3 | Cross sections and decay rates

In the preceding chapter we explained how to obtain a Feynman diagram expansion for Green's functions that describe the interaction between a number of external sources. Eventually, we are, however, interested in experimentally measurable quantities such as cross sections and decay rates. Intuitively it is obvious that the sources can be used to set up the incoming and outgoing parts of a scattering or a decay process. The physics of the interaction is in the invariant amplitude $\mathcal{M}$, which corresponds to the sum of all possible Feynman diagrams contributing to a certain scattering or decay process, with truncated external propagators and sources. In an actual scattering experiment, incoming particles are produced by an injector, they are subsequently accelerated and allowed to scatter, and finally the collision products are registered in a detector. The time scales are such that the collision time is much shorter than the acceleration period or the time of flight from the collision point to the detector. Hence it makes sense to think of a particle, described by a quantum-mechanical wave packet starting at $t = -\infty$, colliding with some other particle at $t \approx 0$, and the collision products being detected at $t = +\infty$.

The interaction between arbitrary sources does not necessarily correspond to such a temporal sequence of events and we have to study what happens to the Green's function when the time scale is expanded to the situation that the initial sources are pushed to $t = -\infty$, while the final "sinks" are pushed to $t = +\infty$. As we have already demonstrated in the previous chapter the field excitations that survive in this limit satisfy the relativistic dispersion law of a free particle. Hence we identify physical particles by the fact that they can propagate during long time intervals, in contrast to "virtual" particles which only exist on an extremely short time scale Δt determined by the uncertainty relation $\Delta E \Delta t \approx \hbar$, where ΔE characterizes the energy difference between the virtual and the physical particle. Once the desired temporal sequence of events is imposed, we must make a proper identification of the scattering amplitude. The final amplitude that is obtained from the Green's function by truncation of external sources and propagators must be normalized appropriately, and so must the external lines in order that they correspond to the absorption or emission of precisely one particle. We start this discussion with a short treatment of relativistic wave packets and conservation laws. Subsequently we examine the large-time limit of localized external sources, and discuss the normalization factors for the probability amplitude. At that point it becomes obvious how to define the correct differential cross section for which we write down a master formula (or relativistic "Golden Rule"). In the other sections of this chapter we discuss the kinematics of scattering reactions, the decay lifetimes for unstable particles and a variety of applications. We remind the reader that we will consistently use units such that $c = \hbar = 1$.

3.1. Relativistic wave packets

In an experiment, electromagnetic confinement fields steer particle bunches through an accelerating device to form a particle beam. For the time scales that we are considering, this incoming beam is described by a wave packet extending over a small region in position space and momentum space. Such a wave packet which we will denote by $f(x)$, consists of a superposition of plane wavcs with *positive* frequencies

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}}\, f(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega(\mathbf{p})t} \,, \tag{3.1}$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$; the motivation for choosing the (conventional) normalization factor $[2\omega(\mathbf{p})]^{-1/2}$ will become clear shortly. This wave function satisfies the free Klein-Gordon equation, i.e.

$$\begin{aligned}(\Box - m^2) f(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}}\, f(\mathbf{p}) \\ &\quad \times \big(- \mathbf{p}^2 + \omega^2(\mathbf{p}) - m^2 \big)\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega(\mathbf{p})t} \\ &= 0 \,. \end{aligned} \tag{3.2}$$

Definition (3.1) can be easily extended to cover the case that the particles carry spin or other internal degrees of freedom by specifying the wave function for each of the corresponding components. For example, massive particles of spin-1 are described by

$$f_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}} \sum_{\lambda=1}^{3} f^{(\lambda)}(\mathbf{p})\, \varepsilon_\mu(\mathbf{p}, \lambda)\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega(\mathbf{p})t} \,, \tag{3.3}$$

where the three independent polarization spinors $\varepsilon_\mu(\mathbf{p}, \lambda)$ are transverse with respect to the four-momentum $p^\mu = (\mathbf{p}, \omega(\mathbf{p})$:

$$p^\mu \varepsilon_\mu(\mathbf{p}, \lambda) = 0 \,. \tag{3.4}$$

To specify that we are discussing wave packets peaked at $\mathbf{p} = \mathbf{P}$, we sometimes use the notation $f_{\mathbf{P}}(x)$ or $f_{\mathbf{P}}(\mathbf{p})$. The packet is then reasonably localized in position and in momentum space and travels with a group velocity $\mathbf{P}/\omega(\mathbf{P})$. The time dependence of the packet is governed by the Klein-Gordon equation, which in the nonrelativistic limit leads to the same result as in Schrödinger quantum mechanics. We note that these wave functions and wave packets have the standard probability interpretation as in regular quantum mechanics. Hence they are complex functions, although the corresponding fields in the Lagrangian may be real. For instance, the field that describes neutral pions is real, but the corresponding wave function is complex. Note also that particle and antiparticle wave functions are defined in the same way.

In order to have a consistent probability interpretation for the above wave functions one must be able to define a conserved probability current $j^\mu(x)$ in terms of the wave function. An obvious candidate is

$$\begin{aligned} j^\mu(x) &= -\,\mathrm{i}f^*(x)[\partial^\mu f(x)] + \mathrm{i}[\partial^\mu f^*(x)]f(x) \\ &\equiv -\,\mathrm{i}f^*(x)\overleftrightarrow{\partial}^\mu f(x)\,, \end{aligned} \tag{3.5}$$

which is conserved by virtue of the Klein Gordon equation (3.2),

$$\begin{aligned} \partial_\mu j^\mu(x) &= -\,\mathrm{i}f*(x)[\Box f(x)] + \mathrm{i}[\Box f^*(x)]f(x) \\ &= 0\,. \end{aligned} \tag{3.6}$$

Although (3.5) is identical in form to the Noether current (1.100) (up to a overall sign) associated with phase transformations on a complex scalar field, its interpretation is fundamentally different. Unlike the general scalar field appearing in (1.100) the wave function (3.1) contains only the *positive* frequency modes. This aspect is crucial in interpreting the current as a probability current. The Noether current (1.100) is *not* a probability current; it can be interpreted as an (electric) charge density current for which the associated charge can take both positive and negative values (see problem 3.1).

The spatial components $\mathbf{j}(x)$ of the current (3.5) define the probability current density, and its fourth component j^0 defines the probability density. Therefore probability is locally conserved owing to (3.6). It is now straightforward to evaluate the current in terms of the wave function,

$$\begin{aligned} j^\mu(x) = (2\pi)^{-3} \int \mathrm{d}^3p\,\mathrm{d}^3p'\, f^*(\mathbf{p}')\, f(\mathbf{p}) \\ \times\, \frac{(p+p')^\mu}{\sqrt{4\,\omega(\mathbf{p})\,\omega(\mathbf{p}')}}\, \mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}-\mathrm{i}(\omega(\mathbf{p})-\omega(\mathbf{p}'))t}\,, \end{aligned} \tag{3.7}$$

where $p^\mu = (\mathbf{p}, \omega(\mathbf{p}))$ and $p'^\mu = (\mathbf{p}', \omega(\mathbf{p}'))$. The probability density j^0 now defines the (positive) probability of finding a particle in a volume element d^3x at position $\mathbf{x}$ and time t. This interpretation is confirmed by integrating this expression over space, where one easily obtains

$$\int \mathrm{d}^3x\, j^0(\mathbf{x},t) = \int \mathrm{d}^3p\, |f(\mathbf{p})|^2\, \frac{p^\mu}{\omega(\mathbf{p})}\,, \tag{3.8}$$

which is independent of time. Thes result for j^0 then identifies the probability of finding a particle with momentum $\mathbf{p}$ in a volume element d^3p as

$$W_{\mathbf{p}}\, \mathrm{d}^3p = |f(\mathbf{p})|^2\, \mathrm{d}^3p\,. \tag{3.9}$$

The normalization factor $[2\omega(\mathbf{p})]^{-1/2}$ in (3.1) was chosen such as to obtain this simple form for (3.9).

Further confrmation of this probability interpretation comes from considering the total energy and momentum carried by the wave packet from the expression for the energy-momentum tensor $T_{\mu\nu}$ for a free complex scalar field (cf. 1.105),

$$T^{\mu\nu} = \partial^\mu f^*\, \partial^\nu f + \partial^\nu f^*\, \partial^\mu f - \eta^{\mu\nu}\big(|\partial_\rho f|^2 + m^2|f|^2\big)\,. \tag{3.10}$$

From the Klein-Gordon equation (3.2) it follows that $T^{\mu\nu}$ is conserved,

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{3.11}$$

Substituting (3.1) into (3.10) one obtains

$$\begin{aligned} T^{\mu\nu}(x) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3p\, \mathrm{d}^3p'}{\sqrt{4\,\omega(\mathbf{p})\,\omega(\mathbf{p}')}}\, f^*(\mathbf{p}')\, f(\mathbf{p})\, \mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}-\mathrm{i}(\omega-\omega')t} \\ \times \big(p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu}\, p\cdot p' - \eta^{\mu\nu} m^2\big)\,, \end{aligned} \tag{3.12}$$

with the same definition of $\omega(\mathbf{p})$ and $\omega(\mathbf{p}')$ as before. Integrating this expression over space we find

$$\int \mathrm{d}^3x\, T^{\mu\nu}(\mathbf{x},t) = \int \mathrm{d}^3p\, |f(\mathbf{p})|^2\, \frac{p^\mu p^\nu}{\omega(\mathbf{p})}\,, \tag{3.13}$$

which is again independent of time. The energy and momentum of the wave packet then follow from

$$\int \mathrm{d}^3x\, T^{\mu 0}(\mathbf{x},t) = \int \mathrm{d}^3p\, |f(\mathbf{p})|^2\, p^\mu\,, \tag{3.14}$$

which is consistent with the probability interpretation defined by (3.9).

3.2. The probability amplitude

The aim of this section is to construct an appropriately normalized expression for the quantum-mechanical probability amplitude in terms of Feynman diagrams. We start from an n-point Green's function that describes the correlation between n external sources. Such a Green's function consists of the sum of all possible Feynman diagrams with n external lines, including both tree diagrams and diagrams with closed loops. According to the description given in the introduction we want to identify this with a scattering process by pushing the external sources to times that are large compared to the typical time scale of the interactions involved. In this description we may safely restrict ourselves to *connected* diagrams only, because there is no causal relation between sources attached to disconnected pieces of a Feynman diagram. A connected Green's function can be truncated by removing the propagators associated with its external lines.

We should point out here that when loop diagrams are included the word propagator will always refer to the full propagator, i.e., the sum of all connected Feynman graphs with two external lines. This aspect is not directly relevant for the discussion in this section, so that one may pretend to be in the context of tree diagrams only. Including of loop diagrams is, however, essential in order to discuss unstable particles, a subject that we shall be dealing with in section 3.6.

Any Green's function can be written as an invariant amplitude $\mathcal{M}$, which depends on the external momenta, multiplied by a propagator for each of the external lines, and an overall δ-function to preserve energy and momentum conservation. Hence we have the factorization

$$G(k) = \Big\{ \prod_j \Big(\int \mathrm{d}^4k_j \, \mathrm{e}^{\mathrm{i}k_j\cdot x_j} \varDelta^{(j)}(k_j) \Big) \Big\} \, \mathrm{i}(2\pi)^4 \, \delta^4(\textstyle\sum_i k_i) \, \mathcal{M}(k) \,, \qquad (3.15)$$

which we have pictorially presented in fig. 3.1, where $\varDelta^{(i)}$ denotes the propagator for the i-th line, with k_i its (incoming) four-momentum and x_i its endpoint. In the previous chapter the effect of moving the endpoint of one of the external lines in G to infinite time was seen to be controlled by the poles at real values of k_0, where k_μ is the momentum assigned to the external line. The generic structure of the Feynman diagrams indicates that such poles originate from the propagator of the external line, so that the large-time behaviour is governed by momenta that satisfy the relativistic dispersion law $k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}$. Depending on whether one considers large positive or large negative times, and on the particular momentum assigned to the propagator, one identifies these contributions with those from either particles or antiparticles. Recall that particles and antiparticles are equivalent as long as both endpoints of a propagator refer to the same fields. This is not always true for complex fields. In that case propagator lines may carry an orientation (charge flow) which must be conserved in the diagram; usually one assigns the momentum along the direction of this orientation.

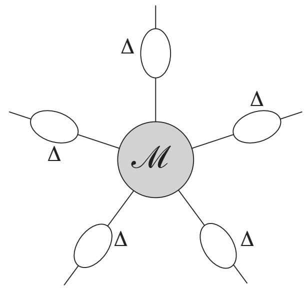


Figure 3.1: A graphical representation of the factorization of the five-point Green's function G into an invariant amplitude $\mathcal{M}$ and five propagators Δ according to (3.15).

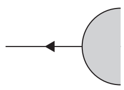
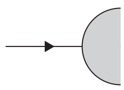
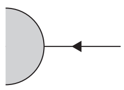
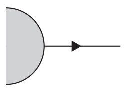
When the endpoint of one of the external lines is moved to infinite time one may distinguish four different situations, which are summarized in table 3.1. In this table the second column indicates the external line and its attachment to the main part of the Feynman diagram. The arrow of the line indicates the assignment of the line momentum (and possibly its intrinsic orientation). The external lines are drawn such that the time direction is to the left. Note that, irrespective of the direction of the momentum assigned to the external line, the energy of the (anti)particle is always positive and equal to $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. In the absence of an intrinsic orientation, particles and anti-particles are equivalent.

Moving the endpoints of all external lines to infinite (positive or negative) times, the momenta assigned to the external lines take values from one of the four categories listed in table 3.1 (assuming that this is consistent with energy-momentum conservation). To discuss this limit in a more rigorous fashion one needs to switch off all interactions adiabatically at large times, but such subtleties will be ignored. Hence in the infinite-time limit all propagators are replaced by their residue factor at the poles, $[(2\pi)^3 2\omega(\mathbf{k})]^{-1}$, irrespective of whether they represent incoming or outgoing (anti)particles, and we find that (3.15) asymptotically approaches the form

$$G^{\mathrm{as}} = \mathrm{i}(2\pi)^4 \delta^4(\textstyle\sum_i k_i)\, \mathcal{M}(\mathbf{k}_i, \pm\omega_i(\mathbf{k}_i)) \prod_j [(2\pi)^3 2\omega_j(\mathbf{k}_j)]^{-1} \,. \tag{3.16}$$

It is this quantity that we would like to identify (modulo a normalization factor to be discussed below) as the quantum-mechanical probability amplitude for a scattering process of (anti)particles. The probability amplitude $\mathcal{A}$ is a

Table 3.1: The four different assignments of external line momenta corresponding to incoming and outgoing particles and anti-particles.[a] The direction of time is from right to left. It is assumed that the line momenta are directed along the intrinsic line orientations. In the absence of such an orientation, particles and anti-particles are equivalent.

