Transverse momentum dependence in structure functions in hard scattering processes

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Chapter 1

General cross sections

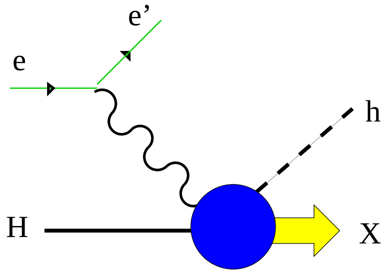
1.1 Introduction

We will consider the following three types of processes,

- The *lepton-hadron* scattering process $\ell H \to \ell' X$
- The *annihilation* process $\ell\bar{\ell} \to X$
- The *lepton-pair production* process $AB \to \ell\bar{\ell}X$ (Drell-Yan process).

All these processes involve electroweak currents, coupling to the leptons in a known way. The basic advantage of electroweak processes lies in the fact that the process is accurately described in terms of the exchange of one photon (for electromagnetic processes), since the coupling, $\alpha = e^2/4\pi \approx 1/137$, is weak. The same is true for the weak vector bosons. On the hadronic side, the coupling to the quarks is known, but the structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the coupling to the quarks is known, however, enables the study of hadron structure.

For lepton-hadron scattering we consider the inclusive measurement $\ell H \to \ell' X$ and the *1-particle inclusive* or *semi-inclusive* measurement $\ell H \to \ell' h X$. The invariants are defined,



$$q^2 = (k - k')^2 \equiv -Q^2 \leq 0 \qquad (1.1)$$

$$2P \cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_B} \qquad (1.2)$$

$$2P_h \cdot q \equiv -z_h\,Q^2 \qquad (1.3)$$

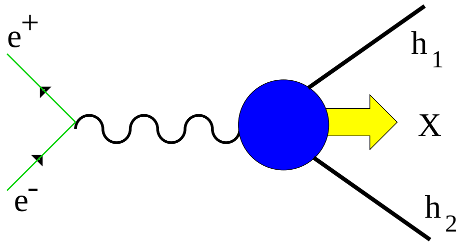
$$P \cdot P_h \equiv z\,P \cdot q \qquad (1.4)$$

The variable x_B is the Bjorken scaling variable. Since the invariant mass squared of the hadronic final state satisfies

$$W_X^2 = (P+q)^2 = \frac{1-x_B}{x_B}\,Q^2 + M^2 \geq M^2, \qquad (1.5)$$

one has $0 \leq x_B \leq 1$, with $x_B = 1$ corresponding to elastic scattering, i.e. $W_X^2 = M^2$. In this case a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ GeV and one probes deep into the nucleon (<0.1 fm) with $Q > 2$ GeV. As we will see, the invariants $z \approx z_h$ for the case of *one* leading jet (to which h belongs) in the limit that $Q^2 \to \infty$.

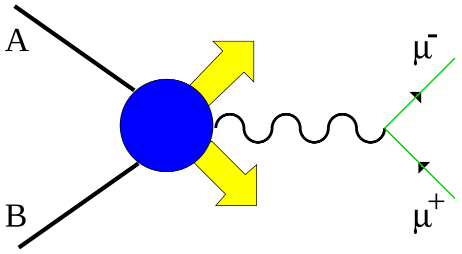
For the annihilation process we distinguish the *inclusive* measurement $\ell\bar{\ell} \to X$, the *1-particle inclusive* measurements $\ell\bar{\ell} \to hX$ and the *2-particle inclusive* measurements $\ell\bar{\ell} \to h_1 h_2 X$ (hadrons belonging to back-to-back jets). The invariants are defined



$$q^2 = (k+k')^2 \equiv Q^2 \geq 0 \quad (1.6)$$
$$2\,P_1 \cdot q \equiv z_1\,Q^2 \quad (1.7)$$
$$2\,P_2 \cdot q \equiv z_2\,Q^2 \quad (1.8)$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon, in which case it is clear that Q is a measure of the excitation energy.

For the Drell-Yan process we only consider the *inclusive* case, which already involves two hadrons. We restrict ourselves to lepton pairs with small transverse momentum (compared to Q), for which we have the invariants



$$q^2 = (k+k')^2 \equiv Q^2 \geq 0 \quad (1.9)$$
$$2P_A \cdot q \equiv \frac{Q^2}{x_A} \quad (1.10)$$
$$2P_B \cdot q \equiv \frac{Q^2}{x_B} \quad (1.11)$$

1.2 The hadronic tensor

1.2.1 Lepton-hadron scattering

Consider the process $\ell + H \to \ell' + h + X$, in which a lepton with momentum k scatters off a hadron H with momentum P and one hadron h with momentum P_h is measured in coincidence with the scattered lepton with momentum k'. The lepton emits a highly virtual photon with momentum

$$q^\mu = k^\mu - k'^\mu, \quad (1.12)$$

with $Q^2 \equiv -q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X^μ. We will consider the most general case of a pure incoming spin state, characterized by the spin vectors S^μ, an observed hadronic spin state characterized by the spin vector S_h and lepton helicities λ and λ'.

We have the following relations

$$P^2 = M^2, \quad P_h^2 = M_h^2, \quad (1.13)$$
$$k^2 = m^2 \approx 0, \quad k'^2 = m'^2 \approx 0, \quad (1.14)$$
$$(k+P)^2 = s, \quad (1.15)$$
$$S^2 = S_h^2 = -1, \quad (1.16)$$
$$P \cdot S = P_h \cdot S_h = 0. \quad (1.17)$$

We will work in the limit where Q^2, $P\cdot q$ and $P_h \cdot q$ are large keeping the ratios $Q^2/2P\cdot q$ and $2P_h \cdot q/Q^2$ finite. The invariant amplitude for the process is given by

$$\mathcal{M} = \bar{u}(k',\lambda')\gamma^\mu u(k,\lambda)\frac{e^2}{Q^2}\langle P_X; P_h S_h|J_\mu(0)|PS\rangle. \quad (1.18)$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(\ell H)} H^{\mu\nu(\ell H)}, \quad (1.19)$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(\ell H)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma\right). \quad (1.20)$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(\ell H)}(P_X;PS;P_hS_h) = \langle PS|J_\mu(0)|P_X;P_hS_h\rangle\langle P_X;P_hS_h|J_\nu(0)|PS\rangle, \tag{1.21}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{1.22}$$

with the flux factor

$$F = 4\sqrt{(P\cdot k)^2 - M^2m^2} \;\approx\; 2s \tag{1.23}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(k+P-k'-P_X-P_h)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k'}{(2\pi)^3 2k'^0}\frac{d^3P_h}{(2\pi)^3 2P_h^0}. \tag{1.24}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$2M\mathcal{W}_{\mu\nu}^{(\ell H)}(q;PS;P_hS_h) = \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X-P_h)H_{\mu\nu}^{(\ell H)}(P_X;PS;P_hS_h). \tag{1.25}$$

The phase space for the scattered lepton can be rewritten as

$$\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{E'\,dE'\,d\Omega}{16\pi^3}, \tag{1.26}$$

where Ω is the lepton scattering angle and E' the energy of the scattered lepton. Thus one gets

$$\frac{2E_h\,d\sigma^{(\ell H)}}{d^3P_h\,d\Omega dE'} = \frac{2ME'}{s}\,\frac{\alpha^2}{Q^4}L_{\mu\nu}^{(\ell H)}\mathcal{W}^{\mu\nu(\ell H)}, \tag{1.27}$$

where the lepton tensor is given by the expression between brackets in Eq. (1.20) and the hadron tensor by Eq. (1.25).

Note that for inclusive scattering one obtains the familiar result

$$\frac{d\sigma^{(\ell H)}}{d\Omega dE'} = \frac{2ME'}{s}\,\frac{\alpha^2}{Q^4}L_{\mu\nu}^{(\ell H)}W^{\mu\nu(\ell H)}, \tag{1.28}$$

with the hadron tensor given by

$$\begin{aligned} 2M\,W_{\mu\nu}^{(\ell H)}(q;PS) &= \frac{1}{2\pi}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X)\langle PS|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|PS\rangle \\ &= \frac{1}{2\pi}\int d^4x\; e^{iq\cdot x}\;\langle PS|[J_\mu(x),J_\nu(0)]|PS\rangle. \end{aligned} \tag{1.29}$$

1.2.2 Electron-positron annihilation

Consider the process $e^- + e^+ \to h_1 + h_2 + X$, where two hadrons belonging to opposite jets emerge with momenta P_1^μ and P_2^μ. The annihilating incoming leptons with momenta k^μ and k'^μ produce a high mass photon with momentum

$$q^\mu = k^\mu + k'^\mu, \tag{1.30}$$

with $Q^2 \equiv q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$. We will consider the general case of polarized leptons with helicities λ and λ' and production of hadrons of which the spin states are characterized by spin vectors S_1^μ and S_2^μ, respectively. We have the following relations

$$P_1^2 = M_1^2, \quad P_2^2 = M_2^2, \tag{1.31}$$

$$k^2 = k'^2 = m^2 \approx 0, \tag{1.32}$$

$$S_1^2 = S_2^2 = -1, \tag{1.33}$$

$$P_1\cdot S_1 = P_2\cdot S_2 = 0. \tag{1.34}$$

We will work in the limit where Q^2, $P_1 \cdot q$ and $P_2 \cdot q$ are large, keeping the ratios $2P_1 \cdot q/Q^2$ and $2P_2 \cdot q/Q^2$ finite. The invariant ampitude for the process is given by

$$\mathcal{M} = \bar{v}(k', \lambda')\gamma^\mu u(k,\lambda)\frac{e^2}{Q^2}\langle P_X; P_1 S_1; P_2 S_2 | J_\mu(0)|0\rangle. \tag{1.35}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(e^+e^-)} H^{\mu\nu(e^+e^-)}, \tag{1.36}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(e^+e^-)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}k^\rho k'^\sigma\right). \tag{1.37}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(e^+e^-)}(P_X; P_1 S_1; P_2 S_2) = \langle 0|J_\mu(0)|P_X; P_1 S_1; P_2 S_2\rangle\langle P_X; P_1 S_1; P_2 S_2|J_\nu(0)|0\rangle, \tag{1.38}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{1.39}$$

with the flux factor

$$F = 4\sqrt{(k\cdot k')^2 - k^2 k'^2} \;\approx\; 2Q^2 \tag{1.40}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(k + k' - P_X - P_1 - P_2)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3P_1}{(2\pi)^3 2P_1^0}\frac{d^3P_2}{(2\pi)^3 2P_2^0}. \tag{1.41}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\mathcal{W}_{\mu\nu}^{(e^+e^-)}(q; P_1 S_1; P_2 S_2) = \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q - P_X - P_1 - P_2)H_{\mu\nu}^{(e^+e^-)}(P_X; P_1 S_1; P_2 S_2). \tag{1.42}$$

One obtains the cross section (including a factor 1/2 from averaging over incoming polarizations)

$$\frac{P_1^0\,P_2^0\;d\sigma^{(e^+e^-)}}{d^3P_1\,d^3P_2} = \frac{\alpha^2}{4\,Q^6}L_{\mu\nu}^{(e^+e^-)}\mathcal{W}^{\mu\nu(e^+e^-)}, \tag{1.43}$$

where the lepton tensor is given by the expression between brackets in Eq. (1.37) and the hadron tensor by Eq. (1.42).

Note that for a single produced hadron one finds

$$E_h\,\frac{d\sigma}{d^3P_h} = \frac{\alpha^2}{2\,Q^6}\,L_{\mu\nu}^{(e^+e^-)}W^{\mu\nu(e^+e^-)}, \tag{1.44}$$

where the hadron tensor is given by

$$W_{\mu\nu}^{(e^+e^-)}(q; P_h S_h) = \frac{1}{(2\pi)}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q - P_X - P_h)\langle 0|J_\mu(0)|P_X; P_h S_h\rangle\langle P_X; P_h S_h|J_\nu(0)|0\rangle. \tag{1.45}$$

For the annihilation cross section one finds

$$\sigma(e^+e^- \to hadrons) = \frac{4\pi^2\,\alpha^2}{Q^6}\,L_{\mu\nu}^{(e^+e^-)}R^{\mu\nu(e^+e^-)}, \tag{1.46}$$

where the tensor $R_{\mu\nu}$ is given by

$$\begin{aligned} R_{\mu\nu}^{(e^+e^-)}(q) &= \int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q - P_X)\langle 0|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|0\rangle \\ &= \int d^4x\; e^{iq\cdot x}\,\langle 0|[J_\mu(x), J_\nu(0)]|0\rangle. \end{aligned} \tag{1.47}$$

1.2.3 The Drell-Yan process

Consider the process $A + B \to \ell + \bar{\ell} + X$, where two spin-$\frac{1}{2}$ hadrons with momenta P_A^μ and P_B^μ interact and two outgoing leptons (considered massless) are measured with momenta k^μ and k'^μ. The leptons are assumed to originate from a high mass photon with momentum

$$q^\mu = k^\mu + k'^\mu, \tag{1.48}$$

with $Q^2 \equiv q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X^μ. We will consider the case of pure incoming spin states, characterized by the spin vectors S_A^μ and S_B^μ, respectively, and observed lepton helicities λ and λ'. We have the following relations

$$P_A^2 = M_A^2, \quad P_B^2 = M_B^2, \tag{1.49}$$
$$(P_A + P_B)^2 = s, \tag{1.50}$$
$$k^2 = k'^2 = m^2 \approx 0, \tag{1.51}$$
$$S_A^2 = S_B^2 = -1, \tag{1.52}$$
$$P_A \cdot S_A = P_B \cdot S_B = 0. \tag{1.53}$$

In the deep inelastic limit $Q^2, s \to \infty$, with the ratio $\tau = Q^2/s$ fixed. The invariant amplitude for the process is given by

$$\mathcal{M} = \bar{u}(k,\lambda)\gamma^\mu v(k',\lambda')\frac{e^2}{Q^2}\langle P_X|J_\mu(0)|P_A S_A; P_B S_B\rangle. \tag{1.54}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(DY)} H^{\mu\nu(DY)}, \tag{1.55}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(DY)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma\right). \tag{1.56}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(DY)}(P_X; P_A S_A; P_B S_B) = \langle P_A S_A; P_B S_B|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|P_A S_A; P_B S_B\rangle, \tag{1.57}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{1.58}$$

with the flux factor

$$F = 4\sqrt{(P_A \cdot P_B)^2 - M_A^2 M_B^2} \;\approx\; 2s \tag{1.59}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(P_A + P_B - P_X - k - k')\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0}. \tag{1.60}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\begin{aligned}\mathcal{W}_{\mu\nu}^{(DY)}(q; P_A S_A; P_B S_B) &= \frac{1}{(2\pi)^4}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(P_A + P_B - P_X - q)\\ &\qquad \times H_{\mu\nu}^{(DY)}(P_X; P_A S_A; P_B S_B)\\ &= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\langle P_A S_A; P_B S_B|[J_\mu(0), J_\nu(x)]|P_A S_A; P_B S_B\rangle.\end{aligned} \tag{1.61}$$

For the last equality completeness of the out states and causality has been used. The remaining phase space is conveniently written as

$$\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{d^4q}{(2\pi)^4}\frac{d\Omega}{32\pi^2}, \tag{1.62}$$

where the angles are those of the lepton axis in the rest frame of the two leptons. In terms of the fine-structure constant $\alpha \equiv e^2/4\pi$, we then obtain the Drell-Yan cross section (including a factor 2 from the summation over the lepton polarizations)

$$\frac{d\sigma^{(DY)}}{d^4q\,d\Omega} = \frac{\alpha^2}{2s\,Q^4} L_{\mu\nu}^{(DY)} \mathcal{W}^{\mu\nu(DY)}, \tag{1.63}$$

where the lepton tensor is given by the symmetric part in Eq. (1.56) and the hadron tensor by Eq. (1.61).

1.3 Deep inelastic kinematics

In order to deal with the hard processes, it is convenient to consider a Cartesian set of vectors constructed from the momenta. These start with defining q^μ as a spacelike or timelike direction depending on the process. Then one proceeds using vectors that are orthogonal to q. Such vectors $\tilde{a}$ are obtained subtracting from a the projection along q,

$$\tilde{a}^\mu \equiv \tilde{g}^{\mu\nu} a_\nu = a^\mu - \frac{a \cdot q}{q^2}\, q^\mu, \tag{1.64}$$

where

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}. \tag{1.65}$$

Another set of vectors that is useful, in particular for the theoretical description of the structure of hadrons, are a set of lightlike vectors, n_-^μ and n_+^μ ($n_+^2 = n_-^2 = 0$, $n_+ \cdot n_- = 1$) that are in essence hadronic momenta divided by the large scale Q. If $P/Q = a\,n_+ + b\,n_-$ and $a \sim 1$ then $b \sim M^2/Q^2 \ll a$. Hadronic momenta divided by Q are thus in essence proportional to one lightlike vector, the hard momentum q/Q, however, involves two lightlike vectors. We will use for four vectors the notations $p = (p^0, p^1, p^2, p^3)$ or $p = [p^-, p^+, p^1, p^2]$ where $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$, depending on the fact if we use a Cartesian set of basisvectors or a set with two lightlike vectors. In the latter case one must be aware of the metric, having e.g. $p^\pm = p \cdot n_\mp$.

We will consider two different sets of frames, the first set (type I) has $\boldsymbol{q}_\perp \equiv (q^1, q^2) = \boldsymbol{0}_\perp$, i.e. the virtual photon has no transverse components. We note that there is still freedom to parametrize q, e.g. in lepton-hadron scattering,

$$q \stackrel{I}{=} \left[-\frac{1}{A}\frac{Q}{\sqrt{2}}, A\,\frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp \right].$$

The quantity A specifies a particular frame. Frames with different A are connected via a simple boost along the z-axis. The second set of frames (type II) are those where the hadrons have no perpendicular momentum, relevant in cases where two hadrons play a role. In these frames the transverse momenta are indicated with $\boldsymbol{p}_T$ and thus $\boldsymbol{P}_T = \boldsymbol{P}_{hT} = \boldsymbol{0}_T$. Note that q in such a frame in general does have transverse components,

$$q \stackrel{II}{=} \left[-\frac{1}{A}\frac{\tilde{Q}}{\sqrt{2}}, A\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{q}_T \right],$$

with $\boldsymbol{q}_T^2 = Q_T^2$ and $Q^2 = \tilde{Q}^2 + Q_T^2$. The connection between frames of type I and II can be made by a Lorentz transformation, e.g. one that leaves the minus component unchanged [1] and involves a parameter b^- and a two-component vector $\boldsymbol{b}$

$$[a^-, a^+, \boldsymbol{a}] \longrightarrow \left[a^-, a^+ - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{b^-} + \frac{\boldsymbol{b}^2\,a^-}{2(b^-)^2}, \boldsymbol{a} - \frac{a^-}{b^-}\,\boldsymbol{b} \right]. \tag{1.66}$$

In most of the following we will assume that all hadrons and the virtual photon are in essence parallel, i.e. $Q_T \ll Q$ and up to $\mathcal{O}(1/Q^2)$ corrections $\tilde{Q} \approx Q$. This implies for semi-inclusive lepton-hadron just one leading jet containing the produced hadron, for 2-particle inclusive lepton annihilation just two back-to-back jets and for Drell-Yan only lepton-pairs with transverse momentum $\ll Q$.

[1] To do this one needs in the two parametrizations boost factors A differing by a factor $Q/\tilde{Q}$.

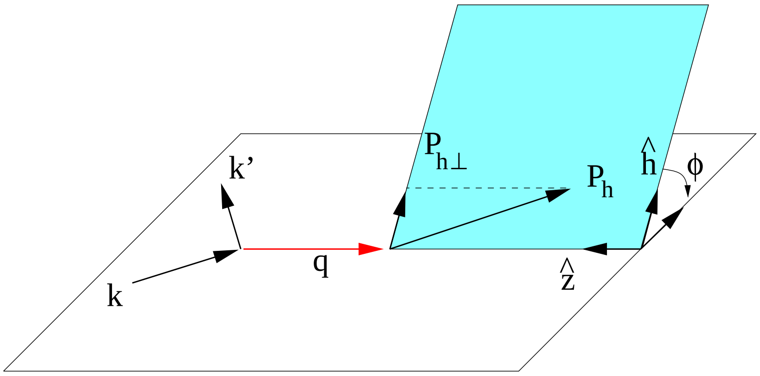


Figure 1.1: Kinematics of lepton hadron scattering in a frame where target hadron and virtual photon momentum are parallel (including target rest frame).

1.3.1 Lepton-hadron scattering

For lepton-hadron scattering the starting point of defining a Cartesian set is a spacelike direction defined by the momentum transfer q^μ. Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\,q^\mu$, which is timelike and satisfies $\tilde{P}^2 = \kappa\, P\cdot q$ with

$$\kappa = 1 + \frac{M^2\,Q^2}{(P\cdot q)^2} = 1 + \frac{4\,M^2 x_{_B}^2}{Q^2}. \tag{1.67}$$

taking into account mass corrections $\propto M^2/Q^2$ which will vanish for large Q^2 ($\kappa \to 1$). Defining

$$Z^\mu \equiv -q^\mu, \tag{1.68}$$

$$T^\mu \equiv \frac{Q^2}{P\cdot q}\,\frac{\tilde{P}^\mu}{\sqrt{\kappa}} = \frac{q^\mu + 2x_{_B}\,P^\mu}{\sqrt{\kappa}}, \tag{1.69}$$

we have $Z^2 = -Q^2$ and $T^2 = Q^2$ and will mostly consider normalized vectors $\hat{z}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/Q$. Note that $P\cdot T = \sqrt{\kappa}\,P\cdot q$ and $\tilde{P}^2 = (P\cdot\hat{t}\,)^2 = \kappa\,(P\cdot\hat{z})^2$. In the space orthogonal to $\hat{z}$ and $\hat{t}$ one has the tensors

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu, \tag{1.70}$$

$$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma \;=\; \frac{1}{(P\cdot q)\sqrt{\kappa}}\,\epsilon^{\mu\nu\rho\sigma}\,P_\rho q_\sigma. \tag{1.71}$$

A relevant vector in the perpendical space appears if we have more than one hadron, e.g. in 1-particle inclusive leptoproduction. For instance $P_{h\perp} = g_\perp^{\mu\nu}\,P_{h\nu}$, defining the orientation of the production plane in semi-inclusive leptoproduction, $\hat{h}^\mu = P_{h\perp}^\mu/|\boldsymbol{P}_{h\perp}|$ (see figure 1.1).

We can also construct a timelike direction using the vector vector P_h. The vector $\tilde{P}_h = P_h^\mu - (P_h\cdot q/q^2)\,q^\mu$ satifies $\tilde{P}_h^2 = \kappa_h\,(P_h\cdot q)^2$ with

$$\kappa_h = 1 + \frac{M_h^2 Q^2}{(P_h\cdot q)^2} = 1 + \frac{4\,M_h^2}{z_h^2\,Q^2}. \tag{1.72}$$

One has $\tilde{P}\cdot\tilde{P}_h = (P\cdot\hat{t}\,)(P_h\cdot\hat{t}\,)$ and $\tilde{P}_h^2 = M_h^2 + (P_h\cdot\hat{z})^2 = (P_h\cdot\hat{t}\,)^2 - \boldsymbol{P}_{h\perp}^2 = \kappa_h\,(P_h\cdot\hat{z})^2$.

One can extend the Cartesian set with

$$X^\mu \;\equiv\; \frac{(P_h\cdot T)\,\tilde{P}^\mu - (P\cdot T)\,\tilde{P}_h^\mu}{(P\cdot\hat{t}\,)(P_h\cdot\hat{z}) - (P\cdot\hat{z})(P_h\cdot\hat{t}\,)} \tag{1.73}$$

$$Y^\mu \;\equiv\; \frac{\epsilon^{\mu\nu\rho\sigma}P_\nu P_{h\rho}q_\sigma}{(P\cdot\hat{t}\,)(P_h\cdot\hat{z}) - (P\cdot\hat{z})(P_h\cdot\hat{t}\,)} \;=\; -\frac{1}{\sqrt{\kappa}}\,\epsilon_\perp^{\mu\nu}X_\nu. \tag{1.74}$$

Explicitly one has

$$\begin{aligned} X^\mu &= q^\mu + x_{\scriptscriptstyle B}\, P^\mu \left[\frac{2}{1-\frac{(P\cdot T)(P_h\cdot q)}{(P\cdot q)(P_h\cdot T)}}\right] - \frac{P_h^\mu}{z_h}\left[\frac{2}{1-\frac{(P\cdot q)(P_h\cdot T)}{(P\cdot T)(P_h\cdot q)}}\right] \\ &\approx q^\mu + x_{\scriptscriptstyle B}\, P^\mu - \frac{P_h^\mu}{z_h}, \end{aligned} \tag{1.75}$$

the last expression being without the mass corrections, valid up to $1/Q^2$ corrections.

