16

Quantum chromodynamics

Quantum chromodynamics (QCD) is the fundamental theory of the strong interactions. According to this theory hadrons consist of quarks and gluons. The quarks were originally proposed to explain the regularities seen in the meson and baryon spectra. They were regarded as subunits or building blocks of the hadrons and were assigned noninteger electric charges. Extensive searches were made to find isolated quarks, so far without any success. However, deep-inelastic scattering showed compelling evidence for fractionally charged constituents in the proton, which at short distances behave like pointlike particles. The discovery that nonabelian gauge theories can be asymptotically free, thus suggested that the binding force between the quarks is provided by such a gauge theory. The degrees of freedom corresponding to the gauge fields are called gluons. These particles have not been observed directly. In this chapter we shall not give a full account of the history of the subject. We start with a brief introduction to the quark model, discussing various implications of 'colour' and 'flavour'. Experimental evidence for quantum chromodynamics is reviewed. The phenomenon of colour confinement, which should explain the absence of quarks and gluons as free particles, is briefly discussed in the context of lattice gauge theory. The main part of this chapter is devoted to perturbative quantum chromodynamics. We discuss its principal starting points and demonstrate a number of characteristic features and applications, such as partonic cross sections, Drell-Yan scattering and jets.

16.1. Quarks and gluons; flavour and colour

It is presently believed that six types of quarks exist, three with electric charge $\frac{2}{3}$ and three with electric charge $-\frac{1}{3}$ (measured in units of the elementary charge). They are denoted by u, c, t and d, s, b, respectively, abbreviations of exotic names such as 'up', 'charm', 'top', and 'down', 'strange', 'bottom'. These quark types are called *flavours*. As they are not detected as separate physical particles, their masses are not really known, but can be estimated from hadron spectroscopy once the hadron composition in terms of quarks is given. For obvious reasons these mass values are called "constituent masses". One commonly introduces quantum numbers to distinguish the type of quark, which then explains the corresponding quantum numbers of the hadronic bound states. The quark flavours and constituent masses are summarized in table 16.1. Of course, there are also corresponding antiquarks $\bar{u}, \bar{c}, \bar{t}$ and $\bar{d}, \bar{s}, \bar{b}$, with opposite charges. The lightest-mass mesons and baryons are bound states of quarks and/or antiquarks with zero angular momentum, in a way that we will discuss later.

Table 16.1: Quarks of different flavours, grouped in three doublets according to the generation structure of the standard model. The mass values correspond to the so-called constituent mass. The quantum numbers corresponding to electric charge, weak isospin, strangeness, charm, bottom and top are denoted by Q, T_3, S, C, B and T.

flavour	mass [GeVc^{-2}]	Q	T_3	S	C	B	T
u	≈ 0	$\frac{2}{3}$	$\frac{1}{2}$	0	0	0	0
d	≈ 0	$-\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	0
c	≈ 0	$\frac{2}{3}$	$\frac{1}{2}$	0	1	0	0
s	≈ 0	$-\frac{1}{3}$	$-\frac{1}{2}$	1	0	0	0
t	≈ 0	$\frac{2}{3}$	$\frac{1}{2}$	0	0	0	1
b	≈ 0	$-\frac{1}{3}$	$-\frac{1}{2}$	0	0	1	0

For the moment let us restrict our attention to a single quark flavour. In order to let a nonabelian gauge group act on the quark field, we are forced to extend the number of fields. According to QCD, this gauge group is SU(3). We shall try to justify this choice for the gauge group in due course and first consider the theory. In order that SU(3) can act nontrivially on the quark field $q(x)$, this field must have at least three components, so we write $q_\alpha(x) = (q_1(x), q_2(x), q_3(x))$. Hence for a given quark flavour, we have three different fields. These three varieties are called *colours* and are denoted by 'red', 'green' and 'blue'. Of course, this assumption tends to make matters worse. We started with one quark for each flavour, which cannot be observed as a free particle; now we have three times as many quarks, none of them observed in experiments. Actually, the problem is even more pressing. Because quarks rotate under an SU(3) symmetry group, one should expect a corresponding degeneracy for the observed bound states. In other words, each hadronic state is in general degenerate and will carry colour, while all other properties such as mass, electric charge and the like are independent of colour. Clearly there is not such an exact degeneracy in Nature. Nevertheless, let us ignore this apparent proliferation of degrees of freedom for the moment and turn to the other ingredients of the model. As the group SU(3) is eight-dimensional, we must have eight gauge fields denoted by ${V_\mu}^a$. Under SU(3) the quark fields transform in the triplet representation, viz.

$$q(x) \to q'(x) = \exp\left(\tfrac{1}{2} i \lambda_a \xi^a(x)\right) q(x) \; , \qquad (16.1)$$

where $\xi^a(x)$ are the eight transformation parameters of SU(3), and $q(x)$

represents the three-component column vector q_α consisting of the three quark colours. The conjugate quark fields are represented by the row vector $\bar{q}_\alpha = (\bar{q}_1, \bar{q}_2, \bar{q}_3)$ and transform according to

$$\bar{q}(x) \to \bar{q}'(x) = \bar{q}(x) \exp\left(-\tfrac{1}{2} i \lambda_a \xi^a(x)\right) . \tag{16.2}$$

The invariant Lagrangian now takes the standard form, as given in chapter 11,

$$L = -\tfrac{1}{4}(G_{\mu\nu}{}^a)^2 - \bar{q}\not{D}q - m\,\bar{q}q , \tag{16.3}$$

with

$$\begin{aligned} G_{\mu\nu}{}^a &= \partial_\mu V_\nu{}^a - \partial_\nu V_\mu{}^a - g f_{bc}{}^a \, V_\mu{}^b V_\nu{}^c , \\ D_\mu q &= \partial_\mu q - \tfrac{1}{2} i g \, V_\mu{}^a \lambda_a \, q . \end{aligned} \tag{16.4}$$

The SU(3) generators $t_a = \frac{1}{2} i\lambda_a$ are expressed in terms of a standard set of matrices λ_a, which are generalizations of the Pauli matrices τ_a (cf. section 11.2). They are listed in appendix G. The SU(3) structure constants $f_{bc}{}^a$ follow from the commutators of these generators, which are antihermitian. Following the definitions introduced previously, our structure constants will differ by a sign from the ones that are often used in the literature, which follow from taking the commutators of the matrices $\frac{1}{2}\lambda_a$ (i.e. without including the factors i into the generators). With this convention the nonlinear term in the field strength $G_{\mu\nu}{}^a$ in (16.3) will then have a different sign.

For other flavours, the QCD Lagrangian takes the same form as in (16.3), except that the actual value for the quark-mass parameter is different. The full Lagrangian thus depends on the QCD coupling constant g and on the mass parameters m, one for each flavour (quarks of different colour but of the same flavour should have the same mass in order to conserve the SU(3) gauge symmetry). Here we stress that the mass parameter in the Lagrangian should *not* be identified directly with the constituent mass listed in table 16.1. The mass parameters in the Lagrangian are sometimes called the 'current masses', for reasons that we will not explain here (we already made this distinction in section 8). Obviously, the QCD interactions leave the flavour of the quarks unchanged. This implies that the strong interactions conserve the quantum numbers listed in table 16.1. However, with the exception of the electric charge, these quantum numbers are not conserved by the weak interactions. In other words, quarks can change their flavour by weak interaction processes. This feature will be discussed fully when turning to the standard model in chapter 20. The gluons do not carry flavour, but they do carry colour, as they transform under the SU(3) gauge group. Hence the quark content of the hadrons can be probed by weak and electromagnetic interactions. At this point we recollect the deep-inelastic scattering results discussed in chapter 6. We found there that approximately 50% of the momentum of the proton is carried by the constituents that do *not* interact with the electromagnetism or weak interactions. According to QCD these constituents are the gluons.

