13 | Feynman rules for gauge fields

Already on several occasions we noted that the use of Feynman diagrams in gauge theories is not entirely straightforward. The reason, to be reviewed in the next section, is that the gauge field propagator, when calculated according to the standard procedure, turns out to be singular. This is a consequence of the fact that the time evolution in a system with gauge invariance is ill defined. In previous chapters we dealt with this problem by employing so-called gauge-fixing terms, which enable one to construct a well-defined propagator. However, this propagator is not unique and depends on the specific form of these gauge-fixing terms. Thus the corresponding formulation of the theory is gauge-dependent, and in order for this procedure to make sense it is of the utmost importance to ascertain that physical quantities calculated in this framework do not depend on the choice made for the gauge-fixing terms. These and other aspects will be studied in this chapter, which will therefore be somewhat technical. The central result will be a Lagrangian from which one can straightforwardly calculate Feynman diagrams, and which contains gauge-fixing terms as well as a set of new fields, called *ghost fields*. Readers who are primarily interested in this result for the quantized gauge field Lagrangian can skip the first three sections and are advised to proceed directly to section 13.4. In the next section we discuss the Feynman rules in a Lorentz class of gauges. Next we examine the gauge fixing procedure in more detail, and then demonstrate the role played by the ghost fields in ensuring the unitarity of the theory in the last section.

13.1. The gauge field propagator

It is easy to see why gauge field propagators cannot be defined directly by means of the standard procedure outlined in chapter 2. Gauge invariance implies that the Lagrangian depends on a smaller set of fields than initially indicated. This should also be the case for the terms in the Lagrangian that are quadratic in the fields, which usually remain if one puts all coupling constants to zero. Therefore the matrix describing these quadratic terms will have zero eigenvalues. As a result it is not possible to construct the propagators, because those are related to the inverse of this matrix.

This phenomenon can easily be demonstrated for the free gauge field action, which after a Fourier decomposition takes the form (cf. 4.38)

$$S[A] = -\tfrac{1}{2}(2\pi)^4 \int \mathrm{d}^4k\, \bar{A}_\mu(k)[k^2\eta^{\mu\nu} - k^\mu k^\nu]A_\nu(k)\,, \tag{13.1}$$

with $\bar{A}_\mu(k)$ the complex conjugate field. The propagator should thus be proportional to the inverse of four-by-four $k^2\eta_{\mu\nu} - k_\mu k_\nu$. However, this inverse does not exist because $k^2\eta_{\mu\nu} - k_\mu k_\nu$ has an eigenvector proportional to k_ν with zero eigenvalue,

$$[k^2\eta_{\mu\nu} - k_\mu k_\nu]k^\nu = 0\,. \tag{13.2}$$

Indeed, this particular eigenvector is precisely related to the gauge degrees of freedom, since under a gauge transformation $A_\mu(k)$ changes by the Fourier transform of $\partial_\mu\xi(x)$, which is just proportional to k_μ.

The above complication can be avoided if we could somehow rewrite the Lagrangian without the spurious fields, i.e., without pretending that it depends on more fields than are actually there. In order to do so it would be sufficient to decompose all the fields into gauge-invariant components and a set of real (single component) fields, one for each gauge group generator, which remain subject to the gauge transformations. When substituted into the Lagrangian only the gauge-invariant fields will appear and the other fields will simply drop out as a result of gauge invariance. After these manipulations one can then proceed in the standard way and determine the propagators to calculate Feynman diagrams. Let us pursue for a while this approach and consider quantum electrodynamics, whose Lagrangian and gauge transformations are

$$\begin{aligned}&\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}(\partial\!\!\!/ - \mathrm{i}qA\!\!\!/ + m)\psi\,,\\ &A_\mu \to A_\mu + \partial_\mu\xi\,, \qquad \psi \to \mathrm{e}^{\mathrm{i}q\xi}\,\psi\,.\end{aligned} \tag{13.3}$$

In view of the presence of the derivative in the transformation law the decomposition of the gauge field A_μ into gauge invariant and gauge noninvariant parts does not take an algebraic form, but involves non-local expressions. To see this, we first express the spatial components of A_μ in terms of a three-vector field a_i and a field B,

$$A_i(x) = a_i(x) + \partial_i B(x)\,. \tag{13.4}$$

This decomposition is made unique by specifying a condition the field a_i. An obvious choice is

$$\partial_i a_i = 0\,, \tag{13.5}$$

but there are other options as well. It is not difficult to determine a_i and B in terms of A_i. Using the well-known Poisson equation for a point charge

$$\Delta\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\,\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{13.6}$$

we can solve the equation $\Delta B = \partial_i A_i$ that follows from (13.4) and (13.5). This leads to a solution for B,

$$B(\mathbf{x},t) = \frac{1}{4\pi}\int \mathrm{d}^3x' \, \frac{\partial_i A_i(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \,, \tag{13.7}$$

from which the expression for a_i follows via (13.4). Observe that we implicitly assumed here that $\partial_i A_i$ is free of singularities and falls off sufficiently fast at spatial infinity in order to make (13.7) well defined. Furthermore we assumed that also B tends to zero at infinity. However, we will disregard such subtleties as they do not play an important role in the context of a perturbative treatment by means of Feynman diagrams.

The above decomposition of A_μ is directly related to the plane wave decomposition of the photon field that we have employed earlier in section 4.2, i.e.,

$$A_\mu(k) = a^\lambda(k)\,\varepsilon_\mu(\mathbf{k},\lambda) + b(k)\,k^*_\mu + c(k)\,k_\mu \,, \tag{13.8}$$

where $\varepsilon_\mu(\mathbf{k},\lambda)$ defines the transverse polarization vectors of the photon and $k^*_\mu = (-k_0,\mathbf{k})$. We recall that, when $A_\mu(k)$ is subject to the free field equations, $b(k) = 0$ and the $a^\lambda(k)$ correspond to the two physical polarizations, while $c(k)$ remains undetermined. Comparing this decomposition to (13.4) gives the following identification,

$$\begin{aligned} a_i(x) &= \int \mathrm{d}^4k\, a^\lambda(k)\,\varepsilon_i(\mathbf{k},\lambda)\ \mathrm{e}^{\mathrm{i}k\cdot x} \,, \\ B(x) &= -\mathrm{i}\int \mathrm{d}^4k\, \big(b(k)+c(k)\big)\ \mathrm{e}^{\mathrm{i}k\cdot x} \,, \\ A_0(x) &= \int \mathrm{d}^4k\, \big(-b(k)+c(k)\big)\,k_0\ \mathrm{e}^{\mathrm{i}k\cdot x} \,. \end{aligned} \tag{13.9}$$

Hence we see that the field a_i is just the part of A_μ that describes physical photons. For this reason a_i is often called the *radiation field*.

From (13.7) it is straightforward to deduce that under a gauge transformation the fields a_i and B transform as

$$a_i(x) \to a_i(x), \quad B(x) \to B(x) + \xi(x) \,, \tag{13.10}$$

by performing a gauge transformation to the right-hand side of (13.7) and integrating by parts. Furthermore we may redefine the fermion field ψ according to

$$\psi(x) = \mathrm{e}^{\mathrm{i}qB(x)}\,\psi'(x) \,, \tag{13.11}$$

so that the newly-defined field ψ' is inert under the gauge transformations. It is also convenient to introduce the gauge invariant combination

$$\phi(x) = A_0(x) - \partial_0 B(x) \,. \tag{13.12}$$

Thus we have now expressed the original fields A_μ and ψ in terms of the gauge-invariant fields a_i, ϕ, ψ', and a single non-invariant field B. Expressing the Lagrangian (13.3) in terms of these new fields it is obvious that the terms containing the field B cancel owing to the gauge invariance of (13.3). Hence we are left with a Lagrangian that only depends on the gauge-invariant fields a_i, ϕ and ψ'. Dropping a total divergence this Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu a_i)^2 - \tfrac{1}{2}(\partial_i \phi)^2 - \bar{\psi}'(\not{\partial} + m)\psi' + \mathrm{i}q\bar{\psi}'(\gamma_i a_i + \gamma^0\phi)\psi' \,. \qquad (13.13)$$

It contains the massless Klein-Gordon Lagrangian for the radiation field a_i and the Dirac Lagrangian for ψ'. However, the kinetic term for ϕ is not of the standard type as it contains only spatial derivatives.

The determination of the propagators is now straightforward. For the fermion field there is the standard Dirac propagator (cf. 5.32), for the radiation field a_i we find,

$$\Delta_{ij} = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right), \qquad (13.14)$$

where we took into account the condition (13.5), and for the field ϕ we have

$$\Delta_\phi = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{|\mathbf{k}|^2}. \qquad (13.15)$$

Obviously (13.14) describes the exchange of virtual photons whereas (13.15) describes an additional interaction, which takes place between charge densities as ϕ couples to $\psi'^\dagger\psi'$ according to (13.13). This is just the instantaneous Coulomb interaction, as is confirmed by taking the Fourier transform of $1/|\mathbf{k}|^2$ which is just proportional to $\delta(t-t')\,|\mathbf{r}-\mathbf{r}'|^{-1}$ (cf. sect. 2.1).

Although it is straightforward to define the propagators after performing the decompositions introduced above, this approach is rather tedious and has a number of obvious disadvantages. First of all the manifest Lorentz invariance is lost, as one can see directly from (13.13). Of course, one expects that physical results derived from (13.13) will remain compatible with Lorentz invariance, but it is not a priori obvious that this will indeed be the case and a more rigorous proof is necessary. Furthermore the propagators are of an unusual type (cf. 13.15), which is rather inconvenient in actual calculations. For these reasons one usually prefers a more flexible formulation in which Lorentz invariance can be kept manifest and the Lagrangian is still of the type encountered so far. In the next section we shall show how this can be achieved by means of Lorentz invariant gauge conditions. However, the problems noted above persist, albeit in a different form. Although the Lorentz invariance can be preserved in this way, the Lagrangian will now depend on the gauge condition that one adopts. Therefore, one must show that any *physical* results computed with the gauge-fixed Lagrangian, such as cross sections, are independent of the gauge condition, which will then guarantee that these results coincide with those based on the Lagrangian (13.13). From the gauge independence it then follows that the latter results are Lorentz invariant, again as far as *physical* quantities are concerned.

