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Relativistic fluid dynamics

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1 Ideal relativistic fluids

1.1 Energy-momentum and fluid current

An introduction to relativistic hydrodynamics can be found in ref. [1]; a recent review of perfect fluids and their generalizations is [2]. We summarize the results here following the convention that the units of space and time are chosen such that $c = 1$. The covariant energy-momentum tensor of an isotropic fluid at rest is

$$T(0) = \left(\begin{array}{cc} \varepsilon & 0 \\ & \\ 0 & p\,\delta_{ij} \end{array} \right), \tag{1}$$

where ε is the proper energy density, and p the hydrostatic pressure. Let $\Lambda(\mathbf{v})$ represent a Lorentz transformation to a system with velocity $\mathbf{v}$ (measured in units of c) relative to the rest frame:

$$\Lambda(\mathbf{v}) \;\; = \;\; \left(\begin{array}{cc} 0 & 0 \\ & \\ 0 & \delta_{ij} - \frac{v_i v_j}{\mathbf{v}^2} \end{array} \right) + \gamma \left(\begin{array}{cc} 1 & -v_j \\ & \\ -v_i & \frac{v_i v_j}{\mathbf{v}^2} \end{array} \right) \tag{2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}}. \tag{3}$$

In this frame the energy-momentum tensor takes the form

$$T(\mathbf{v}) = \Lambda(\mathbf{v})\, T(0)\, \tilde{\Lambda}(\mathbf{v}) = p \left(\begin{array}{cc} -1 & 0 \\ & \\ 0 & \delta_{ij} \end{array} \right) + \gamma^2 (\varepsilon + p) \left(\begin{array}{cc} 1 & -v_j \\ & \\ -v_i & v_i v_j \end{array} \right). \tag{4}$$

Introducing the velocity 4-vector

$$u^{\mu} = \gamma\,(1, \mathbf{v})\,, \qquad u^2 = \eta_{\mu\nu} u^{\mu} u^{\nu} = -1, \tag{5}$$

the energy-momentum tensor in a frame w.r.t. which the fluid moves with velocity $\mathbf{v}$ can be written covariantly as

$$T_{\mu\nu} = p \eta_{\mu\nu} + (\varepsilon + p)\, u_{\mu} u_{\nu}. \tag{6}$$

The conservation laws of energy and momentum take the differential form

$$\partial_{\mu} T^{\mu\nu} = \partial_{\nu} p + \partial_{\mu} \left((\varepsilon + p)\, u^{\mu} u^{\nu} \right) = 0. \tag{7}$$

The content of these equations becomes more explicit by splitting them in space and time components. The equation with $\nu = 0$ becomes

$$\frac{\partial p}{\partial t} - \frac{\partial}{\partial t} \left(\frac{\varepsilon + p}{1 - \mathbf{v}^2} \right) - \nabla \cdot \left(\frac{(\varepsilon + p)\, \mathbf{v}}{1 - \mathbf{v}^2} \right) = 0, \tag{8}$$

whilst the space components with $\nu = i$ become

$$\begin{aligned} 0 &= \nabla_i p + \frac{\partial}{\partial t} \left(\frac{(\varepsilon + p)\, v_i}{1 - \mathbf{v}^2} \right) + \nabla \cdot \left(\frac{(\varepsilon + p)\, \mathbf{v} v_i}{1 - \mathbf{v}^2} \right) \\ &= \nabla_i p + v_i \frac{\partial p}{\partial t} + \frac{\varepsilon + p}{1 - \mathbf{v}^2} \left(\frac{\partial v_i}{\partial t} + \mathbf{v} \cdot \nabla v_i \right). \end{aligned} \tag{9}$$

Here the second line is obtained by substitution of eq. (8) to eliminate the time derivatives of ε and p. The last equation can be rewritten as a relativistic form of the Euler equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = - \frac{1 - \mathbf{v}^2}{\varepsilon + p} \left(\nabla p + \mathbf{v} \frac{\partial p}{\partial t} \right). \tag{10}$$

The fluid itself is described by a fluid-density current j^{μ}; in the rest frame it takes the form

$$j^{\mu} = (\rho,\, \mathbf{0}), \tag{11}$$

with ρ the fluid density at rest. The definition can extended to a moving fluid, taking the general form

$$j^{\mu} = \left(j^0,\, \mathbf{j} \right) = \rho u^{\mu}, \qquad j^2 = \eta_{\mu\nu} j^{\mu} j^{\nu} = -\rho^2. \tag{12}$$

Thus the density can be defined in an invariant way. The physical requirement that the total amount of fluid is conserved, is described by the vanishing of the 4-divergence:

$$\partial_{\mu} j^{\,\mu} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial t} \left(\frac{\rho}{\sqrt{1 - \mathbf{v}^2}} \right) + \nabla \cdot \left(\frac{\rho \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) = 0, \tag{13}$$

the Bernouilli equation in relativistic form.

1.2 Thermodynamical considerations

A complete description of the fluid requires specification of the relation between ε, p and ρ. This is provided by the equation of state. It is often convenient to specify this in the form of expressions for the energy density and pressure in terms of the fluid density:

$$\varepsilon = f(\rho), \qquad p = g(\rho). \tag{14}$$

As we will see shortly, in situations where the entropy per is constant, we can take the function $g(\rho)$ to be the negative of the legendre transform of the energy density:

$$p = g(\rho) = \rho f'(\rho) - f(\rho) \quad \Leftrightarrow \quad \varepsilon + p = \rho f'(\rho) \tag{15}$$

The Euler equation can then be written in the form

$$\frac{\partial(\gamma\rho\mathbf{v})}{\partial t} + \nabla\cdot(\gamma\rho\mathbf{v}\mathbf{v}) = -\frac{1}{\gamma f'(\rho)}\left(\nabla p + \mathbf{v}\,\frac{\partial p}{\partial t}\right). \tag{16}$$

If the fluid obeys thermodynamics (strictly speaking, in conditions of thermal equilibrium), we can derive a useful relation for the flow of entropy starting from the condition of energy-momentum conservation (7), rewritten as

$$\partial_\nu p + \partial_\mu\left(\frac{(\varepsilon+p)}{\rho}\,j^\mu u_\nu\right) = 0. \tag{17}$$

After contraction with u^ν and using the conservation of the current and equation (5) expressing the fact that the four-velocity is a time-like unit vector, we get

$$\begin{aligned} 0 &= u^\mu\partial_\mu p - j^\mu\partial_\mu\left(\frac{\varepsilon+p}{\rho}\right) \\ &= u^\mu\left[\partial_\mu p - \rho\,\partial_\mu\left(\frac{\varepsilon+p}{\rho}\right)\right] = -j^\mu\left[p\,\partial_\mu\left(\frac{1}{\rho}\right) + \partial_\mu\left(\frac{\varepsilon}{\rho}\right)\right]. \end{aligned} \tag{18}$$

Now the first law of thermodynamics for a system with one component states that

$$dU = TdS - pdV + \mu dN. \tag{19}$$

We define the specific energy, entropy and volume as the corresponding quantity per particle:

$$u = \frac{U}{N}, \qquad s = \frac{S}{N}, \qquad v = \frac{V}{N}. \tag{20}$$

Then the first law takes the alternative form

$$\begin{aligned} TdS &= NTds + TsdN = dU + pdV - \mu dN \\ &= N\,(du + p\,dv) + (u + pv - \mu)\,dN, \end{aligned} \tag{21}$$

and therefore

$$Tds = du + p\,dv + \left(\frac{u + pv - Ts - \mu}{N} \right) dN. \tag{22}$$

Now for a 1-component fluid the Gibbs-Duhem relation implies for the Gibbs potential:

$$G = U - TS + pV = \mu N; \tag{23}$$

as a result, the last term in parentheses in eq. (22) vanishes:

$$u + pv - Ts - \mu = 0 \quad \Rightarrow \quad Tds = du + pdv. \tag{24}$$

The energy density ε and particle density ρ are related to the specific energy and volume by

$$\varepsilon = \rho u, \qquad \rho = \frac{1}{v}. \tag{25}$$

Finally it then follows that

$$Tds = p\,d\left(\frac{1}{\rho}\right) + d\left(\frac{\varepsilon}{\rho}\right), \tag{26}$$

and

$$f'(\rho) = \frac{\varepsilon + p}{\rho} = u + pv = Ts + \mu. \tag{27}$$

Thus we infer that at $T = 0$ the chemical potential is $\mu_{T=0} = f'(\rho)$. Equation (18) is now seen to imply that the comoving time-derivative of the specific entropy vanishes:

$$u^\mu \partial_\mu\, s = 0 \quad \Leftrightarrow \quad \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0. \tag{28}$$