For completeness, let us stress that the quarks also carry spin indices, as they are normal Dirac spinor fields. Hence, quark fields carry three different types of indices. One index is the spinor index, which takes four values. Then there is the colour index, denoted above by $\alpha, \beta, \ldots$, which takes three values. Finally we can assign a flavour index, which takes six values corresponding to the different flavours.

By construction the QCD Lagrangian is invariant under local SU(3). However, depending on the values for the mass parameters, there can also be a number of flavour symmetries. The presence of these flavour symmetries has direct consequences for the hadronic bound states. The flavour symmetries are most relevant for the light quarks. As the mass parameters of the u and d quarks are comparable in size the QCD Lagrangian tends to be invariant under (constant) unitary rotations of the u and d quarks. These rotations form the group $\mathrm{U}(1) \otimes \mathrm{SU}())$ (i.e. products of a phase factor with a unitary 2×2 matrix with unit determinant). The invariance under U(1) is related to the conservation of baryon number (quarks carry baryon number $\frac{1}{3}$, antiquarks $-\frac{1}{3}$). The SU(2) transformations are just the isospin transformations, which constitute an approximate invariance in Nature. The breaking of isospin invariance is thus due to the fact that the u and d mass parameters are not quite equal (an additional but small breaking is caused by the electroweak interactions, which are not considered in this chapter). Because the u and d mass parameters are not only equal but are also very small, the Lagrangian has even more approximate flavour symmetries. For vanishing quark mass the Lagrangian is invariant under unitary transformations of the u and d fields that may also contain the matrix γ_5. Such transformations are called *chiral* transformations. Because of the presence of γ_5, the transformations of the quarks will depend on their spin. The effect of these extra transformations is more subtle because the chiral symmetry is realized in a so-called spontaneously broken way. Spontaneous symmetry breaking will be discussed in chapter 18. The fact that the pion mass is so small (as compared to the other hadron masses) can be explained by an approximate chiral symmetry in Nature.

Obviously, we may follow the same strategy when including the s quark and consider extensions of the flavour symmetry group. Apart from the phase transformations one then encounters an SU(3) flavour group (not to be confused with the SU(3) colour group). In view of the fact that the s quark has a much higher mass, flavour SU(3) is not as good a symmetry as isospin. Symmetry breaking effects are usually of the order of 10%. Of course one may consider further extensions by including γ_5 into the tranformation rules or by including even heavier quarks. However, these extensions of the flavour symmetries tend to be less and less useful as they are affected by the large quark masses and thus no longer correspond to approximate symmetries of Nature.

16.2. Colour degeneracy and confinement

As explained in the previous section, in order to realize the SU(3) gauge tranformations on the quark fields, we are forced to introduce three varieties of quarks, prosaically denoted by colours. However, it seems inevitable that the observed hadrons, bound states of quarks and antiquarks, will also exhibit the colour degeneracy. For instance, the pions are thought of as bound states of a u or a d quark with a $\bar{u}$ or a $\bar{d}$ antiquark. Since quarks and antiquarks come in three different colours, one has in principle *nine* types of pions of given electric charge, which must have equal mass. Altogether there are then twenty-seven types of pions, rather then the three found in Nature!

The reason why this colour degeneracy is not observed in Nature is a rather subtle one. To explain this phenomenon, let us start by considering quarks of a single flavour, say u quarks, and construct the possible states consisting of three quarks, each one at rest. Together they form a state with zero angular momentum. Depending on the properties of the forces acting between these quarks, the three quarks may or may not cluster into a hadronic bound state. By comparing the properties of these three-quark states to those of the low-mass hadrons in Nature (in view of the centrifugal barrier one expects that states with nonzero angular momentum acquire higher masses) one may hope to unravel the systematics of quark spectroscopy and understand the nature of the forces that hold the hadrons together.

Although the states that we consider do not carry angular momentum, they will carry spin because the u quarks have spin $\frac{1}{2}$. Therefore the three-quark states can combine into spin-$\frac{3}{2}$ or spin-$\frac{1}{2}$ states. Actually, there are two different ways to form spin-$\frac{1}{2}$ states, as is expressed by the multiplication rule

$$\mathbf{\tfrac{1}{2}} \otimes \mathbf{\tfrac{1}{2}} \otimes \mathbf{\tfrac{1}{2}} = \mathbf{\tfrac{3}{2}} \oplus \mathbf{\tfrac{1}{2}} \oplus \mathbf{\tfrac{1}{2}} \ . \tag{16.5}$$

This product rule is easy to understand. A spin-$\frac{1}{2}$ particle has two different states (distinguished by the value $\pm\frac{1}{2}$ of its spin measured along some direction in space), so that a combination of three such particles leads to $2^3 = 8$ possible spin states. It is easy to write down all these states by distinguishing them according to their total spin directed along a certain direction, say the positive z-axis. Obviously there is one state with $S_z = \frac{3}{2}$ (in units of $\hbar$), corresponding to the situation where the three quark spins are aligned along the positive z-axis. Then there are three different states with $S_z = \frac{1}{2}$, when the spins of two of the quarks are directed along the positive z-axis, while one of the quarks has its spin pointing in the opposite direction. Similarly, one finds three states with $S_z = -\frac{1}{2}$ and one state with $S_z = -\frac{3}{2}$, so that altogether we have eight different states. According to the rule (16.5), they arrange themselves into subsets of either four or two states; under rotations the states in a given subset tranform among themselves as the four states of a spin-$\frac{3}{2}$ particle or the two states of a spin-$\frac{1}{2}$ particle. So the eight states decompose under the action of the rotations into three different spin representations: $4+2+2=8$.

Hence, when considering the possibility of the three quarks forming a bound state, one may expect the emergence of a spin-$\frac{3}{2}$ bound state and/or one or two spin-$\frac{1}{2}$ bound states. We recall that, by construction, these states have zero angular momentum. Of course, whether or not they are actually realized as bound states depends on the properties of the interquark forces.

However, the above conclusions are invalidated as we are dealing with bound states of *identical* spin-$\frac{1}{2}$ particles. These particles are fermions, so they satisfy Pauli's exclusion principle, according to which the resulting state should be *antisymmetric* under the exchange of any two such particles. It turns out that the spin-$\frac{3}{2}$ bound state is, however, *symmetric* under the interchange of two fermions. This is easy to see for the states with $S_z = \pm\frac{3}{2}$, as they correspond to the situation where all three quark spins are aligned in the same direction. Hence a spin-$\frac{3}{2}$ bound state cannot be realized because of Pauli's exclusion principle. However, the spin-$\frac{1}{2}$ states cannot be realized either, as they are neither symmetric nor antisymmetric under the interchange of any two particles, but are of mixed symmetry (i.e., they can be (anti)symmetric under the exchange of two of the quarks, but not with respect to the third quark). Therefore bound states of three identical spin-$\frac{1}{2}$ particles with zero angular momentum do not exist.

Surprisingly enough, when comparing the result of such quark model predictions to the low-mass baryons in Nature, one finds that there is in fact a bound state of three u quarks with spin-$\frac{3}{2}$, namely the Δ^{++} baryon with a mass of 1232 MeV c^{-2}, which is unstable and decays primarily into $p\,\pi^+$ with an average lifetime of 0.6×10^{-23} sec.. On the other hand, no spin-$\frac{1}{2}$ bound states of three u quarks are found. At this point one could of course question the quark interpretation of the Δ^{++}, were it not for the fact that this phenomenon is universal! When comparing the quark model to the data, it turns out that the baryons always correspond to bound states of quarks that are *symmetric* rather than antisymmetric under the interchange of two quarks. Therefore the Pauli principle is violated in the simple quark model.

