SPIN AND ANGULAR MOMENTUM

chapter 5 of Mandl

1 Spin

1.1 Definition

In quantum mechanics spin is introduced as an observable defined via the vector operator $\boldsymbol{s}$. These (three) hermitean operators satisfy commutation relations

$$[s_i, s_j] = i\hbar\epsilon_{ijk}\, s_k, \tag{1}$$

similar to the commutation relations for the angular momentum operator $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$. The spin operators $\boldsymbol{s}$ commute with the operators $\boldsymbol{r}$ and $\boldsymbol{p}$ and thus also with $\boldsymbol{\ell}$. That's it. All the rest follows from these commutation relations.

1.2 Rotation invariance

Earlier we have seen that, without spin, rotation invariance required specific commutation relations with $\boldsymbol{\ell}$. In fact *scalar* quantities S and *vectors* $\boldsymbol{V}$ under rotations behaved like

$$[\ell_i, S] = 0, \tag{2}$$
$$[\ell_i, V_j] = i\hbar\,\epsilon_{ijk}\, V_k, \tag{3}$$

e.g. scalars $S = \boldsymbol{r}^2$, $\boldsymbol{p}^2$, $\boldsymbol{r}\cdot\boldsymbol{p}$ or $\boldsymbol{\ell}^2$ and vectors $\boldsymbol{V} = \boldsymbol{r}$, $\boldsymbol{p}$ or $\boldsymbol{\ell}$.

Including spin vectors $\boldsymbol{s}$, the notion of behavior under rotations has to be altered, e.g. $[\ell_i, s_j] = 0$ and $[\ell_i, \boldsymbol{\ell}\cdot\boldsymbol{s}] = -i\hbar\,(\boldsymbol{\ell}\times\boldsymbol{s})_i$. It is easy to see that the operator

$$\boldsymbol{j} \equiv \boldsymbol{\ell} + \boldsymbol{s}, \tag{4}$$

satisfies

$$[j_i, S] = 0, \tag{5}$$
$$[j_i, V_j] = i\hbar\,\epsilon_{ijk}\, V_k, \tag{6}$$

not only for the above examples, but now also for the vectors $\boldsymbol{s}$ and $\boldsymbol{j}$ and including scalars like $\boldsymbol{s}^2$ and $\boldsymbol{\ell}\cdot\boldsymbol{s}$.

For a system of many particles the operators $\boldsymbol{r}$, $\boldsymbol{p}$ and $\boldsymbol{s}$ for different particles commute. It is easy to see that the operators

$$\boldsymbol{L} = \sum_{n=1}^{N} \boldsymbol{\ell}_n, \qquad \boldsymbol{S} = \sum_{n=1}^{N} \boldsymbol{s}_n, \qquad \boldsymbol{J} = \sum_{n=1}^{N} \boldsymbol{j}_n = \boldsymbol{L} + \boldsymbol{S}, \tag{7}$$

satisfy commutation relations $[L_i, L_j] = i\hbar\,\epsilon_{ijk}\, L_k$, $[S_i, S_j] = i\hbar\,\epsilon_{ijk}\, S_k$, and $[J_i, J_j] = i\hbar\,\epsilon_{ijk}\, J_k$, while only the operator $\boldsymbol{J}$ satisfies

$$[J_i, S] = 0, \tag{8}$$
$$[J_i, V_j] = i\hbar\,\epsilon_{ijk}\, V_k, \tag{9}$$

for any scalar S or vector $\boldsymbol{V}$.

It is easy to show that the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ of two vectors satisfying the commutation relation in Eq. 9 indeed is a scalar quantity, satifying the commutation relation in Eq. 8.

An important property is that rotational invariance is one of the basic symmetries of our world.

Rotation invariance of a system of particles requires

$$[\boldsymbol{J}, H] = 0. \tag{10}$$

Besides the behavior under rotations, also the behavior under parity is considered to classify quantities. Vectors behave as $P\,\boldsymbol{V}\,P^{-1} = -\boldsymbol{V}$, axial vectors as $P\,\boldsymbol{A}\,P^{-1} = +\boldsymbol{A}$, a scalar S behave as $P\,S\,P^{-1} = +S$, and a pseudoscalar S' behaves as $P\,S'\,P^{-1} = -S'$. Examples of specific quantities are

vector	*axial vector*	*scalar*	*pseudoscalar*
$\boldsymbol{r}$	$\boldsymbol{\ell}$	$\boldsymbol{r}^2$	$\boldsymbol{s}\cdot\boldsymbol{r}$
$\boldsymbol{p}$	$\boldsymbol{s}$	$\boldsymbol{p}^2$	$\boldsymbol{s}\cdot\boldsymbol{p}$
	$\boldsymbol{j}$	$\boldsymbol{\ell}^2$	
		$\boldsymbol{\ell}\cdot\boldsymbol{s}$	

