of quantum electrodynamics represents the electron mass in lowest order, but in higher orders we found that the electron mass is expressible in terms of some suitably chosen set of parameters (cf. 9.96). The same is true for the fine-structure constant, which is defined by the *full* amplitude for an on-shell electron to emit or absorb a zero-frequency photon (cf. 9.101). These relations between physical quantities and the parameters of the theory were exhibited in the previous chapter (where we also took care of the ultraviolet divergences).

Since the parameters in the Lagrangian have therefore no intrinsic physical meaning, it is natural to contemplate alternative parametrizations of the theory. One way to do this is to choose a set of Green's functions taken at prescribed values of the momenta and re-express all results of the theory in terms of these quantities. To be specific, consider a theory with a scalar field and single coupling constant g_0, described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - g_0\phi^4 \,. \tag{10.1}$$

Denote the n-point Green's functions of this theory corresponding to the sum of all *connected* Feynman diagrams with n external lines by $\Gamma^{(n)}(p_1,...,p_n;g_0)$ (see fig. 10.1) and note that only $n-1$ momenta are independent due to momentum conservation. Each of the external lines is related to the field of the Lagrangian (10.1). However, the normalization of this field is entirely motivated by lowest-order arguments, and one could alternatively re-express the Green's functions in terms of a field that has been normalized by some other criterion. One normalization condition is based on the two-point function $\Gamma^{(2)}(p;g_0)$ which represents the full propagator of the field ϕ. Due to Lorentz invariance $\Gamma^{(2)}$ must be a function of p^2 (p_μ is the momentum of the external lines) and a normalization prescription follows from requiring that the dimensionless quantity

$$\frac{\mathrm{d}}{\mathrm{d}p^2}[\Gamma^{(2)}(p;g_0)]^{-1}\Big|_{p^2=\mu^2} = Z(g_0,\mu) \tag{10.2}$$

becomes equal to unity (μ represents an arbitrary reference mass). This prescription amounts to a redefinition of the fields with a multiplicative factor $Z^{1/2}$, so that the corresponding Green's functions acquire a factor $Z^{-n/2}$. A similar prescription can be adopted for the coupling constant parameter, where one starts from the four-point Green's function. A new "renormalized" coupling constant parameter can be defined by

$$g(\mu) = \Gamma^{(4)}(p_i;g_0)\Big(\prod_{i=1}^{4}\Gamma^{(2)}(p_i;g_0)\Big)^{-1}\Big|_{\text{symmetric point}\,\mu^2}\,, \tag{10.3}$$

where the four momenta are chosen in a symmetric fashion according to

$$p_i\cdot p_j = \tfrac{1}{3}\mu^2(4\delta_{ij}-1)\,, \qquad i,j=1,2,3,4. \tag{10.4}$$

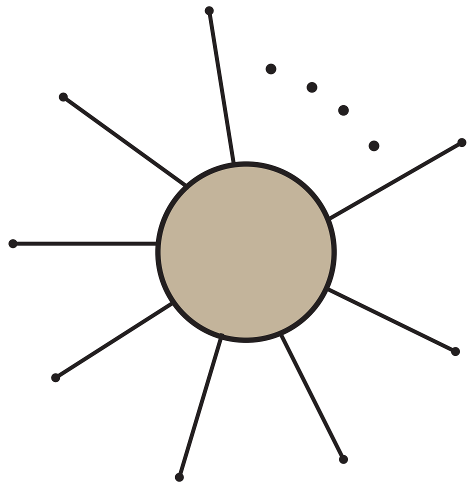


Figure 10.1: The n-point connected Green's functions.

In (10.3), we have divided the four-point function by the propagators of the external lines in order to obtain a dimensionless quantity and to avoid possible complications with propagator poles (hence $g(\mu)$) represents the invariant amplitude for two-particle elastic scattering, taken at (unphysical) values of the momenta characterized by μ). Such details are, however, not very essential in view of the fact that there exists an infinite variety of prescriptions that can be adopted for this purpose. Another prescription, for example, is to choose different reference masses in (10.2) and (10.3), or to choose the momentum configuration in (9.3) in a different fashion.

In a renormalizable field theory (10.2) and (10.3) are often called "renormalization conditions" or "subtractions", because the theory expressed in terms of fields and parameters defined by such conditions will take a finite form. The ultraviolet divergences of the theory are thus absorbed consistently into the newly defined parameters and the field normalization factors. However, for the moment we shall ignore the presence of ultraviolet divergences and proceed as if we are dealing with a finite field theory. It is possible to re-express the theory in terms of the new field and the new coupling constant, and we find straightforwardly:

$$\Gamma_{\mathrm{R}}^{(n)}(p_i; g(\mu), \mu) = Z_{\mathrm{R}}^{-n/2}(g_0, \mu)\Gamma^{(n)}(p_i; g_0)\,. \tag{10.5}$$

Since there are many possible descriptions for reparametrizing the theory, we have attached the label R to indicate that the quantities involved depend on the prescription, i.e., they are *renormalization dependent*. The parameter μ has no physical content, and simply characterizes a continuous set of reparametrizations of the theory; it could, for instance, be identified with the physical mass of the particle associated with ϕ. However, for the moment we wish to adopt the point of view that two theorists who perform calculations in this theory are entitled to their own choice of μ, so that (10.5) gives the relation between two different choices. Of course, any relation between *physical* amplitudes or other quantities that have an intrinsic (i.e. renormalization-independent) definition is not affected by the choice of parametrization; it is only the explicit dependence on the selected set of parameters that will change.

The set of all reparametrizations is called the "renormalization group", and was first studied by Stueckelberg and Petermann in 1953. Soon after, Gell-Mann and Low used this concept to obtain information on the asymptotic behaviour of the photon propagator in quantum electrodynamics. Their work was extended by Bogoliubov and Shirkov and then lay dormant for more than ten years. Around 1970, the subject was revived by Wilson, Callan and Symanzik in their study of the small-distance structure of renormalizable field theories.

The fact that the original theory and thus the original Green's functions $\Gamma^{(n)}$ do not depend on the parameter μ implies that the product $Z_R^{n/2}\Gamma_R^{(n)}$ is μ independent. This statement is expressed by the following partial differential equation

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta^{\rm R}(g,\mu)\frac{\partial}{\partial g}+\tfrac{1}{2}n\gamma^{\rm R}(g,\mu)\Big)\Gamma^n_{\rm R}(p_i;g,\mu)=0\,, \tag{10.6}$$

where

$$\begin{aligned} &g=g(\mu)\,,\\ &\beta^{\rm R}(g,\mu)=\mu\frac{\partial}{\partial\mu}g(\mu)\Big|_{g_0\,\text{fixed}}\,,\\ &\gamma^{\rm R}(g,\mu)=\mu\frac{\partial}{\partial\mu}\ln Z_{\rm R}(g_0,\mu)\Big|_{g_0\text{ fixed}} \end{aligned} \tag{10.7}$$

This equation, which follows straightforwardly from applying the chain differentiation rule to (10.5), is called the *renormalization-group equation*; just as (10.5) it expresses the simple fact that any change in the renormalization point μ amounts to a change in the coupling constant and in the field normalization. In other words, the physical content of the theory is not modified by a mere change of the parametrization. Note that the renormalization-group functions $\beta^{\rm R}$ and $\gamma^{\rm R}$ depend again on the renormalization prescription, which is defined by (10.2) and (10.3).

The presence of the function $\gamma^{\rm R}$ is due to the fact that there is no unique criterion for normalizing the field. Such a term cannot be present for physical quantities, whose normalization follows intrinsically from their definition.

Such quantities change only in their dependence on the coupling constant and renormalization point and therefore satisfy the equation

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta^{\mathrm{R}}(g,\mu)\frac{\partial}{\partial g}\Big)\mathcal{M}(p_i;g,\mu)=0\,, \qquad (10.8)$$

where $\mathcal{M}$ stands for a measurable quantity such as a decay rate or a cross section. Such quantities are called renormalization-group invariant. The function γ^{R} may still occur at intermediate stages of the calculation of physical quantities, but it will always cancel in the final result.

An example of a measurable quantity is the physical mass, identified by the location of the pole in the two-point Green's function (propagator). It is easy to show that the mass is renormalization-group invariant. If the two-point function $\varGamma^{(2)}_{\mathrm{R}}$ exhibits a pole for $p^2=-m^2(g,\mu)$, we may write

$$\varGamma^{(2)}_{\mathrm{R}}(p;g,\mu)=\frac{z(p^2;g,\mu)}{p^2+m^2(g,\mu)}\,, \qquad (10.9)$$

where $z(p^2;g,\mu)$ is finite for $p^2=-m^2(g,\mu)$. Substituting (10.9) into the renormalization-group equation (10.6), multiplying by $[p^2+m^2(g,\mu)]^2$, and taking the limit $p^2=-m^2(g,\mu)$, shows that $m^2(g,\mu)$ satisfies (10.8), which is the desired result.

If the Lagrangian (10.1) defines a finite field theory the renormalization-group equation (9.6) becomes trivial. In that case μ is the only dimensional parameter so that the new coupling constant $g(\mu)$ and the normalization factor $Z_{\mathrm{R}}(g_0,\mu)$, which are dimensionless, cannot depend on it. According to the definitions (10.7) β^{R} and γ^{R} are then zero, and one concludes that the new Green's functions $\varGamma^{(n)}_{\mathrm{R}}$ are also independent of μ. Of course, this example is rather special in that we could have included dimensional parameters in the Lagrangian so that the argument breaks down. However, within the context of this example it is illuminating to see what changes appear when the theory is not finite. Most of the previous result can still be taken over directly by introducing some regularization procedure characterized by a large cut-off mass parameter Λ. Keeping Λ finite for the moment, one simply follows the same arguments as before and derives the renormalization group equation:

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta^{\mathrm{R}}(g,\mu,\Lambda)\frac{\partial}{\partial g}+\tfrac{1}{2}n\gamma^{\mathrm{R}}(g,\mu,\Lambda)\Big)\varGamma^{n}_{\mathrm{R}}(p_i;g,\mu,\Lambda)=0\,, \qquad (10.10)$$

where we have indicated the dependence on the cut-off parameter. The functions β^{R} and γ^{R} are defined as in (10.7), except that also Λ is kept fixed in the partial differentiation:

$$\begin{aligned}\beta^{\mathrm{R}}(g,\mu,\Lambda)&=\mu\tfrac{\partial}{\partial\mu}g(\mu,\Lambda)\Big|_{g_0,\Lambda\,\text{fixed}}\,,\\ \gamma^{\mathrm{R}}(g,\mu,\Lambda)&=\mu\tfrac{\partial}{\partial\mu}\ln Z_{\mathrm{R}}(g_0,\mu,\Lambda)\Big|_{g_0,\Lambda\,\text{fixed}}\end{aligned} \qquad (10.11)$$

Ordinary dimensional analysis reveals that β^{R} and γ^{R} must be functions of Λ/μ. We now invoke the crucial information that the theory is renormalizable. This means that when the cut-off parameter Λ goes to infinity, the Green's functions remain finite when expressed in terms of the renormalized parameters. In other words, the ultraviolet infinities have been completely absorbed into the coupling constant g and the normalization factor Z. In the limit of infinite cut-off, if we consistently express everything in terms of the renormalized parameters, the renormalized Green's functions and therefore also the renormalization-group functions β^{R} and γ^{R} become independent of the cut-off parameter. Since β^{R} and γ^{R} actually depend on Λ through the ratio Λ/μ, they cannot depend on μ either, and we find the equation

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta^{\mathrm{R}}(g)\frac{\partial}{\partial g}+\tfrac{1}{2}n\gamma^{\mathrm{R}}(g)\Big)\mathit{\Gamma}^{n}_{\mathrm{R}}(p_i;g,\mu)=0\,. \tag{10.12}$$

Now β^{R} and γ^{R} need not be zero, and dimensional analysis only reveals the μ-independence of these functions. Of course, the latter result may not hold if we include dimensional parameters in the Lagrangian (in fact, for the theory (10.1), masses are generated by higher-order corrections and the one-loop contribution gives rise to infinite terms such as $\delta m^2 \propto g\Lambda^2, g\Lambda^2\ln\Lambda^2$; therefore it is imperative to include a mass term from the start, in which one can absorb the infinities of the self-energy corrections by renormalization. A similar situation has been discussed for scalar electrodynamics in problem 9.8. An important question is therefore if a subtraction procedure can generally be defined that renders the theory finite and yields μ-independent renormalization group functions. It turns out that this is indeed the case; the procedure that has superior properties from this point of view is the *minimal subtraction procedure* which is based on dimensional regularization. In this way one obtains Green's functions that satisfy the simplest possible form of the renormalization-group equation. As we shall see in section 10.3 this equation can be exploited to derive important consequences for the momentum dependence of Green's functions. Of course, once one has obtained finite results in one subtraction procedure, it is always possible to recast the results in terms of another set of parameters. Before discussing this is more detail we first review the consequences of minimal subtraction.