External	line	Situation
1. $t \to +\infty$		Outgoing particle with four-momentum $P^\mu = k^\mu$, and $k^0 = \omega(\mathbf{k})$
2. $t \to +\infty$		Outgoing anti-particle with four-momentum $P^\mu = -k^\mu$, and $k^0 = -\omega(\mathbf{k})$
3. $t \to -\infty$		Incoming particle with four-momentum $P^\mu = k^\mu$, and $k^0 = \omega(\mathbf{k})$
4. $t \to -\infty$		Incoming anti-particle with four-momentum $P^\mu = -k^\mu$, and $k^0 = -\omega(\mathbf{k})$

[a] P^μ denotes the (anti)particle momentum, while k^μ is the momentum assigned to the external line in the diagram.

function of the momenta of the incoming and outgoing particles involved in a certain scattering process. Given the wave functions for the incoming particles it will determine the probability for producing the outgoing particles with certain momenta. Hence we find the probability for the external particles to be produced in a volume element d^3P at momentum $\mathbf{P}$ by multiplying the amplitude for a certain process by the wave functions of the incoming particles, integrating over the momenta of these particles, and finally squaring, i.e.,

$$\mathrm{d}W(\mathbf{P}^{\text{out}}) = \Big| \int \mathrm{d}^3P^{\text{in}} \mathcal{A}(\mathbf{P}^{\text{in}};\mathbf{P}^{\text{out}}) f^{\text{in}}(\mathbf{P}^{\text{in}}) \Big|^2 \mathrm{d}^3P^{\text{out}}\,, \tag{3.17}$$

where $\mathbf{P}^{\text{in}}$ and $\mathbf{P}^{\text{out}}$ generically denote all incoming and outgoing particle momenta, respectively, and $f^{\text{in}}(\mathbf{P}^{\text{in}})$ is the wave function for all incoming particles, which can be written as a product over single-particle wave functions. Therefore, if there are n incoming (outgoing) particles then $\mathrm{d}^3P^{\text{in(out)}}$ denotes a $3n$-dimensional phase space volume.

Before identifying (3.16) with the probability amplitude we must first discuss its normalization. So far the normalization of the Green's function has been done in an ad hoc way. We have started with the fields as defined in the original Lagrangian, and calculated the corresponding Green's function, where the endpoint of each external line corresponds to the field taken at some point in space-time. If we had assumed different normalization conventions for the fields, then the corresponding Green's functions would acquire a different normalization. To determine the correct normalization of the probability amplitude, we therefore require that the wave function for a single (stable) particle, at the time of emission by a far away source, is the same wave function when the particle is detected at a much later time (provided that there have been no interactions with other particles or external fields). The amplitude that relates the two wave functions is then equal to unity. However, if we identify this amplitude with the asymptotic value of the Green's functions, we find that it corresponds to the propagator whose asymptotic value is equal to the residue at the pole: $[(2\pi)^3 2\omega(\mathbf{k})]^{-1}$. Therefore we conclude that we have been using improperly normalized fields. The correctly normalized amplitude is obtained after rescaling the fields with a factor $[(2\pi)^3 2\omega(\mathbf{k})]^{-1/2}$, so that the pole residue in the propagator becomes equal to unity. Hence we must multiply the invariant amplitude in (3.16) by a factor $[(2\pi)^3 2\omega(\mathbf{k})]^{1/2}$ for each external line; after a partial cancellation with a similar factor in (3.16), we thus identify as the probability amplitude

$$\mathcal{A} = \mathrm{i}(2\pi)^4\, \delta^4\big(\textstyle\sum_i k_i\big)\, \mathcal{M}(\mathbf{k}_i, \pm\omega_i(\mathbf{k}_i)) \displaystyle\prod_j \frac{1}{\sqrt{(2\pi)^3 2\, \omega_j(\mathbf{k}_j)}}\,. \tag{3.18}$$

We should emphasize that the above criterion for the normalization of the external lines is applicable to more general situations. For example, if the particles carry spin or carry certain internal degrees of freedom, then the propagators will take the form of a matrix. Such degrees of freedom have the same mass, so that the propagator at the pole takes the form,

$$\varDelta_{ij}(k) \approx \frac{1}{\mathrm{i}(2\pi)^4}\frac{Z_{ij}}{k^2+m^2}\,, \tag{3.19}$$

where the residue Z_{ij} is now a matrix, and the indices i and j label the various fields. Therefore the residue factors in (3.18) must now contain the matrix $(\sqrt{Z})_{ij}$. Normally, one chooses fields such that Z_{ij} becomes diagonal, so that one simply multiplies (3.18) with the square root of the relevant eigenvalue for each external line.

Another example concerns the normalization of the amplitude when loop diagrams are included. As shown in section 2.6 the full propagator is now a complicated function of the momentum (cf. 2.66). If it exhibits a pole at some value of k^2, this value will no longer coincide with the original location of the pole. Since it is the pole of the full propagator that will determine the large-time behaviour, it is this pole that must correspond to the physical mass. Near the pole the full propagator can be approximated by

$$\varDelta(k) \approx \frac{1}{\mathrm{i}(2\pi)^4}\frac{Z}{k^2+m^2}\,, \tag{3.20}$$

where the residue Z is now a complicated (but in principle calculable) constant. In the calculation of physical amplitudes its square root must be taken into account according to the normalization criterion outlined above. Hence there will be a factor $\sqrt{Z}$ for each external line in (3.18).

3.3. Cross sections

Now that we have defined the probability amplitude in the general case we can make the connection with experimentally measurable quantities. A typical experiment involves the scattering of two particles with momenta p_1 and p_2 into a final state of n particles of momenta $k_j (i = 1, 2, ..., n)$. The incoming particles are characterized by wave functions $f_1(\mathbf{q})$ and $f_2(\mathbf{q})$ with a small momentum spread around the values $\mathbf{p}_1$ and $\mathbf{p}_2$. The amplitude for this process is defined in (3.18)

$$\begin{aligned}\mathcal{A} = 2\pi \int \frac{\mathrm{d}^3q_1\mathrm{d}^3q_2}{\sqrt{4\omega_1(\mathbf{q}_1)\omega_2(\mathbf{q}_2)}}\, f_1(\mathbf{q_1})\, f_2(\mathbf{q_2}) \\ \times\, \mathrm{i}\delta^4(q_1+q_2-\sum_i k_i)\,\mathcal{M}(q_1,q_2;k_i)\,\prod_{j=1}^{n}\frac{1}{\sqrt{(2\pi)^3 2\,\omega_j(\mathbf{k_j})}}\,,\end{aligned} \tag{3.21}$$

where we have already integrated over the wave functions of the incoming particles. The square of the absolute value of this expression will therefore define the transition probability for this process.

Before continuing, it is convenient to first define

$$F(P) = \int \mathrm{d}^4x\, f_1(x)\, f_2(x)\, \mathrm{e}^{-\mathrm{i}P\cdot x}\,, \tag{3.22}$$

where $f_1(x)$ and $f_2(x)$ are the wave functions of the incoming particles transformed to position space. It is not difficult to show, using (3.1), that

$$F(P) = 2\pi \int \frac{\mathrm{d}^3q_1\, \mathrm{d}^3q_2}{\sqrt{4\omega_1(\mathbf{q}_1)\omega_2(\mathbf{q}_2)}}\, f_1(\mathbf{q_1})\, f_2(\mathbf{q_2})\, \delta^4(q_1+q_2-P)\,. \tag{3.23}$$

Comparing this to (3.21) and insisting that the wave functions are sharply peaked wave packets which allow only small variations of $\mathbf{q_1}$ and $\mathbf{q_2}$ around $\mathbf{p}_1$ and $\mathbf{p}_2$, we can write:

$$\mathcal{A} = \mathrm{i}\mathcal{M}(p_1, p_2; k_i)\, F(P) \prod_{j=1}^{n} \frac{1}{\sqrt{(2\pi)^3 2\,\omega_j(\mathbf{k}_j)}}\,, \tag{3.24}$$

with $P = \sum_i p_i$. The transition probability to find the n outgoing particles in the final state in volume elements d^3k_i at momenta $\mathbf{k}_i$, is then given by

$$\mathrm{d}W = |\mathcal{M}(p_1, p_2; k_i)|^2\, |F(P)|^2 \prod_{j=1}^{n} \frac{\mathrm{d}^3k_j}{(2\pi)^3 2\omega_j(\mathbf{k}_j)}\,. \tag{3.25}$$

Let us first consider $|F(P)|^2$. Starting from (3.22) we have

$$|F(P)|^2 = \int \mathrm{d}^4x\, \mathrm{d}^4y\, f_1(x)\, f_1^*(y)\, f_2(x)\, f_2^*(y)\, \mathbf{e}^{-\mathrm{i}P\cdot(x-y)}\,. \tag{3.26}$$

Integrating over P_μ, we find

$$\int \mathrm{d}^4P \left|F(P)\right|^2 = (2\pi)^4 \int \mathrm{d}^4x\, |f_1(x)|^2\, |f_2(x)|^2\,. \tag{3.27}$$

Since the wave functions are sharply peaked, we may express $|f|^2$ in terms of the probability density according to (3.7). The result equals

$$\int \mathrm{d}^4P \left|F(P)\right|^2 = (2\pi)^4 \frac{\int \mathrm{d}^4x\, \rho_1(x)\, \rho_2(x)}{4\,\omega_1(\mathbf{p}_1)\, \omega_2(\mathbf{p}_2)}\,, \tag{3.28}$$

where ρ_1 and ρ_2 are now the probability densities for finding the two particles at a certain point in space-time. The integral $\int d^4x\, \rho_1(x)\, \rho_2(x)$ therefore characterizes the interaction volume, i.e. it measures the volume and the time span within which the incoming particles are sufficiently close to each other to interact. Note that this is the reason why we have been careful in introducing wave functions for the incoming particles, which, although sharply peaked, still have a *finite* momentum spread. For a plane wave $|f(x)| = 1$, so that the probability for finding the particle becomes constant over space. Therefore the overlap (3.27) of the two wave functions would diverge and the interaction volume for this process becomes infinite. Note also that the dimensionless expression (3.27), and therefore (3.28), are Lorentz invariant because we have integrated over all space and time.

Of course the function $|F(P)|^2$ must be sharply peaked around $P = p_1 + p_2$. Therefore we may use (3.28) in order to approximate $|F(P)|^2$ by

$$|F(P)|^2 = (2\pi)^4\,\delta^4\big(P - p_1 - p_2\big)\,\frac{\int \mathrm{d}^4x\,\rho_1(x)\,\rho_2(x)}{4\,\omega_1(\mathbf{p}_1)\,\omega_2(\mathbf{p}_2)}\,. \tag{3.29}$$

Using this formula we can write the probability to find n particles in the final state with momenta $\mathbf{k}_i$ in a volume element d^3k_i as

$$\begin{aligned}\mathrm{d}W = {}&(2\pi)^4\delta^4(P - p_1 - p_2 - \textstyle\sum_i k_i)\,|\mathcal{M}(p_1, p_2; k_i)|^2\\ &\times \frac{\int \mathrm{d}^4x\,\rho_1(x)\,\rho_2(x)}{4\,\omega_1(\mathbf{p}_1)\,\omega_2(\mathbf{p}_2)}\prod_{j=1}^{n}\frac{\mathrm{d}^3k_j}{(2\pi)^3\,2\,\omega_j(\mathbf{k}_j)}\,.\end{aligned} \tag{3.30}$$

In order to compare results from different experiments we must define a quantity which is independent of the wave functions of the incoming particles contained in ρ_1 and ρ_2. This defines the *differential cross section*,

$$\begin{aligned}\mathrm{d}\sigma &= \frac{1}{\sqrt{(p_1\cdot p_2)^2 - m_1^2\,m_2^2}}\,\frac{\omega_1(\mathbf{p_1})\,\omega_2(\mathbf{p_2})}{\int\,\mathrm{d}^4x\,\rho_1(x)\,\rho_2(x)}\,\mathrm{d}W\\ &= \frac{(2\pi)^4\,\delta(p_1 + p_2 - \sum_i k_i)}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}\,|\mathcal{M}(p_1, p_2; k_i)|^2\,\prod_{j=1}^{n}\frac{\mathrm{d}^3k_j}{(2\pi)^3\,2\omega_j(\mathbf{k}_j)}\,,\end{aligned} \tag{3.31}$$

which has the dimenson of area.

Let us first discuss the proportionality factor between $\mathrm{d}\sigma$ and $\mathrm{d}W$. In an actual scattering experiment one has a situation where two particle beams collide, or where one beam scatters off some fixed target. Obviously, not all the particles in the beam or the target are characterized by exactly the same wave function. In practice there is always a certain spread in the momenta of the incoming particles and also in a fixed target the particles are never quite at rest due to Fermi motion. In general one therefore has to integrate (3.30) over a statistical distribution that gives the number of particles characterized by a certain wave function. Assuming that the spread in this distribution is very small, we may ignore this integration. In that case the densities ρ_1 and ρ_2 are equal the particle densities in the beams and/or the target up to a normalization constant. With this normalization the proportionality factor in (3.31) is then the *integrated luminosity* (which is Lorentz invariant),

$$\int \mathrm{d}t\, L = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{\omega_1(\mathbf{p}_1)\, \omega_2(\mathbf{p}_2)} \int \mathrm{d}^4x, \rho_1(x)\, \rho_2(x)\,, \tag{3.32}$$

which is a measure of the number of scatterings that are possible in the experimental situation described by ρ_1 and ρ_2. The transition rate now expresses the number of events with the outgoing particles in the required configuration per unit of time as

$$\mathrm{d}N = \mathrm{d}\sigma\, L\, \mathrm{d}t\,. \tag{3.33}$$

Of course, in an experiment, one prefers to have a stable situation in which the particle beams and/or the target remain unchanged for a long time. In that case, the *luminosity* L will be constant.

For an experimental determination of the differential cross section, one obviously needs to know the luminosity. If one has a good knowledge of the characteristics of the incoming particle beam, one may attempt to calculate L directly. If this is not the case, one uses a well-known interaction cross section to monitor the beam, and compares the number of events under study to the number of events from the known interaction. The latter is often done at experiments with colliding beams of electrons and positrons, where some of the purely electromagnetic cross sections are known to a high degree of accuracy. In colliding beam experiments the magnitude of the luminosity and thus the event rate is very small. Therefore the luminosity is one of the most crucial parameters for a colliding beam storage ring accelerator. It is usually measured in $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. Existing machines have luminosities of the order of $10^{28} - 10^{33}$ $\mathrm{cm}^{-2}\mathrm{s}^{-1}$. Hadronic total cross sections are usually of the order of millibarns ($1\,\mathrm{mb} = 10^{-27}\,\mathrm{cm}^2$); electromagnetic total cross sections are of the order of nanobarns ($1\,\mathrm{nb} = 10^{-33}\,\mathrm{cm}^2$); weak interaction total cross sections are of the order of femtobarns ($1\,\mathrm{fb} = 10^{-39}\,\mathrm{cm}^2$). Typical event rates in these processes, assuming $L \approx 10^{30}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ are, therefore, of order $10^3\,\mathrm{s}^{-1}$, $10^{-3}\,\mathrm{s}^{-1}$, $10^{-9}\,\mathrm{s}^{-1}$, respectively. These numbers illustrate the difficulty of measuring weak interaction effects in colliding beam experiments.

