \xi_1\xi_2 s$. Also we introduce the variable $\tau = Q^2/s$ so that

$$\frac{d\sigma^{(0)}_{AB}(\tau)}{dQ^2} = \sum_{i,j}\int_{\xi_{1,\min}}^1 d\xi_1 \int_{\xi_{2,\min}}^1 d\xi_2 f_{i/A}(\xi_1) f_{j/B}(\xi_2)\frac{d\sigma^{(0)}_{ij}(\xi_1,\xi_2)}{dQ^2} \tag{17.29}$$

where the f's are the parton distribution functions for the quarks and antiquarks in the proton and antiproton. Also the sum runs over all quarks and antiquarks in both the proton and antiproton, while $\xi_{1\rm min} = \tau$, and $\xi_{2\rm min} = \tau/\xi_1$. This explains why we can regard (17.26) as a distribution. It yields finite results only after integration with smooth test functions (the parton distribution functions). We return to this formulae in the next section.

Our problem now is to evaluate the higher order corrections to (17.29) as a power series in α_s. So let us write the partonic cross section as

$$\frac{d\sigma_{ij}}{d\hat{s}} = \sum_n \alpha_s^n \frac{d\sigma_{ij}^{(n)}}{d\hat{s}}\,, \tag{17.30}$$

where the indices run over all possible quark, antiquark and gluon channels. We first concentrate on the quark-antiquark channel and calculate the processes involving the virtual (gluonic) corrections to the Born reaction as well as the higher order processes where a bremsstrahlung gluon is emitted from the incoming quark and from the incoming antiquark. Note that the leptons do not couple directly to the gluon so the trace over the final state leptons always factorizes as in the Born result.

We now outline what happens. We need a regulator procedure to handle the divergences which appear in the calculation of the QCD corrections to $d\sigma_{q\bar{q}}^{(1)}/dQ^2$. As mentioned above these divergences are of three types, ultraviolet, infrared and collinear. We will use n-dimensional regularization to control all singularities, including the infrared and collinear ones. We anticipate that the infrared divergences will cancel when we add the contributions from degenerate states. However we do not know what will happen with the collinear singularities. In general the introduction of any regulator for the infrared and collinear singularities alters the long range properties of the theory. Our fundamental assumption is that we can reliably calculate any quantity which is infrared safe in the sense that it is independent of the long range properties of the theory and therefore of our regularization scheme. As infrared divergent terms only cancel at the level of cross sections we have to evaluate the cross sections themselves in n-dimensions.

The first order QCD corrections to the Born matrix element involve Feynman diagrams similar to those discussed in chapter 9, but now we must evaluate the traces in n-dimensions and add a colour matrix at the quark gluon vertex. We split the correction terms as follows

$$\frac{d\sigma_{q\bar{q}}^{(1)}}{dQ^2} = \frac{d\sigma_{q\bar{q}}^{(1)}}{dQ^2}|_{\rm real} + \frac{d\sigma_{q\bar{q}}^{(1)}}{dQ^2}|_{\rm virtual}\,. \tag{17.31}$$

First consider the virtual corections. The analysis of the virtual graphs in n-dimensions follows what we did for the QED process in Chapter 9. To illustrate the method consider the calculation of the vertex function (9.29). The n-dimensional trace is given in (9.65) and (9.66). We first require the integral

Figure 17.1: The Feynman diagrams for the first order QCD corrections to the partonic Drell-Yan reaction.

$J(t,0,0)$ in (9.29). Remember that we used the notation $t = -(p'-p)^2 = -Q^2$. We wrote the answer for this integral in (9.34) to demonstrate that it has an infrared singularity. If the fermion has no mass then the integrand in (9.31) is a function of the product xyt so the integrals over the parameters x and y now yield Γ-functions. We therefore find

$$J(t,0,0) = i(4\pi)^{-n/2}\left(\frac{-t}{\mu^2}\right)^{n/2-3}(\mu^2)^{n/2-3}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-3)} \times\frac{1}{(n/2-2)^2}\,, \qquad (17.32)$$

where we have inserted a mass μ to make the integral have the correct dimension in n spacetime dimensions. The answer shows a double pole from the overlap of an infrared and a collinear singularity. The same simplification occurs in the two-denominator integral (9.25) so in the massless limit

$$I(-t,0,0) = i(4\pi)^{-n/2}\left(\frac{-t}{\mu^2}\right)^{n/2-2}(\mu^2)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)} \times\frac{1}{2-n/2}\,. \qquad (17.33)$$

Now one can clearly appreciate the difference between the regularization of ultraviolet divergences and the regularization of soft or collinear divergences. The regularization of the former divergence is always made in the n-dimensional integral over the loop momentum giving a factor $(2-n/2)^{-1}$, while the regularization of the latter singularities arise in the integrations over the Feynman parameters and yield factors of $(n/2-2)^{-1}$.

The other integrals $S_\mu(P,Q)$ and $T_{\mu\nu}(P,Q)$ can be done following the method in Appendix F. Using the decomposition in (9.70) and (9.71), we find

$$S(-t) = -i(4\pi)^{-n/2}\Big(\frac{-t}{\mu^2}\Big)^{n/2-3}(\mu^2)^{n/2-3}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)} \times\frac{1}{(n/2-2)}\,, \tag{17.34}$$

$$T_1(-t) = i(4\pi)^{-n/2}\Big(\frac{-t}{\mu^2}\Big)^{n/2-2}(\mu^2)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-1)} \times\frac{1}{2(2-n/2)}\,, \tag{17.35}$$

$$T_2(-t) = i(4\pi)^{-n/2}\Big(\frac{-t}{\mu^2}\Big)^{n/2-3}(\mu^2)^{n/2-3}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-1)} \times\frac{n-3}{2(n/2-2)}\,, \tag{17.36}$$

and

$$T_3(-t) = \frac{1}{2t}I(-t,0,0)\frac{1}{n-2}\,. \tag{17.37}$$

If we now substitute these integrals into the trace then the equivalent of (9.82) is

$$\Lambda(p',p) = e^3\gamma_\mu i(4\pi)^{-2}\Big(\frac{-t}{4\pi\mu^2}\Big)^{n/2-2}(\mu^2)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)} \times\Big[\frac{1}{2}\frac{3-n}{2-n/2}+\frac{2(n-3)}{(n/2-2)^2}-\frac{4}{n/2-2}\Big]\,. \tag{17.38}$$

Note that this answer is exact as we did not need to make any expansion to derive it. If we now substitute $n = 4+\epsilon$ then

$$\Lambda(p',p) = e^3\gamma_\mu i(4\pi)^{-2}\Big(\frac{-t}{4\pi\mu^2}\Big)^{\epsilon/2}(\mu^2)^{\epsilon/2}\frac{\Gamma(1-\epsilon/2)\Gamma^2(1+\epsilon/2)}{\Gamma(2+\epsilon)} \times\Big[\frac{8}{\epsilon^2}+\frac{2}{\epsilon}+1\Big]\,. \tag{17.39}$$

There is another term in (9.82) proportional to the n-dimensional integral of q^{-4} which we will drop according to our general rules for n-dimensional integration. Also the self energy corrections to the fermion propagator are proportional to the same integral so they can be dropped. The calculation of the QED corrections to the reaction $e^+ + e^- \to \mu^+ + \mu^-$ in section 9.6 is now much simpler. If one inserts these integrals into (9.116) and then puts together the pieces of the corresponding bremsstrahlung integral, using (**??**)

for the phase space then the result (9.154) can be shown to be true. This example demonstrates that n-dimensional regularization can be used for the regularization of infrared and collinear singularities when one evaluates the corrections to a physical cross section.

