5 | Particles with spin one-half

Particles with half-integer spin such as electrons, muons, neutrinos and nucleons obey Fermi-Dirac statistics, so they are called fermions. For a Majorana fermion the associated anti-fermion is the same particle (just as the π^0 is its own anti-particle). If this is not so one speaks of a Dirac fermion. Massive, charged, fermions must be Dirac particles, but electrically neutral fermions can be described as Majorana or Dirac particles. In this chapter we will mainly concentrate on Dirac fermions. The Feynman rules are given and polarization spinors are discussed in detail. The rates for several two-particle decays that involve fermions are calculated, and these results are used in a phenomenological discussion of a variety of electromagnetic and weak decay processes.

5.1. Feynman rules for spin-$\frac{1}{2}$ fields

In non-relativistic quantum mechanics spin-$\frac{1}{2}$ particles are described by two-component spinors. The spin is associated with the eigenvalues of 2×2 matrices that are usually defined as

$$S_i = \tfrac{1}{2}\hbar\sigma_i\,, \qquad i = 1, 2, 3, \tag{5.1}$$

where the σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.2}$$

satisfying

$$\sigma_i\,\sigma_j = \delta_{ij}\,\mathrm{I} + \mathrm{i}\varepsilon_{ijk}\,\sigma_k\,. \tag{5.3}$$

The spinor associated with a particle with spin up (down) along the i-th axis is the eigenvector of σ_i with eigenvalue $+1(-1)$.

As emphasized earlier the spin orientation of a particle changes under spatial rotations according to a representation of the rotation group. This representation is determined by the value of the total spin. For spin-$\frac{1}{2}$ the rotations act on two-component spinors $\phi = (\phi_1, \phi_2)$ and can be represented in terms of the σ-matrices. To be specific, a rotation by an angle θ around a unit vector $\hat{\mathbf{n}}$ takes the form

$$\phi \to \phi' = \exp(\tfrac{1}{2}\mathrm{i}\theta\,\hat{\mathbf{n}}\cdot\sigma)\phi\,. \tag{5.4}$$

In a relativistic treatment these transformations must be extended to include the action of full group of Lorentz transformations, which contain spatial rotations and Lorentz boosts. It is possible to do this within the context of a two-component spinor, but a standard approach is to first introduce a second spinor χ also transforming under spatial rotations according to (5.4). Then under Lorentz boosts the two spinors will transform into one another. At this point one has two options. Either one assumes that ϕ and χ are independent spinors; combining them leads to a four-component complex spinor, which will describe a Dirac fermion. Or using the observation that the spinor $(\mathrm{i}\sigma_2\phi^*)$ transforms under spatial rotations as ϕ itself, one may impose a reality condition $\chi = \pm\mathrm{i}\sigma_2\phi^*$ (see problem 5.1). In this way ϕ and χ can be combined into a four-component spinor that is real in the sense that its complex conjugate is linearly dependent on the original spinor. Such a spinor then corresponds to a Majorana particle. In both cases one may thus base the description of spin-$\frac{1}{2}$ fermions on a four-component spinor field,

$$\psi_\alpha(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}\,, \qquad (5.5)$$

where each component is a function of the space-time coordinates x^μ. Note that it is rather accidental that both ψ_α and x^μ have four components. In D space-time dimensions spinors have $2^{D/2}$ components if D is even, and $2^{(D-1)/2}$ components if D is odd (sometimes the spinor dimension can be further reduced; for a discussion of spinors in D dimensions, see appendix E).

At first sight it may seem that the four components of (5.5) will describe four different states corresponding to *two* particles with spin-$\frac{1}{2}$. However, we have already seen for spin-1 particles that such a naive counting argument is not always correct: although spin-1 particles are described in terms of four-component fields, we have established the presence of only 3 (2) independent plane wave solutions, as is appropriate for a massive (massless) spin-1 particle. A similar phenomenon will take place here, so that a Majorana (real) field will describe two states corresponding to a single spin-$\frac{1}{2}$ particle, and a Dirac (complex) field will describe four states corresponding to a spin-$\frac{1}{2}$ particle and a spin-$\frac{1}{2}$ anti-particle.

An important ingredient in dealing with the four-component spinor field (5.5) are the γ-matrices, first introduced by Dirac. There are four such 4×4 matrices γ^μ, where $(\mu = 0,1,2,3)$, which satisfy the key anti-commutation relation,

$$\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu}\,\mathrm{I} \qquad (5.6)$$

where $\mathbf{I}$ is the 4×4 identity matrix. These anti-commutation relations define a so-called Clifford algebra. There are several possible representations for the individual γ-matrices, all of which differ by a similarity transformation. Because the square of each gamma matrix is equal to the unit matrix (for $\mu = 1,2,3$), or equal to minus the unit matrix (for $\mu = 0$), we can choose γ^1, γ^2, γ^3 hermitean, and γ^0 anti-hermitean. The representation that is convenient for subsequent considerations is defined as follows (a general discussion on γ-matrices is given in appendix E).

$$\gamma^k = \begin{pmatrix} 0 & -\mathrm{i}\sigma_k \\ \mathrm{i}\sigma_k & 0 \end{pmatrix}, \qquad k = 1,2,3; \qquad \gamma^0 = \begin{pmatrix} -\mathrm{iI} & 0 \\ 0 & \mathrm{iI} \end{pmatrix}. \tag{5.7}$$

As the reader may easily verify, all gamma matrices are all traceless, a property that can be defined directly from the Clifford algebra relation (5.6) without resorting to an explicit representation. It is useful to introduce certain products of gamma matrices, such as

$$\gamma^5 = \gamma_5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & -\mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \tag{5.8}$$

and

$$\sigma^{\mu\nu} = -\tfrac{1}{2}\mathrm{i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\,, \tag{5.9}$$

where we caution the reader that a variety of definitions for $\sigma^{\mu\nu}$ exists in the literature. Furthermore we note the relations

$$\gamma_5 = \tfrac{1}{24}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,, \qquad {\gamma_5}^\dagger = \gamma_5\,, \tag{5.10}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the fully anti-symmetric Levi-Civita symbol, normalized by $\varepsilon_{0123} = -1$. Observe that, with upper indices, we have $\varepsilon^{0123} = 1$, because the relation between upper and lower indices is effected by contraction with the Minkowski metric $\eta_{\mu\nu}$, so that $\varepsilon^{\mu\nu\rho\sigma} = \det[\eta]\,\varepsilon_{\mu\nu\rho\sigma}$. Furthermore, note that $\{\gamma^\mu,\gamma_5\} = [\sigma^{\mu\nu},\gamma_5] = 0$. All 4×4 matrices can generally be decomposed into the sixteen independent matrices $\mathbf{I}$, γ_5, γ^μ, $\gamma^\mu\gamma_5$ and $\sigma^{\mu\nu}$. We refer to appendix E for further details.

Let us now examine the matrices $\sigma^{\mu\nu}$ more closely. In the representation (5.7) they take the form,

$$\sigma^{ij} = \varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \qquad \sigma^{i0} = -\mathrm{i}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad (i,j,k = 1,2,3)\,. \tag{5.11}$$

The matrices $\sigma^{\mu\nu}$ close under commutation according to

$$\big[\sigma^{\mu\nu},\sigma^{\rho\sigma}\big] = -\mathrm{i}\big(\eta^{\mu\rho}\sigma^{\nu\sigma} - \eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\mu\sigma}\sigma^{\nu\rho} + \eta^{\nu\sigma}\sigma^{\mu\rho}\big)\,, \tag{5.12}$$

The relevance of this relation will be discussed in due course.

Comparing this result to (5.4) we recognize that the matrices σ^{ij} describe the rotations of the spinors around a vector perpendicular to the i-th and j-th directions. This suggests the following definition for the action of a Lorentz transformation on the four-component spinors (see problem 5.2)

$$\psi \to \psi' = \exp\left(\tfrac{1}{4}\mathrm{i}\theta_{\mu\nu}\,\sigma^{\mu\nu}\right)\psi\,, \tag{5.13}$$

where the anti-symmetric, real, parameters $\theta_{\mu\nu}$ characterize the spatial rotations for $\mu,\nu = 1,2,3$, and the Lorentz boosts for μ or $\nu = 0$.

The complex conjugate of ψ, written as a row vector, transforms as

$$\psi^\dagger \to \psi'^\dagger = \psi^\dagger \exp\left(-\tfrac{1}{4}\mathrm{i}\theta_{\mu\nu}\,\sigma^{\mu\nu\,\dagger}\right). \tag{5.14}$$

The matrix on the right-hand side of (5.14) is not equal to the inverse of the matrix occurring in (5.13); therefore $\psi^\dagger$ it is not convenient when constructing Lorentz-invariant expressions from contractions of $\psi^\dagger$ and ψ. For that reason on introduces a slightly modified field, called the conjugate field $\bar\psi$, which is also defined as a row vector,

$$\bar\psi_\alpha = \mathrm{i}\psi^\dagger{}_\beta\,(\gamma^0)_{\beta\alpha}\,, \tag{5.15}$$

or, with the representation (5.7)

$$\bar\psi_\alpha = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)\,. \tag{5.16}$$

Because $(\sigma^{ij})^\dagger = \sigma^{ij}$ and $(\sigma^{i0})^\dagger = -\sigma^{i0}$, one easily verifies that

$$\gamma_0\,\sigma^{ij\,\dagger}\gamma^0 = \sigma^{ij}\,, \qquad \gamma_0\,\sigma^{i0\,\dagger}\gamma^0 = \sigma^{i0}\,. \tag{5.17}$$

As a result one can show that $\bar\psi$ transforms under Lorentz transformations according to

$$\begin{aligned}\bar\psi \to \bar\psi' &= \mathrm{i}\psi^\dagger\;\exp\left(-\tfrac{1}{4}\mathrm{i}\theta_{\mu\nu}\,\sigma^{\mu\nu\,\dagger}\right)\gamma^0\\ &= \bar\psi\,\gamma_0\exp\left(-\tfrac{1}{4}\mathrm{i}\theta_{\mu\nu}\sigma^{\mu\nu\,\dagger}\right)\gamma^0\\ &= \bar\psi\left[\exp\left(\tfrac{1}{4}\mathrm{i}\theta_{\mu\nu}\sigma^{\mu\nu}\right)\right]^{-1}.\end{aligned} \tag{5.18}$$

Let us now consider two fermionic bilinears,

$$\begin{aligned}\bar\psi_\alpha\psi_\alpha &= |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2\,,\\ \bar\psi_\alpha(\gamma_5)_{\alpha\beta}\psi_\beta &= -\psi_1^*\psi_3 - \psi_2^*\psi_4 + \psi_3^*\psi_1 + \psi_4^*\psi_2\,,\end{aligned} \tag{5.19}$$

which transform as Lorentz scalars, as follows directly by making use of (5.13) and (5.18). Since it is complicated to keep writing indices on these quantities we will simply write $\bar\psi\psi$, $\psi\gamma_5\psi$, etc. Altogether one can form five bilinears of this type transforming in a specific way under parity reversal and Lorentz transformations, namely $\bar\psi\psi$ (scalar), $\mathrm{i}\bar\psi\gamma_5\psi$ (pseudoscalar), $\mathrm{i}\bar\psi\gamma^\mu\psi$ (vector), $\mathrm{i}\bar\psi\gamma^\mu\gamma_5\psi$ (axial vector) and $\mathrm{i}\bar\psi\sigma^{\mu\nu}\psi$ (tensor) (see problem 5.2). For instance, under parity reversal $\bar\psi\psi$ and $\mathrm{i}\bar\psi\gamma_5\psi$ transform into $+\bar\psi\psi$ and $-\mathrm{i}\bar\psi\gamma_5\psi$, respectively. Similarly $\mathrm{i}\bar\psi\gamma^\mu\psi$ and $\mathrm{i}\bar\psi\gamma^\mu\gamma_5\psi$ transform with opposite signs. The factors of i have been introduced to make these bilinears real. For instance, $(\bar\psi\gamma_5\psi)^\dagger = -\mathrm{i}\psi^\dagger\gamma^{5\dagger}\gamma^{0\dagger}\psi = -\bar\psi\gamma_5\psi$.

To see how the bilinears transform under Lorentz transformations it suffices to consider the effect of an infinitesimal transformation. e.g.

$$\begin{aligned}\delta(\mathrm{i}\bar\psi\gamma^\mu\psi) &= \mathrm{i}(\delta\bar\psi)\gamma^\mu\psi + \mathrm{i}(\bar\psi)\gamma^\mu\delta\psi\\ &= -\tfrac{1}{4}\theta_{\rho\sigma}\,\bar\psi\big(-\sigma^{\rho\sigma}\gamma^\mu + \gamma^\mu\sigma^{\rho\sigma}\psi\big)\\ &= \theta^\mu{}_\nu\,(\mathrm{i}\bar\psi\gamma^\nu\psi)\,,\end{aligned} \tag{5.20}$$

where we have used $\sigma^{\rho\sigma}\gamma^\mu - \gamma^\mu\sigma^{\rho\sigma} = 2\mathrm{i}\eta^{\mu\rho}\gamma^\sigma - 2\mathrm{i}\eta^{\mu\sigma}\gamma^\rho$ (cf. appendix E).

Now we can write down a free Lagrangian for spin-$\frac{1}{2}$ fields of mass m,

$$\mathcal{L} = -\bar\psi\partial\!\!\!/\psi - m\,\bar\psi\psi\,, \tag{5.21}$$

where the symbol $\partial\!\!\!/$ stands for a 4×4 matrix defined by

$$(\partial\!\!\!/)_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}\,. \tag{5.22}$$

The Lagrangian (5.21) is thus linear in the derivative of the spinor field and, as it involves a four-vector contracted with a derivative, it is Lorentz invariant. Because the action is dimensionless, the field ψ has the dimensions of $[\text{length}]^{-3/2} = [\text{mass}]^{3/2}$ in units where $\hbar = c = 1$. The normalization of the fields in (5.21) is the appropriate one for complex spinors (remember the discussion on this point in chapter 2). In other words $\psi(x)$ and $\bar\psi(x)$ correspond to independent degrees of freedom (Dirac fermions). The standard normalization for Majorana (real) fermions has an overall factor $\frac{1}{2}$.

The equation of motion that follows from (5.21) is the Dirac equation

$$\big(\partial\!\!\!/ + m\big)\,\psi(x) = 0, \tag{5.23}$$

together with the equation for the adjoint spinor

$$\bar\psi(x)\big(-\overleftarrow{\partial\!\!\!/} + m\big) = 0\,. \tag{5.24}$$

Plane wave solutions of (5.23) are easy to construct and can be written as

$$\psi_\alpha(x) = w_\alpha(p)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,, \tag{5.25}$$

where the polarization spinor $w_\alpha(p)$ must satisfy the eigenvalue equation

$$(p\!\!\!/ - \mathrm{i}m)w(p) = 0. \tag{5.26}$$

Multiplying the last equation by $\not{p} + \mathrm{i}m$, and using the fact that $\not{p}^2 = p^2$ (as follows from contracting (5.6) by $p_\mu p_\nu$) we find that p_μ is restricted by $p^2 = -m^2$. Consequently the plane wave solutions describe degrees of freedom with mass m and spin-$\frac{1}{2}$. The solutions of the adjoint equation (5.24) follow from (5.25) by complex conjugation and multiplication by γ^0. However, one can show that these solutions are linearly dependent on those of (5.25). A detailed treatment of linearly independent polarization spinors will be given in section 5.3.