The hamiltonian is a scalar quantity. Therefore, if we have parity invariance, combinations as $\boldsymbol{s}\cdot\boldsymbol{r}$ cannot appear but a tensor operator of the form $(\boldsymbol{s}_1\cdot\boldsymbol{r})(\boldsymbol{s}_2\cdot\boldsymbol{r})$ is allowed. Note, however, that such an operator does not commute with $\boldsymbol{\ell}$ (see Exc. 5.8).

1.3 spin states

As mentioned above, the commutation relations are all that defines spin. As an operator that commutes with all three spin operators (a socalled Casimir operator) we have $\boldsymbol{s}^2 = s_x^2 + s_y^2 + s_z^2$,

$$[s_i, s_j] = i\hbar\,\epsilon_{ijk}\,s_k, \tag{11}$$

$$[\boldsymbol{s}^2, s_i] = 0. \tag{12}$$

Only one of the three spin operators can be used to label states, for which we without loss of generality can take s_z. In addition we can use $\boldsymbol{s}^2$, which commutes with s_z. We write states $\chi_m^{(s)} = |s,m\rangle$ satisfying

$$\boldsymbol{s}^2|s,m\rangle = \hbar^2\,s(s+1)|s,m\rangle, \tag{13}$$

$$\boldsymbol{s}_z|s,m\rangle = m\hbar\,|s,m\rangle. \tag{14}$$

It is of course a bit premature to take $\hbar^2\,s(s+1)$ as eigenvalue. We need to prove that the eigenvalue of $\boldsymbol{s}^2$ is positive, but this is straightforward as it is the sum of three squared operators. Since the spin operators are hermitean each term is not just a square but also the product of the operator and its hermitean conjugate. In the next step, we recombine the operators s_x and s_y into

$$s_\pm \equiv s_x \pm i\,s_y. \tag{15}$$

The commutation relations for these operators are,

$$[\boldsymbol{s}^2, s_\pm] = 0, \tag{16}$$

$$[s_z, s_\pm] = \pm\hbar\,s_\pm, \tag{17}$$

$$[s_+, s_-] = 2\hbar\,s_z, \tag{18}$$

The first two can be used to show that

$$\begin{aligned}
\boldsymbol{s}^2\, s_\pm|s,m\rangle &= s_\pm\boldsymbol{s}^2|s,m\rangle = \hbar^2\, s(s+1)\, s_\pm|s,m\rangle,\\
s_z\, s_\pm|s,m\rangle &= (s_\pm s_z \pm \hbar\, s_\pm)\,|s,m\rangle = (m\pm 1)\hbar\, s_\pm|s,m\rangle,
\end{aligned}$$

hence the name step-operators (raising and lowering operator) which achieve

$$s_\pm|s,m\rangle = c_\pm|s,m\pm 1\rangle.$$

Furthermore we have $s_\pm^\dagger = s_\mp$ and $\boldsymbol{s}^2 = s_z^2 + (s_+s_- + s_-s_+)/2$, from which one finds that

$$\begin{aligned}
|c_\pm|^2 = \langle s,m|s_\pm^\dagger s_\pm|s,m\rangle &= \langle s,m|\boldsymbol{s}^2 - s_z^2 - [s_\pm, s_\mp]/2|s,m\rangle\\
&= \langle s,m|\boldsymbol{s}^2 - s_z^2 \mp \hbar\, s_z|s,m\rangle = s(s+1) - m(m\pm 1).
\end{aligned}$$

It is convention to define

$$\begin{aligned}
s_+|s,m\rangle &= \hbar\sqrt{s(s+1)-m(m+1)}\,|s,m+1\rangle\\
&= =\hbar\sqrt{(s-m)(s+m+1)}\,|s,m+1\rangle \qquad (19)\\
s_-|s,m\rangle &= \hbar\sqrt{s(s+1)-m(m-1)}\,|s,m-1\rangle\\
&= \hbar\sqrt{(s+m)(s-m+1)}\,|s,m-1\rangle. \qquad (20)
\end{aligned}$$