Introducing

$$\kappa_{h\perp} = 1 + \frac{M_{h\perp}^2 Q^2}{(P_h\cdot q)^2} = 1 + \frac{4\,M_{h\perp}^2}{z_h^2\, Q^2}, \tag{1.76}$$

where $M_{h\perp}^2 = M_h^2 + \boldsymbol{P}_{h\perp}^2$, one can write

$$X^\mu = q^\mu + x_{\scriptscriptstyle B}\, P^\mu \left[\frac{2}{1+\sqrt{\frac{\kappa}{\kappa_{h\perp}}}}\right] - \frac{P_h^\mu}{z_h}\left[\frac{2}{1+\sqrt{\frac{\kappa_{h\perp}}{\kappa}}}\right], \tag{1.77}$$

from which one e.g. immediately sees that

$$X^2 = \frac{P_{h\perp}^2}{z_h^2}\left(\frac{2}{1+\sqrt{\frac{\kappa_{h\perp}}{\kappa}}}\right)^2. \tag{1.78}$$

or using z instead of z_h,

$$X^2 = \frac{P_{h\perp}^2}{z^2}\,\frac{\kappa}{\left(1-(1-\kappa)\,\frac{M_{h\perp}^2}{z^2\,Q^2}\right)}. \tag{1.79}$$

Next we introduce two simple lightlike vectors ($n_+\cdot n_- = 1$, $n_+^2 = n_-^2 = 0$), such that we have

$$P^\mu \equiv \frac{\tilde Q}{\xi\sqrt{2}}\, n_+^\mu + \frac{\xi\, M^2}{\tilde Q\sqrt{2}}\, n_-^\mu, \tag{1.80}$$

$$P_h^\mu \equiv \frac{\zeta\tilde Q}{\sqrt{2}}\, n_-^\mu + \frac{M_h^2}{\zeta\tilde Q\sqrt{2}}\, n_+^\mu, \tag{1.81}$$

$$q^\mu \equiv \frac{\tilde Q}{\sqrt{2}}\, n_-^\mu - \frac{\tilde Q}{\sqrt{2}}\, n_+^\mu + q_{\scriptscriptstyle T}^\mu, \tag{1.82}$$

where $q_{\scriptscriptstyle T}^2 \equiv -Q_{\scriptscriptstyle T}^2$ and $\tilde Q^2 = Q^2 - Q_{\scriptscriptstyle T}^2$. Note that the variables ξ and ζ are equal to the invariants $x_{\scriptscriptstyle B}$ and z_h or z up to corrections of order M^2/Q^2, M_h^2/Q^2 and $Q_{\scriptscriptstyle T}^2/Q^2$. Furthermore one has that $X^2 = -Q_{\scriptscriptstyle T}^2 = -\boldsymbol{q}_{\scriptscriptstyle T}^2$.

It is possible to take several kinematic corrections into account by starting with the above parametrization in terms of ξ and ζ and calculate the invariants. Inversion gives

$$\begin{aligned} \xi &= x_{\scriptscriptstyle B}\,\frac{\tilde Q^2}{Q^2}\,\frac{2}{\left(1+\sqrt{1+\frac{\tilde Q^2}{Q^2}\,\frac{4M^2 x_B^2}{Q^2}}\right)}, \\ \zeta &= z_h\,\frac{Q^2}{\tilde Q^2}\,\frac{\left(1+\sqrt{1+\frac{\tilde Q^2}{Q^2}\,\frac{4M_h^2}{z_h^2\,Q^2}}\right)}{2}, \end{aligned}$$

or instead of the last one in terms of z instead of z_h,

$$\zeta = z\,\frac{\left(1+\sqrt{1+\frac{4x_B^2 M^2 M_h^2}{z^2\,Q^4}}\right)}{\left(1+\sqrt{1+\frac{\tilde Q^2}{Q^2}\,\frac{4x_B^2 M^2}{Q^2}}\right)}.$$

$Q_{\scriptscriptstyle T}$ is determined from $\boldsymbol{P}_{h\perp}$ or implicitly from z, z_h and $x_{\scriptscriptstyle B}$ (e.g. equating the two expressions for ζ). For inclusive lepton-hadron scattering one has $Q_{\scriptscriptstyle T} = 0$ and $\xi = 2x_{\scriptscriptstyle B}/(1+\sqrt{\kappa})$. This quantity is referred to as the Nachtmann scaling variable.

We give the hadronic momenta in the frames I and II, including the vector q_T, up to $\mathcal{O}(1/Q^2)$. We do this omitting the 'boost' factors $1/A$ and A multiplying $-$ and $+$ components respectively.

lepton-hadron:			
momentum	frame I	frame II	relations
$q = k - k'$	$\left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \mathbf{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T = -\frac{1}{z}\,\boldsymbol{P}_{h\perp}$
P	$\left[\frac{x_B M^2}{Q\sqrt{2}}, \frac{Q}{x_B\sqrt{2}}, \mathbf{0}_\perp\right]$	$\left[\frac{x_B M^2}{Q\sqrt{2}}, \frac{Q}{x_B\sqrt{2}}, \mathbf{0}_T\right]$	$x_B = -\frac{q^+}{P^+} \approx \frac{Q^2}{2P\cdot q} \approx -\frac{P_h\cdot q}{P_h\cdot P}$
P_h	$\left[\frac{z_h Q}{\sqrt{2}}, \frac{M_{h\perp}^2}{z_h Q\sqrt{2}}, \boldsymbol{P}_{h\perp}\right]$	$\left[\frac{z_h Q}{\sqrt{2}}, \frac{M_h^2}{z_h Q\sqrt{2}}, \mathbf{0}_T\right]$	$z_h = \frac{P_h^-}{q^-} \approx -\frac{2P_h\cdot q}{Q^2} \approx \frac{P\cdot P_h}{P\cdot q}$
q_T	$\left[0\ , -2\,\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$\left[0\ , 0\ , \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T^2 = Q_T^2$

The important thing to notice is that the momentum q_T introduced as the transverse part of q in frame II, produces in frame I a term in the +-direction, which will produce effects only suppressed by $1/Q$, rather than mass effects which always will appear suppressed by $1/Q^2$.

The forms of the vectors in the Cartesian set and the vectors $n_\pm$ and q_T including transverse momentum corrections, but neglecting mass corrections (order $1/Q^2$) are explicitly given below.

ℓH and e^-e^+		
vector	frame I	frame II
T	$\left[\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{Q^2}{\tilde{Q}\sqrt{2}}\ ,\ \mathbf{0}_\perp\right]$	$\left[\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{\tilde{Q}^2+2\,Q_T^2}{\tilde{Q}\sqrt{2}}\ ,\ \boldsymbol{q}_T\right]$
Z	$\left[-\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{Q^2}{\tilde{Q}\sqrt{2}}\ ,\ \mathbf{0}_\perp\right]$	$\left[-\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{\tilde{Q}}{\sqrt{2}}\ ,\ -\boldsymbol{q}_T\right]$
X	$[\ 0\ ,\ 0\ ,\ \boldsymbol{q}_T]$	$\left[\ 0\ ,\ 2\,\frac{Q_T^2}{\tilde{Q}\sqrt{2}},\ \boldsymbol{q}_T\right]$
n_-	$\left[1, \frac{Q_T^2}{\tilde{Q}^2}, -\frac{\boldsymbol{q}_T\sqrt{2}}{\tilde{Q}}\right]$	$[1\ ,\ 0\ ,\ \mathbf{0}_T]$
n_+	$[0\ ,\ 1\ ,\ \mathbf{0}_T]$	$[0\ ,\ 1\ ,\ \mathbf{0}_T]$
q_T	$\left[\ 0\ , -2\,\frac{Q_T^2}{\tilde{Q}\sqrt{2}}\ ,\ \boldsymbol{q}_T\right]$	$[\ 0\ ,\ 0\ ,\ \boldsymbol{q}_T]$

Later on we will need to transform from the theoretically useful vectors $n_\pm$ and q_T to the quantities appearing in the expansion of the hadronic tensor, $\hat{t}$, $\hat{z}$ and $\hat{x}$,

$$n_-^\mu = \frac{1}{\tilde{Q}\sqrt{2}}\left(\left(1+\frac{Q_T^2}{Q^2}\right)T^\mu - \left(1-\frac{Q_T^2}{Q^2}\right)Z^\mu - 2\,X^\mu\right) \approx \frac{T^\mu - Z^\mu - 2\,X^\mu}{Q\sqrt{2}} \tag{1.83}$$

$$n_+^\mu = \frac{\tilde{Q}^2}{Q^2}\,\frac{T^\mu + Z^\mu}{\tilde{Q}\sqrt{2}} \approx \frac{T^\mu + Z^\mu}{Q\sqrt{2}} \tag{1.84}$$

$$q_T^\mu = X^\mu - \frac{Q_T^2}{Q^2}\,(T^\mu + Z^\mu) \tag{1.85}$$

Again inclusion of mass corrections can be done by using the exact inverse of the expressions for P, P_h and q,

$$n_+ = \frac{\xi\sqrt{2}}{\tilde{Q}}\,\frac{\left(P - \frac{\xi M^2}{\zeta\tilde{Q}^2}\,P_h\right)}{\left(1 - \frac{\xi^2 M^2 M_h^2}{\zeta^2 Q^4}\right)},$$

$$n_- = \frac{\sqrt{2}}{\zeta\tilde{Q}}\,\frac{\left(P_h - \frac{\xi M_h^2}{\zeta\tilde{Q}^2}\,P\right)}{\left(1 - \frac{\xi^2 M^2 M_h^2}{\zeta^2 Q^4}\right)},$$

and rewriting q, P, and P_h in terms of Z, T, and X.

Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^{\mu} n_-^{\nu} - n_-^{\mu} n_+^{\nu} \\ &= g_\perp^{\mu\nu} - \frac{Q_T}{Q}\,\hat q^{\{\mu}\hat x^{\nu\}} + \frac{Q_T}{Q}\,\hat t^{\{\mu}\hat x^{\nu\}}, \end{aligned} \tag{1.86}$$

$$g_\perp^{\mu\nu} = g_T^{\mu\nu} - \frac{\sqrt{2}\,n_+^{\{\mu}q_T^{\nu\}}}{Q} \tag{1.87}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0,0,\boldsymbol{a}_T]$ with $|\boldsymbol{a}_T| \sim 1$, i.e. $\sim Q^0$, one has up to $\mathcal{O}(1/Q^2)$ the relation

$$a_T^\mu \approx g_\perp^{\mu\nu} a_{T\nu} - \frac{a_T\cdot q_T}{Q^2}\,q^\mu + \frac{a_T\cdot q_T}{Q^2}\,T^\mu. \tag{1.88}$$

Note that the first term on the righthand side has *in frame I* the form $[0,0,\boldsymbol{a}_T]$. We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu} a_{T\nu}$ for it, but one must be careful not to confuse this with the vector $g_\perp^{\mu\nu}\,a_\nu$. An example of such a vector is S_T, part of the spin vector characterizing the spin of a spin 1/2 hadron, for which one has

$$P^\mu = \frac{M^2}{2P^+}\,n_-^\mu + P^+\,n_+^\mu, \tag{1.89}$$

$$S^\mu = -S_{hL}\,\frac{M}{2P^+}\,n_-^\mu + S_{hL}\,\frac{P^+}{M}\,n_+^\mu + S_T^\mu, \tag{1.90}$$

such that $P\cdot S = 0$ and $-S^2 = 1 = S_{hL}^2 + \boldsymbol{S}_T^2$. Also for the quark transverse momentum vectors k_T this relation will become important.

Also the lepton momenta k and $k' = k - q$ can be expanded in $\hat t$, $\hat z$ and a perpendicular component using the scaling variable $y = P\cdot q/P\cdot k$ (in the target restframe reducing to $y = \nu/E$). The result is

$$\begin{aligned} k^\mu = \frac{2-y}{y}\,\frac{1}{\kappa}\,T^\mu - \frac{1}{2}\,Z^\mu + k_\perp^\mu \quad &= \quad \frac{Q}{2}\,\hat q^\mu + \frac{(2-y)}{2y}\,\frac{Q}{\sqrt{\kappa}}\,\hat t^\mu + \frac{\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}}{y}\,\frac{Q}{\sqrt{\kappa}}\,\hat\ell^\mu \\ &\stackrel{Q^2\to\infty}{\longrightarrow} \quad \frac{Q}{2}\,\hat q^\mu + \frac{(2-y)\,Q}{2y}\,\hat t^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat\ell^\mu, \end{aligned} \tag{1.91}$$

where $\hat\ell^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$, is a spacelike unit-vector in the perpendicular direction lying in the (lepton) scattering plane. The kinematics in the frame where virtual photon and target are collinear (including target rest frame) is illustrated in Fig. 1. With the definition of $\hat\ell$, we have for the leptonic tensor[2] neglecting mass corrections ($\kappa = 1$)

$$\begin{aligned} L_{(\ell H)}^{\mu\nu} = \frac{Q^2}{y^2}\Bigg[&-2\left(1-y+\frac{1}{2}\,y^2\right) g_\perp^{\mu\nu} + 4(1-y)\hat t^\mu \hat t^\nu \\ &+4(1-y)\left(\hat\ell^\mu\hat\ell^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\;\hat t^{\{\mu}\hat\ell^{\nu\}} \\ &-i\lambda_e\,y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda_e\,y\sqrt{1-y}\;\hat t^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat\ell_\rho \Bigg]. \end{aligned} \tag{1.92}$$

[2] A useful relation is

$$\epsilon_{\mu\nu\rho\sigma}\,g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma}\,g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma}\,g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma}\,g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha}\,g_{\sigma\beta}$$

or for a vector $a_\perp$ orthogonal to $\hat t$ and $\hat q$,

$$\epsilon^{\mu\nu\rho\sigma}\hat z_\rho a_{\perp\sigma} = \hat t^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho},$$

$$\epsilon^{\mu\nu\rho\sigma}\hat t_\rho a_{\perp\sigma} = -\hat z^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}.$$

For completeness, we also give the full tensor including mass corrections

$$\begin{aligned} L^{\mu\nu}_{(\ell H)} &= \frac{1}{\kappa}\frac{Q^2}{y^2}\left[-2\left(1-y+\frac{1}{4}(1+\kappa)\,y^2\right)g_\perp^{\mu\nu}+4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\hat{t}^\mu\hat{t}^\nu\right. \\ &\quad +4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\left(\hat{\ell}^\mu\hat{\ell}^\nu+\frac{1}{2}g_\perp^{\mu\nu}\right)+2(2-y)\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}} \\ &\quad \left. -i\lambda_e\sqrt{\kappa}\;y(2-y)\,\epsilon_\perp^{\mu\nu}-2i\lambda_e\sqrt{\kappa}\;y\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\right]. \end{aligned}$$

The lepton-momenta also can be written down in the frames discussed above using that up to $\mathcal{O}(1/Q^2)$ corrections

$$y=\frac{q^-}{k^-}\approx\frac{P\cdot q}{P\cdot k}=\frac{Q^2}{x_B\,s}, \tag{1.93}$$

which in the target rest frame ($A=x_BM/Q$) equals $y=\nu/E$.

lepton-hadron:			
momentum	frame I	frame II	relations
k	$\left[\frac{1}{A}\frac{Q}{y\sqrt{2}},A\,\frac{(1-y)Q}{y\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{1}{A}\frac{yQ}{\sqrt{2}},A\,\frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}},\boldsymbol{k}_T\right]$	$\boldsymbol{k}_T=\boldsymbol{k}_\perp+\frac{\boldsymbol{q}_T}{y}$
k'	$\left[\frac{1}{A}\frac{(1-y)Q}{y\sqrt{2}},A\,\frac{Q}{y\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{1}{A}\frac{(1-y)Q}{y\sqrt{2}},A\,\frac{y(\boldsymbol{k}_T^2-2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}},\boldsymbol{k}_T-\boldsymbol{q}_T\right]$	$\boldsymbol{k}_\perp^2=\frac{1-y}{y^2}Q^2$

The lepton phase space becomes $d^3k'/E'=E'dE'd\Omega=\pi ysdx_Bdy$, and one obtains for Eq. 1.27

$$\frac{d\sigma^{(\ell H)}}{dx_B\,dz\,d^2\boldsymbol{q}_T\,dy}=\frac{\pi\,\alpha^2}{2Q^4}\,yz\,L^{(\ell H)}_{\mu\nu}\,2M\mathcal{W}^{\mu\nu(\ell H)}. \tag{1.94}$$

For the inclusive process one finds

$$\frac{d\sigma^{(\ell H)}}{dx_B\,dy}=\frac{\pi\,\alpha^2}{Q^4}\,y\,L^{(\ell H)}_{\mu\nu}\,2MW^{\mu\nu(\ell H)}. \tag{1.95}$$

1.3.2 The annihilation process

For e^-e^+ annihilation the starting point is the timelike direction defined by q. Then it is often convenient to use one of the hadron momenta, say P_2 to construct an orthogonal spacelike vector proportional to $\tilde{P}_2=P_2-(P_2\cdot q/q^2)q$.

$$T^\mu \;\equiv\; q^\mu \stackrel{I}{=} \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\boldsymbol{0}_\perp\right], \tag{1.96}$$

$$Z^\mu \;\equiv\; \frac{q^2}{P_2\cdot q}\frac{\tilde{P}_2^\mu}{\sqrt{\kappa_2}} = \frac{1}{\sqrt{\kappa_2}}\left(\frac{2}{z_2}P_2^\mu-q^\mu\right) \stackrel{I}{=} \left[-\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\boldsymbol{0}_\perp\right], \tag{1.97}$$

For these vectors we have $T^2=Q^2$ and $Z^2=-Q^2$ while

$$\kappa_2=1+\frac{M_2^2\,Q^2}{(P\cdot q)^2}=1+\frac{4\,M_2^2}{z_2^2\,Q^2}. \tag{1.98}$$

The quantity κ_2 takes into account mass corrections $\propto M_2^2/Q^2$ which will vanish for large Q^2 ($\kappa_2\to 1$). Note that $P_2\cdot Z=$ - $\sqrt{\kappa_2}\,P_2\cdot q$. We will mostly consider normalized vectors $\hat{t}^\mu=T^\mu/Q$ and $\hat{z}^\mu=Z^\mu/Q$. In the space orthogonal to $\hat{z}$ and $\hat{t}$ one has the tensors

$$g_\perp^{\mu\nu}\equiv g^{\mu\nu}-\hat{q}^\mu\hat{q}^\nu+\hat{z}^\mu\hat{z}^\nu, \tag{1.99}$$

$$\epsilon_\perp^{\mu\nu}\equiv-\epsilon^{\mu\nu\rho\sigma}\hat{q}_\rho\hat{z}_\sigma \;=\; \frac{1}{(P_2\cdot q)\sqrt{\kappa_2}}\,\epsilon^{\mu\nu\rho\sigma}P_{2\,\rho}q_\sigma, \tag{1.100}$$

Vectors in the orthogonal space are for instance obtained using the other hadronic momentum P_1 (see figure 1.2). The following Cartesian vectors can be defined,

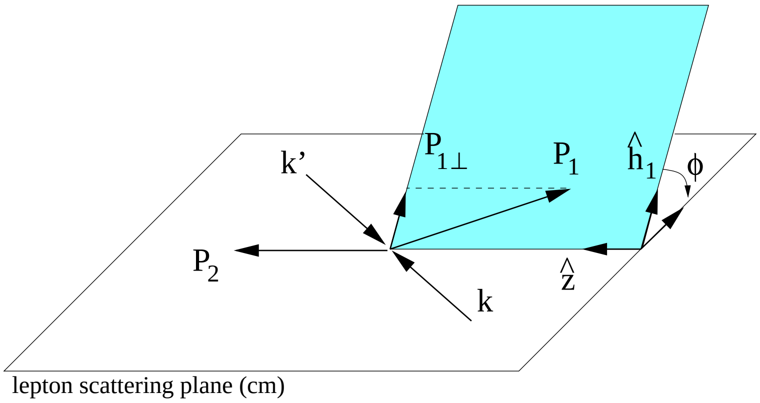


Figure 1.2: Kinematics of the annihilation process in the lepton center of mass frame

$$X^\mu \equiv \frac{-(P_1\cdot Z)\,\tilde{P}_2^\mu + (P_2\cdot Z)\,\tilde{P}_1^\mu}{(P_1\cdot\hat{t})(P_2\cdot\hat{z}) - (P_1\cdot\hat{z})(P_2\cdot\hat{t})}, \tag{1.101}$$

$$Y^\mu \equiv \frac{\epsilon^{\mu\nu\rho\sigma}P_{2\nu}P_{1\rho}q_\sigma}{(P_1\cdot\hat{t})(P_2\cdot\hat{z}) - (P_1\cdot\hat{z})(P_2\cdot\hat{t})}, = -\frac{1}{\sqrt{\kappa_2}}\,\epsilon_\perp^{\mu\nu}X_\nu. \tag{1.102}$$

We define in frame I $X = [0, 0, \boldsymbol{q}_T]$, with length $X^2 = -\boldsymbol{q}_T^2 \equiv -Q_T^2$. The proportionality constant for X is chosen such that $X^\mu = q^\mu + \ldots P_1^\mu + \ldots P_2^\mu \approx q^\mu - P_1^\mu/z_1 - P_2^\mu/z_2$. This choice implies that in frame II, in which P_1 and P_2 do not have transverse momentum, one has the transverse component of q precisely equal to $\boldsymbol{q}_T$. In frame II we, moreover, choose two simple lightlike vectors ($n_+\cdot n_- = 1$), such that we have

$$P_1^\mu \equiv \frac{\zeta_1\tilde{Q}}{\sqrt{2}}\,n_-^\mu + \frac{M_1^2}{\zeta_1\tilde{Q}\sqrt{2}}\,n_+^\mu, \tag{1.103}$$

$$P_2^\mu \equiv \frac{\zeta_2\tilde{Q}}{\sqrt{2}}\,n_+^\mu + \frac{M_2^2}{\zeta_2\tilde{Q}\sqrt{2}}\,n_-^\mu, \tag{1.104}$$

$$q^\mu \equiv \frac{\tilde{Q}}{\sqrt{2}}\,n_-^\mu + \frac{\tilde{Q}}{\sqrt{2}}\,n_+^\mu + q_T^\mu, \tag{1.105}$$

where $\tilde{Q}^2 = Q^2 + Q_T^2$. Note that up to $\mathcal{O}(1/Q^2)$ the variables $\zeta_1 \approx z_1$, $\zeta_2 \approx z_2$ and $\tilde{Q} \approx Q$. Explicitly the hadronic momenta in the frames I and II up to $\mathcal{O}(1/Q^2)$, including the vector q_T are given below.

electron-positron:			
momentum	frame I	frame II	relations
$q = k + k'$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T = -\frac{1}{z_1}\boldsymbol{P}_{1\perp}$
P_2	$\left[\frac{M_2^2}{z_2Q\sqrt{2}}, \frac{z_2Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{M_2^2}{z_2Q\sqrt{2}}, \frac{z_2Q}{\sqrt{2}}, \boldsymbol{0}_T\right]$	$z_2 = \frac{P_2^+}{q^+} \approx \frac{2P_2\cdot q}{Q^2} \approx \frac{P_1\cdot P_2}{P_1\cdot q}$
P_1	$\left[\frac{z_1Q}{\sqrt{2}}, \frac{M_{1\perp}^2}{z_1Q\sqrt{2}}, \boldsymbol{P}_{1\perp}\right]$	$\left[\frac{z_1Q}{\sqrt{2}}, \frac{M_1^2}{z_1Q\sqrt{2}}, \boldsymbol{0}_T\right]$	$z_1 = \frac{P_1^-}{q^-} \approx \frac{2P_1\cdot q}{Q^2} \approx \frac{P_2\cdot P_1}{P_2\cdot q}$
q_T	$\left[0\ , -2\,\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$[0\ , 0\ , \boldsymbol{q}_T]$	$\boldsymbol{q}_T^2 = Q_T^2$

The forms of the vectors T, Z, X, $n_\pm$ and q_T are identical to the case of lepton-hadron scattering. Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu \\ &= g_\perp^{\mu\nu} + \frac{Q_T}{Q}\,\hat{z}^{\{\mu}\hat{x}^{\nu\}} + \frac{Q_T}{Q}\,\hat{q}^{\{\mu}\hat{x}^{\nu\}}, \end{aligned} \tag{1.106}$$

$$g_\perp^{\mu\nu} \quad = \quad g_T^{\mu\nu} - \frac{\sqrt{2}\,n_+^{\{\mu}q_T^{\nu\}}}{Q} \tag{1.107}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0,0,\boldsymbol{a}_T]$ with $|\boldsymbol{a}_T| \sim 1$, i.e. Q^0, one has up to $\mathcal{O}(1/Q^2)$ the relation

$$a_T^\mu \;\approx\; g_\perp^{\mu\nu} a_{T\nu} + \frac{a_T\cdot q_T}{Q^2}\,Z^\mu + \frac{a_T\cdot q_T}{Q^2}\,q^\mu. \tag{1.108}$$

We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu}a_{T\nu}$, but one must be careful not to confuse it with the vector $g_\perp^{\mu\nu}\,a_\nu$.