In order to get an understanding of the meaning of cross sections and luminosity, let us consider as an example the experimental set-up at a fixed-target experiment in the target rest frame. We have a target of volume V_{target} placed in a particle beam (see fig. 3.2). The densities ρ_1 and ρ_2 are taken constant, and the number of target particles is equal to $n_{\text{target}} = \rho_2\, V_{\text{target}}$ where ρ_2 is

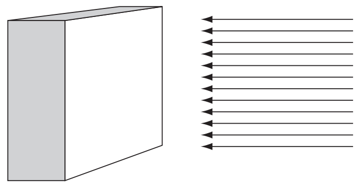


Figure 3.2: A target in a beam of particles.

the particle density of the target. The integral in (3.32) is therefore equal to

$$\begin{aligned}\int \mathrm{d}^4x\,\rho_1(x)\,\rho_2(x) &= \int \mathrm{d}t\,\rho_1\,\rho_2\,V_{\text{target}}\\ &= \int \mathrm{d}t\,\rho_1\,n_{\text{target}}\,. \end{aligned}\tag{3.34}$$

Let us now consider the factor in (3.32). We first evaluate

$$\begin{aligned}(\omega_2\mathbf{p}_1-\omega_1\mathbf{p}_2)^2 &= \omega_2^2\mathbf{p}_1^2+\omega_1^2\mathbf{p}_2^2-2\omega_1\omega_2\mathbf{p}_1\cdot\mathbf{p}_2\\ &= (\mathbf{p}_1\cdot\mathbf{p}_2-\omega_1\omega_2)^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2-\omega_1^2\omega_2^2+\omega_2^2\mathbf{p}_1^2+\omega_1^2\mathbf{p}_2^2\\ &= (p_1\cdot p_2)^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2+\mathbf{p}_1^2\,\mathbf{p}_2^2-m_1^2\,m_2^2\,,\end{aligned}\tag{3.35}$$

so that

$$(p_1\cdot p_2)^2-m_1^2\,m_2^2=(\omega_2\mathbf{p}_1-\omega_1\mathbf{p}_2)^2+(\mathbf{p}_1\cdot\mathbf{p}_2)^2-\mathbf{p}_1^2\mathbf{p}_2^2\,.\tag{3.36}$$

The last two terms in (3.36) may be dropped since in the target rest frame, we have $\mathbf{p}_2=0$ (actually it is sufficient to assume that $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear). The first term is related to the relative velocity between the colliding particles, via:

$$v_{\text{rel}}=\left|\frac{\mathbf{p}_1}{\omega_1}-\frac{\mathbf{p}_2}{\omega_2}\right|,\tag{3.37}$$

so that

$$\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}=\omega_1\,\omega_2\,v_{\text{rel}}\,.\tag{3.38}$$

Combining this result with (3.34) shows that the luminosity is equal to

$$L=\rho_1\,v_{\text{rel}}\,n_{\text{target}}=\text{flux}\cdot n_{\text{target}}\,,\tag{3.39}$$

where the flux is defined as the number of incoming particles passing through a unit area per unit of time. This then identifies the differential cross section as the number of events per second per unit incoming flux per target particle. The luminosity in fixed target experiments is much higher than for colliding beams. Typical flux factors are $10^{10}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ for hadron beams, $10^{8}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ for electron beams and $10^{6}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ for neutrino beams, whereas a target contains of the order of 10^{26-35} protons. The enormous number of protons in the target thus leads to event rates which are much higher than those in colliding beam machines.

We close this section with a few definitions. It is convenient in discussions of two-particle phase space to introduce the function

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\,. \tag{3.40}$$

Using this, the differential cross section can be written as

$$\begin{aligned} \mathrm{d}\sigma = {} & \frac{1}{2\,\lambda^{1/2}(s,m_1^2,m_2^2)}\,(2\pi)^4\,\delta^4\big(p_1+p_2-\textstyle\sum_i k_i\big) \\ & \times |\mathcal{M}(p_1,p_2;k_i)|^2 \prod_{j=1}^{n} \frac{\mathrm{d}^3 k_j}{(2\pi)^3\,2\omega_j(\mathbf{k}_j)}\,. \end{aligned} \tag{3.41}$$

where s is the square of the incoming centre-of-mass energy:

$$s = -(p_1+p_2)^2\,. \tag{3.42}$$

Integrating the differential cross section over the momenta of all outgoing particles gives the total cross section. If the outgoing particles carry more degrees of freedom, such as spin, one may also sum over those. Note that, when some of the outgoing particles are identical, a naive integration and summation leads to overcounting. For each set of m identical particles, one should therefore divide the total cross section by a combinatorial factor $m!$. When other final states can also be produced, then the above procedure gives a partial cross section. Ultimately, one may prefer to sum over all possible final states.

3.4. Kinematics for (quasi-)elastic scattering

In order to get acquainted with the material discussed in the previous section, we consider the scattering process

$$1 + 2 \to 3 + 4\,,$$

which is shown in fig. 3.3. According to (3.41) the corresponding differential cross section is

$$\mathrm{d}\sigma = \frac{\delta(p_1+p_2-p_3-p_4)}{8\pi^2\lambda^{1/2}(s,m_1^2,m_2^2)}\,|\mathcal{M}(p_1,p_2;k_i)|^2\,\frac{\mathrm{d}^3p_3\,\mathrm{d}^3p_4}{4\,\omega_3\,\omega_4}\,. \tag{3.43}$$

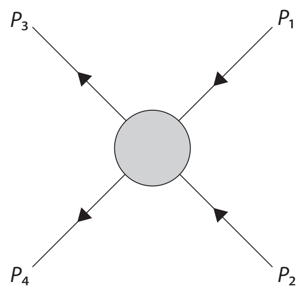


Figure 3.3: The process $1 + 2 \to 3 + 4$.

For given total incoming momentum the process can be characterized in terms of the momenta $\mathbf{p}_3$ and $\mathbf{p}_4$ of the outgoing particles. However, due to energy-momentum conservation, $\mathbf{p}_3$ and $\mathbf{p}_4$ are not linearly independent, and we are only dealing with only two independent parameters. In the centre-of-mass frame those correspond to the angles that specify the direction in which the particles are produced. In order to express the result in terms of these angles one partially performs an integration of (3.43) over the momenta in order to just remove the energy and momentum conserving δ-functions. At this stage one may ignore the invariant amplitude, and consider the two-particle phase-space integral

$$I(s, m_3^2, m_4^2) = \frac{1}{(2\pi)^2} \int \frac{\mathrm{d}^3 p_3 \, \mathrm{d}^3 p_4}{4\, \omega_3\, \omega_4} \, \delta^4(p_1 + p_2 - p_3 - p_4) \,, \tag{3.44}$$

with s defined by (3.42). It is most convenient to evaluate this function in the centre-of-mass frame, where $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}'$. The spatial part of the δ-function then restricts us to $\mathbf{p}_3 = -\mathbf{p}_4 = \mathbf{p}$, and using $\sqrt{s} = \omega_1 + \omega_2$ the integral becomes

$$\begin{aligned} I(s, m_3^2, m_4^2) = \frac{1}{16\pi^2} \int & \frac{\mathrm{d}^3 p}{\sqrt{(\mathbf{p}^2 + m_3^2)(\mathbf{p}^2 + m_4^2)}} \\ & \times \, \delta\big(\sqrt{s} - \sqrt{\mathbf{p}^2 + m_3^2} - \sqrt{\mathbf{p}^2 + m_4^2}\big) \,. \end{aligned} \tag{3.45}$$

To remove the last δ-function, we integrate over $p = |\mathbf{p}|$. Hence, in spherical coordinates, we write

$$\begin{aligned} I(s, m_3^2, m_4^2) = \frac{1}{16\pi^2} \int & \frac{p^2 \, \mathrm{d}p \, \mathrm{d}\Omega_{\mathrm{CM}}}{\sqrt{(p^2 + m_3^2)(p^2 + m_4^2)}} \\ & \times \, \delta\big(\sqrt{s} - \sqrt{p^2 + m_3^2} - \sqrt{p^2 + m_4^2}\big) \,. \end{aligned} \tag{3.46}$$

Note that this integral vanishes unless $s > (m_3 + m_4)^2$, i.e. the incoming energy in the collision must be sufficient to actually produce two physical particles with masses m_3 and m_4. In order to calculate the value of p_2 for which the argument of the δ-function vanishes, one first multiplies by $\sqrt{s} + \sqrt{p^2 + m_3^2} + \sqrt{p^2 + m_4^2}$ to find,

$$s - 2p^2 - m_3^2 - m_4^2 = 2\sqrt{(p^2 + m_3^2)(p^2 + m_4^2)}\,. \tag{3.47}$$

Squaring this result leads to

$$\begin{aligned} p^2 &= a^2 \\ &= \frac{1}{4s}(s - (m_3 + m_4)^2)(s - (m_3 - m_4)^2), \\ &= \frac{1}{4s}\lambda(s, m_3^2, m_4^2)\,., \end{aligned} \tag{3.48}$$

where λ was defined in (3.40). One may verify that $\lambda \geqslant 0$ for $s \geqslant (m_3 + m_4)^2$. We now use the rule for δ-functions,

$$\delta\big(f(x)\big) = \frac{\delta(x - x_0)}{|f'(x)|_{x=x_0}}\,, \tag{3.49}$$

where $f(x)$ is an arbitrary function with one zero at $x = x_0$. Applying this to the integral, where $f(p) = \sqrt{s} - \sqrt{p^2 + m_3^2} - \sqrt{p^2 + m_4^2}$, (3.46) becomes equal to

$$\begin{aligned} I(s, m_3^2, m_4^2) &= \frac{1}{16\pi^2}\frac{a^2}{\sqrt{(a^2 + m_3^2)(a^2 + m_4^2)}} \\ &\quad \times \left(\frac{a}{\sqrt{a^2 + m_3^2}} + \frac{a}{\sqrt{a^2 + m_4^2}}\right)^{-1} \int \mathrm{d}\Omega_{\mathrm{CM}} \\ &= \frac{1}{32}\frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{s}\int \mathrm{d}\Omega_{\mathrm{CM}}\,. \end{aligned} \tag{3.50}$$

At this point, we have removed all the δ-functions, so we return to the original expression (3.43) to find,

$$\mathrm{d}\sigma = \frac{1}{64\pi^2}\,\frac{1}{s}\sqrt{\frac{\lambda(s, m_3^2, m_4^2)}{\lambda(s, m_1^2, m_2^2)}}\,|\mathcal{M}|^2\,\mathrm{d}\Omega_{\mathrm{CM}}\,. \tag{3.51}$$

Assuming that the particles have no spin, $\mathcal{M}$ is a function of the Mandelstam variables s, t and u, defined by

$$\begin{aligned} s &= -\,(p_1 + p_2)^2 = -(p_3 + p_4)^2\,, \\ t &= -\,(p_1 - p_3)^2 = -(p_2 - p_4)^2\,, \\ u &= -\,(p_1 - p_4)^2 = -(p_2 - p_3)^2\,. \end{aligned} \tag{3.52}$$

with $s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$. In the centre-of-mass frame, we have

$$t = 2|\mathbf{p}||\mathbf{p}'|\cos\theta_{CM} - 2\sqrt{(\mathbf{p}'^2+m_1^2)(\mathbf{p}^2+m_3^2)} + m_1^2 + m_3^2\,, \tag{3.53}$$

where $\theta_{\rm CM}$ is the scattering angle between $\mathbf{p}_1$, and $\mathbf{p}_3$ (see fig. 3.4a). Using $\mathrm{d}\Omega_{\rm CM} = \mathrm{d}\cos\theta_{\rm CM}\mathrm{d}\phi$ and integrating over ϕ, one obtains

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta_{\rm CM}} = \frac{1}{32\pi}\,\frac{1}{s}\sqrt{\frac{\lambda(s,m_3^2,m_4^2)}{\lambda(s,m_1^2,m_2^2)}}\,|\mathcal{M}|^2\,, \tag{3.54}$$

or using $\mathrm{d}t = 2|\mathbf{p}||\mathbf{p}'|\mathrm{d}\cos\theta_{CM}$

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{1}{64\pi}\,\frac{1}{s}\sqrt{\frac{\lambda(s,m_3^2,m_4^2)}{\lambda(s,m_1^2,m_2^2)}}\,\frac{|\mathcal{M}|^2}{|\mathbf{p}||\mathbf{p}'|}\\ &= \frac{1}{16\pi}\,\frac{|\mathcal{M}|^2}{\lambda(s,m_1^2,m_2^2)}\,,\end{aligned} \tag{3.55}$$

where we have again used (3.48) to find expressions for $|\mathbf{p}|$ and $|\mathbf{p}'|$.

In a fixed-target experiment the momentum of the target particle in the laboratory frame (see fig. 3.4b) is $p_2 = (\mathbf{0}, m_2)$. The beam particle with momentum $p_1 = (\mathbf{p}_1, E)$ is deflected over an angle θ and carries momentum $p_3 = (\mathbf{p}_3, E')$, whereas the target particle recoils with momentum $p_4 = (\mathbf{p}_4, E_R)$ (we allow different particle masses in the scattering process). In the laboratory frame energy and momentum conservation gives,

$$E + m_2 = E' + E_R\,, \qquad \mathbf{p}_1 = \mathbf{p}_3 + \mathbf{p}_4\,. \tag{3.56}$$

The invariants s and t equal,

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2m_2E\,,\\ t &= -(\mathbf{p}_1-\mathbf{p}_3)^2 + (E-E')^2\\ u &= m_2^2 + m_4^2 - 2m_2E_R\,.\end{aligned} \tag{3.57}$$

The scattering amplitude can be expressed in terms of the energy of the scattered particle E', or the recoil energy E_R. Often a parameter y is used, defined by

$$y = \frac{E-E'}{E} = \frac{E_R-m_2}{E}\,, \qquad \frac{m_4-m_2}{E} < y < \frac{E-m_3}{E}\,, \tag{3.58}$$

whose Lorentz-invariant definition is

$$y = \frac{-t+m_4^2-m_2^2}{s-m_1^2-m_2^2}\,.$$

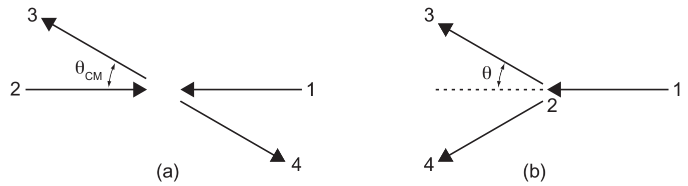


Figure 3.4: The reaction $1+2 \rightarrow 3+4$ in the centre-of-mass frame (a), and the laboratory frame (b), defining the scattering angles $\theta_{\rm CM}$ and θ, respectively.