10.2. Minimal subtraction

We first repeat the observation that in n space-time dimensions the dimension of the Lagrangian must be equal to n units of mass in order that the action be dimensionless; therefore the dimension of the fields and coupling constants must depend on n as well. For instance, the dimension of the field in (10.1) is $\frac{1}{2}(n-2)$. Therefore, the dimension of g_0 equals $n-4\cdot\frac{1}{2}(n-2)=4-n$.

Obviously mass parameters always have dimension 1. Quite generally one can write the dimension of a parameter as

$$\text{dimension} = d + d'\varepsilon\,, \tag{10.13}$$

where d is its dimension at $n = 4$, and $\varepsilon \equiv n - 4$. Hence, for the coupling constant g_0 in (10.1), we have $d = 0$ and $d' = -1$. Renormalizable theories generally have parameters for which d is positive.

When the theory is regularized by dimensional regularization, the infinities take the form of inverse powers of ε. These infinities are then absorbed into the original parameters of the Lagrangian, as was demonstrated in the previous chapters. Taking a one-parameter theory, such as given by (10.1), as an example, one has therefore

$$g_0 = \mu^{d+d'\varepsilon}\Big\{g + \sum_{\nu=1}^{\infty}\frac{a^{(\nu)}(g)}{\varepsilon^{\nu}}\Big\}\,, \tag{10.14}$$

$$Z = 1 + \sum_{\nu=1}^{\infty}\frac{b^{(\nu)}(g)}{\varepsilon^{\nu}}\,, \tag{10.15}$$

where the functions $a^{(\nu)}(g)$ and $b^{(\nu)}(g)$ can be calculated order by order in perturbation expansion by requiring that the theory remains finite for vanishing ε. The parameter g, the renormalized coupling constant, is *defined* by (10.14) and is chosen to remain dimensionless in any number of space-time dimensions. The dimension of the coupling constant g_0 is equal to $d + d'\varepsilon$ and the parameter μ is an arbitrary mass parameter to make (10.14) dimensionally correct. Of course, the field normalization factor (10.15) is always dimensionless so that no power of μ is required in its definition.

The crucial aspect of a renormalizable theory is that coefficient functions $a^{(\nu)}(g)$ and $b^{(\nu)}(g)$ can be found such that the theory remains finite when ε approaches zero. Of course, the subtractions defined by (10.14) and (10.15) are not unique, and it is always possible to introduce additional finite terms. However, we *only* retain the poles (this motivated the name minimal subtraction), so that we have defined a one-parameter set of renormalization conditions characterized by a single mass parameter μ. The subtraction procedure by which g and the renormalized field are defined is thus similar to the conditions (10.2) and (10.3), except that the physical meaning of the renormalized coupling constant g is no longer intuitively clear, because it does not correspond to the value of a Green's function at some given point in momentum space. Changing the value of μ corresponds to adopting a different coupling constant g in accordance with (10.14).

We now simply rederive the renormalization group equation, in $4 + \varepsilon$ dimensions, using the minimal subtraction scheme. We find

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta(g,\varepsilon)\frac{\partial}{\partial g}+\tfrac{1}{2}n\gamma(g,\varepsilon)\Big)\Gamma^{(n)}(p_i;g,\mu,\varepsilon)=0\,, \tag{10.16}$$

where g follows from (10.14) and

$$\beta(g,\varepsilon)=\mu\frac{\partial}{\partial\mu}g(\mu)\Big|_{g_0\,,\varepsilon\,\text{fixed}}\,, \tag{10.17}$$

$$\gamma(g,\varepsilon)=\mu\frac{\partial}{\partial\mu}\ln Z\Big|_{g_0\,,\varepsilon\,\text{fixed}}\,. \tag{10.18}$$

Due to the fact that g was chosen dimensionless, β and γ must be μ independent. Since the theory is assumed to be renormalizable, the limit $\varepsilon\to 0$ can be taken without encountering any infinities. Therefore, we conclude that β and γ must be finite when ε approaches zero, and we can determine them in terms of the coefficient functions in (10.14) and (10.15). To do this we differentiate (10.14) and (10.15) with respect to μ, keeping g_0 and ε fixed. This leads to

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}g_0\Big|_{g_0,\varepsilon\,\text{fixed}} &= (d+d'\varepsilon)\mu^{d+d'\varepsilon}\Big\{g+\sum_{\nu=1}^{\infty}a^{(\nu)}(g)\frac{1}{\varepsilon^{\nu}}\Big\}\\ &\quad+\mu^{d+d'\varepsilon}\Big\{1+\sum_{\nu=1}^{\infty}\frac{\partial a^{(\nu)}(g)}{\partial g}\frac{1}{\varepsilon^{\nu}}\Big\}\beta(g,\varepsilon)=0\,,\end{aligned} \tag{10.19}$$

$$\mu\frac{\partial}{\partial\mu}Z\Big|_{g_0,\varepsilon\,\text{fixed}}=\beta(g,\varepsilon)\sum_{\nu=1}^{\infty}\frac{\partial b^{(\nu)}(g)}{\partial g}\frac{1}{\varepsilon^{\nu}}\equiv Z\gamma(g,\varepsilon) \tag{10.20}$$

where on the right-hand side we substituted (10.17) and (10.18). Both equations can be solved and yield β and γ as a Laurent series in ε:

$$\begin{aligned}\beta(g,\varepsilon) &= -(d+d'\varepsilon)\Big\{g+\sum_{\nu=1}^{\infty}a^{(\nu)}(g)\varepsilon^{-\nu}\Big\}\Big\{1+\sum_{\rho=1}^{\infty}\frac{\partial a^{(\rho)}(g)}{\partial g}\varepsilon^{-\rho}\Big\}^{-1}\\ &= -(d+d'\varepsilon)g-d'\Big(1-g\frac{\partial}{\partial g}\Big)a^{(1)}(g)+O\Big(\frac{1}{\varepsilon}\Big)\,,\\ \gamma(g,\varepsilon) &= \beta(g,\varepsilon)\Big\{\sum_{\nu=1}^{\infty}\frac{\partial b^{(\nu)(g)}}{\partial g}\varepsilon^{-\nu}\Big\}\Big\{1+\sum_{\rho=1}^{\infty}b^{(\rho)}(g)\varepsilon^{-\rho}\Big\}^{-1}\\ &= -d'g\frac{\partial b^{(1)(g)}}{\partial g}+O\Big(\frac{1}{\varepsilon}\Big)\,.\end{aligned}$$

Noting that β and γ are finite for $\varepsilon=0$ we conclude that all higher-order terms in $1/\varepsilon$ must vanish, so that

$$\beta(g,\varepsilon)=-(d+d'\varepsilon)g-d'\Big(1-g\frac{\partial}{\partial g}\Big)a^{(1)}(g)\,, \tag{10.21}$$

$$\gamma(g,\varepsilon) = -d' g \frac{\partial b^{(1)}(g)}{\partial g} \,. \tag{10.22}$$

Substituting (10.21) and (10.22) into (10.19) and (10.20) one finds that the terms of higher order in $1/\varepsilon$ vanish provided that

$$\begin{aligned} d'\Big(1 - g\frac{\partial}{\partial g}\Big) a^{(\nu+1)}(g) &= -d a^{(\nu)}(g) + \Big\{ dg + d'\Big(1 - g\frac{\partial}{\partial g}\Big) a^{(1)}(g) \Big\} \frac{\partial a^{(\nu)}(g)}{\partial g} \,, \\ d' g \frac{\partial b^{(\nu+1)}(g)}{\partial g} &= d' g \frac{\partial b^{(1)}(g)}{\partial g} b^{(\nu)}(g) \\ &\quad - \Big\{ dg + d'\Big(1 - g\frac{\partial}{\partial g}\Big) a^{(1)}(g) \Big\} \frac{\partial b^{(\nu)}(g)}{\partial g} \,, \end{aligned} \tag{10.23}$$

These equations show that the coefficient functions $a^{(\nu)}$ and $b^{(\nu)}$ of the higher-order poles in (10.14) and (10.15) are related to the coefficient functions $a^{(1)}$ and $b^{(1)}$ of the single poles (cf. problem 9.4).

Clearly the minimal subtraction procedure yields functions β and γ that are independent of μ. The above derivation can be straightforwardly generalized to the case of several coupling constants and mass parameters and we merely quote the result. If a theory depends on a set of parameters, either coupling constants or mass parameters, which we generically denote by g_α, then the coefficient functions $a_\alpha^{(\nu)}$ and $b^{(\nu)}$ are usually functions of all these parameters. The renormalization-group equation then takes the form

$$\Big(\mu \frac{\partial}{\partial \mu} + \beta_\alpha(g) \frac{\partial}{\partial g_\alpha} + \tfrac{1}{2} n \gamma(g) \Big) \Gamma^{(n)}(p_i; g, \mu) = 0 \,, \tag{10.24}$$

where each coupling constant has its own β-function defined by

$$\beta_\alpha = -(d_\alpha + d'_\alpha \varepsilon) g_\alpha - \Big(d'_\alpha - \sum_\beta d'_\beta g_\beta \frac{\partial}{\partial g_\beta}\Big) a_\alpha^{(1)} \,. \tag{10.25}$$

Here we have assumed that the unrenormalized coupling constants have dimensions given by $d_\alpha + d'_\alpha \varepsilon$; the renormalized parameters g_α are again made dimensionless by extracting suitable powers of μ. The γ function then takes the form

$$\gamma = -\sum_\beta d'_\beta g_\beta \frac{\partial b^{(1)}}{\partial g_\beta} \,. \tag{10.26}$$

In case there are several fields, each of them has its own normalization constant Z (or the Z's form a matrix if there is mixing between the various fields). This leads to corresponding expressions for the coefficient functions $b^{(\nu)}$ and the renormalization-group functions γ. In addition to (10.25) and (10.26) there are differential equations that govern the relation between the coefficient functions of the multiple poles and the single poles which are direct generalizations of (10.23) (cf. problem 9.4).

It can be shown that the coefficient functions $a^{(\nu)}$ and $b^{(\nu)}$ must be polynomials in the coupling constant and mass parameters at every given order in perturbation theory. This result is obvious for the coupling constants, as one is always performing perturbation theory in terms of coupling constants, but for the mass parameters this is a nontrivial property which follows only after combining all the relevant Feynman graphs and subtracting all divergences for the subgraphs order-by-order in perturbation theory (as we shall see shortly the gauge parameter in quantum electrodynamics, which is neither a coupling constant nor a mass parameter, does not satisfy this polynomial property). Another important observation is that the parameter μ is only required for maintaining the correct dimensionality of the counterterms away from 4 dimensions. Therefore, if we express the coefficient functions in terms of parameters $\mu^{d_\alpha} g_\alpha$ that have the correct dimensionality of the 4-dimensional theory, they should no longer depend on μ. As far as dimensional parameters are concerned the form of the coefficient functions will be strongly restricted by this requirement. This is seen most clearly in a so-called "super-renormalizable" theory, where all parameters have *positive* dimensions (i.e. $d_i > 0$). An example of such a theory is given by

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 - g_0 \phi^3 + c_0 \phi \,. \tag{10.27}$$

The parameters m_0^2, g_0 and c_0 have dimension [2], $[1-\frac{1}{2}\varepsilon]$ and $[3+\frac{1}{2}\varepsilon]$, respectively. A two-loop calculation gives the following result for the counterterms (see problems 9.1-9.3):

$$\begin{aligned} m_0^2 &= \mu^2 \Big\{ m^2 - \frac{9g^2}{4\pi^2}\frac{1}{\varepsilon} \Big\} , \\ g_0 &= \mu^{1-\frac{1}{2}\varepsilon} g \, , \\ c_0 &= \mu^{3+\frac{1}{2}\varepsilon} \Big\{ c + \frac{3gm^2}{8\pi^2}\frac{1}{\varepsilon} - \frac{27g^3}{128\pi^4}\frac{1}{\varepsilon} - \frac{27g^3}{64\pi^4}\frac{1}{\varepsilon^2} \Big\} , \\ Z &= 1 \, . \end{aligned} \tag{10.28}$$

Substituting (10.28) into the Lagrangian (10.27) renders the theory finite in the two-loop approximation. According to (10.28) neither coupling constant nor field renormalization is required. This follows from dimensional analysis: as all coupling constants have positive dimension, one cannot introduce terms in the expressions for g_0 and Z which are polynomials in the renormalized dimensional parameters $\mu^2 m^2$, μg and $\mu^3 c$ without the need for explicit factors of μ to make the result dimensionally correct in 4 dimensions. Moreover, the same dimensional arguments reveal that the expressions (10.28) must in fact suffice to render the theory finite to all orders. The reason is that (10.28) contains already all terms that are allowed on dimensional grounds, and higher-order terms in g cannot be written down.