Also the leptonic momenta can be expanded in the Cartesian directions. Using the scaling variable $y = P_2\cdot k/P_2\cdot q$, we obtain up to $\mathcal{O}(1/Q^2)$ corrections

$$k^\mu = \frac{1}{2}\,q^\mu + \frac{1-2y}{2}\,Z^\mu + k_\perp^\mu \;=\; \frac{Q}{2}\,\hat{q}^\mu + \frac{(1-2y)Q}{2}\,\hat{z}^\mu + Q\sqrt{y(1-y)}\,\hat{\ell}^\mu, \tag{1.109}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$. This leads to the leptonic tensor

$$\begin{aligned} L_{(e^-e^+)}^{\mu\nu} \quad = \quad Q^2\Bigg[& -\left(1-2y+2y^2\right)g_\perp^{\mu\nu} + 4y(1-y)\hat{z}^\mu\hat{z}^\nu \\ & -4y(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) - 2(1-2y)\sqrt{y(1-y)}\;\hat{z}^{\{\mu}\hat{\ell}^{\nu\}} \\ & +i\lambda\,(1-2y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda\,\sqrt{y(1-y)}\;\hat{\ell}_\rho\epsilon_\perp^{\rho\,[\mu}\hat{z}^{\nu]}\Bigg]. \end{aligned} \tag{1.110}$$

The lepton-momenta also can be written down in the frames I and II using

$$y = \frac{k^-}{q^-} \approx \frac{P_2\cdot k}{P_2\cdot q}, \tag{1.111}$$

which in the lepton rest frame equal $y = (1\pm\cos\theta_2)/2$ with θ_2 the angle of hadron (or jet) with respect to the momentum of the incoming leptons.

Electron-positron:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ}{\sqrt{2}},\frac{(1-y)Q}{\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{yQ}{\sqrt{2}},\frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}},\boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + y\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q}{\sqrt{2}},\frac{yQ}{\sqrt{2}},-\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q}{\sqrt{2}},\frac{(\boldsymbol{k}_T^2-2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}},\boldsymbol{q}_T-\boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2 = y(1-y)Q^2$

In the e^-e^+ rest frame $d^3P_1d^3P_2/P_1^0P_2^0 = (dz_1/z_1)(z_2Q^2\,dz_2/4)d\boldsymbol{P}_{1\perp}d\Omega_2 = \pi Q^2\,z_1z_2\,dz_1\,dz_2\,dy\,d^2\boldsymbol{q}_T$, so Eq. 1.43 becomes

$$\frac{d\sigma^{(e^+e^-)}}{dz_1\,dz_2\,d^2\boldsymbol{q}_T\,dy} = \frac{\pi\,\alpha^2}{2Q^4}\,z_1z_2\,L_{\mu\nu}^{(e^+e^-)}\mathcal{W}^{\mu\nu(e^+e^-)}. \tag{1.112}$$

For the production of a single hadron

$$\frac{d\sigma^{(e^+e^-)}}{dz\,dy} = \frac{\pi\,\alpha^2}{Q^4}\,z\,L_{\mu\nu}^{(e^+e^-)}W^{\mu\nu(e^+e^-)}. \tag{1.113}$$

1.3.3 Drell-Yan scattering

For Drell-Yan scattering, for which q^μ is timelike, one can define the following four orthogonal vectors that can be used to expand any vector. Starting with q^μ defining the timelike vector T^μ,

$$T^\mu \quad\equiv\quad q^\mu \;\stackrel{I}{=}\; \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\boldsymbol{0}_\perp\right], \tag{1.114}$$

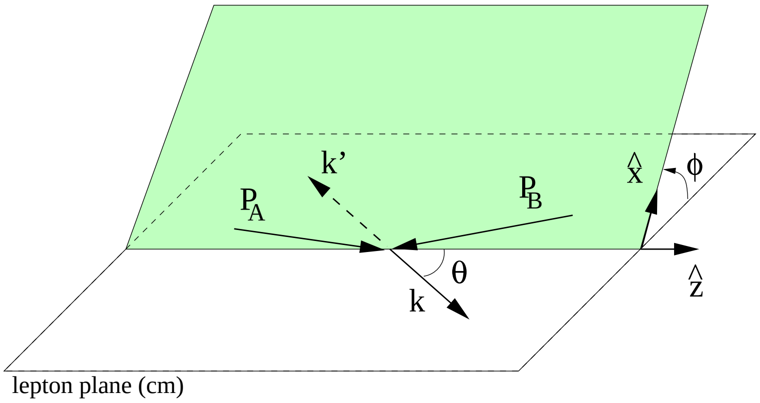


Figure 1.3: Kinematics of the Drell-Yan process in the lepton center of mass frame

$$Z^\mu \equiv \frac{P_B\cdot q}{P_B\cdot P_A}\,\tilde{P}_A^\mu - \frac{P_A\cdot q}{P_A\cdot P_B}\,\tilde{P}_B^\mu = \frac{P_B\cdot q}{P_B\cdot P_A}\,P_A^\mu - \frac{P_A\cdot q}{P_A\cdot P_B}\,P_B^\mu \overset{I}{=} \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\mathbf{0}_\perp\right], \quad (1.115)$$

$$X^\mu \equiv \frac{(P_B\cdot Z)\,\tilde{P}_A^\mu - (P_A\cdot Z)\,\tilde{P}_B^\mu}{(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z}) - (P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})}, \quad (1.116)$$

$$Y^\mu \equiv \frac{\epsilon^{\mu\nu\rho\sigma}P_{A\nu}P_{B\rho}q_\sigma}{(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z}) - (P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})}. \quad (1.117)$$

These vectors satisfy $T^2 = Q^2$ and up to mass corrections $Z^2 \approx -Q^2$. We will use normalized versions $\hat{t}^\mu \equiv T^\mu/Q$ and $\hat{z}^\mu \equiv Z^\mu/\sqrt{-Z^2}$. We note that in this case both hadrons are used to define the spacelike direction, in contrast to e.g. e^+e^- annihilation (compare figs 1.2 and 1.3). In the space transverse to $T = q$ and Z we can use the *perpendicular* tensors

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \hat{t}^\mu\hat{t}^\nu + \hat{z}^\mu\hat{z}^\nu \quad (1.118)$$

$$\epsilon_\perp^{\mu\nu} \equiv -\epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{z}_\sigma = -\frac{1}{Q^2}\epsilon^{\mu\nu\rho\sigma}q_\rho Z_\sigma. \quad (1.119)$$

We define in frame I $X = [0, 0, \boldsymbol{q}_T]$, with length $X^2 = -\boldsymbol{q}_T^2 \equiv -Q_T^2$. The proportionality constant for X is chosen such that $X^\mu = q^\mu + \ldots P_A^\mu + \ldots P_B^\mu \approx q^\mu - x_A P_A^\mu - x_B P_B^\mu$. This choice implies that in frame II, in which P_A and P_B do not have transverse momentum one has the transverse component of q precisely equal to $\boldsymbol{q}_T$. In frame II we, moreover, choose two simple lightlike vectors ($n_+\cdot n_- = 1$), such that

$$P_A^\mu \equiv \frac{\tilde{Q}}{\xi_A\sqrt{2}}\,n_+^\mu + \frac{\xi_A M_A^2}{\tilde{Q}\sqrt{2}},n_-^\mu. \quad (1.120)$$

$$P_B^\mu \equiv \frac{\tilde{Q}}{\xi_B\sqrt{2}}\,n_-^\mu + \frac{\xi_B M_B^2}{\tilde{Q}\sqrt{2}}\,n_+^\mu, \quad (1.121)$$

$$q^\mu \equiv \frac{\tilde{Q}}{\sqrt{2}}\,n_-^\mu + \frac{\tilde{Q}}{\sqrt{2}}\,n_+^\mu + q_T^\mu, \quad (1.122)$$

where $\tilde{Q}^2 = Q^2 + Q_T^2$. Note that up to $\mathcal{O}(1/Q^2)$ the variables $\xi_A \approx x_A$, $\xi_B \approx x_B$ and $\tilde{Q} \approx Q$. Next, we explicitly give the hadronic momenta and q_T in the frames I and II, up to $\mathcal{O}(1/Q^2)$.

Drell-Yan:			
momentum	frame I	frame II	relations
$q = k + k'$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	
P_A	$\left[\frac{x_A M_{A\perp}^2}{Q\sqrt{2}}, \frac{Q}{x_A\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_A}\right]$	$\left[\frac{x_A M_A^2}{Q\sqrt{2}}, \frac{Q}{x_A\sqrt{2}}, \mathbf{0}_T\right]$	$x_A = \frac{q^+}{P_A^+} \approx \frac{Q^2}{2P_A\cdot q} \approx \frac{P_B\cdot q}{P_B\cdot P_A}$
P_B	$\left[\frac{Q}{x_B\sqrt{2}}, \frac{x_B M_{B\perp}^2}{Q\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_B}\right]$	$\left[\frac{Q^2}{x_B Q\sqrt{2}}, \frac{x_B M_B^2}{Q\sqrt{2}}, \mathbf{0}_T\right]$	$x_B = \frac{q^-}{P_B^-} \approx \frac{Q^2}{2P_B\cdot q} \approx \frac{P_A\cdot q}{P_A\cdot P_B}$
q_T	$\left[-\frac{Q_T^2}{Q\sqrt{2}}, -\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$[0\ ,0\ ,\boldsymbol{q}_T]$	$\boldsymbol{q}_T^2 = Q_T^2$

The precise forms of the vectors in the Cartesian set and the vectors $n_\pm$ and q_T are explicitly given below (omitting 'boost' factors $1/A$ and A multiplying $-$ and $+$ components respectively).

DY		
vector	frame I	frame II
T	$\left[\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ \mathbf{0}_\perp\right]$	$\left[\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{\tilde{Q}}{\sqrt{2}}\ ,\ \boldsymbol{q}_T\right]$
Z	$\left[-\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ \mathbf{0}_\perp\right]$	$\left[-\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ -\mathbf{0}_T\right]$
X	$[\,0\ ,\ 0\ ,\ \boldsymbol{q}_T]$	$\left[\frac{Q_T^2}{Q\sqrt{2}},\ \frac{Q_T^2}{Q\sqrt{2}},\ \frac{\tilde{Q}}{Q}\,\boldsymbol{q}_T\right]$
n_-	$\left[\frac{\tilde{Q}+Q}{2\,Q}, \frac{\tilde{Q}-Q}{2\,Q}, -\frac{\boldsymbol{q}_T}{Q\sqrt{2}}\right]$	$[1\ ,\ 0\ ,\ \mathbf{0}_T]$
n_+	$\left[\frac{\tilde{Q}-Q}{2\,Q}, \frac{\tilde{Q}+Q}{2\,Q}, -\frac{\boldsymbol{q}_T}{Q\sqrt{2}}\right]$	$[0\ ,\ 1\ ,\ \mathbf{0}_T]$
q_T	$\left[-\frac{Q_T^2}{Q\sqrt{2}}, -\frac{Q_T^2}{Q\sqrt{2}}\ , \frac{\tilde{Q}}{Q}\,\boldsymbol{q}_T\right]$	$[\,0\ ,\ 0\ ,\ \boldsymbol{q}_T]$

Later on we will need to transform from the theoretically useful vectors $n_\pm$ and q_T to the quantities appearing in the expansion of the hadronic tensor, $\hat{t}$, $\hat{z}$ and $\hat{x}$,

$$n_-^\mu = \frac{1}{Q\sqrt{2}}\left(\frac{\tilde{Q}}{Q}\,T^\mu - Z^\mu - X^\mu\right) \;\approx\; \frac{T^\mu - Z^\mu - X^\mu}{Q\sqrt{2}} \tag{1.123}$$

$$n_+^\mu = \frac{1}{Q\sqrt{2}}\left(\frac{\tilde{Q}}{Q}\,T^\mu - Z^\mu - X^\mu\right) \;\approx\; \frac{T^\mu + Z^\mu - X^\mu}{Q\sqrt{2}} \tag{1.124}$$

$$q_T^\mu = \frac{\tilde{Q}}{Q}\,X^\mu - \frac{Q_T^2}{Q^2}\,T^\mu \;\approx\; X^\mu - \frac{Q_T^2}{Q^2}\,T^\mu \tag{1.125}$$

The last relation is important to keep track of transverse momentum effects at order $1/Q$. Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu \\ &= g_\perp^{\mu\nu} + \frac{Q_T}{Q}\,\hat{q}^{\{\mu}\hat{x}^{\nu\}}, \end{aligned} \tag{1.126}$$

$$g_\perp^{\mu\nu} = g_T^{\mu\nu} - \frac{n_+^{\{\mu} q_T^{\nu\}}}{Q\sqrt{2}} - \frac{n_-^{\{\mu} q_T^{\nu\}}}{Q\sqrt{2}}. \tag{1.127}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0, 0, \boldsymbol{a}_T]$ with $|\boldsymbol{a}_T| \sim 1$, i.e. $\sim\ Q^0$, one has up to $\mathcal{O}(1/Q^2)$ the relation

$$a_T^\mu \;\approx\; g_\perp^{\mu\nu} a_{T\nu} + \frac{a_T\cdot q_T}{Q^2}\,q^\mu. \tag{1.128}$$

We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu} a_{T\nu}$, but one must be careful not to confuse it with the vector $g_\perp^{\mu\nu}\, a_\nu$.

Also the lepton momenta can be expressed in the cartesian vectors. For DY we have

$$k^\mu = \frac{1}{2}\, T^\mu + \frac{1-2y}{2}\, Z^\mu + k_\perp^\mu \;=\; \frac{Q}{2}\,\hat{q}^\mu + \frac{(1-2y)Q}{2}\,\hat{z}^\mu + Q\sqrt{y(1-y)}\,\hat{\ell}^\mu, \tag{1.129}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$. This leads to the leptonic tensor

$$\begin{aligned} L_{(DY)}^{\mu\nu} \;=\; Q^2 \Bigg[& -\left(1-2y+2y^2\right) g_\perp^{\mu\nu} + 4y(1-y)\hat{z}^\mu\hat{z}^\nu \\ & -4y(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\, g_\perp^{\mu\nu}\right) - 2(1-2y)\sqrt{y(1-y)}\;\hat{z}^{\{\mu}\hat{\ell}^{\nu\}} \\ & -i\lambda\,(1-2y)\,\epsilon_\perp^{\mu\nu} + 2i\lambda\,\sqrt{y(1-y)}\;\hat{\ell}_\rho \epsilon_\perp^{\rho\,[\mu}\hat{z}^{\nu]} \Bigg]. \end{aligned} \tag{1.130}$$

The lepton momenta can also be written down in frames I and II using

$$y = \frac{k^-}{q^-} \approx \frac{k'^+}{q^+}, \tag{1.131}$$

which in the lepton rest frame equal $y = (1 \pm \cos\theta)/2$ with θ the angle of the leptons with respect to an axis that is approximately parallel to the momentum of hadrons A and B (the z-direction).

Drell-Yan:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ}{\sqrt{2}}, \frac{(1-y)Q}{\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{yQ}{\sqrt{2}}, \frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}}, \boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + \frac{1}{2}\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q}{\sqrt{2}}, \frac{yQ}{\sqrt{2}}, -\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q}{\sqrt{2}}, \frac{(\boldsymbol{k}_T^2 - 2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}}, \boldsymbol{q}_T - \boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2 = y(1-y)Q^2$

Eq. 1.63 becomes

$$\frac{d\sigma^{(DY)}}{dx_A\, dx_B\, d^2\boldsymbol{q}_T\, dy} = \frac{\pi\,\alpha^2}{Q^4}\, L_{\mu\nu}^{(DY)} \mathcal{W}^{\mu\nu(DY)}. \tag{1.132}$$

1.4 Spin vectors, ...

In the next section it will turn out that the most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -S_{hL}\,\frac{Mx_B}{Q\sqrt{2}}\, n_- + S_{hL}\,\frac{Q}{Mx_B\sqrt{2}}\, n_+ + S_T. \tag{1.133}$$

One has up to $\mathcal{O}(1/Q^2)$ corrections $S_L \approx M\,(S\cdot q)/(P\cdot q)$ and $S_T \approx S_\perp$. For a pure state one has $S_L^2 + \boldsymbol{S}_T^2 = 1$, in general this quantity being less or equal than one.

The final state spin vector S_h in the case of detection of a spin 1/2 hadron (e.g. a Λ-baryon) will be expanded in the same way. This vector can e.g. be determined from the decay products (e.g. the $N\pi$ system in case of a Λ). It satisfies $P_h \cdot S_h = 0$ and is written

$$S_h^\mu = S_{hL}\,\frac{z_h\, Q}{M_h\sqrt{2}}\, n_- - S_{hL}\,\frac{M_h}{z_h\, Q\sqrt{2}}\, n_+ + S_{hT}. \tag{1.134}$$

Up to $\mathcal{O}(1/Q^2)$ corrections one has $S_{hL} \approx M_h\,(S_h\cdot q)/(P_h\cdot q)$, but note that one has $S_{hT} \approx S_{h\perp} - S_{hL}\,P_{h\perp}/M_h$. In general one has $S_{hL}^2 + \boldsymbol{S}_{hT}^2 \leq 1$.

Chapter 2

Quark correlation functions

2.1 Distributions: from hadron to quarks

We consider now the most general form of the two-quark correlation function

$$\Phi_{ij}(k,P,S) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{i\,k\cdot\xi} \langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle, \tag{2.1}$$

where a summation over color indices is implicit, diagrammatically represented in Fig. 2.1. In order to render the definition color gauge-invariant each quark field needs to be accompanied by a path ordered exponential (link operator) of the form

$$\mathcal{U}(a,\xi) = \mathcal{P}\exp\left(-ig\int_a^\xi dz^\mu\, A_\mu(z)\right). \tag{2.2}$$

For the relevant correlation functions Φ in a hard scattering process, we will encounter only those cases in which the link involves gluons of the type $A\cdot n$, where n is a lightlike vector (see Fig. 2.2).

Constraints on the correlation function Φ come from hermiticity, parity and time reversal invariance. We know how the states behave under such transformations and we know how the fields transform. This gives consistency conditions. One finds

$$\Phi^\dagger(k,P,S) = \gamma_0\,\Phi(k,P,S)\,\gamma_0 \quad \text{[Hermiticity]} \tag{2.3}$$

$$\Phi(k,P,S) = \gamma_0\,\Phi(\bar{k},\bar{P},-\bar{S})\,\gamma_0 \quad \text{[Parity]} \tag{2.4}$$

$$\Phi^*(k,P,S) = (-i\gamma_5 C)\,\Phi(\bar{k},\bar{P},\bar{S})\,(-i\gamma_5 C) \quad \text{[Time reversal]} \tag{2.5}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{k} = (k^0, -\boldsymbol{k})$.

We will give the explicit proof of these properties. Starting with hermiticity,

$$\begin{aligned}
(\Phi^\dagger)_{ij} = \Phi^*_{ji} \quad &= \quad \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi^\dagger_k(0)(\gamma_0)_{ki}\psi_j(\xi)|P,S\rangle^* \\
&= \quad \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi^\dagger_j(\xi)(\gamma_0)_{ik}\psi_k(0)|P,S\rangle \\
&= \quad \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\xi)|P,S\rangle \\
&= \quad (\gamma_0)_{ik}\,\Phi_{kl}\,(\gamma_0)_{lj},
\end{aligned}$$

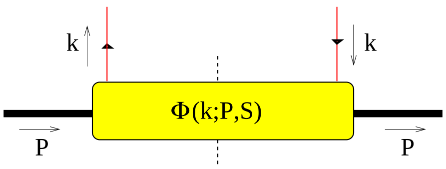


Figure 2.1: The diagrammatic representation of the quark-quark correlation function $\Phi(k,P,S)$.

Figure 2.2: The link as can be obtained from A^+ gluon blobs (see sections 3 and 7) in the case of $\xi^+ = \xi_T = 0$ (left) and the case $\xi^+ = 0, \xi_T \neq 0$ (right).

where from second to third line translation invariance has been used. Next considering parity

$$\begin{aligned}
\Phi_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\mathcal{P}^\dagger\mathcal{P}\overline{\psi}_j(0)\mathcal{P}^\dagger\mathcal{P}\psi_i(\xi)\mathcal{P}^\dagger\mathcal{P}|P,S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle \bar{P},-\bar{S}|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\bar{\xi})|\bar{P},-\bar{S}\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,\bar{k}\cdot\xi}\langle \bar{P},-\bar{S}|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\xi)|\bar{P},-\bar{S}\rangle \\
&= (\gamma_0)_{ik}\,\Phi_{kl}(\bar{p},\bar{P},-\bar{S})\,(\gamma_0)_{lj},
\end{aligned}$$

where $\mathcal{P}\,\psi(\xi)\mathcal{P}^\dagger = \gamma_0\psi(\bar{\xi})$ and from second to third line $k\cdot\xi = \bar{k}\cdot\bar{\xi}$ and $d^4\xi = d^4\bar{\xi}$ has been used. Finally time reversal invariance (with $\mathcal{T}$ anti-unitary),

$$\begin{aligned}
\Phi^*_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\mathcal{T}^\dagger\mathcal{T}\overline{\psi}_j(0)\mathcal{T}^\dagger\mathcal{T}\psi_i(\xi)\mathcal{T}^\dagger\mathcal{T}|P,S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle \bar{P},\bar{S}|\overline{(-i\gamma_5 C\psi)}_j(0)(-i\gamma_5 C\psi)_i(-\bar{\xi})|\bar{P},\bar{S}\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,\bar{k}\cdot\xi}\langle \bar{P},\bar{S}|\overline{(-i\gamma_5 C\psi)}_j(0)(-i\gamma_5 C\psi)_i(\xi)|\bar{P},\bar{S}\rangle \\
&= (-i\gamma_5 C)_{ik}\,\Phi_{kl}(\bar{k},\bar{P},\bar{S})\,(-i\gamma_5 C)_{lj},
\end{aligned}$$

where $\mathcal{T}\,\psi(\xi)\mathcal{T}^\dagger = -i\gamma_5 C\psi(-\bar{\xi})$.

Including the link-operator these properties will be different. For the gauge link one has

$$\mathcal{U}^\dagger(a,\xi) = \mathcal{U}(\xi,a), \tag{2.6}$$
$$\mathcal{P}\,\mathcal{U}(a,\xi)\,\mathcal{P}^\dagger = \mathcal{U}(\bar{a},\bar{\xi}), \tag{2.7}$$
$$\mathcal{T}\,\mathcal{U}(a,\xi)\,\mathcal{T}^\dagger = \mathcal{U}(-\bar{a},-\bar{\xi}), \tag{2.8}$$

for which we used $A^\dagger_\mu = A_\mu$, $\mathcal{P}\,A_\mu(\xi)\,\mathcal{P}^\dagger = \bar{A}_\mu(\bar{\xi})$ and $\mathcal{T}\,A_\mu(\xi)\,\mathcal{T}^\dagger = \bar{A}_\mu(-\bar{\xi})$. This means that the space-reversed (time-reversed) correlation function has a different link structure running from $\bar{a}$ $(-\bar{a})$ respectively. However, if the common point is defined with respect to the two fields in the matrix element, no problem arises. For example the straight line link with path $z^\mu(s) = (1-s)\,0^\mu + s\,\xi^\mu$ gives a path $\bar{z}^\mu$ after applying parity, but after the change of variables one ends up with the same path; similarly for time-reversal.

The most general structure implementing the constraints from hermiticity and parity is

$$\boxed{\begin{aligned}
\Phi(k,P,S) &= M\,A_1 + A_2\,\not{P} + A_3\not{k} + i\,A_4\,\frac{[\not{P},\not{k}]}{2M} + i\,A_5\,(k\cdot S)\gamma_5 + M\,A_6\,\not{S}\gamma_5 \\
&\quad + A_7\,\frac{(k\cdot S)}{M}\,\not{P}\gamma_5 + A_8\,\frac{(k\cdot S)}{M}\,\not{k}\gamma_5 + A_9\,\frac{[\not{P},\not{S}]}{2}\,\gamma_5 + A_{10}\,\frac{[\not{k},\not{S}]}{2}\,\gamma_5 \\
&\quad + A_{11}\,\frac{(k\cdot S)}{M}\,\frac{[\not{P},\not{k}]}{2M}\,\gamma_5 + A_{12}\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k^\rho S^\sigma}{M},
\end{aligned}} \tag{2.9}$$

where the first four terms do not involve the hadron polarization vector. Hermiticity requires all the amplitudes $A_i = A_i(k\cdot P, k^2)$ to be real. The amplitudes A_4, A_5 and A_{12} vanish when also time reversal invariance applies.

One might wonder if the lightlike direction n should not appear in the expansion of the matrix element Φ. We note, however that the matrix element with link, $\Phi^{(n)}$ in the gauge $n \cdot A = 0$ becomes equal to the matrix element $\Phi^{(0)}$ without link. In the expansion of the latter n obviously does not appear. For the fully integrated matrix element, which involves a $d^4\xi$ integration, however, one can consider a different gauge $n' \cdot A = 0$ and perform a change of integration variables such that $\xi \cdot n' = \xi' \cdot n$. One then finds that $\Phi^{(n')} = \Phi^{(n)}$, i.e. the link direction will never appear. Essential in this is the fact that n is not fixed by the momenta k, P or S.