Besides the vertex function we should also consider other virtual contributions, namely the gluon self-energy corrections to the incoming quark and anti-quarks, as well as contributions from counterterms. However, remarkably, none of these contribute to the present calculation. To see why consider first the self-energy contribution for an on-shell massless fermion

$$\Sigma(p) = -\mathrm{i}\not{p} - g^2 \int \frac{d^n q}{(2\pi)^n} \frac{(-\mathrm{i}\not{p})(-\mathrm{i}\not{p} - \mathrm{i}\not{q})(-\mathrm{i}\not{p})}{(p+q)^2 q^2} \tag{17.40}$$

Using $p^2 = 0$ this reduces to

$$\Sigma(p) = -\mathrm{i}\not{p} - \mathrm{i}\not{p}\, g^2 \int \frac{d^n q}{(2\pi)^n} \frac{2p \cdot q}{(p+q)^2 q^2}\,. \tag{17.41}$$

Writing

$$2p.q = (q+p)^2 - q^2 \tag{17.42}$$

we see that the $\mathcal{O}(g^2)$ correction vanishes, by the rules of dimensional regularization.

We now turn to the contribution from the counterterms in the lagrangian. Of course, even though a loop graph gives zero contribution, these counterterm graphs must still be included, for better or for worse. There are in fact three diagrams coming from the counterterms in the lagrangian (ref). Because the lowest order vertex is electromagnetic, we do not need to recall the counterterms of chapter 14, but can illustrate what happens with $\Delta\mathcal{L}_3$ and $\Delta\mathcal{L}_4$ in 8.43 and 8.45. The quark colours, when including initial quark color averaging, lead to a common factor $C_2(R)/N_c$.

Now let us return to the QCD corrections. In this case we have to calculate the differential cross section $d\sigma^{(1)}_{q\bar{q}}/d\hat{s}$ where the virtual graphs are shown in figure 17.1. Therefore we write

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}|_{\mathrm{virtual}} = \frac{1}{8N_c^2\hat{s}} \int_{PS2} dq_1 dq_2\, \delta(\hat{s}+Q^2) \sum |\mathcal{M}|^2\,, \tag{17.43}$$

where we need the amplitude for the interference terms between the Born graph and the first order virtual graphs. The further evaluation of this formula follows from the results above after noting that there is now a minus sign raised to a fractional power. This is the result of going from a channel where the propagator Q^2 is spacelike ($Q^2 = -t > 0$), as we assumed in (9.29), to a channel where Q^2 is timelike ($Q^2 = -\hat{s} < 0$) and yields a factor $(-1)^{\epsilon/2} =$

$1 - \pi^2\epsilon^2/8$. (We drop the term containing the imaginary part since we only need the real part of the interference term.) There are other terms involving π^2 from

$$\Gamma(1-\epsilon/2)\Gamma(1+\epsilon/2) = 1 + \frac{\pi^2}{6}\frac{\epsilon^2}{4} + O(\epsilon^3)\,, \tag{17.44}$$

as well as a colour trace. Putting everything together we find

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}|_{\text{virtual}} &= \sigma^{(0)}_\gamma q_f^2 \frac{1}{2\pi} C_2(R)\Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{-\epsilon/2}\frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \\ &\quad \times\Big[-\frac{8}{\epsilon^2} + \frac{6}{\epsilon} - 8 + \frac{2\pi^2}{3} + O(\epsilon)\Big]\delta\,(1-x)\,.\end{aligned} \tag{17.45}$$

where $x = \hat{s}/s$. Next we consider the real gluon bremsstrahlung graphs which as far as the partonic channel is concerned contribute to the two-to-two body scattering cross section for $q(p_1)+\bar{q}(p_2) \to \gamma(Q)+g(k)$. Now since $s = -(p_1+p_2)^2$ is the square of the total centre-of-mass energy and $\hat{s} = -Q^2 = -(q_1+q_2)^2$ is still the invariant mass of the dilepton pair then $s \neq \hat{s}$. We will therefore have contributions from the region where $x = \hat{s}/s < 1$. Before we begin, let us introduce a convenient shorthand notation for an l-particle n-dimensional phase space measure

$$\int_{\text{PSl}} dq_1 \cdots dq_l = (2\pi)^{n+l(1-n)} \int \frac{d^{n-1}q_1}{2q_1^0} \cdots \frac{d^{n-1}q_l}{2q_l^0}\,\delta^{(n)}(P - \sum_i^l q_i) \tag{17.46}$$

where we have left the masses of each particle unspecified. We need to evaluate

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}|_{\text{real}} = \frac{1}{8N_c^2\hat{s}}\int_{\text{PS3}} dk dq_1 dq_2 \delta^{(n)}(p_1+p_2-k-q_1-q_2)\sum|\mathcal{M}|^2\,, \tag{17.47}$$

where $\mathcal{M}$ is the matrix element of the two-to-three body reaction $q(p_1) + \bar{q}(p_2) \to l(q_1) + \bar{l}(q_2) + g(k)$. The three body phase space integral can be split into two two-body phase space integrals by inserting

$$1 = \int \frac{dQ^2}{2\pi}\int d^n q \delta^{(n)}(q-q_1-q_2)(2\pi)\delta(Q^2-q^2)\,, \tag{17.48}$$

into the integral. Then write the integral over dq as a $n-1$ dimensional integral using $\delta(q^2 - Q^2)$. If we use the notation $p = p_1 + p_2$ then the integrals can be written as

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}|_{\text{real}} = \frac{1}{16\pi N_c^2\hat{s}}\int_{\text{PS2}} dk dq \int_{\text{PS2}} dq_1 dq_2 \sum|\mathcal{M}|^2\,, \tag{17.49}$$

where the first phase space integral has a δ-function $\delta^{(n)}(p-q-k)$ and the second one $\delta^{(n)}(q-q_1-q_2)$.

The leptonic piece (and the leptonic trace) are the same as in the Born result. In fact we could leave these pieces in four-dimensions since they have no divergences. This means that the gauge invariance argument used in (8.115) is valid and we can immediately integrate over the two body phase space of the leptons. The partonic trace which remains is essentially the n-dimensional version of the trace evaluated in four dimensions in (8.122).