The theory based on (10.27) is rather unrealistic in view of the fact that the potential $V(\phi) = \frac{1}{2}m_0^2\phi^2 + g_0\phi^3 - c_0\phi$ is not bounded from below. For that reason, and also for subsequent discussions, we return to quantum electrodynamics. The various parameters are the coupling constant e_0, the electron mass m_0 and the gauge parameter λ_0, which have dimension $[-\frac{1}{2}\varepsilon]$, [1] and [0], respectively. Since there are two fields, A_μ and ψ, we have two field renormalization factors Z_A and Z_ψ. The counterterm expressions can be read off from (8.50)

$$
\begin{aligned}
e_0 &= \mu^{-\frac{1}{2}\varepsilon}\Big\{e - \frac{e^3}{12\pi^2}\frac{1}{\varepsilon} + O(e^5)\Big\}\,,\\
m_0 &= \mu m\Big\{1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + O(e^4)\Big\}\,,\\
\lambda_0 &= \lambda\Big\{1 - \frac{e^2}{12\pi^2}\frac{1}{\varepsilon} + O(e^4)\Big\}\,,\\
Z_A &= 1 + \frac{e^2}{6\pi^2}\frac{1}{\varepsilon} + O(e^4)\,,\\
Z_\psi &= 1 + \frac{e^2\lambda^{-2}}{8\pi^2}\frac{1}{\varepsilon} + O(e^4)\,,
\end{aligned}
\tag{10.29}
$$

where we have only retained the $1/\varepsilon$ terms according to the minimal subtraction procedure, and we have used dimensionless renormalized parameters e, m and λ. Note that the coeffcients of the $1/\varepsilon$ terms are indeed polynomials in e and m, but *not* in λ.

The one-loop renormalization functions follow straightforwardly from (10.25)

$$
\begin{aligned}
\beta_e &= \frac{e^3}{12\pi^2} + O(e^5)\,,\\
\beta_m &= -m\Big\{1 + \frac{3e^2}{8\pi^2} + O(e^4)\Big\}\,,\\
\beta_\lambda &= \frac{\lambda e^2}{12\pi^2} + O(e^4)\,,\\
\gamma_A &= \frac{e^2}{6\pi^2} + O(e^4)\,,\\
\gamma_\psi &= \frac{e^2\lambda^{-2}}{8\pi^2} + O(e^4)\,,
\end{aligned}
\tag{10.30}
$$

where we put $\varepsilon = 0$. The parameters e, m and λ are defined by the minimal subtraction requirement, and do not coincide with those used in chapters

8 and 9, where we have subtracted additional terms. Their relation is easy to determine by comparing (10.29) to (8.50), taking into account that the parameters are now dimensionless. Denoting the renormalized parameters of chapters 8 and 9 by e', m' and λ' the relation with e, m and λ is

$$\begin{aligned} e' &= e + \frac{e^3}{24\pi^2}(\gamma_E - \ln 4\pi) + O(e^5)\,, \\ m' &= \mu m\Big\{1 - \frac{3e^2}{12\pi^2}(\gamma_E - \ln 4\pi) + O(e^4)\Big\}\,, \\ \lambda' &= \lambda\Big\{1 + \frac{e^2}{24\pi^2}(\gamma_E - \ln 4\pi) + O(e^4)\Big\}\,. \end{aligned} \tag{10.31}$$

Also the fields A_μ, and ψ have different normalizations.

$$\begin{aligned} A'_\mu &= \Big\{1 - \frac{e^2}{24\pi^2}(\gamma_E - \ln 4\pi) + O(e^4)\Big\}A_\mu\,, \\ \psi' &= \Big\{1 - \frac{e^2\lambda^{-2}}{32\pi^2}(\gamma_E - \ln 4\pi) + O(e^4)\Big\}\psi\,. \end{aligned} \tag{10.32}$$

With help of (10.31) and (10.32) it is straightforward to cast the one-loop results of chapter 8 into minimally subtracted form. For instance, the photon propagator still has the form (9.92):

$$\varDelta_{\mu\nu} = \frac{1}{i(2\pi)^4}\frac{1}{k^2}\Big\{\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\frac{1}{1+\Pi^f(k^2)} + \lambda^{-2}\frac{k_\mu k_\nu}{k^2}\Big\} \tag{10.33}$$

where $\Pi^f(k^2)$ is now given by (for $k^2 > 0$)

$$\begin{aligned} \Pi^f(k^2) = \frac{-e^2}{12\pi^2}\Big\{ &-\frac{5}{3} + \gamma_E + \ln\frac{m^2}{4\pi} + \frac{4m^2\mu^2}{k^2} \\ &+\Big(1 - \frac{2\mu^2 m^2}{k^2}\Big)\sqrt{1+\frac{4\mu^2m^2}{k^2}}\,\ln\Big|\frac{1+\sqrt{1+4\mu^2m^2/k^2}}{1-\sqrt{1+4\mu^2m^2/k^2}}\Big|\Big\}\,, \end{aligned} \tag{10.34}$$

and we made use of (9.44). Observe that the $\lambda^{-2}k_\mu k_\nu/k^4$ term in (10.33) has not changed, because the A_μ normalization factor cancels against the factor originating from the redefinition of the gauge parameter λ. It is now easy to verify the renormalization-group equation for $\varDelta_{\mu\nu}(k)$,

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_e\frac{\partial}{\partial e} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_m\frac{\partial}{\partial m} + \gamma_A\Big)\varDelta_{\mu\nu}(k) = 0\,, \tag{10.35}$$

by substituting (10.34) and neglecting terms of order e^4.

Rather than expressing the results of chapter 9 in terms of the parameter defined by minimal subtraction one could use the original results and write down a renormalization-group equation in terms of the corresponding field and parameters. To see how the renormalization-group functions change under a reparametrization, consider (10.24) and introduce new parameters and field g'_α and ϕ' according to

$$g_\alpha = g_\alpha(g')\,,\qquad \phi = \sqrt{z(\phi')}\phi' \tag{10.36}$$

where g' in $g_\alpha(g')$ and $z(g')$ generically denotes the full set of parameters g'_α. The renormalization-group equation is then changed accordingly to

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta'_\alpha\frac{\partial}{\partial g'_\alpha} + \tfrac{1}{2}n\gamma'(g')\Big)\Gamma_{\rm R}^{(n)'}(p_i; g', \mu) = 0\,, \tag{10.37}$$

where

$$\begin{aligned}
&\Gamma_{\rm R}^{(n)'}(p_i; g', \mu) = z(g')^{-\frac{1}{2}n}\,\Gamma_{\rm R}^{(n)}(p_i; g(g'), \mu)\,,\\
&\beta'_\alpha(g') = \beta_\beta(g(g'))\frac{\partial g'_\alpha(g)}{\partial g_\beta}\,,\\
&\gamma'(g') = \gamma(g(g')) + \beta_\alpha(g)\frac{\partial \ln z(g')}{\partial g_\alpha}\,.
\end{aligned} \tag{10.38}$$

One may also choose dimensional parameters by reabsorbing the appropriate power of μ into the parameters g_α. In that case one replaces g'_α by $\mu^{-d_\alpha}g'_\alpha$ which leads to an extra term in the new β functions

$$\beta'_\alpha(\mu^{-d}g') = d_\alpha g'_\alpha + \beta_\beta(g(\mu^{-d}g'))\frac{\partial g'_\alpha(g,\mu)}{\partial g_\beta}\,, \tag{10.39}$$

where $\mu^{-d}g'$ generically denotes the $\mu^{-d_\alpha}g'_\alpha$, and $g'_\alpha(g,\mu)$ is the inverse of $g_\alpha(\mu^{-d}g')$ (it can be shown that the conversion to dimensional parameters by absorbing powers of μ still leaves the renormalization-group functions μ-independent; however, further reparametrizations do not necessarily preserve this property). Using (10.31) and (10.32) we can now determine the renormalization-group functions for quantum electrodynamics in the parametrization used in chapter 8 and 9. We find

$$\begin{aligned}
\beta'_m &= m' + \beta_e\frac{\partial m'}{\partial e} + \beta_m\frac{\partial m'}{\partial m} + \beta_\lambda\frac{\partial m'}{\partial\lambda}\\
&= m'\Big(-\frac{3e'^2}{8\pi^2} + O(e'^4)\Big)\,,
\end{aligned} \tag{10.40}$$

while β_e, β_λ, γ_A and γ_ψ, remain unchanged in this order of perturbation theory. It is now straightforward to verify the correctness of these results by applying the renormalization-group equation to the various one-loop results of chapter 9. In particular one may verify that the physical mass and charge given by (9.96) and (9.101) satisfy the renormalization-group equation (10.8).

10.3. The running coupling constant

Now that we have acquainted ourselves with the renormalization-group equation in various parametrizations let us study its consequences for the momentum dependence of Green's functions. We first observe that the subtraction mass μ and the momenta p_i are the only dimensional parameters in (10.24). Using ordinary dimensional analysis it follows that we may rescale μ and p_i by a common factor λ, so that a Green's function with dimension d_Γ will scale uniformly with a factor λ^{d_Γ}. Choosing λ infinitesimally close to unity, this leads to

$$\Big(\mu\frac{\partial}{\partial\mu}+\frac{\partial}{\partial t}\Big)\Gamma^{(n)}(\mathrm{e}^t\,p_i;g,\mu)=d_\Gamma\Gamma^{(n)}(\mathrm{e}^t\,p_i;g,\mu)\,, \tag{10.41}$$

where the ordinary dimension of the Green's function, d_Γ, depends on the dimension of the fields associated with the external lines. Combining (10.24) and (10.41) yields

$$\Big(\frac{\partial}{\partial t}-\beta_\alpha(g)\frac{\partial}{\partial g_\alpha}-d_\Gamma(g)\Big)\Gamma^{(n)}(\mathrm{e}^t\,p_i;g,\mu)=0\,. \tag{10.42}$$

where

$$d_\Gamma(g)=d_\Gamma+\tfrac{1}{2}n\gamma(g)\,. \tag{10.43}$$

At this point the subtraction mass μ no longer plays a role and one may reinstate dimensional parameters again by multiplying with the appropriate power of μ.

According to (10.42) the renormalization-group equation governs the change of the Green's function under a uniform scaling of all momenta. Apart from a multiplicative factor, represented by $d_\Gamma(g)$, the change of scale can be accounted for by a change of the coupling constant by an amount equal to the β function. This suggests that we solve (10.42) by introducing the so-called effective or running coupling constant $g(t)$ as the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g_\alpha(t)=\beta_\alpha(g(t));\quad g_\alpha(0)=g_\alpha\,. \tag{10.44}$$

Consider the expression

$$F(t)=\exp\Big\{\int_{t_0}^{t}\mathrm{d}\tau d_\Gamma(g(\tau))\Big\}\Gamma^{(n)}(\mathrm{e}^{-t}p_i;g(t),\mu)\,, \tag{10.45}$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = \exp\Big\{\int_{t_0}^{t} \mathrm{d}t d_\Gamma(g(\tau))\Big\} \\ \times \Big\{\frac{\partial}{\partial t} + \beta_\alpha(g(t))\frac{\partial}{\partial g_\alpha} + d_\Gamma(g(t))\Big\}\Gamma^{(n)}(e^{-t}p_i; g(t), \mu)\,, \tag{10.46}$$

where the third term proportional to d_Γ arises from the differentiation of the exponential factor. Using (10.42) one concludes that (10.46) vanishes, so $F(t)$ does not depend on t, i.e.

$$\Gamma^{(n)}(\mathrm{e}^{-t_1}p_i; g(t_1), \mu) = \Gamma^{(n)}(\mathrm{e}^{-t_2}p_i; g(t_2), \mu)\exp\Big\{\int_{t_1}^{t_2} \mathrm{d}\tau d_\Gamma g(\tau))\Big\}\,. \tag{10.47}$$

Choosing $t_1 = 0$, $t_2 = t$, and replacing p_i by $\mathrm{e}^t\,p_i$, this result takes the form

$$\Gamma^{(n)}(\mathrm{e}^t p_i; g, \mu) = \Gamma^{(n)}(p_i; g(t), \mu)\exp\Big\{\int_0^{t} \mathrm{d}\tau d_\Gamma g(\tau))\Big\}\,. \tag{10.48}$$

Hence, modulo a multiplicative factor, the Green's function at a different momentum scale $p_i \to \mathrm{e}^t\,p_i$ can be obtained by merely changing the parameter g to $g(t)$, where the latter is the solution of (10.44). For $\beta_\alpha(g) = -d_\alpha g_\alpha$ (so that $g_\alpha(t) = \mathrm{e}^{-d_\alpha t}\,g_\alpha$) and $\gamma(g) = 0$, the values one obtains in the tree approximation, (10.48) is just what one expects on the basis of ordinary dimensional analysis, where the change of the momentum scale is compensated for by a corresponding rescaling of all dimensional parameters. The fact that, in higher orders, the multiplicative factor in (10.48) is no longer equal to $\exp(td_\Gamma)$, where d_Γ is the ordinary dimension of the Green's function, has motivated the name " anomalous" dimension for $\gamma(g)$.