For the applications, it is useful to introduce besides the lightlike vector $n \equiv n_-$ a lightlike vector n_+, such that one has $n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$. The vector n_+ is fixed by the hadronic momentum such that

$$P = \frac{M^2}{2P^+}\, n_- + P^+\, n_+, \tag{2.10}$$

$$S = -\lambda\, \frac{M}{2P^+}\, n_- + \lambda\, \frac{P^+}{M}\, n_+ + S_T, \tag{2.11}$$

$$k = k^-\, n_- + x\, P^+ n_+ + k_T. \tag{2.12}$$

The parametrization satisfies $P^2 = M^2$ and $P \cdot S = 0$. One immediately deduces $k^- = (k^2 + \boldsymbol{k}_T^2)/2xP^+$, while $2x\, k \cdot P = k^2 + \boldsymbol{k}_T^2 + x^2 M^2$. Depending on the use of the soft parts one may need integrations over one or more components of k. At that point the lightlike vector n_- will become relevant.

The fully integrated result leads to a local matrix element, omitting the dependence on hadron momentum and spin vectors $(P,\, S)$,

$$\Phi_{ij} = \int d^4k\, \Phi_{ij}(k, P, S) = \langle P, S | \overline{\psi}_j(0)\, \psi_i(0) | P, S\rangle, \tag{2.13}$$

It is parametrized as

$$\Phi = \frac{1}{2}\left\{ M\, g_S + g_V\, \not{P} + M\, g_A\, \gamma_5\, \not{S} + g_T\, \frac{[\not{S}, \not{P}]\, \gamma_5}{2} \right\}. \tag{2.14}$$

Because the matrix element is local the gauge link will vanish and there will be no dependence on the lightlike vectors. Projecting using a basis of 4×4 Dirac matrices $(\Gamma)$and defining

$$\Phi^{[\Gamma]} \equiv \frac{1}{2}\, \mathrm{Tr}\,(\Phi\Gamma)\,, \tag{2.15}$$

one finds

$$2\,\Phi^{[\gamma^\mu]} = \langle P, S | \overline{\psi}(0)\, \gamma^\mu\, \psi(0) | P, S\rangle = g_V\, 2P^\mu, \tag{2.16}$$

$$2\,\Phi^{[\gamma^\mu\gamma_5]} = \langle P, S | \overline{\psi}(0)\, \gamma^\mu\gamma_5\, \psi(0) | P, S\rangle = g_A\, 2M\, S^\mu, \tag{2.17}$$

$$2\,\Phi^{[i\sigma^{\mu\nu}\gamma_5]} = \langle P, S | \overline{\psi}(0)\, i\sigma^{\mu\nu}\gamma_5\, \psi(0) | P, S\rangle = g_T\, 2\, S^{[\mu}, P^{\nu]} \tag{2.18}$$

$$2\,\Phi^{[1]} = \langle P, S | \overline{\psi}(0)\, \psi(0) | P, S\rangle = g_S\, 2M, \tag{2.19}$$

$$2\,\Phi^{[i\gamma_5]} = \langle P, S | \overline{\psi}(0)\, i\gamma_5\, \psi(0) | P, S\rangle = 0. \tag{2.20}$$

Note that $g_V = n$ (number of quarks minus antiquarks), g_A is the axial charge for quarks and antiquarks of a particular flavor, g_T is the tensor charge. Multiplying g_S with the quark mass one finds precisely the contribution of the quark mass term to the nucleon mass. The first two matrix elements are special because the operators correspond to conserved currents (for the axial current up to mass terms). The anomalous dimensions of these operators vanish.

In inclusive deep inelastic scattering one needs to consider the correlation functions

$$\begin{aligned} \Phi_{ij}(x) \quad &= \quad \int d^2\boldsymbol{k}_T\, dk^-\; \Phi_{ij}(k, P, S)\Big|_{k^+ = xP^+} \\ &= \quad \int \frac{d\xi^-}{2\pi}\; e^{ik\cdot\xi}\, \langle P, S | \overline{\psi}_j(0)\, \mathcal{U}(0,\xi)\, \psi_i(\xi) | P, S\rangle\Big|_{\xi^+ = \xi_T = 0}\,, \end{aligned} \tag{2.21}$$

This correlation function is in general the result of a large number of diagrams, as it not only includes matrix elements that contain only quark fields, $\overline{\psi}\psi$, but also matrix elements of the type $\overline{\psi}A^+ \ldots A^+\psi$. All of these will contribute in leading order in $1/Q$. It will turn out that in a full calculation gluonic matrix elements containing A^+ fields precisely form the for color gauge invariance necessary link, which runs along the minus direction (see Fig. 2.2). Working in the gauge $A^+ = 0$ none of the gluonic matrix elements will appear and the link in Eq. 2.21 becomes unity.

The quantity $\Phi(x)$ can be parametrized as

$$
\boxed{
\begin{aligned}
\Phi(x) \quad = \quad & \frac{1}{2}\left\{ f_1(x)\, \not{n}_+ + \lambda\, g_1(x)\, \gamma_5\, \not{n}_+ + h_1(x)\, \frac{\gamma_5\,[\not{S}_T, \not{n}_+]}{2} \right\} \\
& + \frac{M}{2P^+}\left\{ e(x) + g_T(x)\, \gamma_5\, \not{S}_T + \lambda\, h_L(x)\, \frac{\gamma_5\,[\not{n}_+, \not{n}_-]}{2} \right\} \\
& + \frac{M}{2P^+}\left\{ -\lambda\, e_L(x)\, i\gamma_5 - f_T(x)\, \epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} + h(x)\, \frac{i\,[\not{n}_+, \not{n}_-]}{2} \right\} \\
& + \frac{M^2}{2(P^+)^2}\left\{ f_3(x)\, \not{n}_- + \lambda\, g_3(x)\, \gamma_5 \not{n}_- + h_3(x)\, \frac{\gamma_5\,[\not{S}_T, \not{n}_-]}{2} \right\}, \qquad (2.22)
\end{aligned}
}
$$

The factors of (M/P^+) are the ones required from Lorentz invariance. As we will see, each factor $1/P^+$ gives rise to a suppression factor of $1/Q$ in cross sections. From the structure of the above matrix element, being of the form $\Lambda_H^{t-2}/(P^+)^{t-2}$ one defines the quantity t, the *(operational) twist* of the profile or distribution functions appearing in the parametrization.

To be slightly more specific, using the amplitude expansion for the quark-quark correlation function one can easily analyze the effect of the integration over k^- and $\boldsymbol{k}_T$ for the different Dirac projections of the quark correlation functions (twist analysis). For instance

$$
\begin{aligned}
\Phi^{[1]}(x) \quad &= \quad \frac{1}{2}\int dk^-\, d^2\boldsymbol{k}_T\; Tr(1\, \Phi(k,P,S))\Big|_{k^+ = xP^+} \\
&= \quad \frac{M}{P^+}\int d(2k\cdot P)\, dk^2\; \pi\, A_1(k^2, k\cdot P)\, \theta(2x\, k\cdot P - x^2 M^2 - k^2) \\
&\equiv \quad \frac{M}{P^+}\, e(x), \qquad (2.23)
\end{aligned}
$$

where $x = k^+/P^+$ and the integration over $\boldsymbol{k}_T$ is rewritten as an integration over k^2 using $\boldsymbol{k}_T^2 = 2x\, k\cdot P - x^2\, M^2 - k^2$. In rewriting the matrix element in this way one has separated it in a function $e(x)$ which we refer to as a distribution or *profile function*. This function, containing only hadron and quark momenta and scalar products of them which are of hadronic size (Λ_H), is of $\mathcal{O}(1)$. It is multiplied with some factor that contains some powers of P^+ and momenta of hadronic size of which the consequence has already been mentioned. The functions e_L, f_T and h are expected to vanish because of time-reversal symmetry. They involve the amplitudes A_4, A_5 and A_{12}. We have kept them here, because they will have potential relevance later and furthermore are useful for comparison with fragmentation functions. The functions are referred to as *T-odd*.

Projecting with the various Dirac matrices one finds the 'leading' (twist two) distribution functions

$$\Phi^{[\gamma^+]}(x) = f_1(x), \qquad (2.24)$$

$$\Phi^{[\gamma^+\gamma_5]}(x) = \lambda\, g_1(x), \qquad (2.25)$$

$$\Phi^{[i\sigma^{i+}\gamma_5]}(x) = S_T^i\, h_1(x), \qquad (2.26)$$

the twist three distribution functions

$$\Phi^{[1]}(x) = \frac{M}{P^+}\, e(x), \qquad (2.27)$$

$$\Phi^{[i\gamma_5]} = \frac{M}{P^+}\, e_L(x), \qquad (2.28)$$

$$\Phi^{[\gamma^i]}(x) = -\frac{M\, \epsilon_T^{i\rho} S_{T\rho}}{P^+}\, f_T(x), \qquad (2.29)$$

$$\Phi^{[\gamma^i\gamma_5]}(x) = \frac{M\, S_T^i}{P^+}\, g_T(x), \qquad (2.30)$$

$$\Phi^{[i\sigma^{+-}\gamma_5]}(x) = \frac{M}{P^+}\,\lambda\,h_L(x), \tag{2.31}$$

$$\Phi^{[i\sigma^{ij}\gamma_5]}(x) = \frac{M}{P^+}\,\epsilon_T^{ij}\,\lambda\,h(x), \tag{2.32}$$

and the twist four functions

$$\Phi^{[\gamma^-]}(x) = f_3(x), \tag{2.33}$$

$$\Phi^{[\gamma^-\gamma_5]}(x) = \lambda\,g_3(x), \tag{2.34}$$

$$\Phi^{[i\sigma^{i-}\gamma_5]}(x) = S_T^i\,h_3(x). \tag{2.35}$$

By rewriting Φ in Eq. 2.14 as

$$\begin{aligned}\Phi \;=\; & \frac{P^+}{2}\left\{g_V\,\not n_+ + \lambda\,g_A\,\gamma_5\,\not n_+ + g_T\,\frac{[\not S_T,\not n_+]\,\gamma_5}{2}\right\}\\ & + \frac{M}{2}\left\{g_S + g_A\,\gamma_5\,\not S_T + \lambda\,g_T\,\frac{[\not n_+,\not n_-]\,\gamma_5}{2}\right\}\\ & + \frac{M^2}{2P^+}\left\{\frac{1}{2}\,g_V\,\not n_- + \frac{1}{2}\,\lambda\,g_A\,\gamma_5\,\not n_- + \frac{1}{2}\,g_T\,\frac{[\not S_T,\not n_-]\,\gamma_5}{2}\right\}.\end{aligned} \tag{2.36}$$

it follows from $\Phi = \int dk^+\ \Phi(x) = P^+\int dx\ \Phi(x)$ (assuming convergence and integrating over all x-values, which will be discussed later) that

$$\int dx\ f_1(x) = 2\int dx\ f_3(x) = g_V, \tag{2.37}$$

$$\int dx\ g_1(x) = \int dx\ g_T(x) = -2\int dx\ g_3(x) = g_A, \tag{2.38}$$

$$\int dx\ h_1(x) = \int dx\ h_L(x) = 2\int dx\ h_3(x) = g_T, \tag{2.39}$$

$$\int dx\ e(x) = g_S, \tag{2.40}$$

$$\int dx\ e_L(x) = \int dx\ f_T(x) = \int dx\ h(x) = 0. \tag{2.41}$$

Making use of support properties of the distribution functions ($-1 \le x \le 1$) and symmetry relations between quark and antiquark distributions (the latter to be discussed in the next section), $f_1(x) = -\bar f_1(-x)$, and finally the fact that the vector charge g_V is in fact the definition of the flavor quantum number, i.e. $g_V \equiv n_q$, the first line turns into the number sum rule

$$\int_0^1 dx\ \big(f_1(x) - \bar f_1(x)\big) = n. \tag{2.42}$$

In semi-inclusive deep inelastic scattering or Drell-Yan processes, the matrix elements that are needed for the hadron $\rightarrow$ quark pieces in the hard scattering processes are the ones in which the integration over $\boldsymbol{k}_T$ is not yet performed, namely

$$\begin{aligned}\Phi_{ij}(x,\boldsymbol{k}_T) \;&=\; \int dk^-\ \Phi_{ij}(k,P,S)\Big|_{k^+=xP^+,\ k_T}\\ &=\; \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\ e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}.\end{aligned} \tag{2.43}$$

In this case one is sensitive to transverse separation ξ_T. We consider, however, only gauge links attached to each quark along the minus direction, i.e. built from matrix elements with additional A^+ fields, which contribute at the same order in an $1/Q$ expansion (see Fig. 2.2). That this is the result in an actual diagrammatic expansion needs to be proven. It will turn out that the bilocal quark-quark matrix element will be supplemented with links running from 0 to $[0,\infty,\boldsymbol{0}_T]$ and one running from $[0,\infty,\boldsymbol{\xi}_T]$ to $[0,\xi^-,\boldsymbol{\xi}_T]$ respectively, With the physical condition that any matrix element involving $\overline{\psi}(0)\,A_T(\eta^- = \pm\infty,\eta^+,\eta_T)\,\psi(\xi)$ vanishes, the links can be connected and one has a color gauge invariant quantity, for which after gauge fixing the link becomes unity. At this point it is useful to mention that quark-quark-gluon matrix elements with gluon fields other than A^+ need to be considered separately (see chapter 4).

We write down the expression for $\Phi(x,\boldsymbol{k}_T)$ in terms of n_+, n_- and transverse vectors up to $\mathcal{O}(M/P^+)$, including T-odd parts, but restricting ourselves to twist-two ($\propto 1$) and twist-three ($\propto M/P^+$) parts. Simple kinematic arguments already show that factorization of $\boldsymbol{k}_T$-dependent functions cannot hold beyond twist-three.

$$
\boxed{\begin{aligned}
\Phi(x,\boldsymbol{k}_T) \;=\;& \frac{1}{2}\Bigg\{ f_1(x,\boldsymbol{k}_T)\,\not n_+ + f_{1T}^\perp(x,\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu k_T^\rho S_T^\sigma}{M} + g_{1s}(x,\boldsymbol{k}_T)\,\gamma_5\,\not n_+ \\
&\quad + h_{1T}(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not S_T,\not n_+]}{2} + h_{1s}^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not k_T,\not n_+]}{2M} + h_1^\perp(x,\boldsymbol{k}_T)\,\frac{i\,[\not k_T,\not n_+]}{2M}\Bigg\} \\
&+\frac{M}{2P^+}\Bigg\{ e(x,\boldsymbol{k}_T) + f^\perp(x,\boldsymbol{k}_T)\,\frac{\not k_T}{M} - f_T(x,\boldsymbol{k}_T)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} \\
&\quad -\lambda\, f_L^\perp(x,\boldsymbol{k}_T)\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho k_{T\sigma}}{M} - e_s(x,\boldsymbol{k}_T)\,i\gamma_5 \\
&\quad + g_T'(x,\boldsymbol{k}_T)\,\gamma_5\,\not S_T + g_s^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,\not k_T}{M} + h_T^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not S_T,\not k_T]}{2M} \\
&\quad + h_s(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not n_+,\not n_-]}{2} + h(x,\boldsymbol{k}_T)\,\frac{i\,[\not n_+,\not n_-]}{2}\Bigg\}. \qquad (2.44)
\end{aligned}}
$$

We have here use the shorthand notation

$$
g_{1s}(x,\boldsymbol{k}_T) \equiv \lambda\, g_{1L}(x,\boldsymbol{k}_T) + g_{1T}(x,\boldsymbol{k}_T)\,\frac{(\boldsymbol{k}_T\cdot\boldsymbol{S}_T)}{M}, \qquad (2.45)
$$

and similarly for other functions, e.g. $h_{1s}^\perp$, $g_s^\perp$ and h_s. Included are also T-odd functions $f_{1T}^\perp$, $h_1^\perp$, f_T, $f_L^\perp$, e_s and h.

Again we can analyze the Dirac content of the correlation function (twist analysis). For instance for the unit matrix the effect of the integration over k^- is

$$
\begin{aligned}
\Phi^{[1]}(x,\boldsymbol{k}_T) &= \frac{1}{2}\int dk^-\; Tr(1\,\Phi(k,P,S))\Big|_{k^+=xP^+,\;k_T} \\
&= \frac{M}{P^+}\int d(2k\cdot P)\,dk^2\;A_1(k^2,k\cdot P)\,\delta(\boldsymbol{k}_T^2+k^2-2x\,k\cdot P+x^2M^2) \\
&\equiv \frac{M}{P^+}\,e(x,\boldsymbol{k}_T), \qquad (2.46)
\end{aligned}
$$

where $x = k^+/P^+$ and $\boldsymbol{k}_T$ is the transverse component of the quark momentum k in the frame where P has no transverse components, i.e. frame II discussed in the previous section. The profile function e only depends on x and $\boldsymbol{k}_T^2$. It is expressed as an integral in which all momenta and products thereof are of hadronic size (Λ_H), and is multiplied with a factor of the form $\Lambda_H^{t-4}/(P^+)^{t-2}$, defining the operational twist. It is this factor that will lead in the cross section to a suppression factor $1/Q^{t-2}$.

The various (T-even) Dirac projections $\Phi^{[\Gamma]} = (1/2)\mathrm{Tr}(\Phi\Gamma)$ appearing in here are explicitly

$$
\Phi^{[\gamma^+]}(x,\boldsymbol{k}_T) = f_1(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij}k_{Ti}S_{Tj}}{M}\,f_{1T}^\perp(x,\boldsymbol{k}_T), \qquad (2.47)
$$

$$
\Phi^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) = \lambda\,g_{1L}(x,\boldsymbol{k}_T) + g_{1T}(x,\boldsymbol{k}_T)\,\frac{(\boldsymbol{k}_T\cdot\boldsymbol{S}_T)}{M}, \qquad (2.48)
$$

$$
\begin{aligned}
\Phi^{[i\sigma^{i+}\gamma_5]}(x,\boldsymbol{k}_T) &= S_T^i\,h_{1T}(x,\boldsymbol{k}_T) + \frac{k_T^i}{M}\,h_{1s}^\perp(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij}k_{Tj}}{M}\,h_1^\perp(x,\boldsymbol{k}_T), \\
&= S_T^i\,h_1(x,\boldsymbol{k}_T) + \frac{\lambda k_T^i}{M}\,h_{1L}^\perp(x,\boldsymbol{k}_T) \\
&\quad - \frac{\left(k_T^ik_T^j+\frac{1}{2}\boldsymbol{k}_T^2g_T^{ij}\right)S_{Tj}}{M^2}\,h_{1T}^\perp(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij}k_{Tj}}{M}\,h_1^\perp(x,\boldsymbol{k}_T), \qquad (2.49)
\end{aligned}
$$

and the profile functions that appear multiplied by a factor M/P^+ (twist three) are

$$\Phi^{[1]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\, e(x,\boldsymbol{k}_T) \tag{2.50}$$

$$\Phi^{[\gamma^i]}(x,\boldsymbol{k}_T) = \frac{k_T^i}{P^+}\, f^\perp(x,\boldsymbol{k}_T), \tag{2.51}$$

$$\begin{aligned}\Phi^{[\gamma^i\gamma_5]}(x,\boldsymbol{k}_T) &= \frac{M\,S_T^i}{P^+}\, g_T'(x,\boldsymbol{k}_T) + \frac{k_T^i}{P^+}\, g_s^\perp(x,\boldsymbol{k}_T)\\ &= \frac{M\,S_T^i}{P^+}\, g_T(x,\boldsymbol{k}_T) + \frac{\lambda k_T^i}{P^+}\, g_L^\perp(x,\boldsymbol{k}_T) - \frac{\left(k_T^i k_T^j + \frac{1}{2}\boldsymbol{k}_T^2 g_T^{ij}\right) S_{Tj}}{M^2}\, g_T^\perp(x,\boldsymbol{k}_T),\end{aligned} \tag{2.52}$$

$$\Phi^{[i\sigma^{ij}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{S_T^i k_T^j - k_T^i S_T^j}{P^+}\, h_T^\perp(x,\boldsymbol{k}_T) \tag{2.53}$$

$$\Phi^{[i\sigma^{+-}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\, h_s(x,\boldsymbol{k}_T). \tag{2.54}$$

Note that sometimes it may be useful to work with the functions projected using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$. These are

$$\Phi^{[\sigma^{i+}]}(x,\boldsymbol{k}_T) = \epsilon_T^{ij}\, S_{Tj}\, h_{1T}(x,\boldsymbol{k}_T) + \frac{\epsilon_T^{ij} k_{Tj}}{M}\, h_{1s}^\perp(x,\boldsymbol{k}_T), \tag{2.55}$$

$$\Phi^{[\sigma^{ij}]}(x,\boldsymbol{k}_T) = \frac{M\,\epsilon_T^{ij}}{P^+}\, h_s(x,\boldsymbol{k}_T), \tag{2.56}$$

$$\Phi^{[\sigma^{+-}]}(x,\boldsymbol{k}_T) = \frac{\epsilon_T^{ij}\, k_{Ti} S_{Tj}}{P^+}\, h_T^\perp(x,\boldsymbol{k}_T). \tag{2.57}$$

The integrated results $f_1(x)$ etc. discussed before are obtained from $f_1(x,\boldsymbol{k}_T^2)$ etc., where one must be aware that $g_1 = g_{1L}$, $h_1 = h_{1T} + (\boldsymbol{k}_T^2/2M^2) h_{1T}^\perp \equiv h_{1T} + h_{1T}^{\perp(1)}$ and $g_T = g_T' + (\boldsymbol{k}_T^2/2M^2) g_T^\perp \equiv g_T' + g_T^{\perp(1)}$. Besides the k_T-integrated functions shown before, it is useful to consider k_T^α-weighted functions,

$$\frac{1}{M}\,\Phi_\partial^\alpha(x) \equiv \int d^2\boldsymbol{k}_T\, \frac{k_T^\alpha}{M}\, \Phi(x,\boldsymbol{k}_T), \tag{2.58}$$

$$\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta}(x) \equiv \int d^2\boldsymbol{k}_T\, \frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\, g_T^{\alpha\beta}\right)}{M^2}\, \Phi(x,\boldsymbol{k}_T). \tag{2.59}$$

Note that the operator structure involved is

$$\begin{aligned}\Phi_\partial^\alpha(x,\boldsymbol{k}_T) &= k_T^\alpha\, \Phi(x,\boldsymbol{k}_T)\\ &= \int \frac{d\xi^-\, d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ik\cdot\xi}\, \langle P,S| i\partial_T^\alpha \left(\overline{\psi}(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,\psi(\xi)\right) |P,S\rangle \Big|_{\xi^+=0}\\ &= \int \frac{d\xi^-\, d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ik\cdot\xi}\, \langle P,S|\overline{\psi}(0)\,\mathcal{U}(0,\infty)\, i\partial_T^\alpha\, \mathcal{U}(\infty,\xi)\,\psi(\xi)|P,S\rangle \Big|_{\xi^+=0}.\end{aligned} \tag{2.60}$$

Note that in this case choosing the gauge $A^+ = 0$, one cannot just neglect the link operator, because it contains ξ_T dependence. We will return to this in the chapter 4. Note, however, that in the way defined here, the correlation function Φ_∂ is color gauge invariant.