The square of the partonic matrix element summed over all initial and final spins and polarizations can be written in terms of the Mandelstam invariants for the reaction $q(p_1)+\bar{q}(p_2)\to\gamma(Q)+g(k)$. These we will call $s=-(p_1+p_2)^2$, $t=-(p_1-k)^2$, and $u=-(p_2-k)^2$, which satisfy $s+t+u=-Q^2=\hat{s}$. The four dimensional result can be checked against that in (8.126) after dropping the fermion masses and changing to these variables. Note that a term involving a new mass scale μ will be required because the QCD coupling constant g has mass dimension $(4-n)/2$ in n-dimensions. There is no need to write an n-dimensional generalization for the QED coupling constant therefore we can keep e in four dimensions. The result for this trace is

$$H_{\mu\mu}=4N_cC_2(R)e^2g^2q_f^2(n-2)\mu^{4-n}\Big\{2\frac{Q^2s}{\hat{t}\hat{u}}-\frac{n-2}{2}\Big(\frac{\hat{u}}{\hat{t}}+\frac{\hat{t}}{\hat{u}}\Big)+(4-n)\Big\}\,, \tag{17.50}$$

It is convenient to rewrite the Mandelstam invariants in terms of the two variables $x=\hat{s}/s$ and $y=(1+\cos\theta)/2$ in terms of which we wrote the two-body phase space in (**??**). This yields $s=\hat{s}/x$, $\hat{t}=-\hat{s}(1-x)(1-y)/x$, and $\hat{u}=-\hat{s}(1-x)y/x$. Collecting the pieces together and expressing the above result in terms of the integration variables x and y we find Check normalizations

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}\Big|_{\rm real} &= \sigma^{(0)}_\gamma q_f^2\frac{1}{2\pi}C_2(R)\Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{(4-n)/2}\frac{x^{(4-n)/2}(1-x)^{n-3}}{\Gamma((n-2)/2)}\\ &\times\int_0^1 dy[y(1-y)]^{(n-4)/2}\Big[\frac{2x}{(1-x)^2y(1-y)}\\ &+\frac{(n-2)}{2}\Big(\frac{y}{1-y}+\frac{1-y}{y}\Big)\Big]\,.\end{aligned} \tag{17.51}$$

The integral over y leads to gamma-functions (via the result (13.d11)) which have a pole at $n=4$ reflecting the presence of the collinear divergence. After using the recursion relation for the Γ-function we find

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}\Big|_{\rm real} &= \sigma^{(0)}_\gamma q_f^2\frac{1}{2\pi}C_2(R)\Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{(4-n)/2}\frac{\Gamma((n-2)/2)}{\Gamma(n-3)}x^{(4-n)/2}\\ &\times\Big[4x(1-x)^{n-5}+\frac{(n-2)(n/2-1)}{n-3}(1-x)^{n-3}\Big]\,.\end{aligned} \tag{17.52}$$

This result is exact. Now substitute $n = 4 + \epsilon$ and expand the coefficient in front of the $(1-x)^{n-3}$ term to check that it yields $1 + O(\epsilon^2)$. Using the recursion relation for the Γ-function we can write the answer as

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}|_{\rm real} = \sigma^{(0)}_\gamma q_f^2 \frac{1}{2\pi} C_2(R) \Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{-\epsilon/2} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \frac{4}{\epsilon} \times \Big[2x^{1-\epsilon/2}(1-x)^{-1+\epsilon} + x^{-\epsilon/2}(1-x)^{1+\epsilon}\Big]. \quad (17.53)$$

One collinear pole term in ϵ resulting from the angular integral is now explicit. If we integrate over the variable x, which we have to do to form the hadronic cross section, then there is a second pole when $x \to 1$. The latter is the infrared pole. After integration there are therefore double pole terms from overlapping divergences and single pole terms from the either soft or the collinear singularities. We need to cancel these pole terms against the contributions from the virtual graphs, which only exist for $x = 1$ and it would be convenient to do this before the integration. Then we would immediately have a way of combining the contributions from the virtual and bremsstrahlung graphs.

One way to do this is to split off a small piece in (17.53) between $x = 1 - \delta$ and $x = 1$ and call this the "soft" bremsstrahlung piece. In this small range near unity one can substitute $x = 1$ whenever this is allowed and simply do the integral yielding terms in $\ln\delta$ as well as poles in ϵ. These pieces can then be added to the contributions from the virtual graphs. This is similar to the procedure followed in Chapter 8. The remaining "hard" bremsstrahlung integral over the range 0 to $1 - \delta$ is finite and upon integration will yield a term involving $\ln\delta$ which should cancel with the corresponding $\ln\delta$ term in the virtual graphs. This method is often called the phase-space slicing method.

In this calculation we will introduce another method. Let us find a relation which expresses the double pole terms immediately in terms of $\delta(1-x)$. Such a relation does exist but only in the sense of a distribution, namely when multiplied by a smooth function $F(x)$ and integrated between 0 and 1 (like the δ-function itself). Assume the function $F(x)$ has a Taylor expansion near $x = 1$ so we can write $F(x) = F(1) + F(x) - F(1)$, where the difference between the last two terms is proportional to the finite derivative of $F(x)$ at $x = 1$.

Consider therefore

$$\int_0^1 dx \, \frac{F(x)}{(1-x)^{1-\epsilon}} = F(1)\int_0^1 dx \, \frac{1}{(1-x)^{1-\epsilon}} + \int_0^1 dx \, \frac{F(x) - F(1)}{(1-x)^{1-\epsilon}} \, . \quad (17.54)$$

The first integral yields $F(1)\epsilon^{-1}$. We can rewrite this again as an integral over dx with the argument $\delta(1-x)$. In the second integral we can expand the denominator so it yields

$$\int_0^1 dx\, \frac{F(x)-F(1)}{(1-x)^{1-\epsilon}} = \int_0^1 dx\, \frac{F(x)-F(1)}{(1-x)} + \epsilon \int_0^1 dx\, [F(x)-F(1)] \frac{\ln(1-x)}{(1-x)} + O(\epsilon^2)\,, \tag{17.55}$$

near $\epsilon = 0$. Therefore we have the identity

$$\int_0^1 dx\, \frac{F(x)}{(1-x)^{1-\epsilon}} = \frac{1}{\epsilon}\int_0^1 dx\, F(x)\delta(1-x) + \int_0^1 dx\, \frac{F(x)-F(1)}{1-x} + \epsilon \int_0^1 dx [\, F(x)-F(1)] \frac{\ln(1-x)}{1-x} + O(\epsilon^2)\,. \tag{17.56}$$

This we will write in shorthand notation as

$$\frac{1}{(1-x)^{1-\epsilon}} = \frac{1}{\epsilon}\delta(1-x) + \Big[\frac{1}{1-x}\Big]_+ + \epsilon \Big[\frac{\ln(1-x)}{1-x}\Big]_+ + O(\epsilon^2)\,, \tag{17.57}$$

where the terms on the right hand side are called "plus" distributions. Note that this result is exact for a lower limit $x=0$. If the lower limit is not zero then there are additional terms involving logarithms of this lower limit.

Our final result for the gluon radiation graphs therefore follows by expanding the terms in the square bracket in (17.53) in powers of ϵ and using (17.57). We find

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}}|_{\rm real} = \sigma^{(0)}_\gamma q_f^2 \frac{1}{2\pi}\Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{-\epsilon/2} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \Big[\frac{8}{\epsilon^2}\delta(1-x) + \frac{4}{\epsilon}(1+x^2)\Big[\frac{1}{1-x}\Big]_+ + 4(1+x^2)\Big[\frac{\ln 1-x}{1-x}\Big]_+ - 2(1+x^2)\frac{\ln x}{1-x} + O(\epsilon)\Big]\,. \tag{17.58}$$

Note that the last term is integrable when multiplied by a smooth function of x so we do not need to write it as a plus distribution.

Now we have isolated the term in $\delta(1-x)$ containing the double pole we see that it cancels the corresponding term from the virtual graphs in (17.45). These are the overlap terms containing both soft and collinear divergences and they cancel as expected from the KLN theorem. The single pole term in ϵ cannot possibly cancel as it is proportional to a function of x. Therefore we are left with an uncancelled collinear singularity.

Note also that the mass parameter μ makes the answer dimensionless and will contribute a logarithmic term $\ln(\hat{s}/\mu^2)$ upon expansion in ϵ. Further we have not needed any coupling constant renormalization so the g in $\alpha_s = g^2/4\pi$, which was factored from the above expressions, is the unrenormalized coupling constant.