In order to study the theory at asymptotic momenta it is thus important to first examine (10.44). Since this is a first-order differential equation $g_\alpha(t)$ is uniquely determined in terms of a given set of values for $g_\alpha(0)$, so that $g_\alpha(t)$ describes a trajectory in parameter space. To understand the nature of these trajectories consider a one-parameter theory whose β function is depicted in fig. 10.2a. If $g(0)$ has a value where $\beta(g) > 0$ then $g(t)$ is an increasing function of t, until it reaches a value of g where $\beta(g) = 0$. Alternatively, if $g(0)$ has a value, $\beta(g) < 0$ then $g(t)$ is a decreasing function of t, until again it reaches a value where $\beta(g) = 0$. Trajectories for different values of $g(0)$ are sketched in fig. 10.2b, where the arrows indicate the evolution of $g(t)$ for increasing values of t. Clearly the zeros of the β function separate the one-dimensional parameter space into different regions, and the corresponding coupling constant $g(t)$ remains confined to one such region. Irrespective of the precise value for $g(0)$ in a given region, the asymptotic value of $g(t)$ for

$t \to \pm\infty$ is always the same and given by the nearest points for which $\beta(g)$ vanishes. Therefore these zeros are called *fixed points*. There are two kinds of fixed points; a zero of the β function is called an *ultraviolet attractor* if starting from a coupling constant at a neighbouring value it is approached for $t \to \infty$. Hence in fig. 10.2 g_2 is an ultraviolet attractor, whereas the multiple zero g_3 is an attractor when approached from larger values of g. Alternatively, we have *ultraviolet repulsors*, which are approached when $t \to -\infty$. So g_1 and g_4 are repulsors, and the multiple zero g_3 is a repulsor when approached from the left. For obvious reasons ultraviolet repulsors (attractors) are sometimes called infrared attractors (repulsors).

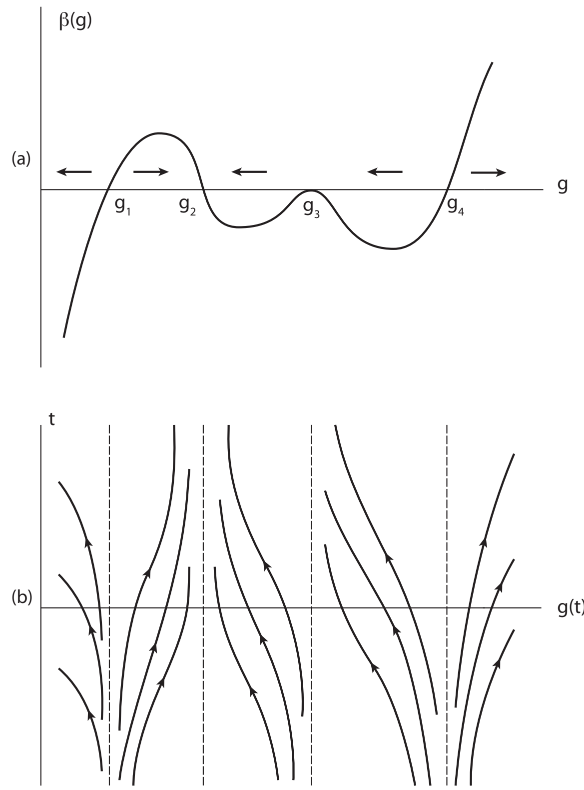


Figure 10.2: Coupling constant flow in a hypothetical one-parameter theory. The β-function shown in (a) has several zeros corresponding to ultraviolet attractors or repulsors. The arrows point towards the ultraviolet attractors. The corresponding "running" coupling constant $g(t)$ is shown in (b); the arrows indicate how $g(t)$ changes for increasing values of t.

If there are more parameters then all β functions must simultaneously vanish at a fixed point. According to the differential equation (10.44) a unique trajectory passes through each point in parameter space, and a fixed point is an ultraviolet attractor (repulsor) if all trajectories in its neighbourhood approach the fixed point as $t \to \infty$ ($t \to -\infty$). However, more complicated situations are possible, where a fixed point is both attractive and repulsive depending on the parameter space domain in which the parameter is initially located. Alternatively one may have a limit contour rather than a simple fixed point. However, we shall confine our discussion to fixed points only.

As an example consider again the superrenormalizable theory described by (10.27). It is straightforward to obtain the corresponding β functions from (10.28) by use of (10.25). The equations (10.44) then read

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} g(t) &= -g(t)\,, \\ \frac{\mathrm{d}}{\mathrm{d}t} m^2(t) &= -2m^2(t) + \frac{9g^2(t)}{4\pi^2}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t} c(t) &= -3c(t) - \frac{3g(t)m^2(t)}{8\pi^2} + \frac{27g^3(t)}{64\pi^4}\,, \end{aligned} \tag{10.49}$$

and have solutions

$$\begin{aligned} g(t) &= g\mathrm{e}^{-t}\,, \\ m^2(t) &= \left\{ m^2 + \frac{9g^2}{4\pi^2} t \right\} \mathrm{e}^{-2t}\,, \\ c(t) &= \left\{ c - \frac{3gm^2}{8\pi^2} t - \frac{27g^3}{64\pi^4} t(t-1) \right\} \mathrm{e}^{-3t}\,, \end{aligned} \tag{10.50}$$

where

$$g = g(0)\,, \quad m^2 = m^2(0)\,, \quad c = c(0)\,. \tag{10.51}$$

There is one fixed point in this case, namely at

$$g = m^2 = c = 0. \tag{10.52}$$

In the limit $t \to \infty$ (10.48) thus gives

$$\lim_{t\to\infty} \mathit{\Gamma}^{(n)}(\mathrm{e}^t p_i; g, m^2, c, \mu) = \mathrm{e}^{d_\Gamma t} \mathit{\Gamma}^{(n)}(p_i; 0, 0, 0, \mu)\,, \tag{10.53}$$

which coincides with naive expectations: in the limit where all momenta are uniformly scaled to infinity, masses and dimensional coupling constants can be ignored. For a superrenormalizable theory, where all parameters have dimension, this implies that one recovers the free-field theory, as demonstrated by (10.25). However, some caution is needed in taking the limit $m^2 \to 0$. For certain momentum configurations the Green's functions will not be finite in this limit due to infrared divergences. Therefore (10.53) is only applicable for momenta (so-called nonexceptionable momenta) where infrared divergences are absent.

10.4. Asymptotic behaviour

We have thus seen that the asymptotic behaviour of Green's functions is determined by fixed points, which are approached asymptotically for any value of the coupling constants located in a corresponding domain in parameter space. Before analyzing the behaviour near a fixed point, it is important to understand how this behaviour depends on the precise definitions of the parameters g_α that one has adopted for calculating the corresponding β functions. To see this, recall (10.38) which prescribes how the $\beta_\alpha(g)$ change under reparametrizations

$$\beta'_\alpha(g) = \beta_\beta(g(g'))\frac{\partial g'_\alpha}{\partial g_\beta}\,. \qquad (10.54)$$

On the other hand reparametrization of (10.44) according to $g_\alpha = g_\alpha(g')$ leads to

$$\frac{\partial g_\alpha}{\partial g'_\beta}\frac{\mathrm{d}}{\mathrm{d}t}g'_\beta(t) = \beta_\beta(g(g'(t)))\,,$$

or, using (9.53), to

$$\frac{\mathrm{d}}{\mathrm{d}t}g'_\alpha(t) = \beta'_\alpha(g'(t))\,.$$

Therefore the asymptotic behaviour of Green's functions can be analyzed in any parametrization (in more formal terms, this is expressed by the statemen that β_α transforms as a covariant vector under reparametrizations, so the (10.44) preserves its form). For a one-parameter theory this fact implies that the derivative of the β function at a fixed point is independent of the parametrization. To see this consider a fixed point g^c, so that $\beta(g^c) = 0$. According to (10.54) there is a corresponding fixed point g'^c in another parametrization given by $g^c = g(g'^c)$ with $\beta'(g'^c) = 0$. Taking derivatives of (10.54) at the fixed point gives

$$\begin{aligned}\frac{\partial}{\partial g'}\beta'(g') &= \frac{\partial\beta(g)}{\partial g'}\frac{\partial g'}{\partial g} + \beta(g)\frac{\partial}{\partial g'}\Big(\frac{\partial g'}{\partial g}\Big)\\ &= \frac{\partial\beta(g)}{\partial g}\frac{\partial(g)}{\partial g'}\frac{\partial(g')}{\partial g}\\ &= \frac{\partial\beta(g)}{\partial g} \qquad \text{for} \qquad g = g^c\,, \qquad g' = g'^c\,,\end{aligned} \qquad (10.55)$$

where we dropped the second term in the first equation because $\beta(g^c) = 0$. It often happens that the first few derivatives vanish at the fixed point, in which case one can show in a similar fashion that the first nonvanishing derivative at the fixed point is independent of the parametrization, Sometimes it is possible to prove that also the second nonvanishing derivative is independent of the parametrization.

Let us now discuss the various types of asymptotic behaviour, where, for simplicity, we restrict ourselves to a theory with one coupling constant g and one mass parameter m. We are interested in the limit of large momenta, and assume an ultraviolet fixed point at $g = g_c$ and $m = 0$. Subsequently we make an expansion of the renormalization-group functions,

$$\begin{aligned}
&\beta_g(g) = -a(g - g^c) + O((g - g^c)^2)\,,\\
&\beta_m(m, g) = -m\{b + c(g - g^c) + O((g - g^c)^2)\}\,,\\
&\gamma(g) = \gamma(g^c) + d(g - g^c) + O((g - g^c)^2)\,,
\end{aligned} \tag{10.56}$$

where $a, b > 0$ in order to have an ultraviolet attractor at $g = g^c$ and $m = 0$. The expression for β_g can be used to find an approximate solution for $g(t)$ by solving (10.44),

$$g(t) = g^c + (g - g^c)\,\mathrm{e}^{-at} + O(\mathrm{e}^{-2at})\,. \tag{10.57}$$

In the case at hand, where β_m is linearly proportional to m, the solution for $m(t)$ is expressed by

$$m(t) = m \exp\Big\{ \int_0^t d\tau \beta_m(1, g(\tau))\Big\}\,, \tag{10.58}$$

which can conveniently be written as

$$m(t) = m\mathrm{e}^{\beta_m(1,g^c)t} \exp\Big\{ \int_g^{g(t)} dg' \,\frac{\beta_m(1, g') - \beta_m(1, g^c)}{\beta_g(g')}\Big\}\,. \tag{10.59}$$

Substituting the β-functions (10.56) gives

$$m(t) = m\mathrm{e}^{-bt}\{1 + O(g(t) - g)\}\,, \tag{10.60}$$

which for large t takes the form

$$m(t) = m_0\mathrm{e}^{-bt}\{1 + O(\mathrm{e}^{-at})\}\,, \tag{10.61}$$

with m_0 an undetermined constant.