We observe, that systems with equations of state satisfying (15) indeed have the special property that the specific entropy is constant:

$$T ds = p d\left(\frac{1}{\rho}\right) + d\left(\frac{\varepsilon}{\rho}\right) = \left(\frac{-\rho f' + f}{\rho^2} + \frac{f'}{\rho} - \frac{f}{\rho^2}\right) d\rho = 0. \tag{29}$$

N.B.: observe, that the specific entropy s is not the same as the entropy density σ:

$$\sigma = \frac{S}{V} = \rho s. \tag{30}$$

Clearly, if the specific entropy is constant, then

$$s = \frac{\sigma}{\rho} = \frac{d\sigma}{d\rho} = \text{constant}. \tag{31}$$

Finally, the Gibbs-Duhem relation in the form

$$G = U - TS + pV = \mu N \quad \Rightarrow \quad T dS - V dp + N d\mu = 0, \tag{32}$$

implies

$$dp = \sigma\, dT + \rho\, d\mu \quad \Leftrightarrow \quad s\, dT = v\, dp - d\mu. \tag{33}$$

1.3 Coupling to gravity and action principle

In this section we will show, that the basic fluid equations (7) and (13), as well as the equation of state (15) can be derived from an action principle. Moreover, this action can be generalized with almost no effort to include coupling to the gravitational field in the contex of general relativity. Therefore we immediately proceed with the general relativistic treatment and define the action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{16\pi G}\, R + \mathcal{L}_{fluid}\right), \tag{34}$$

where

$$\mathcal{L}_{fluid} = -j^\mu \left(\partial_\mu \theta + i\bar{z}\partial_\mu z - iz\partial_\mu \bar{z}\right) - f(\rho). \tag{35}$$

Here θ and $(\bar{z}, z)$ are real and complex scalar potentials respectively, and ρ is considered an composite expression of the metric and current as in (12):

$$\rho^2 = -g_{\mu\nu} j^\mu j^\nu. \tag{36}$$

Variation of the action w.r.t. the (inverse) metric gives the Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = -8\pi G\, T_{\mu\nu}, \tag{37}$$

with the energy momentum tensor the extension of (6):

$$T_{\mu\nu} = p\, g_{\mu\nu} + (\varepsilon + p)\, u_\mu u_\nu = p\, g_{\mu\nu} - (\varepsilon + p)\, \frac{j_\mu j_\nu}{\rho^2}, \tag{38}$$

and the energy density and pressure being defined through (14) and (15). Hence the specific entropy (entropy per particle) is a constant in this model by construction. Furthermore, the Einstein equations imply the covariant conservation of the energy-momentum currents:

$$D_\mu T^{\mu\nu} = 0. \tag{39}$$

Next, varying the action w.r.t. the current leads to an equation expressing the current in terms of the potentials:

$$\frac{f'}{\rho}\, j_\mu = f' u_\mu = \partial_\mu \theta + i\bar{z}\partial_\mu z - iz\partial_\mu \bar{z}, \tag{40}$$

whilst an extremum of the action w.r.t. variation of the scalar potentials requires

$$D_\mu j^\mu = 0, \qquad j^\mu \partial_\mu z = j^\mu \partial_\mu \bar{z} = 0. \tag{41}$$

The first equation is the covariant form of the current conservation (13). Observe, that in 4 space-time dimensions a conserved current has 3 indpendent degrees of freedom, which can be identified with the real and complex scalar potentials in expression (40). Therefore in 4-dimensional space-time eq. (40) represents the most general current one can write down.

A typical equation of state is of the form

$$\varepsilon = f(\rho) = \alpha \rho^{1+\eta}, \qquad p = \rho f' - f = \eta\, \alpha \rho^{1+\eta}, \tag{42}$$

which gives a linear relation between ε and p, as expected on dimensional grounds:

$$p = \eta \varepsilon. \tag{43}$$

Note, that by construction this equation of state satisfies the condition (29)

$$T ds = p\, d\left(\frac{1}{\rho}\right) + d\left(\frac{\varepsilon}{\rho}\right) = 0.$$

For example, the standard equation of state of a gas of massless particles in 4-dimensional space-time is

$$p = \frac{1}{3}\,\varepsilon, \quad \Leftrightarrow \quad \varepsilon = \alpha\rho^{4/3}. \tag{44}$$

Similarly, a gas of cold non-relativistic particles of mass m, with $p \ll \rho m$, can in first approximation be described by $\eta = 0$:

$$p = 0 \quad \Leftrightarrow \quad \varepsilon = \alpha\rho. \tag{45}$$

In the next approximation, at finite temperature and pressure, we have

$$\varepsilon = f(\rho) = \rho m + \frac{3}{2}\,p. \tag{46}$$

With the relations (15) this can be solved for $f(\rho)$ to give:

$$\varepsilon = f(\rho) = m\rho + \kappa\rho^{5/3}, \qquad p = \frac{2\kappa}{3}\,\rho^{5/3}, \qquad \kappa = \frac{3}{2}\,\frac{p_0}{\rho_0^{5/3}}, \tag{47}$$

where p_0 is the pressure at some reference density ρ_0. The result for the pressure is well-known from classical thermodynamics.

1.4 Vorticity

In non-relativistic hydrodynamics one defines the vorticity as the rotation of the velocity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{48}$$

By definition it is divergence-free:

$$\nabla \cdot \boldsymbol{\omega} = 0. \tag{49}$$

The non-relativistic Euler equation is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \ln f', \tag{50}$$

where we have used eqs. (15) and (27). Now we can use the identity

$$\mathbf{v} \times \boldsymbol{\omega} = \mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \left(\frac{1}{2}\,\mathbf{v}^2 \right) - \mathbf{v} \cdot \nabla \mathbf{v}, \tag{51}$$

to rewrite the Euler equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla \left(\ln f' + \frac{1}{2}\, \mathbf{v}^2 \right). \tag{52}$$

It is then straightforward to derive the equation of motion for the vorticity itself

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0. \tag{53}$$

The vorticity is related to the circulation of the fluid along a closed path Γ in the fluid as follows from Gauß' theorem:

$$\Omega = \oint_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \iint_{\Sigma} \omega_n \, d^2\Sigma, \tag{54}$$

where Σ is a 2-dimensional surface in the fluid enclosed by the loop: $\Gamma = \partial\Sigma$, and ω_n is the normal component of $\boldsymbol{\omega}$ on the surface. From the equations of motion (52) and (53) one can then derive the Kelvin-Helmholtz theorem for a perfect fluid, stating that the co-moving time derivative of the circulation vanishes:

$$\frac{\partial \Omega}{\partial t} + \mathbf{v} \cdot \nabla \Omega = 0. \tag{55}$$

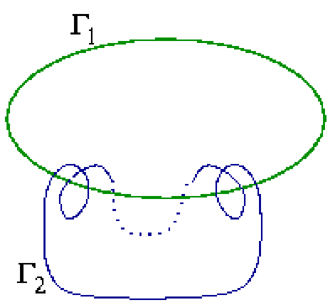

Fig. 3.1 Linked vortex loops with N links.