The propagator associated with the fields ψ and $\bar{\psi}$ follows from the Fourier transform of the action, which equals

$$S[\psi, \bar{\psi}] = -(2\pi)^4 \int \mathrm{d}^4 p\, \bar{\psi}_\alpha(p)(\mathrm{i}\not{p} + m)_{\alpha\beta}\psi(p)\,. \tag{5.27}$$

According to the standard prescription given in section 2.4 the inverse of the 4×4 matrix in the integrand of (5.25) defines the propagator $\mathit{\Delta}_{\alpha\beta}(p)$:

$$\mathit{\Delta}_{\alpha\beta}(p) = \frac{1}{\mathrm{i}(2\pi)^4}\Big(\frac{1}{\mathrm{i}\not{p} + m}\Big)_{\alpha\beta}\,, \tag{5.28}$$

Note that m multiplies the identity matrix, which is usually suppressed. Again using $\not{p}^2 = p^2$ it is easy to show that

Figure 5.1: A graphical representation for the propagator $\mathit{\Delta}_{\alpha\beta}(p)$. The arrow represents the orientation from the endpoints given. The momentum assignment $\mathit{\Delta}_{\alpha\beta}(p)$ is defined in the same direction.

$$\mathit{\Delta}_{\alpha\beta}(p) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{p^2 + m^2 - \mathrm{i}\varepsilon}(-\mathrm{i}\not{p} + m)_{\alpha\beta}\,, \tag{5.29}$$

where we have included the iε-term to give a proper definition of the poles. The Feynman diagram associated with the propagator (5.29) is shown in fig. 5.1. The arrow indicates that the momentum flows from the endpoint associated with $\bar{\psi}_\beta$ to that associated with ψ_α. An outgoing arrow is thus associated with the field $\bar{\psi}$ and an incoming arrow with the field ψ. For Dirac fields ψ and $\bar{\psi}$ are linearly independent. In that case the arrow also indicates an intrinsic orientation as is characteristic for complex fields. As we have already explained in section 2.4 the propagator lines can then only be connected to the vertices such that the orientation arrow flows continuously through the diagram.

As before one may introduce external sources via two new terms in the Lagrangian (5.21)

$$\mathcal{L} = -\bar\psi(x)\,\partial\!\!\!/\psi(x) - m\,\bar\psi(x)\,\psi(x) + \bar J(x)\,\psi(x) + \bar\psi(x)\,J(x)\,, \tag{5.30}$$

and discuss their effective interaction. Since there are only interactions with the external sources, this is described by the propagator (or Green's function) $\varDelta_{\alpha\beta}(x)$, which is the Fourier transform of (5.29):

$$\varDelta_{\alpha\beta}(x) = \frac{1}{\mathrm{i}(2\pi)^4}\int \mathrm{d}^4p\,\frac{(-\mathrm{i}p\!\!\!/ + m)_{\alpha\beta}}{p^2+m^2-\mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}p\cdot x}\,. \tag{5.31}$$

This Green's function satisfies the differential equation

$$(-\partial\!\!\!/ - m)_{\alpha\beta}(x)\varDelta_{\beta\gamma}(x) = \mathrm{i}\delta_{\alpha\gamma}\delta^4(x)\,, \tag{5.32}$$

where we have explicitly indicated the spinor indices. The effective interaction between two sources then takes the form,

$$S^{\mathrm{eff}}[\bar J_1, J_2] = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar J_1(x)_\alpha\,\mathrm{i}\varDelta_{\alpha\beta}(x-y)\,J_2(y)_\beta\,. \tag{5.33}$$

We may examine this expression for sources J_1 and J_2 that are localized and separated by a large time-like distance. Precisely as for spinless fields the large-time limit is dominated by the contributions to the propagator with momenta that satisfy the relativistic dispersion law $p^0 = \pm\sqrt{\mathbf{p}^2+m^2}$. When $t_1 \gg t_2$, then (5.33) describes the emission of a spin-$\frac{1}{2}$ particle by J_2 and its subsequent absorption by J_1. Alternatively, when $t_2 \gg t_1$, then(5.33) describes the emission of an anti-particle by J_1 and its absorption by J_2.

Evaluating $\varDelta_{\alpha\beta}(x)$ for positive and negative time t, just as was done for spinless fields in (2.25)-(2.38), leads to the same physical interpretation as in chapter 2,

$$\varDelta_{\alpha\beta}(x) = \theta(t)\,\varDelta^+_{\alpha\beta}(x) + \theta(-t)\,\varDelta^-_{\alpha\beta}(x)\,, \tag{5.34}$$

where

$$\begin{aligned}
\varDelta^+_{\alpha\beta}(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\omega(\mathbf{p})}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}-\mathrm{i}\omega(\mathbf{p})t}(-\mathrm{i}p_i\gamma^i + \mathrm{i}\omega(\mathbf{p})\,\gamma^0 + m)_{\alpha\beta}\,,\\
\varDelta^-_{\alpha\beta}(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\omega(\mathrm{p})}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}+\mathrm{i}\omega(\mathbf{p})t}(-\mathrm{i}p_i\gamma^i - \mathrm{i}\omega(\mathbf{p})\gamma^0 + m)_{\alpha\beta}\,.
\end{aligned} \tag{5.35}$$

Compared to the propagator for spinless particles there is now an extra residue matrix $(-\mathrm{i}p\!\!\!/ + m)$ in (5.31). At the pole one has $p\!\!\!/^2 = p^2 = -m^2$, so that this matrix acts as a projection operator; this follows from

$$\Big(\frac{-\mathrm{i}p\!\!\!/ + m}{2m}\Big)^2 = \frac{-\mathrm{i}p\!\!\!/ + m}{2m}\,.$$

Because

$$\mathrm{Tr}\Big(\frac{-\mathrm{i}\not{p}+m}{2m}\Big) = 2\,,$$

we conclude that $(-\mathrm{i}\not{p} + m)/2m$ projects out a two-dimensional subspace (the remaining subspace is projected out by $(\mathrm{i}\not{p} + m)/2m$). Consequently the matrix $(-\mathrm{i}\not{p} + m)$ reduces the number of degrees of freedom from four to two, which is the appropriate number for a spin-$\frac{1}{2}$ particle. Note that this reduction is analogous to that caused by the transversal projection operator in the propagator for massive spin-1 fields, which reduces the number of physical degrees of freedom to three.

In principle one can now write down Feynman diagrams by following the general prescription of chapter 2. However, there is an additional complication which is related to the exclusion principle. Fermions obey Fermi-Dirac statistics so that states consisting of fermions are anti-symmetric under the interchange of two identical particles. This has consequences for Feynman diagrams. Diagrams which differ only by the interchange of two fermion lines should have a relative minus sign. In the diagrams of fig. 5.2a, for example, one should take the difference rather than the sum. Of course, the relative sign factors are not explicit in a Feynman diagram representation. Moreover the overall sign factor is ambiguous, but as we are only interested in the absolute value of the total amplitude this aspect is not relevant. A more subtle consideration concerns the closed loops of fermion propagators (such a diagram has already been given in fig. 4.3). It turns out that each such loop acquires an extra minus sign. From the point of view of Feynman diagrams the need for this minus sign can be seen by simply connecting two external fermion lines in an amplitude with several external fermion lines. Some of the resulting diagrams may then have a closed fermion loop whose sign can be compared to diagrams without a loop. For example, connecting the lines labelled by p_1, and p_4 in the diagrams of fig. 5.2a, leads to the two diagrams shown in fig. 5.2b, so that indeed the closed loop carries the extra minus sign.

5.2. Lagrangians for fermions

In order to familiarize ourselves with field theories for fermions we discuss some typical examples.

(a) *Fermion-fermion interactions*

From one point of view the simplest interacting theory would involve no extra fields, so the interaction would be constructed out of a product of $\bar{\psi}$'s and ψ's. Since the ψ's are spinors we cannot form a Lorentz invariant quantity out of the product of an odd number of ψ or $\bar{\psi}$ fields. The simplest interaction con-

Figure 5.2: The interchange of fermion lines in Feynman diagrams. To satisfy Fermi-Dirac statistics, the amplitudes for the the two diagrams in (a) have a relative minus sign. In the diagrams (b), which are formed from (a) by connecting the lines with momentum p_1, and p_4, the amplitude for the closed fermion loop acquires a relative minus sign.

tains at least four fields. One example which does not involve any γ-matrices is

$$G_{\rm S}(\bar\psi(x)\psi(x))(\bar\psi(x)\psi(x))\,. \tag{5.36}$$

Note that the dimension of this interaction term is $[\text{mass}]^6$. In order for the Lagrangian to have the correct dimension the coupling constant $G_{\rm S}$ must have the dimension of $[\text{mass}]^{-2}$. The dimension does not change if we include γ-matrices, as for example in

$$G_{\rm V}(\bar\psi(x)\gamma_\mu\psi(x))(\bar\psi(x)\gamma_\mu\psi(x))\,. \tag{5.37}$$

Feynman rules for these theories are rather simple. For example, the fermion-fermion amplitude follows from the Feynman graphs of the type shown in fig. 5.3 (not all amplitudes of type (5.33) are independent; this can be shown by using so-called Fierz reorderings, which are discussed in appendix E).

To get acquainted with fermion vertices let us calculate the four-point vertex corresponding to (5.33a). In order to indicate how the lines are contracted with γ matrices we denote the vertices as in fig. 5.4. The two diagrams of the lowest-order vertex are shown there, and the corresponding expression takes the form

$$\Gamma_{\beta_2\beta_1\alpha_2\alpha_1} = 2{\rm i}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)G_{\rm V}\,[(\gamma_\mu)_{\beta_2\alpha_2}(\gamma_\mu)_{\beta_1\alpha_1}-(\gamma_\mu)_{\beta_2\alpha_1}(\gamma_\mu)_{\beta_1\alpha_2}]\,.$$

Figure 5.3: Diagrams contributing to the four-fermion amplitude caused by the interactions (5.33).

(5.38)

The factor of 2 arises from the fact that there are two ways to connect the external lines to the vertex. Note that the relative minus sign between the terms is a consequence of Fermi-Dirac statistics. We should emphasize that the negative dimension of the coupling constants implies that higher-order graphs will have additional powers of momenta in the numerator. Therefore quantum corrections in this type of theory tend to be singular and it is not clear how to extract meaningful results from them. We say that these theories are not “renormalizable” based on power counting. As we shall see in chapter 7, theories with dimensionless coupling constants have a better chance to be renormalizable, in which case meaningful results can be obtained. In view of this difficulty we therefore modify our approach and consider two theories where the fermion fields interact with fields of another type.

Figure 5.4: Tree diagrams corresponding to (5.38) showing the different connections of the external lines to the vertex.

(b) *Quantum electrodynamics*
[.1ex] Quantum electrodynamics describes the interactions between electrons

and photons. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x) - \bar{\psi}(x)(\partial\!\!\!/ + m)\psi(x) - \mathrm{i}eA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x)\,, \quad (5.39)$$

where $\psi(x)$ is the spinor field associated with the electron and $A_\mu(x)$ is the vector potential of electromagnetism. The coupling constant is the elementary charge $e = 4.8\times10^{-10}$ esu (so that the electron has charge -e). The Lagrangian is invariant under electromagnetic gauge transformations

$$\begin{aligned} &A_\mu(x) \to A_\mu(x) + \partial_\mu\xi(x)\,, \\ &\psi(x) \to \mathrm{e}^{-\mathrm{i}e\xi(x)}\psi(x)\,, \qquad \bar{\psi}(x) \to e^{\mathrm{i}e\xi(x)}\bar{\psi}(x)\,, \end{aligned} \quad (5.40)$$

where ξ is an arbitrary x-dependent function. The Feynman rules follow from the previous arguments and are summarized in table 5.1. Since the fine-structure constant $\alpha = e^2/4\pi \approx (137)^{-1}$ is very small it makes sense to retain only the first few terms in a perturbation expansion. In subsequent chapters we will be considering several of these perturbative calculations. Here we only want to point out that the comparison between the result of such calculations and the present experimental data is very impressive, as is shown in table 5.2.

Table 5.1 Feynman rules for quantum electrodynamics corresponding to the Lagrangian (5.35)a. Diagram Expression

$$\Delta_{\alpha\beta}(p) \quad = \tfrac{1}{i(2\pi)^4}\tfrac{(-ip+m)_{\alpha\beta}}{p^2+m^2}$$

$$i(2\pi)^4\delta^4(p_1 - p_2 + q)(-ie)(\gamma)_{\alpha\beta}$$

a The arrows on the fermion lines indicate both the orientation (charge flow) and their assigned momenta. The directions of the photon momenta are not indicated; the direction at the vertex follows from the argument in the δ-function. The photon propagator has been derived in section 4.2. (c) *Yukawa couplings*

[.1ex] Another example where bosons mediate the interactions between fermions is a Yukawa coupling. As we have already mentioned in chapter 2, Yukawa was the first to use a spinless boson (which was later identified as the pion) to describe the force between nucleons in analogy with the photon that mediates the electromagnetic interactions. Consider the simplest couplings between a fermion field $\psi(x)$, its adjoint field $\bar{\psi}(x)$ and a spinless field $\phi(x)$, $\bar{\psi}(x)\psi(x)\phi(x)$, or $\bar{\psi}(x)\gamma_\mu\psi(x)\partial_\mu\phi(x)$. The first example is a scalar coupling with dimension [mass]^4 while the second example is a vector coupling with dimension [mass]^5. In the latter case the corresponding coupling constant has negative mass dimension, so we anticipate that there will be problems with renormalizability. Therefore, we concentrate on $\bar{\psi}(x)\psi(x)\phi(x)$.

A full Lagrangian for such a theory is

$$\mathcal{L} = -\bar{\psi}(x)(\partial\!\!\!/ + M)\psi(x) - \tfrac{1}{2}(\partial_\mu\phi(x))^2 - \tfrac{1}{2}m^2\phi^2(x) + g\bar{\psi}(x)\psi(x)\phi(x)\,. \quad (5.41)$$

Quantity	Units	Experimental result	Theoretical prediction
Lamb shift in hydrogen (2Sll2 2P,,2)	MHz	1()57.845(9)	1()57.857(12)
Fine structure in hydrogen (2P3 2 2S, 2)	MHz	9911.17(4)	9911.13(4)
Hyperfine splitting of hydrogen ground state	MHz	142().4()57517667(9)	1420.403444(1278)
Hyperfine splitting in positronium	MHz	2()3389.1()(74)	203402.51
Hyperfine splitting in muonium	MHz	4463.30288(16)	4463.3047(27)
Electron anomalous magnetic moment	$\frac{e}{2m_e}$	1165.911()(1101 x 1()6	1165.9203(20) x 10
Muon anomalous magnetic moment	$\frac{e}{2m_\mu}$	1159.65220()(4()) x lo6	1159.652459(43) x 1()

Table 5.1: Comparison between experiment and theory for quantum electrodynamics. For references see the bibliography at the end of the chapter.