13.2. Gauge fixing

In the previous section we decomposed the gauge field A_μ in terms of a_i, ϕ and B, where a_i was subject to the constraint $\partial_i a_i = 0$. The field B decouples from the theory as a result of local gauge invariance. This (nonlocal) decomposition was somewhat involved, but there is a faster way to obtain the same result, namely by putting B to zero from the start, or equivalently, by imposing a condition on the original fields A_μ. In the case at hand the condition is

$$\partial_i A_i = 0\,. \tag{13.16}$$

This is an example of a so-called *gauge condition* and (13.16) is known as the *Coulomb gauge* or the *radiation gauge*. Such a condition can always be implemented by means of a suitable gauge transformation. Another approach is based on reinserting the field B into the Lagrangian as a free field. In that way the number of field components will coincide with the number of field components indicated in the original Lagrangian (13.3). Of course, the addition of a term $-\frac{1}{2}(\partial_\mu B)^2$ (times some constant of dimension [mass]^2 to make it dimensionally correct) formally breaks the gauge invariance (cf. 13.10), but if one remembers that the field B is just there for practical reasons and does not describe a *physical* particle, then no harm is done at this stage. The addition of the term proportional to $-\frac{1}{2}(\partial_\mu B)^2$ is an example of a *gauge-fixing term*; one introduces the gauge degrees of freedom into the Lagrangian, so that there are now extra terms quadratic in the fields thus enabling one to construct gauge field propagators without encountering singularities.

Although the addition of gauge-fixing terms enables one to construct well-defined propagators, it is clear that these propagators will in general depend on the choice made for the gauge-fixing terms. Therefore the Feynman diagrams will be gauge-dependent, so it may seem that the corresponding physical quantities will also be gauge-dependent, a result that would clearly be unacceptable. In the above example we understand why this undesirable phenomenon does not take place; although we have introduced a kinetic term for the field B, this field does not appear in the interaction terms of the Lagrangian (at least, not after making appropriate field redefinitions, (cf. 13.13), just because of the gauge invariance of the original Lagrangian (13.3). Note that B is defined in a non-local way in terms of the original gauge field.

However, we intend to employ a larger variety of gauge-fixing terms in what follows, for which it is not always so easy to see how the gauge dependence is realized for physical quantities.[1] Rather than to analyze these questions now, we will in this section first study the interaction of fixed external sources with a gauge field in order to get acquainted with a variety of gauge-fixing terms and corresponding propagators and examine the gauge dependence of the amplitudes. In the next section we will then return to the question of gauge dependence for the interacting theory.

Consider external sources $J_\mu(x)$ coupling to a gauge field in the standard fashion. The corresponding Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + A_\mu J^\mu \,. \tag{13.17}$$

As we have discussed in the previous chapter gauge invariance requires that the current is conserved,

$$\partial_\mu J^\mu(x) = 0 \,, \tag{13.18}$$

so that the gauge degrees of freedom do not couple to the source. We now return to the problem of constructing a gauge field propagator. In principle one could choose a basis for the 4×4 matrix $k^2\eta_{\mu\nu} - k_\mu k_\nu$ representing the terms quadratic in the gauge field such that the absence of the gauge degrees of freedom is manifest,

$$D_{\mu\nu}(k) = (k^2\eta_{\mu\nu} - k_\mu k_\nu) \longrightarrow \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{13.19}$$

The upper-left three-by-three submatrix is now invertible, and the source J_μ is restricted to a vector in the corresponding three-dimensional subspace, because of (13.18).[2] Therefore the interaction of the two sources induced by the exchange of gauge field quanta is governed precisely by the inverse of this submatrix. However, as argued above, a formulation in which the gauge degrees of freedom decouple manifestly as in (13.19) is not so easy to implement; hence we want to find a way of determining the inverse of the 3×3 submatrix without the need of going to a particular basis.

In principle this is a problem in linear algebra: given an $n\times n$ symmetric matrix D_{ij} with one zero eigenvalue, one wishes to determine its inverse in the $(n-1)$-dimensional subspace orthogonal to the corresponding null vector.

[1] Obviously there is no principal objection against the gauge dependence of the Green's functions as those do not correspond to quantities that can be measured experimentally, such as cross sections and decay rates.

[2] To see this, write down the field equation $D_{\mu\nu}(k)\, A^\nu(k) \propto J_\mu(k)$ in the basis corresponding to (13.19).

The solution of this problem can conveniently be given as follows (for a proof we refer to problem 13.1). We first add a (degenerate) matrix $\lambda_i\lambda_j$ to D_{ij}, where λ_i is some vector with a nonvanishing component in the direction of the null vector. In that case one can show that the matrix $D_{ij} + \lambda_i\lambda_j$ has no longer a zero eigenvalue and therefore can be inverted. Furthermore, it can be shown that $(D_{ij} + \lambda_i\lambda_j)^{-1}$, when restricted to the $(n-1)$-dimensional subspace orthogonal to the null vector, is independent of λ and coincides with the inverse of the matrix D taken in the same subspace. Furthermore one can prove that $(D_{ij} + \lambda_i\lambda_j)^{-1}$ acting on the vector λ_i is proportional to the null vector. We emphasize that only the inverse matrix elements in the $(n-1)$-dimensional subspace are uniquely determined in this way. The remaining matrix elements of $(D_{ij} + \lambda_i\lambda_j)^{-1}$ will of course depend on the choice for λ_i.

Let us now formulate the above procedure for the case at hand, in which context it is referred to as "quadratic gauge fixing". The propagator for a gauge field is obtained by changing the quadratic term in the action

$$D_{\mu\nu}(k) = k^2\eta_{\mu\nu} - k_\mu k_\nu \,, \tag{13.20}$$

which has a null vector proportional to k_ν, into

$$D_{\mu\nu}(k) \to D_{\mu\nu}(k) + \bar\lambda_\mu(k)\,\lambda_\nu(k) \propto \Delta^{-1}_{\mu\nu}(k)\,, \tag{13.21}$$

where $\lambda_\mu(k)$ satisfies the same reality condition as the gauge field, and is not orthogonal to the null vector of $D_{\mu\nu}(k)$, i.e.,

$$\bar\lambda_\mu(k) = \lambda_\mu(-k), \qquad k_\mu\lambda^\mu(k) \neq 0\,. \tag{13.22}$$

The propagator corresponding to (13.21) is then equal to

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\Big(k^2\eta_{\mu\nu} - k_\mu k_\nu + \bar\lambda_\mu(k)\,\lambda_\nu(k)\Big)^{-1}\,, \tag{13.23}$$

where we have included the standard factor $[\mathrm{i}(2\pi)^4]^{-1}$.

The inverse can be calculated in the usual manner by first assuming a decomposition in terms of the tensors $\eta_{\mu\nu}$, $k_\mu k_\nu$, $\bar\lambda_\mu k_\nu$, $k_\mu\lambda_\nu$ and $\bar\lambda_\mu\lambda_\nu$. As one can easily verify, this leads to the expression

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left\{\eta_{\mu\nu} + \frac{k^2 + \bar\lambda(k)\cdot\lambda(k)}{|k\cdot\lambda(k)|^2}k_\mu k_\nu - \frac{\bar\lambda_\mu(k)k_\nu}{k\cdot\bar\lambda(k)} - \frac{k_\mu\lambda_\nu(k)}{k\cdot\lambda(k)}\right\}. \tag{13.24}$$

When contracted with $\lambda_\nu(k)$ this propagator becomes proportional to the null vector, i.e.,

$$\begin{aligned}\Delta_{\mu\nu}(k)\,\bar\lambda^\nu(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{k_\mu}{k\cdot\lambda(k)}\,,\\ \lambda^\mu(k)\,\Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{k_\nu}{k\cdot\bar\lambda(k)}\,.\end{aligned} \tag{13.25}$$

In the subspace orthogonal to the null vector k_μ, the propagator reduces to

$$\Delta_{\mu\nu}(k) \to \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\,, \tag{13.26}$$

and is indeed independent of λ_μ.

The introduction of the term $\bar\lambda_\mu(k)\,\lambda_\nu(k)$ in (13.21) can be accounted for by modifying the gauge invariant action according to

$$S_{\mathrm{inv}} \to S_{\mathrm{inv}} + S_{\mathrm{g.f.}}\,, \tag{13.27}$$

where $S_{\mathrm{g.f.}}$ represents the quadratic (Fermi) "gauge-fixing" terms (here in momentum space)

$$S_{\mathrm{g.f.}} = (2\pi)^4 \int \mathrm{d}^4k\, \left\{-\tfrac{1}{2}\Big(\bar\lambda^\mu(k)\bar A_\mu(k)\Big)\Big(\lambda^\nu(k)A_\nu(k)\Big)\right\}\,. \tag{13.28}$$

These are the "gauge-fixing" terms that we introduced in chapter 4 in order to derive a propagator for the photon field.

Besides the quadratic gauge-fixing procedure one has the option of imposing so-called singular gauges, in which case one simply restricts certain components of A_μ to zero:

$$\lambda^\mu(k)A_\mu(k) = 0\,. \tag{13.29}$$

The resulting propagator can be obtained directly from (13.24) by replacing $\lambda_\mu(k)$ by $\alpha\lambda_\mu(k)$ and taking the limit $\alpha\to\infty$ (therefore the name "singular" gauge). The analogue of (13.25) then equals

$$\Delta_{\mu\nu}(k)\,\bar\lambda^\nu(k) = \lambda^\mu(k)\,\Delta_{\mu\nu}(k) = 0\,, \tag{13.30}$$

indicating that the gauge field components in the $\lambda_\mu(k)$ direction are indeed suppressed.

Rather than giving a detailed derivation of the algebraic results leading to this procedure we present a number of examples. An important class of gauge conditions are the Lorentz covariant ones, for which $\lambda_\mu(k)$ is proportional to the null vector k_μ. Those are the so-called *Lorentz gauges* with

$$\lambda_\mu(k) = \mathrm{i}\lambda k_\mu\,, \tag{13.31}$$

where λ is an arbitrary parameter. This gauge condition corresponds to introducing the following term in the gauge-invariant Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm inv} - \tfrac{1}{2}\lambda^2(\partial_\mu A^\mu)^2 \,. \tag{13.32}$$

According to (13.24) the propagator corresponding to (13.31) takes the form

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left\{\eta_{\mu\nu} - (1-\lambda^{-2})\frac{k_\mu k_\nu}{k^2}\right\} , \tag{13.33}$$

for which (13.25) is indeed satisfied,

$$\Delta_{\mu\nu}(k)\,(\mathrm{i}\lambda k_\nu) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{\lambda\, k^2}\,\mathrm{i}k_\mu \;. \tag{13.34}$$

There are two special values for λ which are often convenient in calculations. One is the *Feynman gauge*, which is obtained for $\lambda = 1$, so that the propagator reads

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{\eta_{\mu\nu}}{k^2} \,. \tag{13.35}$$

The second one is the *Landau gauge*, obtained in the limit $\lambda \to \infty$; the corresponding propagator is

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) , \tag{13.36}$$

and satisfies the transversality condition

$$\Delta_{\mu\nu}(k)\, k^\nu = 0 \,. \tag{13.37}$$

This corresponds to the condition

$$\partial_\mu A^\mu = 0 \,. \tag{13.38}$$

The Landau gauge is thus a singular gauge, and (13.37) and (13.38) are special cases of (13.30) and (13.29).