Before resolving this puzzle, let us once more exhibit this phenomenon, but now for the slightly more general case of low-mass baryons consisting of u and d quarks. Each quark in the baryon now comes in four varieties: a u quark with spin 'up' or 'down' (measured along some direction in space) or a d quark with spin 'up' or 'down'. Assuming again zero total angular momentum, there are thus $4^3 = 64$ possible spin states, twenty of which are symmetric under the interchange of two particles. These symetric states decompose into sixteen states with both isospin and ordinary spin equal to $\frac{3}{2}$, and four states with both isospin and ordinary spin equal to $\frac{1}{2}$. The first sixteen states correspond to the baryons $\Delta^{++}(uuu)$, $\Delta^{+}(uud)$, $\Delta^{0}(udd)$ and $\Delta^{-}(ddd)$, which carry spin-$\frac{3}{2}$ so that each one of them appears in four possible spin states (we listed the quark content in parentheses). The latter four states correspond to the nucleons $p(uud)$ and $n(udd)$, which carry spin-$\frac{1}{2}$ and thus appear in two varieties.[1] No other states corresponding to bound states of three u or d quarks can be identified with baryons in Nature (of course, for higher masses such bound states can be found, but those will have nonzero angular momentum).

Rather than exploring all the subtleties of the simple quark model in detail, let us turn to quantum chromodynamics. Because the quarks carry colour one can make the three-quark state antisymmetric by postulating total antisymmetry in the three colour indices. In this way the exclusion principle is again preserved. This conjecture may seem rather ad hoc, and one may wonder whether there is an a priori reason for assuming antisymmetry in the colour indices. Indeed, it turns out that there is a principle behind this. When antisymmetrizing over the colour indices of a three-quark state, this state is a singlet under the SU(3) colour group. This follows under the multiplication rule

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,, \tag{16.6}$$

which yields a singlet state under colour SU(3) that is fully antisymmetric.[2] Assuming that all hadrons carry no colour (so that they are *invariant* under the colour gauge group) requires the three-quark states to be antisymmetric in the colour indices. By virtue of Pauli's exclusion principle, they must therefore be *symmetric* with respect to all other quantum numbers, such as spin and isospin.

This principle that hadrons should be colourless, can be put to a test when considering the low-mass mesons. As we mentioned at the beginning of this section, the mesons are bound states of a quark and an antiquark. Because of the three-fold degeneracy of the quarks associated with colour, each meson should appear in nine varieties, which differ in colour, but not in electric charge and mass. However, one particular combination of these states is again colourless. This follows from the product rule

$$\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\,, \tag{16.7}$$

[1] As explained above, the spin-$\frac{1}{2}$ states are of mixed symmetry. However, the mixed symmetry in terms of the spin indices of the quarks can be combined with the mixed symmetry of the isospin indices in such a way that the resulting state becomes symmetric (see problem 16.5). The relevant multiplication rule for isospin coincides with (16.5).

[2] The interpretation of this product rule is as before. The $3^3 = 27$ states formed by all possible products of SU(3) triplet states decompose under the action of SU(3) in four different representations: the singlet representation, which is completely antisymmetric, the **10** representation, which is completely symmetric, and two **8** representations, which have mixed symmetry (the SU(3) representations are denoted by their dimension, unlike the represenions of the rotation group, which are denoted by the value of the spin). To derive such product rules is more complicated for SU(3) than for the SU(2), which is the relevant group for spin and isospin.

according to which the nine colour states decompose into a singlet state and eight states belonging to the octet representation. Only the singlet state is realized as a physical particle, so that the colour degeneracy is avoided. This turns out to be a universal feature of all hadrons. We simply never observe the colour degrees of freedom, but only bound states of quarks that are singlets of the colour symmetry group. In other words if we assign the primary colours to $\alpha = 1, 2, 3$ then the observed hadrons must be white. Of course, this analogy is rather picturesque and by no means necessary.

This rule may also be applied to possible bound states of gluons. Such states are called 'glueballs'. Bound states of two or more gluons could exist provided they are in a colour singlet state. Current estimates put their mass between 1 and 2 GeV/c^2. However, no positive identification of gluonic states has been made, although there are certain candidates. In principle, the gluonic bound states carry no flavour, but they can mix with quark-antiquark states, which makes it harder to define a clear experimental signature.

Although there are still some open questions, it is fair to say that the simple quark model extended with colour and the principle that only colour singlets are realized as physical states, is very successful. While respecting the exclusion principle it is able to predict all the low-mass hadrons, their approximate mass values and their quantum numbers. Also for the heavy quark flavours the model is in good agreement with experiment. Of course, the crucial question is to understand the reasons behind colour confinement. An explanation of this phenomenon must somehow be based on the dynamics of QCD. Some of this will be discussed in section 16.3.

In spite of the fact that colour cannot be observed directly the colour degeneracy has a variety of consequences, some of which can be experimentally observed. The fact that there are precisely three diffent colours often leads to factors of 3 in decay rates or cross sections. For instance, in chapters 4 and 7 we calculated the decay rate for $\pi^0 \to \gamma\gamma$ via a quark loop model with equal mass u and d quarks. After using the Goldberger-Treiman relation for the quarks, we found that the decay rate was proportional to the square of the quark charges multiplied by their axial vector coupling constants. The theoretical $\pi^0 \to \gamma\gamma$ rate turned out to be approximately 0.9 eV whereas the experimental rate is approximately 8 eV, indicating that a factor of roughly 9 is missing. However, because of the colour degeneracy there are three u and d quarks and we will have to sum over those when we calculate the $\pi^0 \to \gamma\gamma$ decay amplitude, so that each type of quark makes a separate contribution. With three different quark colours we thus obtain an extra factor of three in the amplitude and a factor of nine in the $\pi^0 \to \gamma\gamma$ decay rate. Assuming fractionally charged quarks, the colour degeneracy factor is thus needed in order to obtain agreement with experiment.

Another occasion where the colour degeneracy plays a role is the determination of the total cross section for the reaction $e^+e^- \to$ hadrons. At moderate energies (up to 20 GeV per beam), this reaction is predominantly mediated by virtual photon exchange because the weak coupling via intermediate Z bosons is small. Hence the cross section is determined by the electric charges of the quarks. As we saw in Chapter 6, the ratio $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ approaches a constant at sufficiently high energies where the masses of the final particles can be neglected. We calculated R and found the result $R = \sum_i q_i^2$ where q_i is the charge of the quark (in units of the elementary charge e) with flavour i. As the total center-of-mass energy increases, then R should increase in steps as the thresholds for the excitation of new flavour degrees of freedom are passed, because we know that the masses of the c and b quarks are much heavier than the masses of the light quarks. Thus R should be $(2/3)^2 + (1/3)^2 + (-1/3)^2 = 2/3$ below the threshold for charm production and $(2/3)^2 + (1/3)^2 + (1/3)^2 + (2/3)^2 = 10/9$ above it. After subtracting the background contribution from the leptonic reaction $e^+e^- \rightarrow \tau^+\tau^-$, the experimental data show a clear jump from approximately 2 to 10/3 as is shown in fig 6.4. So one sees that a factor of three is missing in both regions. This factor is naturally explained by a threefold quark degeneracy for all the u, d, s and c quarks because we have to sum over the additional degrees of freedom in the final state (just like we sum over photon polarization degrees of freedom after squaring the amplitude).