This shows that given a state $|s,m\rangle$, we have a whole series of states

$$\ldots|s,m-1\rangle,\ |s,m\rangle,\ |s,m+1\rangle,\ldots$$

But, we can also easily see that since $\boldsymbol{s}^2 - s_z^2 = s_x^2 + s_y^2$ must be an operator with positive definite eigenstates that $s(s+1) - m^2 \geq 0$, i.e. $|m| \leq \sqrt{s(s+1)}$ or strictly $|m| < s+1$. From the second expressions in Eqs 19 and 20 one sees that this inequality requires $m_{max} = s$ as one necessary state to achieve a cutoff of the series of states on the upper side, while $m_{min} = -s$ is required as a necessary state to achieve a cutoff of the series of states on the lower side. Moreover to have both cutoffs the step operators require that the difference $m_{max} - m_{min} = 2\,s$ must be an integer, i.e. the only allowed values of spin quantum numbers are

$$\begin{aligned}
&s = 0,\ 1/2,\ 1,\ 3/2,\ldots,\\
&m = s, s-1,\ldots,-s.
\end{aligned}$$

Thus for spin states with a given quantum number s, there exist $2s+1$ states.

1.4 Why is ℓ integer

Purely on the basis of the commutation relations, the allowed values for the quantum numbers s and m have been derived. Since the angular momentum operators $\boldsymbol{\ell} = \boldsymbol{r}\times\boldsymbol{p}$ satisfy the same commutation relations, one has the same restrictions on ℓ and m_ℓ, the eigenvalues connected with $\boldsymbol{\ell}^2$ and ℓ_z. However, we have only found integer values for the quantum numbers in our earlier treatment. This is the consequence of restrictions imposed because for $\boldsymbol{\ell}$ we know *more* than just the commutation relations. The operators have been introduced explicitly working in the space of functions, depending on the angles in R^3. One way of seeing where the constraint is coming from is realizing that we want uni-valued functions. The eigenfunctions of $\ell_z = -i\hbar\, d/d\phi$, were found to be

$$Y_\ell^m(\theta, \phi) \propto e^{i\,m\phi}.$$

In order to have the same value for ϕ and $\phi + 2\pi$ we need $\exp(2\pi\, i\, m) = 1$, hence m (and thus also ℓ) can only be integer.

For spin, there are only the commutation relations, thus the spin quantum numbers s can also take half-integer values. Particles with integer spin values are called *bosons* (e.g. pions, photons), particles with half-integer spin values are called *fermions* (e.g. electrons, protons, neutrinos, quarks). For the angular momenta which are obtained as the sum of other operators, e.g. $\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}$, etc. one can easily see what is allowed. Because the z-components are additive, one sees that for any orbital angular momentum the quantum numbers are integer, while for spin and total angular momentum integer and half-integer are possible.

2 Spin states

2.1 Matrix representations of spin operators

In the space of spin states with a given quantum number s, we can write the spin operators as $(2s+1) \times (2s+1)$ matrices. Let us illustrate this first for spin $s = 1/2$. Define the states

$$\chi^{(1/2)}_{+1/2} \quad \text{or} \quad \chi_\uparrow \quad \text{or} \quad |1/2, +1/2\rangle \equiv \left(\begin{array}{c} 1 \\ 0 \end{array}\right),$$
$$\chi^{(1/2)}_{-1/2} \quad \text{or} \quad \chi_\downarrow \quad \text{or} \quad |1/2, -1/2\rangle \equiv \left(\begin{array}{c} 0 \\ 1 \end{array}\right).$$

Using the definition of the quantum numbers in Eq. 14 one finds that

$$s_z = \hbar \left(\begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array}\right), \quad s_+ = \hbar \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \quad s_- = \hbar \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right),$$

For spin 1/2 we find the familiar spin matrices, $\boldsymbol{s} = \hbar\boldsymbol{\sigma}/2$,

$$\sigma_x = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \quad \sigma_y = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right), \quad \sigma_z = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right).$$

For spin 1 we define the basis states,

$$\chi^{(1)}_{+1} = |1, +1\rangle \equiv \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right), \quad \chi^{(1)}_{0} = |1, 0\rangle \equiv \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right), \quad \chi^{(1)}_{-1} = |1, 0\rangle \equiv \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right).$$

The spin matrices are then easily found,

$$s_z = \hbar \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right), \; s_+ = \hbar \left(\begin{array}{ccc} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{array}\right), \; s_- = \hbar \left(\begin{array}{ccc} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{array}\right),$$

from which also s_x and s_y can be constructed.