The propagator for noncovariant gauges, for which λ_μ is not proportional to k_μ but to some other independent vector, follow straightforwardly from (13.24). In most cases one assumes that λ_μ is either proportional to the three-momentum,

$$\lambda_i(k) = \mathrm{i}\lambda\, k_i \,, \qquad \lambda_0(k) = 0 \,, \tag{13.39}$$

or to some momentum-independent vector,

$$\lambda_\mu(k) = \lambda\, n_\mu \,, \tag{13.40}$$

with $n \cdot k \neq 0$ cf. (13.22). These gauges are obtained by making the following modification of the Lagrangian,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{inv}} - \tfrac{1}{2}\lambda^2(\partial_i A_i)^2 , \\ \mathcal{L} &= \mathcal{L}_{\text{inv}} - \tfrac{1}{2}\lambda^2(n^\mu A_\mu)^2 . \end{aligned} \tag{13.41}$$

The class of gauges defined by (13.39) is of the Coulomb type, and gives rise to the propagator (with $\mu = i, 4$; $\nu = j, 4$; $i, j = 1, 2, 3$)

$$\Delta_{\mu\nu}(k) = \tfrac{1}{\mathrm{i}(2\pi)^4} \begin{pmatrix} \dfrac{1}{k^2}\Big(\delta_{ij} - \dfrac{k_i k_j}{|\mathbf{k}|^2}\Big) + \dfrac{k_i k_j}{\lambda^2|\mathbf{k}|^4} & \dfrac{k_0 k_i}{\lambda^2|\mathbf{k}|^4} \\ \dfrac{k_0 k_i}{\lambda^2|\mathbf{k}|^4} & \dfrac{1}{|\mathbf{k}|^2} - \dfrac{k_0^2}{\lambda^2|\mathbf{k}|^4} \end{pmatrix} ,$$

where i and j specify the spatial components. The condition (13.25) is again satisfied, as

$$\Delta_{\mu j}(k)\,(i\lambda k_j) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{\lambda|\mathbf{k}|^2}\mathrm{i}k_\mu . \tag{13.42}$$

Usually one chooses the singular gauge $\lambda \to \infty$, which is the standard Coulomb (or radiation) gauge cf. (13.14), (13.15), the propagator reads

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4} \begin{pmatrix} \dfrac{1}{k^2}\Big(\delta_{ij} - \dfrac{k_i k_j}{|\mathbf{k}|^2}\Big) & 0 \\ 0 & \dfrac{1}{|\mathbf{k}|^2} \end{pmatrix} ,$$

satisfying

$$\Delta_{\mu i}(k)\, k_i = 0 , \tag{13.43}$$

which corresponds to the gauge condition,

$$\partial_i A_i = 0 . \tag{13.44}$$

The class of gauges (13.40), characterized by an independent four vector n_μ leads to the propagator

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left\{\eta_{\mu\nu} + \frac{k^2 + \lambda^2 n^2}{\lambda^2(n\cdot k)^2}k_\mu k_\nu - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right\} . \tag{13.45}$$

Again one usually takes the singular limit $\lambda \to \infty$, corresponding to the gauge condition

$$n^\mu A_\mu = 0 , \tag{13.46}$$

with propagator

$$\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left\{\eta_{\mu\nu} + \frac{n^2}{(n\cdot k)^2}k_\mu k_\nu - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right\}. \tag{13.47}$$

This gauge is called the *axial gauge*. If n_μ is pointing along the time axis, we have the *temporal gauge* condition $A_0 = 0$, whereas with n_μ pointing along the light cone we have the *light-cone gauge*. An obvious inconvenience in this class of gauges is the presence of singularities whenever $n\cdot k = 0$. The origin of these singularities is related to the fact that the gauge-fixing procedure is only operative for $n\cdot k \neq 0$, while in the Feynman integrals one must integrate over all k.

Clearly the propagators obtained in different gauges are not equivalent, but according to the above arguments the contraction between *conserved* currents should lead to a unique result. Indeed, one always finds

$$\bar{J}^\mu(k)\,\Delta_{\mu\nu}(k)\,J'^\nu(k) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\bar{J}_\mu(k)\,J'^\mu(k)\,, \tag{13.48}$$

for two conserved currents J_μ and J'_μ (so that $k_\mu J^\mu(k) = k_\mu J'^\mu(k) = 0$). This result shows the significance of the fact that an independent tensor proportional to $\bar{\lambda}_\mu\lambda_\nu$ was absent in (13.24); such a term would give rise to a new and gauge-dependent contribution in (13.49). In interpreting the right-hand side of (13.48) one has to keep in mind that the components of J_μ are not linearly independent in view of the transversality condition. On may take this into account by decomposing J_μ into two transverse components with $\mathbf{k}\cdot\mathbf{J}_\perp = 0$, a longitudinal component $|\mathbf{k}|^{-1}\mathbf{k}\cdot\mathbf{J}$ and J_0. Transversality then implies that the longitudinal component is equal to $(k_0/|\mathbf{k}|)J_0$, so that we derive

$$\bar{J}^\mu(k)\,\Delta_{\mu\nu}(k)\,J'^\nu(k) = \frac{1}{\mathrm{i}(2\pi)^4}\left\{\frac{\bar{\mathbf{J}}_\perp(k)\cdot\mathbf{J}'_\perp(k)}{k^2} - \frac{\bar{J}_0(k)\,J'_0(k)}{|\mathbf{k}|^2}\right\}. \tag{13.49}$$

The massless Klein-Gordon propagator occurs only for the interaction of the transverse components $\mathbf{J}_\perp$, confirming that only transverse gauge fields are associated with propagating physical modes with $k^2 = \mathbf{k}^2 - k_0^2 = 0$. The interaction between the remaining component in the time direction no longer depends on k_0. As already explained below (13.15), this term represents the instantaneous Coulomb interaction between the charge densities J_0 and J'_0. Obviously the first term in (13.49) represents the contribution of the radiation field a_i and the second one the contribution from ϕ in the formulation based on the Lagrangian (13.13).

13.3. Ghosts

In the previous subsection we derived a large variety of gauge-field propagators by employing quadratic gauge-fixing terms. Although this procedure re-introduces the missing degrees of freedom associated with the gauge invariance into the Lagrangian, the underlying assumption is that these extra degrees of freedom will still act as free fields. Although the propagators contain the contributions from extra degrees of freedom, these are expected to decouple from the interacting sector of the theory. In previous sections we saw already that this decoupling indeed takes place in many situations. For instance, in chapter 9 we demonstrated the gauge independence of physical quantities in a variety of one-loop calculations for quantum electrodynamics.

Already in chapter 4 we presented an argument why the effect of introducing the gauge-fixing term is presumably rather harmless and will not interfere with the interacting sector (cf. problem 4.5). To recall this argument consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}F^2 + A_\mu J^\mu \,, \tag{13.50}$$

with the following gauge-fixing term

$$F = \lambda\, \partial_\mu A^\mu \,, \tag{13.51}$$

and a conserved current J_μ. When we now write down the equation of motion for A_μ in the presence of the conserved current, and apply a derivative ∂_μ, we find

$$\partial^2(\lambda\, \partial_\mu A^\mu) = 0\,. \tag{13.52}$$

Therefore the gauge-fixing term is just a free, massless, field, which will decouple from the interaction with the conserved current. Although this argument correctly describes the general idea behind the use of gauge-fixing terms, it is really too naive. In order to see this let us consider a more complicated version of the gauge-fixing term, which exhibits many of the characteristic features that arise when considering the quantization of non-abelian gauge theories. Take the following modification of the gauge-fixing term (13.51),

$$F = \lambda\, \partial_\mu A^\mu + \kappa\, A_\mu A^\mu, \tag{13.53}$$

which depends on two arbitrary parameters λ and κ. One may question the presence of nonlinear terms in the gauge-fixing term and wonder whether such terms are allowed. However, it is easy to see that (13.53) is related to (13.52) by means of a field-dependent gauge transformation with parameter

$$\xi(x) = -\mathrm{i}\,\frac{\kappa}{\lambda}\int \mathrm{d}^4y\; \Delta(x-y)\,\big(A_\mu(y)\big)^2, \tag{13.54}$$

where $\Delta(x)$ is the massless Green's function satisfying $\partial^2\Delta(x) = \mathrm{i}\,\delta^4(x)$ (cf. section 2.3). Indeed, upon performing a gauge transformation, $A_\mu \to A_\mu + \partial_\mu\xi$, with ξ as given above, the gauge-fixing term (13.52) changes straightforwardly into (13.53). Therefore, one tends to conclude that nonlinear gauge-fixing terms, such as (13.53), are in principle allowed.

Substituting (13.53) into the Lagrangian (13.50) leads to

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}\lambda^2\big(\partial_\mu A^\mu\big)^2 \\ &\quad -\lambda\kappa\,\partial_\mu A^\mu\, A_\nu^2 - \tfrac{1}{2}\kappa^2\, A_\mu^2\, A_\nu^2 + A_\mu J^\mu\,. \end{aligned} \tag{13.55}$$

In this case the gauge-fixing term no longer leads to a free-field equation. Rather we find

$$\partial^2 F - \frac{2\kappa}{\lambda}\,\partial_\mu\big(A^\mu\, F\big) = 0\,. \tag{13.56}$$

The Lagrangian (13.55) is still expected to describe the coupling of a free abelian gauge field to a conserved external current, in spite of the fact that it takes the form of an interacting field theory. To verify that the apparent interactions are just an artefact of the nonlinear gauge condition (13.53), let us first calculate the amplitude with four external gauge fields in tree approximation, which describes the scattering of the massless particles associated with A_μ. Identifying these particles as photons, we thus calculate the amplitude for light-light scattering in tree approximation. As photons have no self-interactions in the classical Lagrangian, the on-shell amplitude calculated from the Lagrangian (13.55) is expected to vanish.

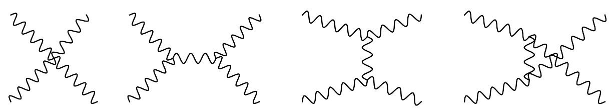

Figure 13.1: Diagrams contributing to light-light scattering in the tree approximation for the theory based on the Lagrangian (13.55).

As it will turn out, this is indeed the case. The diagrams contributing to this process are shown in fig 13.1. The first diagram based on the four-point vertex, yields

$$\mathcal{M} = 8(-\tfrac{1}{2}\kappa^2)\big\{(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4) + (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) + (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\big\}\,. \tag{13.57}$$

where ε_1-ε_4 denote the polarization vectors associated with the four photons. Here the factor 8 counts the number of different ways in which the external lines can be hooked to the gauge fields of the vertex. The next three diagrams give rise to a more complicated expression with a propagator and two three-point vertices. For future use, let us first define the three-point vertex.