Assuming that the coupling constant g is small it makes sense to set up a perturbation expansion. The lowest-order Feynman diagrams for fermion-fermion scattering have already been shown in fig. 5.2a. By analogy with our previous work we can immediately write down the corresponding amplitude from the graphs. Assigning momenta p_1, p_2 and spinor indices α_1, α_2 to the external lines with incoming arrows, and momenta p_3, p_4, and spinor indices α_3, α_4 to the external lines with outgoing arrows, the amplitude is

$$\varGamma_{\alpha_3\alpha_4;\alpha_1\alpha_2} = \mathrm{i}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \tag{5.42}$$

$$\times\left\{g\delta_{\alpha_2\alpha_4}\frac{1}{(p_3-p_1)^2+m^2}g\delta_{\alpha_1\alpha_3} - g\delta_{\alpha_2\alpha_3}\frac{1}{(p_3-p_1)^2+m^2}g\delta_{\alpha_1\alpha_4}\right\}. \tag{5.43}$$

This amplitude describes fermion-fermion, anti-fermion-fermion, fermion-anti-fermion or anti-fermion-anti-fermion scattering depending on how one treats the external lines; this subject will be discussed later. Note again the relative minus sign between the two terms, which is dictated by Fermi-Dirac statistics.

Except that they have indices to express spinor components, fermion lines are no more complicated than spinless boson lines. In Feynman diagrams fermion lines either flow continuously through the whole diagram, or close on themselves to form a loop. In the latter case the spinor indices are joined together and we need to sum over all of them. Furthermore one has an extra minus sign for each closed fermion loop.

To describe the coupling of pseudoscalar mesons one simply changes $\bar{\psi}\psi\phi$ in

(5.41) to $\mathrm{i}\bar{\psi}\gamma_5\psi\phi$. One may further generalize this model to describe nucleons and pions interacting in an isospin invariant way. The bosonic sector of this model has already been given in section 2.5 based on three real pion fields ϕ^a interacting with a scalar meson σ. While pions transform under isospin transformations in the triplet representation (i.e. just as a vector under rotations), the protons and neutrons transform as a doublet (i.e. as a two- component spinor). Hence we combine the proton and neutron fields ψ_p and ψ_n into a doublet

$$N(x) = \begin{pmatrix} \psi_\mathrm{p} \\ \psi_\mathrm{n} \end{pmatrix} . \tag{5.44}$$

Altogether $N(x)$ thus contains eight components, namely four ordinary spin components for each of the fields ψ_p and ψ_n. To construct an isospin invariant pion-nucleon coupling one makes use of the isospin matrices τ_a, which coincide with the Pauli σ-matrices (5.2). The following interaction is invariant (this follows from the result of problem 5.1, where we prove that for doublets ϕ, $\phi^*\sigma_a\phi$ transform as a vector)

$$\mathcal{L}_\mathrm{int} = \mathrm{i}G\bar{N}(x)\gamma_5\tau_a N(x)\phi^a(x) . \tag{5.45}$$

Alternatively (5.40) may be written in terms of proton, neutron and pion fields

$$\mathcal{L}_\mathrm{int} = \mathrm{i}G\sqrt{2}(\bar{\psi}_\mathrm{n}\gamma_5\psi_\mathrm{p}\pi^- + \bar{\psi}_\mathrm{p}\gamma_5\psi_\mathrm{n}\pi^+) + \mathrm{i}G(\bar{\psi}_\mathrm{p}\gamma_5\psi_\mathrm{p} - \bar{\psi}_\mathrm{n}\gamma_5\psi_\mathrm{n})\pi^0 ,$$

where we have used the definition of the π^+, π^- and π^0 fields given in (2.53), and the explicit form of the τ-matrices.

A Lagrangian that contains the coupling (5.45) is

$$\mathcal{L} = -\bar{N}(x)(\partial\!\!\!/ + M)N(x) - \tfrac{1}{2}(\partial\phi(x))^2 - \tfrac{1}{2}m^2\phi(x)^2 \tag{5.46}$$

$$+\mathrm{i}G(\bar{N}(x)\gamma_5\tau N(x)) \cdot \phi(x) . \tag{5.47}$$

In the calculation of Feynman diagrams it is more convenient to use the multicomponent fields N and ϕ, than to write (5.46) in terms of components. The fields then carry a variety of indices so that both the propagators and the vertices involve matrices in spin space and in isospin space. The propagator for the field $N(x)$ is an 8×8 matrix, diagonal in isospin space (since the fields $\psi_\mathrm{p}(x)$ and $\psi_\mathrm{n}(x)$ have the same kinetic energy terms).

$$\Delta^{rs}_{N\alpha\beta}(p) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{p^2+M^2}(-\mathrm{i}p\!\!\!/ + M)_{\alpha\beta}\delta^{rs}, \tag{5.48}$$

where $r, s = 1, 2$ label the isospin components of N and $\alpha, \beta = 1, 2, 3, 4$ the spinor components. Similarly the propagators for the spinless fields are

$$\Delta^{ab}_{\phi}(p) = \frac{1}{\mathrm{i}(2\pi)^4}\frac{\delta^{ab}}{p^2+m^2}, \tag{5.49}$$

where $a, b = 1, 2, 3$ are the isospin labels of the pions. The vertex is a complicated product of matrices, namely

$$\mathrm{i}(2\pi)^4\delta^4(p_1 + p_2 + p_3)\mathrm{i}G(\gamma_5)_{\alpha\beta}(\tau^a)^{rs}\,, \tag{5.50}$$

where the label a refers to the pion component and selects a corresponding τ matrix with indices r and s. Grouping together the isospin and spin labels on the spin-$\frac{1}{2}$ fields we can compare the vertex to the corresponding diagram in fig. 5.5, where we assume that all momenta are incoming.

Figure 5.5: The pion-nucleon vertex corresponding to (5.44).

Figure 5.6: Tree diagrams for pion nucleon scattering.

As an example consider pion-nucleon scattering for which the lowest order Feynman graphs are given in fig. 5.6. The corresponding amplitude is a product of spin and isospin matrices

$$
\begin{aligned}
M^{sr}_{ba,\beta\alpha} \;=\; & \mathrm{i}G(\gamma_5)_{\beta\gamma}(\tau_b)^{st}\Big(\frac{\delta^{tu}}{\mathrm{i}(\not{P}_1+\not{Q}_1)+M}\Big)_{\gamma\delta}\mathrm{i}G(\gamma_5)_{\delta\alpha}(\tau_a)^{ur} \\
& +\mathrm{i}G(\gamma_5)_{\beta\gamma}(\tau_a)^{st}\Big(\frac{\delta^{tu}}{\mathrm{i}(\not{P}_2-\not{Q}_1)+M}\Big)_{\gamma\delta}\mathrm{i}G(\gamma_5)_{\delta\alpha}(\tau_b)^{ur} \\
& = G^2\Big\{(\tau_b\tau_a)^{sr}\Big(\frac{1}{\mathrm{i}(\not{P}_1+\not{Q}_1)-M}\Big)_{\beta\alpha}+(\tau_a\tau_b)^{sr}\Big(\frac{1}{\mathrm{i}(\not{P}_2-\not{Q}_1)-M}\Big)_{\beta\alpha}\Big\} \\
& = G^2\delta_{ba}\delta^{sr}\Big(\frac{1}{\mathrm{i}(\not{P}_1+\not{Q}_1)-M}+\frac{1}{\mathrm{i}(\not{P}_2-\not{Q}_1)-M}\Big)_{\beta\alpha} \\
& -\mathrm{i}G^2\varepsilon_{abc}(\tau)^{sr}\Big(\frac{1}{\mathrm{i}(\not{P}_1+\not{Q}_1)-M}+\frac{1}{\mathrm{i}(\not{P}_2-\not{Q}_1)-M}\Big)_{\beta\alpha}\,.
\end{aligned} \tag{5.51}
$$

From (5.51) it can be shown that the second term anti-symmetric in a and b vanishes if the pion momenta are taken to zero. This follows from crossing: the amplitude must be symmetric under the combined interchange $a \leftrightarrow b$ and $Q_1 \leftrightarrow -Q_2$ (for a further analysis of (5.51) in the soft pion limit, see problem 5.4).

5.3. *Properties of spinors*

In order to explain the relation between physical particles and propagators we can follow the route used in the spin-0 and spin-1 case, and examine the large-time behaviour of Green's functions with fermions. Also the treatment of the external lines and their proper normalization for the invariant amplitude proceeds in a similar fashion as in section 3.3. The fact that ψ_a in (5.5) has four independent components means that there are four independent polarizations which should somehow characterize the spin of the particle and the anti-particle. Naive counting would lead to the erroneous conclusion that we are in fact dealing with four particle and four anti-particle degrees of freedom, but we have already indicated in section 5.1 how this problem resolves itself. The results of the next section, where we present a complete definition of the fermionic amplitudes, will further clarify this. Here we concentrate on the definition of polarization spinors for the external lines. It is convenient to decompose these spinors in terms of the eigenspinors of the matrix $i\not{P}$, where $P_\mu = (\mathbf{P}, \mathrm{i}\omega(P))$ is the (anti-)particle momentum. Because $(\mathrm{i}\not{P})^2 = -P^2 = m^2$ it follows that the eigenvalues of $i\not{P}$ are equal to $\pm m$. Thus we can distinguish eigenspinors $u(\mathbf{P})$ and $v(\mathbf{P})$ according to

$$(\mathrm{i}\not{P})_{\alpha\beta}u_\beta(\mathbf{P}) = -m u_\alpha(\mathbf{P})\,, \tag{5.52}$$

$$(\mathrm{i}\not{P})_{\alpha\beta}v_\beta(\mathbf{P}) = +m v_\alpha(\mathbf{P})\,. \tag{5.53}$$

Let us first analyze the eigenspinors in the rest frame where $P_\mu = (\mathbf{0}, \mathrm{i}m)$ and

$$(\mathrm{i}\not{P})_{\alpha\beta} = -m(\gamma_4)_{\alpha\beta}\,.$$

Choosing the γ-matrices according to (5.7) we have

$$(\mathrm{i}\not{P} \pm m) = -m \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \pm m \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The eigenvalue equation (5.46a) is therefore

$$2m \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} u(\mathbf{0}) = 0, \tag{5.54}$$

which admits two trivial solutions. which we label by $i = 1, 2$:

$$u^1(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^2(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.55}$$

The eigenvalue equation (5.46b) becomes

$$(\mathrm{i}\not{P} - m)v(\mathbf{0}) = -2m \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} v(\mathbf{0}) = 0,$$

with two trivial solutions, which we also label by $i = 1,\ 2$:

$$v^1(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v^2(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.56}$$

Note that we have normalized the eigenspinors according to

$$\begin{aligned} (u^i_\alpha(\mathbf{0}))^* u^j_\alpha(\mathbf{0}) &= (v^i_\alpha(\mathbf{0}))^* v^j_\alpha(\mathbf{0}) = 2m\delta^{ij}\,, \\ (v^i_\alpha(\mathbf{0}))^* u^j_\alpha(\mathbf{0}) &= (u^i_\alpha(\mathbf{0}))^* v^j_\alpha(\mathbf{0}) = 0\,. \end{aligned}$$

Using the definition (5.15) we see that the corresponding adjoint spinors are the row vectors

$$\bar{u}^1(\mathbf{0}) = \sqrt{2m}(1,0,0,0), \quad \bar{u}^2(\mathbf{0}) = \sqrt{2m}(0,1,0,0)\,, \tag{5.57}$$

$$\bar{v}^1(\mathbf{0}) = \sqrt{2m}(0,0,-1,0), \quad \bar{v}^2(\mathbf{0}) = \sqrt{2m}(0,0,0,-1)\,, \tag{5.58}$$

which in the rest frame are just the independent solutions of

$$\bar{u}^i(\mathbf{P})\mathrm{i}\not{P} = -m\bar{u}^i(\mathbf{P})\,, \tag{5.59}$$

$$\bar{v}^i(\mathbf{P})\mathrm{i}\not{P} = +m\bar{v}^i(\mathbf{P})\,. \tag{5.60}$$

It is clear that the spinors and their adjoints are related in the rest frame. In a general frame the spinors can be chosen such that this relationship is expressed by

$$\begin{aligned} u^i(\mathbf{P}) &= \varepsilon^{ij} C \bar{v}^j(\mathbf{P})^{\mathrm{T}}\,, \\ v^i(\mathbf{P}) &= -\varepsilon^{ij} C \bar{u}^j(\mathbf{P})^{\mathrm{T}}\,, \end{aligned} \tag{5.61}$$

where ε^{ij} is an anti-symmetric matrix (with $\varepsilon^{12} = -\varepsilon^{21} = 1$) and the superscript T indicates that we have written the row vectors $\bar{u}$ and $\bar{v}$ as column vectors: the proportionality matrix is $C = \gamma_4\gamma_2$, which in the representation (5.7) takes the form (for an extensive discussion of this so-called charge conjugation matrix, see appendix E)

$$C = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}. \tag{5.62}$$

The spinors in a general Lorentz frame can be obtained by implementing the Dirac equation (5.46). The result can be expressed as follows:

$$u^i(\mathbf{P}) = \sqrt{m+\omega(\mathbf{P})}\begin{pmatrix} \xi^i \\ \frac{\sigma\cdot\mathbf{P}\xi^i}{m+\omega(\mathbf{P})} \end{pmatrix}, \quad v^i(\mathbf{P}) = \sqrt{m+\omega(\mathbf{P})}\begin{pmatrix} \frac{\sigma\cdot\mathbf{P}\xi^i}{m+\omega(\mathbf{P})} \\ \xi^i \end{pmatrix} \tag{5.63}$$

where ξ^i are orthonormal two-component spinors. We caution the reader that there is no uniformity in the literature in defining these spinors, and there are many normaliziltion and phase conventions. The spinors (5.63) are normalized according to

$$\begin{aligned} (u^i(\mathbf{P}))^* u^j(\mathbf{P}) &= 2\omega(\mathbf{P})\delta^{ij}\,, \\ (v^i(\mathbf{P}))^* v^j(\mathbf{P}) &= 2\omega(\mathbf{P})\delta^{ij}\,, \end{aligned} \tag{5.64}$$

which is consistent with the rest-frame normalization adopted in (5.55) (5.56). Note, however, that we do no longer have $(u^i(\mathbf{P}))^* v^j(\mathbf{P}) = (v^i(\mathbf{P}))^* u^j(\mathbf{P}) = 0$. The appropriate form of this equation will be given shortly. For real ξ^i satisfying $\xi^1 = \mathrm{i}\sigma_2\xi^2$ the relation between the spinors (5.63) and the corresponding adjoint spinors is correctly expressed by (5.61).