2.2 Rotated spin states

Instead of the spin states defined as eigenstates of s_z, one might be interested in eigenstates of $\boldsymbol{s}\cdot\hat{\boldsymbol{n}}$, e.g. because one wants to measure it with a Stern-Gerlach apparatus with an inhomogeneous $\boldsymbol{B}$-field in the $\hat{\boldsymbol{n}}$ direction. We can of course just write down these states formally like

$$\chi^{(1/2)}_{+1/2}(\hat{\boldsymbol{n}}) \quad \text{and} \quad \chi^{(1/2)}_{-1/2}(\hat{\boldsymbol{n}})$$

Suppose that we want to write them down in terms of the eigenstates of s_z, given above. To do this we work in the matrix representation discussed in the previous section. Taking $\hat{\boldsymbol{n}} = (\sin\theta, 0, \cos\theta)$, we can easily write down

$$\boldsymbol{s}\cdot\hat{\boldsymbol{n}} = \frac{1}{2}\,\hbar\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{21}$$

We find the following two eigenstates and eigenvalues

$$\chi^{(1/2)}_{+1/2}(\hat{\boldsymbol{n}}) = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \qquad \text{with eigenvalue} \quad +1/2,$$

$$\chi^{(1/2)}_{-1/2}(\hat{\boldsymbol{n}}) = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \qquad \text{with eigenvalue} \quad -1/2.$$

The probability that given a state with spin along the z-direction a measurement of the spin along the $+\hat{\boldsymbol{n}}$-direction yields the value $+\hbar/2$ is thus given by

$$\left|\chi^{(1/2)\,\dagger}_{+1/2}(\hat{\boldsymbol{n}})\chi^{(1/2)}_{+1/2} = \right|^2 = \cos^2(\theta/2).$$

In general the rotated eigenstates are written as

$$\chi^{(s)}_m(\hat{\boldsymbol{n}}) \;=\; \begin{pmatrix} d^{(s)}_{sm}(\theta) \\ \vdots \\ d^{(s)}_{m'm}(\theta) \\ \vdots \\ d^{(s)}_{-sm}(\theta) \end{pmatrix}. \tag{22}$$

where $d_{m'm}(\theta)$ are the d-functions. These are in fact just matrix elements of the spin rotation matrix $\exp(-i\,\theta\,S_y)$ between states quantized along the z-direction. Extended to include azimuthal dependence it is customary to use the rotation matrix $e^{-i\,\phi\,S_z}\,e^{-i\,\theta\,S_y}\,e^{-i\,\chi\,S_3}$ and the functions are called $D_{m'm}(\phi,\theta,\chi)$.

3 Combination of angular momenta

3.1 Quantum number analysis

We consider situations in which two sets of angular momentum operators play a role, e.g.

- An electron with spin in an atomic $(n\ell)$-orbit (spin $\boldsymbol{s}$ and orbital angular momentum $\boldsymbol{\ell}$ combined into a total angular momentum $\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}$).
- Two electrons with spin (spin operators $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, combined into $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2$).
- Two electrons in atomic orbits (orbital angular momenta $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ combined into total orbital angular momentum $\boldsymbol{L} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$).
- Combining the total orbital angular momentum of electrons in an atom ($\boldsymbol{L}$) and the total spin ($\boldsymbol{S}$) into the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

Let us discuss as the generic example

$$\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2. \tag{23}$$

We have states characterized by the direct product of two states,

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle, \tag{24}$$

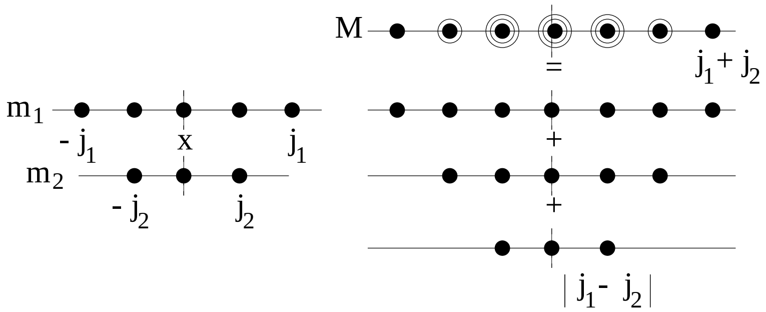
which we can write down since not only $[\boldsymbol{j}_1^2, j_{1z}] = [\boldsymbol{j}_2^2, j_{2z}] = 0$, but also $[j_{1m}, j_{2n}] = 0$. The sum-operator $\boldsymbol{J}$ obviously is not independent, but since the $\boldsymbol{J}$-operators again satisfy the well-known angular momentum commutation relations we can look for states characterized by the commuting operators $\boldsymbol{J}^2$ and J_z, $|\ldots; J, M\rangle$. It is easy to verify that of the four operators characterizing the states in Eq. 24, $[\boldsymbol{J}^2, j_{1z}] \neq 0$ and $[\boldsymbol{J}^2, j_{2z}] \neq 0$ (Note that $\boldsymbol{J}^2$ contains the operator combination $2\boldsymbol{j}_1 \cdot j_2$, which contains operators like j_{1x}, which do not commute with j_{1z}). It is easy to verify that one does have