To determine the asymptotic behaviour of the Green's functions it is convenient to rewrite (10.48) as

$$\begin{aligned}
\Gamma^{(n)}(\mathrm{e}^t p_i; g, m) = \exp\Big\{\tfrac{1}{2} n \int_0^t d\tau[\gamma(g(\tau)) - \gamma(g^c)]\Big\}&\\
\times\, \mathrm{e}^{d_\Gamma(g^c)t}\,\Gamma^{(n)}(p_i; g(t), m(t))\,,&
\end{aligned} \tag{10.62}$$

where from now on we suppress the parameter μ. The factor in the exponent can be evaluated from

$$\int_0^t d\tau[\gamma(g(\tau)) - \gamma(g^c)] = \int_g^{g(t)} dg' \frac{\gamma(g') - \gamma(g^c)}{\beta_g(g')} \,. \tag{10.63}$$

Substituting the expressions for β_g and γ gives

$$\int_g^{g(t)} dg' \Big\{ -\frac{d}{a} + O(g' - g^c) \Big\} = O(g(t) - g) = \ln C + O(\mathrm{e}^{-at}) \tag{10.64}$$

with C some undetermined constant. From (10.57)- (10.60) it follows that

$$\lim_{t \to \infty} \Gamma^{(n)}(\mathrm{e}^t p_i; g, m) = C^{\frac{1}{2}n} \mathrm{e}^{d_\Gamma(g^c)t} \Gamma^{(n)}(p_i; g^c, 0) \,. \tag{10.65}$$

with corrections that are powers of e^t. Again we note that it may be necessary to exclude certain momentum configurations in (10.65) to avoid the appearance of infrared divergences associated with the limit $m \to 0$. Usually the β functions cannot be calculated reliably beyond perturbation theory, so that it is not possible to establish the existence of a fixed point at $g^c \neq 0$. This is not so for a fixed point at the origin. If the origin is an ultraviolet attractor then one can reliably calculate the infinite-momentum limit in perturbation theory, a phenomenon known as *asymptotic freedom*. To see how this works consider the perturbation expansion

$$\begin{aligned} &\beta_g(g) = -ag^3\{1 + bg^2 + O(g^4)\} \,, \\ &\beta_m(g) = -m\{1 + cg^2 + O(g^4)\} \,, \\ &\gamma(g) = dg^2 + O(g^4) \,, \end{aligned} \tag{10.66}$$

where we assume a to be positive in order to have an ultraviolet attractor at $g = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} g^2(t) = -2ag^4(t)\{1 + bg^2(t) + O((g(t))^4)\} \,, \tag{10.67}$$

which leads to

$$\begin{aligned} g^2(t) &= \frac{g^2}{1 + 2ag^2 t} \Big\{ 1 - \frac{bg^2}{1 + 2ag^2 t} \ln(1 + 2ag^2 t) + \cdots \Big\} \\ &\approx \frac{1}{2at} + O\Big(\frac{\ln t}{t^2}\Big) \,. \end{aligned} \tag{10.68}$$

Consequently, $g^2(t)$ vanishes at large t, so that results for the asymptotic behaviour can be obtained in perturbation theory.

Using the expressions (10.66) for β_m and γ we find for (10.59) and (10.63),

$$m(t) = m\,\mathrm{e}^{-t}\exp\Big\{\frac{c}{a}\ln\frac{g(t)}{g} + O((g(t)-g)^2)\Big\}\,,$$
$$\int_g^{g(t)} dg'\,\frac{\gamma(g')-\gamma(0)}{\beta_g(g')} = -\frac{d}{a}\ln\frac{g(t)}{g} + ((g(t)-g)^2)\,, \tag{10.69}$$

which, for large t, gives

$$m(t) = m_0\,\mathrm{e}^{-t}\,t^{-\frac{1}{2}c/a}\Big\{1+O\Big(\frac{\ln t}{t}\Big)\Big\}\,,$$
$$\exp\int_g^{g(t)} dg'\,\frac{\gamma(g')-\gamma(0)}{\beta_g(g')} = Ct^{\frac{1}{2}d/a}\Big\{1+O\Big(\frac{\ln t}{t}\Big)\Big\}\,, \tag{10.70}$$

with m_0 and C undetermined constants that depend on g and m. Using these results one derives for the large-momentum behaviour of the Green's functions

$$\lim_{t\to\infty}\varGamma^{(n)}(\mathrm{e}^t p_i; g, m) = (Ct^{\frac{1}{2}d/a})^{\frac{1}{2}n}\mathrm{e}^{d_\varGamma t}\varGamma^{(n)}(p_i; g(t), m(t)) \tag{10.71}$$

with corrections of order $t^{-1}\ln t$. Since $g(t)$ vanishes at large t one can apply ordinary perturbation theory in g, and replace g by $g(t)$ and m by $m(t)$ in order to calculate the asymptotic behaviour of the Green's functions. Corrections to the leading behaviour will depend *logarithmically* on the momenta (remember that the momenta are scaled according to $p \to \mathrm{e}^t p$).

What makes asymptotic freedom such an important phenomenon is that there are nontrivial field theories exhibiting this property, namely certain non-abelian gauge theories, and furthermore, that the deviations from naive quark-parton behaviour as found in deep-inelastic scattering seem to depend logarithmically on Q^2, as we have been explaining in section 6.6. Indeed, the standard theory of strong interactions, quantum chromodynamics, is an asymptotically-free non-abelian gauge theory. The discovery of asymptotic freedom has opened an entirely new perspective on understanding the strong interactions, as one can now obtain predictions at high energy by the use of perturbation theory. These predictions have been successfully confronted with experimental results.

10.5. Vacuum polarization

To explore the consequences of the renormalization group a little further, consider a theory of charged fields ψ_a with charges e_a. These fields thus couple to the electromagnetic field A_μ, with coupling constants e_a and in addition they may be subject to other interactions with generic coupling constant and mass parameters g, as long as those do not interfere with the electromagnetic gauge invariance of the Lagrangian. In the straightforward way of renormalizing this theory one assigns separate renormalization constants to each of the charges e_a, the parameters g and the fields. In particular we know from the discussion of section 8.3 that making the replacements

$$e_a^0 = e_a Z_e^a \,, \qquad \psi_a^0 = \sqrt{Z_\psi^a}\psi_a, \qquad A_\mu^0 = \sqrt{Z_A}A_\mu \,, \tag{10.72}$$

in the original Lagrangian generates a counterterm for the $A_\mu - \psi_a - \psi_a$ vertex proportional to $e_a Z_e^a Z_\psi^a Z_A^{1/2}$ which can be used to absorb the ultraviolet infinities of this vertex function order-by-order in perturbation theory (here we make the simplifying assumptions that e_a^0 is proportional to e_a and that the value of the index a is not changed in the interaction with the photon by higher-order corrections (i.e. there are no $A_\mu - \psi_a - \psi_b$ couplings with $a \neq b$); however, none of these assumptions are essential for what follows). Since we introduce separate renormalization factors Z_e^a for each of the charges e_a the renormalization-group equation will thus contain separate β functions β_a.

However, the above discussion does not take into account the fact that the photon couples to a conserved current (with a corresponding conserved charge), which imposes important restrictions on the renormalization of the charges e_a. To understand why this must be the case consider a specific theory describing nucleons and pions. Apart from electromagnetic interactions nucleons and pions are subject to strong interactions; hence there are vertices corresponding to $\bar{\mathrm{p}}\mathrm{n}\pi^+$, $\bar{\mathrm{p}}\mathrm{p}\pi^0$, $\bar{\mathrm{n}}\mathrm{p}\pi^-$, etc. (see e.g. 5.47), such that electric charge is conserved. Quantum corrections will obviously cause a renormalization of the nucleon and pion charges, but in order to ensure that charge remains conserved by the strong interactions the nucleon and pion charges must be renormalized in the "same" way. Based on this intuitive argument one may conclude that the charge renormalization factors Z_e^a must be independent of the type of field, so that it will suffice to introduce one common renormalization factor Z_e.

A more rigorous argument leading to the-same conclusion follows from the Ward-Takahashi identity (cf. problem 7.3), which follows from current conservation, and relates the $A_\mu - \psi_a - \psi_a$ vertex corrections to the ψ_a propagator corrections (this identity remains true in the presence of additional interactions, provided that the electromagnetic current remain conserved). Due to this relation, the infinite part of the vertex corrections, which must be cancelled by a counterterm proportional to $e_a(Z_e^a Z_\psi^a Z_A^{1/2} - 1)$, is related to those infinite parts of the propagator corrections that correspond to the infinities to be absorbed by the field renormalization counterterm proportional to $Z_\psi^a - 1$. In this way one can prove that the infinite parts of $Z_e^a Z_\psi^a Z_A^{1/2}$ and Z_ψ^a must be the same, which shows that $Z_e^a Z_A^{1/2}$ must be finite. This result is consistent with our previous conclusion that the infinite part Z_e^a is independent of a.

The finite product $Z_e^a Z_A^{1/2}$ can be chosen equal unity by adopting a suitable definition of renormalized charges. This is the definition that we choose below.

The fact that all charges e_a acquire only a common multiplicative renormalization leads us to write $e_a = q_a e$, and assume that only e will be renormalized bv higher-order corrections, whereas the proportionality constants q_a can be kept fixed. The renormalization of e is now as before

$$e^0 = eZ_e \,, \tag{10.73}$$

and we make the convenient choice indicated above.

$$Z_e = Z_A^{-1/2} \,, \tag{10.74}$$

With these definitions the renormalization group contains only one β-function associated with the various electric charges, and we may write

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_e(e,g)\frac{\partial}{\partial e} + \beta_g(e,g)\frac{\partial}{\partial g} + \tfrac{1}{2}n\gamma(e,g)\Big)\Gamma_{\mathrm{R}}^{(n)}(p_i;e,g,\mu) = 0 \,, \tag{10.75}$$

where all parameters other than e are generically denoted by g, and we have not distinguished between different fields in the term $\frac{1}{2}n\gamma$ (each field has its own anomalous dimension γ); furthermore the dependence on the constants q_a has been suppressed in the Green's functions $\Gamma_{\mathrm{R}}^{(n)}$).

The fact that it is possible to express the charge renormalization in terms of the photon field renormalization factor Z_A implies a relation between β_e and γ_A. Using (10.25) and (10.26) one can easily verify that

$$\beta_e(e,g) = \tfrac{1}{2}e\gamma_A(e,g) \,. \tag{10.76}$$

This useful relation does not change under reparametrization provided (9.72) remains satisfied (to see this use 10.38). Note that (10.76) is indeed satisfied for the β and γ functions given in (10.30).

Let us now examine how quantum corrections modify Coulomb's law. To do this consider the scattering of two particles A and B with charges $e_{\mathrm{A}} = q_{\mathrm{A}}e$ and $e_{\mathrm{B}} = q_{\mathrm{B}}e$, which are only subject to electromagnetic interactions. In the limit that these charges are small we can restrict ourselves to the contribution of single-photon exchange diagrams, shown in fig. 10.3, so that the scattering amplitude is proportional to $e^2\Delta_{\mu\nu}(k)$, where $\Delta_{\mu\nu}(k)$ is the full photon propagator. Here we should caution the reader that this approximation where self-energy and vertex corrections of the particles A and B are neglected, while all quantum corrections are included in the photon propagator, requires some justification. Indeed this approach is not consistent, in general, but for the case at hand where the particles A and B are only subject to electromagnetic interactions, consistency can be established.

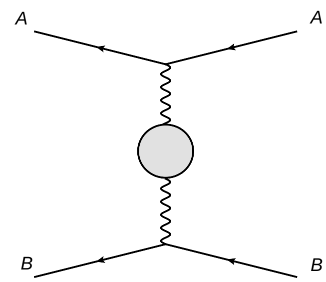


Figure 10.3: Single-photon exchange diagrams contributing to the scattering of two charged particles. The incoming and outgoing particles are treated in tree approximation, while all higher-order corrections are included in the photon propagator, resulting in a deviation from Coulomb's law.

The full photon propagator decomposes into two terms as shown in (9.92). One can prove that the second term proportional to $k_\mu k_\nu k^{-4}\lambda^{-2}$ is not modified by higher-order terms; it also does not contribute to the scattering amplitude since the electromagnetic currents associated with γAA and γBB vertices must be conserved. Therefore we concentrate on the first term, which is proportional to $k^{-2}[1+\Pi^f(k^2)]^{-1}$. As is well-known it is the $1/k^2$ propagator pole that gives rise to the Coulomb potential, so that the momentum-dependent factor $[1+\Pi^f(k^2)]^{-1}$ represents the quantum corrections to the Coulomb potential. In momentum space the modified Coulomb potential thus corresponds to

$$D(k^2) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{e^2}{k^2}\frac{1}{1+\Pi^f(k^2)}\,, \tag{10.77}$$

where we have suppressed the factors q_A and q_B. The reader may be surprised to see the full renormalized charge in (10.77), whereas the γAA and γBB vertices are only treated in tree approximation. However, both e and $[1+\Pi^f(k^2)]^{-1}$ refer to renormalized quantities, which are related to the corresponding unrenormalized quantities by multiplicative factors Z_e and Z_A, respectively. According to (10.74) these factors cancel in (10.77), so that e and $[1+\Pi^f(k^2)]^{-1}$ can be replaced by the unrenormalized charge and vacuum polarization function.