Connected with vorticity is a topological quantity

$$I = \int_V d^3x \, \mathbf{v} \cdot \boldsymbol{\omega} = \int_V d^3x \, \epsilon_{ijk} \, v_i \nabla_j v_k. \tag{56}$$

Note that it is like an abelian Chern-Simons form in 3 dimensions. The integral quantity I is conserved up to boundary terms, as may be checked using the equations of motion for $\mathbf{v}$ and $\boldsymbol{\omega}$. That it has a topological interpretation can be seen from the following argument.

As the divergence of the vorticity vanishes, vortex lines can only end on boundaries; otherwise they form closed loops. In the absence of boundaries, consider two closed vortex loops, each with a circulation

$$\Omega^{(i)} = \int_{\Sigma_i} \boldsymbol{\omega}^{(i)} \cdot d\boldsymbol{\Sigma}^{(i)}, \tag{57}$$

which wind N times around each other, as in fig. 3.1. By continuity of the fluid, we can contract the vortex lines to narrow tubes of a diameter small compared to the smallest distance between the two loops; outside these narrow tubes the vorticity can be taken to vanish, and therefore we can write

$$I = \int_V d^3x \, \mathbf{v} \cdot \boldsymbol{\omega} = I_1 + I_2, \tag{58}$$

where the two terms refer to the two separate volume integrals over the non-overlapping volumes $V_{1,2}$ of each of the tubes separately:

$$I_i = \int_{V_i} d^3x \, \mathbf{v}^{(i)} \cdot \boldsymbol{\omega}^{(i)}, \qquad i = (1,2). \tag{59}$$

In particular we can write a volume element of tube Γ_1 as $dV_1 = dl_1 dA_1$, where dl_1 is a line element in the direction of flow, and dA_1 is a perpendicular area element representing the cross section of the tube; then

$$I_1 = \int_{\Gamma_1} dl_1 \, v^{(1)} \int d^2A_1 \, \omega^{(1)}, \tag{60}$$

with $\omega^{(1)}$ the vorticity in the direction of $v^{(1)}$, and hence perpendicular to dA_2. Now the area integral is just $\Omega^{(1)}$, which is a constant. However, the line integral can be turned into an area integral over the surface Σ_1 enclosed by Γ_1, in the interior of which the only contribution to ω_n comes from the circulation of the second vortex line, which pierces the surface N times:

$$\int_{\Gamma_1} dl_1 v^{(1)} = \int_{\Sigma_1} d^2\Sigma_1 \, \omega_n = N \int_{\Gamma_2} dA_2 \, \omega^{(2)} = N\Omega^{(2)}. \tag{61}$$

Thus we find

$$I_1 = N\Omega^{(1)}\Omega^{(2)}, \tag{62}$$

which is completely symmetric in the circulation of the two loops. By symmetry, the same expression comes from evaluation of I_2; therefore the total result is

$$I = \int_V d^3x \, \mathbf{v} \cdot \boldsymbol{\omega} = 2N\Omega^{(1)}\Omega^{(2)}. \tag{63}$$

As N is an integer, and the $\Omega^{(i)}$ are constant in time, the quantity I is also constant, and measures the number of links (the winding number) of the two vortices.

The definitions of vorticity and linking number have their relativistic generalizations. The starting point of the construction is eq. (40), which we can write as

$$a_\mu = \partial_\mu \theta + i\bar{z}\partial_\mu z - iz\partial_\mu \bar{z} = f'(\rho)\, u_\mu. \tag{64}$$

Then a relativistic analogue of the Chern-Simons form (56) is obtained from the divergenceless axial current

$$A^\mu = \epsilon^{\mu\nu\kappa\lambda}\, a_\nu \partial_\kappa a_\lambda = 2i\, \epsilon^{\mu\nu\kappa\lambda}\, \partial_\nu \theta\, \partial_\kappa\, \bar{z}\partial_\lambda z, \qquad \partial \cdot A = 0. \tag{65}$$

The axial charge A^0 is conserved, and is given by

$$A^0 = \int_V d^3x\, \epsilon_{ijk}\, a_i \nabla_j a_k = 2i \int_V d^3x\, \nabla \cdot \left(\theta\, \nabla \bar{z} \times \nabla z \right). \tag{66}$$

Clearly A^0 reduces to a 2-dimensional boundary term, and its value is independent of the local configurations in the bulk of the fluid.

1.5 Sound waves

Consider a static fluid: $\rho = \rho_0$ and $p = p_0$ are constant, and $\mathbf{v}_0 = 0$; the last condition only implies that the velocity of flow (the average speed of transportation of particles) vanishes, not that the speed of the individual particles vanishes: the particles themselves can be fully relativistic, in the sense that $kT \gg m$. Next consider small fluctuations around the static equilibrium values:

$$\rho = \rho_0 + \delta\rho, \qquad \mathbf{v} = \delta\mathbf{v}. \tag{67}$$

Then of course

$$\varepsilon = \varepsilon_0 + \delta\varepsilon, \qquad p = p_0 + \delta p, \tag{68}$$

with

$$\delta\varepsilon = f_0'\,\delta\rho, \qquad \delta p = \rho_0 f_0''\,\delta\rho = \frac{\rho_0 f_0''}{f_0'}\,\delta\varepsilon. \tag{69}$$

In flat Minkowski space-time the linearized equations of motion for the fluctuations become

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot(\rho_0 \delta\mathbf{v}) = 0, \qquad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{1}{\varepsilon_0 + p_0}\,\nabla(\delta p). \tag{70}$$

Using eq. (69) and recalling the last of eqs. (15), the equation for the velocity fluctuations can be rewritten as

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{f_0''}{f_0'}\,\nabla(\delta\rho). \tag{71}$$

Combining this with the first eq. (70) we get:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \frac{\rho_0 f_0''}{f_0'}\,\nabla^2(\delta\rho) = c_s^2\,\nabla^2(\delta\rho), \tag{72}$$

which is an equation for density fluctuations propagating at a speed c_s defined by

$$c_s^2 = \frac{\rho_0 f_0''}{f_0'} = \left.\frac{\partial p}{\partial \varepsilon}\right|_0. \tag{73}$$

For a non-relativistic gas this using (47) this becomes

$$c_s^2 = \frac{5p_0}{3m\rho_0} \ll 1, \tag{74}$$

whilst for a relativistic gas the result is

$$c_s^2 = \frac{1}{3} < 1. \tag{75}$$

In all cases the sound velocity never exceeds the speed of light.

By far the simplest system is that of pressureless dust. This term refers to gas of non-interacting point particles of mass m moving on geodesics. The action is

$$S = \sum_r \frac{m}{2} \int d\lambda \sqrt{g_{\mu\nu}(\xi_r)\,\frac{d\xi_r^\mu}{d\lambda}\,\frac{\xi_r^\nu}{d\lambda}}, \tag{76}$$

where λ is some affine parameter for the geodesic $\xi_r^\mu(\lambda)$ of the rth particle. The action can be written as a quasi field theory in the form[1]

$$S = \int d^4x \sum_r \frac{m}{2} \int d\lambda \sqrt{g_{\mu\nu}(\xi_r) \frac{d\xi_r^\mu}{d\lambda} \frac{\xi_r^\nu}{d\lambda}} \, \delta^4 \left(x - \xi_r(\lambda)\right). \tag{77}$$

It is the straightforward to compute the energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \sum_r \frac{m}{2} u_{r\mu} u_{r\nu} \frac{1}{\sqrt{-g}} \frac{d\tau_r}{dt} \delta^3 \left(\mathbf{x} - \boldsymbol{\xi}_r(\tau_r)\right). \tag{78}$$

In this expression τ_r denotes the proper time of the rth particle:

$$d\tau_r^2 = -g_{\mu\nu}(\xi_r) \, d\xi_r^\mu \, d\xi_r^\nu, \tag{79}$$

and $u_r^\mu = d\xi_r^\mu / d\tau_i$ is its 4-velocity. Using the four momentum $p^\mu = m u^\mu$ we can alternatively write the energy-momentum tensor as

$$T_{\mu\nu} = \sum_r \frac{p_{r\mu} p_{r\nu}}{2 p_{r0}} \frac{1}{\sqrt{-g}} \delta^3 \left(\mathbf{x} - \boldsymbol{\xi}_r(\tau_r)\right). \tag{80}$$

From the expression (78) it follows immediately, that in the continuum limit we have a fluid with equation of state

$$\varepsilon = m\rho, \qquad p = 0, \tag{81}$$

with

$$\rho(x) \simeq \frac{1}{2\sqrt{-g}} \frac{d\tau}{dt} \delta^3 \left(\mathbf{x} - \boldsymbol{\xi}(\tau)\right). \tag{82}$$

Clearly, to have a non-trivial equation of state one needs particles which interact, at least through elastic collisions.