Combining all possible attachments of the three external lines to the $\partial\cdot A\,A^2$ vertex leads to

$$V_{\mu\nu\rho}(k,p,q) = -\lambda\kappa\big(2\mathrm{i}k_\mu\,\eta_{\nu\rho} + 2\mathrm{i}p_\nu\,\eta_{\rho\mu} + 2\mathrm{i}q_\rho\,\eta_{\mu\nu}\big)\,, \tag{13.58}$$

where the three photon lines carry incoming momenta k, p, q, and indices μ, ν, ρ, respectively. The factor 2 in (13.58) is a combinatoric factor. Note that the standard factor $\mathrm{i}(2\pi)^4$ with the momentum-conserving delta function $\delta^4(k+p+q)$ has been suppressed. The three-point vertex simplifies considerably when contracted with two transverse polarization vectors and only one of the three terms remains. Denoting the polarization vectors and the incoming momenta of the two photons by ε_1, ε_2, and p_1, p_2, respectively, the three-point vertex becomes equal to $-2\mathrm{i}\lambda\kappa\,(\varepsilon_1\cdot\varepsilon_2)\,k_\mu$, where μ is the index associated with the third photon which carries incoming momentum $k=-p_1-p_2$. Both vertices in each of the last three diagrams in fig 13.1 thus give a similar result, but with an opposite sign as the intermediate photon momentum is incoming with respect to one vertex and outgoing with respect to the other. The propagator of the intermediate photon is contracted with two momentum vectors, one from each vertex. This propagator was already determined (cf. 13.33) and satisfies

$$k^\mu\,\Delta_{\mu\nu}(k) = \frac{k_\nu}{\mathrm{i}(2\pi)^4\lambda^2\,k^2}\,, \qquad k^\mu k^\nu\,\Delta_{\mu\nu}(k) = \frac{1}{\mathrm{i}(2\pi)^4\lambda^2}\,. \tag{13.59}$$

Combining the propagator with the vertices, the photon propagator pole cancels by virtue of the second equation in (13.59). After summing over the three graphs in fig 13.1, we obtain the same expression as (13.57), but with opposite overall sign. Therefore the two sets of graphs cancel, so that there is no light-light scattering in tree approximation, just as expected. However, for off-mass-shell photons the cancellation does not take place.

The above result confirms that there is a priori nothing wrong with having nonlinear modifications to a gauge-fixing term. Before we continue, let us show that the above situation has much in common with nonabelian gauge theories, when quantized with a more standard gauge-fixing term. Consider the nonabelian gauge field Lagrangian given in (12.78) and add to it the gauge-fixing terms (one for each gauge field)

$$F^a = \lambda\,\partial_\mu W^{a\mu}\,. \tag{13.60}$$

The Lagrangian then takes the form

$$\mathcal{L} = \mathcal{L}_W - \tfrac{1}{2}\lambda^2\,(\partial_\mu W^{a\mu})^2\,. \tag{13.61}$$

Now write down the field equations for W_μ^a. Using (12.92) it does not require much work to find their explicit form,

$$D^\nu G_{\mu\nu}^a + \lambda^2\partial_\mu(\partial_\nu W^{a\nu}) = 0\,. \tag{13.62}$$

Application of a covariant derivative D^μ then yields the equation

$$D^\mu \partial_\mu F^a = 0\,, \tag{13.63}$$

where we made use of (12.95). This result shows that the gauge-fixing term does *not* satisfy a free-field equation, just as the gauge-fixing term (13.53) in the abelian case. From the experience gained with the example of the abelian case, we feel confident that this feature should not be seen as a signal of further difficulties. However, this expectation is only justified at the tree level; it turns out that difficulties are encountered when going beyond this approximation. Depending on the choice for the gauge condition, these difficulties may arise for both non-abelian and abelian theories.

Before analyzing the precise difficulties beyond the tree approximation, let us first try to understand the possible reason for them from a more physical point of view. As was emphasized already in chapter 4, a massless gauge field describes only two physical states for a given value of its momentum, corresponding to transverse polarization states. This fact was confirmed once more by the analysis in section 13.1, where the transverse radiation field a_i describes just the physical photons. Nevertheless, in certain gauges, the photon propagator exhibits a massless pole for each of the components of the photon field, suggesting that one is dealing with *four* rather than *two* physical polarizations. Our response to this problem so far has been that we only consider transverse polarizations for external photon lines. However, it turns out that this restriction is not sufficient for closed-loop diagrams. To understand this, we remind the reader that amplitudes acquire imaginary parts whenever the momenta associated with a sufficient number of internal lines can be restricted to their mass shell. To be more precise, as soon as one can divide a diagram into two parts by cutting internal lines that carry momenta which are kinematically allowed to be on the mass shell, then the amplitude acquires an imaginary part. We have demonstrated this explicitly in section 3.6, where we derived the definition of the decay rate via precisely this procedure. More generally, the analytic structure of scattering amplitudes as a function of their external momenta is restricted by the requirement of unitarity, which is in turn related to the conservation of probability.

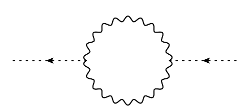

Figure 13.2: Self-energy of the neutral pion due to a $\pi^0\gamma\gamma$-interaction, as discussed in the text.

Perhaps it is best to illustrate this point with a simple example. As extensively discussed in section 4.5, the neutral pion decays into two photons. Now consider the pion self-energy graph in one-loop approximation due to the $\pi^0\gamma\gamma$ interaction (cf. fig 13.2) and determine its imaginary part. This is different from zero when the pion momentum p_{π^0} is timelike. Using a photon propagator that exhibits poles for each of the photon polarizations, all four polarizations will in principle contribute. Therefore it seems that the imaginary part of the propagator is not just related to the physical decay rate, which refers to the decay of the pion into two transversely polarized photons, but includes the contributions from the decay into unphysical photons. Fortunately, there is no discrepancy in this case, as the two unphysical polarizations cancel because of current conservation of the $\pi^0\gamma\gamma$ vertex. The reason for this phenomenon was already discussed in section 4.2. For an amplitude $\mathcal{M}_\mu(k)$, where μ and k denote the polarization index and momentum of an external photon line, respectively, the corresponding cross section or decay rate is proportional to $|\mathcal{M}_\mu\,\epsilon^\mu(\mathbf{k})|^2$, where $\epsilon_\mu(\mathbf{k})$ corresponds to a transverse polarization vector. Summing over all possible incoming or outgoing states implies that we sum over the two transverse polarizations only. However, if $\mathcal{M}_\mu$ is conserved, i.e., $k^\mu\,\mathcal{M}_\mu = 0$, then this is just the same as summing over all polarizations,

$$\sum_{\substack{\text{transverse}\\ \text{polarizations}}} |\mathcal{M}_\mu\,\varepsilon^\mu(\mathbf{k})|^2 = \mathcal{M}^\mu\,\overline{\mathcal{M}}_\mu\,. \tag{13.64}$$

In other words, the contributions from the unphysical polarizations precisely cancel. If the amplitude involves several external photons, such as for instance in Compton scattering, then the multiphoton amplitude $\mathcal{M}_{\mu\nu\cdots}$ should be conserved with respect to each of the photon indices separately, i.e. for any value of the remaining indices. Hence we must have

$$k^\mu\,\mathcal{M}_{\mu\nu\rho\cdots} = 0, \tag{13.65}$$

where all momenta of the external particles are on their respective mass shells, but the uncontracted photon polarization indices $\nu, \rho, \ldots$ are arbitrary. This is a stronger requirement than that derived in section 11.3; there we found that physical amplitudes for processes that involve massless spin-1 particles are Lorentz invariant provided that the amplitudes are conserved whenever all particles are on their mass-shells. In other words, there it was sufficient that (13.64) holds true for only those index values that correspond to transverse polarizations.

The $\pi^0\gamma\gamma$-vertex is conserved for each photon separately, irrespective of the polarization of the other photon. This phenomenon does not occur in general for amplitudes with several nonabelian gauge fields. In that case we only have current conservation if the polarizations of the other gauge fields are transverse. We have already seen an example of this in section 11.6 (cf. 11.76) and we return to this in section 13.6. The nonlinear gauge (13.53) leads to the same phenomenon and a violation of current conservation for amplitudes with several photons whose polarizations are not restricted to be transverse.

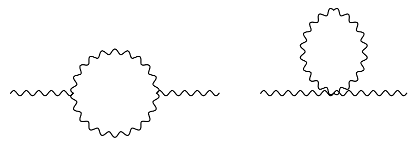

Figure 13.3: One-loop diagrams based on the Lagrangian (13.55) that contribute to the self-energy of the photon

After this digression, let us return to a specific calculation. We determine the interaction between two conserved currents in the one-loop approximation, based on the Lagrangian (13.55). Although we know from the calculation in the linear gauge (13.51), in which the gauge field remains free, that there is no interaction between two external currents beyond the tree approximation, let us check this point explicitly in the gauge (13.53). The one-loop self-energy diagrams that contribute to this process are shown in fig. 13.3. The four-point vertex gives rise to the following contribution (we include a normalization factor $[-i(2\pi)^4]^{-1}$ cf. 2.64)

$$\Pi^{(1)}_{\mu\nu}(k) = \tfrac{1}{2}\kappa^2\Big\{4\,\eta_{\mu\nu}\int \mathrm{d}^4q\;\Delta_\rho{}^\rho(q) + 8\int \mathrm{d}^4q\;\Delta_{\mu\nu}(q)\Big\}\,, \tag{13.66}$$

where $\Delta_{\mu\nu}(q)$ is the photon propagator (13.33).