Now we can sum (17.45) and (17.58), for which we find

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}} &= \sigma^{(0)}_{\gamma} q_f^2 \frac{1}{2\pi} C_2(R) \Big(\frac{4\pi\mu^2}{\hat{s}}\Big)^{-\epsilon/2} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \\ &\times \Big\{ \frac{4}{\epsilon}\Big((1+x^2)\Big[\frac{1}{1-x}\Big]_+ + \frac{3}{2}\delta(1-x)\Big) + 4(1+x^2)\Big[\frac{\ln(1-x)}{1-x}\Big]_+ \\ &-2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2)-8)\delta(1-x) + O(\epsilon)\Big\}, \qquad (17.59)\end{aligned}$$

where $\zeta(2) = \pi^2/6$. The remaining pole term in ϵ demonstrates the failure of the KLN theorem when there are collinear singularities in the initial partonic state. How are we then going to make sense of this result?

First, observe that, if one expands all functions in (17.59) in ϵ one has

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}} &= \sigma^{(0)}_{\gamma} q_f^2 \frac{1}{2\pi} C_2(R) 2 \left(\frac{2}{\epsilon} - \ln 4\pi + \gamma_E\right) \Big((1+x^2)\Big[\frac{1}{1-x}\Big]_+ \\ &+ \frac{3}{2}\delta(1-x)\Big) + \mathcal{O}(\epsilon^0) \\ &= \sigma^{(0)}_{\gamma} q_f^2 \frac{1}{2\pi} C_2(R) 2 \left(\frac{2}{\epsilon} - \ln 4\pi + \gamma_E\right) \Big[\frac{1+x^2}{1-x}\Big]_+ + \mathcal{O}(\epsilon^0) \qquad (17.60)\end{aligned}$$

Next, we realize that this expression should be substituted into the convolution (17.29). At this point we renormalize the parton distributions in (17.29) as follows

$$f_{q/A}(\xi) = \int_0^1 dz \int_0^1 dy f_{q/A}(y,\mu_F) \Phi^{-1}_{qq}(z,\mu_F) \delta(\xi - zy) \qquad (17.61)$$

with μ_F the factorization scale, introduced in the previous section, and Φ_{qq} given in (17.18). To first order the above relation can be written as

$$\begin{aligned}f_{q/A}(\xi) = f_{q/A}(\xi,\mu_F) - \int_\xi^1 \frac{dz}{z} f_{q/A}\left(\frac{\xi}{z},\mu_F\right) \\ \times \left\{\frac{\alpha_s(\mu) C_2(R)}{2\pi} \frac{1}{\epsilon} \left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{-\epsilon/2} \left[\frac{1+z^2}{1-z}\right]_+\right\} \qquad (17.62)\end{aligned}$$

Collecting terms we see indeed, as we announced, the collinear singularities cancel after renormalization of the parton distribution by the transitions, leaving a finite remainder. The result is

$$\begin{aligned}\frac{d\sigma^{(1)}_{q\bar{q}}}{d\hat{s}} &= \sigma^{(0)}_{\gamma} q_f^2 \frac{1}{2\pi} C_2(R) \\ &\times \Big\{ 2\ln\left(\frac{\hat{s}}{\mu_F^2}\right) \left[\frac{1+z^2}{1-z}\right]_+ + 4(1+x^2)\Big[\frac{\ln(1-x)}{1-x}\Big]_+ \\ &-2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2)-8)\delta(1-x)\Big\}, \qquad (17.63)\end{aligned}$$

In the next sections we shall see this cancellation again, but for a different scattering processes.

17.3. Partonic cross section for deep-inelastic scattering

Deep inelastic scattering by electrons and (anti)-neutrinos on target protons and neutrons was previously considered in Sec. 6.5. where we presented the differential scattering cross sections in terms of the energy E and scattering angle θ of the scattered leptons. Formula were given for the results in terms of the variables x and y. Let us change the notation slightly so that the reaction is $l(q_1) + H(p) \to l(q_2) +' X'$, where H is the hadron target and X is the hadron remnant. We also call the space like four vector $q = q_1 - q_2$ the momentum transfer between the incoming lepton and the incoming nucleon. We then define $Q^2 = -q^2 > 0$ so that $x = Q^2/2p.q$ and $y = p \cdot q / p \cdot q_1 = (E - E')/E$. Both x and y range between 0 and 1. The hadronic information in the differential cross sections is contained in the dimensionless structure functions $F_L(x, Q^2)$ (longitudinal) and $F_1(x, Q^2)$ (transverse). Often the combination $F_2(x, Q^2) = 2xF_1(x, Q^2) + F_L(x, Q^2)$ is used in place of F_1. When we consider the weak (parity-violating) reactions, mediated by the charged W and the neutral Z bosons, then there is an additional structure function $F_3(x, Q^2)$. Let us assume a proton target and write the structure functions as $F^{\gamma,Z,W}_{i=1,2,3,L}$. In the leading order parton model description they are given by sums of parton densities which do not depend upon the variable Q^2. These were discussed in Sec. 6.5. In higher order QCD additional gluons and quark/antiquark pairs are radiated so the $F^{\gamma,Z,W}_{i=1,2,3,L}$ become Q^2 dependent. Also their decomposition into quark flavours becomes more complicated. The general formulae which contains these effects can be written as

$$
\begin{aligned}
F_i^{V,V'}(x,Q^2) \quad = \quad & \int_x^1 \frac{dz}{z}[\Sigma_{k=1}^{n_f}(v_k^{(V)}v_k^{(V')}+a_k^{(V)}a_k^{(V')}) \\
& \times\Big[\Sigma(\frac{x}{z},\mu^2)\,\mathcal{C}_{i,q}^S(z,\frac{Q^2}{\mu^2})+G(\frac{x}{z},\mu^2)\,\mathcal{C}_{i,g}(z,\frac{Q^2}{\mu^2}) \\
& +\Sigma_{k=1}^{n_f}(v_k^{(V)}v_k^{(V')}+a_k^{(V)}a_k^{(V')}) \\
& \times\Delta(\frac{x}{z},\mu^2)\,\mathcal{C}_{i,q}^{NS}(z,\frac{Q^2}{\mu^2})\Big]] \quad ,i=1,2,L \qquad (17.64)
\end{aligned}
$$

and

$$
\begin{aligned}
F_3^{V,V'}(x,Q^2) \quad = \quad & \int_x^1 \frac{dz}{z}[\Sigma_{k=1}^{n_f}(v_k^{(V)}v_k^{(V')}+a_k^{(V)}a_k^{(V')}) \\
& \times\Big[V_k(\frac{x}{z},\mu^2)\mathcal{C}_{3,q}^{NS}(z,\frac{Q^2}{\mu^2})\Big]\,. \qquad (17.65)
\end{aligned}
$$

