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Nonabelian gauge theories

In the previous chapter we gave a general introduction to theories with local gauge invariance, discussing the main concepts and their consequences. In order to demonstrate these ideas we stayed primarily within the context of theories, such as electrodynamics, that are invariant under local phase transformations. However, the same framework can be applied to theories that are invariant under more complicated gauge transformations. Such transformations may depend on several parameters and do not necessarily commute, so that the result of two consecutive transformations may depend on the order in which the transformations are performed. Groups that contain noncommuting transformations are called *nonabelian*. This is in contradistinction to phase transformations, which depend on a single parameter ξ and are obviously commuting (and therefore called abelian).

The fact that the gauge transformations depend on several parameters will force us to introduce a corresponding number of independent gauge fields. Also the matter fields must have a certain multiplicity in order that the gauge transformations can act on them. In more mathematical terms, the matter fields constitute so-called *representations* of the gauge group. The fields belonging to a representation transform among themselves, just as the components of a three-dimensional vector transform among themselves under the group of rotations. The coupling of the gauge fields to matter will involve certain matrices, which will also appear in the expressions for the charges. We already considered such charges at the end of the previous chapter, where we discovered that the gauge fields must be self-interacting when these matrices do not commute. In other words, they are not themselves neutral such as the photon, which is electrically neutral and therefore does not couple to itself, at least not classically.

In this chapter we follow the same strategy as in section 11.2, starting from a Lagrangian that is invariant under rigid nonabelian transformations. Subsequently we extend this rigid invariance to a local one. After a brief discussion of nonabelian groups we introduce the corresponding nonabelian gauge fields, derive their transformation properties and construct gauge invariant Lagrangians. Finally, we discuss chiral representations and exhibit how parity reversal and charge conjugation act on gauge fields. The latter topics, which are a little more technical, provide the necessary background material for the construction of realistic gauge-field models, a subject that we deal with in

later chapters.

12.1. Nonabelian gauge fields

The groups that are relevant in the context of gauge theories consist of transformations, usually represented by matrices, that can be parametrized in an analytic fashion in terms of a finite number of parameters.¹ Such groups are called *Lie groups*. The number of independent parameters defines the *dimension* of the group (which is unrelated to the dimension of the matrices). For instance, phase transformations constitute the group $U(1)$, which is clearly of dimension one. As mentioned above, this group is abelian because phase transformations commute. Two important classes of nonabelian groups are the groups $SO(N)$ of real rotations in N dimensions ($N > 2$), and the groups $SU(N)$ of $N \times N$ unitary matrices with unit determinant ($N > 1$). As we shall see in a moment the dimension of these groups is equal to $\frac{1}{2}N(N-1)$ and to $N^2 - 1$, respectively.

Let us generally consider fields that transform according to some representation of a certain Lie group G . This means that, for every element of the group G , we have a matrix U ; these matrices U satisfy the same multiplication rules as the corresponding elements of G . Under a group transformation the fields rotate as follows

$$\psi(x) \rightarrow \psi'(x) = U \psi(x), \quad (12.1)$$

where ψ denotes an array of different fields written as a column vector. More explicitly, we may write

$$\psi_i(x) \rightarrow \psi'_i(x) = U_{ij} \psi_j(x). \quad (12.2)$$

For most groups (see appendix C) the matrices U can be written in exponential form

$$U = \exp(\xi^a t_a), \quad (12.3)$$

where the matrices t_a are called the *generators* of the group defined in the representation appropriate to ψ , and the ξ^a constitute a set of linearly independent *real* parameters in terms of which the group elements can be described. The number of generators, which is obviously equal to the number of independent parameters ξ^a and therefore to the dimension of the group, is unrelated to the dimension of the matrices U and t_a . It is often straightforward to determine the generators for a given group. For example, the generators

¹An elementary introduction to group theory and references to the literature can be found in appendix C.

t_a of the $\text{SO}(N)$ group must consist of the $N \times N$ real and antisymmetric matrices, in order that (12.2) defines an orthogonal matrix: $U^T = U^{-1}$. As there are $\frac{1}{2}N(N-1)$ independent real and antisymmetric matrices the dimension of the $\text{SO}(N)$ group is equal to $\frac{1}{2}N(N-1)$. For the $\text{SU}(N)$ group, the defining relation $U^\dagger = U^{-1}$ requires the generators t_a to be antihermitean $N \times N$ matrices. Furthermore, to have a matrix (12.2) with unit determinant it is necessary that these antihermitean matrices t_a are traceless. There are $N^2 - 1$ independent antihermitean traceless matrices so that the dimension of $\text{SU}(N)$ is equal to $N^2 - 1$. To verify these properties it is usually sufficient to consider infinitesimal transformations, where the parameters ξ^a are small so that $U = 1 + \xi^a t_a + \mathcal{O}(\xi^2)$.

Because the matrices U defined in (12.2) constitute a representation of the group, products of these matrices must be expressible into the same exponential form. This leads to an important condition on the matrices t_a , which can be derived by considering a product of two infinitesimal transformations (see problem 12.1): the matrices t_a generate a group representation if and only if their commutators can be decomposed into the same set of generators. In other words, the generators must close under commutation. These commutation relations define the *Lie algebra* \mathfrak{g} corresponding to the Lie group G ,

$$[t_a, t_b] = f_{ab}^c t_c, \quad (12.4)$$

where the proportionality constants f_{ab}^c are called the *structure constants*, because they define the multiplication properties of the Lie group. As we shall see the Lie algebra relation (12.4) plays a central role in what follows. As the multiplication properties of the group transformations should be the same in different representations, the structure constants are always the same, irrespective of the representation one considers. We refrain from giving further group-theoretical details here and refer the reader to appendix C for a more general and thorough discussion of Lie groups and their representations.

Let us now follow the same approach as in section 11.2 and consider the extension of the group G to a group of *local* gauge transformations. This means that the parameters of G will become functions of the space-time coordinates x^μ . As long as one considers variations of the field at a single point in space-time this extension is trivial, but the local character of the transformations becomes important when comparing changes at different space-time points. In particular this is relevant when considering the effect of local transformations on derivatives of the fields, i.e.,

$$\psi(x) \rightarrow \psi'(x) = U(x) \psi(x), \quad (12.5)$$

$$\partial_\mu \psi(x) \rightarrow (\partial_\mu \psi(x))' = U(x) \partial_\mu \psi(x) + (\partial_\mu U(x)) \psi(x). \quad (12.6)$$

Just as in section 11.2 local quantities such as ψ , which transform according to a representation of the group G at the *same* space-time point, are called

covariant. Due to the presence of the second term on the right-hand side of (12.6), $\partial_\mu\psi$ does not transform covariantly. Although the action of the space-time dependent extension of G is still correctly realized by (12.5), this type of behaviour under symmetry variations is difficult to work with. Therefore one attempts to replace ∂_μ by a so-called *covariant derivative* D_μ , which constitutes a covariant quantity when applied to ψ ,

$$D_\mu\psi(x) \rightarrow (D_\mu\psi(x))' = U(x) D_\mu\psi(x). \quad (12.7)$$

The construction of a covariant derivative has been discussed in the previous chapter for abelian transformations, where it was noted that a covariant derivative can be viewed as the result of a particular combination of an infinitesimal displacement generated by the ordinary derivative and a field-dependent infinitesimal gauge transformation. Such an infinitesimal displacement was called a *covariant translation*. Its form suggests an immediate generalization to the covariant derivative for an arbitrary group. Namely, we take the linear combination of an ordinary derivative and an infinitesimal gauge transformation, where the parameters of the latter *define* the nonabelian gauge fields. Hence

$$D_\mu\psi \equiv \partial_\mu\psi - W_\mu\psi, \quad (12.8)$$

where W_μ is a matrix of the type generated by an infinitesimal gauge transformation. This means that W_μ takes values in the Lie-algebra corresponding to the group G , i.e., W_μ can be decomposed into the generators t_a ,

$$W_\mu = W_\mu^a t_a. \quad (12.9)$$

Indeed W_μ has the characteristic feature of a gauge field, as it can carry information regarding the group from one space-time point to another.