1.6 Hydrostatic equilibrium

The covariant conservation of the fluid energy-momentum tensor in components reads

$$\partial_\mu p + \partial_\nu \left((p + \varepsilon) u_\mu u^\nu\right) = -(p + \varepsilon) \left(\Gamma_{\kappa\lambda}{}^{\lambda} \, u^\kappa u_\mu + \Gamma_{\kappa\lambda\mu} \, u^\kappa u^\lambda\right). \tag{83}$$

[1] Our delta function is a scalar density, satisfying $\int d^4x \, \delta^4(x - x_0) = 1$ in any coordinate system.

In hydrostatic equilibrium, the 4-velocity is a time-like unit vector with components

$$u^\mu = (u^0, 0, 0, 0), \qquad g_{\mu\nu}\, u^\mu u^\nu = g_{00}(u^0)^2 = -1, \tag{84}$$

which fixes the time-component of the velocity to be given by

$$u^0 = (-g_{00})^{-1/2}\,. \tag{85}$$

Simultaneously the pressure, energy density and metric must be time independent: $\partial_0 p = \partial_0 \varepsilon = \partial_0 g_{\mu\nu} = 0$. In particular,

$$\Gamma_{00\mu} = -\frac{1}{2}\,\partial_\mu g_{00}. \tag{86}$$

Then the only non-trivial equations (83) are those with $\mu = i$:

$$\nabla_i\, p = -(p+\varepsilon)\,\Gamma_{00i}\,(u^0)^2 = -\frac{p+\varepsilon}{2g_{00}}\,\nabla_i g_{00}. \tag{87}$$

Equivalently,

$$\frac{\nabla_i\, p}{p+\varepsilon} = \nabla_i \ln f'(\rho) = -\nabla_i \ln \sqrt{-g_{00}}. \tag{88}$$

It follows, that

$$-g_{00}\, f'^{\,2} = \kappa^{-2} = \text{constant} \quad \Leftrightarrow \quad u^0 = (-g_{00})^{-1/2} = \kappa f'(\rho), \tag{89}$$

where κ is a constant. In particular, for the equations of state (42):

$$f(\rho) = \alpha \rho^{1+\eta}$$

the result is

$$\rho^{2\eta} = \frac{C}{-g_{00}} \quad \Leftrightarrow \quad p = \eta\,\varepsilon = \eta\,\alpha'\,(-g_{00})^{-\frac{1+\eta}{2\eta}}\,, \tag{90}$$

where C and α' are constant determined by κ and α. We observe, that in the non-relativistic limit $-g_{00} = 1 + 2\Phi$. with Φ the newtonian potential. It gives the standard result for a relativistic gas

$$\eta = \frac{1}{3} \quad \Rightarrow \quad p \sim (-g_{00})^{-2}, \tag{91}$$

and for a cosmic fluid behaving as a cosmological constant:

$$\eta = -1 \quad \Rightarrow \quad p = p_0 = \text{constant}, \quad \varepsilon = \varepsilon_0 = \text{constant}. \tag{92}$$

Note, that a cold non-relativistic fluid with $\eta = 0$ can exist in hydrostatic equilibrium only if $u^0 = (-g_{00})^{-1/2} = \text{constant}$, for example in Minkowski space. However, a relativistic gas can not be in free hydrostatic equilibrium in a finite region of space (there are no photon or neutrino stars).

2 Stars in equilibrium

2.1 Non-rotating stars

A static, non-rotating star is a spherically symmetric fluid body of finite extent with a time-independent gravitational field both inside and outside. Spherical symmetry and time-translation invariance allow us to choose polar co-ordinates such that the line element is of the form

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{93}$$

where $A(r)$ and $B(r)$ are to be determined. From the metric (93) one can derive parametrized expressions for the components of the connection and the Riemann curvature tensor; they have been collected in appendix 3.

We analyze the inhomogeneous Einstein equations for this space-time geometry with an energy-momentum tensor of the perfect fluid form (6), where the fluid density and the corresponding pressure and energy densities are supposed to vanish beyond some radius R:

$$\rho(r) = p(r) = \varepsilon(r) = 0, \qquad r \geq R. \tag{94}$$

We then know that the metric beyond this radius takes the Schwarzschild-Droste form

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2GM}{r}, \qquad r \geq R, \tag{95}$$

with M the total mass of the star. In order to avoid the presence of a horizon, which would turn the star into a black hole, we require $R > 2GM$. Turning now to the Einstein equations with a prefect fluid source, the tt-component gives

$$-\frac{1}{r^2 B} + \frac{1}{r^2} + \frac{B'}{rB^2} = 8\pi G\varepsilon, \tag{96}$$

where here and in the rest of this section a prime denotes a derivative w.r.t. the radial co-ordinate r. Eq. (96) can be rearranged to give

$$\left(\frac{r}{B}\right)' = 1 - 8\pi G\varepsilon r^2. \tag{97}$$

It is customary to introduce a mass function

$$\mathcal{M}(r) = \int_0^r dr'\,4\pi r'^2 \varepsilon(r'), \tag{98}$$

It then follows, that

$$B(r) = \left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{-1}, \tag{99}$$

which reduces to the Schwarzschild-Droste form (95) for $r > R$ if one identifies the total mass with

$$M = \mathcal{M}(R) = \int_0^R dr\, 4\pi r^2 \varepsilon(r). \tag{100}$$

Next we consider the rr-component of the Einstein equation, to find

$$\frac{1}{r^2 B} - \frac{1}{r^2} + \frac{A'}{rAB} = 8\pi G p. \tag{101}$$

It can be recast in the form

$$\frac{A'}{A} = -\frac{1}{r} + \frac{B}{r} + 8\pi G p\, r B. \tag{102}$$

Substitution of the expression (99) for B turns this equation into

$$(\ln A)' = \frac{2G\left(\mathcal{M} + 4\pi p\, r^3\right)}{r^2\left(1 - \frac{2G\mathcal{M}}{r}\right)}, \tag{103}$$

with the solution

$$A(r) = \exp\left[-2G\int_r^\infty \frac{dr'}{r'^2}\left(\mathcal{M} + 4\pi p\, r'^3\right)\left(1 - \frac{2G\mathcal{M}}{r'}\right)^{-1}\right]. \tag{104}$$

Observe, that we have chosen the limits of integration such that $A(r)$ satisfies the boundary condition $A(\infty) = 1$. More precisely, it is straightforward to check that for $r > R$, where $\mathcal{M} = M$ and $p = 0$:

$$-2G\int_r^\infty \frac{dr'}{r'^2}\left(\mathcal{M} + 4\pi p\, r'^3\right)\left(1 - \frac{2G\mathcal{M}}{r'}\right)^{-1} = \ln\left(1 - \frac{2GM}{r}\right), \tag{105}$$

showing that $A(r)$ assumes the Schwarzschild-Droste form for $r > R$ as well.