$$[\boldsymbol{J}^2, \boldsymbol{j}_1^2] = [\boldsymbol{J}^2, \boldsymbol{j}_2^2] = 0,$$
$$[J_z, \boldsymbol{j}_1^2] = [J_z, \boldsymbol{j}_2^2] = 0,$$

and thus we can relabel the $(2j_1+1)(2j_2+1)$ states in Eq. 24 into states characterized with the quantum numbers

$$|j_1, j_2; J, M\rangle. \tag{25}$$

The basic observation in the relabeling is that $J_z = j_{1z} + j_{2z}$ and hence $M = m_1 + m_2$. This leads to the following scheme, in which in the left part the possible m_1 and m_2-values are given and the upper right part the possible sum-values for M including their degeneracy.



1. Since $|m_1| \leq j_1$ and $|m_2| \leq j_2$, the maximum value for M is $j_1 + j_2$. This state is unique.

2. Since $J_+ = j_{1+} + j_{2+}$ acting on this state is zero, it corresponds to a state with $J = j_1 + j_2$. Then, there must exist other states (in total $2J+1$), which can be constructed via $J_- = j_{1-} + j_{2-}$ (in the scheme indicated as the first set of states in the right part below the equal sign).

3. In general the state with $M = j_1 + j_2 - 1$ is twofold degenerate. One combination must be the state obtained with J_- from the state with $M = j_1 + j_2$, the other must be orthogonal to this state and again represents a 'maximum M'-value corresponding to $J = j_1 + j_2 - 1$.

4. This procedure goes on till we have reached $M = |j_1 - j_2|$, after which the degeneracy is equal to the $\min\{2j_1 + 1, 2j_2 + 1\}$, and stays constant till the M-value reaches the corresponding negative value.

Thus

Combining two angular momenta j_1 and j_2 we find resulting angular momenta J with values

$$J = j_1 + j_2,\; j_1 + j_2 - 1, \ldots, |j_1 - j_2|, \tag{26}$$

going down in steps of one.

Note that the total number of states is (as expected)

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1). \tag{27}$$

Furthermore we have in combining angular momenta:

half-integer with half-integer $\longrightarrow$ integer
integer with half-integer $\longrightarrow$ half-integer
integer with integer $\longrightarrow$ integer

3.2 Clebsch-Gordon coefficients

The actual construction of states just follows the steps outlined above. Let us illustrate it for the case of combining two spin 1/2 states. We have four states according to labeling in Eq. 24,

$$\begin{aligned}
|s_1, m_1\rangle \otimes |s_2, m_2\rangle : \qquad & |1/2, +1/2\rangle \otimes |1/2, +1/2\rangle \equiv |\uparrow\uparrow\rangle, \\
& |1/2, +1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\uparrow\downarrow\rangle, \\
& |1/2, -1/2\rangle \otimes |1/2, +1/2\rangle \equiv |\downarrow\uparrow\rangle, \\
& |1/2, -1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\downarrow\downarrow\rangle.
\end{aligned}$$

1. The highest state has $M = 1$ and must be the first of the four states above. Thus for the labeling $|s_1, s_2; S, M\rangle$
$$|1/2, 1/2; 1, +1\rangle = |\uparrow\uparrow\rangle. \tag{28}$$

2. Using $S_- = s_{1-} + s_{2-}$ we can construct the other $S + 1$ states.
$$S_-|1/2, 1/2; 1, +1\rangle = \hbar\sqrt{2}\,|1/2, 1/2; 1, 0\rangle,$$
$$(s_{1-} + s_{2-})|\uparrow\uparrow\rangle = \hbar(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),$$
and thus
$$|1/2, 1/2; 1, 0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big). \tag{29}$$
Continuing with S_- (or in this case using the fact that we have the lowest nondegenerate M-state) we find
$$|1/2, 1/2; 1, -1\rangle = |\downarrow\downarrow\rangle. \tag{30}$$

3. The state with $M = 0$ is twofold degenerate. One combination is already found in the above procedure. The other is made up of the same two states appearing on the right hand side in Eq. 29. Up to a phase, it is found by requiring it to be orthogonal to the state $|1/2, 1/2; 1, 0\rangle$ or by requiring that $S_+ = s_{1+} + s_{2+}$ gives zero. The result is
$$|1/2, 1/2; 0, 0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big). \tag{31}$$
The convention for the phase is that the higher m_1-value appears with a positive sign.