Yet another example where the colour factor is relevant is in comparing the leptonic and hadronic decays of the W boson. We will elaborate on this elsewhere.

Needless to say, none of the above observations consitute in themselves compelling evidence for colour. Only when considering the experimental facts in a global perspective one comes to the conclusion that QCD must be the underlying theory of hadrons. However, at this point such a conclusion would be premature and it is best to first turn to the dynamical aspects of the theory and discuss its most conspicuous features, asymptotic freedom and confinement.

16.3. Quantum chromodynamics

A unique property of nonabelian gauge theories is that they can be asymptotically free. The phenomenon of asymptotic freedom was already explained in chapters 9 and 13 and implies that the strength of the gauge-field interaction tends to vanish at small distances. Having accepted the idea that hadrons consist of quarks and gluons of three different colours, the assumption that the binding forces are provided by an SU(3) gauge theory thus offers a natural explanation why one observes weakly-bound pointlike constituents in the hadrons when probing them with very energetic leptons. Further support for this idea comes from the fact that the quantum numbers of these constituents are precisely those of the quarks (the gluons are electrically neutral and are not directly observed). The essential point is that for an asymptotically free theory one is able to test these ideas in a quantitative fashion, because one may use perturbation theory for high-energy reactions. This has led to so-called perturbative QCD. Although perturbation theory cannot be applied in all situations, as we shall explain in section 4.4, we can really establish in this way that quantum chromodynamics is the fundamental field theory of hadrons.

The nonperturbative behaviour of quantum chromodynamics is much more difficult to understand. The most intriguing question is how to explain the fact that only colour singlets are realized as hadronic bound states. Apparently, colour is confined in quantum chromodynamics. This is a large-distance phenomenon; when a quark or a gluon tries to break out from a hadron, the force that binds it increases indefinitely with the distance, thus preventing its escape. In scattering processes, colour degrees of freedom cannot be produced on a long enough time scale to become observable as physical particles. Unlike asymptotic freedom, this phenomenon of 'infrared slavery' is nonperturbative in nature. As nonperturbative treatments are cumbersome in field theory it is difficult to demonstrate explicitly that the theory does indeed lead to colour confinement, although there are many indications that this is indeed the case.

When comparing quantum chromodynamics to quantum electrodynamics it is clear that the two theories are crucially different. In the latter theory electric charges are not confined and the theory is not asymptotically free. Furthermore, the gauge fields are not self-interacting as quantum electrodynamics is an abelian gauge theory. The infrared divergences associated with the masslessness of the photon can be dealt with by combining the cross section for a nonradiative process with those for the radiative processes with any number of (soft) photons. Owing to its nonabelian nature quantum chromodynamics is realized in a completely different way. The treatment of the infrared divergences is much more problematic, it is asymptotically free and colour is confined. The self-interactions make the theory much more nonlinear, so that the interactions between quarks are not dominated by simple gluon-exchange diagrams. When gluons are exchanged this nonlinear behaviour suppresses the spread of the gluonic fields in the transverse directions, so that stringlike field configurations dominate. These 'strings' are thought to be responsible for colour confinement. The strings have quarks or antiquarks at their endpoints. When the quarks try to move apart, the strings are stretched leading to an increase in the interquark force. This force will increase until the string breaks by the creation of a quark-antiquark pair. In this way a colourless hadron breaks up into one or more colourless hadrons. The string-like behaviour would also provide an explanation for the existence of straight Regge trajectories for hadronic resonances.

It is illuminating to make the comparison with a phenomenon that takes place in superconductivity and shares characteristic features with the situation described above. In superconductivity one has the Meissner effect, according to which external magnetic fields below a certain critical value are expelled from the superconductor. When the magnetic field is sufficiently increased in strength, the magnetic flux can pierce holes in the (type-II) superconductor, where locally the material ceases to be in a superconducting phase. As the flux cannot spread out, thin flux tubes are formed through the superconductor and as it turns out the flux through these tubes must be quantized in units of elementary flux equal to $2\hbar/q$, where q is the electric charge of the relevant degrees of freedom of the superconductor (the so-called Cooper pairs, whose charge is equal to twice the elementary charge). The energy of the flux tube per unit of length is proportional to q^{-2}. The possibility of these flux configurations was demonstrated by Abrikosov. Later, in the context of relativistic field theory, they became known as Nielsen-Olesen flux tubes. Now suppose that one has a big volume of superconducting material in which magnetic monopoles are present. Since the flux from these monopoles cannot spread out, the monopoles of opposite magnetic charge will be connected by thin flux tubes, whose energy is proportional to their length. This causes a confinement of magnetic charges in precisely the way as described earlier and we may call this superconducting phase the *magnetic confinement* phase. Inside the superconductor the electromagnetic fields are screened, which reflects itself in an effective mass of the photon field; this mass is proportional to q. Clearly the mass generation of the photon can be discussed in the context of perturbation theory in terms of q, while the magnetic confinement mechanism cannot be dealt with in this way.

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Although we are just dealing with electrodynamics, the superconductor provides the necessary ingredient to realize the theory in a different phase. The ordinary phase where the photon is massless, is called the Coulomb phase. Then there is the magnetic confinement phase, where the photon acquires a mass and magnetic charges are confined. The phase that we are interested in is the *electric confinement* phase, which is in some sense dual to the magnetic confinement phase. In that case, electric charges will be confined but now magnetic charges can exist in asymptotic states. Already at the semiclassical level, the duality between electric and magnetic phenomena is a characteristic ingredient of the physics of magnetic monopoles and *dyons* , particles that carry both electric and magentic charges.

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Our hope is now that QCD can be realize in similar phases. The confinement phase of QCD should thus correspond to *colour* electric confinement, while the classical theory with massless gluons obviously relates to the Coulomb phase. Magnetic confinement is not immediately relevant for QCD. The magnetic confinement phase, also called the Higgs phase, turns out to be relevant for the electroweak interactions. We shall return to the description of this phenomenon in later chapters. One may wonder whether possible alternative phases of QCD can be realized. In that case it may be possible to have the theory in a deconfined phase. There are many speculations that the confinement mechanism will break down under extreme circumstances such as high temperatures or high densities, which could for instance be realized in extremely energetic heavy-ion collisions and it seems likely that the deconfined phase was realized at very early stages of the expansion of the universe. In that case free quarks and gluons will exist. It should be extremely exciting to detect signals of deconfinement in laboratory experiments.

There is another aspect of QCD that is nonperturbative in nature and should be mentioned at this place. Naively, in perturbation theory, we consider an expansion of the action about a minimum. As we are interested in translationally and Lorentz invariant situations, this usually implies that all fields are expanded about zero, with the exception of the scalar fields, which could have a constant value. A situation where this is certainly too naive, is when the potential has a periodic structure. In that case, it is questionable whether one should set up perturbation theory by just expanding about one particular minimum. As an example, consider, in ordinary quantum mechanics, a periodic potential $V(x) = \alpha(1 - \cos\beta x)$, which has minima at $x_n = 2\pi n/\beta$ with n an odd integer. Near a minimum x_n the potential behaves as $V(x) \approx \omega(x - x_n)^2$, where $\omega = \alpha\beta^2$.

Standard: shift $x \to 2\pi/\beta$ invariance. shift operator commutes with the Hamiltonian, eigenfunctions of this operator. Unitary operator $\exp i\theta$ Tunneling.

Turns out: QCD the same degeneracy. Massless fermions. PQ symmetry. strong CP violation.