The other two Einstein equations for $\theta\theta$- and $\varphi\varphi$-components of the Einstein tensor are degenerate, and are satisfied automatically provided

$$G_{\theta\theta} = g_{\theta\theta}\, g^{rr}\, G_{rr} = \frac{r^2}{B}\, G_{rr}, \tag{106}$$

which explicitly reads

$$\frac{A''}{A} = \left(\frac{A'}{2A} + \frac{1}{r}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{2}{r^2}(1 - B)\,. \tag{107}$$

This one can check, using the results

$$\begin{aligned}
\frac{A'}{A} &= \frac{2G}{r^2}\left(\frac{\mathcal{M} + 4\pi p r^3}{1 - \frac{2G\mathcal{M}}{r}}\right), \\
\frac{B'}{B} &= \frac{2G}{r^2}\left(\frac{-\mathcal{M} + 4\pi \varepsilon r^3}{1 - \frac{2G\mathcal{M}}{r}}\right), \\
\frac{1}{r}(B - 1) &= \frac{2G\mathcal{M}}{r^2}\frac{1}{1 - \frac{2G\mathcal{M}}{r}},
\end{aligned} \tag{108}$$

and observing that

$$\frac{A''}{A} = \left(\frac{A'}{A}\right)' + \left(\frac{A'}{A}\right)^2. \tag{109}$$

Thus we have a complete solution for the hydrostatic equations for a spherically symmetric star in terms of the pressure $p(r)$ and energy density $\varepsilon(r)$, which in turn can be expressed in terms of the fluid density $\rho(r)$ by eqs. (14) and (15).

We also recall form the conditions for hydrostatic equilibrium (88) and (89) that

$$\frac{A'}{A} = \frac{-2p'}{p + \varepsilon} = \left[-2\ln\left(\frac{df}{d\rho}\right)\right]' \quad \Rightarrow \quad A(r) = \frac{1}{\kappa^2 (df/d\rho)^2}. \tag{110}$$

We can interpret the constant κ as follows. Outside the star ($r > R$), where $p = 0$, we have

$$\frac{df}{d\rho} = \frac{p + \varepsilon}{\rho} = \frac{\varepsilon}{\rho} = u, \tag{111}$$

the energy per particle. Then

$$\frac{1}{\kappa} = \frac{df}{d\rho}\sqrt{A} = u(r)\sqrt{1 - \frac{2GM}{r}}. \tag{112}$$

In the limit $r \to \infty$ the energy per particle is just its restmass m. Therefore

$$\kappa = \frac{1}{m}. \tag{113}$$

Eq. (110) is an implicit condition on the equation of state. This can be made more explicit by substitution of (110) into the first equation (108):

$$p' = -\frac{G}{r^2}\,(p+\varepsilon)\left(\frac{\mathcal{M} + 4\pi p r^3}{1 - \frac{2G\mathcal{M}}{r}}\right). \tag{114}$$

This equation is known as the Tolman-Oppenheimer-Volkov (TOV) equation. Using the equation of state to express the pressure p in terms of the energy or fluid density $\varepsilon = f(\rho)$ and the definition (98) of $\mathcal{M}$, we see that (114) represents an integro-differential equation for $f(\rho)$ from which the metric components have been eliminated as independent degrees of freedom.

A special simplification occurs for equations of state of the type (43):

$$p = \eta\varepsilon.$$

First observe, that from the definition (98)

$$\mathcal{M}' = 4\pi r^2 \varepsilon \quad \Leftrightarrow \quad p = \eta\varepsilon = \frac{\eta\mathcal{M}'}{4\pi r^2} \tag{115}$$

Inserting these relations into the fundamental equation (114) gives

$$\eta\left(\mathcal{M}'' - \frac{2}{r}\,\mathcal{M}'\right) = -\,(1+\eta)\,\frac{G\mathcal{M}'}{r^2}\left(\frac{\mathcal{M} + \eta r \mathcal{M}'}{1 - \frac{2G\mathcal{M}}{r}}\right). \tag{116}$$

This complicated non-linear second-order differential equation for $\mathcal{M}(r)$ has 2 simple special solutions, a linear one:

$$\mathcal{M}(r) = \mu r, \tag{117}$$

for any value of η, and a cubic one:

$$\mathcal{M}(r) = \frac{\mu r^3}{6}, \tag{118}$$

with either $\eta = -1$ or $\eta = -1/3$. However, it is difficult to turn these solutions into consistent stellar models with finite pressure and density, as well as continuous metric co-efficients across the surface, unless one introduces additional features like extra boundary layers on the surface of the star. We discuss these special solutions in some more detail in appendix B.

Constant density solution

There is a well-known simple solution of the TOV equation (114) for stars with constant energy density $\varepsilon = 3M/4\pi R^3 = \text{constant}$ [3], provided we allow the equation of state to be r-dependent:

$$p = \eta(r)\,\varepsilon. \tag{119}$$

At the same time

$$\mathcal{M}(r) = \frac{4\pi\varepsilon r^3}{3} = M\left(\frac{r}{R}\right)^3. \tag{120}$$

Indeed, for constant ε we can rewrite eq. (114) as

$$\frac{p'}{(p+\varepsilon)\left(p+\frac{\varepsilon}{3}\right)} = \frac{-4\pi G r}{1-\frac{8\pi G\varepsilon r^2}{3}}. \tag{121}$$

Eliminating the constant ε in terms of M and R we can write this as

$$\frac{\eta'}{(\eta+1)(3\eta+1)} = \frac{-GMr/R^2}{1-2GMr^2/R^3}. \tag{122}$$

By definition, R is the radial distance at which $p = 0$; with this boundary condition the solution of eqs. (121), (122) is given by (119) with

$$\eta(r) = \frac{\sqrt{1-2GM/R}-\sqrt{1-2GMr/R^2}}{\sqrt{1-2GMr/R^2}-3\sqrt{1-2GM/R}}. \tag{123}$$

Equivalently, we can write

$$\frac{1+\eta(r)}{1+3\eta(r)} = \sqrt{\frac{1-2GM/R}{1-2GMr/R^2}}. \tag{124}$$

Obviously, $\eta(R) = 0$, and the pressure vanishes at the surface $r = R$ as required. At this surface we must match it with the vacuum Schwarzschild solution. For values $0 \leq r < R$ the expression for $\eta(r)$ is real provided

$$\frac{2GM}{R} \leq 1. \tag{125}$$

Generally, the l.h.s. of eq. (124) is non-negative for values

$$\eta(r) \leq -1 \quad \vee \quad \eta(r) \geq -\frac{1}{3}. \tag{126}$$

In this range eq. (121) implies, that $\eta(r)$ decreases monotonically. However, as for $r \in [0, R]$ we have

$$\frac{1 - 2GM/R}{1 - 2GMr/R^2} \leq 1, \tag{127}$$

we can be more precise and infer that

$$\eta(r) \leq -1 \quad \vee \quad \eta(r) \geq 0. \tag{128}$$

The case $\eta = -1$ corresponds to the gravastar model, a stellar model of the type eq. (118) with $\eta = -1$, with an extra layer between the stellar interior and the vacuum where the pressure profile can be modified. For positive values of $\eta(r)$ the central value is a maximum:

$$\eta(r) \leq \eta(0) \equiv \eta_0, \qquad \frac{1 + \eta_0}{1 + 3\eta_0} = \sqrt{1 - \frac{2GM}{R}}. \tag{129}$$

The last equation can be rewritten in the form

$$\frac{2GM}{R} = 1 - \left(\frac{1 + \eta_0}{1 + 3\eta_0} \right)^2. \tag{130}$$

The r.h.s. of this equation takes values between 0 for $\eta_0 = 0$, and 8/9 for $\eta_0 \rightarrow \infty$. Hence such stars have a maximal radius and mass:

$$\frac{2GM_\infty}{R_\infty} = \frac{8}{9}, \qquad R_\infty^2 = \frac{1}{3\pi G\varepsilon}. \tag{131}$$