The calculation of the remaining three diagrams in fig 13.3 is more involved. Using the three-point vertex (13.58) the expression corresponding to the second diagram reads

$$\begin{aligned}\Pi^{(2)}_{\mu\nu}(k) \quad &= \quad -\tfrac{1}{2}\mathrm{i}(2\pi)^2\int \mathrm{d}^4p\,\mathrm{d}^4q\;\delta^4(k+p+q)\;V_{\mu\rho\sigma}(k,p,q)\\ &\qquad \times V_{\nu\rho'\sigma'}(-k,-p,-q)\,\Delta_{\rho\rho'}(p)\,\Delta_{\sigma\sigma'}(q)\,,\end{aligned} \tag{13.67}$$

where the overall factor $\frac{1}{2}$ in the first line is inserted to avoid overcounting the number of ways in which one constructs the diagram by connecting two lines from each of the two vertices (see chapter 2). After substituting the explicit expressions for the vertex (cf. 13.58) we divide the terms into two groups,

$$\Pi^{(2)}_{\mu\nu}(k) = \Pi^{(a)}_{\mu\nu}(k) + \Pi^{(b)}_{\mu\nu}(k)\,, \tag{13.68}$$

such that the first set of the terms are explicitly proportional to k_μ or k_ν,

$$\begin{aligned}\Pi^{(a)}_{\mu\nu}(k) &= -2\mathrm{i}(2\pi)^2\lambda^2\kappa^2 \int \mathrm{d}^4p\,\mathrm{d}^4q\;\delta^4(k+p+q) \\ &\quad\times\Big\{k_\mu k_\nu\,\big(\Delta_{\rho\sigma}(p)\,\Delta^{\rho\sigma}(q)\big) + 2k_\mu\big(p_\sigma\Delta^{\sigma\rho}(p)\,\Delta_{\rho\nu}(q)\big) \\ &\qquad + 2k_\nu\big(p_\sigma\Delta^{\sigma\rho}(p)\,\Delta_{\rho\mu}(q)\big)\Big\}\,. \end{aligned} \tag{13.69}$$

Because the integrals in the last two terms depend only on the external momentum k, this expression must take the following form,

$$\Pi^{(a)}_{\mu\nu}(k) = k_\mu k_\nu\,\Pi(k^2)\,, \tag{13.70}$$

where $\Pi(k^2)$ is some unspecified, but calculable, function (see problem 13.12). The second expression reads,

$$\begin{aligned}\Pi^{(b)}_{\mu\nu}(k) &= -4\mathrm{i}(2\pi)^4\lambda^2\kappa^2 \int \mathrm{d}^4p\,\mathrm{d}^4q\;\delta^4(k+p+q) \\ &\quad\times\Big\{\Delta_{\mu\nu}(q)\,\big(p^\rho p^\sigma\,\Delta^{\rho\sigma}(p)\big) + \big(p^\rho\,\Delta_{\rho\mu}(p)\big)\,\big(q^\sigma\,\Delta_{\sigma\nu}(q)\big)\Big\}\,, \end{aligned} \tag{13.71}$$

where we interchanged p and q for several terms in the integrands in order to combine them. Due to this interchange the last term in (13.69) is not manifestly symmetric in μ and ν. After using (13.59) we can now combine the previous results and find

$$\begin{aligned}\Pi_{\mu\nu}(k) &= \Pi^{(1)}_{\mu\nu}(k) + \Pi^{(2)}_{\mu\nu}(k) \\ &= 2\kappa^2\,\eta_{\mu\nu}\int \mathrm{d}^4q\;\Delta_\rho{}^\rho(q) \\ &\quad - \frac{4}{\mathrm{i}(2\pi)^4}\frac{\kappa^2}{\lambda^2}\int \mathrm{d}^4p\,\mathrm{d}^4q\;\delta^4(k+p+q)\;\frac{p_\mu\,q_\nu}{p^2\,q^2} \\ &\quad + k_\mu k_\nu\,\Pi(k^2)\,. \end{aligned} \tag{13.72}$$

The first term is constant and proportional to the integral $\int \mathrm{d}^4q\,q^{-2}$, so it does not contribute to the imaginary part of the propagator. At any rate, within the context of dimensional regularization this integral vanishes! Therefore, we shall ignore it here[3]. The third term is proportional to $k_\mu k_\nu$ and vanishes when we contract the one-loop propagator with conserved currents. So the only term that concerns us here is the second one, which contributes to the imaginary part of the propagator provided that k_μ is timelike ($k^2 < 0$). Clearly the

[3] Note that this term would give rise to a photon mass. However, it is possible to show by independent arguments that this term should cancel without the need for resorting to dimensional regularization. See problem 13.5.

second term does not vanish, as we expected on the basis of the calculation in the linear gauge (13.51), so that the result is gauge dependent. To make matters worse, the above result also leads to an imaginary part of the photon propagator, thus signaling a breakdown of unitarity!

It turns out that there is a convenient way to characterize the contribution from the second term. Namely, consider a complex scalar field interacting with photons and described by the following Lagrangian

$$\mathcal{L} = -\lambda\,\partial_\mu\phi^*\,\partial^\mu\phi + 2\kappa\,A^\mu\,\phi^*\,\partial_\mu\phi\,. \tag{13.73}$$

This Lagrangian gives rise to a self-energy diagram for the photon and leads to precisely the same result as the second term in (13.72): the propagators yield the terms $(\lambda\,p^2)^{-1}$ and $(\lambda\,q^2)^{-1}$, whereas the two vertices give rise to the terms $\kappa\,p_\mu$ and $\kappa\,q_\nu$. As the reader may verify, also the overall sign is precisely as in the second term in (13.72). Of course we cannot exclude that there are other Lagrangians than (13.73) leading to an identical answer, as we are only interested in the result modulo terms proportional to $k_\mu k_\nu$. Therefore we can add terms that depend on the gauge field exclusively through its divergence $\partial^\mu A_\mu$. However, there are good reasons to write the Lagrangian in the form (13.73), as we shall discuss below.

Of course, even when we were able to prove that all undesirable terms are generated by closed-loop diagrams based on (13.73), the theory remains inconsistent as we are merely isolating the troublesome terms in this way. There is indeed no way out: the ad-hoc gauge-fixing procedure that we have employed is incomplete! The decoupling of the gauge degree of freedom that we have introduced in order to enable us to define propagators and set up perturbation theory, does *not* take place. As it turns out, this degree of freedom can be described by the Lagrangian (13.73). Hence, from our prevous discussion it is clear how the situation could, at least in principle, be cured. Namely, let us add to the theory certain unphysical fields, called *ghost* fields, whose contribution to closed-loop diagrams cancels the unwanted terms. This means that the Lagrangian for the ghost fields should resemble the Lagrangian (13.73), except that its closed-loop contributions should somehow acquire the opposite sign. It may not be immediately clear how to achieve an overall sign change for all the closed-loop diagrams; for instance, one can easily verify that it cannot be done by introducing sign changes into the two terms of the Lagrangian, as this will only change the sign of closed-loop graphs with an odd number or external lines. Yet the answer is obvious: if we assume that the new fields have Fermi-Dirac statistics, just like fermions, then the corresponding closed loops acquire precisely an extra minus sign. For fields corresponding to physical particles this would be disasterous as there is an important theorem in quantum field theory, the so-called *spin-statistics theorem*, according to which particles with integer spin should exhibit Bose-Einstein, and particles with half-integer spin Fermi-Dirac statistics. However, the ghost fields do *not* correspond to physical particles. Although they are spinless, they can nevertheless satisfy Fermi-Dirac statistics.

Therefore the ghost Lagrangian has the same form as (13.73), except that we assign Fermi-Dirac statistics to the scalar fields. We shall define the ghost Lagrangian for more general gauge conditions in the next section, but at this point we already stress that the ghost Lagrangian depends sensitively on the choice of the gauge. Clearly, for the linear gauge condition (i.e., $\kappa = 0$), the ghosts are noninteracting. However, as we shall exhibit later, in certain gauges the ghosts are absent, or do not exhibit the standard kinetic terms that give rise to standard Klein-Gordon propagators. One always has to realize that the ghosts are just there to ensure that physical (on-shell) amplitudes are independent of the gauge condition and that the quantum theory satisfies unitarity, provided that the theory does not have additional shortcomings of some other nature. But for standard gauge theories we know that this is not the case.

13.4. Ghost Lagrangian

In the previous section we discovered that the gauge field Lagrangian, quantized with a quadratic gauge-fixing term, is in general incomplete. There is a mismatch in the number of degrees of freedom, which, in closed-loop diagrams, causes a breakdown of unitarity and gauge independence. In order to correct for this one has to introduce unphysical fields, whose couplings are such that they precisely compensate for the defects. These ghost fields, first introduced by Feynman, DeWitt, Faddeev and Popov, have rather unusual properties, which would be inadmissable for fields describing physical degrees of freedom. We already argued that they violate the connection between spin and statistics. Furthermore the reader may have noticed that the Lagrangian (13.73) that we wrote down in the previous section, is not real although this shortcoming can be avoided. We shall shortly rewrite the ghost Lagrangian so that it takes a real form. Nevertheless it should be clear that the usual rules that one imposes on Lagrangians do not necessarily apply to the ghost fields.

Here we shall not derive the ghost Lagrangian from first principles and a full derivation is clearly outside the scope of this book. Instead we concentrate on those features of (13.73) that are generic for these Lagrangians. Once the underlying systematics is clear, the reader will hopefully appreciate the general prescription for constructing the ghost Lagrangians for arbitrary gauge conditions that we shall present in this section. In the next section the correctness of this procedure shall be confirmed once more, when we verify the unitarity in the one-loop approximation for nonabelian gauge theories in a class of Lorentz gauges.

We start from (13.73), which we first rewrite as

$$\mathcal{L} = \phi^* \big\{ \lambda\, \partial^\mu \partial_\mu \phi + 2\kappa\, A^\mu\, \partial_\mu \phi \big\}\,, \tag{13.74}$$

by dropping a total divergence. Now (13.74) has a suggestive form, as the term multiplying ϕ^* is precisely the effect of an infinitesimal gauge transformation on the gauge-fixing term (13.53), with the gauge parameter $\xi(x)$ replaced by the ghost field $\phi(x)$. Furthermore it is convenient to treat ϕ and ϕ^* as independent fields, not related by complex conjugation, so that the ghost Lagrangian becomes real. To that end we replace ϕ by a ghost field c, and ϕ^* by a so-called antighost field b, multiplied by a factor i. In this way we obtain the ghost action

$$\begin{aligned} \mathcal{L}^{\text{ghost}} &= \mathrm{i} b \Big(\frac{\partial(\delta(\xi)\, F)}{\partial \xi}\, c \Big) \\ &= \mathrm{i} b \,\big\{ \lambda\, \partial^\mu \partial_\mu c + 2\kappa\, A^\mu\, \partial_\mu c \big\}\,, \end{aligned} \tag{13.75}$$

which is equivalent to (13.74). Here $(\partial(\delta(\xi)F)/\partial\xi)\, c$ denotes the change under an infinitesimal gauge transformation of the gauge-fixing term F, with the parameter replaced by the ghost field. More precisely, for the case at hand the gauge-fixing term (13.53) changes under a infinitesimal gauge transformation according to

$$\delta(\xi)\, F = \lambda\, \partial^\mu \partial_\mu \xi + 2\kappa\, A^\mu\, \partial_\mu \xi\,, \tag{13.76}$$

where $\xi(x)$ is the function that characterizes the transformation. Taking the derivative of (13.76) with respect to ξ and multiplying with the ghost field amounts to simply replacing the function $\xi(x)$ by the ghost field $c(x)$.

As b and c are now regarded as independent *real* anticommuting fields, the above Lagrangian is now real; the factor i is required because under an hermitian conjugation of anticommuting fields, the order of the fields should be interchanged. For a ghost bilinear we have therefore $(\mathrm{i}\, b\, c)^\dagger = -\mathrm{i}\, c\, b = \mathrm{i}\, b\, c$. This is similar to hermitean conjugation for fermion fields; the reader may for instance verify that the current $\mathrm{i}\, \bar\psi \gamma_\mu \psi$ is real provided one interchanges the order of the fields under conjugation.