Figure 16.1: Periodic potential

16.4. *Perturbative QCD*

The starting point for perturbative QCD is the observation that at zero coupling constant the beta function vanishes while its first vanishing derivative is *negative*, at least when the number of quarks is not too large. Therefore one has in principle the possibility for obtaining reliable results at high energies by means of perturbation theory. In chapter 13 we evaluated the beta function in the one-loop approximation, which suffices to establish asymptotic freedom. Higher-order contributions can be found in the literature and we quote the following result for a nonabelian gauge theory coupled to $n_{\rm f}$

fermions transforming in a represention denoted by R,

$$\begin{aligned}\frac{\beta(g)}{g} \quad &= \quad -\tfrac{1}{3}\Big(11C_2(G) - 4n_\mathrm{f}\, C_2(R)\,\tfrac{\dim R}{\dim G}\Big)\,\frac{g^2}{16\pi^2} \\ &\quad -\tfrac{1}{3}\Big(34C_2^2(G) - n_\mathrm{f}\,\big[20C_2(G) + 12C_2(R)\big]C_2(R)\tfrac{\dim R}{\dim G}\Big)\,\Big(\frac{g^2}{16\pi^2}\Big)^2 \\ &\quad +\mathrm{O}\big(g^6\big)\,. \end{aligned} \tag{16.8}$$

Even higher orders in g^2 are known, but they depend on the renormalization scheme, unlike the first two terms, which are scheme-independent, at least in a perturbative context (see the discussion in section 9.4).

If we now specialize to the case of QCD with the gauge group SU(3), fermions in the fundamental (triplet) representation and Casimir invariants equal to $C_2(G) = 3$ and $C_2(R) = 4/3$, then (14.d1) becomes

$$\frac{\beta(g)}{g} = -\frac{33 - 2n_\mathrm{f}}{3}\,\frac{\alpha_s}{4\pi} - \frac{306 - 38n_\mathrm{f}}{3}\,\Big(\frac{\alpha_s}{4\pi}\Big)^2 + \mathrm{O}\big(\alpha_s^3\big)\,. \tag{16.9}$$

Here we introduced the notation $\alpha_s \equiv g^2/4\pi$.

Let us briefly summarize the situation. As explained in section 9.3, one can use the renormalization group equation to compare Green's functions at different momentum configurations differing by an overall scale factor e^t. The generic result for the n-point Green's function is given in (9.47),

$$G^{(n)}(\mathrm{e}^{-t_1}p_i;\alpha_s(t_1),\mu) = G^{(n)}(e^{-t_2}p_i;\alpha_s(t_2),\mu)\;\exp\Big\{\int_{t_1}^{t_2}\mathrm{d}\tau\, d_G(\alpha_s(\tau))\Big\}\,, \tag{16.10}$$

where the various momenta are denoted by p_i and d_G is the anomalous dimension of the Green's function. Furthermore $\alpha_s(t)$ is the so-called running coupling constant (for the moment we ignore other possible coupling constants or masses for reasons of simplicity), which satisfies the differential equation

$$\frac{\mathrm{d}\alpha_s(t)}{\mathrm{d}t} = 2\alpha_s(t)\,\frac{\beta(g(t))}{g(t)} = -\beta_0\,\alpha_s^2(t) - \beta_1\,\alpha_s^3(t) + \mathrm{O}\big(\alpha_s^4(t)\big)\,. \tag{16.11}$$

This relation uniquely describes the relation between $\alpha_s(t_1)$ and $\alpha_s(t_2)$. From (14.d4) it follows that the constants β_0 and β_1 are given by

$$\beta_0 = \frac{33 - 2n_\mathrm{f}}{6\pi}\,, \qquad \beta_1 = \frac{153 - 19n_\mathrm{f}}{12\pi^2}\,. \tag{16.12}$$

For less than nine flavours, both constants are positive.

The differential equation (14.d6) is straightforward to solve. By integrating we find

$$t - t_0 = \int_{\alpha_s(t_0)}^{\alpha_s(t)}\frac{\mathrm{d}x}{\beta(x)}\,, \tag{16.13}$$

where t_0 is an arbitrary integration constant. The above equation leads to the algebraic relation (as can be directly verified by differentiation with respect to t),

$$\frac{1}{\alpha_s(t)} + \frac{\beta_1}{\beta_0} \ln \frac{\beta_0\, \alpha_s(t)}{\beta_0 + \beta_1\, \alpha_s(t)} - \beta_0\, t = \text{constant}\,, \tag{16.14}$$

which can be solved numerically or by a series expansion (cf. 9.67). A common approach is to choose the constant equal to $-\beta_0\, t_0 - (\beta_1/\beta_0)\ln(\beta_1/\beta_0)$, so that (14.d9) reads

$$\frac{1}{\alpha_s(t)} + \frac{\beta_1}{\beta_0} \ln \frac{\beta_1\, \alpha_s(t)}{\beta_0 + \beta_1\, \alpha_s(t)} = \beta_0(t - t_0)\,, \tag{16.15}$$

and t_0 characterizes the point where $\alpha_s(t_0)$ becomes infinite. Obviously for large t the running coupling constant vanishes proportional to $1/t$, in accord with asymptotic freedom. We can solve the equation now by expanding in inverse powers of $\beta_0(t - t_0)$, which leads to the result

$$\alpha_s(t) = \frac{1}{\beta_0(t - t_0)} \Big\{ 1 - \frac{\beta_1}{\beta_0^2\,(t - t_0)} \ln \frac{\beta_0^2\,(t - t_0)}{\beta_1} \Big\} + \mathrm{O}\big([\beta_0(t - t_0)]^{-3}\big) \tag{16.16}$$

The quantity $t - t_0$ represents the logarithm of the ratio of two momentum scales, say $\frac{1}{2}\ln s/\Lambda^2$, where s is the square of a characteristic momentum or momentum transfer in the process one is considering and Λ is the QCD scale parameter (often written as $\Lambda_{\rm QCD}$) at which the running coupling constant diverges. Note that the expansion is reliable for large $t - t_0$, while the actual value for Λ is sensitive to what happens for $t \approx t_0$ and characterizes the scale at which perturbation theory will break down. Experiment allows us to fit the solution for large t, but the way in which $\alpha_s(t)$ tends to infinity for $t \to t_0$ depends on the approximation. In first-order, where β_1 is suppressed, $\alpha_s(t)$ diverges as $(t - t_0)^{-1}$ and in the next order $\alpha_s(t)$ diverges as $(t - t_0)^{-2}$, as the reader can easily verify from (14.d10). When fitting the above formula to experiment, the value for Λ will thus depend on the approximation that is being used. Of course all results depend on the number of relevant flavours, i.e., those flavours whose mass is lower than the characteristic energy scale of the process. When this energy is increased one eventually crosses thresholds where additional flavours become relevant. The coefficients in (14.d6) have to be changed accordingly, which has the effect of decreasing the rate by which the running coupling constant tends to zero at large t. Of course, one must impose some continuity requirement across the thresholds, which also influence the experimental value of Λ. [3]

[3] We [may] return to this in due course. Check with decoupling section.

While an asymptotically free theory is in principle amenable to perturbative calculations as far as its behaviour at short distances is concerned, the long-distance behaviour cannot be reliably calculated in perturbation theory (at least not in the continuum formulation). The central problem in perturbative QCD is therefore to disentangle the short- from the long-distance effects. The perturbative calculations are based on the conventional methods of perturbative field theory applied to the QCD Lagrangian. Here one works in terms of gluons and quarks so that it is not directly obvious how to make contact with experiments, as those refer to physical hadrons. The relation between hadrons and the QCD degrees of freedom as described by gluons and quarks, involves the long-distance properties of QCD. In order to make the transition from quarks and gluons to hadrons one must ensure that the perturbative calculations are not sensitive to the (nonperturbative) long-distance features of the theory. In the perturbative results these features manifest themselves as divergences, caused by the fact that the gluons are massless and the quarks have masses that become negligable at asymptotic energies. These divergences are generically called *infrared* divergences and will be encountered already at the level of quark and gluon cross sections. Sensible results can be obtained for quantities that are free of these divergences. Such quantities are called *infrared safe*.