This radius becomes smaller if ε becomes larger, as might be expected. The metric co-efficient $B(r)$ now becomes

$$B(r) = \left(1 - \frac{2GMr^2}{R^3} \right)^{-1} = \left(1 - \frac{r^2}{a^2} \right)^{-1}, \tag{132}$$

where a is the radius of curvature of the spherical hypersurface $t =$ constant ($dt = 0$):

$$a^2 = \frac{9}{8} R_\infty^2. \tag{133}$$

2.2 Particle number

The fluid current is a density current for the amount of fluid (the number of fluid particles); in applications it could be associated e.g. with the (practically) conserved baryon density current. The total number of baryons in a star is then obtained from the expression

$$N = \int_V d^3x\, \sqrt{-g} J^0, \tag{134}$$

where J^μ is the baryon current. The covariant current conservation

$$D_\mu J^\mu = \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g} J^\mu \right) = 0, \tag{135}$$

implies that the total number of particles is conserved:

$$\partial_0 N = - \int_V d^3x\, \nabla_i (\sqrt{-g} J_i) = 0. \tag{136}$$

In the case of a spherically symmetric body, like a non-rotating star, this expression takes the form

$$N = \int_0^R dr\, 4\pi r^2 \sqrt{AB}\, J^0. \tag{137}$$

Now it follows form eq. (85) that

$$\sqrt{A}\, J^0 = -u_0 J^0 = -u_\mu J^\mu = \rho. \tag{138}$$

Substitution of the expression (99) for B then finally leads to the result

$$N = \int_0^R dr\, 4\pi r^2 \frac{\rho(r)}{\sqrt{1 - \frac{2G\mathcal{M}(r)}{r}}}. \tag{139}$$

Knowing the total number of particles, it is possible to address the question of the internal energy of a star. The total rest energy of N non-interacting particles of mass m is

$$E_N = Nm. \tag{140}$$

The total rest energy of the star is its mass M (100), as seen by an observer at infinity at rest w.r.t. the start at infinity. Therefore the total internal energy is

$$\Delta E = M - Nm. \tag{141}$$

We can also introduce the internal energy density in a similar from the local quantities:

$$e(r) = \varepsilon(r) - \rho(r)\, m. \tag{142}$$

With the help of this definition we can split the total internal energy into thermal (kinetic) enery T and potential energy V, by defining

$$\begin{aligned} T &= \int_0^R dr\, 4\pi r^2\, \frac{e(r)}{\sqrt{1 - \frac{2G\mathcal{M}(r)}{r}}} \\ V &= \int_0^R dr\, 4\pi r^2\, \varepsilon(r) \left(1 - \frac{1}{\sqrt{1 - \frac{2G\mathcal{M}(r)}{r}}} \right). \end{aligned} \tag{143}$$

Using the expressions for M and N, these quantities are easily seen to satisfy the equation

$$T + V = \Delta E. \tag{144}$$

An expansion in Newton's constant gives

$$T = \int_0^R dr\, 4\pi r^2\, e(r) \left(1 + \frac{G\mathcal{M}(r)}{r} + ... \right), \tag{145}$$

and similarly for the potential energy:

$$V = \int_0^R dr\, 4\pi r^2 \varepsilon(r) \left(\frac{G\mathcal{M}(r)}{r} + \frac{3G^2\mathcal{M}^2(r)}{2r^2} + ... \right). \tag{146}$$

2.3 A variational principle

The stellar structure equation (114) expressing the pressure gradient in terms of the pressure and energy density can be derived in an alternative way from a variational priniciple. The starting point is provided by the expressions for total energy M (100) and particle number N (139), interpreted as functionals of the density $\rho(r)$. From these expressions we can construct the functional

$$E(\rho; \lambda) = M(\rho) - \lambda \left(N(\rho) - N_0 \right), \tag{147}$$

where N_0 is the fixed value of the total particle number, and λ is a global parameter used as a lagrange multiplier to impose particle number conservation on variations of E; indeed

$$\frac{\partial E}{\partial \lambda} = 0 \quad \Leftrightarrow \quad N = \int_0^R dr\, 4\pi r^2 \frac{\rho}{\sqrt{1 - \frac{2G\mathcal{M}}{r}}} = N_0, \tag{148}$$

We now show, that eq. (114) for the pressure gradient follows from requiring E to be stationary under local variations of the density ρ for some value of the parameter $\lambda = \lambda_0$:

$$\delta E = 0 \quad \Leftrightarrow \quad \delta M = \lambda_0\, \delta N \tag{149}$$

under *local* varations $\delta\rho(r)$. It then follows that

$$\delta E = \frac{\partial E}{\partial \lambda}\, \delta\lambda + \int_0^R dr\, \frac{\delta E}{\delta \rho(r)}\, \delta\rho(r) = 0 \tag{150}$$

under independent variations of λ and ρ at the point $N = N_0$ and $\lambda = \lambda_0$. In particular this then implies, that M itself is stationary under variations $\delta\rho(r)$ such that $\delta N = 0$.

Considering some arbitrary variation $\delta\rho(r)$, we straightforwardly find

$$\begin{aligned} \delta E &= \int_0^R dr\, 4\pi r^2 \delta\rho(r) \left(\left.\frac{df}{d\rho}\right|_r - \frac{\lambda}{\sqrt{1 - \frac{2G\mathcal{M}(r)}{r}}} \right) \\ &\quad - \lambda G \int_0^R dr \left(\frac{4\pi r \rho(r)}{\left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{3/2}} \int_0^r dr'\, 4\pi r'^2\, \delta\rho(r')\, \left.\frac{df}{d\rho}\right|_{r'} \right). \end{aligned} \tag{151}$$

By interchange of the integrations in the last term, we get

$$\begin{aligned} \delta E &= \int_0^R 4\pi r^2 \delta\rho(r) \left(\left.\frac{df}{d\rho}\right|_r - \frac{\lambda}{\sqrt{1 - \frac{2G\mathcal{M}(r)}{r}}} \right. \\ &\quad \left. - \lambda G \left.\frac{df}{d\rho}\right|_r \int_r^R dr'\, \frac{4\pi r' \rho(r')}{\left(1 - \frac{2G\mathcal{M}(r')}{r'}\right)^{3/2}} \right) = 0. \end{aligned} \tag{152}$$

This fixes the value of the lagrange multiplier

$$\frac{1}{\lambda} = \frac{1}{\frac{df}{d\rho}\sqrt{1-\frac{2G\mathcal{M}}{r}}} + G\int_r^R dr'\, \frac{4\pi r'\rho(r')}{\left(1-\frac{2G\mathcal{M}}{r'}\right)^{3/2}}. \tag{153}$$

Now if λ is to be independent of r, we need

$$\frac{d}{dr}\left(\frac{1}{\lambda}\right) = 0, \tag{154}$$

which implies that

$$\frac{4\pi rG\rho}{\left(1-\frac{2G\mathcal{M}}{r'}\right)^{3/2}} = \frac{d}{dr}\left(\frac{1}{\frac{df}{d\rho}\sqrt{1-\frac{2G\mathcal{M}}{r}}}\right). \tag{155}$$

Working out the derivative on the r.h.s., and multiplying the complete equiation by a factor $(df/d\rho)^2\sqrt{1-(2G\mathcal{M})/r}$ gives

$$-\frac{d}{dr}\frac{df}{d\rho} - \frac{df}{d\rho}\frac{G}{r^2}\frac{\left(\mathcal{M}-4\pi r^3 f(\rho)\right)}{1-\frac{2G\mathcal{M}}{r}} = \frac{4\pi rG\rho\left(\frac{df}{d\rho}\right)^2}{1-\frac{2G\mathcal{M}}{r}}. \tag{156}$$

Now

$$\frac{df}{d\rho} = \frac{1}{\rho}\left(p+\varepsilon\right), \qquad \frac{d}{dr}\frac{df}{d\rho} = \frac{1}{\rho}\frac{dp}{dr}. \tag{157}$$

Then we indeed reobtain (114):

$$-r^2 p' = G\left(p+\varepsilon\right)\frac{\left(\mathcal{M}+4\pi r^3 p\right)}{1-\frac{2G\mathcal{M}}{r}}.$$

It remains to calculate the stationary value of the lagrange multiplier λ. The integral on the r.h.s. of (153) can be evaluated using eq. (155), which states that the integrand is a total derivative. As a result we find

$$\lambda = \left.\frac{df}{d\rho}\right|_R \sqrt{1-\frac{2GM}{R}} = m, \tag{158}$$

where we have used the results (112) and (113) to establish the last equality. It follows, that we can rewrite the functional $E(\rho;\lambda)$ for the stationary value of λ as

$$E(\rho;m) = M(\rho) - mN(\rho) - \Delta E_0, \tag{159}$$

with ΔE_0 the total internal energy (141) in the equilibrium configuration with N_0 particles.