There is an intuitive way to understand how the vacuum polarization corrections [contained in $\Pi^f(k^2)$] modify Coulomb's law. If two electric charges exchange a virtual photon, then this photon may create (virtual) particle antiparticle pairs, thus "polarizing" the vacuum in the same way an electric field polarizes a material medium as described in macroscopic electrostatics. Due to this effect the Coulomb force may decrease or increase. If there is a decrease one speaks of *screening*. This is the standard situation that arises in material media, where an electric field separates the opposite charges in the material, thus inducing some charge distribution in the medium. The electric field caused by this charge distribution partly compensates for the electric field applied from outside, so that a test charge inside the medium measures a smaller value of the electric field than a test charge at the same position but without the medium present. Two charges in the medium thus feel a smaller Coulomb force than outside the medium, but the screening becomes less effective if the distance between the two charges is reduced. Therefore the correction to the Coulomb force caused by the screening depends on the distance between the charges, such that the force *increases* (as compared to the standard $1/r^2$ force) at short distances. As the modification to Coulomb's law is represented by the factor $[1+\Pi^f(k^2)]^{-1}$ in (10.77) one may conclude that screening implies that $1+\Pi^f(k^2)$ should decrease at large k^2. Indeed, this behaviour is observed in quantum electrodynamics in the one-loop approximation (cf. 9.46 and fig. 9.3), where $1+\Pi^f(k^2)$ decreases logarithmically for large k^2, i.e.

$$1+\Pi^f(k^2) \approx 1 - \frac{e^2}{12\pi^2}\Big(\ln\frac{k^2}{\mu^2} - \frac{5}{3}\Big) + O\Big(\frac{m^2}{k^2}\Big)\,. \tag{10.78}$$

Note that this result is only reliable if k^2 is not too large, as perturbation theory becomes invalid when $e^2 \ln k^2/\mu^2 \approx 1$. Therefore no significance should be attached to the fact that $1+\Pi^f(k^2)$ vanishes at some very large value of k^2 in this order of perturbation theory.

One might also envisage the alternative situation of *antiscreening*, where the vacuum polarization causes an increase of the electromagnetic field. When two charges are brought together this effect will diminish, so that the Coulomb force will decrease as compared to the standard $1/r^2$ behaviour. This corresponds to an increase of $\Pi^f(k^2)$ at large k^2. Antiscreening thus seems related to the phenomenon of asymptotic freedom that may arise in non-abelian gauge theories as discussed in the previous section. However, we should caution the reader that some of the assumptions that we have made above cannot be justified for such theories, and it is not quite possible to discuss asymptotic freedom in the same way on the basis of the vacuum polarization alone.

The k^2-dependence of (10.77) can be studied by means of the renormalization-group equation. There is no anomalous dimension associated with (10.77), so that

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_e(e,g)\frac{\partial}{\partial e} + \beta_g(e,g)\frac{\partial}{\partial g}\Big) D(k^2;e,g,\mu) = 0\,. \tag{10.79}$$

The reason why the anomalous dimension is absent in (10.79) is that we are treating the particles A and B in tree approximation; alternatively one may derive (10.79) directly from the renormalization-group equation for the photon propagator and the relation

$$\beta_e \frac{\partial}{\partial e} e^2 = e^2 \gamma_A \,, \tag{10.80}$$

which follows from (10.76). Using (10.80) once more we may also write a renormalization-group equation for the vacuum polarization function, which is *inhomogeneous*,

$$\Big(\mu \frac{\partial}{\partial \mu} + \beta_e(e,g) \frac{\partial}{\partial e} + \beta_g(e,g) \frac{\partial}{\partial g}\Big) \frac{\Pi^f(k^2)}{e^2} = \frac{\gamma_A(e,g)}{e^2} \,, \tag{10.81}$$

where we observe that Π^f and γ_A are both proportional to e^2 since they involve at least two photon vertices.

Let us first apply (10.81) to pure quantum electrodynamics, where the only parameter besides e is the electron mass m. In that case (10.81) reads

$$\Big(\mu \frac{\partial}{\partial \mu} + \beta_e(e) \frac{\partial}{\partial e} + \beta_m(m,e) \frac{\partial}{\partial m}\Big) \frac{\Pi^f(k^2)}{e^2} = \frac{\gamma_A(e)}{e^2} \,, \tag{10.82}$$

To simplify matters we examine this equation for $m = 0$, which can be done provided that Π^f is finite in that limit. For the unrenormalized theory it can be shown that vacuum polarization diagrams contain no mass divergences for positive k^2 (mass divergences arise if a “sufficient” number of internal line momenta associated with massless particles can simultaneously become arbitrarily small; the fact that $k^2 > 0$ prevents this from happening so that the external momentum provides a lower cut-off). Therefore if one is sufficiently careful in renormalizing the theory and avoids introducing mass singular terms into the definition of the renormalized parameters (like for instance the $\ln m^2/\mu^2$ terms that appear in the physical parameters 9.96 and 9.101) the renormalized vacuum polarization function will remain free of mass singularities as well. This result holds also for more general Feynman diagrams, although not for all theories, provided the external momentum configuration is not “exceptional” (meaning that all partial sums of incoming momenta must be spacelike). In addition there is a result due to Kinoshita, related to a general theorem of Lee and Nauenberg, according to which the total transition probability remains finite if the masses of some, or all, of the particles in the final state tend to zero. From this result it follows that the imaginary part of the vacuum polarization function, which equals the total decay probability for a massive virtual photon, has no mass singularities either (here it is essential that one sums over different diagrams). Again this result is derived for the unrenormalized theory, and only holds for the corresponding renormalized quantities if the renormalization procedure does not introduce spurious mass divergences.

In the limit $m \to 0$ (10.82) simplifies because β_m is proportional to m and may be dropped. Hence we obtain

$$\Big(\mu\frac{\partial}{\partial\mu} + e^4\gamma(e^2)\frac{\partial}{\partial e^2}\Big)\Pi\Big(\frac{k^2}{\mu^2}, e^2\Big) = \gamma(e^2)\,, \tag{10.83}$$

where we have used the substitutions

$$\beta_e = \tfrac{1}{2}e\gamma_A(e) = \tfrac{1}{2}e^3\gamma(e^2)\,, \qquad \Pi^f(k^2) = e^2\Pi(\frac{k^2}{\mu^2}, e^2)\,. \tag{10.84}$$

From previous results (cf. 9.44 and 10.30) we know the lowest-order terms of γ and Π, namely

$$\begin{aligned}\gamma(e^2) &= \frac{1}{6\pi^2} + O(e^2)\,,\\ \Pi\Big(\frac{k^2}{\mu^2}, e^2\Big) &= \frac{1}{12\pi^2}\Big(-\ln\frac{k^2}{\mu^2} + \frac{5}{3}\Big) + O(e^2)\,, \qquad k^2 > 0\,,\end{aligned} \tag{10.85}$$

based on the subtraction procedure of section 9.4, which does not introduce mass singularities. Writing γ and Π as a series expansion in e^2,

$$\begin{aligned}\gamma(e^2) &= \sum_{n=0}^{\infty} c_n e^{2n}\,,\\ \Pi\Big(\frac{k^2}{\mu^2}, e^2\Big) &= \sum_{n=0}^{\infty} p_n(\tau)e^{2n}\,,\end{aligned} \tag{10.86}$$

where for convenience we use a parameter $\tau = \ln k^2/\mu^2$, (10.83) implies

$$\frac{d}{d\tau}p_n(\tau) - \tfrac{1}{2}\sum_{q=1}^{n-1} q c_{n-q-1} p_q(\tau) + \tfrac{1}{2}c_n = 0\,. \tag{10.87}$$

From this equation it follows that the $p_n(\tau)$ are polynomials in τ of degree n (with the exception of $p_0(\tau)$ which is a polynomial of first degree). Furthermore, the leading powers in τ are determined entirely in terms of the first two coefficients, c_0 and c_1, of the function γ, and thus of the β function, i.e.

$$\begin{aligned}p_n(\tau) &= -\frac{1}{n}\frac{c_1}{c_0}\Big(\frac{c_0\tau}{2}\Big)^n + O(\tau^{n-1})\,, \qquad n \geqslant 1\,,\\ p_0(\tau) &= -\frac{c_0\tau}{2} + \frac{5}{36\pi^2}\,.\end{aligned} \tag{10.88}$$

Let us now concentrate on the first two terms in the expansions (10.86), which yield

$$\Pi\Big(\frac{k^2}{\mu^2}, e^2\Big) = -\tfrac{1}{2}(c_0 + c_1 e^2)\ln\frac{k^2}{\mu^2} + \text{const.} + O(e^4)\,. \tag{10.89}$$

At this point we can make contact with the calculation of section 8.6, where we determined the imaginary part of the vacuum polarization diagrams in the two-loop approximation. According to (9.154) and (9.156) the result for zero fermion mass is

$$\mathrm{Im}\Pi^f(k^2) = \frac{e^2}{12\pi} + \frac{e^4}{64\pi^3} + O(e^6)\,, \tag{10.90}$$

where we should point out that e in (10.90) is defined as the physical electron charge, while e in the preceding formulae was defined according to a mass-independent subtraction scheme in order to avoid the introduction of spurious mass singularities. The same comment applies to the normalization of the photon field which is also different in (10.90). However, it is not difficult to verify that (10.90) holds for both parametrizations, because, in this order of perturbation theory, the effect of the redefinition of e cancels against the effect of adopting a new normalization of the photon field by virtue of the condition (10.74). Hence we can ignore this subtlety and conclude from (10.90) that

$$\mathrm{Im}\Pi(\frac{k^2}{\mu^2}, e^2) = \frac{1}{12\pi} + \frac{e^2}{64\pi^3} + O(e^4)\,. \tag{10.91}$$

This result can be compared to (10.89) after analytic continuation of the latter to timelike k^2. According to the iε-prescription of the propagators one has $\ln k^2 = \ln|k^2| - i\pi\theta(-k^2)$, so that (10.89) yields

$$\mathrm{Im}\Pi(\frac{k^2}{\mu^2}, e^2) = \tfrac{1}{2}\pi(c_0 + c_1 e^2) + O(e^4)\,. \tag{10.92}$$

Comparison of this to (10.91) shows that

$$c_0 = \frac{1}{6\pi^2}\,, \qquad c_1 = \frac{1}{32\pi^4}\,. \tag{10.93}$$

Hence we have now determined the first two coefficients of the β function in quantum electrodynamics

$$\beta_e(e) = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + O(e^7)\,, \tag{10.94}$$

and the leading logarithmic terms of the vacuum polarization function; the latter follow from substitution of (10.93) into (10.88), and can be summed to

$$\Pi^f(k^2) \approx -\frac{e^2}{12\pi^2}\ln\frac{k^2}{\mu^2} + \frac{3e^2}{16\pi^2}\ln\Big(1 - \frac{e^2}{12\pi^2}\ln\frac{k^2}{\mu^2}\Big)\,. \tag{10.95}$$

This last result fully demonstrates the power of the renormalization group for determining the asymptotic behaviour. However, for sufficiently large values of k^2 perturbation theory breaks down and (10.95) is no longer reliable.

By the same techniques we may examine the vacuum polarization function in quantum chromodynamics, where the virtual photon produces a quark-antiquark pair which is subject to interactions with vector gluons. Ignoring higher-order terms in e, so that

$$\Pi(k^2, g, m, \mu) \equiv \frac{\Pi^f(k^2)}{e^2}\,,$$

$$\gamma(g) \equiv \frac{\gamma_A(e,g)}{e^2} \tag{10.96}$$

depend only on the quark-gluon coupling constant g and the quark mass m, (10.81) now reads

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_g(g)\frac{\partial}{\partial g} + \beta_m(m,g)\frac{\partial}{\partial m}\Big)\Pi(k^2, g, m, \mu) = \gamma(g)\,. \tag{10.97}$$

Just as in section 10.3 one can use dimensional analysis to find that

$$\Pi(\mathrm{e}^{2t}\, k^2, g, m, \mu) = \Pi(k^2, g(t), m(t), \mu) - \int_0^t d\tau\gamma(g(\tau))\,, \tag{10.98}$$

where $g(t)$ and $m(t)$ are determined by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = \beta_g(g(t))\,, \qquad g(0) = g\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t) = \beta_m(m(t), g(t))\,, \qquad m(0) = m\,. \tag{10.99}$$

Quantum chromodynamics is an asymptotically free gauge theory, so we can make use of (10.67)- (10.70) to derive an asymptotic prediction for Π. To do that it is convenient to rewrite (10.98) as

$$\Pi(\mathrm{e}^{2t}\, k^2, g, m, \mu) = \Pi(k^2, g(t), m(t), \mu) - \gamma(0)t - \int_g^{g(t)} \mathrm{d}g'\,\frac{\gamma(g') - \gamma(0)}{\beta(g')}\,. \tag{10.100}$$

Substitution of (10.67)-(10.70) then shows that

$$\lim_{t\to\infty} \Pi(\mathrm{e}^{2t}\, k^2, g, m, \mu) = -\gamma(0)t + \Pi(k^2, 0, 0, \mu) + O(\ln t)\,. \tag{10.101}$$

Both $\gamma(0)$ and $\Pi(k^2, 0, 0, \mu)$ follow from the one-loop graphs in quantum electrodynamics, but, as there are different types of quarks with different electric charges, there is an additional factor equal to the sum of the squares of the quark charges (expressed in units of the elementary charge e), i.e.

$$\gamma(0) = \frac{1}{6\pi^2}(\textstyle\sum q^2)\,,$$

$$\Pi(k^2, 0, 0, \mu) = -\frac{1}{12\pi^2}(\textstyle\sum q^2)\Big(\ln\frac{k^2}{\mu^2} - \frac{5}{3}\Big)\,. \tag{10.102}$$

However, in this case we are particularly interested in the imaginary part of Π, since it is related to the probability of a virtual photon to decay into quarks and gluons (which constitute the physical hadrons). This is a quantity that can be measured in the reaction $e^+e^- \to$ hadrons as a function of the centre-of mass energy $\sqrt{s}$ of the electron-positron pair (which equals the mass of the virtual photon). In principle, one can obtain the imaginary part of Π by analytic continuation to timelike k^2, analogously to what we did for pure quantum electrodynamics. However, in quantum chromodynamics the use of perturbation theory may be questioned in the timelike region, as the physical spectrum of quantum chromodynamics consists of physical hadrons, and not of multi-quark and -gluon states as one would expect on the basis of perturbation theory. The latter are confined inside the hadrons, and it seems that nonperturbative effects are crucial to understand this phenomenon. Nevertheless let us disregard such questions and rely on perturbation theory to determine the imaginary part and see the implications of asymptotic freedom for the process $e^+e^- \to$ hadrons.