It is easy to summarize the results in a table, where one puts the states $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ in the different rows and the states $|j_1, j_2; J, M\rangle$ in the different columns, i.e.

$$\begin{array}{c|ccc} j_1 \times j_2 & \vdots & \mathrm{J} & \vdots \\ & \vdots & \mathrm{M} & \vdots \\ \hline \ldots\ \ldots & & & \\ m_1\ m_2 & & & \\ \ldots\ \ldots & & & \end{array}$$

For the above case we have

$$\begin{array}{cc|cccc} \multicolumn{2}{c|}{1/2 \times 1/2} & 1 & 1 & 0 & 1 \\ & & 1 & 0 & 0 & \text{-1} \\ \hline +1/2 & +1/2 & 1 & & & \\ +1/2 & -1/2 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\ -1/2 & +1/2 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\ -1/2 & -1/2 & & & & 1 \end{array}$$

Note that the recoupling matrix is block-diagonal because of the constraint $M = m_1 + m_2$. The coefficients appearing in the matrix are the socalled Clebsch-Gordan coefficients. We thus have
$$|j_1, j_2; J, M\rangle = \sum_{m_1, m_2} C(j_1, m_1, j_2, m_2; J, M)\,|j_1, m_1\rangle \otimes |j_2, m_2\rangle. \tag{32}$$

Represented as a matrix as done above, it is unitary (because both sets of states are normed). Since the Clebsch-Gordan coefficients are choosen real, the inverse is just the transposed matrix, or

$$|j_1, m_2\rangle \otimes |j_2, m_2\rangle = \sum_{J,M} C(j_1, m_1, j_2, m_2; J, M)\, |j_1, j_2; J, M\rangle. \tag{33}$$

In some cases (like combining two spin 1/2 states) one can make use of symmetry arguments. If a particular state has a well-defined symmetry under permutation of states 1 and 2, then all M-states belonging to a particular J-value have the same symmetry (because $j_{1\pm} + j_{2\pm}$ does not alter the symmetry. This could have been used for the $1/2 \times 1/2$ case, as the highest total M is symmetric, all $S = 1$ states are symmetric. This is in this case sufficient to get the state in Eq. 29.

We will give two other examples. The first is

$1 \times 1/2$		3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1	+1/2	1					
+1	−1/2		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
0	+1/2		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
0	−1/2				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
−1	+1/2				$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
−1	−1/2						1

for instance needed to obtain the explicit states for an electron with spin in an (2p)-orbit coupled to a total angular momentum $j = 3/2$ (indicated as $2p_{3/2}$) with $m = 1/2$ is

$$\phi(\boldsymbol{r}, t) = \frac{u_{2p}(r)}{r} \left(\sqrt{\frac{1}{3}}\, Y_1^1(\theta, \phi)\, \chi_\downarrow + \sqrt{\frac{2}{3}}\, Y_1^0(\theta, \phi)\, \chi_\uparrow \right).$$

The second is

1×1		2 +2	2 +1	1 +1	2 0	1 0	0 0	2 −1	1 −1	2 −2
+1	+1	1								
+1	0		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$						
0	+1		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						
+1	−1				$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	0				$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$			
+1	−1				$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	−1							$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
−1	0							$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
−1	−1									1

This example, useful in the combination of two spin 1 particles or two electrons in p-waves, illustrates the symmetry of the resulting wave functions.

3.3 An example: the Helium atom

As a first-order description of the helium atom, one can consider the independent-electron approximation, starting with a hamiltonian in which the electron-electron interaction is neglected, as well as any interactions involving the spin of the electrons. In that case one has a separable hamiltonian and for each of the electrons the solutions are given by hydrogen-like states ($Z = 2$), characterized by $(n\ell)$. Let us investigate the possible ground-state configurations, $(1s)^2$ and the first excited levels $(1s)(2p)$ and $(1s)(2s)$.