The spinors ξ^i characterize the spin content of the solutions (5.63). To exhibit this more explicitly let us apply a rotation by an angle θ around some unit vector $\hat{\mathbf{n}}$ on the spinors (5.63). From (5.13) it follows that such a rotation acts according to

$$\psi \to \psi' = \exp(\tfrac{1}{4}\mathrm{i}\theta\varepsilon_{ijk}\hat{\mathbf{n}}_i\sigma_{jk})\psi\,, \tag{5.65}$$

which by using (5.11) takes the form

$$\psi \to \psi' = \begin{pmatrix} \exp(\frac{1}{2}\mathrm{i}\theta\hat{\mathbf{n}}\cdot\sigma) & 0 \\ 0 & \exp(\frac{1}{2}\mathrm{i}\theta\hat{\mathbf{n}}\cdot\sigma) \end{pmatrix}\psi\,. \tag{5.66}$$

Choosing $\hat{\mathbf{n}}$ along the direction of the particle motion we see that such a rotation amounts to a change of ξ

$$\xi \to \xi' = \exp\left(\tfrac{1}{2}\mathrm{i}\theta\frac{\mathbf{P}\cdot\sigma}{|\mathbf{P}|}\right)\xi\,.$$

A particle with spin up or down along the direction of motion is thus characterized by

$$\begin{aligned} &\sigma\cdot\mathbf{P}\xi^{\mathrm{up}} = |\mathbf{P}|\xi^{\mathrm{up}} \qquad (\text{helicity} \quad +1/2)\,,\\ &\sigma\cdot\mathbf{P}\xi^{\mathrm{down}} = -|\mathbf{P}|\xi^{\mathrm{down}} \qquad (\text{helicity} \quad -1/2)\,. \end{aligned} \tag{5.67}$$

The spinors $u(\mathbf{P})$ and $v(\mathbf{P})$ corresponding to (5.67) are the eigenstates of the helicity operator which measures the spin in the direction of motion. Under rotations around $\mathbf{P}$ they transform with phase factors $\exp(\pm\mathrm{i}\frac{1}{2}\theta)$. The helicity operator follows directly from (5.66). It takes the form

$$h(\mathbf{P}) = \tfrac{1}{2}\begin{pmatrix} \frac{\sigma\cdot\mathbf{P}}{|\mathbf{P}|} & 0 \\ 0 & \frac{\sigma\cdot\mathbf{P}}{|\mathbf{P}|} \end{pmatrix}\,, \tag{5.68}$$

and has eigenvalues $\pm\frac{1}{2}$. Its representation in γ-matrices reads

$$h(\mathbf{P}) = -\tfrac{1}{8}\mathrm{i}\varepsilon_{ijk}[\gamma_j,\gamma_k]\frac{P_i}{|\mathbf{P}|}\,. \tag{5.69}$$

which, after some straightforward manipulation of γ-matrices, may also be written as

$$h(\mathbf{P}) = \frac{-\mathrm{i}\gamma_5\gamma_4\gamma_i P_i}{2|\mathbf{P}|}\,. \tag{5.70}$$

Let us discuss a few more properties of the spinors u and v. Using the eigenvalue equation (5.46) which corresponds to the Dirac equation, we work out the following expression

$$\begin{aligned} (u^i(\mathbf{P}))^* u^j(\mathbf{P}) &= \bar{u}^i(\mathbf{P})\gamma_4 u^j(\mathbf{P}) \\ &= \frac{1}{2\mathrm{i}m}\bar{u}^i(\mathbf{P})(\not{P}\gamma_4 + \gamma_4\not{P})u^j(\mathbf{P}) \\ &= \frac{\omega(\mathbf{P})}{m}\bar{u}^i(\mathbf{P})u^j(\mathbf{P})\,. \end{aligned} \tag{5.71}$$

Using the normalization condition (5.64) we derive

$$\bar{u}^i(\mathbf{P})u^j(\mathbf{P}) = 2m\delta^{ij}\,. \tag{5.72}$$

By similar manipulation we find

$$\bar{v}^i(\mathbf{P})v^j(\mathbf{P}) = -2m\delta^{ij}\,, \tag{5.73}$$

$$\bar{u}^i(\mathbf{P})v^j(\mathbf{P}) = \bar{v}^i(\mathbf{P})u^j(\mathbf{P}) = 0\,, \tag{5.74}$$

$$\bar{u}^i(\mathbf{P})\gamma_\mu u^j(\mathbf{P}) = -2\mathrm{i}P_\mu\delta^{ij}\,, \tag{5.75}$$

$$\bar{v}^i(\mathbf{P})\gamma_\mu v^j(\mathbf{P}) = -2\mathrm{i}P_\mu\delta^{ij}\,. \tag{5.76}$$

It is sometimes advantageous to rewrite (5.46) as

$$\begin{aligned} (iP_j\gamma_4\gamma_j + m\gamma_4)u^i(\mathbf{P}) &= \omega(\mathbf{P})u^i(\mathbf{P})\,, \\ (iP_j\gamma_4\gamma_j + m\gamma_4)v^i(-\mathbf{P}) &= -\omega(\mathbf{P})v^i(-\mathbf{P})\,, \end{aligned} \tag{5.77}$$

because these equations demonstrate that $u^i(\mathbf{P})$ and $v^i(-\mathbf{P})$ are the eigenspinors of a *hermitean* matrix $(\mathrm{i}P_j\gamma_4\gamma_j + m\gamma_4)$ with eigenvalues $\pm\omega(\mathbf{P})$ (note that $\mathrm{i}\not{P}$ is *not* a hermitean matrix). Therefore we know that $u^i(\mathbf{P})$ and $v^j(-\mathbf{P})$ are orthonormal vectors

$$(\bar{u}^i(\mathbf{P}))^* v^j(-\mathbf{P}) = (\bar{v}^i(\mathbf{P}))^* u^j(-\mathbf{P}) = 0\,, \tag{5.78}$$

which can also be verified directly by using similar manipulations as in (5.71). Furthermore $u^i(\mathbf{P})$ and $v^j(-\mathbf{P})$ form a complete orthonormal set of eigenvectors

$$\sum_{i=1,2}\{u^i_\alpha(\mathbf{P})(u^i_\beta(\mathbf{P}))^* + v^i_\alpha(-\mathbf{P})(v^i_\beta(-\mathbf{P}))^*\} = 2\omega(\mathbf{P})\delta_{\alpha\beta}\,. \tag{5.79}$$

Using (5.77) once more we can project out the terms in (5.79) involving either u's or v's. In that way we derive

$$\sum_{i=1,2} u^i_\alpha(\mathbf{P})\bar{u}^i_\beta(\mathbf{P}) = (-\mathrm{i}\not{P} + m)_{\alpha\beta}\,, \tag{5.80}$$

$$\sum_{i=1,2} v^i_\alpha(\mathbf{P})\bar{v}^i_\beta(\mathbf{P}) = (-\mathrm{i}\not{P} - m)_{\alpha\beta}\,, \tag{5.81}$$

These identities are extremely useful for calculating sums over spin polarizations, we shall see shortly.

5.4. Invariant amplitudes for spinors

Let us consider a Green's function that contains external fermionic lines and examine its behaviour when the external sources are moved to large times. Just as for bosons the momenta for which the external line propagators exhibit poles isolate the dominant term of this Green's function, and it is this term that we again wish to identify with the quantum-mechanical probability amplitude for the corresponding scattering process (after appropriate normalization). Ignoring the standard $(p^2 + m^2)$ terms in the denominators (which play the same role as for bosonic external lines), we concentrate on the residue matrix $(-\mathrm{i}\not{p}+m)$ in the numerator of the propagators. Depending on whether the incoming or outgoing momentum $P_\mu = (\mathbf{P}, \mathrm{i}\omega(\mathbf{P}))$ is equal or opposite to the momentum p_μ assigned to the external line we are dealing with particles or anti-particles, and the residue matrix associated with each line is equal to $(\mp\mathrm{i}\not{P} + m)$. Because the spinors u, v, $\bar{u}$ and $\bar{v}$ are defined as the right and left, i.e. column and row eigenspinors of the matrix $\not{P}$ they can be used to characterize the incoming and outgoing particles.

We now consider the four cases of incoming and outgoing (anti-)particles in detail. Let us first move the endpoint of a fermion line with incoming orientation to *negative* infinite time. This corresponds to an incoming particle with $P_\mu = p_\mu$, $p_0 = \omega(\mathbf{P})$. Decomposing the spinor associated with the external linc into u^i and v^i, we find that the propagator residue leads to

$$\begin{aligned}(-\mathrm{i}\not{p}+m)u(\mathbf{P}) &= (-\mathrm{i}\not{P}+m)u(\mathbf{P}) = 2mu(\mathbf{P})\,,\\ (-\mathrm{i}\not{p}+m)v(\mathbf{P}) &= (-\mathrm{i}\not{P}+m)v(\mathbf{P}) = 0\,.\end{aligned} \tag{5.82}$$

Hence the spinors v_i vanish when multiplied with the propagator residue, so that *incoming fermions* are characterized only by the spinors u^i. Consequently there are just two polarizations for incoming fermions.

If we take the opposite direction for the line orientation, we have an incoming anti-particle with momentum P_μ equal $-p_\mu$. The spinors associated with the line can now be decomposed in terms of $\bar{u}^i$ and $\bar{v}^i$, but this time the contribution of $\bar{u}^i$ vanishes

$$\begin{aligned}\bar{u}(\mathbf{P})(-\mathrm{i}\not{p}+m) &= \bar{u}(\mathbf{P})(\mathrm{i}\not{P}+m) = 0\,,\\ \bar{v}(\mathbf{P})(-\mathrm{i}\not{p}+m) &= \bar{v}(\mathbf{P})(\mathrm{i}\not{P}+m) = 2m\bar{v}(\mathbf{P})\,,\end{aligned} \tag{5.83}$$

Therefore *incoming anti-fermions* are characterized entirely by the spinors $\bar{v}^i$, so that there are again two polarizations.

The case of outgoing particles and anti-particles can be analyzed along the same lines, and we summarize all four cases in table 5.3.

Hence we have established that the asymptotic value of the Green's functions with all sources moved to large times is given by the invariant amplitude (i.e., the truncated Green's functions with the propagators of the external lines removed), contracted with the appropriate spinors u, v, $\bar{u}$ or $\bar{v}$ There is still a factor $2m$ that we have picked up from the residue of the propagator (cf. 5.82 and 5.83) which, however, turns out to cancel the proper normalization factor of the external fermion lines. To see this consider the amplitude that relates the wave function of the particle at $t = -\infty$ to its wave function at $t = +\infty$, which is given by the asymptotic value at large time of the propagator. The propagator leads to the standard factor $[(2\pi)^3 2\omega(\mathbf{P})]^{-1}$, which we have also encountered for bosons, times

$$\begin{aligned} \bar{u}^i(\mathbf{P})(-\mathrm{i}\not{P} + m)u^j(\mathbf{P}) &= (2m)^2\delta^{ij} \quad \text{(fermion)}\,, \\ \bar{v}^i(\mathbf{P})(\mathrm{i}\not{P} + m)v^j(\mathbf{P}) &= -(2m)^2\delta^{ij} \quad \text{(anti}-\text{fermion)}\,. \end{aligned} \tag{5.84}$$

Therefore the external lines require an extra normalization factor $(2m)^{-1}$, which cancels against the previous factor $2m$ (because the mass drops out massless fermions are also included in this treatment). Hence the correctly normalized probability amplitude is equal to the invariant amplitude contracted with the spinors u, v, $\bar{u}$ or $\bar{v}$ times the same normalization factors $[(2\pi)^3 2\omega(\mathbf{P})]^{-1/2}$ that one has for bosons (cf. 3.18). With this result one can now evaluate cross sections and decay rates according to the prescription derived in chapter 3. Table 5.3

The four different assignments of external line momenta corresponding to incoming and outgoing particles and anti-particles.

Limit External line Situation

1. $T \to \infty$

Outgoing fermion with momentum $P_\mu = p_\mu$, $p_0 = \omega(P)$, and polarization spinors $\bar{u}^i(P)$

Outgoing anti-fermion with momentum $P_\mu = -p_\mu$, $p_0 = -\omega(P)$, and polarization spinors $\bar{v}^i(P)$

Incoming fermion with momentum $P_\mu = p_\mu$, p_0, $p_0 = \omega(P)$, and polarization spinors $u^i(P)$

Incoming anti-fermion with momentum $P_\mu = -p_\mu$, $p_O = -\omega(P)$, and polarization spinors $\bar{v}^i(P)$

P_μ denotes the (anti-)particle momentum, while p_μ is the momentum assigned to the external line in the diagram.

Now that we have identified the fermion polarization spinors it is possible to consider a fermion wave function

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 P}{\sqrt{2\omega(\mathbf{P})}} \sum_{i=1,2} f^{(i)}(\mathbf{P}) u_\alpha^{(i)}(\mathbf{P}) \mathbf{e}^{\mathrm{i}\mathbf{P}\cdot x - \mathrm{i}\omega(\mathbf{P})t}\,. \tag{5.85}$$

Note that this wave function satisfies the Dirac equation (5.23), and that we have restricted the polarizations to u-spinors only in view of the results found above. This restriction does not imply that we are exclusively dealing with particles; also anti-particles can be described by (5.85) by using the linear relation (5.61) between u-spinors and the row spinors $\bar{v}$.

Just as for bosons there is a conserved probability current (see problem 5.6). It reads

$$j_\mu(x) = \mathrm{i}\bar{f}(x)\gamma_\mu f(x)\,. \tag{5.86}$$

Indeed

$$\partial_\mu j_\mu = \mathrm{i}(\bar{f}(x)\overleftarrow{\not\partial})f(x) + \mathrm{i}\bar{f}(x)\not\partial f(x)) = 0 \tag{5.87}$$

by virtue of the Dirac equation. Inserting the wave function (5.85) into (5.87), and integrating over space gives

$$\int \mathrm{d}^3x j_\mu(\mathrm{x},t) = \int \mathrm{d}^3P \sum_{i=1,2} |f^{(i)}(\mathbf{P})|^2 \frac{P_\mu}{\omega(\mathbf{P})}\,. \tag{5.88}$$

The integral over the probability density is

$$\int \mathrm{d}^3x j_0(\mathbf{x},t) = \int \mathrm{d}^3P \sum_{i=1,2} |f^{(i)}(\mathbf{P})|^2\,, \tag{5.89}$$

which is the direct analogue of (3.8), and identifies the probability for finding a fermion with momentum $\mathbf{P}$ in a volume element d^3P and with spin characterized by u^i as

$$W_{\mathbf{P}}^{(i)}\mathrm{d}^3P = |f^{(i)}(\mathbf{P})|^2\mathrm{d}^3P\,. \tag{5.90}$$

One can also determine the total energy and momentum carried by the wave function (5.85) by inserting it into the energy-momentum tensor. For free massive fermions this tensor is given by (see problem 5.6)

$$T_{\mu\nu} = 1/4\bar{f}(x)(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu)f(x)\,. \tag{5.91}$$

Owing to the Dirac equation this tensor is conserved

$$\begin{aligned} \partial_\mu T_{\mu\nu} &= \tfrac{1}{4}(\bar{f}(x)\overleftarrow{\not\partial})\overleftrightarrow{\partial}_\nu f(x) + \tfrac{1}{4}\bar{f}(x)\overleftrightarrow{\partial}_\nu(\not\partial f(x)) \\ &\quad + \tfrac{1}{4}\bar{f}(x)\gamma_\nu(\Box f(x) - \tfrac{1}{4}(\bar{f}(x)\Box)\gamma_\nu f(x) = 0 \end{aligned} \tag{5.92}$$

Substituting (5.85) into (5.91) and taking the integral over space one finds

$$\int \mathrm{d}^3x T_{\mu\nu}(\mathbf{x},t) = \int \mathrm{d}^3P \sum_{i=1,2} |f^{(i)}(\mathbf{P})|^2 \frac{P_\mu P_\nu}{\omega(\mathbf{P})}\,, \tag{5.93}$$

which may be compared to (3.13). The energy momentum carried by the wave function (5.85) is therefore expressed by

$$\int \mathrm{d}^3x T_{\mu 0}(\mathbf{x},t) = \int \mathrm{d}^3P \sum_{i=1,2} |f^{(i)}(\mathbf{P})|^2 P_\mu \,, \tag{5.94}$$

in accordance with the probability interpretation (5.90).