Rather than to perform an analytic continuation for Π it is easier to apply (10.97) directly to the imaginary part itself. As the right-hand side of (10.97) is real, we recover again the homogeneous renormalization group equation, whose implication is given by (10.71). In this case there are no terms related to the (anomalous) dimension, so that we have

$$\mathrm{Im}\Pi(\mathrm{e}^{2t}\,k^2, g, m, \mu) = \mathrm{Im}\Pi(k^2, g(t), m(t), \mu)\,. \tag{10.103}$$

In order to obtain an asymptotic prediction for $\mathrm{Im}\Pi$ one calculates in perturbation theory to a certain order in g, and subsequently replaces g and m by $g(t)$ and $m(t)$. As $m(t)$ decreases exponentially for large t (cf. 10.70) it can be dropped, so that it suffices to perform the perturbative calculations directly for the massless theory (obviously this requires the absence of mass singularities, as we have discussed before).

Asymptotically terms of order g^2 can be neglected, so that the result coincides with the imaginary part of the one-loop graphs of quantum electrodynamics, multiplied by the sum of the squares of the quark charges. It is convenient to factor out the lowest-order result of quantum electrodynamics and to consider the ratio (cf. 6.127)

$$R = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons})}{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)}\,.$$

From the above reasoning we thus conclude that in quantum chromodynamics

$$R \underset{s\to\infty}{\longrightarrow} \left(\sum q^2\right)[1 + O(g^2(t)]\,, \tag{10.104}$$

where $s = -e^{2t}k^2$ and $s_0 = -k^2$ is an arbitrary reference energy (often identified with the characteristic mass scale Λ^2 of quantum chromodynamics).

Consequently we have

$$t = \tfrac{1}{2} \ln \frac{s}{s_0} \,. \tag{10.105}$$

In order to calculate the corrections proportional to $g^2(t)$ it is necessary to determine the imaginary part of the diagrams shown in fig. 10.4. Because the quark gluon vertices resemble the electron (or muon) photon vertex, this can be done by comparing with the result for quantum electrodynamics, which follows from (9.154) and (9.156),

$$\mathrm{Im}\Big(\frac{\Pi^f}{e^2}\Big)^{\mathrm{QED}} = \frac{1}{12\pi}\Big\{1 + \frac{3e^2}{16\pi^2} + O(e^4)\Big\} \,. \tag{10.106}$$

Figure 10.4: Diagrams whose imaginary part contributes to the ratio R. The external line denotes the virtual photon, the internal curly lines correspond to gluons and the solid lines to quarks.

The difference between the diagrams of fig. 9.8, which are responsible for (10.106), and those of fig. 10.4 is that the internal photon lines have been replaced by gluon lines and that the quark charges are not equal to e but to qe (with $q = \frac{2}{3}$ or -$\frac{1}{3}$). Therefore a precise comparison between the interaction vertices will yield a result analogous to (10.106) for quantum chromodynamics, which will be proportional to q^2 and contain a term of second order in the quark-gluon coupling constant g.

As explained previously there are u, d, c, s, t and b quarks. These quark types are called quark "flavours". The flavour of a quark determines its weak and electromagnetic interactions, but not its strong interactions mediated by gluons. In addition each quark of a certain flavour appears in a three-fold degeneracy, conventionally denoted by "colours": red, green and blue. Therefore one can assign three fermion fields to each quark, one for each colour, which can be arranged into a three-component colour vector,

$$q(x) = (q_{\mathrm{r}}(x), q_{\mathrm{g}}(x), q_{\mathrm{b}}(x)) \qquad \mathrm{r, g, b} = \mathrm{red, green, blue} \,.$$

The gluons carry colour as well, so that the quarks may change their colour in the interactions with a gluon. There are eight types of gluons, with corresponding fields $V^a_\mu(x)(a = 1, 2, ..., 8)$, which couple to the quark according to the interaction Lagrangian

$$\mathcal{L}_{\mathrm{q\bar{q}g}} = \mathrm{i}g\bar{q}\gamma_\mu \frac{\lambda_a}{2} q V^a_\mu \,. \tag{10.107}$$

The λ_a are 3×3 matrices, defined in appendix G where the full QCD Lagrangian is given, which determine how the quarks and gluons change their colour in the interaction. The important observation for what follows is that, apart from the colour matrix $\frac{1}{2}\lambda_a$, the quark-gluon interaction takes precisely the same form as the quark photon interaction. This feature will now be exploited to determine the expression for the imaginary part of the diagrams of fig. 10.4 from the corresponding result (10.106) in quantum electrodynamics.

The flavour of a quark is not changed in the interaction with the gluons and (9.105) holds for all flavours with the same coupling constant g. Therefore let us first consider the diagrams of fig. 10.4 for a quark of given flavour. Obviously, all diagrams acquire a factor q^2 as compared to the corresponding graphs in quantum electrodynamics. Furthermore we must sum over all the colours of quarks and gluons. Therefore the one-loop graphs acquire a factor Tr(I), where I denotes the identity matrix in the quark colour space, so that Tr(I) is just the number of quark colours. For the two-loop graphs we must replace e^2 in the corresponding expression for quantum electrodynamics by g^2, and sum over both quark and gluon colours, giving rise to an overall factor of $\mathrm{Tr}(\frac{1}{2}\lambda_a \frac{1}{2}\lambda_a)$. We now sum over all possible quark flavours, so the quantum chromodynamics result corresponding to (10.106) is

$$\mathrm{Im}\,\Pi(k^2, g, 0, \mu) = \frac{1}{12\pi}\left(\sum q^2\right)\left\{1+\frac{3g^2}{16\pi^2}\frac{\mathrm{Tr}(\frac{1}{2}\lambda_a \frac{1}{2}\lambda_a)}{\mathrm{Tr(I)}}+O(g^4)\right\}, \quad k^2 < 0\,, \tag{10.108}$$

which is proportional to the total rate for the annihilation of a virtual photon into massless quarks, antiquarks and gluons, up to order g^4. The λ-matrices are normalized according to (cf. appendix G)

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}\,, \tag{10.109}$$

so that three quark colours and eight gluons give rise to the factor

$$\frac{\mathrm{Tr}(\frac{1}{2}\lambda_a \frac{1}{2}\lambda_a)}{\mathrm{Tr(I)}} = \frac{4}{3}\,. \tag{10.110}$$

The sum in $\sum q^2$ extends both over quark colours and flavours. Hence quark flavours with charge $-\frac{1}{3}$ (d, s, and b quarks) contribute $\frac{1}{3}$, and quark flavours with charge $\frac{2}{3}$ (u, c and t quarks) contribute $\frac{4}{3}$.

It is now easy to give the order-g^2 correction to the ratio R:

$$R \underset{s\to\infty}{\longrightarrow} \left(\sum q^2\right)\left(1 + \frac{g^2(t)}{4\pi^2} + O(g^4(t))\right), \tag{10.111}$$

which shows that the asymptotic result is approached from above. The asymptotic expression for $g(t)$ follows from (10.68), using the lowest-order term in the β function of quantum chromodynamics

$$\beta_g(g) = -\frac{1}{24\pi^2}\Big(\frac{33}{2} - n_f\Big)g^3 + O(g^5)\,. \qquad (10.112)$$

Here n_f is the number of quark flavours (note that for $n_f \geqslant 17$ asymptotic freedom is lost because (10.112) changes sign). Combining (10.68) and (10.112) shows that $g(t)$ approaches zero according to

$$\frac{g^2(t)}{4\pi^2} \sim \frac{12}{33 - 2n_f}\frac{1}{\ln s/s_0}\,, \qquad (10.113)$$

so that (9.109) becomes

$$R \underset{s\to\infty}{\longrightarrow} \big(\sum q^2\big)\Big(1 + \frac{12}{33 - 2n_f}\frac{1}{\ln s/s_0} + \cdots\Big)\,. \qquad (10.114)$$

This result can be compared to experiment, as we have discussed in section 6.7.

Problems

10.1. Consider a self-coupled scalar field theory with Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - g\phi^3 + c\phi \qquad (1)$$

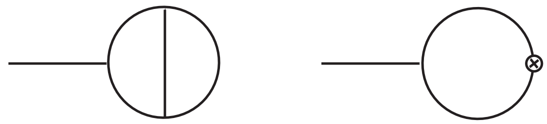
in n space-time dimensions. Show that the dimensions of ϕ, m^2, g and c are $[\frac{1}{2}n - 1]$, $[2]$, $[3 - \frac{1}{2}n]$ and $[1 + \frac{1}{2}n]$, respectively. Examine the one-loop tadpole, self-energy and vertex diagrams for the theory defined by (1). Show that only the tadpole and self-energy contributions have ultraviolet-divergent terms which are cancelled by the contributions from the counterterm Lagrangian

$$\Delta\mathcal{L} = \frac{3gm^2\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}\phi + \frac{9g^2\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}\phi^2\,. \qquad (2)$$

10.2. Consider the order-g^3 contribution to the tadpole diagram for the Lagrangian in problem 10.1. There are two diagrams shown below where the insertion in the second diagram denotes the vertex corresponding to the ϕ^2 part of the one-loop counterterm (2) found in the previous problem.

Using p and q to parametrize the internal momenta, show that the contribution from diagram (a) is

$$I^{(a)} = \frac{54g^3}{(2\pi)^{2n}}\int \mathrm{d}^n q\frac{1}{q^2 + m^2}\int \mathrm{d}^n p\frac{1}{(p^2 + m^2)^2}\frac{1}{(p + q)^2 + m^2}\,, \qquad (1)$$



(*a*) (*b*)

where we have extracted the usual factor $\mathrm{i}(2\pi)^n$ to enable a direct comparison with the lowest-order vertex. To evaluate the integrals apply (9.27) to combine the denominators in the p integral and evaluate it using (F.l) in appendix F. Then apply (9.27) again to combine the new denominators and evaluate the q-integral to find

$$I^{(a)} = -\frac{27g^3\mu^{2\varepsilon}}{128\pi^4}\left(\frac{m^2}{4\pi\mu^2}\right)^{\varepsilon}\Gamma(1-\varepsilon)\int_0^1 \mathrm{d}x(1-x)\int_0^1 \mathrm{d}y y^{-\frac{1}{2}\varepsilon}[1-y+x(1-x)y]^{-2-\frac{1}{2}\varepsilon}\,. \tag{2}$$

When $\varepsilon = n-4$ tends to zero the parametric integral diverges as ε^{-1}. This pole can be extracted by adding and subtracting the term

$$\int_0^1 \mathrm{d}x\,(1-x)x^{-1-\varepsilon/2} = \frac{\Gamma(-\frac{1}{2}\varepsilon)}{\Gamma(1-\frac{1}{2}\varepsilon)}\,.$$

Verify by explicitly evaluating the integrals that

$$\int_0^1 \mathrm{d}x\,(1-x)\int_0^1 \mathrm{d}y\,\{[1-y+x(1-x)y]^{-2} - x^{-1}\} = 1$$

and show that

$$I^{(a)} = -\frac{27g^3\mu^{2\varepsilon}}{64\pi^4}\left[\frac{1}{\varepsilon^2} + \left(\gamma_E - \tfrac{1}{2} + \ln\frac{m^2}{4\pi\mu^2}\right)\frac{1}{\varepsilon} + \text{finite}\quad\text{terms}\right]. \tag{3}$$

Now evaluate the contribution from the diagram (b), which yields

$$I^{(b)} = \frac{27g^3\mu^{2\varepsilon}}{64\pi^4}\left[\frac{2}{\varepsilon^2} + \left(\gamma_E + \ln\frac{m^2}{4\pi\mu^2}\right)\frac{1}{\varepsilon} + \text{finite}\quad\text{terms}\right]. \tag{4}$$

Combine (3) and (4) and verify that the counterterm Lagrangian whose contribution cancels the infinities is

$$\Delta\mathcal{L} = -\frac{27g^3\mu^{2\varepsilon}}{64\pi^4}\left(\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\right)\phi\,. \tag{5}$$

Note that the $\ln m^2/\mu^2$ terms have cancelled in accordance with the claim made in section 10.2.