Let us now examine the consequences of (12.7). Combining (12.5) and (12.6) shows that $W_\mu\psi$ must transform under gauge transformations as

$$\begin{aligned} (W_\mu\psi)' &= \partial_\mu\psi' - (D_\mu\psi)' \\ &= U(\partial_\mu\psi) + (\partial_\mu U)\psi - U(D_\mu\psi) \\ &= \{UW_\mu U^{-1} + (\partial_\mu U)U^{-1}\}\psi'. \end{aligned} \quad (12.10)$$

This implies the following transformation rule for W_μ ,

$$W_\mu \rightarrow W'_\mu = UW_\mu U^{-1} + (\partial_\mu U)U^{-1}. \quad (12.11)$$

Clearly the gauge fields do not transform covariantly. Before verifying the consistency of this result we note that (12.11) defines a group, because successive application of two transformations U_1 and U_2 on W_μ gives the same

result as directly applying a single transformation $U_3 = U_1 U_2$,

$$\begin{aligned} W_\mu \rightarrow W'_\mu &= U_2(U_1 W_\mu U_1^{-1} + (\partial_\mu U_1)U_1^{-1})U_2^{-1} + (\partial_\mu U_2)U_2^{-1} \\ &= (U_2 U_1)W_\mu (U_2 U_1)^{-1} + \partial_\mu (U_2 U_1)(U_2 U_1)^{-1}. \end{aligned} \quad (12.12)$$

The consistency of (12.11) requires that the right-hand side is also Lie-algebra valued. To verify this for the term $(\partial_\mu U)U^{-1}$ we note that $U(x)$ displaced over an infinitesimally small distance dx^μ will differ from $U(x)$ by an infinitesimal gauge transformation, because we assume $U(x)$ to be differentiable as a function of the space-time coordinates. Hence we may write

$$U(x^\mu + dx^\mu) = (I + c_\mu^a(x) dx^\mu t_a + \dots) U(x). \quad (12.13)$$

On the other hand we can use a Taylor expansion for the left-hand side,

$$U(x^\mu + dx^\mu) = U(x) + \partial_\mu U(x) dx^\mu + \dots \quad (12.14)$$

Comparing (12.13) and (12.14) shows that

$$(\partial_\mu U)U^{-1} = c_\mu^a t_a, \quad (12.15)$$

which is indeed Lie-algebra valued. For an infinitesimal transformation one easily verifies that

$$\begin{aligned} (\partial_\mu U)U^{-1} &= (\partial_\mu \xi^a t_a + O(\xi^2))(I - \xi^a t_a + O(\xi^2)) \\ &= \partial_\mu \xi^a(x) t_a + O(\xi^2), \end{aligned} \quad (12.16)$$

while for an abelian gauge group (i.e., a group where all the generators commute) this result becomes exact by virtue of

$$\partial_\mu U = \partial_\mu \xi^a t_a \exp(\xi^b t_b) = \partial_\mu \xi^a t_a U. \quad (12.17)$$

Also the first term in (12.11) is consistent with the ansatz (12.8). This follows simply from the observation that a group element X , when sandwiched between the transformation matrix U and its inverse, is again an element of the group, i.e.,

$$Y = U X U^{-1} \in \text{in } G. \quad (12.18)$$

Near the identity X and Y can be parametrized according to

$$\begin{aligned} X &= I + \xi^a t_a + O(\xi^2), \\ Y &= I + \tilde{\xi}^a t_a + O(\tilde{\xi}^2). \end{aligned} \quad (12.19)$$

Substituting this into (12.18) shows that $\tilde{\xi}^a = O(\xi)$. Therefore, expanding $\tilde{\xi}$ in a power series in ξ and comparing terms linear in ξ , it follows that

Ut_aU^{-1} can again be decomposed in terms of the matrices t_a . Consequently the term $UW_\mu U^{-1}$ in (12.11) is also Lie-algebra valued; in fact this term specifies that the fields W_μ^a transform in the so-called *adjoint representation*, whose dimension is equal to the dimension of the group itself! (cf. appendix C; for an alternative proof, see problem C.2).

The result (12.11) thus shows that the gauge fields W_μ^a transform according to the adjoint representation of the group, modified by an inhomogeneous term. It is easy to evaluate (12.11) for infinitesimal transformations by using (12.3) and we find

$$W_\mu^a \rightarrow (W_\mu^a)' = W_\mu^a + f_{bc}{}^a \xi^b W_\mu^c + \partial_\mu \xi^a + O(\xi^2). \quad (12.20)$$

This result differs from the transformation law of abelian gauge fields by the presence of the term $f_{bc}{}^a \xi^b W_\mu^c$. Note that (12.20) can be written formally as

$$\delta W_\mu^a = D_\mu \xi^a + O(\xi^2), \quad (12.21)$$

if we regard ξ^a as a quantity that transforms in the adjoint representation of the gauge group. This way of expressing the infinitesimal gauge transformations is often convenient in actual calculations.

We have already made use of the observation that the D_μ can be viewed as the generators of covariant translations, which consist of infinitesimal space-time translations combined with infinitesimal field-dependent gauge transformations in order to restore the covariant character of the translated quantity. Since both these infinitesimal transformations satisfy $\delta(\phi\psi) = (\delta\phi)\psi + \phi(\delta\psi)$, Leibniz' rule applies to covariant derivatives,

$$D_\mu(\phi\psi) = (D_\mu\phi)\psi + \phi(D_\mu\psi). \quad (12.22)$$

Note that covariant derivatives depend on the representation of the fields on which they act through the choice of the generators t_a . Hence each of the three terms in (12.22) may contain a different representation for the generators, (see the simple abelian example in (11.14)).

Unlike ordinary differentiations, two covariant differentiations do not necessarily commute. It is easy to see that the commutator of two covariant derivatives D_μ and D_ν , which is obviously a covariant quantity, is given by

$$\begin{aligned} [D_\mu, D_\nu]\psi &= D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) \\ &= -(\partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu])\psi. \end{aligned} \quad (12.23)$$

This result leads to the definition of a covariant antisymmetric tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu], \quad (12.24)$$

which is called the *field strength*. Observe that this field strength is not in any way equal to the covariant derivative of the gauge field. Such a covariant

derivative is simply ill defined in view of the fact that the gauge field transforms not covariantly. The covariant object associated with the gauge field is just the field strength. As ψ and $D_\mu D_\nu \psi$ transform identically under the gauge transformations the field strength must transform covariantly according to

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{-1}. \quad (12.25)$$

Because W_μ is Lie-algebra valued and the quadratic term in (12.24) is a commutator, the field strength is also Lie-algebra valued, i.e., $G_{\mu\nu}$ can also be decomposed in terms of the group generators t_a ,

$$G_{\mu\nu} = G_{\mu\nu}^a t_a, \quad (12.26)$$

with

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - f_{bc}^a W_\mu^b W_\nu^c. \quad (12.27)$$

Note that (12.27) differs from the abelian field strength derived in the previous chapter (cf. 11.18) by the presence of the term quadratic in W_μ^a .

Under an infinitesimal transformation $G_{\mu\nu}$ transforms as

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = G_{\mu\nu} + [\xi^a t_a, G_{\mu\nu}], \quad (12.28)$$

or, equivalently,

$$G_{\mu\nu}^a \rightarrow (G_{\mu\nu}^a)' = G_{\mu\nu}^a + f_{bc}^a \xi^b G_{\mu\nu}^c. \quad (12.29)$$

For abelian groups the structure constants vanish (i.e., the adjoint representation is trivial for an abelian group), so that the field strengths are invariant in that case.

The result (12.23) can now be expressed in a representation independent form,

$$[D_\mu, D_\nu] = -G_{\mu\nu}, \quad (12.30)$$

implying that the commutator of two covariant derivatives is equal to an infinitesimal gauge transformation with $-G_{\mu\nu}^a$ as parameters. This equation is known as the *Ricci identity*. Precisely as for the abelian case we may apply further covariant derivatives to (12.30). In particular, consider

$$[D_\mu[D_\nu, D_\rho]] + [D_\nu[D_\rho, D_\mu]] + [D_\rho[D_\mu, D_\nu]], \quad (12.31)$$

which vanishes identically because of the Jacobi identity. Inserting (12.30) we obtain the result

$$D_\mu G_{\nu\rho} + D_\nu G_{\rho\mu} + D_\rho G_{\mu\nu} = 0, \quad (12.32)$$

where, according to (12.25), the covariant derivative of $G_{\mu\nu}$ equals

$$D_\mu G_{\nu\rho} = \partial_\mu G_{\nu\rho} - [W_\mu, G_{\nu\rho}], \quad (12.33)$$

or, in components,

$$D_\mu G_{\nu\rho}^a = \partial_\mu G_{\nu\rho}^a - f_{bc}{}^a W_\mu^b G_{\nu\rho}^c. \quad (12.34)$$

The relation (12.32) is called the *Bianchi identity*; in the abelian case the Bianchi identity corresponds to the homogeneous Maxwell equations. As we have shown in section 1.3, the integral form of the latter implies the conservation of magnetic flux. For a nonabelian gauge theory the situation is different because (12.32) contains a covariant rather than an ordinary derivative. Therefore the nonabelian analogue of the magnetic flux is not conserved and can be carried away by the nonabelian gauge fields.