We use the following properties for integrating $\boldsymbol{k}_T^2$ dependent functions over $\boldsymbol{k}_T$,

$$\int d^2\boldsymbol{k}_T\; k_T^\alpha k_T^i \ldots = -\frac{1}{2}\int d^2\boldsymbol{k}_T\; \boldsymbol{k}_T^2\, g_T^{\alpha i} \ldots$$

$$\int d^2\boldsymbol{k}_T\; k_T^\alpha k_T^\beta\, k_T^i k_T^j \ldots = \frac{1}{8}\int d^2\boldsymbol{k}_T\; \boldsymbol{k}_T^4\, \left(g_T^{\alpha i} g_T^{\beta j} + g_T^{\alpha j} g_T^{\beta i} + g_T^{\alpha\beta} g_T^{ij}\right) \ldots,$$

$$\int d^2\boldsymbol{k}_T\; k_T^\alpha k_T^\beta\, \left(k_T^i k_T^j + \frac{1}{2}\,\boldsymbol{k}_T^2\, g_T^{ij}\right) \ldots = \frac{1}{8}\int d^2\boldsymbol{k}_T\; \boldsymbol{k}_T^4\, \left(g_T^{\alpha i} g_T^{\beta j} + g_T^{\alpha j} g_T^{\beta i} - g_T^{\alpha\beta} g_T^{ij}\right) \ldots,$$

$$\int d^2\boldsymbol{k}_T\; \left(k_T^\alpha k_T^\beta + \frac{1}{2}\,\boldsymbol{k}_T^2\, g_T^{\alpha\beta}\right) \left(k_T^i k_T^j + \frac{1}{2}\,\boldsymbol{k}_T^2\, g_T^{ij}\right) \ldots = \frac{1}{8}\int d^2\boldsymbol{k}_T\; \boldsymbol{k}_T^4\, \left(g_T^{\alpha i} g_T^{\beta j} + g_T^{\alpha j} g_T^{\beta i} - g_T^{\alpha\beta} g_T^{ij}\right) \ldots,$$

Nonvanishing for twist two are

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^+\gamma_5]}(x) = S_T^\alpha \int d^2\boldsymbol{k}_T\; \frac{\boldsymbol{k}_T^2}{2M^2}\, g_{1T}(x,\boldsymbol{k}_T) \equiv S_T^\alpha\, g_{1T}^{(1)}(x), \tag{2.61}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{i+}\gamma_5]}(x) = -g_T^{\alpha i}\,\lambda\int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,h_{1L}^\perp(x,\boldsymbol{k}_T) \equiv -g_T^{\alpha i}\,\lambda\,h_{1L}^{\perp(1)}(x), \tag{2.62}$$

$$\begin{aligned}\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta[i\sigma^{i+}\gamma_5]}(x) &= -\frac{1}{2}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\int d^2\boldsymbol{k}_T\,\left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^2 h_{1T}^\perp(x,\boldsymbol{k}_T)\\ &\equiv -\frac{1}{2}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\,h_{1T}^{\perp(2)}(x),\end{aligned} \tag{2.63}$$

and for twist three

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^i]}(x) = -g_T^{\alpha i}\,\frac{M}{P^+}\,f^{\perp(1)}(x), \tag{2.64}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^i\gamma_5]}(x) = -g_T^{\alpha i}\,\frac{M}{P^+}\,\lambda\,g_L^{\perp(1)}(x) \tag{2.65}$$

$$\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta[\gamma^i\gamma_5]}(x) = -\frac{M}{2P^+}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\,g_T^{\perp(2)}(x) \tag{2.66}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{ij}\gamma_5]}(x) = \frac{M}{P^+}\left(g_T^{\alpha i}S_T^j - g_T^{\alpha j}S_T^i\right)\,h_T^{\perp(1)}(x), \tag{2.67}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{+-}\gamma_5]}(x) = \frac{M}{P^+}\,S_T^\alpha\,h_T^{(1)}(x). \tag{2.68}$$

We obtain now besides the integrated results, the k_T-weighted results

$$\boxed{\begin{aligned}\Phi_\partial^\alpha(x) &= \frac{1}{2}\left\{-g_{1T}^{(1)}(x)\,S_T^\alpha\,\not{n}_+\gamma_5 - \lambda\,h_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\right\}\\ &\quad+\frac{M}{2P^+}\left\{-f^{\perp(1)}(x)\,\gamma^\alpha + \lambda\,g_L^{\perp(1)}(x)\,\gamma^\alpha\gamma_5\right.\\ &\qquad\left.+h_T^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{S}_T]\gamma_5}{2} + h_T^{(1)}(x)\,S_T^\alpha\,\frac{[\not{n}_+,\not{n}_-]\gamma_5}{2}\right\} &(2.69)\\ \Phi_{\partial\partial}^{\alpha\beta}(x) &= -\frac{1}{2}\,h_{1T}^{\perp(2)}(x)\left(\frac{S_T^{\{\alpha}[\gamma^{\beta\}},\not{n}_+]\gamma_5 - g_T^{\alpha\beta}\,[\not{S}_T,\not{n}_+]\gamma_5}{4}\right)\\ &\quad+\frac{M}{2P^+}\,g_T^{\perp(2)}(x)\left(\frac{S_T^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not{S}_T\gamma_5}{2}\right) &(2.70)\end{aligned}}$$

For the matrix elements $\Phi_\partial^+(x)$ no new functions come in. Working in $A^+ = 0$ gauge (or using $\partial^+\mathcal{U}(0,\xi) = \mathcal{U}(0,\xi)D^+$, see chapter 4) one sees that

$$\Phi_\partial^+(x) = xP^+\,\Phi(x) = \int\frac{d\xi^-}{2\pi}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\xi)\,iD^+\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{2.71}$$

and hence using the parametrization of $\Phi^{[\gamma^+]}$,

$$\begin{aligned}2\,\Phi_\partial^{+[\gamma^+]} &= P^+\int dx\;\mathrm{Tr}\left(\Phi_\partial^+(x)\,\gamma^+\right) = (P^+)^2\int dx\;x\,\mathrm{Tr}\left(\Phi\,\gamma^+\right)\\ &= 2(P^+)^2\underbrace{\int dx\;xf_1(x)}_{\epsilon_q} = \langle P,S|\underbrace{\overline{\psi}(0)\,\gamma^+\,iD^+\psi(0)}_{\theta_q^{++}}|P,S\rangle.\end{aligned} \tag{2.72}$$

Realizing that θ_q^{++} is only part of the energy-momentum tensor, it leads to the momentum sumrule $\int dx\;xf_1(x) = \epsilon_q \le 1$. Using the support properties of the distribution functions ($-1 \le x \le 1$) and the symmetry relation $f_1(x) = -\bar{f}_1(-x)$, the sum rule reads

$$\int_0^1 dx\;x\left(f_1(x) + \bar{f}_1(x)\right) = \epsilon_q \le 1. \tag{2.73}$$

2.2 Antiquark distribution functions

The profile functions for antiquarks in a hadron are obtained from the matrix elements

$$\begin{aligned}\overline{\Phi}_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\mathcal{U}(0,\xi)\psi_i(\xi)\overline{\psi}_j(0)|P,S\rangle \\ &= -\frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\mathcal{U}(0,\xi)\psi_i(\xi)|P,S\rangle. \end{aligned} \tag{2.74}$$

For a definition of the profile functions that is consistent with the definition of free particle and antiparticle states, one needs the correlation function Φ^c that is defined analogous to Φ but using the conjugate spinors $\psi^c = C\,\overline{\psi}^T$, where $C\,\gamma_\mu^T\,C^\dagger = -\gamma_\mu$,

$$\Phi^c_{ij}(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}^c_j(0)\mathcal{U}(0,\xi)\psi^c_i(\xi)|P,S\rangle. \tag{2.75}$$

The relation between these quantities is $\Phi^c = -C\,\overline{\Phi}^T C^\dagger$. Using $\Phi^{c[\Gamma]}$ to define the antiquark profile functions, $\overline{f}(x,\boldsymbol{k}_T)$, etc., one must be aware of the relative sign ($\pm$) between $\overline{\Phi}^{[\Gamma]}$ and $\Phi^{c[\Gamma]}$ depending on $\Gamma = \mp C\Gamma^T C^\dagger$. Explicitly,

$$\begin{aligned}\overline{\Phi}^{[\Gamma]} &= +\Phi^{c[\Gamma]} \qquad \text{for } \Gamma = \gamma_\mu, \sigma_{\mu\nu} \text{ or } i\sigma_{\mu\nu}\gamma_5, \\ \overline{\Phi}^{[\Gamma]} &= -\Phi^{c[\Gamma]} \qquad \text{for } \Gamma = 1,\ \gamma_\mu\gamma_5 \text{ and } i\gamma_5,\end{aligned}$$

We note also that (at the twist two and twist three level) the anticommutation relations for fermions can be used to obtain the symmetry relation

$$\overline{\Phi}_{ij}(k,P,S) = -\Phi_{ij}(-k,P,S). \tag{2.76}$$

For the profile functions this gives the symmetry relations

$$\overline{f}_1(x,\boldsymbol{k}_T^2) = -f_1(-x,\boldsymbol{k}_T^2) \tag{2.77}$$

and identically for g_{1T}, h_{1T}, $h_{1T}^\perp$, $g_L^\perp$ and h_L (C-even functions), while

$$\overline{g}_{1L}(x,\boldsymbol{k}_T^2) = g_{1L}(-x,\boldsymbol{k}_T^2) \tag{2.78}$$

and identically for $h_{1L}^\perp$, e, $f^\perp$, g_T', $g_T^\perp$, $h_T^\perp$ and h_T (C-odd functions).

Explicitly we get (fixing k^- and integrating over k^+)

$$\begin{aligned}&\frac{1}{2}\int dk^+\;\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-,\ \boldsymbol{k}_T} = \\ &\quad\frac{1}{4}\left\{\overline{f}_1(x,\boldsymbol{k}_T)\,\not{n}_- + \overline{g}_{1s}(x,\boldsymbol{k}_T)\,\not{n}_-\gamma_5 - \overline{h}_{1T}(x,\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5 S_T^\mu n_-^\nu - \overline{h}_{1s}^\perp(x,\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5 k_T^\mu n_-^\nu}{M}\right\} \\ &\quad+\frac{M}{4P^-}\left\{-\overline{e}(x,\boldsymbol{k}_T) + \overline{f}^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M} + \overline{g}'_T(x,\boldsymbol{k}_T)\,\not{S}_T\gamma_5 + \overline{g}_s^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T\gamma_5}{M}\right. \\ &\qquad\qquad\left. -\overline{h}_T^\perp(x,\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5 S_T^\mu k_T^\nu}{M} - \overline{h}_s(x,\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5 n_-^\mu n_+^\nu\right\}. \end{aligned} \tag{2.79}$$

and the projections for twist two are

$$\overline{\Phi}^{[\gamma^-]}(x,\boldsymbol{k}_T) = \overline{f}(x,\boldsymbol{k}_T), \tag{2.80}$$

$$\overline{\Phi}^{[\gamma^-\gamma_5]}(x,\boldsymbol{k}_T) = -\overline{g}_{1s}(x,\boldsymbol{k}_T), \tag{2.81}$$

$$\overline{\Phi}^{[i\sigma^{i-}\gamma_5]}(x,\boldsymbol{k}_T) = S_T^i\,\overline{h}_{1T}(x,\boldsymbol{k}_T) + \frac{k_T^i}{M}\,\overline{h}_{1s}^\perp(x,\boldsymbol{k}_T), \tag{2.82}$$

while those of twist three are

$$\overline{\Phi}^{[1]}(x,\boldsymbol{k}_T) = -\frac{M}{P^-}\,\overline{e}(x,\boldsymbol{k}_T), \tag{2.83}$$

$$\overline{\Phi}^{[\gamma^i]}(x,\boldsymbol{k}_T) = \frac{k_T^i}{P^-}\,\overline{f}^\perp(x,\boldsymbol{k}_T), \tag{2.84}$$

$$\overline{\Phi}^{[\gamma^i\gamma_5]}(x,\boldsymbol{k}_T) = -\frac{M\,S_T^i}{P^-}\,\overline{g}_T'(x,\boldsymbol{k}_T) - \frac{k_T^i}{P^-}\,\overline{g}_s^\perp(x,\boldsymbol{k}_T), \tag{2.85}$$

$$\overline{\Phi}^{[i\sigma^{ij}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{S_T^i k_T^j - k_T^i S_T^j}{P^-}\,\overline{h}_T^\perp(x,\boldsymbol{k}_T), \tag{2.86}$$

$$\overline{\Phi}^{[i\sigma^{-+}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{M}{P^-}\,\overline{h}_s(x,\boldsymbol{k}_T). \tag{2.87}$$

Using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$ one has

$$\overline{\Phi}^{[\sigma^{i-}]}(x,\boldsymbol{k}_T) = -\epsilon_T^{ij}\,S_{Tj}\,\overline{h}_{1T}(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij}k_{Tj}}{M}\,\overline{h}_{1s}^\perp(x,\boldsymbol{k}_T), \tag{2.88}$$

$$\overline{\Phi}^{[\sigma^{ij}]}(x,\boldsymbol{k}_T) = -\frac{M\,\epsilon_T^{ij}}{P^-}\,\overline{h}_s(x,\boldsymbol{k}_T), \tag{2.89}$$

$$\overline{\Phi}^{[\sigma^{-+}]}(x,\boldsymbol{k}_T) = -\frac{\epsilon_T^{ij}\,k_{Ti}S_{Tj}}{P^-}\,\overline{h}_T^\perp(x,\boldsymbol{k}_T). \tag{2.90}$$

The integrated results for antiquarks are

$$\begin{aligned}\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\,\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-} &= \frac{1}{4}\left\{\overline{f}_1(x)\,\not n_- + \lambda\,\overline{g}_1(x)\,\not n_-\gamma_5 + \overline{h}_1(x)\,\frac{[\not S_T,\not n_-]\gamma_5}{2}\right\}\\ &+\frac{M}{4P^-}\left\{-\overline{e}(x) + \overline{g}_T(x)\,\not S_T\gamma_5 + \lambda\,\overline{h}_L(x)\,\frac{[\not n_-,\not n_+]\gamma_5}{2}\right\},\end{aligned} \tag{2.91}$$

$$\begin{aligned}\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\,\frac{k_T^\alpha}{M}\,\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-} &= \frac{1}{4}\left\{\overline{g}_{1T}^{(1)}(x)\,S_T^\alpha\,\not n_-\gamma_5 - \lambda\,\overline{h}_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not n_-]\gamma_5}{2}\right\}\\ &+\frac{M}{4P^-}\left\{-\overline{f}^{\perp(1)}(x)\,\gamma^\alpha - \lambda\,\overline{g}_L^{\perp(1)}(x)\,\gamma^\alpha\gamma_5\right.\\ &\left.+\overline{h}_T^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not S_T]\gamma_5}{2} + \overline{h}_T^{(1)}(x)\,S_T^\alpha\,\frac{[\not n_-,\not n_+]\gamma_5}{2}\right\}\end{aligned} \tag{2.92}$$

$$\begin{aligned}\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\,\frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)}{M^2}\,\overline{\Phi}(k,P,S)\Bigg|_{k^-=xP^-} &=\\ -\frac{1}{4}\,\overline{h}_{1T}^{\perp(2)}(x)&\left(\frac{S_T^{\{\alpha}[\gamma^{\beta\}},\not n_-]\gamma_5 - g_T^{\alpha\beta}\,[\not S_T,\not n_-]\gamma_5}{4}\right)\\ -\frac{M}{4P^-}\,\overline{g}_T^{\perp(2)}(x)&\left(\frac{S_T^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not S_T\gamma_5}{2}\right)\end{aligned} \tag{2.93}$$

2.3 Result for an ensemble of free quarks (parton model)

It is instructive to calculate the correlation function for a free quark. This is given by

$$\phi_{ij}(p,s;k) = u_i(k,s)\overline{u}_j(k,s)\delta^4(k-p) = \tfrac{1}{2}\big((\not k + m)(1+\gamma_5\not s)\big)_{ij}\delta^4(k-p), \tag{2.94}$$

where the momentum and spin of the quark are parametrized as

$$k = \left[\frac{\boldsymbol{k}_T^2+m^2}{2k^+},\ k^+,\ \boldsymbol{k}_T\right], \tag{2.95}$$

$$s = \left[-\frac{m\,\lambda_q}{2k^+}+\frac{\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^+}+\frac{\lambda_q\,\boldsymbol{k}_T^2}{2m\,k^+},\ \frac{\lambda_q\,k^+}{m},\ \boldsymbol{s}_{qT}+\frac{\lambda_q}{m}\boldsymbol{k}_T\right] \tag{2.96}$$

in terms of a quark lightcone helicity λ_q and a quark lightcone transverse polarization s_{qT}, such that $k\cdot s = 0$ and $\lambda_q^2+\boldsymbol{s}_{qT}^2 = -s^2 = 1$. Note that this helicity only is a true helicity for a quark with infinite momentum. It is then straightforward to calculate the projections for a free quark target. For twist two

$$\phi^{[\gamma^+]}(k) = \delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\uparrow/q\lambda}+f_{q\downarrow/q\lambda}, \tag{2.97}$$

$$\phi^{[\gamma^+\gamma_5]}(k) = \lambda_q\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\uparrow/q\lambda}-f_{q\downarrow/q\lambda}, \tag{2.98}$$

$$\phi^{[i\sigma^{i+}\gamma_5]}(k) = \boldsymbol{s}_{qT}^i\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\rightarrow/q\boldsymbol{s}_T}-f_{q\leftarrow/q\boldsymbol{s}_T}, \tag{2.99}$$

where $\xi = k^+/p^+$, and we have indicated the intuitive interpretation in terms of probabilities of finding quarks in a quark with spin given by λ and $\boldsymbol{s}_T$ (see below). For twist three we get

$$\phi^{[1]}(k) = \frac{m}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.100}$$

$$\phi^{[\gamma^i]}(k) = \frac{\boldsymbol{k}_T^i}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.101}$$

$$\phi^{[\gamma^i\gamma_5]}(k) = \frac{(m\,\boldsymbol{s}_{qT}^i+\lambda_q\,\boldsymbol{k}_T^i)}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.102}$$

$$\phi^{[i\sigma^{ij}\gamma_5]}(k) = \frac{\boldsymbol{s}_{qT}^i\boldsymbol{k}_T^j-\boldsymbol{k}_T^i\boldsymbol{s}_{qT}^j}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.103}$$

$$\phi^{[i\sigma^{+-}\gamma_5]}(k) = \frac{m\,\lambda_q-\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T). \tag{2.104}$$

In a parton model description of the target, one uses the expansion for the free quark field to get

$$\Phi_{ij}(k) = 2\,\delta(k^2-m^2)\left[\theta(x)u_i^{(\beta)}(k)\mathcal{P}_{\beta\alpha}(k)\bar{u}_j^{(\alpha)}(k)-\theta(-x)v_i^{(\beta)}(-k)\overline{\mathcal{P}}_{\beta\alpha}(-k)\bar{v}_j^{(\alpha)}(-k)\right], \tag{2.105}$$

where $x = k^+/P^+$. The use of lightcone coordinates is convenient because of the integration over k^- that is needed in deep inelastic processes. The functions $\mathcal{P}$ and $\overline{\mathcal{P}}$ are given by

$$\mathcal{P}_{\beta\alpha}(k) = \mathcal{P}_{\beta\alpha}(x,\boldsymbol{k}_T^2) \equiv \frac{1}{2(2\pi)^3}\int\frac{dx'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2x'}\,\langle PS|b_\alpha^\dagger(k')b_\beta(k)|PS\rangle, \tag{2.106}$$

$$\overline{\mathcal{P}}_{\beta\alpha}(k) = \overline{\mathcal{P}}_{\beta\alpha}(x,\boldsymbol{k}_T^2) \equiv \frac{1}{2(2\pi)^3}\int\frac{dx'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2x'}\,\langle PS|d_\beta^\dagger(k')d_\alpha(k)|PS\rangle. \tag{2.107}$$

Note that $\mathcal{P}_{\beta\alpha}(k)$ is a production matrix in the quark spin-space of which the trace is the quark density operator evaluated in the target. The Dirac structure can be parametrized as

$$u^{(\beta)}(k,s)\mathcal{P}_{\beta\alpha}(k)\bar{u}^{(\alpha)}(k,s) = \mathcal{P}(k)(\not{k}+m)\left(\frac{1+\gamma_5\not{s}(k)}{2}\right) \tag{2.108}$$

$$v^{(\beta)}(k,s)\overline{\mathcal{P}}_{\beta\alpha}(k)\bar{v}^{(\alpha)}(k,s) = \overline{\mathcal{P}}(k)(\not{k}-m)\left(\frac{1+\gamma_5\not{\bar{s}}(k)}{2}\right) \tag{2.109}$$

in terms of a positive definite quark and antiquark densities $\mathcal{P}(k)$ and $\overline{\mathcal{P}}(k)$ and spin densities $s(k)$ and $\bar{s}(k)$ which satisfy $0\le -s^2(k)\le 1$ or $0\le\lambda_q^2(k)+\boldsymbol{s}_{qT}^2(k)\le 1$. Inserting the free field expansion in the current expectation value $\langle PS|\overline{\psi}(0)\gamma^\mu\psi(0)|PS\rangle = 2P^\mu(N-\overline{N})$, where N and $\overline{N}$ are total number of quarks and antiquarks respectively one obtains from the +-component the normalizations $\int_0^1 dx\int d^2\boldsymbol{k}_T\mathcal{P}(x,\boldsymbol{k}_T^2) = N$ and $\int_0^1 dx\int d^2\boldsymbol{k}_T\overline{\mathcal{P}}(x,\boldsymbol{k}_T^2) = \overline{N}$.

Integrating over k^- one obtains the result

$$\frac{1}{2}\int dk^- \Phi(k) = \theta(x)\frac{\mathcal{P}(x,\boldsymbol{k}_T^2)}{2k^+}(\not k + m)\left(\frac{1+\gamma_5 \not s(x,\boldsymbol{k}_T)}{2}\right) - \theta(-x)\frac{\overline{\mathcal{P}}(-x,\boldsymbol{k}_T^2)}{2k^+}(\not k + m)\left(\frac{1+\gamma_5 \not{\overline{s}}(-x,-\boldsymbol{k}_T)}{2}\right). \tag{2.110}$$

This gives (for $x > 0$)

$$\Phi^{[\gamma^+]}(k) = \mathcal{P}(x,\boldsymbol{k}_T^2) = \mathcal{P}_{RR} + \mathcal{P}_{LL} = \mathcal{P}_{\alpha\alpha} + \mathcal{P}_{\beta\beta}, \tag{2.111}$$

$$\Phi^{[\gamma^+\gamma_5]}(k) = \lambda_q(x,\boldsymbol{k}_T)\,\mathcal{P}(x,\boldsymbol{k}_T^2) = \mathcal{P}_{RR} - \mathcal{P}_{LL}, \tag{2.112}$$

$$\Phi^{[i\sigma^{i+}\gamma_5]}(k) = \boldsymbol{s}_{qT}^i(x,\boldsymbol{k}_T)\,\mathcal{P}(x,\boldsymbol{k}_T^2) = \mathcal{P}_{\alpha\alpha} - \mathcal{P}_{\beta\beta}, \tag{2.113}$$

where the indices R/L and α/β are particular (chiral and transverse spin) projections for the fermion fields or spinors, obtained using the projection operators $P_{R/L} = (1 \pm \gamma_5)/2$ and $Q^i_{\alpha/\beta} = (1 \pm \gamma^i\gamma_5)/2$. The interpretation of the first equation is that of the probability for finding quarks in a target. The second equation is interpreted as the probability for righthanded quarks minus that for lefthanded quarks. The third equation is interpreted as the probability for quarks with spin parallel to the transverse direction i minus that for quarks with spin opposite. The chiral structure of these functions is $\overline{R}L + \overline{L}R$ and they are referred to as *chirally odd* distribution functions. Equating these unpolarized and polarized densities to the distribution functions defined in the previous section,

$$\mathcal{P}(x,\boldsymbol{k}_T^2) = f_1(x,\boldsymbol{k}_T), \tag{2.114}$$

$$\lambda_q(x,\boldsymbol{k}_T)\,\mathcal{P}(x,\boldsymbol{k}_T^2) = \lambda\, g_{1L}(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\, g_{1T}(x,\boldsymbol{k}_T) = g_{1s}(x,\boldsymbol{k}_T), \tag{2.115}$$

$$\begin{aligned}\boldsymbol{s}_{qT}^i(x,\boldsymbol{k}_T)\,\mathcal{P}(x,\boldsymbol{k}_T^2) &= \boldsymbol{S}_T^i\, h_{1T}(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T^i}{M}\left[h_{1L}^\perp(x,\boldsymbol{k}_T)\lambda + h_{1T}^\perp(x,\boldsymbol{k}_T)\frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\right]\\ &= \boldsymbol{S}_T^i\, h_{1T}(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T^i}{M}\, h_{1s}^\perp(x,\boldsymbol{k}_T),\end{aligned} \tag{2.116}$$

shows how the functions g_{1L}, g_{1T}, h_1, $h_{1L}^\perp$ and $h_{1T}^\perp$ are to be interpreted as longitudinal and transverse spin distributions given the spin of the hadron (λ and $\boldsymbol{S}_T$). For the antiquarks the same relations hold between the antiquark helicity $\overline{\lambda}_q$ and transverse polarization $\overline{\boldsymbol{s}}_T$ on the one hand and the antiquark distributions on the other hand. Extending to all x, results are obtained in accordance with the symmetry relations in the previous section, e.g. $f_1(x,\boldsymbol{k}_T) = \theta(x)\mathcal{P}(x,\boldsymbol{k}_T^2) - \theta(-x)\overline{\mathcal{P}}(x,\boldsymbol{k}_T^2)$.

We note that at the twist two level this parton interpretation can be made rigorous as the distribution functions can be expressed as densities involving the socalled good components of ψ, $\psi_+ \equiv P_+\psi$ obtained with the projection operator $P_+ = \frac{1}{2}\gamma^-\gamma^+$. In lightfront quantization a Fourier expansion for the good components (at $x^+ = 0$) can be written down in which the Fourier coefficients can be interpreted as particle and antiparticle creation and annihilation operators The different spin-distributions involve projection operators ($P_{R/L}$ and $Q^1_{\alpha/\beta}$) that commute with P_+. At twist three the analysis of the quark - quark correlation functions lead to a number of new distribution functions. For an ensemble of free quarks they can also be expressed in the quark densities and in this way related to the (six) twist two distribution functions. Explicitly one has for the ensemble of free quarks,

$$e(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, f_1(x,\boldsymbol{k}_T), \tag{2.117}$$

$$f^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, f_1(x,\boldsymbol{k}_T), \tag{2.118}$$

$$g_T'(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, h_{1T}(x,\boldsymbol{k}_T), \tag{2.119}$$

$$g_s^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, g_{1s}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\, h_{1s}^\perp(x,\boldsymbol{k}_T), \tag{2.120}$$

$$h_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, h_{1T}(x,\boldsymbol{k}_T), \tag{2.121}$$

$$h_s(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, g_{1s}(x,\boldsymbol{k}_T) + \frac{k_T\cdot S_T}{Mx}\, h_{1T}(x,\boldsymbol{k}_T) + \frac{k_T^2}{M^2x}\, h_{1s}^\perp. \tag{2.122}$$

As we will show in one of the next sections, the above results are not generally true; the presence of nonvanishing quark - quark - gluon correlation functions causes deviations from these naive parton model results.