The differential cross section can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_R} = -2m_2 \frac{\mathrm{d}\sigma}{\mathrm{d}t}\,, \quad \text{or} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}y} = -2m_2 E \frac{\mathrm{d}\sigma}{\mathrm{d}t}\,. \tag{3.59}$$

Alternatively one may use the angle θ between $\mathbf{p_1}$ and $\mathbf{p}_3$. We quote the result without proof (see problem 3.4):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\rm lab}} = \frac{(E^2 - m_1^2)^{1/2}(E'^2 - m_3^2)^{3/2}}{m_2(EE' - m_3^2) + m_3^2(E' - E) + \frac{1}{2}E'(m_1^2 + m_2^2 - m_3^2 - m_4^2)} \times \frac{m_2}{\pi}\,\frac{\mathrm{d}\sigma}{\mathrm{d}t}\,. \tag{3.60}$$

However, to express $\mathrm{d}\sigma/\mathrm{d}t$ in terms of E and θ is a rather tedious exercise in general.

3.5. On polarization vectors

Many elementary particles carry more degrees of freedom than those characterized by their momentum: particles with spin are a typical example. In addition to the momentum also the orientation of their spin needs to be specified, and as is well-known there is a $(2s + 1)$-fold degeneracy of spin states with the same energy and momentum. We should point out that this degeneracy is always related to an invariance law. In the case of spin it is rotational invariance that implies that the $2s + 1$ spin states must have the same mass.

In this section we want to discuss some of the complications that arise in the case of degenerate particle states. This will be done for the model of section 2.5, where the pions have a three-fold degeneracy because of isospin invariance. The advantage of this example is that we still remain in the context of scalar particles, where the complications due to the extra degrees of freedom take a relatively simple form. We recall that there are three types of pions described by real fields $\phi^a(x)(a = 1, 2, 3)$, which transform as a vector under isospin rotations. The differential cross section for pion-pion scattering follows from the amplitude (2.61),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}} = \frac{1}{64\pi^2 s}\left|\delta^{ab}\delta^{cd}F(s) + \delta^{ac}\delta^{bd}F(t) + \delta^{ad}\delta^{bc}F(u)\right|^2, \tag{3.61}$$

where the function F was defined in (2.62).

The presence of the indices in (3.61) makes it complicated to deal with this expression. In fact, the derivation of section 3.3 leading to the formula for the cross section must be modified, because the wave function of the incoming particles should now carry an isospin index. The standard approach is therefore to factorize the wave function into a momentum part and an isospin “polarization” vector, i.e.

$$f^a(\mathbf{p}) = f(\mathbf{p})\,\varepsilon^a\,,$$

where ε^a is a three-component vector normalized to unity. It is not always possible to choose ε^a real, although an overall phase can be absorbed into the wave function $f(\mathbf{p})$. For instance, the field associated with the charged pions is a complex linear combination of the fields ϕ^a, so for describing an incoming or outgoing $\pi^\pm$ one needs a complex vector ε^a. The appropriate polarization vectors for $\pi^\pm$ and π^0 follow from a decomposition of ϕ

$$\phi^a(x) = \pi^+\varepsilon^a(+) + \pi^-\varepsilon^a(-) + \pi^0\varepsilon^a(0)\,, \tag{3.62}$$

where $\pi^\pm$ have been defined in (2.53), so that

$$\varepsilon^a(\pm) = \tfrac{1}{2}\sqrt{2}(1, \pm\mathrm{i}, 0), \qquad \varepsilon^a(0) = (0,0,1)\,. \tag{3.63}$$

Note that the right-hand side of (3.62) is real.

Just as for complex fields one assigns vectors ε to incoming particles and their complex conjugates ε^* to outgoing particles. Hence for incoming pions characterized by ε_1 and ε_2, and outgoing pions characterized by ε_3 and ε_4 the formula for the $\pi-\pi$ cross section takes the form

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}} = \frac{1}{64\pi^2 s}&\big|(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3^*\cdot\varepsilon_4^*)F(s)\\ &+ (\varepsilon_1\cdot\varepsilon_3^*)(\varepsilon_2\cdot\varepsilon_4^*)F(t) + (\varepsilon_1\cdot\varepsilon_4^*)(\varepsilon_2\cdot\varepsilon_3^*)F(u)\big|^2\,. \end{aligned}\tag{3.64}$$

A convenient way of writing (3.64) makes use of projection operators, three-by-three matrices that characterize the incoming and outgoing pions,

$$(P_i)^{ab} = ({\varepsilon_i}^a)^*\,{\varepsilon_i}^b\,, \tag{3.65}$$

where no summation is implied over the index $i = 1,2,3,4$ that specifies each of the incoming or outgoing particles. The cross section (3.64) can then be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}} = \frac{1}{64\pi^2 s}\Big\{ F^2(s)\,\mathrm{Tr}(P_1P_2^*)\,\mathrm{Tr}(P_3P_4^*) + F^2(t)\,\mathrm{Tr}(P_1P_3)\,\mathrm{Tr}(P_2P_4) + F^2(u)\,\mathrm{Tr}(P_1P_4)\,\mathrm{Tr}(P_3P_4) + F(s)\,F(t)\,\mathrm{Tr}(P_1P_2^*P_4^*P_3 + P_1^*P_2P_4P_3^*) + F(s)\,F(u)\,\mathrm{Tr}(P_1P_2^*P_3^*P_4 + P_1^*P_2P_3P_4^*) + F(t)\,F(u)\,\mathrm{Tr}(P_1P_3P_2P_4 + P_1^*P_3^*P_2^*P_4^*)\Big\}\,, \tag{3.66}$$

where there are various ways of ordering the projection operators in the traces. Associated with each type of pion we have a projection operator, i.e.

$$P(\pi^\pm) = \tfrac{1}{2}\begin{pmatrix} 1 & \pm\mathrm{i} & 0 \\ \mp\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad P(\pi^0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{3.67}$$

In order to calculate the cross section for a particular process, one thus substitutes the corresponding projection operators into (3.66). If one wants to sum over all possible pions in the incoming or outgoing states, one simply replaces P by the identity matrix since

$$P(\pi^+) + P(\pi^-) + P(\pi^0) = \mathrm{I}. \tag{3.68}$$

This identity is a consequence of the fact that the vectors ε associated with π^+, π^- and π^0 form a complete orthonormal set. If an incoming pion beam is not pure, i.e. if there is a mixture of $\pi^\pm$ and π^0 particles with probabilities $p_\pm$ and p_0 respectively, then the corresponding projection operator must be replaced by a so-called density matrix

$$\rho = p_+P(\pi^+) + p_-P(\pi^-) + p_0P(\pi^0)\,. \qquad p_+ + p_- + p_0 = 1\,, \qquad p_\pm\,, p_0 > 0\,. \tag{3.69}$$

Note that ρ is a hermitean matrix, which satisfies

$$\mathrm{Tr}(\rho) = 1, \qquad \mathrm{Tr}(\rho^2) \leqslant 1\,. \tag{3.70}$$

Only for a pure pion beam do we have $\mathrm{Tr}(\rho^2) = 1$.

3.6. Unstable particles and decay rates

In chapter 2 physical particles were identified with fluctuations of the fields that are able to propagate over asymptotically large time scales. Those fluctuations satisfy the relativistic dispersion law $k^0 = \sqrt{\mathbf{k}^2 + m^2}$, as a result of the corresponding propagator poles. Strictly speaking it is not necessary that the propagation time be infinite, only that the time (or distance) of particle propagation from the source to the target and that from the target to the detector be large compared to the time (or distance) during which the particles interact. Thus, even the interactions of short-lived particles such as pions, whose mean life time is $\sim 10^{-8}$ s, can be described in the terms above, as long as they interact before they decay. The latter is usually the case, since their decay lifetime is lengthened at relativistic velocities; beams of relativistic pions can travel hundreds of meters in actual laboratory experiments, while they interact in targets which have the dimensions of a few centimeters. These distances satisfy the requirements above, so (3.41) can be used to calculate the differential scattering cross section for these reactions.

However, it is not sufficient to consider only the scattering of unstable particles, but one should also understand their decay properties. To examine such questions, let us reconsider the model of section 2.5, which describes the coupling of scalar sigma mesons to pions. In tree approximation the propagators for the sigma and the pion fields differ only by the value of the mass (cf. table 3.2). How then can one account for the fact that the sigma can obviously decay into two pions (assuming that the sigma mass μ is larger than twice the pion mass m), whereas the pion remains stable (in the context of this model)? The propagation characteristics of the sigma must therefore be radically different from those of the pions. This difference must arise from higher-order contributions to the propagator. Let us therefore return to a discussion of the quantum effects.

In section 2.6 we have described the higher-order Feynman diagrams as arising from a continuous emission and absorption of virtual particles. Virtual particles with energy differing by an amount ΔE from the energy of the corresponding real particle can exist during a characteristic time Δt, subject to the quantum-mechanical uncertainty relation $\Delta E\,\Delta t > \hbar$. Because of the higher-order corrections the mass of a physical particle is changed by self-energy effects, which are governed by the Dyson equation, (2.66),

$$\Delta(p) = \frac{1}{\mathrm{i}(2\pi)^4}\,\frac{1}{p^2 + m^2 - \frac{1}{\mathrm{i}(2\pi)^4}\Sigma(p)}\,.$$

In this equation $\Sigma(p)$ represents the sum of all the irreducible self-energy diagrams.

The presence of $\Sigma(p)$ causes a shift in the position of the propagator pole. If the pole is shifted only along the real p^0-axis, there is no qualitative change in the asymptotic behaviour of the propagator. The only effect is that the actual physical mass will be changed by the self-energy graphs. In general, however, the function $\Sigma(p)$ becomes complex for certain values of the external momenta, and it is thus conceivable that the pole will be displaced into the complex p^0-plane.

Let us first examine the consequences of a complex pole. For that purpose we assume that near the pole the propagator behaves as

$$\varDelta(p) \approx \frac{1}{\mathrm{i}(2\pi)^4}\frac{Z}{p^2+m^2-\mathrm{i}\gamma}\,. \tag{3.71}$$

The fact that the residue factor Z at the poles is no longer equal to unity does not play a role here. We will assume that $\gamma > 0$, so that for infinitesimally small γ we make contact with the standard iϵ-prescription for the propagator poles that we have discussed in the previous chapter.

Consider now a reaction in which a particle is produced, and at a later time is absorbed by some detector. The particle must therefore propagate from the interaction region to the detector, and the characteristics of the propagation are determined by the propagator. As we have learned in chapter 2 the leading contribution at large time originates from the momenta associated with the propagator pole in the lower complex p^0-plane; hence from

$$p^0 \sim \omega(\mathbf{p}) - \mathrm{i}\frac{\gamma}{2\omega(\mathbf{p})}\,, \tag{3.72}$$

where we have assumed that $\gamma \ll \omega^2(\mathbf{p}) = \mathbf{p}^2 + m^2$. When a particle is produced in some reaction then the time dependence of the dominant part of the amplitude contains an exponential factor $\exp(-\mathrm{i}p^0\,t)$, where p^0 is the pole position given by (3.72). For a stable particle one should take the limit $\gamma \to 0$, so that the exponential factor yields just the characteristic time dependence $\exp\big(-\mathrm{i}\omega(\mathbf{p})\,t\big)$ of a plane wave associated with a free stable particle. For unstable particles the same exponential factor now gives

$$\exp(-\mathrm{i}\omega(\mathbf{p})\,t)\exp\Big(-\frac{\gamma\,t}{2\omega(\mathbf{p})}\Big)\,.$$

Hence the imaginary part in (3.72) leads to an exponential damping factor. Therefore the *probability* for the detection of the particle decreases exponentially with a characteristic decay time given by

$$\tau(\mathbf{p}) = \frac{\omega(\mathbf{p})}{\gamma} = \frac{1}{\varGamma(\mathbf{p})}\,. \tag{3.73}$$

The quantity $\varGamma(\mathbf{p})$ is called the *decay width* (or decay rate) of the unstable particle. The *mean lifetime* $\tau(\mathbf{p})$ and the width $\varGamma(\mathbf{p})$ depend on the Lorentz frame in which they are measured. This is also obvious from (3.71) since $\gamma = \omega(\mathbf{p})\,\varGamma(\mathbf{p})$ is a Lorentz-invariant quantity.

It is now clear why we have insisted on positive γ; a negative value of γ would instead lead to an amplitude that would *increase* in time, thus violating the conservation of probability. A theory that conserves probability is called *unitary* because the *scattering operator* whose matrix elements describe all possible scattering reactions must be a unitary operator. It is clear that it is a subtle matter to ensure that a field theory is unitary. One of the most crucial ingredients is again the iϵ-prescription for dealing with the poles in the propagators associated with the internal lines in Feynman diagrams. A general proof of this is outside the scope of this book. Instead we will use a specific example to exhibit some of the essential features of the proof.

Consider the typical self-energy diagram shown in fig. 3.5. In order to obtain the corresponding expression for the decay width, we determine the imaginary part of of this diagram. Subsequently the result will be generalized to a formula for the width in terms of the corresponding invariant amplitude.

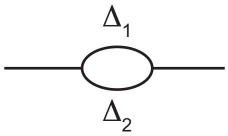


Figure 3.5: Self-energy diagram corresponding to (3.74).