Here the electromagnetic, vector and axial vector electroweak couplings of the photon, W and Z bosons are given by $v_k^{(V)}$ and $a_k^{(V)}$ where $V=\gamma,Z$ and W. The parton densities are given a numerical label so that $k=-5,-4,-3,-2,-1,0,1,2,3,4,5$ stand for the $\bar{b}$, $\bar{c}$, $\bar{s}$, $\bar{d}$, $\bar{u}$, g, u, d, s, c, b partons respectively. The top quark is too heavy to be regarded as a partonic constituent of a proton. The flavor singlet (Σ) and flavour non-singlet (Δ_k, V_k) denote combinations of parton densities

$$
\begin{aligned}
&\Sigma(z,\mu^2)=\frac{1}{n_f}\Sigma_{k=1}^{n_f}(f_k(z,\mu^2)+f_{\bar{k}}(z,\mu^2)) \\
&\Delta_k(z,\mu^2)=f_k(z,\mu^2)+f_{\bar{k}}(z,\mu^2)-\Sigma(z,\mu^2) \\
&V_k(z,\mu^2)=f_k(z,\mu^2)-f_{\bar{k}}(z,\mu^2)\,, \qquad (17.66)
\end{aligned}
$$

where f_k and $f_{\bar{k}}$ are quark and antiquark densities of species k respectively. Note also that the parton densities satisfy the evolution equations we discussed in the previous section,

$$
\frac{d}{d\ln\mu}f_p(x,\mu^2)=\Sigma_{p'}\int_x^1\frac{dz}{z}P_{pp'}(\frac{x}{z},\alpha_s)f_{p'}(z,\mu) \qquad (17.67)
$$

where p and p' refer to the quarks, antiquarks and gluons. These are matrix equations as we discused previously.

The coefficient functions $\mathcal{C}$, which appear in the flavour singlet and non-singlet terms, can be calculated in higher order perturbation theory. They depend on both the variables z and Q^2/μ^2, where μ^2 is the renormalization/mass factorization scale. In leading order (LO) perturbation theory they are either $\delta(1-z)$ functions or do not appear at all as in the case of the gluon and have no Q^2 dependence.

The above formulae are for spin summed/averaged reactions. There are also structure functions (usually denoted by a prefix Δ) for deep inelastic processes with incoming polarized leptons and/or polarized targets. We will not discuss them here.

Note that one way to solve the above formulae is to rewrite it to contain double integrals using the general formula

$$F(x) = \int_0^1 dy \int_0^1 dz \delta(x - yz) G(y) H(z) \,, \tag{17.68}$$

and there is a method of reducing the latter equation to an algebraic equation by taking the Mellin transform. This involves multiplying both sides of the equation by x^{N-1} where N is a positive integer and defining

$$F_N = \int_0^1 dx x^{N-1} F(x) \,, \tag{17.69}$$

with corresponding results for G and H. Then the right-hand-side of the equation splits into a product

$$F_N = G_N H_N \,, \tag{17.70}$$

which can be solved algebraically. To reconstruct $F(x)$ one takes the inverse Mellin transform, namely

$$F(x) = \int \frac{dN}{2\pi \mathrm{i}} x^{-N} F_N \,, \tag{17.71}$$

where the integration contour is a vertical path in the complex N plane from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ with all the singularities on the left of the contour.

The discussion of the corrections to the Drell-Yan reaction can be modified to cover the deep-inelastic case. One splits off the leptonic piece using trace tricks as before and then removes numerical factors to normalize the LO coefficient functions so that they contain $\delta(1-z)$. Then one calculates the amplitudes for the scattering of a virtual space like photon on a quark to produce a quark and a gluon namely $\gamma(q) + q(p_1) \to q(p_2) + g(k)$. This reaction is the crossed amplitude from the Drell-Yan one. When the gluon momentum is small or parallel to the momenta of the quarks there are soft and collinear singularites in the square of this amplitude. Therefore one has to compute all cross sections in n dimensions and extract the delta function terms when the integral over two body phase space becomes singular. When one adds the iinterference terms between the LO amplitude and the one loop correction together with the square of the bremsstralung graphs then all pole terms in $\epsilon = n - 4$ cancel apart from one term whose residue contains the same factor $P_{qq}(z)$ which arose in the Drell-Yan calculation. This is the quark splitting function and it is removed by a renormalization of the transition amplitude. Therefore this splitting function is seen to be universal in next-to-leading order processes, also $P_{\bar{q}\bar{q}} = P_{qq}$ The renormalization group arguments given previously show that these splitting functions control the scale dependence of the parton densities. There is a remainder which yields the $C^S_{i,q}$, $C^{NS}_{i,q}$ functions in this order in perturbation theory. The reactions where the virtual photon interacts with the anti-quarks yield the same answers. There there is a contribution from gluon initial sates in this order from the reaction $\gamma(q) + g(k) \to q(p_1) + \bar{q}(p_2)$. Now the final quark or antiquark can become soft or collinear to the initial state gluon leading to poles in ϵ with residue factors $P_{qg}(z)$ and $P_{\bar{q}g}(z)$ which are part of the evolution equations for the parton densities. This gives contributions to the coefficient function $C_{i,g}$ which are listed below. Finally to complete the set one needs the contributions from the neutrino anti-neutrino deep inelastic scattering reactions to $F_3(x, Q^2)$. These next order contributions are also listed below.

Look up the results from Bardeen Buras et al for all the coefficient functions $C^{(1)}_{1,q}$ and $C^{(1)}_{1,g}$ and maybe put in a table?

17.4. Jets

We have seen that the comparison between experiment and theory for the ratio R is an excellent test of the application of QCD and the ideas of asymptotic freedom to hadron physics. Also the analysis of QCD corrections to deep inelastic scattering and dilepton production are in agreement with theoretical predictions. However these tests are indirect. They do not provide direct physical evidence for the existence of the non-abelian gauge boson. To do this one must study more exclusive processes where gluons are produced. Since the gluon carries colour we will never see it directly. Rather we will detect some colourless hadrons which result from the many gluons and quarks that in turn were form by a fragmentation process from the gluons and quarks produced in the hard scattering. The process that converts these final gluons and quarks into hadrons is called hadronization and involves nonperturbative physics. If there is some hope of correlating the original partons with the final hadrons then we must examine measureable distributions which are insensitive to the details of the hadronization process. Said differently, this means that they are independent of long distance or soft physics effects.

For example consider the reaction $e^+e^- \to q\bar{q}g$. While the amplitude for this process is calculable from the QCD and QED lagrangians, (it is part of the expression for $\mathrm{Im}\Psi^{(4)}$ in (9.154) and of R in (10.111)), we recognize immediately that it is both infrared divergent and mass divergent because the gluon is massless. The Kinoshita-Lee-Nauenberg theorem states that mass singularities are absent in physical processes if all degenerate initial and final states are summed over.

The total cross section for e^+e^- collisions requires a sum over the final states with a $q\bar{q}$ pair (with virtual gluon exchanges) and with a $q\bar{q}g$ with a zero energy gluon the KLN theorem states, and is therefore free from mass singularities, as we have seen in chapters 8 and 9. For other observables, one may not have a sum over degenerate states, so that these singularities do not cancel. In many cases one can then absorb the singularity, if it is unverisal, it in some nonperturbative part of the problem, as we did for initial state collinear singularities in the previous sections.

In the reaction $e^+e^- \to$ "γ" $\to q\bar{q}g$ the virtual photon does not couple to the gluon, while, if we insist on detecting the gluon, the amplitude is infrared divergent. There is simply no nonperturbative aspect of the initial state where we can bury a mass singularity. So we must find an observable that shows the presence of an extra gluon and yet allows a sum over generate states.

To this end, let us consider the notion of a jet. In low energy e^+e^- collisions hadronic resonance states are produced at low s, which do not have any particular structure so particles are emitted uniformly in phase space. At sufficiently high energy, however, when the photon couples directly to a quark-antiquark pair. The latter are produced back-to-back in the e^+e^- cm frame. As the quarks begin to fly apart they undergo a complicated fragmentation or hadronization process to produce colourless hadronic final states. One could therefore expect the final hadrons to follow the line of flight of the quarks to produce two showers of back to back particles. This may be further motivated by the fact that large angle, hard emissions from the primary quarks are suppressed, due to asymptotic freedom, relative to soft and collinear emissions.