The first line of (13.75) can now be generalized and leads to the following prescription. The full Lagrangian that one needs for the calculation of quantum-mechanical corrections (in perturbation theory) consists of the classical (gauge-invariant) Lagrangian modified according to

$$\mathcal{L}^{\text{quantum}} = \mathcal{L}^{\text{classical}} + \mathcal{L}^{\text{g.f.}}, \tag{13.77}$$

where $\mathcal{L}^{\text{g.f.}}$ contains both the square of the gauge-fixing terms *and* the corresponding ghost Lagrangian,

$$\mathcal{L}^{\text{g.f.}} = -\tfrac{1}{2}(F^a)^2 + \mathrm{i}\, b_a\, \frac{\partial(\delta(\xi)\, F^a)}{\partial \xi^b}\, c^b\,, \tag{13.78}$$

where the indices $a, b, \ldots$ label the generators of the gauge group[4].

Let us emphasize once more that shall not derive this result from first principles, at least not within the context of the presentation here. The addition of the quadratic gauge-fixing terms to the Lagrangian was admittedly somewhat ad hoc, so the reader should not be surprised that further modifications are necessary. On the other hand, one could have foreseen that the addition of nonlocal terms such as given by the ghost Lagrangian are necessary. In section 13.1 we saw that the decomposition of the fields into gauge invariant components and a component that changes under the gauge transformations, was complicated and involved nonlocal field redefinitions (cf. 13.7). The introduction of ghost fields ensures that this decomposition is correctly implemented for general gauges. Although this introduction of the ghosts remains equally ad hoc, we hope that the reader will accept this prescription on the basis of the sytematic features that we exhibit. Needless to say, the result is consistent with everything we found so far, and, as we go along, we will show that the quantum Lagrangian (13.77) gives indeed the desired result for a large variety of theories.

Before continuing, let us get acquainted with the Feynman rules for the ghost fields associated with the abelian theory (cf. 13.75), as given in table 13.1 The propagator has an unusual factor i and furthermore carries an orientation. In chapter 2 we have repeatedly stressed that such an orientation is usually associated with an invariance, such as conservation of electric charge. The corresponding invariance here is *ghost number*; the standard convention is to assign ghost number +1 to the ghosts and ghost number −1 to the antighosts. The arrow on the ghost lines thus indicates the flow of ghost number. The interaction with the ghosts always involves an equal number of ghosts and antighosts; therefore ghost number is conserved. The lines associated with the antighosts have an outgoing arrow, those associated with the ghosts have an incoming arrow. The propagator and vertices corresponding the Lagrangian (13.75) are summarized in table 13.1.

As an illustration, let us consider the diagrams with one closed loop of ghosts and two or three external gauge fields. After inserting all propagators and vertices we are left with the combinatorial factors. These are restricted by the fact that the ghost orientation must be preserved. Denoting the incom-

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[4]This is one of the instances where a description is as accurate as, and yet simpler than, an actual formula. The proper way to write the gauge variation of the gauge-fixing term with the parameter replaced by the ghost field is somewhat complicated and involves a functional derivative

$$\Big(\frac{\partial(\delta(\xi)\,F^a)}{\partial\xi^b}\,c^b\Big)(x) \equiv \int \mathrm{d}^4y\;\frac{\partial(\delta(\xi)\,F^a(x))}{\partial\xi^b(y)}\,c^b(y)\,. \tag{13.79}$$

where we generalized to the nonabelian case with an certain number of independent gauge-fixing terms, gauge parameters and ghost fields. The left-hand side should thus be viewed as a useful mnemonic.

ing photon momenta by $k_1, k_2, \ldots$ (where momentum conservation requires that the sum of the incoming momenta vanishes) and polarization indices by $\mu_1, \mu_2, \ldots$, we find

$$\Gamma_{\mu_1\mu_2}(k_1,k_2) = -(-2\kappa)^2 \int \mathrm{d}^4 q \; \frac{q_{\mu_1}}{i\lambda\, q^2}\,\frac{(q+k_1)_{\mu_2}}{\mathrm{i}\lambda\,(q+k_1)^2}\,, \tag{13.80}$$

where $k_1 + k_2 = 0$, and

$$\begin{aligned}\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) &= -(-2\kappa)^3 \int \mathrm{d}^4 q \; \frac{q_{\mu_1}}{\mathrm{i}\lambda\, q^2} \\ &\times \left\{ \frac{(q+k_1)_{\mu_2}}{\mathrm{i}\lambda\,(q+k_1)^2}\,\frac{(q-k_3)_{\mu_3}}{\mathrm{i}\lambda\,(q-k_3)^2} + \frac{(q+k_1)_{\mu_3}}{\mathrm{i}\lambda\,(q+k_1)^2}\,\frac{(q-k_2)_{\mu_2}}{\mathrm{i}\lambda\,(q-k_2)^2} \right\},\end{aligned} \tag{13.81}$$

where $k_1+k_2+k_3=0$. In the second expression we had to add two diagrams. The overall minus sign originates from the fermionic statistics of the ghost fields, so that each closed loop acquires an extra minus sign. Note that there are no combinatorial factors, as the orientation of the ghosts implies that there is precisely one way to construct each diagram by joining the propagators and the vertices. Finally, one may verify that (13.80) cancels precisely against the second term in (13.72), after taking into account the same normalization factor $[-i(2\pi)^4]^{-1}$ for the self-energy diagram.

The definition (13.78) for the gauge-fixing Lagrangian is applicable to any generic gauge theory (including general relativity!). So let us now see what it gives for nonabelian gauge theories in two typical gauges defined by

$$F^a = \lambda\,\partial^\mu W_\mu^a\,, \qquad F^a = \lambda\, n^\mu W_\mu^a\,. \tag{13.82}$$

The corresponding results for the gauge-fixing Lagrangians are

$$\begin{aligned}\mathcal{L}^{\text{g.f.}} &= -\tfrac{1}{2}\big(\lambda\,\partial^\mu W_\mu^a\big)^2 - \mathrm{i}\lambda\,\partial^\mu b^a\big\{\partial_\mu c^a - f_{bc}{}^a\, W_\mu^b\, c^c\big\}\,,\\ \mathcal{L}^{\text{g.f.}} &= -\tfrac{1}{2}\big(\lambda\, n^\mu W_\mu^a\big)^2 + \mathrm{i}\lambda n^\mu\, b^a\big\{\partial_\mu c^a - f_{bc}{}^a\, W_\mu^b\, c^c\big\}\,,\end{aligned} \tag{13.83}$$

where in the first case we suppressed a total divergence by writing the derivative on the antighost field. Clearly the ghosts are interacting in this case and the interactions are directly related to the nonabelian nature of the theory. The ghost interactions can only be suppressed in certain singular gauges. An example where this happens is the second gauge in (13.82), where the interaction term with the ghosts is proportional to the gauge-fixing term itself: $F^a = \lambda n^\mu W_\mu^a$. In the limit $\lambda \to \infty$ the gauge-fixing term is put to zero, and so are therefore the interactions. It is now also clear why in quantum electrodynamics, where we consistently ignored the presence of the ghost fields, we still obtained the correct results. In those calculations (e.g. in chapter 9) we always worked with the gauge-fixing term $F = \lambda\,\partial_\mu A^\mu$. In the abelian theory the ghosts are then noninteracting, so that they never contribute to connected diagrams with ordinary external fields.

In the first gauge (13.83) the gauge field propagator exhibits four independent poles (cf. 13.33), of which only two correspond to physical photons, while the ghost field propagator exhibits a standard Klein-Gordon pole, which seem to indicate the existence of two corresponding particles (one corresponding to the ghost, the other to the antighost). Obviously the number of unphysical particles associated with the gauge field matches the number of unphysical particles associated with the ghosts. This is not a coincidence, because the ghosts must compensate in the closed-loop diagrams for the effect of the unphysical gauge-field polarizations. This argument is sometimes referred to as "ghost counting". In other gauges with nonstandard propagators, this relation may be less obvious. Nevertheless, for the second gauge (13.83), one can easily establish that the gauge-field propagator contains only two poles (cf. 13.46). This follows, for instance, from the observation that neither $k^\mu\Delta_{\mu\nu}(k)$ nor $n^\mu\Delta_{\mu\nu}(k)$ exhibit a $1/k^2$ pole. As k_μ and n_μ are assumed to be linearly independent (otherwise we have a Lorentz gauge condition, which we already discussed), this implies that there are at most two $1/k^2$ poles left, while the ghost propagator has no $1/k^2$ pole at all. Of course, there are $1/k\cdot n$ poles, but their interpretation is different and less clear in this context.

This suggests that there must be a rather direct relation between the gauge-field and ghost propagators. In the general gauge discussed in section 13.2, these two propagators take the form

$$\begin{aligned}\Delta_{\mu\nu}(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\Big\{\eta_{\mu\nu} + \frac{k^2+\bar\lambda(k)\cdot\lambda(k)}{|k\cdot\lambda(k)|^2}k_\mu k_\nu - \frac{\bar\lambda_\mu(k)k_\nu}{k\cdot\bar\lambda(k)} - \frac{k_\mu\lambda_\nu(k)}{k\cdot\lambda(k)}\Big\},\\ \Delta(k) &= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k\cdot\lambda(k)}\,. \end{aligned}\tag{13.84}$$

Indeed, according to (13.25) there is a linear relation between these two propagators, which reads

$$\Delta_{\mu\nu}(k)\,\lambda^\nu(k) = \Delta(k)\,k_\mu\,. \tag{13.85}$$

This identity plays an important role in establishing that physical quantities do not depend on the choice of the gauge condition[5].