Summarizing we get for $\Phi^{[\Gamma]}(x, \boldsymbol{k}_T)$ and $\Phi^{[\Gamma]}(x)$, ordered according to twist and chirality for unpolarized (U), longitudinally polarized (L) and transversely polarized (T) (spin 1/2) hadrons:

DISTRIBUTIONS			
		chirality	
$\Phi^{[\Gamma]}(x, \boldsymbol{k}_T)$		even	odd
twist 2	U	f_1	
	L	g_{1L}	$h_{1L}^\perp$
	T	g_{1T}	h_{1T} $h_{1T}^\perp$
twist 3	U	$f^\perp$	e
	L	$g_L^\perp$	h_L
	T	g'_T $g_T^\perp$	h_T $h_T^\perp$

DISTRIBUTIONS			
		chirality	
$\Phi^{[\Gamma]}(x)$		even	odd
twist 2	U	f_1	
	L	g_1	
	T		h_1
twist 3	U		e
	L		h_L
	T	g_T	

2.4 Fragmentation: from quarks to hadron

For the fragmentation of quarks into hadrons we need the correlation function

$$\begin{aligned}\Delta_{ij}(k, P_h, S_h) &= \sum_X \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \langle 0|\mathcal{U}(0,\xi)\psi_i(\xi)|P_h, X\rangle\langle P_h, X|\overline{\psi}_j(0)|0\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \langle 0|\mathcal{U}(0,\xi)\psi_i(\xi) a_h^\dagger a_h \overline{\psi}_j(0)|0\rangle, \end{aligned} \tag{2.123}$$

where an averaging over color indices is implicit. We note, however that fragmentation is into a hadron in a specified spin state.

The use of intermediate states X and in addition one specified state with momentum P_h needs some explanation. First note that the unit operator can be written as

$$\mathcal{I} \equiv \sum_X |X\rangle\langle X| = \sum_{p=0}^{\infty} \mathcal{I}_p \tag{2.124}$$

with

$$\mathcal{I}_p = \frac{1}{p!} \int d\tilde{k}_1 \ldots d\tilde{k}_p \; a^\dagger(k_1)\ldots a^\dagger(k_p)|0\rangle\langle 0|a(k_1)\ldots a(k_p) \tag{2.125}$$

containing the p-particle states (with $d\tilde{k}$ being the invariant one-particle phase-space). Thus

$$\begin{aligned}\sum_X |P_h, X\rangle\langle P_h, X| &= |P_h\rangle\langle P_h| + \int d\tilde{k}_1 \; |P_h, k_1\rangle\langle P_h, k_1| \\ &\quad + \frac{1}{2!}\int d\tilde{k}_1 d\tilde{k}_2 \; |P_h, k_1, k_2\rangle\langle P_h, k_1, k_2| + \ldots \\ &= a_h^\dagger \mathcal{I} a_h = a_h^\dagger a_h. \end{aligned} \tag{2.126}$$

After integrating over P_h one obtains

$$\int d\tilde{P}_h \sum_X |P_h, X\rangle\langle P_h, X| = \sum_{p=0}^{\infty} p\,\mathcal{I}_p, \tag{2.127}$$

which is the number operator N_h. This will become relevant when one integrates over the phase-space of particles in the final state to go from 1-particle inclusive to inclusive scattering processes.

For the Dirac structure the same expansion as before can be written down,

$$\boxed{\begin{aligned}\Delta(k, P_h, S_h) = \; & M_h\, B_1 + B_2\, \not{P}_h + B_3 \not{k} + (B_4/M_h)\, \sigma^{\mu\nu} P_{h\mu} k_\nu \\ & + i\, B_5 (k\cdot S_h)\gamma_5 + M_h\, B_6 \not{S}_h \gamma_5 + (B_7/M_h)(k\cdot S_h)\not{P}_h\gamma_5 + (B_8/M_h)(k\cdot S_h)\not{k}\gamma_5 \\ & + i\, B_9\, \sigma^{\mu\nu}\gamma_5\, S_{h\mu} P_{h\nu} + i\, B_{10}\, \sigma^{\mu\nu}\gamma_5\, S_{h\mu} k_\nu + i\,(B_{11}/M_h^2)\,(k\cdot S_h)\,\sigma^{\mu\nu}\gamma_5\, k_\mu P_{h\nu} \\ & + (B_{12}/M_h)\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P_h^\nu k^\rho S_h^\sigma, \end{aligned}} \tag{2.128}$$

where the amplitudes B_i depend on $P_h \cdot k$ and k^2. As the states $|P_h, X\rangle$ in the above expression for Δ_{ij} are not plane waves, one cannot apply time-reversal invariance. So the amplitudes B_4, B_5 and B_{12} do not vanish.

The fact that the time-reversal-odd amplitudes do not vanish is a consequence of the final state interactions of the produced hadron h. To see this, it is instructive (even necessary) to treat $|P_h, X\rangle$ as physical states. In that case they should be labeled as outstates. Consider the multi-channel matrix

$$\Delta_{ij}^{(ff')} = \sum_X \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_i(\xi)|f;\text{out}\rangle\langle f';\text{out}|\overline{\psi}_j(0)|0\rangle \tag{2.129}$$

(Note that $\Delta^{(f'f)} = \Delta^{(ff')\dagger}$). The behavior of $\Delta^{(ff')}$ under time reversal involves

$$\begin{aligned} \langle f';\text{out}|\overline{\psi}_j(\xi)|0\rangle &= \langle \tilde{f}';\text{in}|(\overline{-i\gamma_5 C\psi})_j(-\bar{\xi})|0\rangle^* \\ &= \langle \tilde{f}';\text{out}|S^\dagger \, (\overline{-i\gamma_5 C\psi})_j(-\bar{\xi})|0\rangle^* \end{aligned} \tag{2.130}$$

from which one finds that the combination

$$\tilde{\Delta}^{ff} = (\sqrt{S})^{ff_1}\Delta^{f_1 f_2}(\sqrt{S^\dagger})^{f_2 f} \tag{2.131}$$

behaves as

$$\tilde{\Delta}^*(k, P_h, S_h) = (-i\gamma_5 C)\,\tilde{\Delta}(\bar{k}, \bar{P}_h, \bar{S}_h)\,(-i\gamma_5 C). \tag{2.132}$$

If the final state would be one unique channel obtains zero because $S = e^{2i\delta}$. For two channels the S-matrix can e.g. be parametrized as

$$S = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_1} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \tag{2.133}$$

Working in channel space with the diagonal S-matrix one has

$$\tilde{\Delta}^{ff} = \begin{pmatrix} \Delta^{(11)} & e^{i(\delta_1-\delta_2)}\,\Delta^{(12)} \\ e^{-i(\delta_1-\delta_2)}\,\Delta^{(21)} & \Delta^{(22)} \end{pmatrix}. \tag{2.134}$$

From the hermiticity condition one knows that for Δ the amplitude analysis requires real amplitudes $B_i^{(ff')}$ for $i = 1, 12$. The time-reversal invariance condition applied to $\tilde{\Delta}$ requires $\tilde{B}_i^* = \tilde{B}_i$ for $i = 1, 2, 3, 6, \ldots, 11$ and $\tilde{B}_i^* = -\tilde{B}_i$ for $i = 4, 5, 12$. Thus one finds

$$B_i^{(11)} = B_i^{(22)} = 0 \qquad (\text{i} = 1,\ 2,\ 3,\ 6, \ldots,\ 11) \tag{2.135}$$

$$\sin(\delta_1 - \delta_2)\, B_i^{(12)} = 0 \qquad (\text{i} = 1,\ 2,\ 3,\ 6, \ldots,\ 11) \tag{2.136}$$

$$\cos(\delta_1 - \delta_2)\, B_i^{(12)} = 0 \qquad (\text{i} = 4,\ 5,\ 12), \tag{2.137}$$

In general one can make a partial wave expansion of the final state in states $|J, M\rangle$ and one finds

$$\sum_{J,J'} \sin\left(\delta_J - \delta_{J'}\right)\, B_i^{(JJ')} = 0 \qquad (\text{i} = 1,\ 2,\ 3,\ 6, \ldots,\ 11) \tag{2.138}$$

$$\sum_{J,J'} \cos\left(\delta_J - \delta_{J'}\right)\, B_i^{(JJ')} = 0 \qquad (\text{i} = 4,\ 5,\ 12), \tag{2.139}$$

The first equation is a constraint between partial waves which is trivially satisfied in the absence of final state interactions. The second equation implies in the absence of final state interactions that the amplitudes B_4, B_5 and B_{12} vanish.

For the fragmentation a twist analysis of Δ considering the projections

$$\begin{aligned} \Delta^{[\Gamma]}(z, \boldsymbol{k}_T) &= \frac{1}{4z}\int dk^+ \; Tr(\Delta\,\Gamma)\Big|_{k^- = P_h^-/z,\; k_T} \\ &= \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{4z\,(2\pi)^3}\; e^{ik\cdot\xi}\; Tr\langle 0|\mathcal{U}(0,\xi)\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=0}, \end{aligned} \tag{2.140}$$

leads to the following set of twist two profile functions, which depend on $z = P_h^-/k^-$ and $\boldsymbol{k}_T^2 = -k_\perp^2$,

$$\Delta^{[\gamma^-]}(z,\boldsymbol{k}_T) = D_1(z,-z\boldsymbol{k}_T) + \frac{\epsilon_T^{ij}\,k_{Ti}S_{hTj}}{M_h}\,D_{1T}^\perp(z,-z\boldsymbol{k}_T), \tag{2.141}$$

$$\Delta^{[\gamma^-\gamma_5]}(z,\boldsymbol{k}_T) = G_{1s}(z,-z\boldsymbol{k}_T) \tag{2.142}$$

$$\Delta^{[i\sigma^{i-}\gamma_5]}(z,\boldsymbol{k}_T) = S_{hT}^i\,H_{1T}(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{M_h}\,H_{1s}^\perp(z,-z\boldsymbol{k}_T) + \frac{\epsilon_T^{ij}\,k_{Tj}}{M_h}\,H_1^\perp(z,-z\boldsymbol{k}_T), \tag{2.143}$$

and those of twist three are

$$\Delta^{[1]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\,E(z,-z\boldsymbol{k}_T), \tag{2.144}$$

$$\Delta^{[\gamma^i]}(z,\boldsymbol{k}_T) = \frac{k_T^i}{P_h^-}\,D^\perp(z,-z\boldsymbol{k}_T) + \frac{\lambda_h\,\epsilon_T^{ij}\,k_{Tj}}{P_h^-}\,D_L^\perp(z,-z\boldsymbol{k}_T) + \frac{M_h\,\epsilon_T^{ij}\,S_{hT\,j}}{P_h^-}\,D_T(z,-z\boldsymbol{k}_T), \tag{2.145}$$

$$\Delta^{[i\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\,E_s(z,-z\boldsymbol{k}_T), \tag{2.146}$$

$$\Delta^{[\gamma^i\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h\,S_{hT}^i}{P_h^-}\,G_T'(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{P_h^-}\,G_s^\perp(z,-z\boldsymbol{k}_T), \tag{2.147}$$

$$\Delta^{[i\sigma^{ij}\gamma_5]}(z,\boldsymbol{k}_T) = \frac{S_{hT}^i k_T^j - k_T^i S_{hT}^j}{P_h^-}\,H_T^\perp(z,-z\boldsymbol{k}_T) + \frac{M_h\,\epsilon_T^{ij}}{P_h^-}\,H(z,-z\boldsymbol{k}_T), \tag{2.148}$$

$$\Delta^{[i\sigma^{-+}\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\,H_s(z,-z\boldsymbol{k}_T). \tag{2.149}$$

Again it may sometimes be useful to work with the functions projected using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$. These are

$$\Delta^{[\sigma^{i-}]}(z,\boldsymbol{k}_T) = -\epsilon_T^{ij}\,S_{hTj}\,H_{1T}(z,-z\boldsymbol{k}_T) - \frac{\epsilon_T^{ij}\,k_{Tj}}{M_h}\,H_{1s}^\perp(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{M_h}\,H_1^\perp(z,-z\boldsymbol{k}_T), \tag{2.150}$$

$$\Delta^{[\sigma^{ij}]}(z,\boldsymbol{k}_T) = -\frac{M_h\,\epsilon_T^{ij}}{P_h^-}\,H_s(z,-z\boldsymbol{k}_T), \tag{2.151}$$

$$\Delta^{[\sigma^{-+}]}(z,\boldsymbol{k}_T) = -\frac{\epsilon_T^{ij}\,k_{Ti}S_{hTj}}{P_h^-}\,H_T^\perp(z,-z\boldsymbol{k}_T) + \frac{M_h}{P_h^-}\,H(z,-z\boldsymbol{k}_T). \tag{2.152}$$

The shorthand notations G_{1s} etc. stands for

$$G_{1s}(z,-z\boldsymbol{k}_T) = \lambda_h\,G_{1L}(z,-z\boldsymbol{k}_T) + G_{1T}(z,-z\boldsymbol{k}_T)\,\frac{(\boldsymbol{k}_T\cdot\boldsymbol{S}_{hT})}{M_h}. \tag{2.153}$$

The integrated profile functions are defined as

$$\begin{aligned}\Delta^{[\Gamma]}(z) \quad &= \quad z^2\int d^2\boldsymbol{k}_T\Delta^{[\Gamma]}(z,\boldsymbol{k}_T) = \frac{z}{4}\int dk^+\,d^2\boldsymbol{k}_T\;Tr(\Delta\,\Gamma)\Big|_{k^-=P_h^-/z} \\ &= \quad \frac{z}{8\pi}\int d\xi^+\;e^{ik\cdot\xi}\,Tr\langle 0|\mathcal{U}(0,\xi)\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=\xi_T=0},\end{aligned} \tag{2.154}$$

Besides these we have nonvanishing k_T-weighted functions in analogy to the distribution functions.

The Dirac structure of the fragmentation correlation function integrated over k^+ then becomes up to twist three

$$\frac{1}{4z}\int dk^+\;\Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z,\;k_T} =$$

$$\frac{1}{4}\Bigg\{D_1(z,-z\boldsymbol{k}_T)\,\not{n}_- + D_{1T}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_-^\nu k_T^\rho S_{hT}^\sigma}{M_h} - G_{1s}(z,-z\boldsymbol{k}_T)\,\not{n}_-\gamma_5$$

$$
\begin{aligned}
&\quad -H_{1T}(z,-z\boldsymbol{k}_T)\, i\sigma_{\mu\nu}\gamma_5\, S_{hT}^\mu n_-^\nu - H_{1s}^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\, k_T^\mu n_-^\nu}{M_h} + H_1^\perp(z,-z\boldsymbol{k}_T)\,\frac{\sigma_{\mu\nu}k_T^\mu n_-^\nu}{M_h}\Bigg\}\\
&+\frac{M_h}{4P_h^-}\Bigg\{E(z,-z\boldsymbol{k}_T) + D^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not k_T}{M_h} + D_T(z,-z\boldsymbol{k}_T)\,\epsilon_{\mu\nu\rho\sigma}n_+^\mu n_-^\nu\gamma^\rho S_{hT}^\sigma\\
&\quad +\lambda_h\, D_L^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}n_+^\mu n_-^\nu\gamma^\rho k_T^\sigma}{M_h} - E_s(z,-z\boldsymbol{k}_T)\, i\gamma_5\\
&\quad -G_T'(z,-z\boldsymbol{k}_T)\,\not S_{hT}\gamma_5 - G_s^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not k_T\gamma_5}{M_h} - H_T^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\, S_{hT}^\mu k_T^\nu}{M_h}\\
&\quad -H_s(z,-z\boldsymbol{k}_T)\, i\sigma_{\mu\nu}\gamma_5\, n_-^\mu n_+^\nu + H(z,-z\boldsymbol{k}_T)\,\sigma_{\mu\nu}n_-^\mu n_+^\nu\Bigg\}. \qquad (2.155)
\end{aligned}
$$

The integrated results are

$$
\boxed{
\begin{aligned}
&\frac{z}{4}\int dk^+\, d^2\boldsymbol{k}_T\;\Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z} = \frac{1}{4}\Bigg\{D_1(z)\,\not n_- - \lambda_h\, G_1(z)\,\not n_-\gamma_5 + H_1(z)\,\frac{[\not S_{hT},\not n_-]\gamma_5}{2}\Bigg\}\\
&\qquad +\frac{M_h}{4P_h^-}\Bigg\{D_T(z)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{hT\sigma} + E(z) - \lambda_h\, E_L(z)\, i\gamma_5\\
&\qquad\qquad - G_T(z)\,\not S_{hT}\gamma_5 + \lambda_h\, H_L(z)\,\frac{[\not n_-,\not n_+]\gamma_5}{2} + i\,H(z)\,\frac{[\not n_-,\not n_+]}{2}\Bigg\}, \qquad (2.156)\\
&\frac{z}{4}\int dk^+\, d^2\boldsymbol{k}_T\;\frac{k_T^\alpha}{M_h}\,\Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z} = \frac{1}{4}\Bigg\{-G_{1T}^{(1)}(z)\, S_{hT}^\alpha\,\not n_-\gamma_5 - \lambda_h\, H_{1L}^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not n_-]\gamma_5}{2}\Bigg\}\\
&\qquad +\frac{1}{4}\Bigg\{D_{1T}^{(1)}(z)\,\epsilon^{\mu\nu\rho\alpha}\gamma_\mu n_{-\nu}S_{hT\rho} + \lambda_h\, H_1^{\perp(1)}(z)\,\frac{i[\not n_-,\gamma^\alpha]}{2}\Bigg\}\\
&\qquad +\frac{M_h}{4P_h^-}\Bigg\{-D^{\perp(1)}(z)\,\gamma^\alpha + \lambda_h\, G_L^{\perp(1)}(z)\,\gamma^\alpha\gamma_5 + H_T^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not S_{hT}]\gamma_5}{2}\\
&\qquad\qquad + H_T^{(1)}(z)\, S_{hT}^\alpha\,\frac{[\not n_-,\not n_+]\gamma_5}{2}\Bigg\}, \qquad (2.157)\\
&\frac{z}{4}\int dk^+\, d^2\boldsymbol{k}_T\;\frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\, g_T^{\alpha\beta}\right)}{M_h^2}\,\Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z} =\\
&\qquad -\frac{1}{4}\,H_{1T}^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}[\gamma^{\beta\}},\not n_-]\gamma_5 - g_T^{\alpha\beta}\,[\not S_{hT},\not n_-]\gamma_5}{4}\right)\\
&\qquad +\frac{M_h}{4P_h^-}\,G_T^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not S_{hT}\gamma_5}{2}\right). \qquad (2.158)
\end{aligned}
}
$$

The appropriate normalization of the fragmentation functions can be obtained via a momentum sum rule. For this consider the following integral for the fragmentation function D_1,

$$
\begin{aligned}
\int dz\; z\, D_1(z) &= \int dz\, d^2P_{h\perp}\; z\, D_1(z,\boldsymbol{P}_{h\perp})\\
&= \int dz\, d^2P_{h\perp}\int\frac{d\xi^+ d^2\boldsymbol{\xi}_T}{4\,(2\pi)^3}\; e^{ik\cdot\xi}\; Tr\langle 0|\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} \qquad (2.159)
\end{aligned}
$$

(we have omitted the links along the n_+-direction, but including them all subsequent arguments remain valid). Considering the integrand for fixed quark momentum k and choosing this to have no perpendicular component, the $P_{h\perp}$ dependence is only contained in $a_h^\dagger a_h$ and summing over all produced hadrons (for a given quark) one obtains

$$\sum_h \int dz\ z\,D_1(z) = \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\ \mathrm{Tr}\langle 0|\psi(\xi)\sum_h \int \frac{dP_h^-\, d^2P_{h\perp}}{(2\pi)^3\, 2P_h^-}\, a_h^\dagger P_h^- a_h \overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0}$$
$$= \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\ \mathrm{Tr}\langle 0|\psi(\xi)P^-\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} \qquad (2.160)$$

Inserting a complete set of quark states one obtains

$$\sum_h \int dz\ z\,D_1(z) = \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\ \mathrm{Tr}\langle 0|\psi(\xi)\int \frac{dk'^-\, d^2k'_T}{(2\pi)^3\, 2k'^-}\sum_{s'}|k',s'\rangle k'^-\langle k',s'|\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0}$$
$$= \int \frac{dk'^-\, d^2k'_T}{(2\pi)^3}\int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{4k^-}\ e^{i(k-k')\cdot\xi}\sum_{s'}\bar{u}(k',s')\gamma^- u(k',s')\Big|_{\xi^-=0} = \frac{1}{2}, \qquad (2.161)$$

and hence

$$\sum_h \sum_{S_h}\int dz\ z\,D_1(z) = 1. \qquad (2.162)$$

2.5 Antiquark fragmentation functions

For the fragmentation of antiquarks the profile functions are obtained from

$$\overline{\Delta}_{ij}(k,P_h,S_h) = \sum_X \frac{1}{(2\pi)^4}\int d^4\xi\ e^{-ik\cdot\xi}\ \langle 0|\overline{\psi}_j(0)|P_h,X\rangle\langle P_h,X|\mathcal{U}(0,\xi)\psi_i(\xi)|0\rangle, \qquad (2.163)$$

considering the projections

$$\overline{\Delta}^{[\Gamma]}(z,\boldsymbol{k}_T) = \frac{1}{4z}\int dk^-\ Tr(\overline{\Delta}\,\Gamma)\Big|_{k^+=P_h^+/z,\ \boldsymbol{k}_T}$$
$$= \pm\int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{4z\,(2\pi)^3}\ e^{-ik\cdot\xi}\ \langle 0|\overline{\psi}(0)\Gamma\, a_h^\dagger a_h\,\mathcal{U}(0,\xi)\psi(\xi)|0\rangle\Big|_{\xi^+=0}, \qquad (2.164)$$

and they are denoted $\overline{D}(z,-z\boldsymbol{k}_T)$, etc. The functions $\overline{E}$ and $\overline{G}$ acquire a sign compared with the quark profile functions. For the antiquark fragmentation functions we have

$$\frac{1}{4z}\int dk^-\ \overline{\Delta}(k,P_h,S_h)\Big|_{k^+=P_h^+/z,\ k_T} =$$
$$\frac{1}{4}\Bigg\{\overline{D}_1(z,-z\boldsymbol{k}_T)\,\not{n}_+ + \overline{D}_{1T}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_-^\nu k_{T\rho}S_{hT\sigma}}{M_h} + \overline{G}_{1s}(z,-z\boldsymbol{k}_T)\,\not{n}_+\gamma_5$$
$$-\overline{H}_{1T}(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu n_+^\nu - \overline{H}_{1s}^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,k_T^\mu n_+^\nu}{M_h} + \overline{H}_1^\perp(z,-z\boldsymbol{k}_T)\,\frac{\sigma_{\mu\nu}k_T^\mu n_+^\nu}{M_h}\Bigg\}$$
$$+\frac{M_h}{4P_h^+}\Bigg\{-\overline{E}(z,-z\boldsymbol{k}_T) + \overline{D}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not{k}_T}{M_h} + \overline{D}_T(z,-z\boldsymbol{k}_T)\,\epsilon_{\mu\nu\rho\sigma}n_-^\mu n_+^\nu\gamma^\rho S_{hT}^\sigma$$
$$+\lambda_h\,\overline{D}_L^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho k_{T\sigma}}{M_h} + \overline{E}_s(z,-z\boldsymbol{k}_T)\,i\gamma_5$$
$$+\overline{G}_T'(z,-z\boldsymbol{k}_T)\,\not{S}_{hT}\gamma_5 + \overline{G}_s^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not{k}_T\gamma_5}{M_h} - \overline{H}_T^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu k_T^\nu}{M_h}$$
$$-\overline{H}_s(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,n_+^\mu n_-^\nu + \overline{H}(z,-z\boldsymbol{k}_T)\,\sigma_{\mu\nu}n_+^\mu n_-^\nu\Bigg\}. \qquad (2.165)$$