10.3. Denoting the parameters in the Lagrangian (2) of problem 10.1 by g_0, m_0^2 and c_0 verify that the counterterms found in the previous problems are generated by making the replacements

$$g_0 = g\,, \quad m_0^2 = m^2 - \frac{9g^2\mu^\varepsilon}{4\pi^2}\frac{1}{\varepsilon}\,,$$
$$c_0 = c + \Big(\frac{3gm^2\mu^\varepsilon}{8\pi^2} - \frac{27g^3\mu^{2\varepsilon}}{128\pi^4}\Big)\frac{1}{\varepsilon} - \frac{27g^3\mu^{2\varepsilon}}{64\pi^4}\frac{1}{\varepsilon^2}\,.$$

Now define dimensionless quantities m, g and c by rescaling with suitable powers of μ and verify that the relation between the original and the renormalized parameters is given by (10.28).

10.4. Consider a renormalizable field theory with several parameters g. In order to absorb the ultraviolet divergences into the parameters g_0 of the Lagrangian, we write them in terms of renormalized parameters according to a generalization of (10.14),

$$g_{0\alpha} = \mu^{d_0 + d_0'\varepsilon}\Big\{g_\alpha + \sum_{\nu=1}^{\infty}\frac{a_\alpha^{(\nu)}(g)}{\varepsilon^\nu}\Big\}\,. \tag{1}$$

Following the same argument as in section 10.2 prove the generalization of (10.19)

$$\beta_\alpha(g,\varepsilon) + \sum_{\nu=1}^{\infty}\sum_\gamma \beta_\gamma(g,\varepsilon)\frac{\partial a_\alpha^{(\nu)}(g)}{\partial g_\gamma}\frac{1}{\varepsilon^\nu} = -(d_\alpha + d_\alpha'\varepsilon)\Big\{g_\alpha + \sum_{\nu=1}^{\infty}\frac{a_\alpha^{(\nu)}(g)}{\varepsilon^\nu}\Big\}\,, \tag{2}$$

where

$$\mu\frac{\partial}{\partial\mu}g_\alpha\Big|_{g_0,\varepsilon\ \text{fixed}} = \beta_\alpha(g,\varepsilon)\,. \tag{3}$$

By using the fact that $\beta_\alpha(g,\varepsilon)$ must be finite for $\varepsilon = 0$ derive the result (10.25), and demonstrate that the cancellation of the $\varepsilon^{-\nu}$ terms in (2) leads to a generalization of the first equation in (10.23), i.e.

$$D_\alpha a_\alpha^{(\nu+1)}(g) = -d_\alpha a_\alpha^{(\nu)}(g) + \sum_\gamma (d_\gamma g_\gamma - D_\gamma a_\gamma^{(1)}(g))\frac{\partial a_\alpha^{(\nu)}(g)}{\partial g_\gamma}\,, \tag{4}$$

where use is made of the definition

$$D_\alpha = d_\alpha' - \sum_\gamma d_\gamma' g_\gamma \frac{\partial}{\partial g_\gamma}\,. \tag{5}$$

In section 10.2 it was claimed that the coefficient functions $a_\alpha^{(\nu})$ should all have the correct dimensionality in four space-time dimensions, i.e. if we use dimensional parameters $\mu^{d_\alpha}g_\alpha$ the μ-dependence in $a_\alpha^{(\nu)}(g)$ should be contained in an overall factor μ^{-d_α}. Show that this implies (cf. 10.41)

$$d_\alpha a_\alpha^{(\nu)}(g) = \sum_\gamma d_\gamma g_\gamma \frac{\partial a_\alpha^{(\nu)}(g)}{\partial g_\gamma}\,. \tag{6}$$

so that (4) takes the form

$$D_\alpha a_\alpha^{(\nu+1)}(g) = \sum_\nu D_\gamma a_\gamma^{(1)}(g)) \frac{\partial a_\alpha^{(\nu)}(g)}{\partial g_\gamma}\,, \tag{7}$$

As an example consider the theory defined by the Lagrangian (10.27), which yields the relations (10.28) that we have calculated in the previous problems. Identify g_1, g_2 and g_3 with g, m^2 and c, respectively, and find the values for d_α and d'_α. Verify that (6) is satisfied for the coefficient functions, and show that

$$D_1 = \tfrac{1}{2}\Big(-1+g\frac{\partial}{\partial g}-c\frac{\partial}{\partial c}\Big), \quad D_2 = \tfrac{1}{2}\Big(g\frac{\partial}{\partial g}-c\frac{\partial}{\partial c}\Big), \quad D_1 = \tfrac{1}{2}\Big(1+g\frac{\partial}{\partial g}-c\frac{\partial}{\partial c}\Big)\,. \tag{8}$$

Find the coefficient functions $a_\alpha^{(\nu)}$ from (10.28) and prove that

$$D_1 a_1^{(1)} = 0\,, \qquad D_2 a_2^{(1)} = -\frac{9g^2}{4\pi^2}\,, \qquad D_3 a_1^{(1)} = \frac{3gm^2}{8\pi^2} - \frac{27g^3}{64\pi^4}\,. \tag{9}$$

Calculate the β functions and show that they are consistent with (10.49). Noting that none of the $a_\alpha^{(\nu)}$ depend on c, write (7) in the form

$$D_\alpha a_\alpha^{(\nu+1)} + \frac{9g^2}{4\pi^2}\frac{\partial}{\partial m^2} a_\alpha^{(\nu)} = 0 \tag{10}$$

and verify that this equation is satisfied for the coefficient functions (equations (4) were first derived by G. 't Hooft; see the references quoted at the end of this chapter).

10.5. Find the β and γ functions for scalar electrodynamics with a $|\phi|^4$ coupling by first taking the results from problem 9.9 and defining dimensionless quantities e, m^2 and g by rescaling with suitable powers of μ. Use (10.25) and (10.26) to check that, in the one-loop approximation,

$$\begin{aligned}
\beta_e &= \frac{e^3}{48\pi^2}\,,\\
\beta_g &= \frac{3}{2\pi^2}[g^2 - \tfrac{1}{2}ge^2 + \tfrac{1}{4}e^4]\,,\\
\beta_{m^2} &= -2m^2\Big[g^2 - \frac{1}{4\pi^2}(\tfrac{3}{4}e^2 - g)\Big]\,,\\
\gamma_A &= \frac{e^2}{24\pi^2}\,, \qquad \gamma_\phi = -\frac{e^2}{8\pi^2}(3-\lambda^{-2})\,,
\end{aligned} \tag{1}$$

and verify that (10.76) is satisfied.
Prove that the physical mass M and charge $e_{\rm P}$ calculated in problem 9.10 satisfy a generalization of (10.8), so that they vanish under the action of the differential operator

$$D = \mu\frac{\partial}{\partial\mu} + \beta_e\frac{\partial}{\partial e} + \beta_g\frac{\partial}{\partial g} + \beta_{m^2}\frac{\partial}{\partial m^2}\,. \tag{2}$$

Solve the equations for the running coupling constants $e(t)$ and $g(t)$. The solution for $e(t)$ follows from (10.68), i.e.

$$e^2(t) = \frac{e^2}{1 - e^2 t/24\pi^2}\,, \tag{3}$$

where $e \equiv e(0)$. To solve for $g(t)$, let

$$h(t) = \Big(1 - \frac{e^2 t}{24\pi^2}\Big) g(t)\,, \tag{4}$$

and introduce a new variable τ defined by

$$\tau = -\frac{24\pi^2}{e^2} \ln \Big(1 - \frac{e^2 t}{24\pi^2}\Big)\,. \tag{5}$$

Show that $h(\tau)$ satisfies the differential equation

$$\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \frac{1}{2\pi^2}(3h^2(\tau) - \frac{19}{12} e^2 h(\tau) + \tfrac{3}{4} e^4)\,. \tag{6}$$

Check that a solution of this equation satisfying $h(\tau) = h(\tau_0)$ at $\tau = \tau_0$ is

$$\tau - \tau_0 = \frac{12\pi^2}{\sqrt{14}e^2}\Big[\arctan\Big(\frac{18h(\tau) - 5e^2}{2\sqrt{14}e^3}\Big) - \arctan\Big(\frac{18h(\tau) - 5e^2}{2\sqrt{14}e^3}\Big)\Big]\,. \tag{7}$$

Defining $g = g(0)$, the solution for $g(t)$ therefore takes the form

$$g(t) = a \tag{8}$$

Using the initial condition $g = 0$, identify the structure of the trajectories for $e^2(t)$ and $g(t)$. Note the presence of a pole at $t = 24\pi^2/e^2$; beyond this value $e^2(t)$ turns negative. Discuss the qualitative difference between the trajectory defined by $e^2 = 0, g < 0$, and the other trajectories.

References

Original references to the renormalization group and the related Callan-Symanzik equation

E.G.C. Stueckelberg and A. Petermann, Helv. Phys. Acta 26 (1953) 499.

M. Gell-Mann and F.E. Low, Phys. Rev. 95 (1954) 1300.

N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, 3rd Ed. Wiley, New York, 1979).

C. Callan, Phys. Rev. D2 (1970) 1541.

K. Symanzik, Comm. Math. Phys. 18 (1970) 227; 23 (1971) 49.

For reviews on the renormalization-group equation

S. Coleman, in: Properties of the Fundamental Interactions, 1971 International School of Subnuclear Physics, Erice, ed. A. Zichichi (Editrice Compositore, Bologna,(1973) p. 358. (1973) p. 358.

H.D. Politzer, Phys. Rep. 14 (1974) 130.

A. Peterman, Phys. Rep. 53 (1979) 157. J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. 47 (1975) 165.

D.L. Gross, in: Methods in Field Theory, Les Houches 1975, eds R. Balian and J. Zinn-Justin, North-Holland, Amsterdam, 1976) p. 140.

R.J. Crewther, in: Weak and Electromagnetic Interactions at High Energies, Part A, Cargese, 1975, eds M. Levy, J.-L. Basdevant, D. Speiser and R. Gastmans (Plenum, New York, 1976) p. 345.

For an early discussion of asymptotic freedom

K. Symanzik, Nuovo cim. Lett. 6 (1973) 77.

G. Parisi, Nuovo cim. Lett. 7 (1973) 84.

The introduction of the renormalization-group equation in the context of dimensional regularization

G. 't Hooft, Nucl. Phys. B61 (1973) 455.

Related references are: M.J. Holwerda, W.L. van Neerven and R.P. Van Royen, Nucl. Phys. B75 (1974) 302.

J.C. Collins and A.J. Macfarlane, Phys. Rev. D10 (1974) 1201.

J.C. Collins, Phys. Rev. D10 (1974) 1213.

S.Y. Lee, Phys. Rev. D10 (1974) 1103.

L.-P. Yu, Nucl. Phys. B81 (1974) 458.

A different approach is discussed by: S. Weinberg, Phys. Rev. D8 (1973) 3497.

Discovery of asymptotic freedom

H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.

D.J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.

G. 't Hooft, Marseille meeting 1973 (unpublished), see comments in Nucl. Phys. B254 (1985) 11.

Only nonabelian gauge theories can be asymptotically free:

S. Coleman and D.J. Gross, Phys. Rev. Lett. 31 (1973) 851.

For a general discussion of the role of renormalization group fixed points in strong interactions

K.G. Wilson, Phys. Rev. D3 (1971) 1818.

For mass divergences

T. Kinoshita, J. Math. Phys. 3 (1962) 650. T.D. Lee and M. Nauenberg, Phys. Rev. 133B (1964) 1549.

The asymptotic result (9.93) for the photon propagator can be found in

N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, 3rd Ed. Wiley, New York, 1979).

Consult also

B. Lautrup and E. de Rafael, Nucl. Phys. B70 (1974) 317, where this result was used to determine the leading contributions to the anomalous magnetic moment of the muon.