- The ground state configurations $(1s)^2$.
  Knowing the two angular momenta involved is sufficient to know the parity of these states, $\Pi = (-)^{\ell_1}(-)^{\ell_2} = +$. The angular momentum recoupling works in the following way.
  - Combining $\ell_1 = 0$ and $\ell_2 = 0$, the only possibility is $L = 0$. The orbital wave function then is symmetric under the interchange of the two electrons 1 and 2.
  - Combining the spins $s_1 = 1/2$ and $s_2 = 1/2$ gives two possibilities, $S = 0$ or $S = 1$. The first possibility is antisymmetric under the interchange of the electrons, the second is symmetric.
  - The total wave function (product of orbital and spin parts) must be antisymmetric for fermions according to the Pauli principle, hence $L = 0$ can only be combined with $S = 0$. This leaves only one possibility for the total angular momentum, $J = 0$. The notation for the only allowed ground state configuration is

$$(n_1\,\ell_1)(n_2\,\ell_2)\ {}^{2S+1}L_{J^\Pi} = (1s)^2\ {}^1S_{0^+}.$$

- The configurations $(1s)(2p)$ with parity $\Pi = -$.
  - We have $L = 1$, but appearing twice. We can construct the symmetric and antisymmetric combinations,

$$\phi^{s/a}_{LM_L} = \frac{1}{\sqrt{2}}\left[\frac{u_{1s}(r_1)}{r_1}\,Y_0^0(\Omega_1)\,\frac{u_{2p}(r_2)}{r_2}\,Y_1^{M_L}(\Omega_2) \pm \frac{u_{2p}(r_1)}{r_1}\,Y_1^{M_L}(\Omega_1)\frac{u_{1s}(r_2)}{r_2}\,Y_0^0(\Omega_2)\right]$$

    for the spatial part.
  - The combination of the spins gives again an antisymmetric $S = 0$ and a symmetric $S = 1$ wave function.
  - The allowed configurations are thus obtained by the appropriate antisymmetric combinations of orbital and spin parts,

$$(1s)(2p)\ {}^1P_{1^-} \qquad \text{and} \qquad (1s)(2p)\ {}^3P_{0^-,1^-,2^-}.$$

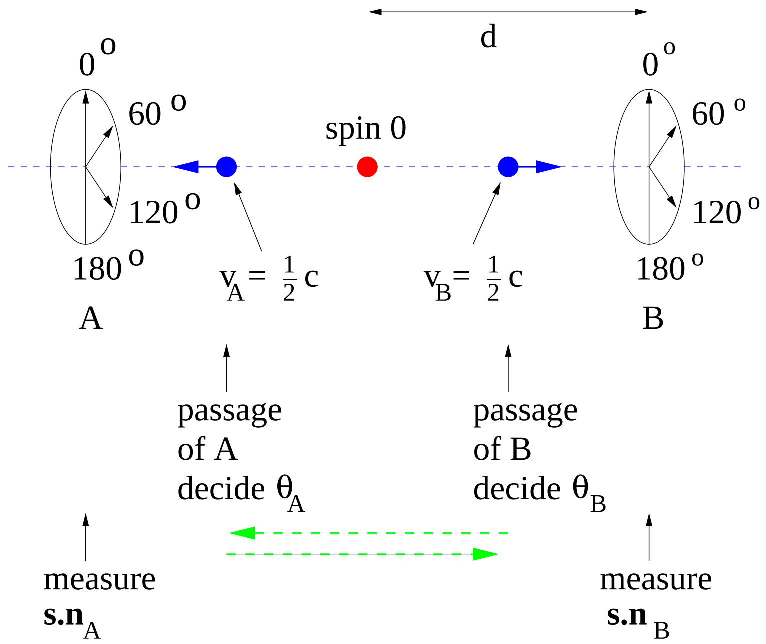
- The configurations $(1s)(2s)$ with parity $\Pi = +$.
  - We have $L = 0$, but now also appearing twice in a symmetric and antisymmetric combination.
  - As above, antisymmetric $S = 0$ and symmetric $S = 1$.
  - This gives the allowed configurations

$$(1s)(2s)\ {}^1S_{0^+} \qquad \text{and} \qquad (1s)(2s)\ {}^3S_{1^+}.$$

4 The EPR experiment

4.1 The 'experiment'

One of the best ways to illustrate the 'absurdness' of quantum mechanics when one tries to understand it with classical means is the EPR experiment, proposed by Einstein, Podolsky and Rosen (1935) to show that quantum mechanics should be wrong. A simplified version of the experiment is the decay of a spin 0 system into two spin 1/2 particles with opposite spin. The spin is measured along a direction perpendicular to the motion of the two particles, which are moving apart at (say) half of the speed of light. The angles θ_A and θ_B, along which the spin is measured can be varied, but the actual decision which angles are measured is only taken (at random) when the particles are halfway[1].