The symmetry relations in z are $\overline{D}_1(z,\boldsymbol{k}_T'^2) = D_1(-z,\boldsymbol{k}_T'^2)$ and identically for D_T, G_{1T}, H_{1T}, $H_{1T}^\perp$, E_T, $G_L^\perp$, H and H_L, while $\overline{G}_{1L}(z,\boldsymbol{k}_T'^2) = -G_{1L}(-z,\boldsymbol{k}_T'^2)$ and identically for $D_{1T}^\perp$, $H_{1L}^\perp$, $H_1^\perp$, E, $D^\perp$, $D_L^\perp$, E_L, G_T', $G_T^\perp$, $H_T^\perp$ and H_T. The integrated results are

$$
\frac{z}{4}\int dk^-\, d^2\boldsymbol{k}_T\; \overline{\Delta}(k,P_h,S_h)\Big|_{k^+=P_h^+/z} = \frac{1}{4}\left\{\overline{D}_1(z)\,\not n_+ + \lambda_h\,\overline{G}_1(z)\,\not n_+\gamma_5 + \overline{H}_1(z)\,\frac{[\not S_{hT},\not n_+]\gamma_5}{2}\right\}
$$
$$
+\frac{M_h}{4P_h^+}\left\{-\overline{D}_T(z)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{hT\sigma} - \overline{E}(z) + \lambda_h\,\overline{E}_L(z)\,i\gamma_5 + \overline{G}_T(z)\,\not S_{hT}\gamma_5 + \lambda_h\,\overline{H}_L(z)\,\frac{[\not n_+,\not n_-]\gamma_5}{2} + i\,\overline{H}(z)\,\frac{[\not n_+,\not n_-]}{2}\right\}, \tag{2.166}
$$

$$
\frac{z}{4}\int dk^-\, d^2\boldsymbol{k}_T\; \frac{k_T^\alpha}{M_h}\,\overline{\Delta}(k,P_h,S_h)\Big|_{k^+=P_h^+/z} = \frac{1}{4}\left\{\overline{G}_{1T}^{(1)}(z)\,S_{hT}^\alpha\,\not n_+\gamma_5 - \lambda_h\,\overline{H}_{1L}^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not n_+]\gamma_5}{2}\right\}
$$
$$
+\frac{M_h}{4P_h^+}\left\{-\overline{D}^{\perp(1)}(z)\,\gamma^\alpha - \lambda_h\,\overline{G}_L^{\perp(1)}(z)\,\gamma^\alpha\gamma_5 + \overline{H}_T^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not S_{hT}]\gamma_5}{2} + \overline{H}_T^{(1)}(z)\,S_{hT}^\alpha\,\frac{[\not n_+,\not n_-]\gamma_5}{2}\right\}, \tag{2.167}
$$

$$
\frac{z}{4}\int dk^-\, d^2\boldsymbol{k}_T\; \frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)}{M_h^2}\,\overline{\Delta}(k,P_h,S_h)\Big|_{k^+=P_h^+/z} =
$$
$$
-\frac{1}{4}\,\overline{H}_{1T}^{\perp(2)}(z)\left(\frac{S_{hT}^{\{\alpha}[\gamma^{\beta\}},\not n_+]\gamma_5 - g_T^{\alpha\beta}\,[\not S_{hT},\not n_+]\gamma_5}{4}\right)
-\frac{M_h}{4P_h^+}\,\overline{G}_T^{\perp(2)}(z)\left(\frac{S_{hT}^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not S_{hT}\gamma_5}{2}\right). \tag{2.168}
$$

2.6 The parton picture for fragmentation functions

Similar as in the case of distribution functions, it is instructive to consider the correlation function for the case of a free quark, given by

$$
\delta_{ij}(p,s;k) = u_i(k,s)\overline{u}_j(k,s)\delta^4(k-p) = \tfrac{1}{2}\big((\not k + m)(1+\gamma_5\not s)\big)_{ij}\delta^4(k-p), \tag{2.169}
$$

where the momentum and spin of the quark are parametrized as

$$
k = \left[k^-,\ \frac{\boldsymbol{k}_T^2+m^2}{2k^-},\ \boldsymbol{k}_T\right], \tag{2.170}
$$
$$
s = \left[\frac{\lambda_q\,k^-}{m},\ -\frac{m\,\lambda_q}{2k^-} + \frac{\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^-} + \frac{\lambda_q\,\boldsymbol{k}_T^2}{2m\,k^-},\ \boldsymbol{s}_{qT} + \frac{\lambda_q}{m}\boldsymbol{k}_T\right] \tag{2.171}
$$

in terms of a quark lightcone helicity λ_q and a quark lightcone transverse polarization s_{qT}. The projections become for twist two

$$
\delta^{[\gamma^-]}(k) = \tfrac{1}{2}\delta(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\lambda/q\uparrow} + \tfrac{1}{2}D_{q\lambda/q\downarrow}, \tag{2.172}
$$
$$
\delta^{[\gamma^-\gamma_5]}(k) = \tfrac{1}{2}\lambda_q\,\delta(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\lambda/q\uparrow} - \tfrac{1}{2}D_{q\lambda/q\downarrow}, \tag{2.173}
$$
$$
\delta^{[i\sigma^{i-}\gamma_5]}(k) = \tfrac{1}{2}\boldsymbol{s}_{qT}^i\,\delta(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\boldsymbol{s}_T/q\rightarrow} - \tfrac{1}{2}D_{q\boldsymbol{s}_T/q\leftarrow}, \tag{2.174}
$$

where $\zeta = p^-/k^-$ and we have given the intuitive interpretation in terms of probabilities for quarks to fragment into quarks with spin characterized by λ and $\boldsymbol{s}_T$. For twist three we get

$$
\delta^{[1]}(k) = \frac{m}{2k^-}\,\delta(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.175}
$$
$$
\delta^{[\gamma^i]}(k) = \frac{\boldsymbol{k}_T^i}{2k^-}\,\delta(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{2.176}
$$

$$\delta^{[\gamma^i\gamma_5]}(k) = \frac{(m\,\boldsymbol{s}_{qT}^i + \lambda_q\,\boldsymbol{k}_T^i)}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T - \boldsymbol{p}_T), \tag{2.177}$$

$$\delta^{[i\sigma^{ij}\gamma_5]}(k) = \frac{\boldsymbol{s}_{qT}^i\boldsymbol{k}_T^j - \boldsymbol{k}_T^i\boldsymbol{s}_{qT}^j}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T - \boldsymbol{p}_T), \tag{2.178}$$

$$\delta^{[i\sigma^{-+}\gamma_5]}(k) = \frac{m\,\lambda_q - \boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T - \boldsymbol{p}_T). \tag{2.179}$$

Inserting the expansion for the free quark field gives

$$\Delta_{ij}(k) = 4z\,\delta(k^2-m^2)\left[\theta(z)u_i^{(\beta)}(k)\mathcal{D}_{\beta\alpha}(k)\bar{u}_j^{(\alpha)}(k) - \theta(-z)v_i^{(\beta)}(-k)\overline{\mathcal{D}}_{\beta\alpha}(-k)\bar{v}_j^{(\alpha)}(-k)\right], \tag{2.180}$$

where $z = P_h^-/k^-$. The use of lightcone coordinates is convenient because of the integration over k^+ that is needed in deep inelastic processes. The functions $\mathcal{D}$ and $\overline{\mathcal{D}}$ are given by

$$\mathcal{D}_{\beta\alpha}(k) = \mathcal{D}_{\beta\alpha}(z, z^2\boldsymbol{k}_T^2) \equiv \frac{1}{4z(2\pi)^3}\int \frac{dz'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2z'}\,\langle 0|b_\beta(k')a_h a_h^\dagger b_\alpha^\dagger(k)|0\rangle, \tag{2.181}$$

$$\overline{\mathcal{D}}_{\beta\alpha}(k) = \overline{\mathcal{D}}_{\beta\alpha}(z, z^2\boldsymbol{k}_T^2) \equiv \frac{1}{4z(2\pi)^3}\int \frac{dz'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2z'}\,\langle 0|d_\alpha(k')a_h a_h^\dagger d_\beta^\dagger(k)|0\rangle. \tag{2.182}$$

Note that $\mathcal{D}_{\beta\alpha}(k)$ is a decay matrix in the quark spin-space. Most easily to deal with is the 'momentum sum rule'

$$\begin{aligned}\sum_h\int zdz\,d^2\boldsymbol{P}_{hT}\,\mathcal{D}_{\beta\alpha}(z,\boldsymbol{P}_{hT}^2) &= \frac{1}{2}\int\frac{dz'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2z'}\,\frac{1}{k^-}\,\langle 0|b_\beta(k')\sum_h\int\frac{dP_h^-\,d^2\boldsymbol{P}_{hT}}{(2\pi)^3\,2P_h^-}\,a_h P_h^- a_h^\dagger b_\alpha^\dagger(k)|0\rangle \\ &= \frac{1}{2}\int\frac{dz'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2z'}\,\frac{1}{k^-}\,\langle 0|b_\beta(k')\,\mathcal{P}_{\rm op}^- b_\alpha^\dagger(k)|0\rangle = \frac{1}{2}\,\delta_{\beta\alpha}.\end{aligned} \tag{2.183}$$

Integrating over k^- one obtains the twist two results (for $z > 0$)

$$\Delta^{[\gamma^-]}(k) = \tfrac{1}{2}\mathcal{D}_{RR} + \tfrac{1}{2}\mathcal{D}_{LL} = \tfrac{1}{2}\mathcal{D}_{\alpha\alpha} + \tfrac{1}{2}\mathcal{D}_{\beta\beta}, \tag{2.184}$$

$$\Delta^{[\gamma^-\gamma_5]}(k) = \tfrac{1}{2}\mathcal{D}_{RR} - \tfrac{1}{2}\mathcal{D}_{LL}, \tag{2.185}$$

$$\Delta^{[i\sigma^{i-}\gamma_5]}(k) = \tfrac{1}{2}\mathcal{D}_{\alpha\alpha} - \tfrac{1}{2}\mathcal{D}_{\beta\beta}, \tag{2.186}$$

using the same projection operators as for the distribution functions. In this way an interpretation in terms of quark decay functions is obtained, again rigorous for the twist two functions. In this 'parton picture' the twist three results can also for the fragmentation functions be expressed in the twist two functions.

Summarizing we get for $\Delta^{[\Gamma]}(z, -z\boldsymbol{k}_T)$ and $\Delta^{[\Gamma]}(z)$, ordered according to twist and chirality for production of unpolarized (U), longitudinally polarized (L) and transversely polarized (T) (spin 1/2) hadrons:

FRAGMENTATION			
		chirality	
$\Delta^{[\Gamma]}(z,-z\boldsymbol{k}_T)$		even	odd
twist 2	U	D_1	$H_1^\perp$
	L	G_{1L}	$H_{1L}^\perp$
	T	$G_{1T}\ \ D_{1T}^\perp$	$H_{1T}\ \ H_{1T}^\perp$
twist 3	U	$D^\perp$	$E\ \ H$
	L	$G_L^\perp\ \ D_L^\perp\ \ E_L$	H_L
	T	$G'_T\ \ G_T^\perp\ \ D_T\ \ E_T$	$H_T\ \ H_T^\perp$

FRAGMENTATION			
		chirality	
$\Delta^{[\Gamma]}(z)$		even	odd
twist 2	U	D_1	
	L	G_1	
	T		H_1
twist 3	U		$E\ \ H$
	L	E_L	H_L
	T	$G_T\ \ D_T$	

2.7 Structure of $H \to q$ profile functions

In order to study the structure of the profile functions, we will first give the expressions in terms of the amplitudes $A_i(\sigma,\tau)$, where $\sigma = 2\,k\cdot P$ and $\tau = k^2$,

$$f_1(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_2 + xA_3]\,, \tag{2.187}$$

$$g_{1L}(x,\boldsymbol{k}_T^2) = \int \ldots \left[-A_6 - \left(\frac{\sigma - 2xM^2}{2M^2}\right)(A_7 + xA_8)\right], \tag{2.188}$$

$$g_{1T}(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_7 + xA_8\right], \tag{2.189}$$

$$h_{1T}(x,\boldsymbol{k}_T^2) = \int \ldots \left[-(A_9 + xA_{10})\right], \tag{2.190}$$

$$h_{1L}^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_{10} - \left(\frac{\sigma - 2xM^2}{2M^2}\right) A_{11}\right], \tag{2.191}$$

$$h_{1T}^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_{11}\right], \tag{2.192}$$

$$e(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_1\right], \tag{2.193}$$

$$f^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_3\right], \tag{2.194}$$

$$g_T'(x,\boldsymbol{k}_T^2) = \int \ldots \left[-A_6\right], \tag{2.195}$$

$$g_L^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[-\left(\frac{\sigma - 2xM^2}{2M^2}\right) A_8\right], \tag{2.196}$$

$$g_T^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[A_8\right], \tag{2.197}$$

$$h_T^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots \left[-A_{10}\right], \tag{2.198}$$

$$h_L(x,\boldsymbol{k}_T^2) = \int \ldots \left[-(A_9 + xA_{10}) - \left(\frac{\sigma - 2xM^2}{2M^2}\right) A_{10} + \left(\frac{\sigma - 2xM^2}{2M^2}\right)^2 A_{11}\right], \tag{2.199}$$

$$h_T(x,\boldsymbol{k}_T^2) = \int \ldots \left[-\left(\frac{\sigma - 2xM^2}{2M^2}\right) A_{11}\right], \tag{2.200}$$

where

$$\begin{aligned}\int \ldots &= \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma) \\ &= \int d\sigma d\tau\, \delta(x\eta M^2 + \tau - x\sigma),\end{aligned} \tag{2.201}$$

with

$$\eta = x + \frac{\boldsymbol{k}_T^2}{xM^2}. \tag{2.202}$$

The integrated functions also can be expressed as an integral over amplitudes, e.g.

$$f_1(x) = \pi \int d\sigma d\tau\; \theta(x^2M^2 + \tau - x\sigma)\left[A_2 + xA_3\right], \tag{2.203}$$

now involving a θ-function to ensure contributions from the physical region $\boldsymbol{p}_T^2 \geq 0$, indicated in the figure. Implicitly, there are several relations which can be traced back to the fact that there are less amplitudes than profile functions. Useful are the following relations for $\boldsymbol{k}_T^2$-weighted functions: If we have

$$\begin{aligned}f^{(n)}(x,\boldsymbol{k}_T^2) &= \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma)\left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n F(x,\sigma,\tau), \\ f'^{(n)}(x,\boldsymbol{k}_T^2) &= \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma)\left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n \frac{\partial F}{\partial x}(x,\sigma,\tau), \\ g^{(n)}(x,\boldsymbol{k}_T^2) &= -\int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma)\left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n \frac{\sigma - 2x\,M^2}{2M^2} F(x,\sigma,\tau),\end{aligned} \tag{2.204}$$

one easily proves the relations

$$\frac{d}{dx} f^{(1)}(x,\boldsymbol{k}_T^2) = -g(x,\boldsymbol{k}_T^2) + f'^{(1)}(x,\boldsymbol{k}_T^2) + 2M^2 \frac{d}{d\boldsymbol{k}_T^2} g^{(1)}(x,\boldsymbol{k}_T^2),$$

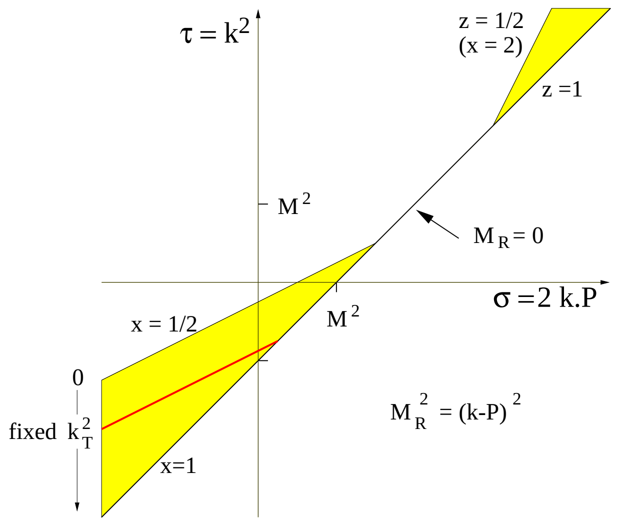


Figure 2.3: The region $\boldsymbol{k}_T^2 \geq 0$ which contributes in the integration over σ and τ in the expressions for the distribution functions and fragmentation functions

$$\frac{d}{dx}\, f^{(1)}(x) = -g(x) + f'^{(1)}(x) + 2M^2\, g^{(1)}(x,0), \tag{2.205}$$
$$\frac{d}{dx}\, f^{(2)}(x,\boldsymbol{k}_T^2) = -2g^{(1)}(x,\boldsymbol{k}_T^2) + f'^{(2)}(x,\boldsymbol{k}_T^2) + 2M^2\,\frac{d}{d\boldsymbol{k}_T^2}\, g^{(2)}(x,\boldsymbol{k}_T^2),$$
$$\frac{d}{dx}\, f^{(2)}(x) = -2g^{(1)}(x) + f'^{(2)}(x) + 2M^2\, g^{(2)}(x,0). \tag{2.206}$$

For the five g-functions this leads to the two relations (assuming $g^{(n\geq 1)}(x,0)$ to vanish)

$$g_T(x) = g_1(x) + \frac{d}{dx}\, g_{1T}^{(1)}, \tag{2.207}$$
$$g_L^\perp(x) = -\frac{d}{dx}\, g_T^{\perp(1)} \qquad \text{and} \qquad 2\, g_L^{\perp(1)}(x) = -\frac{d}{dx}\, g_T^{\perp(2)}, \tag{2.208}$$

while for the six h-functions three relations are found, where the one for $h_T^\perp$ is trivial,

$$h_L(x) = h_1(x) - \frac{d}{dx}\, h_{1L}^{\perp(1)}, \tag{2.209}$$
$$h_T(x) = -\frac{d}{dx}\, h_{1T}^{\perp(1)} \qquad \text{and} \qquad 2\, h_T^{(1)}(x) = -\frac{d}{dx}\, h_{1T}^{\perp(2)}, \tag{2.210}$$
$$h_T^{\perp(n)}(x) = h_T^{(n)}(x) - h_{1L}^{\perp(n)}(x). \tag{2.211}$$

Similarly we have for the fragmentation functions expressions of the profile functions in terms of the amplitudes B_1 to B_{12}. This gives relations analogous to the distribution functions,

$$2z\, D_1(z, z^2\boldsymbol{k}_T^2) = \int d\sigma d\tau\ \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right)\left[B_2 + \frac{1}{z}\, B_3\right], \tag{2.212}$$

etc. with now in addition some *time reversal odd* functions,

$$2z\, D_{1T}^\perp(z, -z\boldsymbol{k}_T) = \int \ldots\ [B_{12}]\,, \tag{2.213}$$
$$2z\, H_1^\perp(z, -z\boldsymbol{k}_T) = \int \ldots\ [-B_4]\,, \tag{2.214}$$
$$2z\, D_L^\perp(z, -z\boldsymbol{k}_T) = \int \ldots\ [-B_{12}]\,, \tag{2.215}$$

$$2z\, D_T(z,-z\boldsymbol{k}_T) = \int \ldots \left[-\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right) B_{12}\right]. \tag{2.216}$$

$$2z\, E_L(z,-z\boldsymbol{k}_T) = \int \ldots \left[-\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right) B_5\right]. \tag{2.217}$$

$$2z\, E_T(z,-z\boldsymbol{k}_T) = \int \ldots \left[B_5\right], \tag{2.218}$$

$$2z\, H(z,-z\boldsymbol{k}_T) = \int \ldots \left[\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right) B_4\right]. \tag{2.219}$$

The integrated functions are given by

$$\frac{2}{z}\, D_1(z) = \pi \int d\sigma d\tau\; \theta\left(\frac{\sigma}{z} - \tau - \frac{M_h^2}{z^2}\right) \left[B_2 + \frac{1}{z}\, B_3\right], \tag{2.220}$$

with an integration region also indicated in the figure. The following relations can be derived in the case of fragmentation functions: If we have

$$\begin{aligned}
2z\, D^{(n)}(z,z^2\boldsymbol{k}_T^2) &= \int d\sigma d\tau\, \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right) \left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n F(z,\sigma,\tau), \\
2z\, D'^{(n)}(z,z^2\boldsymbol{k}_T^2) &= \int d\sigma d\tau\, \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right) \left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n \frac{\partial F}{\partial z}(z,\sigma,\tau), \\
2z\, G^{(n)}(z,z^2\boldsymbol{k}_T^2) &= -\int d\sigma d\tau\, \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right) \left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n \frac{\sigma - 2M_h^2/z}{2M_h^2}\, F(z,\sigma,\tau),
\end{aligned} \tag{2.221}$$

one easily proves the relations

$$\begin{aligned}
z^2 \frac{d}{dz}\left[z\, D^{(1)}(z,z^2\boldsymbol{k}_T^2)\right] &= z\, G(z,z^2\boldsymbol{k}_T^2) + z^3\, D'^{(1)}(z,z^2\boldsymbol{k}_T^2) + 2M_h^2\, \frac{d}{d\boldsymbol{k}_T^2}\left[z\, G^{(1)}(z,z^2\boldsymbol{k}_T^2)\right], \\
z^2 \frac{d}{dz}\left[\frac{D^{(1)}(z)}{z}\right] &= \frac{G(z)}{z} + z\, D'^{(1)}(z) + 2M_h^2\, \frac{G^{(1)}(z,0)}{z},
\end{aligned} \tag{2.222}$$

$$\begin{aligned}
z^2 \frac{d}{dz}\left[z\, D^{(2)}(z,z^2\boldsymbol{k}_T^2)\right] &= 2z\, G^{(1)}(z,z^2\boldsymbol{k}_T^2) + z^3\, D'^{(2)}(z,z^2\boldsymbol{k}_T^2) + 2M_h^2\, \frac{d}{d\boldsymbol{k}_T^2}\left[z\, G^{(2)}(z,z^2\boldsymbol{k}_T^2)\right], \\
z^2 \frac{d}{dz}\left[\frac{D^{(2)}(z)}{z}\right] &= 2\, \frac{G^{(1)}(z)}{z} + z\, D'^{(1)}(z) + 2M_h^2\, \frac{G^{(1)}(z,0)}{z}.
\end{aligned} \tag{2.223}$$

For the five G-functions this leads to two relations (assuming $G^{(n\geq 1)}(z,0)$ to vanish,

$$\frac{G_T(z)}{z} = \frac{G_1(z)}{z} - z^2 \frac{d}{dz}\left[\frac{G_{1T}^{(1)}(z)}{z}\right], \tag{2.224}$$

$$\frac{G_L^{\perp}(z)}{z} = z^2 \frac{d}{dz}\left[\frac{G_T^{\perp(1)}(z)}{z}\right] \quad \text{and} \quad 2\, \frac{G_L^{\perp(1)}(z)}{z} = z^2 \frac{d}{dz}\left[\frac{G_T^{\perp(2)}(z)}{z}\right]. \tag{2.225}$$

For the eight H-functions this leads to 4 relations,

$$\frac{H_L(z)}{z} = \frac{H_1(z)}{z} + z^2 \frac{d}{dz}\left[\frac{H_{1L}^{\perp(1)}(z)}{z}\right], \tag{2.226}$$

$$\frac{H_T(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_{1T}^{\perp(1)}(z)}{z}\right] \quad \text{and} \quad 2\, \frac{H_T^{(1)}(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_{1T}^{\perp(2)}(z)}{z}\right], \tag{2.227}$$

$$H_T^{\perp(n)}(z) = H_T^{(n)}(z) - H_{1L}^{\perp(n)}(z), \tag{2.228}$$

$$\frac{H(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_1^{\perp(1)}(z)}{z}\right]. \tag{2.229}$$

The last equation relates time reversal odd functions. Other relations between time reversal odd functions are,

$$\frac{E_L(z)}{z} = z^2 \frac{d}{dz}\left[\frac{E_T^{(1)}(z)}{z}\right], \tag{2.230}$$

and

$$\frac{D_T(z)}{z} = z^2 \frac{d}{dz}\left[\frac{D_{1T}^{\perp(1)}(z)}{z}\right], \tag{2.231}$$

$$D_L^{\perp(n)}(z) = -D_{1T}^{\perp(n)}(z). \tag{2.232}$$

Using the splitting of the twist three profile functions in a piece that is expressed in terms of partonic (twist two) functions and a remainder ('true' twist three piece) that as we will see in the next chapter can be expressed in terms of 'good' quark and gluon fields. This is achieved by using the 'free' quark results or equivalently the results for quark-quark-gluon correlation functions using the QCD equations of motion,

$$e(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,f_1(x,\boldsymbol{k}_T) + \tilde{e}(x,\boldsymbol{k}_T), \tag{2.233}$$

$$f^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,f_1(x,\boldsymbol{k}_T) + \tilde{f}^\perp(x,\boldsymbol{k}_T), \tag{2.234}$$

$$g_T'(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,h_{1T}(x,\boldsymbol{k}_T) + \tilde{g}_T'(x,\boldsymbol{k}_T), \tag{2.235}$$

$$g_L^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,g_{1L}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\,h_{1L}^\perp(x,\boldsymbol{k}_T) + \tilde{g}_L^\perp(x,\boldsymbol{k}_T), \tag{2.236}$$

$$g_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,g_{1T}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\,h_{1T}^\perp(x,\boldsymbol{k}_T) + \tilde{g}_T^\perp(x,\boldsymbol{k}_T), \tag{2.237}$$

$$g_T(x,\boldsymbol{k}_T) = \frac{1}{x}\,g_{1T}^{(1)}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\,h_1(x,\boldsymbol{k}_T) + \tilde{g}_T(x,\boldsymbol{k}_T), \tag{2.238}$$

$$h_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,h_{1T}(x,\boldsymbol{k}_T) + \tilde{h}_T^\perp(x,\boldsymbol{k}_T), \tag{2.239}$$

$$h_L(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,g_{1L}(x,\boldsymbol{k}_T) - \frac{\boldsymbol{k}_T^2}{M^2x}\,h_{1L}^\perp + \tilde{h}_L(x,\boldsymbol{k}_T), \tag{2.240}$$

$$h_T(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,g_{1T}(x,\boldsymbol{k}_T) - \frac{h_{1T}(x,\boldsymbol{k}_T)}{x} - \frac{\boldsymbol{k}_T^2}{M^2x}\,h_{1T}^\perp + \tilde{h}_T(x,\boldsymbol{k}_T). \tag{2.241}$$

The functions $\tilde{e}$, etc. can be immediately seen as the functions appearing in the twist three projections of the quark-quark-gluon correlation functions.