In previous chapters we already encountered infrared divergences, mainly in the context of quantum electrodynamics. They occur both in loop integrals from virtual photons and in phase-space integrals from incoming or outgoing real photons. In principle infrared divergences are of two types. One type of divergence is called *soft* and arises from contributions of zero-momentum photons. Of course, at zero-momentum one cannot distinguish between a real and a virtual photon; a zero-momentum virtual photon is just a real photon that is emitted and reabsorbed. A second type of divergence (to be discussed in section 14.5) arises from massless particles moving with finite momenta in nearly parallel directions. These are the so-called *collinear* singularities. In this section we shall not concern ourselves with the details but discuss the generic features of the treatment of both types of infrared divergences. As it turns out, one of the crucial aspects of massless particles is that we are dealing with degenerate states. For instance, in quantum electrodynamics, it is not meaningful to distinguish between a single electron and an electron accompanied by a number of zero-momentum photons, as the corresponding states carry the same electric charge, energy and momentum. An (infinite) degeneracy of states implies that a naive application of perturbation theory may run into difficulties, a phenomenon that is, for instance, also known from applications in ordinary quantum mechanics. According to the Kinoshita-Lee-Nauenberg theorem the divergences that are in principle present in partial transition probabilites, must cancel when averaging over a suitable set of degenerate states. This theorem encompasses the so-called Bloch-Nordsieck theorem, which was invoked in section 4.6 in the treatment of K_S decay. There the bremsstrahlung of soft photons was shown to give rise to divergences, which cancel against the infrared divergences due to virtual photons (i.e. loop corrections) in the nonradiative decay rate. In order that this cancellation could take place, we had to sum over two kinds of final states, namely states consisting of two charged pions and states consisting of two charged pions accompanied by a soft photon. (More soft photons will be encountered when extending the calculations to higher orders of perturbation theory.) Such a cancellation was explicitly established in section 8.6 for the decay of a massive virtual photon, again after combining the radiative and the nonradiative decays. However, in that calculation we also verified that the combined result remains finite in the limit of vanishing muon mass, so that (in that order of perturbation theory) we deal with a decay into two and three *massless* particles. All these cancellations can be understood as a consequence of the Kinoshita-Lee-Nauenberg theorem.

Hence it is clear that infrared safe quantities can be obtained provided one averages over a sufficiently large ensemble of degenerate states. How large this ensemble should be depends on the experimental process that one is considering. For instance, in electron-positron annihilation at high energies, the *total* cross section (where one sums over *all* possible finite states) consitutes an infrared safe quantity, for which one can obtain reliable predictions by means of perturbative QCD. In fact we have already demonstrated this in section 9.5 (see, in particular (9.106), where we determined the leading two terms in the total cross section). Strictly speaking this calculation yields the annihilation cross section into states consisting of quarks, antiquarks and gluons. After their production these degrees of freedom should somehow neutralize their colour and recombine into physical hadrons. This is a nonperturbative process, which is hard to describe theoretically. On the other hand, at high centre-of-mass energies, the time scale that is relevant for the creation of (anti)quarks and gluons is very short as compared to the typical time scale for the formation of physical hadrons. Therefore we can reliably calculate the short-distance process by perturbative methods and rely on the fact that the conversion into physical hadrons has probability unity, so that it will not modify our prediction for the total cross section.

When we are interested in more exclusive reactions in electron-positron annihilation, the situation is considerably more complicated. In order to obtain an infrared safe result we must sum over at least a subset of the final states. This is done by considering *jets* : we sum over all the (anti)quarks and gluons that move nearly parallel in a cone of a small but finite angle (see later discussion). In this way the infrared divergences cancel. Then we have to describe the conversion of this spray of (anti)quarks and gluons into the physical hadrons. Again, this process is assumed to take place at a much longer time scale than the initial perturbative process, and involves the long-distance nonperturbative aspects of QCD. In practice the conversion into the various hadronic final states is described by so-called fragmentation functions, which are determined from experiment.

Electron-positron annihilation is relatively simple because there are only hadrons in the final states and not in the incoming one. This is not so for deep-inelastic scattering, where we scatter a lepton off a hadron, and for Drell-Yan reactions, where two hadrons collide and form (among others) a virtual photon or weak vector boson that decays into leptons. The presence of hadrons in the incoming state gives further complications. For deep-inelastic scattering there are two approaches. The first makes use of the so-called operator product expansion, in which the product of two operators at neighbouring space-time points is expressed into a sum of local operators taken at one of these space-time points multiplied by coefficient functions that depend on their difference. These coefficient functions are in general singular in the limit that both space-time points coincide. The coefficient functions, or at least their singular parts, can be calculated in perturbation theory. The operators in this approach are expressed in terms of the QCD fields. and the nonperturbative part resides in the matrix elements of these operators. Again those are not accessable to theoretical determination and they are taken from experiment.

The second approach, which is applicable both to deep-inelastic and to Drell-Yan reactions, makes use of the parton model. The partonic cross sections are determined in perturbation theory in such a way that they are infrared safe, and are used as input in the theoretical expressions for these reactions. A crucial ingredient here is the factorization theorem, which ensures that the infrared divergences in the partonic cross sections can be factored out in a uniform manner order-by-order in perturbation theory. For the incoming parton states, these divergences are then absorbed by defining so-called parton distribution functions, which are fitted to experiment. Therefore one experiment, such as deep inelastic scattering is used to fix these functions. After this is done one can make predictions for a different reaction, for instance the Drell-Yan reaction discussed in section 6.6, so long as the same parton distribution functions are used as input[4]. For deep inelastic scattering the result of the parton approach agrees with that obtained from the operator-product expansion, which gives a reliable theoretical basis to the whole subject.

A key feature of perturbative QCD is thus the existence of two different time scales, one for the 'hard' scattering processes between the quarks and gluons, which yields the large scale in α_s, and one relevant for the 'soft' nonperturbative processes responsible for the conversion of hadrons into quarks and leptons, and vice versa. While the time scale for the hard process is short in the limit where a characteristic energy or momentum transfer becomes

[4] A somewhat troublesome feature in the context of Drell-Yan reactions is that the quarks in the incoming states must be massless in order that the required factorization takes place.

large, the time scale relevant for the rearrangement into hadrons is more or less insensitive to the value of the energy or momentum transfer and thus remains much longer. Under those conditions the experimental reaction can be decomposed into a part referring to the hard scattering and a part referring to the conversion into hadrons. The hard reaction can be evaluated in perturbation theory, up to certain unknown (and often infrared-divergent) factors. These factors are uniform and do not depend on the process one is studying, provided one remains within the same calculational scheme. They contain the nonperturbative long-distance physics that is hard to access by theoretical calculations, and can be absorbed into quantities that can be measured in different experiments.

In the remainder of this section we present QCD results for the total cross section of electron-positron annihilation and introduce some of the characteristic features of perturbative QCD calculations in the context of dimensional regularization. The latter with a view towards the applications in subsequent sections, where we discuss the calculation of partonic cross sections, giving a more detailed account of the infrared divergences in the initial and final states. We will show how these divergences are extracted in order to obtain finite results. Furthermore we study the Drell-Yan reaction, making use of the partonic cross sections, describe the comparison of different reactions with experiment and the results for the parton distribution functions. We also discuss the treatment of exclusive reactions in terms of jet cross sections.