This completes the introduction of nonabelian gauge fields and their corresponding field-strengths and transformation rules. In this presentation the number of calculations was kept to a minimum. Needless to say, that results such as (12.29) or (12.31) can also be verified directly by straightforward but laborious calculations, using the Jacobi identity for the structure constants. The reader may wonder why we have not yet encountered a coupling constant that measures the interaction strength of the gauge fields to matter and to themselves. In order to introduce a coupling constant, one may rescale the gauge fields W_μ^a to gW_μ^a . When the gauge group is a product of different subgroups, then one rescales the gauge fields corresponding to each one of these subgroups by an independent coupling constant. The rescaling seems a trivial exercise at this stage, but when introducing both matter Lagrangians and a separate gauge-field Lagrangian, the coupling constant corresponds to the parameter in front of the gauge-field Lagrangian, so that it does represent a physically relevant quantity. The number of relevant coupling constants is thus equal to the number of independent gauge-invariant terms that depend exclusively on the gauge fields. We shall be more explicit in section 12.3 when discussing the gauge-field Lagrangian.

Assuming we are dealing with a simple gauge group (a group without invariant subgroups, so that we have only one coupling constant; for the precise definition of a simple group, see appendix C), let us explicitly perform the rescaling and present the relevant formulae, as those will be the ones that we shall be using henceforth. We perform the following substitutions,

$$W_\mu^a \rightarrow gW_\mu^a, \quad G_{\mu\nu}^a \rightarrow gG_{\mu\nu}^a, \quad \xi^a \rightarrow g\xi^a. \quad (12.35)$$

Hence the covariant derivative (12.8) changes into

$$D_\mu \psi \equiv \partial_\mu \psi - gW_\mu \psi, \quad (12.36)$$

while the Ricci identity (12.30) now reads

$$[D_\mu, D_\nu] = -g G_{\mu\nu}, \quad (12.37)$$

with the field strength tensor

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g [W_\mu, W_\nu], \quad (12.38)$$

or, from $G_{\mu\nu} = G_{\mu\nu}^a t_a$,

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f_{bc}^a W_\mu^b W_\nu^c. \quad (12.39)$$

Because the gauge-transformation parameters are rescaled as well, we have

$$U = \exp(g \xi^a t_a), \quad (12.40)$$

so that W_μ^a , $G_{\mu\nu}^a$ and the matter field ψ transform under infinitesimal gauge transformations according to

$$\delta W_\mu^a = \partial_\mu \xi^a - g f_{bc}^a W_\mu^b \xi^c = D_\mu \xi^a, \quad (12.41)$$

$$\delta G_{\mu\nu}^a = g f_{bc}^a \xi^b G_{\mu\nu}^c, \quad (12.42)$$

$$\delta \psi = g \xi^a t_a \psi. \quad (12.43)$$

Observe that in rewriting the gauge field transformation as a covariant derivative, we are formally treating ξ^a as a quantity that transforms in the adjoint representation of the group.

12.2. Gauge theory of $SU(2)$

By making use of the covariant derivatives constructed in the previous section it is rather straightforward to construct gauge-invariant Lagrangians for spin-0 and spin- $\frac{1}{2}$ fields. To demonstrate this, consider a set of N spinor fields ψ_i transforming under transformations U belonging to a certain group G according to ($i, j = 1, \dots, N$)

$$\psi_i \rightarrow \psi'_i = U_{ij} \psi_j, \quad (12.44)$$

or, suppressing indices i, j , and writing ψ as an N -dimensional column vector,

$$\psi \rightarrow \psi' = U \psi. \quad (12.45)$$

Conjugate spinors $\bar{\psi}_i$ then transform as

$$\bar{\psi}_i \rightarrow \bar{\psi}'_i = U_{ij}^* \bar{\psi}_j, \quad (12.46)$$

or, regarding $\bar{\psi}$ as a row vector and again suppressing indices, as

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} U^\dagger. \quad (12.47)$$

Obviously, if U is unitary, i.e., if $U^\dagger = U^{-1}$, the massive Dirac Lagrangian,

$$\mathcal{L} = -\bar{\psi}_i \not{\partial} \psi_i - m \bar{\psi}_i \psi_i, \quad (12.48)$$

is invariant under G . This Lagrangian thus describes N spin- $\frac{1}{2}$ (anti)particles of equal mass m .

We now require that the Lagrangian be invariant under *local* G transformations. To achieve this we simply replace the ordinary derivative in (12.48) by a covariant derivative (here and henceforth we will suppress indices i, j etc.),

$$\begin{aligned} \mathcal{L} &= -\bar{\psi} \not{D} \psi - m \bar{\psi} \psi \\ &= -\bar{\psi} \not{\partial} \psi - m \bar{\psi} \psi + g \bar{\psi} \gamma^\mu W_\mu \psi, \end{aligned} \quad (12.49)$$

where $W_\mu = W_\mu^a t_a$ is the Lie-algebra valued gauge field introduced in the previous section. Hence the gauge field interactions are given by

$$\mathcal{L}_{\text{int}} = g W_\mu^a \bar{\psi} \gamma^\mu t_a \psi, \quad (12.50)$$

where t_a are the parameters of the gauge group G in the representation appropriate to ψ . Observe that the matrices t_a are antihermitean in order that the gauge transformations are unitary. The matrices $(t_a)_{ij}$ can be regarded as the nonabelian charges and are just proportional to the charges $F_a^{ij}(0)$ discussed in section 11.2, up to a proportionality factor i . The commutation relation (11.83) that was derived for the charges can now be understood as a consequence of the Lie algebra relation (12.3) which in turn was required in order that the nonabelian gauge transformations constitute a group. The constants F_{abc} that appear in the three- W vertex are related to the structure constants f_{bc}^a of the gauge group. This can be verified directly from a gauge invariant Lagrangian for the gauge fields that will be constructed in section 12.3.

To illustrate the above construction, let us explicitly construct a gauge invariant Lagrangian for fermions transforming as doublets under the group $SU(2)$. This group consists of all 2×2 unitary matrices with unit determinant. Such matrices can be written in exponentiated form,

$$U(\xi) = \exp(g \xi^a t_a), \quad (a = 1, 2, 3) \quad (12.51)$$

where we have included the coupling constant g in accordance with (12.36). The three generators of $SU(2)$ are expressed in terms of the isotopic spin matrices τ_a ,

$$t_a = \frac{1}{2} i \tau_a, \quad (12.52)$$

which coincide with the Pauli matrices used in the context of ordinary spin,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12.53)$$

Consequently the generators t_a satisfy the commutation relations

$$[t_a, t_b] = -\varepsilon_{abc} t_c, \quad (12.54)$$

ensuring that the matrices (12.51) form a group.

Historically the first construction of a nonabelian gauge field theory was based on $SU(2)$ and presented by Yang and Mills and independently by Shaw.² It was motivated by the existence of approximate isospin invariance (cf. sections 2.5 and 5.2), based on the observation that the proton and neutron are almost degenerate in mass, and play an identical role in strong interaction processes. According to the notion of isospin (or isobaric spin) invariance the proton and the neutron can be regarded as an isospin doublet. Therefore one introduces a doublet field

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \quad (12.55)$$

analogous to the $s = \frac{1}{2}$ doublet of ordinary spin. Conservation of isospin is just the requirement of invariance under isospin rotations

$$\psi \rightarrow \psi' = U\psi, \quad (12.56)$$

where U is an $SU(2)$ matrix as defined in (12.51). If isospin invariance were an exact symmetry then it is a matter of convention which component of ψ would correspond to the proton and which one to the neutron. If one insists on being able to define this convention at any space-time point separately, then one is led to the construction of a gauge field theory based on local isospin transformations.