The results of the measurements (which in recent years have actually been performed with a variety of particles over macroscopic distances) are as follows:

- Considering only the measurement at A, the probability to measure a spin along $\hat{\boldsymbol{n}}_A$ is $P(\theta_A, +) = 1/2$ and the probability to measure a spin opposite to this is $P(\theta_A, -) = 1/2$. Similarly at B.

- Given a '+'-measurement at A at a given angle θ_A, the following results are found:

$\lvert\theta_A - \theta_B\rvert =$	0°	60°	120°	180°
$P(\theta_A, +; \theta_B, +)$	0	1/4	3/4	1
$P(\theta_A, +; \theta_B, -)$	1	3/4	1/4	0

[1] In this way information on the setting of B cannot reach A at the time of measurement

4.2 A classical explanation?

In any 'realistic' theory the information on the potential measurements for particle A should be a property of particle A and hence stored in the particle. Thus to simplify things, only concentrate on three angles (0°, 60° and 120°). Any event has particles A and B flying apart with known prescriptions for what to do if they encounters a Stern-Gerlach apparatus under a given angle θ_A. E.g.

$$\begin{array}{lll} \text{Event 1:} & \text{memory\_A} = (+-+) & \text{memory\_B} = (-+-) \\ \text{Event 2:} & \text{memory\_A} = (++-) & \text{memory\_B} = (--+) \end{array}$$

etc. For instance in the first example the prescription tells A that if $\theta_A = 0^\circ$, then a spin measurement yields $+\hbar/2$, if $\theta_A = 60^\circ$, then a spin measurement yields $-\hbar/2$, while if $\theta_A = 120^\circ$, then a spin measurement yields $+\hbar/2$. In each of the events the common origin of A and B is reflected in the complementary prescriptions. This is the only way the first and last colums of the 'experiment' can be explained. Let's now concentrate on the correlations and write some explicit probabilities $P(\theta_A, +; \theta_B, +)$ in terms of the fractions of events with particular memories. We have e.g.

$$\begin{aligned} P(0^\circ, +; 120^\circ, +) &= \sum_{\sigma=-\tau} f(+\sigma-; -\tau+) \\ &= f(++-;--+) \ + \ f(+--;-++) \end{aligned} \tag{34}$$

$$P(60^\circ, +; 120^\circ, +) = f(++-;--+) \ + \ \underbrace{f(-+-;+-+)}_{\geq 0} \tag{35}$$

$$P(0^\circ, +; 60^\circ, +) = \underbrace{f(+-+;-+-)}_{\geq 0} + f(+--;-++), \tag{36}$$

and hence we must have in a realistic theory

$$P(60^\circ, +; 120^\circ, +) + P(0^\circ, +; 60^\circ, +) \geq P(0^\circ, +; 120^\circ, +), \tag{37}$$

which is an explicit example of a Bell inequality. But clearly our 'experiment' above violates this inequality! Thus a realistic theory cannot explain the results in the EPR experiment.

4.3 The quantum-mechanical explanation!

Quantum mechanics can explain the results. Using e.g. the spin states defined with respect to the $\theta = 0$ directions, we know that the AB system is in an antisymmetric (spin zero) state,

$$|AB\rangle = \frac{1}{\sqrt{2}} \left(|A\uparrow\rangle \otimes |B\downarrow\rangle - |A\downarrow\rangle \otimes |B\uparrow\rangle \right) \tag{38}$$

Using the rotated spin states along an arbitrary direction $\hat{\boldsymbol{n}}$, it is straightforward to check the results of the EPR experiment. Given a '+'-measurement at A, the wave function becomes

$$|A\uparrow\rangle \otimes |B\downarrow\rangle$$

and the probability

$$P(0^\circ,+;\theta_B,+) = \left|\langle A\uparrow||A\uparrow\rangle\,\langle\theta_B,+||B\downarrow\rangle\right|^2 = \sin^2(\theta_B/2). \tag{39}$$

For any other angle for θ_A, simply choose this as the quantization axis, to see that the result only depends on $\theta_A - \theta_B$.