Finally we explain how to square a spinor amplitude and sum over the polarizations of incoming and/or outgoing fermions. Consider a typical exprcssion for the amplitude with an incoming and an outgoing fermion

$$\mathcal{M}_{ij} = \bar{u}^i_{1\alpha}(\mathbf{P}_1)\mathit{\Gamma}_{\alpha\beta} u^j_{2\beta}(\mathbf{P}_2)\,, \tag{5.95}$$

where $\mathit{\Gamma}_{\alpha\beta}$ is some matrix in spinor space. The complex conjugate of (5.95) can be written as

$$\begin{aligned} (\mathcal{M}^{ij})^* &= [(u^i_1(\mathbf{P}_1))^*\gamma_4 \mathit{\Gamma} u^j_2(\mathbf{P}_2)]^* \\ &= u^j_2(\mathbf{P}_2)^* \mathit{\Gamma}^\dagger \gamma_4 u^i_1(\mathbf{P}_1) \\ &= \bar{u}^j_2(\mathbf{P}_2)\gamma_4 \mathit{\Gamma}^\dagger \gamma_4 u^i_1(\mathbf{P}_1)\,, \end{aligned} \tag{5.96}$$

where we have used the fact that γ_4 is hermitean and that $\gamma_4^2 = 1$ Combining (5.95) and (5.96) gives

$$|\mathcal{M}^{ij}|^2 = [\bar{u}^i_1(\mathbf{P}_1)\mathit{\Gamma} u^j_2(\mathbf{P}_2)][\bar{u}^j_2(\mathbf{P}_2)\gamma_4 \mathit{\Gamma}^\dagger \gamma_4 u^i_1(\mathbf{P}_1)]\,. \tag{5.97}$$

It is now easy to sum over the fermion polarizations. Writing the spinor indices explicitly we find

$$\begin{aligned} \sum_{i,j=1,2} |\mathcal{M}^{ij}|^2 &= \mathit{\Gamma}_{\gamma\delta}\Big(\sum_{j=1,2} u^j_{2\delta}(\mathbf{P}_2)\bar{u}^j_{2\alpha}(\mathbf{P}_2)\Big)(\gamma_4 \mathit{\Gamma}^\dagger \gamma_4)_{\alpha\beta} \\ &\qquad \Big(\sum_{i=1,2} u^j_{1\beta}(\mathbf{P}_1)\bar{u}^i_{1\gamma}(\mathbf{P}_1)\Big)\,. \end{aligned}$$

Using (5.72) this can be conveniently written as

$$\sum_{i,j=1,2} |\mathcal{M}^{ij}|^2 = \mathrm{Tr}[\mathit{\Gamma}(-i\not{P}_2 + m_2)\gamma_4 \mathit{\Gamma}^\dagger \gamma_4(-i\not{P}_1 + m_1)]\,. \tag{5.98}$$

This result can now be further evaluated by inserting the expression for $\mathit{\Gamma}$ and using the properties of the γ-matrices. Corresponding expressions involving the spinors v can be found similarly.

5.5. *An application: boson decays into two spin-$\frac{1}{2}$ fermions*

As an application of the preceding results we shall now work out the decay of a boson of spin 0 or 1 into a fermion-anti-fermion pair. Rather than starting from some (phenomenological) Lagrangian we immediately concentrate on the decay amplitude. First consider the decay of a spinless particle $\phi_0(P) \rightarrow \mathrm{F}_1(P_1) + \bar{\mathrm{F}}_2(P_2)$ where the momenta are indicated in parentheses. The decay amplitude can be written as

$$\mathcal{M}^{ij} = \bar{u}^i(\mathbf{P}_1)(g_{\mathrm{S}} + \mathrm{i}\gamma_5 g_{\mathrm{P}})v^j(\mathbf{P}_2)\,, \tag{5.99}$$

where i and j characterize the spins of F_1, and $\bar{\mathrm{F}}_2$, respectively. Because the spinors satisfy the Dirac equation one can show that (5.99) is the most general Lorentz-invariant parametrization (for example, terms such as $\bar{u}Pv$ or $\bar{u}P\gamma_5 v$ can be rewritten into the form (5.99) by writing $P = P_1 + P_2$ and subsequently using the Dirac equation on the spinors). Because all relativistic invariants formed from the external momenta in a three-point interaction are expressible in terms of the masses of the external particles, the parameters g_{S} and g_{P} are constants. These constants are not necessarily real (unless the two fermions are identical).

For the decay of a spin-1 particle $\phi_1(P) \rightarrow F_1(P_1) + F_2(p_2)$ the situation is more complicated. Here the amplitude may depend on four parameters. For simplicity, however, we restrict ourselves to the amplitude

$$\varepsilon_\mu(\mathbf{P})\mathcal{M}^{ij}_\mu = \varepsilon_\mu(\mathbf{P})\bar{u}^i(\mathbf{P}_1)\gamma_\mu(\mathrm{i}g_{\mathrm{V}} + \mathrm{i}\gamma_5 g_{\mathrm{A}})v^j(\mathbf{P}_2)\,, \tag{5.100}$$

where $\varepsilon_\mu(\mathbf{P})$ is the polarization vector associated with the decaying particle.

The decay probability for the process is proportional to the square of the amplitude summed over the fermion spins. Since both amplitudes (5.99) and (5.100) are of the same generic form, let us first determine the probability for an amplitude

$$\mathcal{M}_{ij} = \bar{u}^i(\mathbf{P}_1)\mathit{\Gamma} v^j(\mathbf{P}_2)\,. \tag{5.101}$$

where $\mathit{\Gamma}$ is an arbitrary matrix in spinor space. Following the same steps as those leading to (5.96)- (5.98) we have

$$(\mathcal{M}^{ij})^* = \bar{v}^j(\mathbf{P}_2)\gamma_4\mathit{\Gamma}^\dagger\gamma_4 u^i(\mathbf{P}_1)\,, \tag{5.102}$$

so that

$$\begin{aligned}\sum_{i,j=1,2}|\mathcal{M}^{ij}|^2 &= \sum_{i,j=1,2}(\bar{u}^i_\alpha(\mathbf{P}_1)\mathit{\Gamma}_{\alpha\beta}v^j_\beta(\mathbf{P}_2))(\bar{v}^j_\gamma(\mathbf{P}_2)(\gamma_4\mathit{\Gamma}^\dagger\gamma_4)_{\gamma\delta}u^i_\delta(\mathbf{P}_1)) \\ &= \mathit{\Gamma}_{\alpha\beta}\Big(\sum_{j=1,2}v^j_\beta(\mathbf{P}_2)\bar{v}^j_\gamma(\mathbf{P}_2)\Big)(\gamma_4\mathit{\Gamma}^\dagger\gamma_4)_{\gamma\delta}\Big(\sum_{i=1,2}u^i_\delta(\mathbf{P}_1)\bar{u}^i_\delta(\mathbf{P}_1)\Big)\end{aligned} \tag{5.103}$$

Using (5.80) and (5.81) the expression (5.103) is simply the trace of a product of matrices

$$\sum_{i,j=1,2}|\mathcal{M}^{ij}|^2 = \mathrm{Tr}[\mathit{\Gamma}(-\mathrm{i}\not{P}_2 - m^2)\gamma_4\mathit{\Gamma}^\dagger\gamma_4(-\mathrm{i}\not{P}_1 + m_1)]\,. \tag{5.104}$$

For the case $\Gamma = g_{\rm S} + {\rm i}\gamma_5 g_{\rm P}$ we have

$$\begin{aligned}\gamma_4\Gamma^\dagger\gamma_4 &= \gamma_4 g_{\rm S}^*\gamma_4 + {\rm i}\gamma_4\gamma_5 g_{\rm P}^*\gamma_4 \\ &= g_{\rm S}^* + {\rm i}\gamma_5 g_{\rm P}^*\,,\end{aligned} \tag{5.105}$$

Similarly for $\Gamma = \varepsilon_\mu\gamma_\mu({\rm i}g_{\rm V} + {\rm i}\gamma_5 g_{\rm A})$ the same operation leads to

$$\gamma_4\Gamma^\dagger\gamma_4 = \gamma_4(-{\rm i}g_{\rm V}^*\gamma_\mu - {\rm i}g_{\rm A}^*\gamma_5\gamma_\mu)\gamma_4\varepsilon_\mu^*\,. \tag{5.106}$$

Commuting γ_4 through γ_μ and using the anti-commutation relation (5.6), one finds for (5.106)

$$\gamma_4\Gamma^\dagger\gamma_4 = \bar\varepsilon_\mu({\rm i}g_{\rm V}^*\gamma_\mu + {\rm i}g_{\rm A}^*\gamma_5\gamma_\mu)\,. \tag{5.107}$$

Substituting these results into (5.104) one obtains for the decay amplitude of a spin-0 boson

$$\sum_{i,j=1,2}|\mathcal{M}^{ij}|^2 = {\rm Tr}[(g_{\rm S}+{\rm i}\gamma_5 g_{\rm P})({\rm i}\not{P}_2+m_2)(g_{\rm S}^*+{\rm i}\gamma_5 g_{\rm P}^*)({\rm i}\not{P}_1-m_1)]\,, \tag{5.108}$$

and for the decay amplitude of a spin-1 boson

$$\begin{aligned}\sum_{i,j=1,2}|\varepsilon_\mu\mathcal{M}_\mu^{ij}|^2 &= \varepsilon_\mu\bar\varepsilon_\nu{\rm Tr}[({\rm i}g_{\rm V}+{\rm i}g_{\rm A}\gamma_\mu\gamma_5)({\rm i}\not{P}_2+m_2)(g_{\rm S}^*+{\rm i}\gamma_5 g_{\rm P}^*)({\rm i}\not{P}_1-m_1)]\,, \\ &\quad \times[({\rm i}g_{\rm V}^*+{\rm i}g_{\rm A}^*\gamma_\nu\gamma_5)({\rm i}\not{P}_1-m_1)]\,.\end{aligned} \tag{5.109}$$

In order to evaluate these expressions note that $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$ and $\gamma_5^2 = 1$, so that we may simplify our task and distinguish between terms with and without a single γ_5 matrix. Using the cyclicity property of the trace (i.e. ${\rm Tr}(ABC) = {\rm Tr}(BCA)$) and retaining only nonvanishing products of γ-matrices, (5.108) and (5.109) reduce to

$$\sum_{i,j=1,2}|\mathcal{M}^{ij}|^2 = {\rm Tr}[-\not{P}_1\not{P}_2 - m_1m_2)|g_{\rm S}|^2 + (-\not{P}_1\not{P}_2 + m_1m_2)|g_{\rm P}|^2] \tag{5.110}$$

$$\begin{aligned}\sum_{i,j=1,2}|\varepsilon_\mu\mathcal{M}_\mu^{ij}|^2 &= \varepsilon_\mu\bar\varepsilon_\nu{\rm Tr}[|g_{\rm V}|^2(\gamma_\mu\not{P}_2\gamma_\nu\not{P}_1 + m_1m_2\gamma_\mu\gamma_\nu) \\ &+ \ |g_{\rm A}|^2(\gamma_\mu\not{P}_2\gamma_\nu\not{P}_1 - m_1m_2\gamma_\mu\gamma_\nu) - (g_{\rm V}g_{\rm A}^* + g_{\rm V}^*g_{\rm A})\gamma_\mu\not{P}_2\gamma_\nu\not{P}_1\gamma_5]\end{aligned}$$

The traces can be evaluated by means of the results derived in appendix E:

$$\begin{aligned}{\rm Tr}(\mathbf{I}) &= 4\,, \\ {\rm Tr}(\gamma_\mu\gamma_\nu) &= 4\delta_{\mu\nu}\,, \\ {\rm Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= 4(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma})\,, \\ {\rm Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) &= 4\varepsilon_{\mu\nu\rho\sigma}\,,\end{aligned}$$

which gives

$$\sum_{i,j=1,2} |\mathcal{M}^{ij}|^2 = 4|g_{\rm S}|^2(-P_1\cdot P_2 - m_1 m_2) + 4|g_{\rm P}|^2(-P_1\cdot P_2 + m_1 m_2)\,, \quad (5.111)$$

$$\begin{aligned}\sum_{i,j=1,2} |\varepsilon_\mu \mathcal{M}^{ij}_\mu|^2 &= 4\varepsilon_\mu\bar{\varepsilon}_\nu\{(|g_{\rm V}|^2 + |g_{\rm A}|^2)(P_{1\mu}P_{2\nu} + P_{1\nu}P_{2\mu} - P_1\cdot P_2\delta_{\mu\nu})\\ &+ (|g_{\rm V}|^2 - |g_{\rm A}|^2)m_1 m_2\delta_{\mu\nu} - (g_{\rm V}g^*_{\rm A} + g_{\rm la}V^* g_{\rm A})\varepsilon_{\mu\nu\rho\sigma}P_{1\rho}P_{2\sigma}\}\end{aligned} \quad (5.112)$$

For unpolarized spin-1 bosons one must also average over the three independent polarizations. Using the polarization sum (4.21)

$$\sum_{\rm pol} \varepsilon_\mu\bar{\varepsilon}_\nu = \delta_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}\,,$$

(5.104) gives

$$\begin{aligned}\tfrac{1}{3}\sum_{\rm pol,spin} |\varepsilon_\mu \mathcal{M}^{ij}_\mu|^2 &= \frac{4}{3}(|g_{\rm V}|^2 + |g_{\rm A}|^2)[-(P_1\cdot P_2) + 2(P\cdot P_1)(P\cdot P_2)M^{-2}]\\ &+ 4(|g_{\rm V}|^2 - |g_{\rm A}|^2)m_1 m_2\,.\end{aligned} \quad (5.113)$$

At this point we note that it is easy to obtain the amplitude for the inverse decays $\mathrm{F}_2(P_2) \to \mathrm{F}_1(P_1) + \phi(P)$, or $\bar{\mathrm{F}}_1(P_1) \to \bar{\mathrm{F}}_2(P_2) + \phi(P)$, assuming that these are kinematically possible, by simply changing the spinor $v^j(\mathbf{P}_2)$ into $u^j(\mathbf{P}_2)$, or $\bar{u}^i(\mathbf{P}_1)$ into $\bar{v}^i(\mathbf{P}_1)$ in (5.99) and (5.100). After summing over spin polarizations the term proportional to $m_1 m_2$ in (5.111) and (5.113) will change sign.