Starting from (12.56) it is straightforward to construct the covariant deriva-

²An interesting early attempt motivated by Yukawa's conjecture of the existence of the pion was presented by O. Klein (1938). It was based on a, somewhat ad-hoc, extension of gravity in five space-time dimensions, and led to theories which have features very similar to those of $SU(2)$ and $SU(2) \times U(1)$ gauge theories.

tive on ψ ,

$$\begin{aligned}
D_\mu \psi &= \begin{pmatrix} \partial_\mu \psi_p \\ \partial_\mu \psi_n \end{pmatrix} - \frac{1}{2}ig W_\mu^a \tau_a \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \\
&= \begin{pmatrix} \partial_\mu \psi_p \\ \partial_\mu \psi_n \end{pmatrix} - \frac{1}{2}ig \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \\
&= \begin{pmatrix} \partial_\mu \psi_p - \frac{1}{2}ig W_\mu^3 \psi_p - \frac{1}{2}ig (W_\mu^1 - iW_\mu^2) \psi_n \\ \partial_\mu \psi_n + \frac{1}{2}ig W_\mu^3 \psi_n - \frac{1}{2}ig (W_\mu^1 + iW_\mu^2) \psi_p \end{pmatrix}. \tag{12.57}
\end{aligned}$$

A locally $SU(2)$ invariant Lagrangian is then obtained by replacing the ordinary derivative by a covariant one in the Lagrangian of a degenerate doublet of spin- $\frac{1}{2}$ fields (cf. (12.36)),

$$\begin{aligned}
\mathcal{L} &= -\bar{\psi} \not{D} \psi - m \bar{\psi} \psi \\
&= -\bar{\psi} \not{\partial} \psi - m \bar{\psi} \psi + \frac{1}{2}ig W_\mu^a \bar{\psi} \gamma^\mu \tau_a \psi. \tag{12.58}
\end{aligned}$$

The field strength tensors follow straightforwardly from the $SU(2)$ structure constants exhibited in (12.58)

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon_{abc} W_\mu^b W_\nu^c, \tag{12.59}$$

while under infinitesimal $SU(2)$ transformations the gauge fields transform according to

$$W_\mu^a \rightarrow (W_\mu^a)' = W_\mu^a + g \epsilon_{abc} W_\mu^b \xi^c + \partial_\mu \xi^a. \tag{12.60}$$

It turns out that, contrary to initial expectations, *local* $SU(2)$ transformations have no role to play in the strong interactions. Instead these forces are governed by an $SU(3)$ gauge theory called quantum chromodynamics because one has introduced the term *colour* for the degrees of freedom transforming under $SU(3)$. This theory will be extensively discussed in chapter 16. The corresponding gauge fields are called gluon fields because they are assumed to bind the elementary hadronic constituents, called quarks, into hadrons. The quarks transform as triplets under $SU(3)$ and can thus be viewed as a straightforward extension of the $SU(2)$ doublet (12.55). However, theories based on $SU(2)$ gauge transformations are relevant for the weak interactions. We shall come back to these aspects in due course.

Gauge invariant Lagrangians with spin-0 fields are constructed in the same way. For instance, consider an array of complex scalar fields transforming under transformations U as in (12.44). If we regard ϕ as a column vector and the complex conjugate fields as a row vector ϕ^* , we may write,

$$\phi \rightarrow \phi' = U\phi, \quad \phi^* \rightarrow (\phi^*)' = \phi^* U^\dagger. \tag{12.61}$$

Covariant derivatives read

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi - g W_\mu^a t_a \phi, \\ D_\mu \phi^* &= \partial_\mu \phi^* - g \phi^* t_a^\dagger W_\mu^a. \end{aligned} \quad (12.62)$$

Provided that the transformation matrices U in (12.61) are unitary (so that $t_a^\dagger = -t_a$) the following Lagrangian is gauge invariant,

$$\mathcal{L} = -|D_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4, \quad (12.63)$$

where we have used the inner product $|\phi|^2 = \phi_i^* \phi_i$. Substitution of (12.62) leads to

$$\begin{aligned} \mathcal{L} &= -(\partial^\mu \phi^* + g \phi^* t_a W^{a\mu})(\partial_\mu \phi - g W_\mu^b t_b \phi) - m^2 |\phi|^2 - \lambda |\phi|^4 \\ &= -|\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4 \\ &\quad - g W^{a\mu} (\phi^* t_a \partial_\mu \phi - (\partial_\mu \phi^*) t_a \phi) + g^2 W^{a\mu} W_\mu^b (\phi^* t_a t_b \phi), \end{aligned} \quad (12.64)$$

where in the gauge field interaction terms ϕ^* and ϕ are written as row and column vectors. This result once more exhibits the role played by the generators t_a as matrix generalizations of the charge. Using (12.52) it is easy to give the corresponding Lagrangian invariant under $SU(2)$. In that case (12.64) reads

$$\begin{aligned} \mathcal{L} &= -|\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4 \\ &\quad - \frac{1}{2} i g W^{a\mu} (\phi^* \tau_a \overleftrightarrow{\partial}_\mu \phi) - \frac{1}{4} g^2 (W_\mu^a)^2 |\phi|^2, \end{aligned} \quad (12.65)$$

where we used $\tau_a \tau_b + \tau_b \tau_a = 2I \delta_{ab}$.

12.3. The gauge field Lagrangian

In the preceding section we have discussed how to construct locally invariant Lagrangians for matter fields. Starting from a Lagrangian that is invariant under the corresponding rigid transformations, one replaces ordinary derivatives by covariant ones. Until that point the gauge fields are not yet treated as new dynamical degrees of freedom. For that purpose one must also specify a Lagrangian for the gauge fields, which must be separately locally gauge invariant. A transparent construction of such invariants makes use of the Lie-algebra valued field strength tensor $G_{\mu\nu}$. Let us recall that $G_{\mu\nu}$ transforms according to (cf. 12.25),

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{-1}, \quad (12.66)$$

so that for any product of these tensors we have

$$G_{\mu\nu} G_{\rho\sigma} \cdots G_{\lambda\tau} \rightarrow G'_{\mu\nu} G'_{\rho\sigma} \cdots G'_{\lambda\tau} = U (G_{\mu\nu} G_{\rho\sigma} \cdots G_{\lambda\tau}) U^{-1}. \quad (12.67)$$

Consequently the trace of arbitrary products of the form (12.67) is gauge invariant, i.e.,

$$\mathrm{Tr}(G_{\mu\nu} \cdots G_{\lambda\tau}) \rightarrow \mathrm{Tr}(UG_{\mu\nu} \cdots G_{\lambda\tau}U^{-1}) = \mathrm{Tr}(G_{\mu\nu} \cdots G_{\lambda\tau}), \quad (12.68)$$

by virtue of the cyclicity of the trace operation. The simplest Lorentz invariant and parity conserving Lagrangian can therefore be expressed as a quadratic form in $G_{\mu\nu}$,

$$\mathcal{L}_G = \frac{1}{4g^2} \mathrm{Tr}(G^{\mu\nu} G_{\mu\nu}). \quad (12.69)$$

Observe that when we introduce the scaling (12.35), the dependence of the coupling constant g will be contained in the g -dependent version of the field strengths, given in (12.39). When the gauge group allows more invariant Lagrangians, which is for instance the case when the gauge group factorizes into separate groups, we may introduce the corresponding number of coupling constants. Note that two alternative Lorentz invariant forms, G_μ^μ and $G_{\mu\nu}G_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$, are excluded; the first one vanishes by antisymmetry of $G_{\mu\nu}$, and the second one is not parity conserving. Actually one can show (see problem 12.6) that the second term is equal to a total divergence

$$\varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(G_{\mu\nu}G_{\rho\sigma}) = 4 \partial_\mu \left\{ \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(W_\nu \partial_\rho W_\sigma - \frac{2}{3}g W_\nu W_\rho W_\sigma) \right\}, \quad (12.70)$$

so it may be ignored in the action.³ Writing out all the terms the Lagrangian (12.69) acquires the form

$$\begin{aligned} \mathcal{L}_W = \mathrm{Tr}(t_a t_b) & \left\{ \frac{1}{4} (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) (\partial_\mu W_\nu^b - \partial_\nu W_\mu^b) \right. \\ & \left. - g f_{cd}^a W_\mu^c W_\nu^d \partial_\mu W_\nu^b + \frac{1}{4} g^2 f_{cd}^a f_{ef}^b W_\mu^c W_\mu^e W_\nu^d W_\nu^f \right\}, \end{aligned} \quad (12.71)$$

where one may distinguish a kinetic term, which resembles the abelian Lagrangian (11.25), $(W)^3$ -interaction terms which, as we have been alluding to above, are proportional to the structure constants of the gauge group, and $(W)^4$ -interaction terms.