We close this section by briefly discussing a more formal property of the quantized theory based on (13.77). Obviously, the gauge-fixing procedure followed above is intended to break the gauge invariance of the original classical

[5]We do *not* intend to use arbitrary functions $\lambda_\mu(k)$ and work in momentum space, as the nonabelian tranformation rules in momentum space are rather complicated. In practice we shall take $\lambda_\mu(k)$ to be a constant vector or linearly proportional to the momentum components and work in coordinate space, so that the gauge-fixing terms are linear combinations of the gauge-field components or of their derivatives.

theory. Nevertheless there is another symmetry that emerges, which is *not* a local symmetry, although it can be regarded as a remnant of the original gauge symmetry. This symmetry is called BRST symmetry, after its discoverers, Becchi, Rouet, Stora and Tyutin. In the context of this book it is not possible to discuss it in much detail, but we should stress that it plays a principal role in establishing important properties of the theory, such as its unitarity, gauge independence of physical results and its possible renormalizability. To see how this symmetry emerges, consider a special gauge transformation with the gauge parameter proportional to the ghost field. So we have (for the moment, let us restrict ourselves to the abelian theory) $\delta A_\mu \propto \partial_\mu c$. As the classical Lagrangian is invariant under arbitrary local gauge transformations, this transformation still leaves the classical Lagrangian in (13.77) invariant. However, the gauge-fixing Lagrangian (13.78) is not invariant under this transformation. First the variation of $-\frac{1}{2}F^2$ gives rise to a term $-F^a\,(\partial(\delta(\xi)F)/\partial\xi)\,c$, which contains the same factor as the ghost Lagrangian. Therefore this variation can be cancelled by assigning a transformation to the antighost field proportional to the gauge-fixing term F (obviously, gauge transformations are not defined for the (anti)ghost fields, so that the BRST variations for these fields must be separately defined). Then there is a second variation due to the change of A_μ in the ghost Lagrangian. As it turns out this term vanishes owing to the anticommuting character the ghost fields. To see this we recall that the ghost Lagrangian contains the variation of the gauge-fixing term F under an infinitesimal gauge transformation with its parameter proportional to the ghost field. Of this term we must now determine the effect of a second gauge variation with again the ghost field as its parameter. So we determine the variation under two consecutive infinitesimal gauge transformations, i.e., $\delta(\xi_1)\,\delta(\xi_2)\,F$, and subsequently replace the parameters ξ_1 and ξ_2 by the ghost field c. However, because of the fact that the gauge symmetry is abelian, the product of two infinitesimal symmetry variations does not depend on the order in which they are applied,

$$\delta(\xi_1)\,\delta(\xi_2)\,F = \delta(\xi_1)\,\delta(\xi_2)\,F\,, \tag{13.86}$$

so that the variation is symmetric in ξ_1 and ξ_2. Therefore, substituting the anticommuting ghost field c for both ξ_1 and ξ_2 gives zero.

It thus follows that the full Lagrangian (13.77) is invariant under the so-called BRST transformations

$$\delta A_\mu = \partial_\mu(\mathrm{i}\Lambda\, c)\,, \qquad \delta b = \Lambda\, F\,, \qquad \delta c = 0\,. \tag{13.87}$$

Here we have introduced a (constant) anticommuting parameter Λ to give the field and its variation the same statistics. The factor i is added in order to have the correct reality properties.

The BRST invariance is a general feature of all gauge theories, irrespective of the symmetry group and the precise choice for the gauge condition. For completeness we give the transformation for the nonabelian case as well,

$$\delta A_\mu = D_\mu(\mathrm{i}\Lambda\, c^a)\,, \qquad \delta b^a = \Lambda\, F^a\,, \qquad \delta c^a = -\tfrac{1}{2}\mathrm{i}\, f_{bc}{}^a\, c^b\, \Lambda\, c^c\,. \tag{13.88}$$

The transformation of the gauge field is just an infinitesimal gauge transformation with its parameter ξ^a replaced by $(\mathrm{i}\Lambda\, c^a)$. On possible matter fields the BRST tranformations take the same form of a ghost-dependent gauge transformation. For the proof of the nonabelian invariance we refer to problem 13.8.

13.5. Feynman rules in the Lorentz gauge

In this section we summarize the Feynman rules for a generic theory of nonabelian gauge fields coupled to spin-$\frac{1}{2}$ and spin-0 fields transforming in some representation of the gauge group. The reason for choosing the Lorentz gauge is that it is of the renormalizable type and therefore very suitable for dealing with the ultraviolet divergences in the Feynman diagrams. On the basis of the discussion in the previous sections, the reader should have no difficulty in writing down the Feynman rules in some alternative gauges.

The Lagrangian consists of four different parts,

$$\mathcal{L} = \mathcal{L}_W + \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}^{\mathrm{g.f.}}\,. \tag{13.89}$$

The first one is the classical gauge-field Lagrangian discussed in chapter 12 (cf. 12.71),

$$\begin{aligned}\mathcal{L}_W \;\; = \;\; & -\tfrac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)^2 + g\, f_{abc}\, W^{\mu\, a}\, W_\nu^b\, \partial_\mu W^{c\,\nu} \\ & -\tfrac{1}{4}g^2 f_{abc} f_{ade}\, W_\mu^b\, W^{d\,\mu}\, W_\nu^c\, W^{e\,\nu}.\end{aligned} \tag{13.90}$$

Then we have the gauge-invariant Lagrangians for the matter fields (cf. 12.49) and (12.64),

$$\begin{aligned}\mathcal{L}_\psi \;\; &= \;\; -\bar{\psi}_i \not{\partial} \psi_i - m\, \bar{\psi}_i\psi_i + g\,(t_a)_{ij}\, W_\mu^a\, \bar{\psi}_i\gamma^\mu\psi_j\,, \\ \mathcal{L}_\phi \;\; &= \;\; -\partial_\mu\phi_i^*\, \partial^\mu\phi_i - m^2\, \phi_i^*\, \phi_i - g\,(t_a)_{ij}\, W_\mu^a(\phi_i^*\, \partial_\mu\phi_j - (\partial_\mu\phi_i^*)\, \phi_j) \\ & \qquad + \tfrac{1}{2}g^2\, (t_a t_b + t_b t_a)_{ij}\, W_\mu^a\, W^{b\mu}\, \phi_i^*\, \phi_j\,,\end{aligned} \tag{13.91}$$

where we have assumed that the fermions and the bosons constitute a single irreducible representation of the gauge group and suppressed possible matter interactions. Fermion and boson fields were labeled by the same indices $i, j, \ldots$ and we used the same notation for the generators t_a. Note also that we have assumed complex spinless fields and that the gauge fields couple to exclusively to vector fermionic currents (axial-vector couplings were discussed in section 7.1).

Finally we have the Lagrangian for the gauge-fixing terms corresponding to the Lorentz gauge $F^a = \lambda\,\partial^\mu W_\mu^a$,

$$\mathcal{L}^{\text{g.f.}} = -\tfrac{1}{2}\lambda^2\big(\partial_\mu W^{a\,\mu}\big)^2 - \mathrm{i}\lambda\,\partial^\mu b^a\,\partial_\mu c^a - \mathrm{i}\lambda g\, f_{abc}\,W_\mu^a\,(\partial^\mu b^b)\,c^c\,. \qquad (13.92)$$

We restricted ourselves to a compact simple gauge group. The gauge-field indices $a, b, \ldots$ can therefore be raised and lowered without further ado and the structure constants f_{abc} are fully antisymmetric. We return to this aspect in section 14.1, where we will further discuss normalization conventions.

The Feynman rules are now easily written down. The propagators follow in the standard way from the terms quadratic in the fields. They have all been encountered before and are listed in table 13.2.

The vertices are given in table 13.3. Here it is convenient to sum over all possible attachments of the external lines. This is the reason why the gauge-field vertices exhibit so many terms. When considering closed-loop diagrams this may require to divide by certain combinatorial factors to avoid overcounting the independent ways of forming the diagram. We shall encounter examples of this in the subsequent chapters.

13.6. Ward identities and one-loop unitarity

Consider amplitudes for on-shell particles with external gauge fields that are themselves not necessarily on their respective mass shells. The gauge field interactions are treated in lowest order, but other possible interactions leading to closed loops, are included to some arbitrary order. For our purpose it is not necessary to know what precisely the on-shell particles are, but for definiteness on may consider the situation discussed in section 11.6, where we have two external scalar particles coupling to the gauge fields.

The amplitude that describes one gauge field interacting with a certain configuration of on-shell particles will be denoted by $\mathcal{M}_\mu^a(k)$. We indicate only the indices of the gauge field and its incoming four-momentum. The gauge-field momentum must balance the total momentum carried by the various particles. Likewise, the amplitude for *two* gauge fields interacting with the on-shell particles is denoted by $\mathcal{M}_{\mu\nu}^{ab}(k^a, k^b)$. Unlike in section 11.6 we keep *all* gauge-field momenta incoming, so that the amplitude is symmetric under the simultaneous interchange of the gauge-field indices and momenta. The amplitude contains all the relevant diagrams, in particular, the diagram where a single gauge field interacts with the particles and subsequently splits into two other gauge fields. However, we repeat that the gauge fields are not contained in closed loops.

As we have stressed repeatedly, these amplitudes must satisfy certain Ward identities. The amplitude with one gauge field should vanish when contracting the four-vector index associated with the external gauge field with the corresponding momentum,

$$k^\mu \mathcal{M}^a_\mu(k) = 0\,. \tag{13.93}$$

For the amplitude with more gauge fields there are corresponding identities. In section 11.6 we considered such identities for the case of two gauge fields with two scalar on-shell particles in the Born approximation. It turns out that these identities have a more general validity. For two gauge fields the corresponding identies can be found from comparison with (11.79). For our purposes they can be written as follows (to do this we made use of (13.93)),

$$k^\mu \mathcal{M}^{a\,b}_{\mu\nu}(k,p) = -if^{ab}{}_c\,\frac{p^2\delta^\rho_\nu - p_\nu\, p^\rho}{(p+k)^2}\,\mathcal{M}^c_\rho(p+k)\,. \tag{13.94}$$

As explained in chapter 11, the right-hand side vanishes when the second gauge-field momentum is on its mass shell, $p^2 = 0$, and carries transverse polarizations.

Similarly the identity for the amplitude with three gauge fields takes the form

$$\begin{aligned} k^\mu \mathcal{M}^{a\,b\,c}_{\mu\nu\rho}(k,p,q) \quad = \quad & -if^{ab}{}_d\,\frac{p^2\delta^\sigma_\nu - p_\nu\, p^\sigma}{(p+k)^2}\,\mathcal{M}^{d\,c}_{\sigma\rho}(p+k,q) \\ & -if^{ac}{}_d\,\frac{q^2\delta^\sigma_\rho - q_\rho\, q^\sigma}{(q+k)^2}\,\mathcal{M}^{b\,d}_{\nu\sigma}(p,q+k)\,. \end{aligned} \tag{13.95}$$

The structure of these Ward identities is rather clear. They connect the amplitude with n gauge fields (and a given number of on-shell fields) to the corresponding amplitude with $n-1$ gauge fields. The pole terms originate from the graphs in which a smaller number of gauge fields couples to the on-shell fields; the gauge fields then branch out by means of tree diagrams so that one ends up with precisely n gauge fields. Again it is clear that these amplitudes are conserved once the $n-1$ gauge fields are on shell and carry tranverse polarizations.

Figure 13.4: Some typical diagrams with gauge-field propagator poles

The derivation of these Ward identities is beyond the scope of this book. Here it suffices to say that they can be worked out for special cases in tree approximation, as we did in section 11.6 and in problem 13.9. A rigorous derivation is based on the BRST invariance of the quantized action.