Since the angular distribution of the quarks is $1 + \cos^2\theta$ where θ is the polar angle between a quark and the beam direction, the angular distribution of the hadrons should have roughly the same dependence on $\cos\theta$. We may ask how this angular distribution is changed by the emission of an additional gluon. Remember the amplitude for gluon emission contains soft and collinear divergences. However, intuitively, if we integrate the the gluon emission rate over a small angle close to the quark or antiquark direction the collinear divergence should cancel with the divergence contained in the virtual contribution, leaving a contribution depending on the size of the angular range. Also, if we allow for a soft gluon to be emitted and add this contribution to that from the virtual contribution we expect that the infrared divergences will cancel too. The result will still depend on one angle and one energy. One therefore defines a two jet event as one where almost all of the energy, namely $(1-\epsilon)\sqrt{s}$, is contained in two small cones of semiangle δ, where ϵ and δ are fixed, and can be reasonably large, as shown in the Figure.

Figure 17.2: The two-jet process

An explicit calculation of the corrections to the (anti)quark angular distribution (see problem) shows that the angular distribution is not changed but the coefficient is modified by the factor $1 - \alpha_s(q^2)/\pi C_2(R)\Big[(4\ln 2\epsilon + 3)\ln\delta + \pi^2/3 - 7/3 + 0(\epsilon) + 0(\delta)\Big]$. If one would take ϵ and $\delta \to 0$ divergences would show up again, so one must be careful not to choose ϵ or δ so small that the term in α_s is still a relatively small correction. The jet angular distribution is therefore well defined and has been successfully compared with experiment.

It is possible to generalize the definition of jet such that singularities still cancel, but that their definition are more easily implemented in both experimental measurements and theoretical calculations, the latter then most profitably in the form of a Monte Carlo program, see section 7.3. In practice one has to invent an iterative algorithm for combining the detected hadrons into jets. Starting point is a list of of the detected particles with their observed energies and angles. Then, in one method, for all particle pairs i and j one calculates the quantity $y_{ij} = 2E_iE_j(1-\cos\theta_{ij})/s$. All y_{ij}'s are then compared with a chosen value $y_{\rm cut}$. If a y_{ij} is smaller than $y_{\rm cut}$ then the two particle momenta are combined according some prescription, for instance "add the four-momenta", and the result is then returned to the list of particle as a new 'particle'. The procedure is then repeated until it stops. This subdivides the hadronic events into a number of 'jets', of which one can study the properties. One should be aware however that not all algorithms are safe for all collider types.

The above result triggered a search for new variables which would be infrared safe from the theoretical viewpoint and easy to measure experimentally. These new variables, called *event shapes*, describe properties of final state configurations in e^+e^- collisions but have also been generalized to other reactions. A fundamental assumption here is that the characteristics of the perturbative QCD result are not lost in the nonperturbative hadronization process. If a jet-like group of hadrons is detected one can, instead of a jet-algorithm, define an energy flow direction in the hope that it is in direction of the initial quark.

Since singularities from the collinear splitting of a massless quark or gluon into another two massless quarks and/or gluons, we have to define a quantity at the quark (gluon) level which is free of this singularity. Such a definition should, at the hadron level, be insensitive to inclusion of a soft or nearly-collinear hadron. The latter requirement is based on plausibility arguments, since nothing about the hadronization mechanism can be proven without an understanding of confinement

For instance the sum of the longitudinal momenta of a set of hadrons still respects a cancellation of mass singularities at each vertex in the chain. In contrast, a sum of squares of longitudinal momentum distributions would probably not allow the cancellation of the singularities and is therefore infrared divergent. Check squares statement

As an example of such infrared safe event shape, one can study the maximum directed momentum, or thrust T, in e^+e^- collisions. The thrust is defined by T=max($\hat{n}$) $\sum_i |\vec{p}_i \cdot \vec{n}| / \sum_i |\vec{p}_i|$ where p_i are the momenta of the particles and the three-vector $\vec{n}$ is varied until a maximum value of T is obtained. Thus a thrust value can be given to every event measured in e^+e^- interactions. It varies between $T = \frac{1}{2}$ for a spherical energy flow and $T = 1$ for a pencil-like linear energy flow. If a detector is divided into two hemispheres individual thrust determinations can be done one the part of the event in each hemisphere..

Let us illustrate this discussion by the calculation of the thrust distribution for the reaction $e^+e^- \to q\bar{q}g$. The Feynman diagrams are the same as in Fig. – . The kinematical situation is that of a heavy photon decaying into three massless particles so we can use Dalitz plot variables to describe the final state. From our discussion in Chapter 3 we know that we can choose two energies to specify the allowed region in a Dalitz plot and would be flat for a constant matrix element. Here it is more convenient to choose invariant mass variables rather than energies, so, using the particle name to represent its four momenta, we introduce $s_{12} = -(q+g)^2$ and $s_{23} = -(\bar{q}+g)^2$. Since the three final particles are coplanar, the whole kinematics is specified by s_1, s_2 and 3 angular variables. One angular variable θ specifies the polar angle between the beam axis and a line in the three particle plane. Another azimuthal angular variable χ specifies the orientation of the plane will respect to this line and finally there is an overall azimuthal angle. These variables are depicted in Fig. – where the χ axis is chosen in some convenient direction

The phase space for the final three particles therefore becomes

$$\frac{1}{(2\pi)^5} \int \frac{d^3q}{2E_q} \int \frac{d^3\bar{q}}{2E_{\bar{q}}} \int \frac{d^2g}{2E_g} = \frac{1}{(2\pi)^5} \int \frac{d^4q}{32q^2} ds_1 ds_2 d\chi d\sin\theta d\phi \qquad (17.72)$$

The expression for the cross-section after squaring the matrix element for $e^+e^- \to q\bar{q}g$ and integrating over χ and ϕ gives

$$\frac{d^3\sigma}{ds_1 ds_2 d\sin\theta} = \frac{\alpha_e^2}{8} \frac{\alpha_s}{q^2} \left({x_1}^2 + {x_2}^2\right) \left(2 + \cos^2\theta\right) \frac{1}{s_1 s_2} \qquad (17.73)$$

where the variables $x_i = E_i/E$ are related to the invariant mass variables by $s_{12} = \theta^2(1-x_3), s_{23} = \theta^2(1-x_1), s_{31} = \theta^2(1-x_2)$. From (–) we see that the angular distribution of the normal to the plane with respect to the beam line is given by $3 - \cos^2\theta$. An integration over $\sin\theta$ yields the two equivalent expressions.

$$\sigma_T^{-1}\frac{d^2\sigma}{ds_1 ds_2} = \frac{2}{3\pi}\alpha_s\frac{x_1^2 - x_2^2}{s_{13}s_{23}} \tag{17.74}$$

or

$$\sigma_T^{-1}\frac{d^2\sigma}{dx_1 dx_2} = \frac{2}{3\pi}\alpha_s\frac{{x_1}^2 + {x_2}^2}{(1-x_1)(1-x_2)} \tag{17.75}$$

where $\sigma_T = \frac{4}{3}\pi\alpha_e^2/s$, the $e^+e^- \to \mu^+\mu^-$ cross section used to set the scale. These distributions diverge for small invariant masses, or, equivalently as the scaled energies of the quark and antiquark tend to unity. A plot of the allowed region is shown in Fig. —, noting that $x_1 + x_2 + x_3 = 2$