Let us now deal with the tensor that multiplies the terms in (12.71),

$$g_{ab}^R = \mathrm{Tr}(t_a t_b), \quad (12.72)$$

where we have introduced the suffix R to indicate that it depends on the representation taken for the generators t_a . When these generators are in the adjoint representation, (12.72) defines the so-called Cartan-Killing metric,

$$g_{ab} = f_{ad}^c f_{bc}^d. \quad (12.73)$$

³At least in perturbation theory, but not when one is dealing with nontrivial boundary conditions. The term in curly brackets, $\varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(W_\nu \partial_\rho W_\sigma - \frac{2}{3}g W_\nu W_\rho W_\sigma)$ is known as the Chern-Simons term.

By means of (12.72) one can write down a tensor

$$f_{abc}^R \equiv -f_{ab}{}^d g_{dc}^R, \quad (12.74)$$

which is *antisymmetric* in all the indices. To see this, write (12.74) by means of (12.72) as

$$f_{abc}^R = -f_{ab}{}^d \text{Tr}(t_d t_c) = -\text{Tr}(t_a t_b t_c - t_b t_a t_c), \quad (12.75)$$

which is antisymmetric under the interchange of any two indices by virtue of the cyclicity of the trace. For any representation, g_{ab}^R constitutes a group-invariant tensor. This follows from the observation that a rank-2 tensor T_{ab} in the adjoint representation transforms under infinitesimal gauge transformations as (cf. 12.29)

$$T_{ab} \rightarrow T_{ab} - \xi^c (f_{ca}{}^d T_{db} + f_{cb}{}^d T_{ad}) + O(\xi^2). \quad (12.76)$$

Substituting g_{ab}^R for T_{ab} and using the fact that g_{ab}^R is symmetric because $\text{Tr}(t_a t_b) = \text{Tr}(t_b t_a)$, one finds

$$g_{ab}^R \rightarrow g_{ab}^R - \xi^c (f_{cab}^R + f_{cba}^R) + O(\xi^2). \quad (12.77)$$

Owing to the total antisymmetry of f_{abc}^R the variation of g_{ab}^R vanishes, which proves that the tensors g_{ab}^R are indeed invariant. Furthermore, because the structure constants are real, the complex conjugate matrices t_a^* also satisfy (12.4) and thus generate a group representation. Therefore, if g_{ab}^R is complex, both its real and imaginary parts will separately constitute an invariant tensor.

As was mentioned above, the invariant tensors g_{ab}^R depend on the representation. This feature is easily demonstrated for an abelian group, for which the Cartan-Killing metric (12.73) vanishes trivially because the structure constants are zero. On the other hand, abelian groups have obviously nontrivial representations for which (12.72) will not vanish. Only for so-called *simple* groups can it be shown that the invariant tensors g_{ab}^R must be equal up to a proportionality constant (see appendix C).

For any given representation it is possible to bring g_{ab}^R (or at least its real or imaginary part) in diagonal form with eigenvalues equal to 1, -1 or 0, by suitably redefining the gauge fields (and thus the group parameters ξ^a and corresponding structure constants). For our purposes it is important that g_{ab}^R has only nonzero eigenvalues of equal sign, so that the first term in the Lagrangian (12.71) leads to kinetic terms for each of the gauge fields of the same normalization. This is the case for so-called *compact* Lie groups, where $t_a^\dagger = -t_a$, which implies that g_{ab}^R has nonpositive eigenvalues. The gauge groups that we will be dealing with are always compact; therefore in the Lagrangian (12.71) we will from now on replace $\text{Tr}(t_a t_b)$ by $-\delta_{ab}$, without

specifying the representation that is used, so that

$$\begin{aligned}\mathcal{L}_W &= -\frac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)^2 + gf_{abc}W^{a\mu}W^{b\nu}\partial_\mu W_\nu^c \\ &\quad -\frac{1}{4}g^2f_{abc}f_{ade}W^{b\mu}W_\mu^dW^{c\nu}W_\nu^e.\end{aligned}\quad (12.78)$$

In this particular case, for (12.78), we make no distinction between upper and lower indices and $f_{abc} \equiv f_{ab}{}^c$ is totally antisymmetric. Nevertheless we should emphasize that there is no problem of principle to write down gauge-field Lagrangians for noncompact groups, and it is possible to evaluate Feynman diagrams as long as g_{ab}^R has no zero eigenvalues. Actually noncompact gauge theories do play a role in the theory of gravity. There it is sometimes convenient to consider the gauge theory associated with the Lorentz group (which is noncompact). However, in that context one uses a Lagrangian which is not of same type as (12.71). Because of that, the role played by the corresponding gauge fields is rather different.

One may combine the gauge-field Lagrangian (12.78) with a gauge invariant Lagrangian for the matter fields and derive the corresponding Euler-Lagrange equations from Hamilton's principle. To be specific let us choose the gauge-invariant Lagrangian (12.49) for fermions and combine it with (12.78). Hence

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_W + \mathcal{L}_\psi \\ &= \frac{1}{4}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) - \bar{\psi}\overleftrightarrow{D}\psi - m\bar{\psi}\psi.\end{aligned}\quad (12.79)$$

The field equations for the fermions are obviously a covariant version of the Dirac equation, i.e.,

$$(\overleftrightarrow{D} + m)\psi = 0, \quad \bar{\psi}(\overleftarrow{D} - m) = 0, \quad (12.80)$$

or (using $t_a^\dagger = -t_a$),

$$\begin{aligned}(\overleftrightarrow{D} + m)\psi &= gW^a t_a \gamma^\mu \psi, \\ \bar{\psi}(\overleftarrow{D} + m) &= -g\bar{\psi} \gamma^\mu t_a W_\mu^a.\end{aligned}\quad (12.81)$$

These equations are the nonabelian generalizations of (11.28) and (11.29).

To derive the field equations for the gauge fields is more complicated. First consider an infinitesimal variation

$$W_\mu \rightarrow W_\mu + \delta W_\mu. \quad (12.82)$$

A convenient way to write the corresponding change of the field strength is

$$G_{\mu\nu} \rightarrow G_{\mu\nu} + D_\mu(\delta W_\nu) - D_\nu(\delta W_\mu), \quad (12.83)$$

where we regard δW_μ as a covariant quantity transforming in the adjoint representation of the group. Here we note again that the field strength itself

can not be identified with the dcovariant derivative of the gauge field, as the latter is not defined. The covariant derivative on δW_μ is then defined in the standard way

$$D_\mu(\delta W_\nu) = \partial_\mu(\delta W_\nu) - g[W_\mu, \delta W_\nu]. \quad (12.84)$$

The variation of the gauge-field action now takes the form

$$\delta \left(\frac{1}{4} \int d^4x \operatorname{Tr} (G^{\mu\nu} G_{\mu\nu}) \right) = \int d^4x \operatorname{Tr} (G^{\mu\nu} D_\mu(\delta W_\nu)), \quad (12.85)$$

where we have made use of the fact that $G_{\mu\nu}$ is antisymmetric in μ and ν . The integrand on the right-hand side can now be written as

$$D_\mu \{ \operatorname{Tr} (G^{\mu\nu} \delta W_\nu) \} - \operatorname{Tr} ((D_\mu G^{\mu\nu}) \delta W_\nu). \quad (12.86)$$

In previous manipulations we treated δW_μ as a covariant object, so that the expression $\operatorname{Tr} (G_{\mu\nu} \delta W_\nu)$ is gauge invariant. Therefore the covariant derivative in the first term can be replaced by an ordinary derivative; by Gauss' theorem this term leads to a surface integral over the boundary of the integration domain used in (12.85). However, the variation δW_μ should vanish at the boundary according to Hamilton's principle. Hence only the second term in (12.86) contributes to the variation of the action, so that (12.85) becomes

$$\delta \left(\int d^4x \mathcal{L}_W \right) = - \int d^4x \operatorname{Tr} ((D^\mu G_{\mu\nu}) \delta W^\nu). \quad (12.87)$$

Following we may now write the variation of the matter action due to (12.82) as

$$\delta \left(\int d^4x \mathcal{L}_\psi(\psi, W) \right) = - \int d^4x \operatorname{Tr} (\delta W^\mu J_\mu), \quad (12.88)$$

so that the field equation for the gauge field corresponding to the Lagrangian (12.79) takes the form

$$D^\mu G_{\mu\nu} + J_\nu = 0, \quad (12.89)$$

where $G_{\mu\nu}$ and J_μ are Lie-algebra valued matrices.