Although the amplitudes are thus conserved for on-shell tranverse gauge fields, this is not sufficient when considering gauge fields in closed loops, as the polarizations of the gauge field are then arbitrary. The remainder of this section is devoted to the discussion of a toy model in which we demonstrate that ghost fields are necessary in order to have a unitary and gauge-independent theory. In this discussion the Ward identities play a central role. Consider two types of particles, which we call "quarks" and "leptons". Just as in the real world leptons and quarks have no direct interactions other from those via the gauge fields (in this toy model there are only massless gauge fields). It thus makes sense to discuss two types of amplitudes, depending on the type of particles involved, coupled to a certain number of gauge fields, just as before. We will denote these amplitudes by $\mathcal{Q}$ and $\mathcal{L}$. Now consider the transitions of quarks into leptons by the exchange of a pair of gauge fields, pictorially represented in fig 13.5. The corresponding expression is given by

$$\mathcal{M} = \mathcal{L}^{a}_{\rho\sigma} b(k,p)\Delta^{\mu\rho}\Delta^{\mu\rho}\mathcal{Q}^{a\,b}_{\mu\nu}(k,p)\,. \tag{13.96}$$

Figure 13.5: Transition of "quarks" into "leptons" by the exchange of two gauge fields.

Problems

13.1. In section 12.2 we described the technique of quadratic gauge fixing, whose justification can be given in the context of ordinary linear algebra. Suppose we are given an $n \times n$ symmetric matrix D_{ij} with one zero eigenvalue. By a suitable orthogonal transformation such a matrix can be brought in the form

$$D_{ij} = \begin{pmatrix} \Lambda_{IJ} & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

where indices i, j run from 1 to n and indices I, J from 1 to $n-1$. The submatrix Λ is regular, i.e. $\det\Lambda \neq 0$. We are interested in the inverse of Λ, which we want to determine without first bringing D into the form (1). According to the text in section 12.2, this can be achieved by introducing an arbitrary array of n parameters λ_i with $\lambda_n \neq 0$, and by considering the inverse of $D_{ij} + \lambda_i\,\lambda_j$. In the $(n-1)$-dimensional subspace orthogonal to the null vector of D, this matrix is claimed to coincide with Λ^{-1}_{IJ}. To see this, show that the inverse of $D_{ij}+\lambda_i\,\lambda_j$, when decomposed as in (1), takes the form

$$\big(D_{ij} + \lambda_i\,\lambda_j\big)^{-1} = \begin{pmatrix} \Lambda^{-1}_{IJ} & -\dfrac{1}{\lambda_n}\,\Lambda^{-1}_{IK}\,\lambda_K \\ -\dfrac{1}{\lambda_n}\,\lambda_K\Lambda^{-1}_{KJ} & \dfrac{1}{\lambda_n^2}\big(1+\lambda_K\Lambda^{-1}_{KL}\lambda_L\big) \end{pmatrix} \tag{2}$$

Indeed in the subspace orthogonal to the null vector (the space of vectors with vanishing component along the n-th direction), the matrix (2) is equal to Λ^{-1}_{IJ}. It is obvious that this result can also be applied straightforwardly without first bringing D in the special form (1), as this amounts only to a rotation of the parameters λ_i, which were arbitrary from the start (apart from the condition that the vector λ_i is not orthogonal to the null vector, a condition that is invariant under rotations).

13.2. Derive the analogue of the Lagrangian (13.13) for scalar electrodynamics. Construct the Feynman rules for the interaction of the complex scalar field with the a_i and ϕ fields. Reproduce the scattering amplitude

$$\mathcal{M} = \frac{1}{t}\Big\{e_1 e_2\,(s-u)\Big\} \tag{1}$$

of problem 11.3.

13.3. Consider the various gauges introduced in subsection 13.2, and write down the corresponding ghost Lagrangians, both for the abelian and for the non-abelian case. Consider ghost-field loops and argue that their result does not depend on the overall scale of the gauge parameters λ_μ, so that these diagrams remain for singular gauges. Argue how they are still irrelevant for the singular gauge condition $n^\mu W^a_\mu = 0$.

13.4. Verify that the nonlinear terms in the gauge (13.53) lead to contributions to the pion-Compton scattering amplitude (cf. 4.65), which vanish on-shell. Show that the amplitude is no longer conserved (with the pions on the mass shell) for arbitrary polarizations of the second photon.

13.5. As the gauge-fixing term F should somehow decouple from the physical subsector of the theory, its vacuum-expectation value, defined as the sum of all possible tadpole graphs emanating from F, should be zero (this result can be derived in a more rigorous fashion from the BRST invariance of the theory). Consider now the gauge-fixing term (13.53) modified by an additive constant c,

$$F = \lambda\,\partial_\mu A^\mu + \kappa\, A_\mu A^\mu + c\,. \tag{1}$$

Show that c must vanish in tree approximation. Show also that a nonzero value for c would lead to a gauge-dependent mass term for the photon field.
Now evaluate the vacuum-expectation value for F in the one-loop approximation. As a single photon field cannot disappear into the vacuum because of Lorentz invariance, the first term in F does not contribute. The second term contributes in the one-loop approximation, as both fields can be connected by a propagator. The last term is just given by the constant. Show that we obtain

$$\langle F\rangle = \kappa \int \mathrm{d}^4 q\; \Delta_\mu{}^\mu(q) + c\,.$$

This result should vanish and this condition leads to an expression for the constant c. Demonstrate now that this particular value for c gives rise to a photon mass term that precisely cancels the first term in (13.72). For references, see B. de Wit, Phys. Rev. D12 (1973) 1843, B. de Wit and N. Papanicolaou, Nucl. Phys. B133 (1976) 261.

13.6. Consider scalar electrodynamics with the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \tfrac{1}{4}F_{\mu\nu}^2 - \mathrm{i}eA_\mu(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)\\ &- e^2\,A_\mu A^\mu\,\phi^*\phi - g(\phi^*\phi)^2 - \tfrac{1}{2}\big(\lambda\partial_\mu A^\mu +\\ &\qquad\qquad \kappa_1 A_\mu A^\mu + \kappa_2\phi^*\phi + c\big)^2 - \mathrm{i}\lambda\,\partial_\mu b\,\partial^\mu c + 2\mathrm{i}\kappa\,A^\mu b\,\partial_\mu c\,.\end{aligned}$$

Write down the one-loop self-energy diagrams for the ϕ field. Show that, besides the usual diagrams, there are new diagrams that contain vertices proportional to $\lambda\kappa_2$, κ_2^2 and $\kappa_1\kappa_2$. Fix again the value of c by requiring the vacuum-expectation of the gauge-fixing term to vanish. With this result, show that the mass of the field ϕ is gauge independent in the one-loop approximation. Likewise, show that the photon field remains massless in this approximation.

13.7. Verify explicitly the invariance of (13.77) under the transformations (13.87) for the abelian theory with the gauge fixing term (13.53).

13.8. Prove the invariance of (13.78) for a non-abelian gauge theory. The transformations of the ordinary fields are the infinitesimal gauge transformations with the gauge parameters ξ^a proportional to c^a as indicated in (13.87). As in the abelian case the variation of the square of the gauge-fixing term cancels against the variation of the antighost field b^a. In the nonabelian case the expression

$$\frac{\partial(\delta(\xi)\,F^a)}{\partial\xi^b}\,c^b \tag{1}$$

is no longer invariant but cancels against a variation of the ghost field. To see this, note that the variation of the ordinary fields leads to the change of F^a under two consecutive infinitesimal gauge transformations, i.e., $\delta(\xi_1)\,\delta(\xi_2)\,F^a$, with the parameters ξ_1 and ξ_2 replaced by the ghost fields. However, the gauge transformations do *not* commute. Instead the commutator of two nonabelian infinitesimal transformations is again equal to an infinitesimal gauge transformation, say with a parameter ξ_3^a, and we must have uniformly on all fields,

$$\delta(\xi_1)\,\delta(\xi_2) - \delta(\xi_2)\,\delta(\xi_1) = \delta(\xi_3)\,, \tag{2}$$

where $\xi_3^a = f_{bc}{}^a\,\xi_2^b\,\xi_1^c$. Verify the correctness of (2) for the gauge fields and for matter fields transforming in some representation of the gauge group with generators t_a. Subsequently, apply this result to the gauge-fixing terms F^a and show that the BRST transformations leave the Lagrangian (13.78) invariant.

13.9. In subsection 11.6 we derived the Ward identities (11.79) by requiring that the amplitude for two different gauge fields to on-shell spinless fields be conserved, at least when the second gauge field is on the mass shell and carries a physical (i.e. transverse) polarization. On the basis of this result we found that the gauge fields

must couple to charges that are in general nonabelian, i.e., they should satisfy the commutation relation (11.80). When setting up the nonabelian gauge theories in chapter section 12, we found indeed the same result for the charges (cf. subsection 12.3). In this problem we will repeat the derivation of the Ward identity, but now for fermions and in the context of the nonabelian theories constructed in chapter 12 Hence consider the Lagrangians (12.b37) and (12.c71) and consider the amplitude for two gauge fields interacting with two fermions in some arbitrary representation of the gauge group characterized by the matrices t_a. In tree approximation there are three diagrams, two of them containing a fermion propagator and one a gauge field propagator.
Show that the amplitude with incoming fermion with momentum $\mathbf{p}_1$ and outgoing fermion with momentum $\mathbf{p}_2$, and two off-shell gauge fields with incoming momenta q_1 and q_2, polarization indices μ and ν and group indices a and b, respectively, is given by

$$\begin{aligned}\mathcal{M} \;=\;& t_b\, t_a\, \frac{\bar{u}(\mathbf{p}_2)\big(\gamma_\nu(-\mathrm{i}(\not{p}_1+\not{q}_1)+m)\gamma_\mu\big)u(\mathbf{p}_1)}{q_1^2+2\,q_1\cdot p_1}\\ &+t_a\, t_b\, \frac{\bar{u}(\mathbf{p}_2)\big(\gamma_\mu(-\mathrm{i}(\not{p}_2+\not{q}_1)+m)\gamma_\mu\big)u(\mathbf{p}_1)}{q_2^2+2\,q_2\cdot p_1}\\ &+\mathrm{i}f_{abc}\, t_c\, \frac{\bar{u}(\mathbf{p}_2)\big(\eta_{\mu\nu}(\not{q}_2-\not{q}_1)+2q_{1\nu}\,\gamma_\mu-2q_{2\mu}\,\gamma_n\big)u(\mathbf{p}_1)}{(q_1+q_2)^2}\end{aligned}$$

etc.
Compare to the divergence equation (13.94).

13.10. Derive the Ward identity for two and three gauge fields for amplitudes that are 1PI with respect to the gauge field. In this way one could presumably set up an iterative proof of all identies, but probably this goes too far. (the poles must cancel for 1PI !)

13.11. Consider two scalars with three gauge fields. Give a little guidance to avoid long calculations and check the Ward identity. OR, do this only for the 1PI parts and make use of the previous problem. (work iteratively according to $3 = 2 + 1$ etc.)

13.12. Show that the function $\Pi(k^2)$ in (13.70) equals,

$$\Pi(k^2) = \tag{1}$$

by using that the function can only depend on the external momentum k_μ and must be Lorentz invariant.