For boson decay one has $P = P_1 + P_2$; squaring this relation leads to

$$\begin{aligned}(P_1\cdot P_2) &= -\tfrac{1}{2}(M^2 - m_1^2 - m_2^2)\,,\\ (P\cdot P_1)(P\cdot P_2) &= -\tfrac{1}{4}(m_1^2 - m_2^2 - M^2)(m_1^2 - m^2 + M^2)\,.\end{aligned} \quad (5.114)$$

For the inverse (fermion) decays, where $P = P_2 - P_1$ or $P = P_1 - P_2$, one finds the result (5.114) with opposite signs. Therefore the final results for boson and fermion decays, when expressed entirely in terms of masses, differ only by an overall sign.

Combining (5.111) and (5.113) with (5.114) one finds for spin-0 decay

$$\sum_{\rm spins} |\mathcal{M}|^2 = 2|g_{\rm S}|^2(M^2 - (m_1+m_2)^2) + 2|g_{\rm P}|^2(M^2 - (m_1-m_2)^2)\,, \quad (5.115)$$

and for spin-1 decay

$$\begin{aligned}\tfrac{1}{3}\sum_{\rm spins,spins} |\mathcal{M}|^2 &= \tfrac{2}{3}|g_{\rm V}|^2 M^{-2}(M^2 - (m_1 - m_2)^2)(2M^2 + (m_1 + m_2)^2)\\ &+ 2|g_{\rm A}|^2 M^{-2}(M^2 - (m_1 + m_2)^2)(2M^2 + (m_1 - m_2)^2)\end{aligned} \quad (5.116)$$

As mentioned above the overall sign changes for the inverse decays such that the results (5.115) and (5.116) remain positive for the allowed range of corresponding mass values.

To obtain the corresponding decay rates one uses (3.82). Since there are no identical particles in the final state one finds

$$\begin{aligned} \Gamma(\phi_0 \to \mathrm{F}_1\bar{\mathrm{F}}_2) &= \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{8\pi M^3} \\ &\times \{|g_\mathrm{S}|^2(M^2 - (m_1+m_2)^2) + |g_\mathrm{P}|^2(M^2 - (m_1-m_2)^2)\} \\ &\approx \frac{M}{8\pi}(|g_\mathrm{S}|^2 + |g_\mathrm{P}|^2)\,, \quad M \gg m_1\,, m_2\,, \end{aligned} \tag{5.117}$$

and

$$\begin{aligned} \Gamma(\phi_1 \to \mathrm{F}_1\bar{\mathrm{F}}_2) &= \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{24\pi M^5} \\ &\times \{|g_\mathrm{V}|^2(M^2 - (m_1-m_2)^2)(2M^2 + (m_1+m_2)^2) \\ &+ |g_\mathrm{A}|^2(M^2 - (m_1+m_2)^2)(2M^2 + (m_1-m_2)^2)\} \\ &\approx \frac{M}{12\pi}(|g_\mathrm{V}|^2 + |g_\mathrm{A}|^2)\,, M \gg m_1, m_2\,. \end{aligned} \tag{5.118}$$

5.6. *Weak and electromagnetic two-particle decays and universality*

The results of the previous section enable us to examine a number of two-particle decays and their phenomenological consequences. We begin with a discussion of weak and electromagnetic decays of spin-1 particles. In table 5.4 we have listed some information on masses, total decay rates, decay modes and branching ratios for such particles, where we distinguish the strongly interacting vector mesons ρ^0, ω, ϕ , J/ψ and Υ, from the weak intermediate vector bosons W and Z. Let us first consider the decays of the former into $e^+ e^-$, which are mediated either by a virtual photon or a virtual Z-boson (because leptons are only subject to electromagnetic and weak interactions). However, the weak interaction contribution is much smaller than the electromagnetic one so we may ignore parity violating interactions and set $g_\mathrm{A} = 0$. A straightforward comparison of (5.118) with the data then gives the values of $|g_\mathbf{V}|$ as presented in the table. Table 5.4 Properties of vector bosons'.

a Note that we have not assigned isospin and parity quantum numbers to the W and Z since the weak interactions do not respect these symmetries. b For the Z and W^+ we quote the results from both of the two CERN collider experiments that have measured the decay parameters. c Theoretical values.

To calculate g_V from a more fundamental theory one may view the strongly interacting vector mesons as bound states of quark-anti-quark pairs. The pho-

Figure 5.7: The decay $V_0 \to e^+e^-$ mediated by a virtual photon.

ton that mediates the decay is then emitted by annihilation of the q$\bar{\text{q}}$ pair; the relevant Feynman diagram is shown in fig. 5.7. However, we do not know how to calculate from first principles the amplitude for coupling an on-mass-shell vector meson to a virtual photon, so it is customary to parametrize it in terms of a gauge-invariant phenomenological Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\frac{e}{f_V}(\partial_\mu V_\nu - \partial_\nu V_\mu)F_{\mu\nu}\,, \tag{5.119}$$

where e is the elementary electric charge, V_μ the field associated with the vector meson and $F_{\mu\nu}$ the electromagnetic field strength. In momentum space this leads to a $V-\gamma$ transition amplitude equal to $eM_\text{V}^2 f_\text{V}^{-1}$. In the full decay amplitude the factor M_V^2 then cancels against the denominator of the photon propagator, so that $g_\text{V} = e^2 f_\text{V}^{-1} = 4\pi\alpha f_\text{V}^{-1}$, where α is the fine-structure constant. The experimental rates now yield $f_\rho^2/4\pi = 2.1$, $f_\omega^2/4\pi = 19.5$, $f_\phi^2/4\pi = 13.9$, $f_{\text{J}/\phi}^2/4\pi = 11.7$, $f_\Upsilon^2/4\pi = 1.43$.

To understand the disparity between these numbers we first note that quarks fall into two categories, namely the u, c and t quarks with charge $\frac{2}{3}$, and the d, s and b quarks with charge -$\frac{1}{3}$. On the basis of the quark model one therefore expects a factor of $\frac{1}{9}$ for q$\bar{\text{q}}$ bound states of charge -$\frac{1}{3}$ quarks and a factor $\frac{4}{9}$ for q$\bar{\text{q}}$ bound states of charge -$\frac{2}{3}$ quarks. This argument leads to a factor of $\frac{1}{9}$ for $\phi(s\bar{s})$ and $\Upsilon(b\bar{b})$ and $\frac{4}{9}$ for J$/\psi(c\bar{c})$. The light-mass mesons ρ and ω are built from admixtures of u$\bar{\text{u}}$ and d$\bar{\text{d}}$; to good approximation these admixtures are given by

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\,, \qquad \omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\,.$$

Therefore the corresponding transition amplitudes are proportional to $\sqrt{\frac{1}{2}}(\frac{2}{3}+\frac{1}{3})$ and $\sqrt{\frac{1}{2}}(\frac{2}{3}-\frac{1}{3})$, respectively; squaring these factors leads to $\frac{1}{2}$ and $\frac{1}{18}$.

Taking the different quark charges into account the relevant comparison is

$$\frac{f_\rho^2}{2} : \frac{f_\omega^2}{18} : \frac{f_\phi^2}{9} : \frac{f_{J/\psi}^2}{9} : \frac{f_\Upsilon^2}{9} = 1 : 1.03 : 1.47 : 4.95 : 15.13\,.$$

Remarkably enough these numbers are rather close to the corresponding ratios of vector boson masses

$$M_\rho : M_\omega : M_\phi : M_{J/\phi} : M_\Upsilon = 1 : 1.02 : 1.32 : 4.02 : 12.39\,,$$

which is consistent with the mass dependence one expects for the probability of a two-particle bound state to annihilate into a photon (using the non- relativistic van Royen-Weisskopf formula). This confirms that the differences in the decay rates are mainly due to the different electric charges of the quark constituents in these vector mesons, and may be interpreted as another justification for assigning fractional charges to the quarks.

The fact that these decays are mediated by a virtual photon makes it straightforward to calculate the corresponding decays into $\mu^+\mu^-$ without introducing new parameters. As the photon couples universally to all charged leptons, one expects the same result; only for the light vector mesons can there be a slight difference because the muon mass is much larger than the electron mass, but for the heavier meson decays there must be *lepton universality.* Of course, the same comment applies to the decay into $\tau^+\tau^-$-pairs, provided that the vector mesons are sufficiently massive.

The experimental numbers in table 5.4 are best determined from the inverse process, namely the production and decay of the vector meson in e^+e^- colliding beam experiments. Near the resonance peak we use the narrow-width approximation (3.91) to write the cross section for this reaction as (ignoring the electron mass)

$$\sigma(e^+\,e^- \to \mathbf{X}) = 12\pi \frac{\Gamma(\mathbf{V} \to X)\Gamma(V \to e^+\,e^-)}{(s_V - m_V^2)^2 + m_V^2\Gamma^2} \tag{5.120}$$

where $\Gamma(V \to X)$, $\Gamma(V \to e^+e^-)$ and Γ are the partial and total decay rates in the V rest frame. In (5.120) we have assumed that the electron and positron beams are not polarized and we have summed over the spins and integrated over the angles of the particles contained in X. This leads to a factor $\frac{3}{4}$ as compared to (3.91) (see the comments following 3.92). In this way one can thus determine the branching ratios for other decays than into e^+e^- by a comparison of (5.120) for different final states X. This is the cleanest way to study the J/ψ and Υ vector mesons. Unfortunately there are not yet colliding e^+e^- beams with sufficient energy to produce the Z boson but some of its properties have been determined at the CERN SPS collider via the reaction $p\bar{p} \to Z + Y \to e^+e^-Y$ where Y denotes a final hadronic state.

The decay amplitudes for the weak intermediate vector bosons Z and W must violate parity reversal, so they contain both γ_μ and $\gamma_\mu\gamma_5$ couplings. The strength of these two terms is a model-dependent quantity, unless one of the decay products is a neutrino. Classic experiments have shown that only the negative helicity components of the neutrinos couple through the weak interactions; in other words, neutrinos (anti-neutrinos) are left-handed (right-handed), i.e. their spins are anti-parallel (parallel) to their motion (assuming that neutrinos are massless). Therefore, if (anti-)neutrinos are involved the decay amplitude is proportional to $(1+\gamma_5)$, so that $g_V = g_A$ (see problem 5.5).

Conservation of angular momentum along the direction of the fermions in the rest frame of the decaying boson implies that only spin-l particles can decay into $\nu\bar{\nu}$. But the measurement of Z$\rightarrow$ $\nu\bar{\nu}$ is not possible other than through its contribution to the total width, so let us first discuss the decay Z$\rightarrow$ e^+e^-. For this decay both g_V and g_A are present in (5.118). The standard model of electroweak interactions predicts that $|g_V| \ll |g_A|$; the decay Z$\rightarrow$ e^+e^- can thus be used to extract a value for $|g_A|$. Some events of this type have been identified at the CERN SPS collider. The production cross section at sufficiently high energies is given by

$$\sigma(p\bar{p} \rightarrow e^+e^-Y) = \sigma(p\bar{p} \rightarrow ZY)\frac{\mathit{\Gamma}(Z \rightarrow e^+e^-}{\mathit{\Gamma}(Z \rightarrow \text{all})}\,. \tag{5.121}$$

where Y is any hadronic final state. The right-hand side of this equation can only be determined within the framework of a model, so that the branching ratio in (5.121) cannot be accurately determined experimentally. Nevertheless the total width can be determined from a Breit-Wigner shape near the resonance peak.

The leptonic decay modes for the charged vector bosons $W^\pm$ always have a neutrino in the final state. The decay amplitudes for $W^+ \rightarrow e^+\nu_e$ and $W^- \rightarrow e^-\bar{\nu}_e$ are therefore proportional to $\gamma_\mu(1+\gamma_5)$, so that $g_V = g_A$. These decays (as well as their muonic counterparts $W^+ \rightarrow \mu^+\nu_\mu$ and $W^- \rightarrow \mu^-\bar{\nu}_\mu$) have been identified at the CERN SPS collider via the analogue of (5.121), i.e.

$$\sigma(p\bar{p} \rightarrow e^+e^-Y) = \sigma(p\bar{p} \rightarrow ZY)\frac{\mathit{\Gamma}(Z \rightarrow e^+e^-)}{\mathit{\Gamma}(Z \rightarrow \text{all})}\,. \tag{5.122}$$

However, the production cross section is again model dependent so we cannot unambiguously determine the branching ratio even though an estimate of the total width can be made.

The leptonic decays of the W and Z bosons are in agreement with the predictions of the standard model whose Lagrangian is given in appendix G. For leptons this model contains three fundamental parameters, g, ,g' and

$\theta_{\rm W}$, which are related to the charge of the electron (-e), the Fermi coupling constant for μ-decay ($G = 1.03 \times 10^{-5}/m_{\rm P}^2 = 1.166 \times 10^{-5}$ GeV^{-2} and the masses of the W and Z bosons, by the relations

$$e = g \sin\theta_{\rm W} = g' \cos\theta_{\rm W}\,, \tag{5.123}$$

$$M_{\rm W}^2 = M_{\rm Z}^2 \cos^2\theta_{\rm W} = \frac{\sqrt{2}g^2}{8G}\,. \tag{5.124}$$

The theoretical values of $g_{\rm V}$ and $g_{\rm A}$ are presented in table 5.5. since the W and Z masses have been measured there is an experimental result for the electroweak mixing angle, namely $\sin^2\theta_{\rm W} = 0.226 \pm 0.013$. The decay rates can thus be determined and are quoted in the same table. According to the standard electroweak theory the W and Z bosons couple identically to the various lepton generations, i.e. the weak interactions also exhibit lepton universality. The fact that the μ and τ masses are much larger than the e mass does not play a role here in view of the much larger mass values for W and Z. Therefore each lepton generation leads to the result shown in the table. The total (leptonic and hadronic) decay rate is expected to be approximately 2.2 GeV c^2 for the Z and 3.1 GeV c^2 for the W, so we have assumed these values in table 5.5. Table 5.5 Leptonic coupling constants and decay widths for W and

Z bosons based on the standard model.