The lines parallel to the sides of the large triangle, at 1/2 the height namely, EF, FD and DE are the lines $x_1 = 1, x_2 = 1$ and $x_3 = 1$ respectively. The thrust variable is the largest energy quark or gluon for this 3 body final state so for every event T=max (x_1, x_2, x_3) where $\frac{2}{3} \leqslant T \leqslant 1$. If we subdivide the final phase space into three regions depending on which particles has the larges x value then, for $T = x_{\bar{q}}$

$$\begin{aligned}
\sigma_T^{-1}\frac{d\sigma}{dT} &= \frac{2\alpha_s}{3\pi}\int dx_q dx_g \delta(2 - x_q - x_g - T)\theta(T - x_q)\theta(T - x_g) \\
&\frac{T^2 + {x_q}^2}{(1-x_q)(1-T)} \\
&= \frac{2}{3}\frac{\alpha_s}{\pi}\int_{2(1-T)}^{T} dx\frac{T^2 + x^2}{(1-T)(1-x)} \\
&= \frac{2}{3}\frac{\alpha}{\pi}\left\{\frac{1+T^2}{1-T}\ln\frac{2T-1}{1-T} + \frac{3T^2 - 14T + 8}{2(1-T)}\right\},
\end{aligned} \tag{17.76}$$

with an identical result for $T = x_q$. The $T = x_g$ case is different, and following the same method we find

$$\sigma_T^{-1}\frac{d\sigma}{dT} = \frac{4\alpha_s}{3\pi}\left\{\frac{1 + (1-T)^2}{T}\ln\frac{2T-1}{1-T} + 2 - 3T\right\} \tag{17.77}$$

Note that the thrust distributions for the quark and antiquark are singular as $T \to 1$. In that region non-perturbative effects are important so our results should be smeared by an intrinsic $< P_T >$. Since the integral of the gluon T distribution is finite at $T = 1$, we can integrate (–) from $2/3 \leqslant T \leqslant 1$ to find the probability that the gluon is the most energetic particle. This yields

$$\sigma_T^{-1}\int_{2/3}^{1}\frac{d\sigma}{dT}dT = 0.61\frac{\alpha_s}{\pi} \tag{17.78}$$

where α_s is a function of q^2. Thus the total probability that the gluon is the most energetic particle decreases with increasing q^2. The probability that the quark or the antiquark is the most energetic particle is therefore given by $(1 - 0.61\alpha_s/\pi)$.

The total thrust distribution is therefore two times (–) plus (–) namely

$$\sigma_T^{-1}\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi}\left[\frac{2(3T^2 - 3T + 2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] \quad (17.79)$$

The average value of $(1 - T)$ follows from this formula because the integrand is integrable at $T = 1$. We find

$$\sigma_T^{-1}\int(1-T)\frac{d\sigma}{dT}dT = 1.05\frac{\alpha_s(q^2)}{\pi} \quad (17.80)$$

so the average value of (1-T) diminishes as q^2 increases. Note that the Dalitz plot in Fig. (–) has specific symmetries, if we cannot tell which particle is which, so the events can be ordered in the sequence $x_1 > x_2 > x_3$ and then the integrations are performed over a restricted region of the plot. Since the thrust variable only specifies the jettiness in one direction called the thrust axis, it is usual to define the geometry of an event by setting up two orthogonal axes to the direction of maximum energy flow. The second axis, called the major one, is found by making the energy flow i.e., $\sum_i |\vec{p}_i \cdot \vec{e}| \sum_i |\vec{p}_i|$ a maximum for $\vec{e} \perp \{\vec{e}_1\}$. Finally the third direction or minor axis $\vec{e}_3$ is prependicular to both $\vec{e}_1$ and $\vec{e}_2$. These axes set up a geometrical system in space for each e^+e^- event and distinguish the topology of the final energy flow. In the thrust-major plane the 3 jet nature of the event is singled out, whereas in the thrust-minor plane the event looks like a 2 jet event because we have rotated the event to make it flat. A measure of the difference in the energy flow along the major and minor axes is called the oblateness defined by

$$0_B = (\sum_i |\vec{p}_i \cdot \vec{e}_2| - \sum_i |\vec{p}_i \cdot \vec{e}_3|)/\sum |\vec{p}_I| \quad (17.81)$$

There are several other parameters which are used by experimental groups to study the structure of the energy flow. However it is not necessary to go into more details here. In fig. – we show typical results from the high energy e^+e^- colliding beam machine PETRA in Hamburg. The three yet nature of the event is clearly visible and this three jet structure has been widely heralded as the discovery of the gluon.

The 3 jet structure does not immediately check the spin properties of the gluon. To do this one has to look at an angular distribution which is sensitive to the gluon spin. Since it is the events at smaller T values which indicate the presence of gluons one can take these events and boost them backwards along the thrust axis into the rest frame of the other two jets. Since this new system has a different thrust axis the distribution in the angle between it and the boost direction is a quantity which is sensitive to the spin of the gluon. In Fig. – we show the experimental data and the theoretical distribution in the cosine of this angle for scalar and vector gluons. Obviously the data favour vector gluons.

To summarize the QCD predictions for gluon radiation are in agreement with experiment and therefore provides evidence for the existence of a massless spin-1 gluon.

Problems

17.1. Work out the corrsesponding results when the divergences are regulated by introducing a small gluon mass m_g. Show that the corresponding two and three particle cross sections are

$$\sigma_2^{(m_g)}(Q^2) = \sigma_0 C_2(R)\left(\frac{\alpha_s(Q^2)}{\pi}\right)[-2\ln^2(Q/m_g) + 3\ln(Q/m_g) - \frac{7}{4} + \frac{\pi^2}{6}]\,, \quad (1)$$

$$\sigma_3^{(m_g)}(Q^2) = \sigma_0 C_2(R)\left(\frac{\alpha_s(Q^2)}{\pi}\right)[2\ln^2(Q/m_g) + 3\ln(Q/m_g) - \frac{5}{2} + \frac{\pi^2}{6}]\,, \quad (2)$$

respectively. Notice that, although these two expressions are similar to the n-dimensional results, they really have a different dependence upon the unphysical regulator parameter m_g. This is a sign that there are differences in the long-distance behaviors of the theories.