The first-order QCD corrections to $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ were already derived in section 9.5. The results are given in (9.109). The calculation made use of the renormalization-group equation applied to the photon vacuum-polarization function to lowest order in the fine-structure constant, modified by QCD corrections to order α_s. Owing to the the Kinoshita-Lee-Nauenberg theorem the quark mass could be suppressed without encountering infrared divergences (we remind the reader that in order for this theorem to apply, it is necessary that the ultraviolet renormalization procedure does not introduce spurious mass singularities). Let us now quote the first three terms in the perturbation series for R, in the limit of zero fermion mass,

$$\begin{aligned} R(t) \;&= (\textstyle\sum q^2)\Big\{1 \quad +\frac{\alpha_s(t)}{4\pi}3C_2(R) \\ &\qquad +\Big(\frac{\alpha_s(t)}{4\pi}\Big)^2\Big[-\tfrac{1}{2}C_2^2(R) + (\tfrac{123}{2} - 44\zeta(3))\,C_2(G)\,C_2(R) \\ &\qquad\qquad +n_{\rm f}(-22 + 16\zeta(3))\,C_2^2(R)\,\tfrac{\dim R}{\dim G}\Big]\Big\} \qquad (16.17) \end{aligned}$$

where $(\sum q^2)$ denotes the square of the electric charges of the fermions, summed over all colours and flavours. As before, t is represents the logarithm of the ratio of two energy scales, one the total centre-of-mass energy of the incoming electron-positron pair and the other the some low-energy reference point, usually characterized by the QCD scale parameter Λ. Obviously, the running coupling constant should be evaluated to the same order as the cross section. Through $\alpha_s(t)$ this result thus depends on the number of relevant quark species.

The Riemann zeta function typically appears in higher-loop calculations. It is defined by $\zeta(z) = \sum_{n=0}^{\infty} 1/n^z$. The specific value encountered above is $\zeta(3) \approx 1.2020569$. Using these values, the numerical coefficients for SU(3) with five quark flavours yield

$$R(t) = (\sum q^2)\Big(1 + \frac{\alpha_s(t)}{\pi} + 1.409\Big(\frac{\alpha_s(t)}{\pi}\Big)^2\Big)\,. \tag{16.18}$$

As discussed in chapter 9.5 the result (14.d12) should not be applied in the vicinity of heavy flavour thresholds, because bound states appear in $R(t)$ which are not describable in perturbation theory.

A derivation of these results can be given in the context of dimensional regularization. Because the quantities that one calculates are infrared safe, infrared divergences must cancel at the end of the calculation. However, this requires one to determine the full cross section in n dimensions and take the limit $n \to 4$ only at the end. In particular, also the phase-space integrals should be evaluated in n dimensions. Another complication is that the mass shell for massless particles is not well defined, not even in dimensional regularization. The reason is the occurence of singularities that extend all the way to the would-be mass shell. This is is a physical effect caused by long-distance effects, which prevent the emergence of massless free particles as asymptotic states. Of course, conceptually, this poses no real problem here, since in QCD the partons do not correspond to physical states, but for the calculations this is a cumbersome feature, at least at an intermediate level.

Before closing this section we demonstrate two aspects of the calculations. First we briefly discuss the two-particle phase-space integral in n dimensions. Secondly we evaluate a typical self-energy diagram to exhibit the complications when approaching the mass shell for massless particles.

In n dimensions the two-particle phase-space integral (3.44) is defined by

$$I^{(n)}(s, m_3^2, m_4^2) = \int \frac{\mathrm{d}^{n-1}p_3}{(2\pi)^{n-1}2\omega_3}\frac{\mathrm{d}^{n-1}p_4}{(2\pi)^{n-1}2\omega_4}\,(2\pi)^n\,\delta^{(n)}(p_1+p_2-p_3-p_4)\,, \tag{16.19}$$

where $s = -(p_1+p_2)^2$ and $\omega_{3,4} = (\mathbf{p}_{3,4}^2 + m_{3,4}^2)^{1/2}$. Following the derivation of (3.49), we choose the centre-of-mass frame and decompose the integral in an integral over the $n-2$ angular variables (which we leave) and an integral over the length of the $(n-1)$-dimensional momentum vector $\mathbf{p}_3 = -\mathbf{p}_4$, which contains a delta function. Integrating out this delta function yields

$$I^{(n)}(s,m_3^2,m_4^2) = \frac{1}{8\pi\sqrt{s}}\Big[\frac{\lambda(s,m_3^2,m_4^2)}{16\pi^2 s}\Big]^{\frac{n-3}{2}} \int \mathrm{d}\Omega_{CM}\,. \tag{16.20}$$

As usual this expression must be combined with the square of the invariant amplitude to yield a cross section or decay rate. Assuming that the invariant amplitude depends only on the deflection angle θ_{CM} between $\mathbf{p}_1$ and $\mathbf{p}_3$, we can integrate over the remaining $n-3$ angles, using the result (8.9-10) and/or problem 8.1; the relevant formula is

$$\int \mathrm{d}\Omega_{CM} = \frac{2\pi^{-1+\frac{1}{2}n}}{G(\frac{1}{2}n-1)}\int_0^\pi \mathrm{d}\theta_{CM}\,\big[\sin\theta_{CM}\big]^{n-3}\,, \tag{16.21}$$

where the integral on the left-hand side runs over all $n-2$ angles, while the integral on the right-hand side contains only the deflection angle. In the next section we need the integral for $m_4=0$. Replacing m_3 by m and introducing the variables $x=m^2/s$ and $y=\frac{1}{2}(1+\cos\theta)$ the combined result for the phase-space integral reads

$$I^{(n)}(s,m^2,0) = \frac{1}{8\pi}\Big(\frac{m^2}{4\pi}\Big)^{-2+\frac{1}{2}n}\,\frac{x^{2-\frac{1}{2}n}(1-x)^{n-3}}{\Gamma(\frac{1}{2}n-1)}\int_0^1 \mathrm{d}y\;[y(1-y)]^{-2+\frac{1}{2}n}\,, \tag{16.22}$$

which has the correct dimension of a mass to the power $n-4$. For $n>2$ this expression is free of singularities. Obviously corresponding expressions can be derived for multi-particle phase-space integrals. Formulae like (14.d16) are obviously needed for calculating decay rates and cross sections in arbitrary dimension.