As it turns out, to investigate the reality properties of Feynman diagrams is subtle in the momentum representation, because the integrations over the momenta are subject to the way one integrates in the neighbourhood of the propagator poles. The analysisis much more straightforward in the coordinate representation. Then the self-energy diagram of fig. 3.5 takes the form

$$\frac{1}{\mathrm{i}(2\pi)^4}\Sigma(p) = \mathrm{i}g^2 \int \mathrm{d}^4x \, \Delta_1(x) \, \Delta_2(x) \, \mathrm{e}^{-\mathrm{i}p\cdot x} \,, \tag{3.74}$$

where Δ_1 and Δ_2 are standard propagators with masses m_1 and m_2 for the two internal lines, and g is the coupling constant associated with the vertex. For convenience we take a vertex that involves three inequivalent fields, so that there are no combinatorial factors to keep track of. Now recall relation (2.26),

$$\Delta(x) = \theta(x^0) \, \Delta^+(x) + \theta(-x^0) \, \Delta^-(x) \,,$$

which we can use to rewrite (3.74) as

$$\begin{aligned} \frac{1}{\mathrm{i}(2\pi)^4}\Sigma(p) = \mathrm{i}g^2 &\int \mathrm{d}^4x \, \mathrm{e}^{-\mathrm{i}p\cdot x} \\ &\times \left[\theta(x^0) \, \Delta_1^+(x) \, \Delta_2^+(x) + \theta(-x^0) \, \Delta_1^-(x) \, \Delta_2^-(x)\right] . \end{aligned} \tag{3.75}$$

Using $[\Delta^{\pm}(x)]^* = \Delta^{\pm}(-x)$ (c.f. 2.29), it is now easy to write down the imaginary part of (3.75),

$$\begin{aligned}
&\mathrm{Im}\Big(\frac{1}{\mathrm{i}(2\pi)^4}\Sigma(p)\Big) = \\
&= \tfrac{1}{2}g^2 \int \mathrm{d}^4x \,\{\theta(x^0)[\Delta_1^+(x)\Delta_2^+(x)\mathrm{e}^{-\mathrm{i}p\cdot x} + \Delta_1^+(-x)\Delta_2^+(-x)\mathrm{e}^{\mathrm{i}p\cdot x}] \\
&\qquad + \theta(-x^0)[\Delta_1^-(x)\Delta_2^-(x)\mathrm{e}^{-\mathrm{i}p\cdot x} + \Delta_1^-(-x)\Delta_2^-(-x)\mathrm{e}^{\mathrm{i}p\cdot x}]\} \\
&= \tfrac{1}{2}g^2 \int \mathrm{d}^4x \,\big[\Delta_1^+(x)\Delta_2^+(x) + \Delta_1^-(x)\Delta_2^-(x)\big]\mathrm{e}^{-\mathrm{i}p\cdot x}\,, \qquad (3.76)
\end{aligned}$$

where the last line is obtained by changing the integration variable x to $-x$ in the second and fourth terms in the expression above. Substituting the expressions for $\Delta^{\pm}$ given in (2.28), one obtains,

$$\begin{aligned}
&\mathrm{Im}\Big(\frac{1}{\mathrm{i}(2\pi)^4}\Sigma(p)\Big) = \\
&= \tfrac{1}{2}(2\pi)^4 g^2 \int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_1(\mathbf{k}_1)}\,\frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_2(\mathbf{k}_2)}\,\delta^3(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2) \\
&\qquad \times [\delta(p^0-\omega_1(\mathbf{k}_1)-\omega_2(\mathbf{k}_2)) + \delta(p^0+\omega_1(\mathbf{k}_1)+\omega_2(\mathbf{k}_2))]\,. \qquad (3.77)
\end{aligned}$$

Expression (3.77) is Lorentz invariant and thus a function of p^2. As one can verify by going to a frame where $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2 = 0$, the δ-functions contribute only for $-p^2 \geqslant (m_1+m_2)^2$, corresponding to the range of momenta where the decay of a particle with mass $\sqrt{-p^2}$ into two particles with momenta k_1 and k_2 and masses m_1 and m_2 becomes possible. Therefore self-energy diagrams of the type shown in fig. 3.5 always become complex when $-p^2$ is sufficiently large. The characteristic feature of a *stable particle* is that the propagator is still real near the position of the pole. The absence of an imaginary part implies that it is not possible to cut the self-energy graphs into two parts, each containing one of the external lines, such that all internal lines that have been cut have momenta that are on the corresponding mass shell (i.e. satisfy $k_i^2 = -m_i^2$) and have all positive (or all negative) energy $(k^0)_i$. In other words, the masses of the various (external and internal) lines are such that the kinematical configuration required for a decay process does not exist. On the other hand, for an *unstable particle* it is kinematically possible to have such momentum configurations. Therefore the propagator will be complex in the neighbourhood of the pole, which then moves away from the real axis.

By combining (3.73) and (3.77) it is now straightforward to find the decay rate. In this order in perturbation theory the imaginary part in the inverse propagator, γ, is given by (3.77) (the real part of the self-energy diagram is not relevant, since, near the pole, it can be absorbed into the mass and the overall normalization of the propagator). Hence we find

$$\Gamma(\mathbf{p}) = \frac{(2\pi)^4}{2\omega(\mathbf{p})} \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3\, 2\omega_1(\mathbf{k}_1)} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\, \omega_2(\mathbf{k}_2)}\, \delta^3(\mathbf{p} - \mathbf{k}_1 - \mathbf{k}_2)\, g^2\,, \quad (3.78)$$

which is indeed positive. We note that this result has the same structure as the differential cross section given in (3.41). We have $(2\pi)^4$ times the square of the invariant amplitude for the process in question, which in this case is just the coupling constant g. Furthermore, there is an energy momentum conserving δ-function, and we have the same normalization factor $[(2\pi)^3 2\omega(\mathbf{k})]^{-1}$ for each of the outgoing particles. Only the terms that refer to the incoming particles are different, but this is to be expected because we are now describing particle decay rather than particle scattering.

The above considerations make it easy to generalize the result (3.78) to a decay into a final state of n particles with momenta in a volume element $\mathrm{d}^3 k_i$ at $\mathbf{k}_i$, (see problem 3.6). We thus postulate

$$\mathrm{d}\Gamma(\mathbf{p}) = \frac{(2\pi)^4}{2\omega(\mathbf{p})}\, \delta^4\big(p - \textstyle\sum_i k_i\big)\, |\mathcal{M}(p;k_i)|^2 \displaystyle\prod_{j=1}^{n} \frac{\mathrm{d}^3 k_j}{(2\pi)^3 2\omega_j(\mathbf{k}_j)}\,, \qquad (3.79)$$

where $\mathcal{M}(p;k_i)$ is the invariant amplitude for the decay. One usually integrates over the momenta of the outgoing particles; in case the decay products contain identical particles one must include a *combinatorial factor* to avoid overcounting identical configurations in the final state. We now define the partial decay rate $\Gamma^n(\mathbf{p})$ of a particle as the width associated with a specific set of final state particles integrated over all final state variables. Since we are only dealing with scalar particles here, the partial decay rate is simply the integral of (3.79) over the momenta of the decay products. Thus the total rate is the sum of the partial rates, i.e.

$$\Gamma^{\mathrm{tot}}(\mathbf{p}) = \sum_n \Gamma^n(\mathbf{p})\,. \qquad (3.80)$$

As an example consider the decay of a particle A into two particles B and C with masses m_{A}, m_{B} and m_{C}. We find for the partial width,

$$\mathrm{d}\Gamma(\mathrm{A} \to \mathrm{B} + \mathrm{C}) = \frac{N}{2\omega_{\mathrm{A}}(\mathbf{p}_{\mathrm{A}})} \frac{1}{(2\pi)^2}\, \delta^4(p_{\mathrm{A}} - k_{\mathrm{B}} - k_{\mathrm{C}})\, |\mathcal{M}|^2\, \frac{\mathrm{d}^3 k_{\mathrm{B}}\, \mathrm{d}^3 k_{\mathrm{C}}}{4\, \omega_{\mathrm{B}}\, \omega_{\mathrm{C}}}\,, \qquad (3.81)$$

where N is equal to $\frac{1}{2}$ when the particles B and C are identical. The integral in (3.81) is precisely the phase space integral (3.44) that we have evaluated in section 3.4 (except that the centre-of-mass energy $\sqrt{s}$ is now equal to m_{A}).

Hence in the rest frame of the decaying particle we may write,

$$\begin{aligned}\Gamma(\mathrm{A}\to\mathrm{B}+\mathrm{C}) &= \frac{N}{2\omega_{\mathrm{A}}(\mathbf{p}_{\mathrm{A}})}\,\frac{1}{32\pi^2}\,\frac{\lambda^{1/2}(m_{\mathrm{A}}^2,m_{\mathrm{B}}^2,m_{\mathrm{C}}^2)}{m_{\mathrm{A}}^2}\int \mathrm{d}\Omega_{\mathrm{CM}}\,|\mathcal{M}|^2 \\ &= \frac{N}{16\pi}\,\frac{\lambda^{1/2}(m_{\mathrm{A}}^2,m_{\mathrm{B}}^2,m_{\mathrm{C}}^2)}{m_{\mathrm{A}}^2}\,|\mathcal{M}|^2\,, \end{aligned} \tag{3.82}$$

where $\omega_{\mathrm{A}} = m_{\mathrm{A}}$ in the rest system of A. Note that in this case it was trivial to do the angular integration, because unlike in the case of quasi-elastic scattering, the invariant amplitude is just a constant. This lack of angular dependence follows from Lorentz invariance; a spinless particle at rest has no preferred direction in which to decay into two other particles. The general formula (3.82) can be applied to the case where a sigma meson decays into two pions. In the model of section 2.5 the amplitudes for $\sigma\to\pi^0\pi^0$ and $\sigma\to\pi^+\pi^-$ are equal to 2λ (cf. 2.56; note that $\phi^2 = \pi^0\pi^0 + 2\pi^+\pi^-$). Including a combinatoric factor $N = \frac{1}{2}$ for the first decay, we have

$$\begin{aligned}&\Gamma(\sigma\to\pi^0\pi^0) = \tfrac{1}{2}\Gamma(\sigma\to\pi^+\pi^-) = \frac{1}{32\pi}\,\frac{\sqrt{\mu^2-4m^2}}{\mu^2}\,(2\lambda)^2\,, \\ &\Gamma^{\mathrm{tot}} = \Gamma(\sigma\to\pi^0\pi^0) + \Gamma(\sigma\to\pi^+\pi^-) = 3\Gamma(\sigma\to\pi^0\pi^0)\,. \end{aligned} \tag{3.83}$$

The inverse of Γ^{tot} then measures the lifetime of the sigma particle in its rest frame according to (3.73). Note that the second equation can be derived from isospin invariance alone without referring to a specific model.

We will now present an independent justification for associating (3.77) with the probability for the decay process. Consider a scattering experiment in which an unstable particle A is produced together with a number of stable particles denoted by Y. Subsequently the unstable particle decays into a final state X consisting of n particles. This situation is described in fig. 3.6. Strictly speaking one should consider the total amplitude for the scattering process $\mathrm{a}+\mathrm{b}\to\mathrm{X}+\mathrm{Y}$, but the process shown in fig. 3.6 will often give the dominant contribution. In that case we may write the total amplitude as

$$\begin{aligned}\mathcal{M}^{\mathrm{tot}}(\mathrm{a}+\mathrm{b}\to\mathrm{X}+\mathrm{Y}) = &\,\mathcal{M}^{\mathrm{decay}}(\mathrm{A}\to\mathrm{X})\,\frac{1}{p^2+m^2-\mathrm{i}\gamma} \\ &\times \mathcal{M}^{\mathrm{prod}}(a+b\to\mathrm{A}+\mathrm{Y})\,. \end{aligned} \tag{3.84}$$

Note that we have assumed that the kinematical configuration is such that the momentum p associated with the propagator for A is near the pole. The residue factor Z that may be present (cf. 3.71) has been absorbed into the definition of the amplitudes associated with the production and decay of the particle A. When we are in the neighbourhood of the propagator pole we indeed expect (3.84) to dominate all other contributions to $\mathcal{M}^{\mathrm{tot}}(\mathrm{a}+\mathrm{b}\to\mathrm{X}+\mathrm{Y})$.

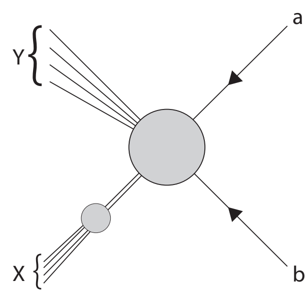


Figure 3.6: A graphical representation of the formula (3.84).

The square of the absolute value of (3.84) can now be written as

$$|\mathcal{M}^{\text{tot}}|^2 = \frac{|\mathcal{M}^{\text{decay}}|^2}{2\omega_{\text{A}}(\mathbf{p})} \left| \frac{2\omega_{\text{A}}(\mathbf{p})}{-s_{\text{A}} + m_{\text{A}}^2 - \mathrm{i} m_{\text{A}} \Gamma} \right|^2 \frac{|\mathcal{M}^{\text{prod}}|^2}{2\omega_{\text{A}}(\mathbf{p})}\,, \tag{3.85}$$

where Γ is the total decay rate in the rest system and $\sqrt{s_{\text{A}}}$ is the invariant mas of the virtual A particle (i.e. $s_{\text{A}} = -p^2$). Multiplying both sides of (3.85) with the appropriate normalization factors for a cross section (cf. 3.41) the expression for the differential cross section takes the form

$$\begin{aligned} \frac{\mathrm{d}\sigma(\text{a}+\text{b}\to\text{X}+\text{Y})}{\mathrm{d}\mathbf{p}\,\mathrm{d}s_{\text{X}}} &= \frac{\mathrm{d}\Gamma(\text{A}\to\text{X})}{\Gamma(\text{A}\to\text{all})}\,\frac{\mathrm{d}\sigma(\text{a}+\text{b}\to\text{A}+\text{Y})}{\mathrm{d}\mathbf{p}\,\mathrm{d}s_{\text{A}}} \\ &\times \left\{ \frac{1}{\pi}\,\frac{m_{\text{A}}\Gamma}{(s_{\text{A}} - m_{\text{A}}^2)^2 + m_{\text{A}}^2\Gamma^2} \right\} \sqrt{\frac{\mathbf{p}^2 + m_{\text{A}}^2}{\mathbf{p}^2 + s_{\text{A}}}}\,, \end{aligned} \tag{3.86}$$

where $\mathrm{d}\sigma/\mathrm{d}\mathbf{p}\,\mathrm{d}s_{\text{X}}$ and $\mathrm{d}\sigma/\mathrm{d}\mathbf{p}\,\mathrm{d}s_{\text{A}}$ denote the inclusive cross sections in which the four-momentum of the dacaying/produced resonance is kept fixed. Furthermore we have used $\mathrm{d}^4p = \frac{1}{2}(\mathbf{p}^2 + s_{\text{A}})^{-1/2}\,\mathrm{d}^3p\,\mathrm{d}s_{\text{A}}$. In (3.86), $\mathrm{d}\Gamma(\text{A}\to\text{X})$ defines the partial rate (3.79) for the decay of the virtual particle A with mass $\sqrt{s_{\text{A}}}$ and momentum $\mathbf{p}$, $\mathrm{d}\sigma(\text{a}+\text{b}\to\text{A}+\text{Y})$ is the cross section for producing A, and $\Gamma(\text{A}\to\text{all})$ defines the *total* decay rate, but now in the frame where A has momentum $\mathbf{p}$ (hence we used the notation $\omega_{\text{A}}(\mathbf{p})\Gamma(\text{A}\to\text{all}) = m_{\text{A}}\Gamma$).

The term in parentheses,

$$W(s_{\text{A}}) = \frac{1}{\pi}\,\frac{m_{\text{A}}\Gamma}{(s_{\text{A}} - m_{\text{A}}^2)^2 + m_{\text{A}}^2\Gamma^2} \tag{3.87}$$

is the relativistic Breit-Wigner resonance formula, which, if Γ is not too large gives an enhancement of the cross section near $s_{\text{A}} \approx m_{\text{A}}^2$. Measuring the invariant mass of a subset of particles (usually consisting of two or three particles) produced in some scattering reaction one may thus detect dramatic increases in the cross section which signal the presence of unstable particle (resonances). A nice example of such a measurement is shown in fig. 3.7. In practice the symmetry of the peak at $s_{\rm A} = m_{\rm A}^2$ is often distorted by quantum mechanical interference with background reactions, which were ignored in (3.85). Also complications may arise if X and Y contain identical particles, so that the identification of the decay products of A becomes ambiguous.