17.2. Consider the scattering process

$$q(p_1) + \bar{q}(p_2) \to V(q) + g(k) \quad (1)$$

in which a massless quark and anti-quark annihilate to form a vector boson V of mass Q and a massless gluon. In this problem we compute the relevant phase-space integral in n dimensions. It is given by

$$I(n, Q^2, 0) = \int \frac{d^{n-1}q}{(2\pi)^{n-1}2\omega_q}\int \frac{d^{n-1}k}{(2\pi)^{n-1}2\omega_k}(2\pi)^n\delta^{(n)}(p_1 + p_2 - q - k) \quad (2)$$

where $\omega_q^2 = Q^2 + \vec{q}^2$, $\omega_k^2 = \vec{k}^2$, and $\vec{q}$, $\vec{k}$ the $(n-1)$-dimensional spatial parts of the n-dimensional vectors q^μ, k^μ. With phase space integral one can compute infrared divergent cross sections in dimensional regularization.
To simplify this expression, argue that one may choose a convenient Lorentz frame in which

$$p_1^\mu = \frac{\sqrt{s}}{2}(1, 0, \ldots, 0, 1), \quad p_2^\mu = \frac{\sqrt{s}}{2}(1, 0, \ldots, 0, -1)\,, \quad (3)$$

were $s = -(p_1 + p_2)^2$. In this $n-1$-dimensional phase-space we may choose our angular variables as we did for loop integrals in (9.9) and (9.10). Assuming that the squared invariant amplitude will only depend on the angle θ_{n-2} and on $|\vec{k}|$, show that

$$I(n,Q^2,0) = \frac{1}{4\sqrt{s}}(4\pi)^{(4-n)/2}\frac{1}{2\pi\Gamma((n-2)/2)} \times \int_0^\infty d|\vec{k}||\vec{k}|^{n-3}\delta(|\vec{k}|f\frac{s-Q^2}{2\sqrt{s}})\int_0^\pi d\theta_{n-2}\sin^{n-3}\theta_{n-2} \quad (4)$$

17.3. Show that the squared invariant amplitude for the scattering process in Problem 17.2 in $n = 4+\epsilon$ dimensions, summed over all spins and polarizations, as well as number of colors N_c is given by

$$|\mathcal{M}|^2 = 16g^2e^2q_f^2N_cC_2(R) \times\left(\frac{s^2/2}{(p_1\cdot k)(p_2\cdot k)} + \frac{p_1\cdot k}{p_2\cdot k} + \frac{p_2\cdot k}{p_1\cdot k} - \frac{2p_1\cdot p_2}{p_2\cdot k} - \frac{2p_1\cdot p_2}{p_2\cdot k} + \mathcal{O}(\epsilon)\right) \quad (1)$$

where eq_f is the electric charge of the quark, and g is the quark-gluon coupling constant.
We now concentrate on the dominant term in the soft gluon limit (this is known as the *eikonal approximation*), which is the first term in (1). To compute the cross section according to the discussion in section 3.3, including spin averaging, we must compute

$$\frac{1}{2^2}\frac{1}{2s}\int\frac{d^{n-1}q}{(2\pi)^{n-1}2\omega_q}\int\frac{d^{n-1}k}{(2\pi)^{n-1}2\omega_k}(2\pi)^n\delta^{(n)}(p_1+p_2-q-k) \times 8g^2e^2q_f^2N_cC_2(R)\left(\frac{(2p_1\cdot p_2)^2}{(p_1\cdot k)(p_2\cdot k)}\right) \quad (2)$$

Use the result of problem 17.2 to show that this is proportional to

$$\int_0^\infty d|\vec{k}||\vec{k}|^{n-5}\delta(|\vec{k}|-\frac{s-Q^2}{2\sqrt{s}})\int_0^\pi d\theta_{n-2}\sin^{n-5}\theta_{n-2}\,. \quad (3)$$

Argue now that the cross section for this process contains both infrared and collinear divergences, and compute these divergent terms.

17.4. In this problem we analyse infrared and collinear divergences in a one-loop integral

$$J(s,0,0) = \frac{1}{(2\pi)^n}\int\frac{\mathrm{d}^nq}{((p_1+q)^2-\mathrm{i}\varepsilon)((p_2-q)^2-\mathrm{i}\varepsilon)(q^2-\mathrm{i}\varepsilon)}\,, \quad (1)$$

where $s = -(p_1+p_2)^2$ (we have followed the notation of (9.29)). Use Feynman parameters to show that J can be written as

$$\frac{2}{(2\pi)^n}\int_0^1 dx\int_0^{1-x}dy\int\frac{\mathrm{d}^nq'}{\left[(q^2+K^2-\mathrm{i}\varepsilon)\right]^3}\,,\quad K^2 = -2xyp_1\cdot p_2 = -xys \quad (2)$$

Show that the result may be written as

$$\frac{\mathrm{i}}{(4\pi)^{n/2}}\left(\frac{-s}{\mu^2}\right)^{\frac{n}{2}-3}(\mu^2)^{\frac{n}{2}-3}\frac{4}{(n-4)^2}\frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}n-1\right)^2}{\Gamma(n-3)}\,. \tag{3}$$

and that the divergences so obtain are not ultraviolet in origin.

17.5. The loop integral in problem 17.4 can be written as

$$J(s,0,0)=\frac{1}{(2\pi)^n}\int d^nq\frac{1}{D_1D_2D_3} \tag{1}$$

$$=\frac{2}{(2\pi)^n}\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\delta(1-x_1-x_2-x_3)$$

$$\times\int d^nq\frac{1}{[x_1D_1+x_2D_2+x_3D_3]^3} \tag{2}$$

$$D_1=q^2-\mathrm{i}\varepsilon,\quad D_2=(p_1+q)^2-\mathrm{i}\varepsilon,\quad D_3=(p_2-q)^2-\mathrm{i}\varepsilon\,. \tag{3}$$

To examine infrared and collinear divergences we use the second form of the integral and analyse where the denominator, viewed as a function of the integration variables $\{q^\mu, x_i\}$, is zero. Using that all integrals, on account of the iε prescription, may be considered as integrals over complex variables, argue that this happens for values of the Feynman parameters near their endpoints 0, while for q^μ the conditions

$$D\equiv x_1D_1+x_2D_2+x_3D_3=0,\quad \frac{\partial D}{\partial q^\mu}=0 \tag{4}$$

must be satisfied. Together, these conditions are known as the Landau equations. Show now that, for our case, the following three solutions exist for the Landau equations

$$C_1: x_3=0,\quad q^\mu=zp_1^\mu,\quad x_1z=x_2(1-z)\,, \tag{5}$$

$$C_2: x_2=0,\quad q^\mu=-z'p_2^\mu,\quad x_1z'=x_3(1-z')\,, \tag{6}$$

$$S: q^\mu=0,\quad x_2/x_1=x_3/x_1=0\,. \tag{7}$$

Construct now parametrizations for the integration variables near these singular regions, and demonstrate that in dimensional regularization each produces a $1/\epsilon$ divergence. Those from solutions C_1, C_2 are collinear divergences, and from solution S a soft divergence. Argue that when these overlap, a $1/\epsilon^2$ divergence arises. Show that they all have a superficial degree of divergence that is logarithmic.

17.6. *(Note to students: while the draft of this chapter in general needs more editing, the equations referred to below are correct).* Use the plus-distribution identities in (17.57) to derive the expression (17.58) for the real contribution to the Drell-Yan partonic cross section from (17.53). Then, add the virtual contribution in (17.45) and show that (17.60) results.

In the next step we verify the cancellation of the remaining divergence in (17.60). Starting point is the factorization that expresses the hadronic cross section as a convolution of partonic densities and the partonic cross section

$$\frac{d\sigma_{AB}(\tau)}{d\hat{s}} = \sum_{i,j} \int_{\xi_{1,\min}}^{1} d\xi_1 \int_{\xi_{2,\min}}^{1} d\xi_2 f_{i/A}(\xi_1) f_{j/B}(\xi_2) \frac{d\sigma_{ij}^{(0)}(\xi_1, \xi_2)}{d\hat{s}} \quad (1)$$

where $\hat{s}$ is the square of the off-shell photon mass, and $\tau = \hat{s}/s$. Use the fact that all parton distributions $f_{i/A}(\xi)$ are non-zero only for $0 < \xi < 1$ to show that the substitution in (17.62) does indeed remove the collinear divergence.