Alternatively, one may write (12.81- 12.83) in components. Using $\operatorname{Tr}(t_a t_b) = -\delta_{ab}$, the variation of the gauge field action reads

$$\delta \left(\int d^4x \mathcal{L}_W \right) = \int d^4x \delta W^{a\nu} D^\mu G_{\mu\nu}^a, \quad (12.90)$$

while the current J_μ^a associated with W_μ^a is defined by

$$\delta \left(\int d^4x \mathcal{L}_\psi(\psi, W) \right) = \int d^4x \delta W^{\mu a} J_\mu^a, \quad (12.91)$$

(observe that (12.88) and (12.91) are related by $J_\mu = J_\mu^a t_a$). Evaluating (12.90) and (12.91) gives the field equation

$$D^\nu G_{\mu\nu}^a = J_\mu^a, \quad (12.92)$$

which is obviously a nonabelian extension of (11.27). The explicit form of this equation is, however, rather complicated,

$$\begin{aligned} \partial^\nu (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) + g f_{bc}^a (W^{b\nu} \partial_\mu W_\nu^c + W_\mu^b \partial_\nu W^{c\nu} - 2W^{b\nu} \partial_\nu W_\mu^c) \\ + g^2 f_{bc}^a f_{de}^c W_\mu^d W_\nu^b W^{e\nu} = J_\mu^a, \end{aligned} \quad (12.93)$$

For the case of the Lagrangian (12.79) the current J_μ^a thus equals

$$J_\mu^a = g \bar{\psi} \gamma_\mu t_a \psi. \quad (12.94)$$

To examine whether J_μ^a is conserved, we apply a covariant derivative D_μ to (12.92). On the left-hand side this leads to

$$\begin{aligned} D_\mu D_\nu G^{a\mu\nu} &= \frac{1}{2} [D_\mu, D_\nu] G^{a\mu\nu} \\ &= -\frac{1}{2} g G_{\mu\nu}^b f_{bc}^a G^{c\mu\nu}, \end{aligned} \quad (12.95)$$

where we have used the antisymmetry of $G^{a\mu\nu}$ and the Ricci identity (12.28). Because f_{bc}^a is antisymmetric in b and c the right-hand side of (12.95) vanishes. Therefore the current satisfies a covariant divergence equation

$$D^\mu J_\mu^a = 0, \quad (12.96)$$

or, explicitly,

$$\partial^\mu J_\mu^a - g f_{bc}^a W^{b\mu} J_\mu^c = 0. \quad (12.97)$$

The above result implies that gauge fields can only couple consistently to currents that are covariantly constant. According to (12.96) the charges associated with the current are not quite conserved. The reason is obviously that the gauge fields are not neutral. Their contribution must be included in order to define charges that are conserved.

Just as for the abelian case it is possible to give an alternative derivation of (12.96) based on the field equations of the matter fields. The fact that both set of field equations lead to identical results can again be understood from the gauge invariance of the combined action (cf. section 12.3).

12.4. Chiral gauge groups and discrete symmetries

So far we did not pay much attention to the action of discrete symmetries on gauge fields. Discrete symmetries such as parity reversal and charge conjugation were already discussed in section 7.1, but there the emphasis was

primarily on the spinor fields. At the same time we introduced chiral spinor fields, eigenspinors of the γ_5 matrix, which we will also employ below.

In this section we intend to briefly discuss the action of discrete symmetries on nonabelian gauge fields. As a prelude let us first consider nonabelian gauge transformations that may give rise to fermionic vector and/or axial-vector currents. A simple abelian version of this was already presented in section 7.1, but for a nonabelian group it is no longer possible to arbitrarily include γ_5 matrices. In order to appreciate this, consider a nonabelian group of transformations based on generators t_a . Then the matrices $t_a\gamma_5$, which act on both the gauge-group and the spinor indices, do *not* generate a group of transformations. The reason is that the matrices $t_a\gamma_5$ do not close under commutation, simply because

$$[t_a\gamma_5, t_b\gamma_5] = f_{ab}{}^c t_c \neq f_{ab}{}^c t_c\gamma_5. \quad (12.98)$$

In order to avoid introducing both t_a and $t_a\gamma_5$ as independent generators, thereby doubling the dimension of the group and thus the number of independent gauge fields, it is convenient to include γ_5 via chiral projection operators. These operators, $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$, were defined in (7.17). Because projection operators satisfy $P_{\pm}^2 = P_{\pm}$, the matrices $t_a P_{\pm}$ do generate a representation of the group. Indeed, we have

$$[(t_a P_{\pm}), (t_b P_{\pm})] = f_{ab}{}^c (t_c P_{\pm}). \quad (12.99)$$

Using the chiral projection operators one can decompose every fermion field into chiral components (cf. (7.20)), $\psi_L = P_+\psi$ and $\psi_R = P_-\psi$. It is obvious from (12.99) that ψ_L and ψ_R can now independently be assigned to representations of the gauge group. Therefore it makes sense to define two sets of gauge group generators, t_a^R and t_a^L , which are a priori unrelated but satisfy the same commutation relations (12.99) associated with the gauge group and act on independent arrays of fields denoted by ψ_R and ψ_L , respectively. The coupling of the gauge fields to the fermions will therefore contain the chiral projectors P_{\pm} and be proportional to $\gamma_{\mu}(1 + \gamma_5)t_a^L + \gamma_{\mu}(1 - \gamma_5)t_a^R$. Depending on the actual form of the generators we thus obtain vector or axial-vector gauge field couplings to fermions, or combinations thereof.

One may distinguish a number of possibilities here. First of all, the generators t_a^L or t_a^R could be inequivalent, in which case the numbers of left- and right-handed fields do not have to be equal. One particular example is the case that there are only fields present of one given chirality, but not of the other one. Then either the t_a^L or the t_a^R will vanish and the gauge fields couple all to the same linear combination of a vector and an axial vector, i.e. either to $\gamma_{\mu}(1 + \gamma_5)$ or to $\gamma_{\mu}(1 - \gamma_5)$. A second possibility is that ψ_L and ψ_R transform equivalently, meaning that the matrices satisfy $t_a^L = t_a^R$. (possibly up to a similarity transformation which can be removed by a suitable field redefinition). This always leads to a purely *vectorlike* coupling. A third, somewhat

intermediate, option is that the generators t_a^L or t_a^R are each others complex conjugates, $t_a^L = (t_a^R)^*$. Here it is important to realize that there is always a complex conjugate representation, as follows from the fact that the complex conjugate generators satisfy the same commutator algebra,

$$[(t_a)^*, (t_b)^*] = f_{ab}^c (t_c)^*. \quad (12.100)$$

When the representation is real this is a triviality and one returns to the previous case. One may wonder whether it is possible for all generators to be imaginary, but this can only be the case if the group is abelian (this can be shown by an argument similar to (12.98)). So in this case the gauge fields associated with real generators have vectorlike and those with imaginary generators have axial-vectorlike couplings to fermions.

Observe that the generators relevant to the charge-conjugated fields defined in (7.28), ψ_L^c and ψ_R^c , are the complex conjugate matrices $(t_a^R)^*$ and $(t_a^L)^*$, respectively. So when choosing the left-handed fields, ψ_L and ψ_L^c as a basis of independent fermion fields, we are dealing with a *reducible* representation with generators,

$$\begin{pmatrix} t_a^L & 0 \\ 0 & (t_a^R)^* \end{pmatrix}. \quad (12.101)$$

The reader can easily convince himself that these matrices still satisfy the relevant commutation relations associated with the gauge group. Of course, a similar decomposition holds for the right-handed fields ψ_R and ψ_R^c . As mentioned above, when the gauge transformations act identically on right- and left-handed components, the representation is called *vectorlike*, because the corresponding fermionic currents contain no $\gamma_\mu \gamma_5$ terms. The representation based on generators (12.101) is called *real*, because the generators can be written in real form by an appropriate change of basis. For real representation one can write down a standard gauge-invariant mass term, which involves products of right- and left-handed spinors. Another important property of vectorlike representations is that they are free of so-called anomalies. These are quantum-mechanical inconsistencies of the theory, which will be discussed in chapter 22.

A minimal requirement for a gauge theory to be invariant under discrete transformations, such as P , C or the combined transformation CP , is that covariant derivatives of the spinors transform just as ordinary derivatives under the discrete transformations. The covariant derivatives of ψ_L and ψ_R contain the Lie-algebra valued matrices $W_\mu^L = W_\mu^a t_a^L$ and $W_\mu^R = W_\mu^a t_a^R$, and read

$$D_\mu \psi_L = \partial_\mu \psi_L - W_\mu^L \psi_L, \quad D_\mu \psi_R = \partial_\mu \psi_R - W_\mu^R \psi_R. \quad (12.102)$$

Imposing the above requirement then determines the possible transformation rules for the gauge fields.