When the width of the unstable particle is very small as compared to the rate of change of $\mathrm{d}\mathit{\Gamma}(\mathrm{A} \to \mathrm{X})$ and $\mathrm{d}\sigma^{\rm prod}(\mathrm{a}+\mathrm{b} \to \mathrm{A}+\mathrm{Y})$ as a function of energy, we can use the representation for the δ-function,

$$\lim_{\epsilon\to 0}\left\{\frac{1}{\pi}\frac{\epsilon}{\eta^2+\epsilon^2}\right\} = \delta(\eta)\,, \tag{3.88}$$

to show that in the narrow-width approximation

$$W(s_{\rm A}) \approx \delta(s_{\rm A} - m_{\rm A}^2)\,.$$

Thus, if $\mathit{\Gamma}$ is small, and therefore the lifetime of the unstable particle is large, we find for the total cross section

$$\sigma(\mathrm{a}+\mathrm{b}\to\mathrm{X}+\mathrm{Y}) = \sigma^{\rm prod}(\mathrm{a}+\mathrm{b}\to\mathrm{A}+\mathrm{Y})\,B(A\to X)\,, \tag{3.89}$$

where B is the *branching ratio* defined by

$$B(A\to X) \equiv \frac{\mathit{\Gamma}(\mathrm{A}\to\mathrm{X})}{\mathit{\Gamma}(\mathrm{A}\to\mathrm{all})}\,. \tag{3.90}$$

This result agrees with our physical intuition that the probability for the combined reaction equals the production probability multiplied by a dimensionless number B that measures the relative probability for the particle A to decay into the final state X.

The result (3.86) takes a particularly useful form if only a single *unstable* particle is formed in the initial scattering process. The momentum of the virtual particle A is then fixed in terms of the incoming particle momenta, and the cross section $d\sigma(\mathrm{a}+\mathrm{b}\to\mathrm{A}+\mathrm{Y})$ in (3.86) can be expressed in terms of the inverse process $\bar{\mathrm{A}} \to \bar{\mathrm{a}} + \bar{\mathrm{b}}$, where $\bar{\mathrm{A}}$, $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ are the antiparticles associated with A, a and b (which may again be the same particle). Following the same arguments as above, inserting normalization factors as in (3.41) and (3.82), we find that the cross section near the resonance is given by

$$\sigma(\mathrm{a}+\mathrm{b}\to\mathrm{X}) = \frac{16\pi\, m_{\rm A}^4}{N\,\lambda(m_{\rm A}^2, m_{\rm a}^2, m_{\rm b}^2)}\,\frac{\mathit{\Gamma}(\mathrm{A}\to\mathrm{X})\,\mathit{\Gamma}(\bar{\mathrm{A}}\to\bar{\mathrm{a}}+\bar{\mathrm{b}})}{(s-m_{\rm A}^2)^2 + m_{\rm A}^2\mathit{\Gamma}^2}\,, \tag{3.91}$$

where $\mathit{\Gamma}(\mathrm{A}\to\mathrm{X})$, $\mathit{\Gamma}(\bar{\mathrm{A}}\to\bar{\mathrm{a}}+\bar{\mathrm{b}})$ and $\mathit{\Gamma}$ are the partial and total decay rates in the *rest frame* of the particle A. On the peak at $s_{\rm A} = m_{\rm A}^2$ we thus find the

Figure 3.7: The schematic yield of dilepton pairs in proton-nucleus collisions. The plot shows the differential cross section $\mathrm{d}\sigma/\mathrm{d}M$, where M is the invariant mass of the dilepton pair, and is a compilation of data from several experiments with different acceptances and systematics. There is some uncertainty in the relative normalization of the portions of the curve. Enhancements are seen at 0.77, 1.03, 3.10, 3.59, 9.46 and 10.04 GeV/c^2, corresponding to the masses of the ρ ϕ, J/ϕ, Υ and Υ' resonances, respectively. [C.N. Brown, in: Workshop on Lepton Pair Production, Moriond (1981)87, ed.J.Tran Thanh Van(Editions Frontiéres,Dreux,France).]

analogue of (3.89),

$$\sigma(\mathrm{a}+\mathrm{b}\to\mathrm{X}) = \frac{16\pi\, m_{\mathrm{A}}^4}{N\,\lambda(m_{\mathrm{A}}^2, m_{\mathrm{a}}^2, m_{\mathrm{b}}^2)}\,\frac{\Gamma(\mathrm{A}\to\mathrm{X})\,\Gamma(\bar{\mathrm{A}}\to\bar{\mathrm{a}}+\bar{\mathrm{b}})}{\Gamma^2(\mathrm{A}\to\mathrm{all})}\,, \tag{3.92}$$

where $N = \frac{1}{2}$ when the particles a and b are identical. For particles with spin there may be extra factors in (3.91) and (3.92). For instance, in the cross section $\sigma(\mathrm{a}+\mathrm{b}\to\mathrm{A})$ one must average over the spins of a and b (at least for unpolarized beams and targets) and sum over polarizations of A. The converse is the case for $\Gamma(\bar{\mathrm{A}}\to\bar{a}+\bar{b})$, where one averages over the polarizations of A and sums over the polarizations of a and b. To compensate for this difference in summing and averaging, (3.91) and (3.92) then acquires a factor $(2S_{\mathrm{A}}+1)(2S_{\mathrm{a}}+1)^{-1}(2S_{\mathrm{b}}+1)^{-1}$, where S_{A}, S_{a} and S_{b} denote the spin of the particles A, a and b, respectively.

3.7. Three-particle phase space

In the previous sections we have calculated reactions in which we had to integrate over the two-particle phase-space. In many situations, however, one considers processes that involve three or more particles in the final state. We then have to use (3.41) or (3.79) and integrate over all final-state variables. It is important to note that the momenta of the outgoing particles always cover a finite domain, because they are restricted by the amount of incoming energy in the decay or scattering reaction.

A three-particle state involves three momenta $\mathbf{k}_i$ $(i = 1,2,3)$ with corresponding energies $\omega_i = \sqrt{\mathbf{k}_i^2 + m_i^2}$, which are restricted by energy momentum conservation. Hence for given incoming momentum and energy the final state is described in terms of $9-4=5$ independent variables. There are many ways to parametrize the momentum variables, and it usually requires tedious algebra to work out the boundary of the parameter domain. As variables one may choose the energies of two of the particles, say ω_1 and ω_2, in the centre-of-mass frame; ω_1 and ω_2 thus determine the length of the momentum vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Since the three momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ define a plane in this frame, momentum conservation determines the orientation of $\mathbf{k}_3$ relative to $\mathbf{k}_1$, and $\mathbf{k}_2$ and the length of $\mathbf{k}_3$, for given ω_1, ω_2 and the angle θ between $\mathbf{k}_1$, and $\mathbf{k}_2$. Energy conservation then restricts the angle θ in terms of ω_1 and ω_2. Three angular variables, to determine the overall orientation of the $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ tripod with respect to a fixed coordinate frame, remain: one may for instance choose two angles denoted by Ω to fix the direction of $\mathbf{k}_1$, and one angle ϕ for the rotations of the $(\mathbf{k}_2, \mathbf{k}_3)$ system around $\mathbf{k}_1$, as is shown in fig. 3.8).

Since we are only partially performing the integration over the final state momenta in order to just remove the energy-momentum conserving δ-functions in (3.41) or (3.79), we ignore the invariant amplitude and start from the Lorentz invariant phase-space integral

$$I((P-k_i)^2, m_i^2, M^2) = \frac{1}{(2\pi)^5} \times \int \frac{\mathrm{d}^3k_1}{2\omega_1}\frac{\mathrm{d}^3k_2}{2\omega_2}\frac{\mathrm{d}^3k_3}{2\omega_3}\,\delta^4(P-k_1-k_2-k_3)\,, \tag{3.93}$$

where P_μ, is the total incoming momentum with $P^2 = -M^2$. Note that the three variables $(P-k_i)^2$ are the direct analogue of the Mandelstam variables introduced for quasi-elastic scattering in (3.52). It is easy to verify the relation $\sum_i (P-k_i)^2 = -M^2 - \sum_i m_i^2$.

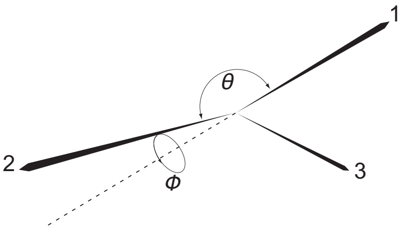


Figure 3.8: The definition of the angles θ and ϕ for the decay of a particle at rest into a three-particle final state.

Choosing parameters as described above the expression (3.93) can be written as

$$I((P-k_i)^2, m_i^2, M^2) = \frac{1}{(2\pi)^5}\int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\phi \times \int_0^\infty \frac{|\mathbf{k}_1|^2\,\mathrm{d}|\mathbf{k}_1|}{2\omega_1}\int_0^\infty \frac{|\mathbf{k}_2|^2\,\mathrm{d}|\mathbf{k}_2|}{2\omega_2}\int_0^\pi \frac{\sin\theta\,\mathrm{d}\theta}{2\omega_3}\,\delta(M-\omega_1-\omega_2-\omega_3)\,, \tag{3.94}$$

where ω_3 takes the form

$$\begin{aligned}\omega_3 &= \sqrt{|\mathbf{k}_1|^2 + |\mathbf{k}_2|^2 + 2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta + m_3^2} \\ &= \sqrt{\omega_1^2 + \omega_2^2 + m_3^2 - m_1^2 - m_2^2 + 2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta}\,.\end{aligned} \tag{3.95}$$

We remove the δ-function by performing the θ-integration. Using again (3.49) we write

$$\delta(M-\omega_1-\omega_2-\omega_3) = \left|\frac{\partial\omega_3}{\partial\cos\theta}\right|^{-1}\delta(\cos\theta-\alpha)\,, \tag{3.96}$$

where α is given by

$$\sqrt{\omega_1^2 + \omega_2^2 + m_3^2 - m_1^2 - m_2^2 + 2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta} + \omega_1 + \omega_2 = M\,,$$

or

$$2|\mathbf{k}_1||\mathbf{k}_2|\alpha = (M - \omega_1 - \omega_2)^2 - \omega_1^2 - \omega_2^2 - m_3^2 + m_1^2 + m_2^2\,. \tag{3.97}$$

Using

$$|\mathbf{k}_1|\,\mathrm{d}|\mathbf{k}_1| = \omega_1\,\mathrm{d}\omega_1\,, \quad |\mathbf{k}_2|\,\mathrm{d}|\mathbf{k}_2| = \omega_2\,\mathrm{d}\omega_2\,, \quad \frac{\partial\omega_3}{\partial\cos\theta} = \frac{|\mathbf{k}_1||\mathbf{k_2}|}{\omega_3}\,, \tag{3.98}$$

and performing the θ-integration, the integral (3.94) takes the form

$$I((P-k_i)^2, m_i^2, M^2) = \frac{1}{256\pi^3}\int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\omega_1 \int \mathrm{d}\omega_2\,, \tag{3.99}$$

with the restriction that $\alpha = \cos\theta$ in (3.97) must obey $-1 < \alpha < 1$ (otherwise the θ-integral in (3.94) vanishes). This implies that ω_1 and ω_2 are restricted according to

$$4(\omega_1^2 - m_1^2)(\omega_2^2 - m_2^2) < \left[(M - \omega_1 - \omega_2)^2 - \omega_1^2 - \omega_2^2 - m_3^2 + m_1^2 + m_2^2\right]^2. \tag{3.100}$$

Hence the energies ω_1 and ω_2 cover a closed region in the ω_1-ω_2 plane.

The invariant amplitude for a three-body decay only depends on the relative orientation of the momenta $\mathbf{k}_i$ in the centre-of-mass frame, which is completely fixed at this point; therefore the integral over the angular variables Ω and ϕ becomes trivial (provided the decaying particle has no spin). The result (3.99) then shows that for a constant amplitude the events will be distributed uniformly in the ω_1-ω_2 plane. On the other hand a specific structure of the decay amplitude will immediately show a characteristic density of events in this so-called Dalitz plot. Sometimes such a plot makes it straightforward to identify certain properties of a decaying particle, such as parity and spin. Furthermore, if two of the decay products originate from an unstable particle formed at an intermediate stage, i.e. in a sequential decay such as

$$\mathrm{A} \to \mathrm{B}^* + \mathrm{C} \to \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{C}\,,$$

the data points will cluster along a line in the Dalitz plot where ω_{C} or equivalently $\omega_{\mathrm{B}_1} + \omega_{\mathrm{B}_2}$, is constant. An example of this phenomenon is shown in fig. 3.9.

To exhibit some of the complexities of the three-particle phase-space let us further evaluate (3.99). First we introduce a variable $x = \frac{1}{2}(\omega_2 - \omega_3)$, and parametrize

$$\omega_1 = \omega\,, \quad \omega_2 = x + \tfrac{1}{2}(M - \omega)\,, \quad \omega_3 = -x + \tfrac{1}{2}(M - \omega)\,. \tag{3.101}$$

Figure 3.9: Dalitz plot distribution for the decay $\mathrm{J}/\psi \to \mathrm{K}^+\mathrm{K}^-$ plotted as a function of $M^2(K^+\gamma) = M^2_{\mathrm{J}/\psi} + M^2_{\mathrm{K}} - 2M_{\mathrm{J}/\psi}\omega_{K^-}$ and $M^2(\mathrm{K}^-\gamma) = M^2_{\mathrm{J}/\psi} + M^2_{\mathrm{K}} - 2M_{\mathrm{J}/\psi}\omega_{\mathrm{K}^+}$. Three bands with a 45^0 slope can be distinguished corresponding to (from the edge inwards) the $\mathrm{K}^+\mathrm{K}^-$ resonances $f'(1.60)$, $\theta(1.72)$ and $\xi(2.20)$, respectively. [K. Einsweiler, Thesis, Stanford University, SLAC-Report 292 (1984), as reported by D. Hitlin, Proceedings of the Cornell Conference (1983).]