The discussion of purely leptonic decays of charged pseudoscalar mesons is similar to that of the electromagnetic decays of vector mesons. Here a virtual W boson is created, which subsequently annihilates into a charged lepton and its corresponding neutrino. To be specific consider the decay $\pi^+ \to {\rm e}^+\nu_e$; the corresponding diagram is shown in fig. 5.8. The π^+-$\rm W^+$ transition in this diagram can again be characterized by a phenomenological Lagrangian,

$$\mathcal{L} = \tfrac{1}{4}\sqrt{2}gf_\pi \cos\theta_C \partial_\mu \pi^+ W_\mu^-\,, \tag{5.125}$$

where g is the universal coupling constant of W bosons (as defined in the standard model), f_π is a phenomenological parameter with the dimension of a mass and θ_C is the so-called Cabibbo angle which will be discussed shortly. The amplitude corresponding to (5.125) is proportional to the pion momentum: contracting it with the W propagtor yields a multiplicative factor ${M_{\rm W}}^{-2}$ (cf. 4.15). The W-coupling to the leptons is

$$\mathcal{L} = \tfrac{1}{4}\sqrt{2}{\rm i}gW_\mu^+ \bar{\nu}_e \gamma_\mu (1+\gamma_5) e\,, \tag{5.126}$$

where e and ν_e denote the electron and neutrino fields. The amplitude corresponding to (5.126) is proportional to $\bar{u}_\nu \gamma_\mu (1+\gamma_5) v_e$; contracting this with the momentum of the decaying particle leads to $-{\rm i}m_e \bar{u}_\nu (1+\gamma_5) v_e$ (by using the Dirac equation for the spinors; $m_{\rm e}$ is the electron mass). Combining all the

Figure 5.8: The decay $\pi^+ \to e^+\nu_e$ mediated by a virtual W boson.

factors we find that the parameters $g_{\rm S}$; and $g_{\rm P}$ of the parametrization (5.99) are given by

$$\mathrm{i}g_{\rm S} = -ig_{\rm P} = \frac{G}{\sqrt{2}} m_{\rm e} f_\pi \cos\theta_{\rm C}\,, \tag{5.127}$$

where we have used (5.123).

The decay rate for $\pi^+ \to \mathrm{e}^+ + \nu_{\rm e}$ can now be read off from (5.117) and we find

$$\Gamma(\pi^+ \to \mathrm{e}^+\nu_{\rm e}) = \frac{G^2 f_\pi^2 \cos^2\theta_{\rm C}}{8\pi} \frac{m_{\rm e}^2(m_\pi^2 - m_{\rm e}^2)^2}{m_\pi^3}\,. \tag{5.128}$$

As shown in table 5.6 a comparison with the experimental result yields $|g_{\rm S}| = |g_{\rm P}| = 1.1 \times 10^{-7}$, so $f_\pi \cos\theta_{\rm C} \approx 128$ MeV. Using the fact that the W couples universally to muons and electrons, we may compare the decays $\pi^+ \to \mathrm{e}^+\nu_{\rm e}$ and $\pi^+ \to \mu^+\nu_\mu$. Note, however, that the effective decay constants $g_{\rm S}$ and $g_{\rm P}$ are proportional to the lepton mass, even though (5.125) and (5.126) contain no explicit mass factors. Furthermore phase space considerations are important since the muon and pion masses are comparable. Invoking lepton universality, we can straightforwardly use the above results (5.128) and conclude that

$$\frac{\Gamma(\pi^+ \to \mathrm{e}^+\nu_{\rm e})}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2}{m_\mu^2}\Big(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\Big)^2 = 1.23 \times 10^{-4}\,. \tag{5.129}$$

which agrees with the experimental results quoted in table 5.6.

For the leptonic decay of K mesons a similar situation holds. The effective coupling between K^+ and a virtual W^+ boson is written as in (5.125) except that $f_\pi \cos\theta_C$ is replaced by $f_{\rm K} \sin\theta_{\rm C}$ for reasons explained below. The experimental rate can be used to determine $f_{\rm K} \sin\theta_{\rm C} \approx 35$ MeV. Of course, the analogue of (5.129) should also hold

$$\frac{\Gamma(K^+ \to \mathrm{e}^+\nu_\mathrm{e})}{\Gamma(K^+ \to \mu^+\nu_\mu)} = \frac{m_\mathrm{e}^2}{m_\mu^2}\Big(\frac{m_K^2 - m_\mathrm{e}^2}{m_K^2 - m_\mu^2}\Big)^2 = 0.3 \times 10^{-4}\,. \tag{5.130}$$

as is confirmed by the results shown in table 5.6. We may apply the same arguments and examine decays of D or F mesons, but the branching ratios for their purely leptonic decays are small, so that they have not yet been identified experimentally. For this reason we have left blanks in the appropriate columns in table 5.6. Note that the charmed mesons can also decay into τ leptons, so that lepton universality can in principle be tested for this lepton generation by considering ratios such as

$$\frac{\Gamma(D^+ \to \tau^+\nu_\tau)}{\Gamma(D^+ \to \mu^+\nu_\mu)} = \frac{m_\tau^2}{m_\mu^2}\Big(\frac{m_D^2 - m_\tau^2}{m_D^2 - m_\mu^2}\Big)^2 = 1.23 \times 10^{-4}\,. \tag{5.131}$$

Universality for the τ lepton can also be verified by examining the inverse decay rates such as for $\tau^- \to \nu_\tau\pi^-$ or $\tau^- \to \nu_\tau\mathrm{K}^-$, which are now determined in terms of known parameters. Also the decay $\tau^- \to \rho^-\nu$ has been measured, but unfortunately decays such as $\rho^- \to \mu^-\bar{\nu}_\mu$ and $\rho^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}$ have unobservably small branching ratios, because ρ mesons decay predominantly into hadrons via the strong interactions. Therefore a comparison with the ρ decay mode is not feasible. On the other hand the comparison of purely leptonic three-body decays provides another test, since all three decays $\tau^- \to \mu^-\bar{\nu}_\mu\nu_\tau$, $\tau^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}\nu_\tau$ and $\mu^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}\nu_\mu$ have been identified. Table 5.7 lists some information on various τ decay modes.

Now that we have reviewed the evidence for lepton universality, let us examine whether the W bosons also couple universally to hadrons. At first sight this does not seem to be the case, since we have already found $f_\pi \cos\theta_\mathrm{C} \approx$ 128 MeV and $f_K \sin\theta_\mathrm{C} \approx$ 35 MeV. Because π and K mesons are strongly interacting one should take into account that the strong interactions affect the W boson coupling. The factorization of the π-W and K-W coupling in terms of $f_\pi \cos\theta_\mathrm{C}$ and $f_\mathrm{K} \sin\theta_\mathrm{C}$ should be seen as an attempt to separate between a difference in strength of the primary coupling of W to the mesons (parametrized by the Cabibbo angle θ_C and the effect of strong interaction corrections which modify the coupling. These corrections are contained in the phenomenological decay constants f_π and f_K. However, on the basis of the approximate SU(3) symmetry that holds for the low-mass hadrons f_π and f_K are expected to differ by at most 20 %. Therefore, one is forced to conclude that $\theta_\mathrm{C} \neq 0$, so that the primary coupling of W to the mesons is not universal.

However, the situation is more subtle. Whereas the leptons differ primarily by the fact that their masses are not equal, there are more profound differences between hadrons. These differences are characterized by additional quantum numbers, such as strangeness and isospin, and with respect to these quantum numbers the π and K mesons are rather different. By investigating a large number of weak semileptonic decays it has been found that the W coupling to hadrons in which the strangeness quantum number does not change, differs in strength from the universal coupling constant g by a factor $\cos\theta_C$ whereas the W coupling in which the strangeness quantum number changes (by one unit), is reduced by a factor $\sin\theta_C$. The important point is that the reduction of the W coupling by $\cos\theta_C$ or $\sin\theta_C$ is universal, i.e. it holds for all couplings of the W boson to hadrons.

The phenomenon of "Cabibbo universality" can be tested in a large number of semileptonic processes, but it should be realized that the W coupling to hadrons is always somewhat modified by the strong interactions. Therefore one must rely on additional symmetry properties of the strong interactions, such as isospin, SU(3), CVC (Conserved Vector Current hypothesis) and others. Nevertheless there is a striking agreement with experiment. The Cabibbo angle can for instance be measured by comparing neutron β-decay, $\mathrm{n} \to \mathrm{pe}^-\bar{\nu}_\mathrm{e}$, with μ-decay, $\mu^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}\nu_\mu$ or with other baryon decays such as $\Lambda \to \mathrm{pe}^-\bar{\nu}_\mathrm{e}$ or $\Sigma^- \to \mathrm{ne}^-\bar{\nu}_\mathrm{e}$. In the latter two decays strangeness changes by one unit, so that these processes require a $\sin\theta_C$ suppression; in neutron β-decay strangeness does not change so that one has $\cos\theta_C$. A similar comparison can be made for three-body meson decays, such $\pi^+ \to \pi^0\mathrm{e}^+\nu_\mathrm{e}$ and Kwhich are suppressed by $\cos\theta_C$ and $\sin\theta_C$ respectively. At present an average of all experimental data indicates the following value for the Cabibbo angle

$$\cos\theta_C = 0.9737 \pm 0.0025\,. \tag{5.132}$$

Hence we conclude that W couples universally to leptons and hadrons, but for a reason that we do not yet understand, the hadronic couplings have been rotated by a small amount characterized by θ_C. This rotation is responsible for the existence of strangeness-changing processes. The obvious question is whether this is just a simple rotation between particles of different strangeness, or whether other hadronic quantum numbers such as charm or bottom are involved as well. The latter turns out to be the case; according to what is presently known the rotation discussed above is a three-dimensional complex rotation parametrized by three angles and one phase factor. These mixing angles are best described in the context of the quark model on the basis of the standard electroweak gauge theory (cf. appendix G).

Having observed the presence of mixing angles for hadrons one might wonder why such angles seem absent for leptons. This fact is best explained in the standard model if one assumes that neutrinos are truly massless. This is one of the reasons why the measurement of neutrino masses is important. Unfortunately experiments in this field are very diffficult, and at present it is not clear whether the neutrino masses are finite or not.

Problems

5.1. Prove that the spin matrices (5.2) satisfy $\sigma_2\sigma_i\sigma_2 = -(\sigma_i)^*$, and use (5.4) to show that $\sigma_2\phi^*$ transforms as ϕ under rotations. Show, by expanding the exponential, that ($\mathbf{n}$ is a unit vector)

$$U(\theta,\hat{\mathbf{n}}) = \exp(\tfrac{1}{2}\mathrm{i}\theta\hat{\mathbf{n}}\cdot\sigma) = \cos\tfrac{1}{2}\theta\,\mathrm{I} + \mathrm{i}\sin\tfrac{1}{2}\theta\,\hat{\mathbf{n}}\cdot\sigma\,, \tag{1}$$

Prove also that,

$$U^{-1}(\theta,\hat{\mathbf{n}})\,\sigma_i\,U(\theta,\hat{\mathbf{n}}) = \sigma_j R^{-1}(\theta,\hat{\mathbf{n}})^j{}_i\,, \tag{2}$$

and find the three-dimensional rotation matrix $R(\theta,\hat{\mathbf{n}})$ with angle θ around a vector $\hat{\mathbf{n}}$. Compare your result with (C.50) in appendix C. Show that $\phi^\dagger\sigma_i\phi$ transforms as a vector under rotations.
By applying two successive rotations U_1, and U_2 show that (2) holds for the product $U_3 = U_1U_2$ and corresponding $R_3 = R_1R_2$. Hence we note a relation between the product rules of corresponding matrices U and matrices R. More precisely,

$$\begin{aligned} U(\theta_1,\hat{\mathbf{n}}_1)U(\theta_2,\hat{\mathbf{n}}_2) &= U(\theta_3,\hat{\mathbf{n_3}})\,,\\ R(\theta_1,\hat{\mathbf{n}}_1)R(\theta_2,\hat{\mathbf{n}}_2) &= R(\theta_3,\hat{\mathbf{n_3}})\,. \end{aligned} \tag{3}$$

Show that this correspondence implies that the matrices S_i and $\frac{1}{2}\sigma_i$ defined by the respective expansions of $R(\theta,\hat{\mathbf{n}}) = \mathrm{I}+\mathrm{i}\theta\,\hat{\mathbf{n}}\cdot\mathbf{S}+O(\theta^2)$ and $U(\theta,\hat{\mathbf{n}}) = \mathrm{I}+\frac{1}{2}\mathrm{i}\theta\,\hat{\mathbf{n}}\cdot\sigma+O(\theta^2)$, satisfy the same commutation relations, e.g.,

$$\left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right] = \mathrm{i}\varepsilon_{ijk}\frac{\sigma_k}{2}\,,\qquad [S_i,S_j] = \mathrm{i}\varepsilon_{ijk}S_k\,. \tag{4}$$

5.2. A Lorentz transformation on a four-vector $x^\mu \to x'^\mu = L^\mu{}_\nu\,x^\nu$ can he written infinitesimally as

$$L^\mu{}_\nu = \delta^\mu{}_\nu + \theta^\mu{}_\nu + \mathcal{O}(\theta^2)\,, \tag{1}$$

with $\theta^\rho{}_\mu\,\eta_{\rho\nu} + \eta_{\mu\rho}\,\theta^\rho{}_\mu$. Specify the $\theta^\mu{}_\nu$ that describe infinitesimal rotations and boosts. The Lorentz group has other representations than (1) where the transformations are matrices of a different dimensionality. A representation independent notation is therefore (cf. appendix A)

$$L = \mathrm{I} + \tfrac{1}{2}\theta_{\mu\nu}\,M^{\mu\nu} + \mathcal{O}(\theta^2)\,, \tag{2}$$

where the dimensionality of the matrices L and M is left unspecified. The $M^{\mu\nu}$ are called the *generators* of the Lorentz group. In the four-vector representation L and M are four-dimensional matrices and

$$(M^{\mu\nu})^\rho{}_\sigma = \eta^{\mu\rho}\,\delta^\nu{}_\sigma - \eta^{\nu\rho}\,\delta^\mu{}_\sigma\,. \tag{3}$$

Verify that the $M^{\mu\nu}$ must satisfy the commutation relation

$$[M^{\mu\nu},M^{\rho\sigma}] = -\eta^{\mu\rho}\,M^{\nu\sigma} + \eta^{\nu\rho}\,M^{\mu\sigma} - \eta^{\nu\sigma}\,M^{\mu\rho} + \eta^{\mu\sigma}\,M^{\nu\rho}\,. \tag{4}$$

Lorentz transformations acting on spinors take the form $U = \exp(\frac{1}{4}\mathrm{i}\theta_{\mu\nu}\sigma^{\mu\nu})$ so that the generators $M_{\mu\nu}$ for spinors are defined by

$$M^{\mu\nu} = \tfrac{1}{2}\mathrm{i}\sigma^{\mu\nu}\,, \tag{5}$$

Show that (5) also satisfies the commutation relation (4) (c.f. 5.12) thus confirming that the transformation (5.13) is a representation of the Lorentz group. Prove the relation

$$\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} = \tfrac{1}{2}\big[\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\big]\mathbf{I} - \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\,\gamma_5\,, \tag{6}$$

and use it to obtain an explicit expression for $U(\theta)$. Prove for infinitesimal transformations $U(\theta)$ and $L(\theta)$ that

$$U^{-1}(\theta)\,\gamma_\mu\,U(\theta) = \gamma_\nu\,(L^{-1})^\nu{}_\mu(\theta)\,. \tag{7}$$

Compare this result with (2) in the previous problem. Note that (7) can also be proven for finite transformations by using the results of appendix E. Further background material can be found in appendices A and C.
Using (7) show that $\bar\psi\psi$, $\mathrm{i}\bar\psi\gamma_5\psi$ transform as Lorentz scalars, $\mathrm{i}\bar\psi\gamma^\mu\psi$, $\mathrm{i}\bar\psi\gamma^\mu\gamma_5\psi$ as Lorentz vectors, and $\bar\psi\sigma^{\mu\nu}\psi$ as a Lorentz tensor.