To see this, let us start with parity reversal under which left- and right-handed spinors are interchanged. We will be assuming that the intrinsic parity phase of the spinors are identical for fields transforming in an irreducible gauge-group representation. In order that the covariant derivatives of ψ_L and ψ_R transform into each other (cf. 7.27), the gauge fields must transform as follows under parity reversal,

$$W_\mu^a(\mathbf{x}, x^0) t_a^L \xrightarrow{P} -(1 - 2\delta_{\mu 0}) W_\mu^a(-\mathbf{x}, x^0) t_a^R, \quad (12.103)$$

and vice versa. Observe that, once the gauge-group representations have been chosen, the action of P on the gauge fields is completely fixed, with no possibility of adding an extra phase factor. However, for (12.103) to make sense, it is necessary that the generators t_a^L and t_a^R are linearly dependent. When that is not the case the theory cannot be invariant under parity reversal. When it is the case, then the theory can be invariant under parity reversal and (12.103) will uniquely determine the action of P on the gauge fields. There are two obvious situations where the left- and right-handed generators are linearly dependent. In one of them the representations are equivalent and in the other one they are each other's complex conjugate (up to a similarity transformation which can be absorbed into the fields). In the first case all the gauge fields transform as vectors under parity reversal and the theory is vectorlike. In the second case the gauge fields associated with the real generators transform as vectors while gauge fields associated with imaginary generators transform as axial-vectors under parity reversal.

Likewise, in order to have invariance under charge conjugation, we analyze the behaviour of the covariant derivatives. Here we also need relations such as,

$$D_\mu \bar{\psi}_L^T = \partial_\mu \bar{\psi}_L^T - \bar{W}_\mu^L \bar{\psi}_L^T, \quad \text{or} \quad \bar{\psi}_L \overleftarrow{D}_\mu = \bar{\psi}_L (\overleftarrow{\partial}_\mu - W_\mu^{L\dagger}).$$

In order to obtain the correct transformations of the covariant derivatives (cf. 7.28), it then follows that gauge fields must transform under charge conjugation as

$$W_\mu^a(\mathbf{x}, x^0) t_a^L \xrightarrow{C} W_\mu^a(\mathbf{x}, x^0) (t_a^R)^*. \quad (12.104)$$

Again this result can only be meaningful for equivalent left- and right-handed representations and for left- and right-handed representations that are each other's complex conjugates. In the first case this implies that the gauge fields associated with real generators are even and gauge fields with purely imaginary generators are odd under charge conjugation. In the second case all fields are even under charge conjugation.

While P and C can only be defined if spinors of both chiralities are present, this is not so for CP conjugation. Combining the previous results we find for

the CP transformation of the gauge fields,

$$W_\mu^a(\mathbf{x}, x^0) t_a \xrightarrow{CP} -(1 - 2\delta_{\mu 0}) W_\mu^a(-\mathbf{x}, x^0) (t_a)^*, \quad (12.105)$$

with no possibility for assigning an arbitrary phase factor. In order to have invariance under CP the relation (12.105) should hold for all irreducible representations that are relevant for the theory in question. Clearly the CP phase factor depends on whether the gauge field corresponds to a real or purely imaginary generator.

Problems

12.1. Consider the n -parameter Lie group G with elements $g(\xi_1, \dots, \xi_n)$ and generators $t_a (a = 1, \dots, n)$. According to (12.2) we may write $g(\xi^1, \dots, \xi^n) = \exp(\xi^a t_a)$. In order to define a group, the product of two group elements should again be an element of the group, so that

$$g(\xi^1, \dots, \xi^n) g(\zeta^1, \dots, \zeta^n) = \exp(\eta^a t_a), \quad (1)$$

with the parameters η^a functions of ξ^a and ζ^a . Expand the left-hand side of (1) to first order in ξ^a and ζ^a and verify that, in this approximation, the result can only be written as a group element provided that the generators satisfy $[t_a, t_b] = f_{ab}{}^c t_c$ with η^a given by $\eta^a = \xi^a + \zeta^a + \frac{1}{2} f_{bc}{}^a \xi^b \zeta^c + \dots$.

12.2. Consider the vertex for the terms in the Lagrangian (12.78) with three gauge fields. Assuming that the gauge fields have incoming four-momenta k_1, k_2, k_3 group indices d, e, f and Lorentz indices ρ, σ, τ respectively, show that the interaction vertex in momentum space is

$$V_{\rho\sigma\tau}^{def}(k_1, k_2, k_3) = i(2\pi)^4 (-ig) f_{def} [\eta_{\rho\sigma}(k_1 - k_2)_\tau + \eta_{\sigma\tau}(k_2 - k_3)_\rho + \eta_{\tau\rho}(k_3 - k_1)_\sigma]. \quad (1)$$

12.3. Consider the term in the Lagrangian (12.78) with four gauge fields. Assuming that the gauge fields have incoming four momenta k_1, k_2, k_3, k_4 group indices a, b, c, d and Lorentz indices ρ, σ, τ, ν show that the interaction vertex in momentum space is

$$V_{\rho\sigma\tau\nu}^{abcd}(k_1, k_2, k_3, k_4) = -\frac{1}{4} i(2\pi)^4 g^2 [f_{abe} f_{ecd} (\eta_{\rho\sigma} \eta_{\tau\nu} - \eta_{\rho\tau} \eta_{\sigma\nu}) + f_{ade} f_{ebc} (\eta_{\rho\tau} \eta_{\sigma\nu} - \eta_{\rho\nu} \eta_{\tau\sigma}) + f_{ace} f_{ebc} (\eta_{\rho\nu} \eta_{\sigma\tau} - \eta_{\rho\sigma} \eta_{\tau\nu})]. \quad (1)$$

12.4. Consider the Lagrangian (12.64) for the interaction between an $SU(2)$ gauge field W_μ and an $SU(2)$ charged scalar field ϕ . Extract a coupling constant g by

rescaling W_μ to gW_μ . Write down the interaction vertices and the order- g^2 terms in the amplitude for ϕ - W_μ scattering. Add the SU(2) version of the gauge-field Lagrangian (12.78) and calculate the tree graph involving the three gauge field vertex. Check that the resulting expression is gauge invariant when the generators obey the SU(2) commutation relation.

12.5. Show how $[D_\mu, D_\nu]$ acts on a fermion ψ in the fundamental representation of the gauge group, and how this can be interpreted as a infinitesimal gauge transformation. Now prove (12.95) by considering the action on the Lie-algebra valued expression $G_{\rho\sigma}$.

12.6. Nonabelian gauge theories form a special class of nonlinear field theories which have interesting classical solutions. However, it is known that, for a four-dimensional Minkowski space-time, a pure gauge theory (i.e., with only gauge fields and no matter fields) has no static solutions of finite energy. In Euclidean four-dimensional space such solutions exist and are known as ‘instantons’. Instantons can be regarded as static solutions in a five-dimensional space-time, but their main interest lies in their relevance for describing quantum-mechanical tunneling between the topologically inequivalent ground states of a gauge theory.

Here we study the instanton solution for SU(2). Rather than solving the full field equation (cf. (12.93) with $J_\mu^a = 0$) it is easier to examine (anti-)selfdual field configurations. Such solutions satisfy the restriction,

$$\tilde{G}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma} = \pm G^{\mu\nu}, \quad (1)$$

or, in components, $G_{12} = \pm G_{34}$, $G_{13} = \mp G_{24}$ and $G_{14} = \pm G_{23}$. We note that for real field strengths (anti-)selfduality can only be imposed in Euclidean but not in Minkowski space. Because we formulate the solution in terms of gauge fields, the Bianchi identity, $D_\mu \tilde{G}^{\mu\nu} = 0$, is satisfied. Hence (anti-)selfdual solutions are therefore automatically solutions of the field equation $D_\mu G^{\mu\nu} = 0$.

As we are interested in solutions with finite action, the field strength must vanish at infinity. This implies that in the limit where the four-dimensional Euclidean length $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$ tends to infinity, the gauge fields must take the form,

$$W_\mu = [\partial_\mu U(x)]U^{-1}(x), \quad (r \rightarrow \infty) \quad (2)$$

where $U \in \text{SU}(2)$. Obviously, when the above expression holds everywhere, then the solution is trivial as it is equal to zero modulo a gauge transformation. To ensure that we are not dealing with such trivial solutions, the transformation $U(x)$ must be such that it cannot be extended smoothly over the interior of the four-dimensional space. A typical transformation that is not defined everywhere, is given by

$$U(x) = \frac{x^4 \mathbf{I} + i \vec{\sigma} \cdot \vec{x}}{r}, \quad (3)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual sigma matrices, and $\vec{x} = (x^1, x^2, x^3)$. Prove that (3) defines indeed an element of SU(2).