Obviously the minimal value of ω is at m_1. The maximal value of ω follows from the fact that $-(k_2 + k_3)^2 > (m_2 + m_3)^2$, which can easily be proven in the centre-of-mass frame for particles 2 and 3. since

$$(k_2 + k_3)^2 = (P - k_1)^2 = -M^2 - m_1^2 + 2M\omega$$

we find the inequality,

$$M^2 + m_1^2 - 2M\omega \geqslant (m_2 + m_3)^2 \, .$$

Hence

$$\omega \leqslant \omega_m = \tfrac{1}{2}M^{-1}(M^2 - (m_2 + m_3)^2 + m_1^2) \, . \tag{3.102}$$

For given ω we must now establish the boundary values of x. To that order we rewrite (3.100) as

$$\begin{aligned} &\Big(x + \tfrac{1}{2}(M - \omega)\frac{m_2^2 - m_3^2}{2M\omega - M^2 - m_1^2}\Big)^2 (2M\omega - M^2 - m_1^2)^2 \\ &\quad \leqslant M^2(\omega^2 - m_1^2)(\omega_m - \omega)\Big(\omega_m - \omega + \frac{2m_2m_3}{M}\Big) \, , \end{aligned} \tag{3.103}$$

where we have used the fact that $2M\omega - M^2 - m_1^2$ is negative. From this quadratic equation we conclude that $x_- < x < x_+$, with $x_\pm$ given by

$$x_\pm = \frac{1}{\omega_m - \omega + \frac{(m_2+m_3)^2}{2M}} \times\left\{ \frac{m_2^2 - m_3^2}{4M}(M-\omega) \pm \tfrac{1}{2}\sqrt{(\omega^2 - m_1^2)(\omega_m - \omega)\Big(\omega_m - \omega + \frac{2m_2m_3}{M}\Big)}\right\}. \tag{3.104}$$

So the result for the phase-space integral is

$$I((P-k_i)^2, m_i^2, M^2) = \frac{1}{256\pi^5}\int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\phi \int_{m_1}^{2\pi} \mathrm{d}\omega \int_{x_-}^{x_+} \mathrm{d}x\,. \tag{3.105}$$

All that remains is to insert the invariant amplitude, which for a scattering process depends in general on ω and x and two of the angles. For a decay reaction there is no angular dependence (unless the decaying particle has spin).

In order to see how much a three-particle decay rate is suppressed compared to the two-particle rate, consider the decay of a spinless particle of mass M into spinless particles of negligible masses. The phase space integrals then follow from (3.50) and (3.105), and are equal to $I = (8\pi)^{-1}$ and $I = (256\pi^3)^{-1}M^2$, respectively. Assuming that the two-particle invariant amplitude, which has the dimension of a mass, is given by gM, where g is a dimensionless parameter, one finds

$$\Gamma(M \to 2) = \frac{1}{16\pi} g^2 M\,.$$

If, for reason of comparison, we assume that g is just the three-particle decay amplitude, then the rate for the three-particle decay is

$$\Gamma(M \to 3) = \frac{1}{512\pi^3} g^2 M\,,$$

so that

$$\frac{\Gamma(M\to 3)}{\Gamma(M\to 2)} = \frac{1}{32\pi^2} \approx 3\times 10^{-3}\,. \tag{3.106}$$

In realistic situations where the decay products do not necessarily have negligible masses, the suppression factor in (3.106) is even smaller because there is relatively less phase space for decays into massive particles. For instance, when $M = m_1 + m_2 + m_3$ and $M > m_1 + m_2$, the three-body decay is forbidden, whereas the two-body decay is still kinematically possible.

3.8. *Hadronic decays of neutral K-mesons*

Interesting decay processes that involve only spinless particles are those of a neutral K-meson into two or three pions. There are four pseudoscalar K-mesons, namely the charged K^+ and K^- and the neutral K^0 and $\bar{\mathrm{K}}^0$ which are almost degenerate in mass. Their corresponding fields can be arranged into a complex two-component vector,

$$K = (K^+, K^0)\,, \tag{3.107}$$

transforming as a doublet under isospin transformations, so that its isospin equals $I = 1/2$. To classify the strongly interacting particles it is customary to assign a "strangeness" quantum number; nucleons and pions have $S = 0$, and K^+- and K^0-mesons have $S = 1$. The complex conjugate of the above doublet,

$$K^* = (K^-, \bar{K}^0), \tag{3.108}$$

has $S = -1$. In the context of strong interactions, where isospin and strangeness are conserved quantum numbers, K^0 and $\bar{\mathrm{K}}^0$ must correspond to two physical particles of equal mass. Just as K^+ and K^- are each others antiparticle, so are K^0 and $\bar{\mathrm{K}}^0$.

If the effect of the weak interactions is taken into account, however, isospin and strangeness are no longer conserved quantum numbers. Consequently the treatment of the neutral K-mesons is no longer intrinsically related to the charged K-mesons and one may prefer to describe the former in terms of two real fields

$$K_1^0 = \frac{\mathrm{i}}{\sqrt{2}}(\bar{K}^0 - K^0)\,, \quad K_2^0 = \frac{1}{\sqrt{2}}(\bar{K}^0 + K^0)\,, \tag{3.109}$$

which no longer carry any definite strangeness. In the absence of strangeness conservation it is no longer clear which field representation is most appropriate, but one may exploit another symmetry, namely CP. This symmetry is the product of a parity reversal (P) and a charge conjugation (C) transformation. The standard weak interactions conserve CP, although they break C and P separately. Apart from changing space-time coordinates $(\mathbf{x}, t)$ into $(-\mathbf{x}, t)$ they act on neutral K-meson fields as

$$K^0 \xrightarrow{\mathrm{CP}} -\bar{K}^0\,, \qquad \bar{K}^0 \xrightarrow{\mathrm{CP}} -K^0\,.$$

Therefore K_1^0 and K_2^0 are precisely eigenvectors under CP, viz.

$$K_1^0 \xrightarrow{\mathrm{CP}} K_1^0\,, \qquad \bar{K}_2^0 \xrightarrow{\mathrm{CP}} -K_2^0\,. \tag{3.110}$$

If CP were an exact symmetry then K_1^0 and K_2^0 would be associated with neutral particles whose mass difference is small and induced by the strangeness-violating component of the weak interactions. The lifetimes of K_1^0 and K_2^0 would also be different because their CP assignments imply that they should decay into different final states. To elucidate this difference consider their decay into pions. Because pions have $S = 0$ this decay violates strangeness. Isospin is violated as well, because pions have $I = 1$ so that they can only form states of integer isospin, whereas K-mesons have $I = 1/2$. Actually it turns out that the two-pion state that arises in K-decay is predominantly $I = 0$.[1] To see this we note that the experimental result for this decay gives the branching ratio,

$$\frac{\Gamma(\mathrm{K}_1^0 \to \pi^0\pi^0)}{\Gamma(\mathrm{K}_1^0 \to \pi^+\pi^-)} \approx 4.6\,,$$

which is close to the value $\frac{1}{2}$ characteristic for an $I = 0$ two-pion state (cf. 3.83). Furthermore, an $I = 0$ state is electrically neutral, so that the decay of a charged K-meson cannot lead to an $I = 0$ state. Indeed, experimentally one finds a suppression,

$$\frac{\Gamma(\mathrm{K}^+ \to \pi^+\pi^0)}{\Gamma(\mathrm{K}_1^0 \to \pi^0\pi^0)} \approx 0.01\,.$$

The fact that the pions appear mainly in an $I = 0$ state lends support to the so-called $\Delta I = 1/2$ rule, according to which non-leptonic weak decays occur mainly through a $\Delta I = 1/2$ isospin-violating component (see problem 3.9).

If CP is conserved, only K_1^0 can decay into two pions, as the resulting two-pion final state is even under CP irrespective of isospin considerations. . On the other hand a final state of three neutral pions is CP odd, so that K_2^0 can decay in that way, whereas the decay $\mathrm{K}_1^0 \to \pi^0\pi^+\pi^-$ can only take place for odd values of the angular momentum of the $\pi^+\pi^-$ pair. Non-zero angular momenta are, however, suppressed, so that K_1^0 will mainly decay into two, and K_2^0 into three pions. Because the phase-space volume for the three-pion decay mode is considerably less than for the two-pion mode (cf. 3.106), one expects the $\mathrm{K}_1^0 \to 2\pi$ decay rate to be much larger than the rate for $\mathrm{K}_2^0 \to 3\pi$. Consequently K_1^0 must have a shorter lifetime than K_2^0. Experiments show that this is indeed the case; both the partial and the total decay rates differ by almost three orders of magnitude (see table 3.2, where the short-lived and long-lived mesons are called $\mathrm{K_S}$ and $\mathrm{K_L}$ for reasons discussed below).

Although the above considerations are thus consistent with experimental observations, the situation is actually more complicated. In 1964 it was established that the long-lived neutral K-meson also decays into two pions with a

[1] Two-pion states can have $I = 2$, or $I = 0$.

Particle	$l(JP)$	Mass [MeV/c^2]	Mean lifetime [s]	Decay mode	Branching ratio
$K^{\pm}$	$\frac{1}{2}(0^-)$	493.7	$1.237 \times l0^{-8}$	$\pi^+\pi^0$	0.21
				$\pi^+\pi^+\pi^-$	0.065
				$\pi^+\pi^0\pi^0$	0.017
K_S	$\frac{1}{2}(0^-)$	497.67	0.892×10^{10}	$\pi^+\pi^-$	0.686
				$\pi^+\pi^0\pi^0$	0.314
K_L	$\frac{1}{2}(0^-)$	$\frac{1}{2}(0^-)$	497.67	$\pi^0\pi^0\pi^0$	0.22
				$\pi^+\pi^-\pi^0$	0.12

Table 3.2: Properties of K-meson decays. Here $I(J^P)$ denote the isospin, spin and parity of the meson. Note that the mass difference $m_{K_L} - m_{K_S} = 0.535 \times 10^{10}\, \hbar\, c^{-1} = 3.52 \times 10^{-12}\,\mathrm{MeV}\, c^{-2}$.

small branching ratio $B \approx 2 \times 10^{-3}$. Therefore one is forced to conclude that CP is slightly broken in Nature. Because there are then transitions between the CP-even and CP-odd states, the physical particles do not directly correspond to the fields K_1^0 and K_2^0. There is instead a mixing phenomenon, which we will now discuss in terms of a simple model.

We should start from (2.66) which expresses the full propagator in terms of the sum of all irreducible self-energy diagrams. Due to CP-violating reactions there will be self-energy diagrams with an incoming K_1^0, and an outgoing K_2^0, and vice versa, so that we are dealing with a 2×2 matrix problem for the (K_1^0,K_2^0) system. Let us make the following parametrization for $\Sigma(p)$:

$$\frac{\mathrm{i}}{(2\pi)^4}\Sigma(p) = \begin{pmatrix} m^2(1-\delta) & \mu^2 \\ \mu^2 & m^2(1+\delta) \end{pmatrix}, \tag{3.111}$$

where μ characterizes the CP-violating K_1^0-K_2^0 transitions; the quantity δ measures the difference between the K_1^0-K_1^0 and K_2^0-K_2^0 diagrams, and is induced by strangeness violating interactions (because, under strangenss, $K_2^0 \leftrightarrow K_2^0$). Here we should emphasize that m, δ and μ are, in general, complex functions of p^2. We are only interested in the full propagator near the poles, and as the K_L-K_S mass difference is small we may make the simplifying assumption that m, δ and μ are constants. The mass term of the lowest-order propagator has already been absorbed into $\Sigma(p)$, so that the full propagator is

$$\mathrm{i}(2\pi)^4\Delta(p) = \begin{pmatrix} p^2+m^2(1-\delta) & \mu^2 \\ \mu^2 & p^2+m^2(1+\delta) \end{pmatrix}^{-1}. \tag{3.112}$$

It is easy to diagonalize (3.112). Since μ is small we find the inverse eigenvalues

$p^2 + m^2(1-\delta) + O(\mu^4)$ and $p^2 + m^2(1+\delta) + O(\mu^4)$, with corresponding eigenvectors

$$K_{\rm S} = \frac{1}{\sqrt{1+\varepsilon^2}}(K_1^0 - \varepsilon K_0^2)\,, \qquad K_{\rm L} = \frac{1}{\sqrt{1+\varepsilon^2}}(K_2^0 - \varepsilon K_1^0)\,, \tag{3.113}$$

where

$$\varepsilon = \frac{\mu^2}{2m^2\delta} + O(\mu^4)\,. \tag{3.114}$$

Because ε is small K_S and K_L correspond to the fields associated with the short-and long-lived neutral K-mesons, respectively.

We may now decompose $m^2(1\pm\delta)$ into real and imaginary parts

$$\begin{aligned} m^2(1-\delta) &= m_{\rm S}^2 - {\rm i}m_{\rm S}\Gamma_{\rm S}\,, \\ m^2(1+\delta) &= m_{\rm L}^2 - {\rm i}m_{\rm L}\Gamma_{\rm L}\,, \end{aligned} \tag{3.115}$$

where $m_{\rm S,L}$ and $\Gamma_{\rm S,L}$ are the masses and (rest frame) decay rates of $\rm K_{S,L}$. Therefore we have

$$\varepsilon = \frac{\mu^2}{m_{\rm L}^2 - m_{\rm S}^2 - {\rm i}(m_{\rm L}\Gamma_{\rm L} - m_{\rm S}\Gamma_{\rm S})}\,. \tag{3.116}$$

Due to the K_1^0 - K_2^0 mixing, $\rm K_L$ can now decay into two pions. If mixing is the only mechanism responsible for this decay, one must find

$$\frac{\Gamma({\rm K_L}\to 2\pi)}{\Gamma({\rm K_S}\to 2\pi)} \approx |\varepsilon|^2\,. \tag{3.117}$$

Similarly, the process $\rm K_S \to \pi^0\pi^0\pi^0$ is also possible through this effect, and we find the same result

$$\frac{\Gamma({\rm K_S}\to 3\pi)}{\Gamma({\rm K_L}\to 3\pi)} \approx |\varepsilon|^2\,. \tag{3.118}$$

Unfortunately, it is very difficult to measure the decay $\rm K_S \to 3\pi$. Moreover, mixing is not the only mechanism responsible for these decays. The CP-violating interactions will also induce direct transitions of $\rm K_2^0$ into two and $\rm K_1^0$ into three pions.

The mixing between $\rm K^0$ and $\rm \bar{K}^0$ gives rise to the interesting phenomenon of strangeness oscillations. In a strong interaction process one often has a situation where only $\rm K^0$ but not $\rm \bar{K}^0$ is produced (or the other way around). According to the above arguments this leads to a particular mixture of long and short lived K-mesons. As the $\rm K_L$-$\rm K_S$ mass difference is very small, this mixture is then essentially determined by the linear combination

$$K^0 = \frac{1}{\sqrt{2}}\frac{1+{\rm i}\varepsilon}{\sqrt{1+\varepsilon^2}}(K_{\rm L} + {\rm i}K_{\rm S})\,, \tag{3.119}$$

However, K_L and K_S decay at different rates, so that the production amplitude for K_L and K_S evolves differently in time. Generalizing the arguments given in section 3.6 one finds the ratio (here $\mathit{\Gamma}_S$ and $\mathit{\Gamma}_L$, denote the decay widths in an arbitrary frame)

$$\frac{\mathcal{A}(X \to K_S^0 + Y)}{\mathcal{A}(X \to K_L^0 + Y)} = -i \, \frac{\exp(-i\omega_S t - \frac{1}{2}\mathit{\Gamma}_S t)}{\exp(-i\omega_L t - \frac{1}{2}\mathit{\Gamma}_L t)} \, . \tag{3.120}$$

(the factor $-i$ arises because the K^0-meson is outgoing). If we assume that only $\bar{K}^0$ is produced at $t = 0$ then (3.120) changes sign. For stable mesons, the same sign change would take place at times t where $\exp(i(\omega_S - \omega_L)t) = -1$. In other words, the probability for detecting K-mesons with strangeness $S = +1$ and $S = -1$ oscillates in time with oscillation period $\mathit{\Delta} t = 2\pi(\omega_L - \omega_S)^{-1}$. These oscillations are the same quantum-mechanical phenomenon as the spin oscillation of a particle moving in a magnetic field, as measured in the Stern-Gerlach experiment. The wave function is then in general a superposition of spin eigenstates which evolve differently in time because they have different energy and the oscillation time is inversely proportional to the magnetic field. What is different in the case of strangeness oscillations is that the particles are not stable, so that there are exponential damping factors in (3.120).