The observation that solutions of the structure equation (114) define the stationary points of the functional $M(\rho)$ at fixed number of particles $N = N_0$ is important for an analysis of the stability of the solution. Indeed, it follows directly, that a solution is stable only if the stationary point is a minimum.

2.4 Newtonian stars

Newtonian star models describe stars in terms of a non-relativistic fluid under the influence of newtonian gravity. Non-relativistic fluids are described by the equation of continuity and the Euler equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla (\rho \mathbf{v}) = -\frac{\mathbf{f}}{m}, \tag{160}$$

with $\mathbf{f}$ the local force density (force per unit of volume) acting on a fluid element with mass density $m\rho$ and velocity $\mathbf{v}$. The equations (160) represent the non-relativistic limit of equations (13) and (10), taking into account that in that limit the energy density is dominated by the mass density, and that the pressure is negligeable in comparison with the energy density:

$$p + \varepsilon \approx \varepsilon = m\rho. \tag{161}$$

Now the situation of hydrostatic equilibrium is characterized simply by the conditions

$$\frac{\partial \rho}{\partial t} = 0, \qquad \mathbf{v} = 0, \qquad \mathbf{f} = 0. \tag{162}$$

In the case of a fluid dubject only to internal forces and newtonian gravity, the force balance states simply that the gravitational force must be balanced by the pressure gradient:

$$\mathbf{f}_{grav} = \nabla p. \tag{163}$$

Consider a spherical body of fluid with density distribution $\rho(r)$. Consider a fluid element of mass

$$dM = m\rho \, dV, \tag{164}$$

at a distance r from the center of the body. The gravitational force on this fluid element is simple to calculate: the force from all fluid elements inside the sphere of radius r adds up to an effective force per unit of volume

$$f_{grav} = -\frac{G\mathcal{M}(r) m\rho}{r^2}, \tag{165}$$

where $\mathcal{M}(r)$ is the total mass inside radius r:

$$\mathcal{M}(r) = m \int_0^r dr' \, 4\pi r'^2 \rho(r'). \tag{166}$$

In contrast, the total gravitational force of spherically distributed matter outside radius r has no net effect on the fluid element: the $1/r^2$ behaviour guarantees that all such forces cancel. Therefore the equilibrium condition (163) becomes

$$r^2 p'(r) = r^2 \left. \frac{dp}{dr} \right|_r = -G\mathcal{M}(r)\, m\rho(r). \tag{167}$$

This is indeed the non-relativistic limit of eq. (114).

To get rid of the integral defining $\mathcal{M}$ we can differentiate the equation once more, to find

$$\frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dp}{dr} \right) = -4\pi G m^2 r^2 \rho. \tag{168}$$

This equation has to be supplemented by an equation of state $p(\rho)$ and appropriate boundary conditions:

$$p'(0) = 0, \qquad \rho'(0) = 0. \tag{169}$$

These conditions, implying that $\rho(0)$ is finite, guarantee that there is no singularity in the center of the star.

2.5 Polytropes

A newtonian polytrope is a newtonian star described by an equation of state of the form

$$p = \sigma \rho^\gamma, \tag{170}$$

with σ a (dimensionful) constant. Recalling the definition of the local internal energy (142):

$$e(r) = \varepsilon(r) - m\rho(r), \qquad |e(r)| \ll m\rho,$$

the equation of state (170) can be seen to be equivalent to the proportionality of $p(r)$ and $e(r)$:

$$p = (\gamma - 1)\, e. \tag{171}$$

To establish the connection, observe that the condition of constant entropy per particle allows us to write this as a differential equation

$$\rho \frac{df}{d\rho} - f = (\gamma - 1)\,(f - m\rho). \tag{172}$$

Now with $\varepsilon = f(\rho) = e(\rho) + m\rho$, this gives

$$\rho \frac{de}{d\rho} = \gamma e \quad \Rightarrow \quad e(\rho) = \frac{\sigma}{(\gamma - 1)}\, \rho^{\gamma}, \tag{173}$$

with σ a constant of integration; eq. (170) then follows immediately.

If the polytrope equation of state is used in the structure equation (168), this equation becomes an ordinary second order differential equation

$$\frac{\gamma\sigma}{\gamma - 1} \frac{d}{dr} \left(r^2 \frac{d}{dr}\, \rho^{\gamma - 1} \right) = -4\pi G m^2 r^2 \rho. \tag{174}$$

To get rid of the various dimensionful constants, we redefine the dependent and independent variables as follows:

$$\theta(r) = \left[\frac{\rho(r)}{\rho(0)} \right]^{\gamma - 1}, \qquad \xi^2 = \left(\frac{\gamma - 1}{\gamma\sigma} \right) \frac{4\pi G m^2}{[\rho(0)]^{\gamma - 2}}\, r^2. \tag{175}$$

Eq. (174) then takes the simple form

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^{\frac{1}{\gamma - 1}} = 0, \tag{176}$$

with the boundary conditions

$$\theta(0) = 1, \qquad \theta'(0) = 0. \tag{177}$$

The solution is known as the Lane-Emden function of index $1/(\gamma - 1)$. A series expansion of this function takes the form

$$\theta(\xi) = 1 - \frac{1}{6}\,\xi^2 + \frac{1}{120(\gamma - 1)}\,\xi^4 - \frac{1}{15120} \frac{(5\gamma - 7)}{(\gamma - 1)^2}\,\xi^6 + ... \tag{178}$$

For values $\xi > 6/5$ the function has a zero at finite $\xi = \xi_1$:

$$\theta(\xi_1) = 0. \tag{179}$$

This value of ξ then corresponds to the radius of the star:

$$R = \xi_1 \sqrt{\left(\frac{\gamma\sigma}{\gamma - 1}\right) \frac{[\rho(0)]^{\gamma-2}}{4\pi G m^2}}. \tag{180}$$

One can now also compute the mass of the star:

$$\begin{aligned} M &= m \int_0^R 4\pi r^2 \rho(r) \\ &= \left[\left(\frac{\gamma\sigma}{\gamma - 1}\right) \frac{[\rho(0)]^{(3\gamma-4)/3}}{(4\pi m^4)^{1/3}\, G}\right]^{3/2} \int_0^{\xi_1} d\xi\, \xi^2 \left[\theta(\xi)\right]^{\frac{1}{\gamma-1}} \\ &= \left[\left(\frac{\gamma\sigma}{\gamma - 1}\right) \frac{[\rho(0)]^{(3\gamma-4)/3}}{(4\pi m^4)^{1/3}\, G}\right]^{3/2} \int_0^{\xi_1} d\xi\, \frac{d}{d\xi}\left(-\xi^2 \frac{d\theta}{d\xi}\right) \\ &= \left[\left(\frac{\gamma\sigma}{\gamma - 1}\right) \frac{[\rho(0)]^{(3\gamma-4)/3}}{(4\pi m^4)^{1/3}\, G}\right]^{3/2} \left|\xi_1^2\, \theta'(\xi_1)\right|. \end{aligned} \tag{181}$$

The absolute value in the last expression comes about, because the slope of $\theta(\xi)$ at ξ_1 is negative: the pressure and density decrease to zero at the surface of the star. Note, that for $\gamma = 4/3$ the mass M becomes independent of the central density $\rho(0)$.