Near the origin this transformation is not well defined, while at infinity, one obtains a different element of SU(2) for every possible direction in the Euclidean space along

which one moves to infinity. In other words, when considering a three-dimensional sphere of large radius $r = R$, then to every point on the sphere we assign a different element of $SU(2)$ such that the whole sphere precisely covers the group. Of course, transformations $U(x)$ that cover the group precisely n times, are obtained by taking the n -th power of (3). None of these transformations can be defined globally over the whole four-dimensional space, so we can only assume that the value (2) is approached asymptotically at large distances, while at finite distances the solution takes a more complicated form, which is determined by the field equations.

At this point let us introduce some notation. Define four independent 2×2 matrices τ_μ by $\tau_\mu = i\sigma_a$ for $\mu = a = 1, 2, 3$ and $\tau_4 = \mathbf{I}$. Verify that $\tau_\mu^\dagger \tau_\nu + \tau_\nu^\dagger \tau_\mu = \tau_\mu \tau_\nu^\dagger + \tau_\nu \tau_\mu^\dagger = 2\delta_{\mu\nu} \mathbf{I}$, and show that (3) can be written as $U(x) = \tau_\mu x^\mu / r$. Note that the hermitean conjugate of $U(x)$ takes the same form but with τ_μ replaced by τ_μ^\dagger . This form is simply obtained by changing $\vec{x} \rightarrow -\vec{x}$. To appreciate the implications of (3), calculate the gauge fields (2), corresponding to $U(x)$ and $U^\dagger(x)$, denoted by $W_\mu^{(+)}$ and $W_\mu^{(-)}$, respectively,

$$W_\mu^{(+)}(x) = 2 \Sigma_{\mu\nu} \frac{x^\nu}{r^2}, \quad W_\mu^{(-)}(x) = 2 \bar{\Sigma}_{\mu\nu} \frac{x^\nu}{r^2}. \quad (4)$$

where $\Sigma_{\mu\nu}$ and $\bar{\Sigma}_{\mu\nu}$ are 2×2 matrices defined by

$$\Sigma_{\mu\nu} \equiv \frac{1}{4}(\tau_\mu \tau_\nu^\dagger - \tau_\nu \tau_\mu^\dagger), \quad \bar{\Sigma}_{\mu\nu} \equiv \frac{1}{4}(\tau_\mu^\dagger \tau_\nu - \tau_\nu^\dagger \tau_\mu).$$

Both $\Sigma_{\mu\nu}$ and $\bar{\Sigma}_{\mu\nu}$ are proportional to the three generators of $SU(2)$, equal to $\frac{1}{2}i\sigma_a$, so that the gauge fields W_μ^\pm take their values in the $SU(2)$ Lie algebra. To show this, verify that the nonvanishing matrices $\Sigma_{\mu\nu}$ satisfy,

$$\Sigma_{12} = \Sigma_{34} = \frac{1}{2}i\sigma_3, \quad \Sigma_{13} = -\Sigma_{24} = -\frac{1}{2}i\sigma_2, \quad \Sigma_{23} = \Sigma_{14} = \frac{1}{2}i\sigma_1, \quad (5)$$

from which it follows that $\Sigma_{\mu\nu}$ is selfdual. The following properties of the $\Sigma_{\mu\nu}$ are now straightforward to prove,

$$\begin{aligned} (\Sigma_{\mu\nu})^\dagger &= -\Sigma_{\mu\nu} = \Sigma_{\nu\mu}, \\ \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Sigma_{\rho\sigma} &= \Sigma^{\mu\nu}, \\ [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] &= -\delta_{\mu\rho}\Sigma_{\nu\sigma} + \delta_{\mu\sigma}\Sigma_{\nu\rho} - \delta_{\nu\sigma}\Sigma_{\mu\rho} + \delta_{\nu\rho}\Sigma_{\mu\sigma}, \\ \text{Tr}(\Sigma_{\mu\nu}\Sigma_{\rho\sigma}) &= -\frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}). \end{aligned} \quad (6)$$

Corresponding properties hold for the $\bar{\Sigma}_{\mu\nu}$, which are anti-selfdual.

To obtain a nontrivial solution, we replace (4) by a function that is regular and for large r tends to the same result,

$$W_\mu^{(+)}(x) = 2 \Sigma_{\mu\nu} \frac{(x-a)^\nu}{(x-a)^2 + \lambda^2}, \quad W_\mu^{(-)}(x) = 2 \bar{\Sigma}_{\mu\nu} \frac{(x-a)^\nu}{(x-a)^2 + \lambda^2}. \quad (7)$$

Show that the field strength corresponding to the first gauge field reads (from now on we suppress the index (+); the other case can be treated along the same lines)

$$\begin{aligned} G_{\mu\nu} &\equiv \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu] \\ &= \frac{4\lambda^2}{[(x-a)^2 + \lambda^2]^2} \Sigma_{\mu\nu}, \end{aligned} \quad (8)$$

which is indeed selfdual. This is the one-instanton solution. The vector a^μ denotes the position of the instanton and λ^2 its size. Note that a^μ and λ^2 are not determined by the field equations, simply because the gauge field equations are invariant under translations and scale transformations, so that a solution exists for any a^μ and λ^2 .

12.7. For a compact group, such as SU(2) the trace over the generators leads to a group-invariant metric that is negative definite. Use the identity $-\text{Tr}[(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2] \geq 0$ to show that the Euclidean gauge field action is bounded by

$$\begin{aligned} S[A_\mu] &= -\frac{1}{2g^2} \int d^4x \text{Tr}[G_{\mu\nu} G^{\mu\nu}] \\ &\geq \left| \frac{1}{4g^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \text{Tr}[G_{\mu\nu} G_{\rho\sigma}] \right|, \end{aligned} \quad (1)$$

where the trace is taken in the 2-dimensional representation of SU(2). The last term is a so-called topological invariant, which only depends on the boundary values of the fields of the four-dimensional integration domain. To see this, take its variation with respect to changes of the gauge field and use the Bianchi identity to show that it reduces to a boundary integral. The expression is known as the Pontryagin index which is defined by

$$Q = -\frac{1}{32\pi^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \text{Tr}[G_{\mu\nu} G_{\rho\sigma}]. \quad (2)$$

Evaluate the index for the one-instanton and one-anti-instanton solutions found above and show by performing the integral over the Euclidean four-space that it is equal to +1 and -1, respectively. (Use that the volume of a three-sphere with radius R is equal to $2\pi^2 R^3$.) Instanton solutions can be characterized by the fact that they saturate the above bound, so that their action is a multiple of $8\pi^2/g^2$. We can write the integrand directly as a total derivative, but of a quantity which is not gauge invariant, by using (12.70). Prove the validity of (12.70) and show that we can write the index as

$$Q = -\frac{1}{8\pi^2} \int d^4x \partial_\mu \left\{ \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \left[W_\nu \partial_\rho W_\sigma - \frac{2}{3} W_\nu W_\rho W_\sigma \right] \right\}, \quad (3)$$

Under the assumption that the fields are regular in the interior we can substitute the boundary values (2) for $r \rightarrow \infty$ and show that the index can be written as a functional of the transformation $U(x)$ taken at large r ,

$$Q[U] = -\frac{1}{12\pi^2} \int_{S^3} \varepsilon^{\mu\nu\rho} \text{Tr} [\partial_\nu U U^{-1} \partial_\rho U U^{-1} \partial_\sigma U U^{-1}]. \quad (4)$$

Here the indices refer to the coordinates on the sphere S^3 , which represents the boundary at infinity of the four-dimensional Euclidean space. This term is also known as the Wess-Zumino-Novikov-Witten term. It measures the 'winding' number of the transformation U , defined as the number of times that the the group SU(2) is mapped on the sphere. Show that the index is additive, i.e. $Q[U_1 U_2] = Q[U_1] + Q[U_2]$.

12.8. Construct the gauge-field Lagrangian for $SL(2)$ and E_2 and compare the result with that for the group $SU(2)$ (cf. Appendix C). Explain why the first two theories are physically unacceptable.