Now let us turn to the evaluation of a typical propagator diagram. In section 8.3 we evaluated (cf. 8.22)

$$I(k^2,m_1^2,m_2^2) = \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^n q}{\big((q+\frac{1}{2}k)^2+m_1^2\big)\big((q-\frac{1}{2}k)^2+m_1^2\big)} \tag{16.23}$$

Using Feynman parameters and putting $m_1=m_2=0$, one obtains

$$I(k^2,0,0) = \frac{i}{16\pi^2}\,\Gamma(2-\tfrac{1}{2}n)\Big(\frac{k^2}{4\pi}\Big)^{-2+\frac{1}{2}n}\int_0^1 \mathrm{d}x\;[x(1-x)]^{-2+\frac{1}{2}n} \tag{16.24}$$

The x-integral can be evaluated by using (8.27) and yields $\Gamma^2(\frac{1}{2}n-1)/\Gamma(n-2)$, so that

$$I(k^2,0,0) = \frac{i}{16\pi^2}\,\frac{\Gamma(2-\frac{1}{2}n)\Gamma^2(\frac{1}{2}n-1)}{\Gamma(n-2)}\Big(\frac{k^2}{4\pi}\Big)^{-2+\frac{1}{2}n}. \tag{16.25}$$

This expressions exhibits poles for both large and small values of n, thus signaling ultraviolet and infrared singularities respectively. Obviously, the result is ambiguous when approaching the mass shell, $k^2 \to 0$. On the other hand, in a realistic calculation of infrared safe quantities, one knows that these ambiguities cancel. Therefore a standard approach is to include the counterterms, which depend on the short-distance behaviour, and drop all the ambiguous mass shell factors already at an intermediate stage of the calculation.

discuss electron-positron data including LEP

Problems

16.1. Write down the chiral transformations for the u-d system in terms of chiral components and derive the $U(2) \otimes U(2)$ structure. Discuss the consequences of spontaneous breakdown of chiral symmetry for the pions. By the same argument the mass of the η-meson should be comparable to the pion mass. This is in obvious disagreement with the data since ${m_\eta}^2/{m_\pi}^2 > 15$. The resolution of this puzzle is related to the so-called Adler-Bell-Jackiw triangle anomalies, discussed in chapter 22.

16.2. Construct the number of independent components of an n-component symmetric tensor with indices taking k possible values.

16.3. Consider representations of arbitrary spin. Use multiplicity 1 of J_z eigenstates to construct product rules like $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}$ etc.

16.4. To construct the three-quark states corresponding to proton and neutron in the simple quark model, consider first the two-quark state consisting of u and d quarks with zero spin and isospin. There are four different quark states: $|u\uparrow\rangle$, $|u\downarrow\rangle$, $|d\uparrow\rangle$ and $|d\downarrow\rangle$, where $\uparrow$ and $\downarrow$ denote the spin equal to $+\frac{1}{2}$ or $-\frac{1}{2}$, respectively, when measured along some direction in space. There are thus sixteen possible two-quark states, such as $|u\uparrow\rangle_1\,|u\uparrow\rangle_2$, etc.. To construct the state with zero spin and isospin, we start from $|u\uparrow\rangle_1\,|u\uparrow\rangle_2$ and antisymmetrize in both spin and isospin indices. In this way one obtains a linear combination of four such product states. Write down this combination and show that it is *symmetric* under the exchange of the two quarks. This state can be combined with a third quark to yield states with total spin and isospin equal to $\frac{1}{2}$. Symmetrizing these states over all three quarks leads to the expression for the proton and neutron states in the simple quark model.

16.5. Splitting functions in phi^3 theory.
To demonstrate the above procedure, we shall first compute the transition function for massless scalar field ϕ^3 theory in $D = 6-2\epsilon$ dimensions, whose Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{6}\phi^3\,. \tag{1}$$

This theory is asymptotically free, and is much simpler to calculate with. At lowest order we have

$$\phi^{(0)}(z) = N_\phi \left(A^{(0)}\right)^* A^{(0)} \,\delta(1-z)\frac{1}{p^+} \tag{2}$$

At this order, in this theory $A^{(0)} = 1$, so that $N_\phi = p^+$. At next order, we first consider the radiative contribution to ϕ.

$$\phi^{(1),R}(z) = N_\phi \int \frac{d^D k}{(2\pi)^D}\,(2\pi)\delta_+(k^2)\,\left(A^{(1),R}\right)^* A^{(1),R}\,\delta(p^+ - zp^+ - k^+) \tag{3}$$

The diagram for $A^{(1),R}$ is shown in Fig., and yields, using that p^μ has only a non-zero plus component

$$A^{(1),R} = \frac{g}{(p-k)^2} = \frac{-g}{2\,p^+k^-} \tag{4}$$

Note that we have also omitted any explicit factors of i, as they cancel in expressions such as (3) anyway. Let us simplify these expressions somewhat. First, the phase space measure may be written as

$$d^D k = dk^+ dk^- d^{D-2} k_T \tag{5}$$

which, upon doing all angular integrals in the transverse space, reads

$$\frac{2\pi^{(D-2)/2}}{\Gamma((D-2)/2)} dk^+ dk^- \left(k_T^2\right)^{(D-4)/2}\; dk_T^2 \tag{6}$$

We shall perform the $k^\pm$ integrals using the δ functions, but in order to perform the k_T^2 integrals we must restrict it (needs better discussion).

$$\begin{aligned}
\phi^{(1),R}(z) &= p^+ \frac{g^2}{4} \int \frac{2\pi^{(D-2)/2}}{\Gamma((D-2)/2)} dk^+ dk^- \left(k_T^2\right)^{(D-4)/2} \int_0^{\mu_F^2} dk_T^2 \\
&\times \delta\left(-2k^+k^- + k_T^2\right) \delta\left(k^+ - (1-z)p^+\right) \frac{1}{\left((p-k)^2\right)^2} \\
= &\quad \frac{g^2}{4p^+} \frac{2\pi^{(D-2)/2}}{(2\pi)^{D-1}} \frac{1}{\Gamma((D-2)/2)} \int_0^{\mu_F^2} dk_T^2 \\
&\left(k_T^2\right)^{(D-4)/2} \frac{1}{2(1-z)p^+} \frac{\left(2(1-z)p^+\right)^2}{\left(k_T^2\right)^2} \\
= &\quad \frac{g^2 \pi^{(D-2)/2}}{(2\pi)^{D-1}} \frac{1}{\Gamma((D-2)/2)} (1-z)\, \frac{2}{D-6} (\mu_F^2)^{(D-6)/2}\,.
\end{aligned} \tag{7}$$

Substitute now $D = 6 - 2\epsilon$, $g^2 = 4\pi(\mu^2)^\epsilon \alpha_\phi$ and find

$$\phi^{(1),R}(z) = \frac{\alpha_\phi}{8\pi^2} \left(\frac{4\pi\mu^2}{\mu_F^2}\right)^\epsilon \frac{1}{\Gamma(2-\epsilon)} \left(\frac{-1}{\epsilon}\right) (1-z) \tag{8}$$

The $1/\epsilon$ pole corresponds to a collinear singularity. Note that when $z \to 1$ (i.e. the soft limit), this singularity is suppressed. This is particular to this field theory, we will see that for QCD the collinear singularity is in this limit further enhanced. The one-loop virtual contribution $\phi^{(1),V}(z)$ we determine by the requirement that

$$\int_0^1 dz \left(\phi^{(0)}(z) + \phi^{(1),R}(z) + \phi^{(1),V}(z)\right) = 1 \tag{9}$$

The right hand side of this equation is already saturated by the lowest order term. Since, by definition $\phi^{(1),V}(z) \propto \delta(1-z)$, we readily arrive at

$$\phi^{(1),V}(z) = -\frac{\alpha_\phi}{16\pi^2}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^\epsilon \frac{1}{\Gamma(2-\epsilon)}\left(\frac{-1}{\epsilon}\right)\delta(1-z) \tag{10}$$

The combination of the real and virtual contribution may be expressed most efficiently in terms of so-called plus-distributions. Like the Dirac delta-function, they are defined in integrals with sufficiently smooth test functions as follows:

$$\int_a^1 dz[f(z)]_+ g(z) = \int_a^1 dz(f(z) - f(1))\, g(z) - f(1)\int_0^a dz g(z) \tag{11}$$

Then we have

$$\phi^{(1)}(z) = \frac{\alpha_\phi}{8\pi^2}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^\epsilon \frac{1}{\Gamma(2-\epsilon)}\left(\frac{-1}{\epsilon}\right)(1-z)_+ \tag{12}$$