It is also possible to eliminate the central density between equations (180) and (181) so as to obtain the mass-radius relation

$$M = \left[\left(\frac{\gamma\sigma}{\gamma - 1}\right) \frac{1}{4\pi G m^\gamma}\right]^{\frac{-1}{\gamma-2}} 4\pi \left(\frac{R}{\xi_1}\right)^{\frac{3\gamma-4}{\gamma-2}} \left|\xi_1^2\, \theta'(\xi_1)\right|. \tag{182}$$

Clearly, for $\gamma = 4/3$ the mass becomes also independent of the radius. This is of course a sign of instability. Indeed, it turns out that for $\gamma < 4/3$ polytropes are unstable; in this region $dM/dR > 0$. In contrast, for $\gamma > 4/3$ polytropes are are stable; therefore physical polytropes get smaller when they get heavier: $dM/dR < 0$. One can show, that both the lightest and heaviest white dwarfs are in this class. The limits are described by the equations of state (44) for extremely relativistic matter:

$$p = \frac{1}{3}\,\varepsilon = \sigma\rho^{4/3},$$

such that $\gamma = 4/3$; and the equation of state (47) for non-relativistic matter:

$$\varepsilon = m\rho + \frac{3}{2}\,p = m\rho + \frac{3}{2}\,\sigma\rho^{5/3} \quad \Rightarrow \quad p = \sigma\rho^{5/3},$$

such that $\gamma = 5/3$. Thus the range of values $4/3 \leq \gamma \leq 5/3$ is the physical range, and the region $\gamma > 2$, where $dM/dR > 0$ again, is never reached.

3 Appendix A
Static spherically symmetric space-time

Static spherically symmetric space-time can be described by a line element of the parametrized form (93):

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

with A and B functions of r only. From this metric one can compute the Riemann-Christoffel connection; its non-zero components are:

$$\begin{aligned}
\Gamma_{tt}^{\;\;r} &= \frac{A'}{2B}, & \Gamma_{rt}^{\;\;t} &= \frac{A'}{2A}, \\
\Gamma_{rr}^{\;\;r} &= \frac{B'}{2B}, & & \\
\Gamma_{\theta\theta}^{\;\;r} &= -\frac{r}{B}, & \Gamma_{r\theta}^{\;\;\theta} &= \frac{1}{r}, \\
\Gamma_{\varphi\varphi}^{\;\;r} &= -\frac{r\sin^2\theta}{B}, & \Gamma_{r\varphi}^{\;\;\varphi} &= \frac{1}{r}, \\
\Gamma_{\varphi\varphi}^{\;\;\theta} &= -\sin\theta\,\cos\theta, & \Gamma_{\theta\varphi}^{\;\;\varphi} &= \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{183}$$

where a prime denotes a derivative w.r.t. the radial co-ordinate r. The next step is to compute the components of the Riemann tensor. Again, we only write down the non-zero components, up to the usual symmetry degeneracies of the components:

$$\begin{aligned}
R_{trtr} &= -\frac{1}{2}\,A'' + \frac{1}{4}\,A'\left(\frac{A'}{A} + \frac{B'}{B}\right), & & \\
R_{t\theta t\theta} &= -\frac{rA'}{2B}, & R_{t\varphi t\varphi} &= -\frac{r\sin^2\theta\,A'}{2B}, \\
R_{r\theta r\theta} &= -\frac{rB'}{2B}, & R_{r\varphi r\varphi} &= -\frac{r\sin^2\theta\,B'}{2B}, \\
R_{\theta\varphi\theta\varphi} &= r^2\sin^2\theta\left(\frac{1}{B} - 1\right). & &
\end{aligned} \tag{184}$$

Contracting the components, we obtain the non-zero diagonal elements of the Ricci tensor:

$$
\begin{aligned}
R_{tt} &= -\frac{A''}{2B} - \frac{A'}{rB} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right), \\
R_{rr} &= \frac{A''}{2A} - \frac{B'}{rB} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right), \\
R_{\theta\theta} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \qquad R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta},
\end{aligned}
\tag{185}
$$

whilst the Riemann curvature scalar becomes

$$
R = -\frac{2}{r^2} + \frac{2}{r^2 B} + \frac{A''}{AB} - \frac{A'}{2AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{2}{rB}\left(\frac{A'}{A} - \frac{B'}{B}\right). \tag{186}
$$

Finally, we give the corresponding expressions for the Einstein tensor

$$
\begin{aligned}
G_{tt} &= -\frac{A}{B}\left(-\frac{1}{r^2} + \frac{B}{r^2} + \frac{B'}{rB}\right), \\
G_{rr} &= -\frac{1}{r^2} + \frac{B}{r^2} - \frac{A'}{rA} \\
G_{\theta\theta} &= -\frac{r^2}{2B}\left[\frac{A''}{A} - \frac{A'}{2A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right], \quad G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta}.
\end{aligned}
\tag{187}
$$

4 Appendix B
Special solutions of the TOV equation

In this appendix we discuss in some more detail the two special solutions of the TOV equation mentioned in sect. 2.1. We first consider the solution (117):

$$\mathcal{M}(r) = \mu r \quad \Rightarrow \quad \mathcal{M}' = \mu, \qquad \mathcal{M}'' = 0. \tag{188}$$

It follows that the energy and pressure grow without bound in the center of the star:

$$p = \eta\varepsilon = \frac{\eta\mu}{4\pi r^2} \tag{189}$$

However, the singularity is integrable, and the total mass of the star is finite:

$$M = \int_0^R dr\, 4\pi r^2 \varepsilon = \mu R \quad \Rightarrow \quad \frac{dM}{dR} = \mu. \tag{190}$$

For these stars the mass increases with the radius, as for black holes. Substitution of the expression for $\mathcal{M}(r)$ in the TOV equation gives

$$\frac{4\eta}{(1+\eta)^2} = \frac{2G\mu}{1-2G\mu}, \tag{191}$$

which implies that the parameters η and μ are not independent. Observe, that the l.h.s. of eq. (191) is invariant under the substitution $\eta \rightarrow 1/\eta$; hence any value of μ corresponds to two values of η, unless $\eta = 1$, which is a fixed point of the transformation. At the fixed point

$$\eta = 1 \quad \Leftrightarrow \quad 2G\mu = \frac{2GM}{R} = \frac{1}{2}. \tag{192}$$

Hence the radius of these stars with $\eta = 1$ is twice the Schwarzschild radius. For $\eta = 0$ the transformation is singular, but as it corresponds to $\mu = 0$ it is uninteresting. For other values of μ

$$\eta = \frac{1}{G\mu}\left(1 - 3G\mu \pm \sqrt{(1-2G\mu)(1-4G\mu)}\right), \tag{193}$$

provided either $2G\mu \geq 1$ or $4G\mu \leq 1$. This shows that there is another fixed point:

$$\eta = -1 \quad \Leftrightarrow \quad 2G\mu = \frac{2GM}{R} = 1. \tag{194}$$

In this case the radius of the star is equal to the Schwarzschild radius. The interior metric (inside the star) has co-efficients

$$A(r) = \left(\frac{r}{R}\right)^{4\eta/(1+\eta)} \left(1 - \frac{2GM}{R}\right), \quad B(r) = \frac{1}{1 - 2G\mu} = \frac{1}{1 - \frac{2GM}{R}}, \tag{195}$$

where the normalization of both co-efficients is determined by the continuity of the metric at the stellar surface; note that the interior $B(r)$ is constant. The speed of sound inside the star is

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \eta, \tag{196}$$

which is physically sensible only if $0 \leq \eta \leq 1$. For negative η no sound waves are possible (imaginary c_s), whilst for $\eta > 1$ the speed of sound exceeds the speed of light.

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