Strangeness oscillations can be measured. For instance, leptonic decays of K-mesons follow the so-called $\mathit{\Delta} Q = \mathit{\Delta} S$ rule, which implies that the electric charge and the strangeness of the mesons in these decays are changed, by the same unit. Hence one has

$$K^0 \to \pi^- e^+ \nu_e \, , \qquad \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e \, . \tag{3.121}$$

From (3.119) one concludes that the decay amplitudes have ratios

$$\frac{\mathcal{A}(K_S \to \pi^- e^+ \nu_e)}{\mathcal{A}(K_L \to \pi^- e^+ \nu_e)} = i \, , \qquad \frac{\mathcal{A}(K_S \to \pi^+ e^- \bar{\nu}_e)}{\mathcal{A}(K_L \to \pi^+ e^- \bar{\nu}_e)} = -i \, . \tag{3.122}$$

Combining (3.120) and (3.122) one concludes that the number of positrons detected at time t is proportional to

$$N_+(t) \propto \left| \exp(-i\omega_S t - \tfrac{1}{2}\mathit{\Gamma}_S t) + \exp(-i\omega_L t - \tfrac{1}{2}\mathit{\Gamma}_L t) \right|^2 ,$$

or

$$N_+(t) = \tfrac{1}{4} N_+(0) \left[e^{-\mathit{\Gamma}_S t} + e^{-\mathit{\Gamma}_L t} + 2 e^{-\frac{1}{2}(\mathit{\Gamma}_S + \mathit{\Gamma}_L)t} \cos(\omega_S - \omega_L)t \right] . \tag{3.123}$$

Likewise one has

$$N_-(t) = \tfrac{1}{4} N_-(0) \left[e^{-\mathit{\Gamma}_S t} + e^{-\mathit{\Gamma}_L t} - 2 e^{-\frac{1}{2}(\mathit{\Gamma}_S + \mathit{\Gamma}_L)t} \cos(\omega_S - \omega_L)t \right] . \tag{3.124}$$

Figure 3.10: Measured values for $N_+(\tau)/N_+(0)$ and $N_-(\tau)/N_-(0)$. The curves drawn correspond to (3.123) and (3.124). [F. Niebergall, M. Regler, H.H. Stier, K. Winter, J.J. Aubert, X. De Bouard, V. Lepeltier, L. Massonet, H. Pessard, M. Vivargent, T.R. Willitts, M. Yvert, W. Bartl, G. Neuhofer and M. Steuer, Phys. Lett. 49B (1974) 103.]

In this way observing the number of positrons and/or electrons as a function of the distance from a monochromatic K-meson source makes it possible to determine the $\mathrm{K_L}$-$\mathrm{K_S}$ mass difference and the decay rates. An experimental plot of the time dependence of $N_+(t)/N_+(0)$ and $N_-(t)/N_-(0)$ is shown in fig. 3.10, where the time is measured in the K-meson rest frame (so that ω_S, ω_L, Γ_S and Γ_L take their rest frame values).

Problems

3.1. Calculate the probability current (3.5) and energy-momentum tensor (3.10) for a wave function that contains both positive and negative energy components,

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}} \{ f(\mathbf{p}) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega(\mathbf{p})t} + g(\mathbf{p}) \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x} + \mathrm{i}\omega(\mathbf{p})t} \} \,.$$

Determine the analogues of (3.8) and (3.13). Investigate their time dependence and their signs. Note the relevance of these results in view of the text following (3.6). In addition, calculate the analogue of (3.14) and explain the time independence of (3.8) and (3.14).

3.2. Consider the scattering reaction $1+2 \to 3+4$ where the particles have masses $m_1, \ldots, m_4$. Show that the variable t defined in (3.52) ranges between t^+ and t^- equal to

$$\begin{aligned} t^\pm =& \, m_1^2 + m_3^2 \\ & - \frac{1}{2s} \big[(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) \mp \lambda^{1/2}(s, m_1^2, m_2^2) \lambda^{1/2}(s, m_3^2, m_4^2) \big] \,. \end{aligned}$$

Show that, when s is large compared to the masses,

$$\begin{aligned} t^+ \to & - (m_1^2 - m_3^2)(m_2^2 - m_4^2) s^{-1} \\ & - m_1^2 - m_3^2 + m_2^2 - m_4^2)(m_1^2 m_2^2 - m_3^2 m_4^2) s^{-2} + \mathcal{O}(s^{-3}) \,, \\ t^- \to & - s + \mathcal{O}(s^0) \,. \end{aligned} \qquad (1)$$

3.3. In section 2.5 the amplitude for pion-pion scattering was calculaed in a simple model. Use the result (2.61) to determine the amplitude for the elastic scattering process $\pi^+ + \pi^- \to \pi^+ + \pi^-$, using the isospin polarization vectors as defined in section 3.5. Observe that only two of the three terms contribute and argue that this is in accord with charge conservation.
Determine now the differential cross section in the centre-of-mass frame as a function of the scattering angle between the incoming and outgoing π^+ and the total energy s, using the relation $t = -(s - 4m^2)\sin^2 \frac{1}{2}\theta_\mathrm{CM}$ and assuming that the coupling constant λ satisfies $\lambda^2 = 2\, g\, \mu^2$. Compare the result,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_\mathrm{CM}} = \frac{g^2}{\pi^2\, s} \left[\frac{\mu^2 - 2s}{\mu^2 - s} - \frac{\mu^2}{\mu^2 + (s - 4m^2)\sin^2 \frac{1}{2}\theta_\mathrm{CM}} \right]^2 ,$$

with the result for the same scattering reaction but mediated by a photon, which we shall derive in due course (c.f. (4.49)), and note the qualitative differences.

3.4. A particle with mass M and laboratory energy E decays into two particles with masses m_1, and m_2. The particle with mass m_1 moves at angle θ with respect to the parent's original direction.
Provided that θ is *small* and E is *large*, determine the range of energies that the particle with mass m_1 may have. As θ increases, find the particular θ for which there is only one energy value for the particle and argue that this is the largest angle possible.
Show that $E = \frac{1}{2}(M^2 + m_1^2 - m_2^2)/m_1$ corresponds to that E below which there are no longer two values of energy for particle 1 at any θ. What is its significance in terms of the allowed range for θ?

3.5. Consider the scattering reaction $1+2 \to 3+4$ as discussed in section 3.4. Show that

$$2p_2 \cdot (p_1 - p_3) = m_2^2 + m_1^2 - m_4^2 + m_3^2 + 2p_1 \cdot p_3 \,. \tag{1}$$

In the laboratory frame (cf. 3.56 and 3.57) show that (1) reduces to

$$2m_2(E' - E) = m_2^2 + m_1^2 - m_4^2 + m_3^2 + 2p_1 \cdot p_3 \,, \tag{2}$$

while

$$t = -2m_2(E - E') - m_2^2 + m_4^2. \tag{3}$$

For fixed E, E' is a function of the scattering angle θ as defined by (2). Determine $\mathrm{d}E'/\mathrm{d}\cos\theta$ and $\mathrm{d}t/\mathrm{d}\cos\theta$ in terms of E and E'. Use this result to derive (3.60).

3.6. Show that

$$\int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}} f(p) = \int \mathrm{d}^4 p\, \delta(p^2 + m^2)\, \theta(p^0)\, f(p) \,,$$

by making use of (3.49). Subsequently argue that the integral

$$I(k) = \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega(\mathbf{p})}}\, f(k^2, p^2, k \cdot p) \,,$$

is Lorentz invariant, i.e., it is a function of k^2 only.

3.7. Consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 - g^4\phi^4 \,.$$

Find the one-and two-loop irreducible self-energy diagrams. Show that the one-loop diagram is a *real* constant. Write the two-loop diagram in the form suggested by (3.74). By following the same manipulations that lead to (3.77), show that its imaginary part is related to the three-particle decay rate defined via (3.79). Pay particular attention to the combinatorial factors in the self-energy graph, the decay amplitude, and the decay rate.

3.8. Consider the three-particle phase space integral (3.93) and insert the equality $\int \mathrm{d}s_{23} \int \mathrm{d}^4 q\, \delta^4(q-k_2-k_3)\, \delta(s_{23}+q^2) = 1$ under its integral sign. Subsequently, factorize the integral into a product of two-particle phase space integrals and demonstrate that (3.93) equals

$$I = \frac{1}{128\pi^3} \int_{(m_2+m_3)^2}^{(M-m_1)^2} \mathrm{d}s_{23}\, \frac{\lambda^{1/2}(M^2, s_{23}, m_1^2)}{M^2}\, \frac{\lambda^{1/2}(s_{23}, m_2^2, m_3^2)}{s_{23}}\,.$$

This reveals the existence of a recursion relation linking multiparticle phase-space integrals.

3.9. From the discussion of K-mesons in section 3.8 we know that the doublet $(\mathrm{K}^+, \mathrm{K}^0)$ transforms under isospin transformations just as a doublet of spin-$\frac{1}{2}$ states under ordinary rotations. Hence

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \rightarrow \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}' = \exp\!\left(\tfrac{1}{2}\mathrm{i}\theta\cdot\tau\right) \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \tag{1}$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ are the angles specifying the isospin rotation and $\tau = (\tau_1, \tau_2, \tau_3)$ are the usual Pauli spin matrices (which in the context of ordinary spin are denoted by σ_1, σ_2, σ_3)

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

which satisfy

$$\left[\frac{\tau_a}{2}, \frac{\tau_b}{2}\right] = \mathrm{i}\varepsilon_{abc}\frac{\tau_c}{2}\,. \tag{3}$$

Hence K^+ and K^0 are eigenstates of the isospin operator $I_3 = \frac{1}{2}\tau_3$, which is the analogue of the angular momentum operator L_z. Consequently, K^+ has $I_3 = \frac{1}{2}$ and K^0 has $I_3 = -\frac{1}{2}$. The complex conjugate of (1) shows that K^- has therefore $I_3 = -\frac{1}{2}$ and $\bar{\mathrm{K}}^0$ has $I_3 = \frac{1}{2}$.
Verify that the electric charges of K^+, K^-, K^0 and $\bar{\mathrm{K}}^0$ satisfy the relation

$$Q = I_3 + \tfrac{1}{2}S\,. \tag{4}$$

Consult the Review of Particle Properties to show that this relation holds for all light-mass mesons (for higher mass other quantum numbers such as charm are necessary). The relation (4) is a special case of the so-called Gell-Mann-Nishijima relation,

$$Q = I_3 + \tfrac{1}{2}Y\,, \tag{5}$$

where the "hypercharge" $Y \equiv S + B$. Here B denotes "baryon number", which equals $+1$ for baryons, -1 for antibaryons and 0 for mesons. To check the validity of (5) for protons and neutrons, note that nucleons have $S = 0$ (so that $Y = B = 1$), that the proton has $I_3 = \frac{1}{2}$ and that the neutron has $I_3 = -\frac{1}{2}$. The validity of (5) for light-mass baryons and antibaryons can again be verified in the Review of Particle Properties.

3.10. As mentioned in section 3.8 the (hadronic) decays of K-mesons satisfy the $\varDelta I = \frac{1}{2}$ rule. This rule is equivalent to asserting that an extra unobserved "particle" is produced in the decay with $I = \frac{1}{2}$ and $S = 1$ which carries no energy or momentum. This isodoublet is called a "spurion", and, according to the $\varDelta I = \frac{1}{2}$ rule, the K-decay amplitudes conserve both isospin and strangeness if the spurion doublet is included.
The amplitude for pionic K-decay is thus proportional to the K-field, the spurion field ϕ and a sufficient number of π-fields. As the pions have $S = 0$, we start by considering all products of K and ϕ fields that have $S = 0$. There are the following four combinations and their complex conjugates

$$\begin{aligned} X_0 &= K_i^* \phi_i \,, \\ X_a &= K_i^* (\tau_a)_{ij} \phi_j \,,\ i,j = 1,2\,;\ a = 1,2,3, \end{aligned} \tag{1}$$

where K_i denotes the (K^+, K^0) doublet. Show, by considering infinitesimal transformations

$$K_i \to \quad K_i' = K_i + \tfrac{1}{2}\mathrm{i}(\theta^a \tau_a)_{ij} K_j + O(\theta^2)\,, \tag{2}$$

$$\phi_i \to \quad \phi_i' = \phi_i + \tfrac{1}{2}\mathrm{i}(\theta^a \tau_a)_{ij} \phi_j + O(\theta^2)\,, \tag{3}$$

that X_0 is invariant and X_a transforms as a vector (just as the pion field) under isospin transformations. We now need products of π-fields which are either invariant or transform as a vector under isospin transformations, and which can be contracted with (1) to give invariant mteractions. If we exclude derivatives of the fields (meaning that we consider only states of lowest angular momentum, which is a good approximation in these decays) we have two obvious combinations

$$Y^0 = \pi \cdot \pi\,, \qquad Y^a = (\pi \cdot \pi)\, \pi^a\,, \tag{4}$$

Hence we find the isospin invariants

$$\begin{aligned} L_1 &= \mathrm{i} g_1 X_0 Y^0 + \text{complex conjugate}\,, \\ L_2 &= g_2 X_a Y^a + \text{complex conjugate}\,. \end{aligned} \tag{5}$$

Now we must deal with the spurion field. Since this field is only a vehicle for carrying away strangeness and isospin in the decay, it must be put equal to a constant, thus inducing strangeness and isospin violations in (5), according to the $\varDelta I = 1/2$ rule. However, electric charge must be conserved, so that we are forced to choose $\phi = (0, 1)$. Show that CP conservation implies that the coupling constants g_1 and g_2 must be real (see the discussion following 3.109).
Write (5) in terms of $K^\pm$, K_1^0, K_0^2, $\pi^\pm$ and π^0 fields and calculate the decay rates to show that

$$\begin{aligned} &\varGamma(\mathrm{K}_1^0 \to \pi^+\pi^-) : \varGamma(\mathrm{K}_1^0 \to \pi^0\pi^0) = 2:1\,, \\ &\varGamma(\mathrm{K}^+ \to \pi^+\pi^+\pi^-) : \varGamma(\mathrm{K}_2^0 \to \pi^0\pi^0\pi^0) : \varGamma(\mathrm{K}_2^0 \to \pi^+\pi^-\pi^0) : \varGamma(\mathrm{K}^+ \to \pi^+\pi^0\pi^0) \\ &= 4:3:2:1\,, \end{aligned} \tag{6}$$

where we have ignored the fact that the phase space integrals should be evaluated for unequal kaon masses and unequal pion masses.

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