5.3. Show, by using (5.11), that matrices which commute with spatial rotations in spinor space must be a linear combination of I, γ_4, γ_5 and $\gamma_5\gamma_4$. Show that only I and γ_5 commute with the Lorentz transformations. Under parity reversal a four-vector changes as $x_\mu \to x'_\mu = (2\delta_{\mu4} - 1)x_\mu$. Argue that the corresponding operator for spinors must satisfy

$$P^{-1}\gamma_\mu P = (2\delta_{\mu4} - 1)\gamma_\mu\,, \tag{1}$$

in order to have a parity-invariant Dirac equation.
Since parity reversal commutes with spatial rotations the above result demonstrates that P must be a linear combination of I, γ_4, γ_5 and $\gamma_5\gamma_4$. Show that the only unitary matrix satisfying (1) is $P = \mathrm{e}^{\mathrm{i}\alpha}\,\gamma_4$ where α is a phase factor. We will choose $P = \mathrm{i}\gamma_4$ so that under parity spinors transform as

$$\psi(\mathbf{x}, t) \xrightarrow{P} \psi'(\mathbf{x}, t) = \mathrm{i}\gamma_4\psi(-, t)\,. \tag{2}$$

To justify the choice of the phase factor in P note that a Majorana field, which in the conventions of this chapter satisfies

$$\psi^* = \mathrm{i}\gamma_2\psi\,, \tag{3}$$

transforms consistently under parity if the matrix on the right-hand side of (2) is $\pm\mathrm{i}\gamma_4$. Note also that with this choice of phase $P^2 = -1$. This is consistent with the fact that angular momentum states have intrinsic parity $P = (-)^l$ so that we can now generalize this result to half-integer spins (to appreciate this answer it is important to know that under a rotation over an angle 2π a spinor changes its sign; this double-valuedness of the spinor representation is discussed in appendix C). Argue from (2) that a parity transformation changes spinors $u(\mathbf{P})$ and $v(\mathbf{P})$

into $i\gamma_4 u(-\mathbf{P})$ and $i\gamma_4 v(-\mathbf{P})$ respectively. Use (5.63) to show that $u(\mathbf{P})\overset{P}{\to}+iu(-\mathbf{P})$, and likewise $v(\mathbf{P})\overset{P}{\to}iv(-\mathbf{P})$, $u(\mathbf{P})\overset{P}{\to}-iu(-\mathbf{P})$ and $v(\mathbf{P})\overset{P}{\to}+iv(-\mathbf{P})$. Hence, $u(\mathbf{P})$, $(\bar{u}(\mathbf{P}))$ and $\bar{v}(\mathbf{P})$, $(v(\mathbf{P}))$ describe an incoming (outgoing) particle and anti-particle, respectively, with intrinsic parity $+i(-i)$.

5.4. Calculate the πN scattering amplitude from (5.51) by attaching fermion spinors $\bar{u}(\mathbf{P}_2)$ and $u(\mathbf{P}_1)$. Consider the case of massless pions and the soft-pion limit where $Q_2 \to 0$. Show that in this limit $Q^2 \to 0$, $P_1 \to P_2$, and that the πN amplitude approaches

$$\bar{u}_\beta(\mathbf{P}_2)\mathcal{M}^{sr}_{ba,\beta\alpha}u_\alpha(\mathbf{P}_1) \to iG^2\delta_{ab}\delta^{rs}\frac{\bar{u}(\mathbf{P}_2)\not{Q}_2 u(\mathbf{P}_1)}{P_2\cdot Q_2} = -\frac{G^2}{M}\delta_{ab}\delta^{rs}\bar{u}(\mathbf{P}_2)u(\mathbf{P}_1)\,, \tag{1}$$

where in the last equation we have used (5.72) and (5.75).
We will attempt to construct an extension of the σ model of section 2.5 and problem 2.5 which yields a vanishing πN amplitude in the soft-pion limit. Obviously the term which cancels (1) above must come from interactions of the σ field with the nucleon. Parity and isospin considerations show that this coupling is proportional to $\sigma\bar{\mathrm{N}}\mathrm{N}$. This leads to a new diagram contributing to πN scattering in which the σ is exchanged between the nucleon and the pions.
From the Lagrangian in problem 2.6 show that the $\sigma\pi\pi$ term in $\mathcal{L}$ is

$$\mathcal{L}_{\sigma\pi\pi} = -\frac{\mu^2}{2v}\phi^2\sigma\,. \tag{2}$$

Construct the new contribution to the πN amplitude and determine the $\sigma\pi\pi$ coupling constant by requiring that (1) is now cancelled. Show that the fermionic part of the Lagrangian now reads

$$\mathcal{L} = -\bar{N}\not{\partial}N - \Big(M - \frac{G^2v^2}{M}\Big)\bar{N}N - G\bar{N}\Big(\frac{Gv}{M}\sigma' - i\gamma_5\tau\cdot\phi\Big)N\,, \tag{3}$$

where $\sigma' = \sigma + v$. This result is obviously invariant under isospin rotations under which

$$\delta N = \tfrac{1}{2}i\Lambda\cdot\tau N\,, \tag{4}$$

but we will now insist that the interaction term in (3) is invariant under an extension of the transformation (3) of problem 2.6. Show that this can be done provided $M = Gv$ and the combined transformations are

$$\delta\sigma = -\phi\cdot\xi\,,\quad \delta\phi = (\sigma+v)\xi\,,\quad \delta N = \tfrac{1}{2}i\xi\cdot\tau\gamma_5 N\,. \tag{5}$$

Use the fact that the latter transformation implies that

$$\delta\bar{N} = \tfrac{1}{2}i\bar{N}\gamma_5\xi\cdot\tau\,, \tag{6}$$

and observe that the mass term in (3) now vanishes. Combining the transformations (4) and (5) show that the result can be expressed in terms of chiral components $N_L = \frac{1}{2}(1+\gamma_5)N$ and $N_R = \frac{1}{2}(1-\gamma_5)N$ as

$$\delta N_L = \tfrac{1}{2}i\xi_L\cdot N_L \quad \delta N_R = \tfrac{1}{2}i\xi_R\cdot\tau N_R\,, \tag{7}$$

where $\xi_{\rm L}$ and $\xi_{\rm R}$ are equal to $\Lambda + \xi$ and $\Lambda - \xi$ respectively. Hence, the combined transformations (7) constitute two independent SU(2) transformations acting on different chiral components of the fermions.

5.5. Prove that the helicity operator (5.70) satisfies

$$2h(\mathbf{P}) = -\gamma_5 + O(m/|\mathbf{P}|)$$

for spinors $u(\mathbf{P})$ and $v(\mathbf{P})$ that satisfy the Dirac equation. This shows that for massless fermions, or fermions moving at relativistic velocities, the chiral projections $(1\pm\gamma_5)u(\mathbf{P})$ and $(1\pm\gamma_5)v(\mathbf{P})$ are helicity eigenstates with eigenvalues $\pm\frac{1}{2}$. Therefore the $1+\gamma_5$ components are often called left-handed and the $1-\gamma_5$ components are called right-handed.

5.6. Consider the Lagrangian (5.21). Show that it is invariant under the transformation $\psi \to \psi' = \exp({\rm i}\xi)\,\psi$ and, if $m = 0$, under $\psi \to \psi' = \exp({\rm i}\xi\gamma_5)\,\psi$ and that the corresponding Noether currents are $V_\mu = {\rm i}\bar\psi\gamma_\mu\psi$ and $A_\mu = {\rm i}\bar\psi\gamma_\mu\gamma_5\psi$, respectively. Verify that these currents transform as a vector or an axial vector under parity reversal, respectively.
Prove that the energy-momentum tensor (the Noether current associated with the translations $x^\mu \to x'^\mu = x^\mu - \xi^\mu$, (c.f. section 1.5) equals

$$T^{\mu\nu} = \bar\psi\gamma^\mu\partial^\nu\psi + \eta^{\mu\nu}\mathcal{L} \qquad (1)$$

and verify that it is indeed conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$. Show that (1) can be made symmetric in μ and ν by adding the improvement term,

$$t^{\mu\nu} = \tfrac{1}{4}{\rm i}\partial_\rho\big[\bar\psi(\gamma^\mu\sigma^{\nu\rho} + \gamma^\nu\sigma^{\mu\rho} + \gamma^\rho\sigma^{\mu\nu})\psi\big]\,, \qquad (2)$$

and using the equation of motion. Compare the result with (5.91). The term (2) is known as the Belinfante term. [F.J. Belinfante, Physica 7 (1940) 449.]

5.7. Consider the decay amplitude for a scalar particle ϕ decaying into a fermion $\rm F_1$, and an anti-fermion $\rm F_2$ given in (5.115). Assume that $M \approx m_1 + m_2$, so that in the rest frame $\rm F_1$ and $\rm F_2$ are produced almost at rest. Show that only the pseudoscalar transition, characterized by $g_{\rm P}$, contributes. Consider also the decay amplitude for a fermion $\rm F_1$ into a fermion $\rm F_2$ and a scalar ϕ. Assume $m_1 \approx m_2 + M$ so the $\rm F_2$ and ϕ are produced almost at rest. Show that only the scalar transition, characterized by $g_{\rm S}$, contributes. Explain these results by noting that a fermion-anti-fermion state has negative intrinsic parity (see problem 5.3). Repeat this analysis for the vector decay and explain the result.

5.8. Show for fermions how the irreducible self-energy diagrams can generally be decomposed in terms of four functions of p^2. Use the Dyson equation to write the full propagator as

$$\mathit{\Delta}(p) = \frac{1}{{\rm i}(2\pi)^4}\,\frac{1}{{\rm i}\not{p}[B(p^2) + \gamma_5 D(p^2)] + m[A(p^2) + {\rm i}\gamma_5 C(p^2)]} \qquad (1)$$

Below threshold any propagator should satisfy certain reality conditions. In this case $\gamma_0 \Delta^\dagger(p)\gamma^0 = -\Delta(p)$. Compare this result to the reality condition for a scalar field. Show that the functions A, B, C and D are real. In addition the functions A, B, C and D should satisfy certain positivity constraints which follow from unitarity. Prove that the propagator has poles for $p^2 = -M^2 = -m_0^2(A_0^2 + C_0^2)/(B_0^2 - D_0^2)$ where A_0, B_0, C_0 and D_0 denote $A(-M^2)$, $B(-M^2)$, $C(-M^2)$ and $D(-M^2)$, respectively. Show that near the pole (1) can be written as

$$\Delta(p) = \frac{1}{\mathrm{i}(2\pi)^4} \frac{-\mathrm{i}\not{p}(B_0 + \gamma_5 D_0) + m(A_0 - \mathrm{i}\gamma_5 C_0)}{E(p^2 + M^2)}\,, \tag{2}$$

with $E = B_0^2 - D_0^2 - 2M^2(B_0B_0' - D_0D_0') + 2m^2(A_0A_0' + C_0C_0')$, where A_0', B_0', C_0' and D_0' denote the derivatives of A, B, C and D with respect to p^2 taken at $p^2 = M^2$. Show that (2) can be written as

$$\Delta(p) = \frac{1}{\mathrm{i}(2\pi)^4}(a + b\gamma_5)\frac{1}{\mathrm{i}\not{P} + M}(a^* - b^*\gamma_5), \tag{3}$$

where

$$\begin{aligned}
|a|^2 &= \tfrac{1}{2}(B_0 + \frac{m}{M}A_0)E^{-1}\,, \\
|b|^2 &= \tfrac{1}{2}(B_0 - \frac{m}{M}A_0)E^{-1}\,, \\
ab^* &= -\,\tfrac{1}{2}(D_0 + \mathrm{i}\frac{m}{M}C_0)E^{-1}\,,
\end{aligned} \tag{4}$$

provided certain positivity requirements on the functions A, B, C and D are satisfied. Redefine the fermion fields in such a way that $a = 1$ and $b = 0$. Show that the field renormalization factor now contains γ_5.

5.9. Using the interaction (4.71) and the Lagrangian (5.39) for quantum electrodynamics one can calculate the decay rate for $\rho^0 \to \gamma(k) + \mathrm{e}^-(P_2) + \mathrm{e}^+(P_1)$. Write down the square of the amplitude and sum over the spins of the outgoing particles. Show that the result takes the form

$$\sum_{\text{pol}} |\mathcal{M}|^2 = 64e^2\frac{\alpha^2C^2}{q^4}(2k\cdot q q_\mu k_\nu - (k\cdot q)^2\delta_{\mu\nu} - q^2k_\mu k_\nu)W_{\mu\nu}\,, \tag{1}$$

where $q = P_1 + P_2$ and

$$\begin{aligned}
W_{\mu\nu} &= \sum_{ij}\bar{u}^i(\mathbf{P}_2)\gamma_\mu v^j(\mathbf{P}_1)\bar{v}^j(\mathbf{P}_1)\gamma_\nu u^i(\mathbf{P}_2) \\
&= -4[P_{1\mu}P_{2\nu} + P_{2\mu}P_{1\nu} - \delta_{\mu\nu}(P_1\cdot P_2 - m^2)]\,,
\end{aligned} \tag{2}$$

where the last result can be taken from (5.112). Evaluate in the rest frame of the pion and express it in terms of the photon energy ω and the parameter x defined in (3.101) as the difference of the fermion energies. Show that (1) yields

$$\sum \mathcal{M}|^2 = 512\pi C^2\alpha^3\frac{4m^2\omega^2 + m_\pi(\omega^2 + 4x^2)(m_\pi - 2\omega)}{(m_\pi - 2\omega)^2}\,. \tag{3}$$

Show that $0 < \omega < \omega_m$, where $\omega_m = \frac{1}{2}m_\pi^{-1}(m_\pi^2 - 4m^2)$ and $x_- \leqslant x \leqslant x_+$, where $x_\pm = \pm\frac{1}{2}\omega\beta$ with

$$\beta = \sqrt{\frac{m_\pi^2 - 2m_\pi\omega - 4m^2}{m_\pi^2 - 2m_\pi\omega}}\,. \tag{4}$$

Perform the x and ω integrations dropping terms proportional to m^2/m_π^2. Using the result (4.81) find that

$$\frac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = \frac{\alpha}{3\pi}\Big(2\ln\frac{m_\pi^2}{m^2} - 7\Big)\,. \tag{5}$$

This result for Dalitz decay is in excellent agreement with experiment. [R.H. Dalitz, Proc. Phys. Soc. London A 64 (1951) 667.]

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