Comparing the above expression with the second relations found for the g-functions one has

$$g_T = \frac{g_{1T}^{(1)}}{x} + \frac{m}{M}\,\frac{h_1}{x} + \tilde{g}_T = g_1 + \frac{d}{dx}\,g_{1T}^{(1)}, \tag{2.242}$$

from which one obtains the relation

$$x^2\,\frac{d}{dx}\left(\frac{g_{1T}^{(1)}}{x}\right) = -x\,g_1 + \frac{m}{M}\,h_1 + x\,\tilde{g}_T, \tag{2.243}$$

that can be used to eliminate $g_{1T}^{(1)}$,

$$\frac{d}{dx}\,(g_T - g_1) = \frac{d}{dx}\,g_2 = -\frac{1}{x}\,\frac{d}{dx}\,(xg_1) + \frac{m}{M}\,\frac{1}{x}\,\frac{d}{dx}\,h_1 + \frac{1}{x}\,\frac{d}{dx}\,(x\tilde{g}_T), \tag{2.244}$$

or (provided sufficient convergent behavior at the endpoints)

$$g_2(x) = -\left[g_1(x) - \int_x^1 dy\,\frac{g_1(y)}{y}\right] + \frac{m}{M}\left[\frac{h_1(x)}{x} - \int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \left[\tilde{g}_T(x) - \int_x^1 dy\,\frac{\tilde{g}_T(y)}{y}\right]. \tag{2.245}$$

Similarly we obtain from the expression for h_L above and the second of the h-relations

$$h_L = \frac{m}{M}\,\frac{g_1}{x} - 2\,\frac{h_{1L}^{\perp(1)}}{x} + \tilde{h}_L = h_1 - \frac{d}{dx}\,h_{1L}^{\perp(1)}, \tag{2.246}$$

from which one obtains the relation

$$x^3 \frac{d}{dx}\left(\frac{h_{1L}^{\perp(1)}}{x^2}\right) = x\,h_1 - \frac{m}{M}\,g_1 - x\,\tilde{h}_L, \tag{2.247}$$

which can be used to eliminate $h_{1L}^{\perp(1)}$,

$$\frac{d}{dx}\left(\frac{h_L - h_1}{x}\right) = \frac{d}{dx}\left(\frac{h_2}{2x}\right) = -\frac{1}{x^2}\,\frac{d}{dx}\,(xh_1) + \frac{m}{M}\,\frac{1}{x^2}\,\frac{d}{dx}\,g_1 + \frac{1}{x^2}\,\frac{d}{dx}\,(x\tilde{h}_L), \tag{2.248}$$

or (provided sufficient convergent behavior at the endpoints)

$$\begin{aligned} \frac{1}{2}\,h_2(x) \quad &= \quad -\left[h_1(x) - 2x\int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \frac{m}{M}\left[\frac{g_1(x)}{x} - 2x\int_x^1 dy\,\frac{g_1(y)}{y^3}\right] \\ &\qquad + \left[\tilde{h}_L(x) - 2x\int_x^1 dy\,\frac{\tilde{h}_L(y)}{y^2}\right]. \end{aligned} \tag{2.249}$$

Continuing with the twist-three functions we have

$$h_T = -\frac{h_1}{x} - \frac{h_{1T}^{\perp(1)}}{x} + \frac{m}{M}\,\frac{g_{1T}}{x} + \tilde{h}_T = -\frac{d}{dx}\,h_{1T}^{\perp(1)}, \tag{2.250}$$

from which one obtains the relation

$$x^2\,\frac{d}{dx}\left(\frac{h_{1T}^{\perp(1)}}{x}\right) = h_1 - \frac{m}{M}\,g_{1T} + x\,\tilde{h}_T, \tag{2.251}$$

which can be used to eliminate $h_{1T}^{\perp(1)}$,

$$\frac{d}{dx}\,h_T = -\frac{1}{x}\,\frac{d}{dx}\,h_1 + \frac{m}{M}\,\frac{1}{x}\,\frac{d}{dx}\,g_{1T} + \frac{1}{x}\,\frac{d}{dx}\,(x\tilde{h}_T), \tag{2.252}$$

or (provided sufficient convergent behavior at the endpoints)

$$h_T(x) = -\left[\frac{h_1(x)}{x} - \int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \frac{m}{M}\left[\frac{g_{1T}(x)}{x} - \int_x^1 dy\,\frac{g_{1T}(y)}{y^2}\right] + \left[\tilde{h}_T(x) - \int_x^1 dy\,\frac{\tilde{h}_T(y)}{y}\right]. \tag{2.253}$$

For the second $\boldsymbol{k}_T^2$-moment one obtains

$$h_T^{(1)} = -\frac{h_1^{(1)}}{x} - \frac{h_{1T}^{\perp(2)}}{x} + \frac{m}{M}\,\frac{g_{1T}^{(1)}}{x} + \tilde{h}_T^{(1)} = -\frac{1}{2}\,\frac{d}{dx}\,h_{1T}^{\perp(2)}, \tag{2.254}$$

from which one obtains the relation

$$x^2\,\frac{d}{dx}\left(\frac{h_{1T}^{\perp(2)}}{x^2}\right) = 2\,\frac{h_1^{(1)}}{x} - 2\,\frac{m}{M}\,\frac{g_{1T}^{(1)}}{x} - 2\,\tilde{h}_T^{(1)}, \tag{2.255}$$

which can be used to eliminate $h_{1T}^{\perp(2)}$,

$$\frac{d}{dx}\left(\frac{h_T^{(1)}}{x}\right) = -\frac{1}{x^2}\,\frac{d}{dx}\,h_{1T}^{(1)} + \frac{m}{M}\,\frac{1}{x^2}\,\frac{d}{dx}\,g_{1T}^{(1)} + \frac{1}{x^2}\,\frac{d}{dx}\,(x\tilde{h}_T^{(1)}), \tag{2.256}$$

or (provided sufficient convergent behavior at the endpoints)

$$\begin{aligned} h_T^{(1)}(x) \quad &= \quad -\left[\frac{h_1^{(1)}(x)}{x} - 2x\int_x^1 dy\,\frac{h_1^{(1)}(y)}{y^3}\right] + \frac{m}{M}\left[\frac{g_{1T}^{(1)}(x)}{x} - 2x\int_x^1 dy\,\frac{g_{1T}^{(1)}(y)}{y^3}\right] \\ &\qquad + \left[\tilde{h}_T^{(1)}(x) - 2x\int_x^1 dy\,\frac{\tilde{h}_T^{(1)}(y)}{y^2}\right]. \end{aligned} \tag{2.257}$$

It is sometimes useful to realize that the moments of combinations appearing above are

$$\int_0^1 dx\;x^{n-1}\left[f(x) - k\,x^{k-1}\int_x^1 dy\,\frac{f(y)}{y^k}\right] = \frac{n-1}{n+k-1}\int_0^1 dx\;x^{n-1}\,f(x) \qquad (\text{for } n \geq 1,\ k \geq 1). \tag{2.258}$$

Therefore all first moments ($n = 1$) of the expressions between brackets above vanish, at least if the first moments of the function f is finite.

Chapter 3

Deep inelastic processes

3.1 The point cross sections

In section 1 we have discussed the formalism for three types of hard processes, the Drell-Yan process, e^+e^- annihilation and lepton-hadron scattering,

$$\begin{aligned} A+B &\rightarrow \ell+\overline{\ell}+X, \qquad (DY) \\ e^-+e^+ &\rightarrow h_1+h_2+X, \qquad (e^-e^+) \\ \ell+H &\rightarrow \ell'+h+X. \qquad (\ell H) \end{aligned}$$

The underlying processes are:

$$DY: \qquad q+\overline{q} \rightarrow \ell+\overline{\ell}, \tag{3.1}$$

$$e^-e^+: \qquad e^-+e^+ \rightarrow q+\overline{q}, \tag{3.2}$$

$$\ell H: \qquad \ell+q \rightarrow \ell'+q. \tag{3.3}$$

The cross sections for these processes would be quite similar to the (observable) lepton cross sections. The annihilation cross section $e^-e^+ \rightarrow \mu^-\mu^+$, neglecting lepton masses is given by

$$\frac{d\sigma}{dt}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{2\pi\,\alpha^2\,(t^2+u^2)}{s^4}, \tag{3.4}$$

or the equivalent expressions in terms of $y = -t/s = (1+\cos\theta_{cm})/2$,

$$\frac{d\sigma}{d\Omega_{cm}}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{\alpha^2}{4s}\left(1+\cos^2\theta_{cm}\right), \tag{3.5}$$

$$\frac{d\sigma}{dy}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{4\pi\,\alpha^2}{s}\left(\frac{1}{2}-y+y^2\right). \tag{3.6}$$

The total cross section becomes

$$\sigma(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{4\pi\,\alpha^2}{3\,s}. \tag{3.7}$$

The cross section for $e^-\mu^+ \rightarrow e^-\mu^+$, neglecting lepton masses is given by

$$\frac{d\sigma}{dt}(e^-\mu^+ \rightarrow e^-\mu^+) = \frac{2\pi\,\alpha^2\,(s^2+u^2)}{s^2\,t^2}, \tag{3.8}$$

or using $y = -t/s$,

$$\frac{d\sigma}{dy}(e^-\mu^+ \rightarrow e^-\mu^+) = \frac{4\pi\,\alpha^2\,s}{Q^4}\left(\frac{y^2}{2}+1-y\right). \tag{3.9}$$

Also useful is the cross section for a virtual (transverse) photon, given by

$$\sigma(\gamma_T^*\mu^+ \rightarrow \mu^+) = \frac{4\pi^2\alpha}{Q^2}\,\delta\left(\frac{s}{Q^2}-1\right). \tag{3.10}$$

3.2 The Drell-Yan process

The hadronic tensor in this case is rewritten as

$$\begin{aligned}\mathcal{W}_{\mu\nu}(q;P_AS_A;P_BS_B) &= \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(P_A+P_B-P_X-q)\\ &\quad\times\langle P_AS_A;P_BS_B|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|P_AS_A;P_BS_B\rangle,\\ &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A;P_BS_B|J_\mu(x)\;J_\nu(0)|P_AS_A;P_BS_B\rangle,\end{aligned}$$

which in tree approximation (Born terms) becomes

$$\begin{aligned}\mathcal{W}_{\mu\nu} &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A;P_BS_B|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\\ &\qquad\times:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|P_AS_A;P_BS_B\rangle\\ &= \frac{1}{(2\pi)^4}\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A|\overline{\psi}_j(x)\psi_i(0)|P_AS_A\rangle(\gamma_\mu)_{jk}\\ &\qquad\langle P_BS_B|\psi_k(x)\overline{\psi}_l(0)|P_BS_B\rangle(\gamma_\nu)_{li}\\ &\quad+\frac{1}{(2\pi)^4}\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A|\psi_k(x)\overline{\psi}_l(0)|P_AS_A\rangle(\gamma_\nu)_{li}\\ &\qquad\langle P_BS_B|\overline{\psi}_j(x)\psi_i(0)|P_BS_B\rangle(\gamma_\mu)_{jk},\\ &= \frac{1}{3}\int d^4p\,d^4k\,\delta^4(p+k-q)\;\mathrm{Tr}\left(\Phi(p)\gamma_\mu\overline{\Phi}(k)\gamma_\nu\right)+\left\{\begin{array}{c}q\leftrightarrow -q\\ \mu\leftrightarrow\nu\end{array}\right\},\end{aligned}\tag{3.11}$$

where we have used

$$\begin{aligned}\Phi_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ip\cdot x}\,\langle P_AS_A|\overline{\psi}_j(x)\psi_i(0)|P_BS_B\rangle,\\ \overline{\Phi}_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ik\cdot x}\,\langle P_BS_B|\psi_k(x)\overline{\psi}_l(0)|P_BS_B\rangle,\end{aligned}$$

and its symmetry properties (see section 2). Note that since in both Φ (quark production) and $\overline{\Phi}$ (antiquark production) summations over colors are assumed, a factor 1/3 appears in the result in Eq. 3.11

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements Φ are limited, i.e. p^2, $p\cdot P_A$ are of hadronic scale and similarly in the matrix elements $\overline{\Phi}$ for k^2 and $k\cdot P_B$, one can write the delta function up to $\mathcal{O}(1/Q^2)$ as

$$\delta^4(p+k-q)\approx\delta(p^+-q^+)\,\delta(k^--q^-)\,\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T),\tag{3.12}$$

The result in leading order is then

$$\mathcal{W}_{\mu\nu} = \frac{1}{3}\int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\mathrm{Tr}\left([\textstyle\int dp^-\Phi(p)]\gamma_\mu[\int dk^+\overline{\Phi}(k)]\gamma_\nu\right)\Big|_{\substack{p^+=x_AP_A^+\\ k^-=x_BP_B^-}}$$

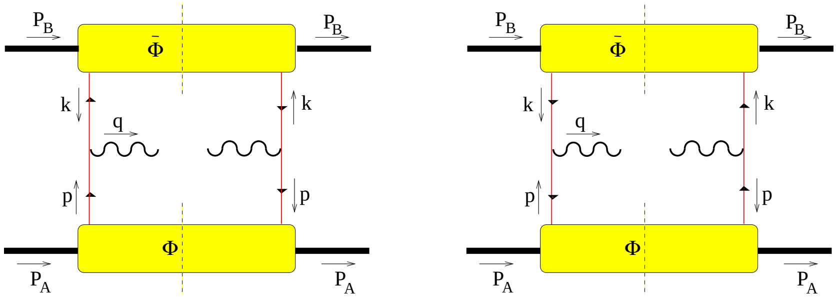


Figure 3.1: Born diagrams for Drell-Yan scattering

$$= \quad \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{Z_\mu Z_\nu}{Z^2}\right) \frac{1}{3}\, I[f_1\,\overline{f}_1], \tag{3.13}$$

where

$$I[f_1\,\overline{f}_1] = \int d^2p_T\, d^2k_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\, f_1(x_A, \boldsymbol{p}_T)\overline{f}_1(x_B, \boldsymbol{k}_T). \tag{3.14}$$

Using the contraction with the leptonic tensor,

$$\begin{aligned} L^{\mu\nu}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{Z_\mu Z_\nu}{Z^2}\right) &= 4Q^2\left(\frac{1}{2} - y + y^2\right) \\ &= Q^2\left(1 + \cos^2\theta_{\mu\mu}\right), \end{aligned} \tag{3.15}$$

the cross section becomes

$$\frac{d\sigma(AB \to \mu^+\mu^- X)}{dx_A\, dx_B\, d^2\boldsymbol{q}_T\, d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\left(1 + \cos^2\theta_{\mu\mu}\right)\, I[f_1\,\overline{f}_1]. \tag{3.16}$$

Integrated over the transverse momenta of the produced mu pair,

$$\frac{d\sigma(AB \to \mu^+\mu^- X)}{dx_A\, dx_B\, d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\left(1 + \cos^2\theta_{\mu\mu}\right)\, f_1(x_A)\,\overline{f}_1(x_B), \tag{3.17}$$

and integrated over the muon angular distribution,

$$\frac{d\sigma}{dx_A\, dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\, f_1(x_A)\,\overline{f}_1(x_B), \tag{3.18}$$

or including the summation over quarks and antiquarks,

$$\begin{aligned} \frac{d\sigma(AB \to \mu^+\mu^- X)}{dx_A\, dx_B} &= \frac{4\pi\,\alpha^2}{9\,Q^2}\sum_a e_a^2\, f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B) \\ &= \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\hat{\sigma}(a\overline{a} \to \mu^-\mu^+), \end{aligned} \tag{3.19}$$

where the quark-antiquark annihilation cross section is given by

$$\hat{\sigma}(a\overline{a} \to \mu^-\mu^+) = \frac{4\pi\alpha^2}{3Q^2}\, e_a^2. \tag{3.20}$$

and the factor 1/3 multiplying the summation is the color factor that can be naturally understood because only quarks of the same color can annihilate and we have seen that the definitions of the quark distribution functions included a summation over colors.

Introducing the virtuality of the photon (i.e. the invariant mass $\hat{s}$ of the produced mu pair) as a variable one can consider the Drell-Yan cross section as a function of s. Writing

$$\frac{d\hat{\sigma}}{dQ^2}\,(a\overline{a} \to \mu^-\mu^+) = \frac{4\pi\,\alpha^2}{3\,Q^2}\, e_a^2\, \delta(\hat{s} - Q^2), \tag{3.21}$$

one has

$$\begin{aligned} \frac{d\sigma(AB \to \mu^-\mu^+ X)}{dQ^2\, dx_A\, dx_B} &= \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\frac{d\hat{\sigma}}{dQ^2}(a\overline{a} \to \mu^-\mu^+) \\ &= \frac{4\pi\alpha^2}{9Q^4}\sum_a e_a^2\, f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\delta\left(\frac{s}{Q^2}\,x_A x_B - 1\right), \end{aligned} \tag{3.22}$$

which exhibits explicitly the scaling in $\tau = s/Q^2$ for the cross section $Q^4\,d\sigma/dQ^2$.

3.3 Electron-positron annihilation

The hadronic tensor in this case is rewritten as

$$\begin{aligned}\mathcal{W}_{\mu\nu}(q;P_1S_1;P_2S_2) &= \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q-P_X-P_1-P_2)\\ &\qquad\times\langle 0|J_\mu(0)|P_X;P_1S_1;P_2S_2\rangle\langle P_X;P_1S_1;P_2S_2|J_\nu(0)|0\rangle,\\ &= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle 0|J_\mu(x)\sum_X|X;P_1S_1;P_2S_2\rangle\langle X;P_1S_1;P_2S_2|J_\nu(0)|0\rangle,\end{aligned}$$

which in tree approximation (Born terms) becomes

$$\begin{aligned}\mathcal{W}_{\mu\nu} &= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle 0|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X|X;P_1S_1;P_2S_2\rangle\\ &\qquad\times\langle X;P_1S_1;P_2S_2|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|0\rangle\\ &= \frac{1}{(2\pi)^4}\,3\int d^4x\; e^{iq\cdot x}\,\langle 0|\overline{\psi}_j(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\psi_i(0)|PS\rangle(\gamma_\mu)_{jk}\\ &\qquad\langle 0|\psi_k(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li}\\ &\quad+\frac{1}{(2\pi)^4}\,3\int d^4x\; e^{iq\cdot x}\,\langle 0|\psi_k(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li}\\ &\qquad\langle 0|\overline{\psi}_j(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\psi_i(0)|0\rangle(\gamma_\mu)_{jk},\\ &= 3\int d^4p\,d^4k\,\delta^4(q-k-p)\;\mathrm{Tr}\left(\overline{\Delta}(p)\gamma_\mu\Delta(k)\gamma_\nu\right)+\left\{\begin{array}{c}q\leftrightarrow -q\\ \mu\leftrightarrow\nu\end{array}\right\},\end{aligned}\tag{3.23}$$

where we have used

$$\begin{aligned}\Delta_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{ik\cdot x}\,\langle 0|\psi_k(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\overline{\psi}_l(0)|0\rangle,\\ \overline{\Delta}_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{ip\cdot x}\,\langle PS|\overline{\psi}_j(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\psi_i(0)|PS\rangle,\end{aligned}$$

and its symmetry properties (see section 2). Note that since both in Δ (quark decay) and $\overline{\Delta}$ (antiquark decay) an averaging over colors is assumed, we get a color factor of 3 in Eq. 3.23. We have only considered two hadrons in different jets, i.e. no fragmentation parts involving matrix elements of the form $\langle 0|\psi_j(x)\sum_X|X;P_1S_1;P_2S_2\rangle\;\langle X;P_1S_1;P_2S_2|\psi_i(0)|0\rangle$ are considered.

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements $\overline{\Delta}$ are limited, i.e. p^2, $p\cdot P_2$ are of hadronic scale and similarly in the matrix elements Δ for k^2 and $k\cdot P_1$, one can write the delta function up to $\mathcal{O}(1/Q^2)$ as

$$\delta^4(q-k-p)\approx\delta(q^--k^-)\,\delta(q^+-p^+)\,\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T),\tag{3.24}$$

The result in leading order is then

$$\begin{aligned}\mathcal{W}_{\mu\nu} &= 3\int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T)\mathrm{Tr}\left([\textstyle\int dp^-\overline{\Delta}(p)]\gamma_\mu[\int dk^+\Delta(k)]\gamma_\nu\right)\Big|_{\begin{subarray}{l}k^-=P_1^-/z_1\\ p^+=P_2^+/z_2\end{subarray}}\\ &= \left(\frac{q_\mu q_\nu}{q^2}-g_{\mu\nu}+\frac{Z_\mu Z_\nu}{Z^2}\right)12\,z_1z_2\,I[D_1\,\overline{D}_1],\end{aligned}\tag{3.25}$$

where

$$I[D_1\,\overline{D}_1]=\int d^2k_T\,d^2p_T\,\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T)\,D_1(z_1,-z_1\boldsymbol{k}_T)\overline{D}_1(z_2,-z_2\boldsymbol{p}_T).\tag{3.26}$$

Using the contraction with the leptonic tensor,

$$L^{\mu\nu}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}+\frac{Z_\mu Z_\nu}{Z^2}\right) = 4Q^2\left(\frac{1}{2}-y+y^2\right) = Q^2\left(1+\cos^2\theta\right), \tag{3.27}$$

where θ is the angle of the produced hadrons h_2 in the e^-e^+ rest frame. The cross section becomes

$$\frac{d\sigma(e^-e^+\to h_1h_2X)}{d\Omega\,dz_1\,dz_2\,d^2\boldsymbol{q}_T} = \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)z_1^2z_2^2\,I[D_1\,\overline{D}_1]. \tag{3.28}$$

Integrated over transverse momenta of hadron h_1 one finds,

$$\frac{d\sigma(e^-e^+\to h_1h_2X)}{d\Omega\,dz_1\,dz_2} = \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)D_1(z_1)\,\overline{D}_1(z_2), \tag{3.29}$$

Including the summation over quarks and antiquarks one obtains

$$\begin{aligned}\frac{d\sigma(e^-e^+\to h_1h_2X)}{d\Omega\,dz_1\,dz_2} &= \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)\sum_a e_a^2 D_{1\,h_1/a}(z_1)\,\overline{D}_{1\,h_2/\overline{a}}(z_2)\\ &= 3\sum_a\frac{d\hat\sigma}{d\Omega}(e^-e^+\to a\overline{a})\,D_{1\,h_1/a}(z_1)\,\overline{D}_{1\,h_2/\overline{a}}(z_2),\end{aligned} \tag{3.30}$$

where the annihilation cross section into a quark-antiquark pair is given by

$$\frac{d\hat\sigma}{d\Omega}(e^-e^+\to a\overline{a}) = \frac{\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right). \tag{3.31}$$

The factor 3 multiplying the cross section can be naturally understood as the definitions of each fragmentation function includes an averaging over color and the annihilation can be into a quark-antiquark pair of any of the three colors. The result for the production of a single hadron is obtained by considering hadron 2 as the jet with $\overline{D}_1 = \delta(1-z_2)$, thus

$$\begin{aligned}\frac{d\sigma(e^-e^+\to hX)}{d\Omega\,dz} &= \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)\sum_a e_a^2\,D_{1\,h/a}(z)\\ &= 3\sum_a\frac{d\hat\sigma}{d\Omega}(e^-e^+\to a\overline{a})\,D_{1\,h/a}(z).\end{aligned} \tag{3.32}$$

Integrating over the jet direction gives

$$\begin{aligned}\frac{d\sigma(e^-e^+\to hX)}{dz} &= \frac{4\pi\,\alpha^2}{Q^2}\sum_a e_a^2\,D_{1\,h/a}(z)\\ &= 3\sum_a\hat\sigma(e^-e^+\to a\overline{a})\,D_{1\,h/a}(z).\end{aligned} \tag{3.33}$$

Finally the jet cross section is found by taking $D_1(z) = \delta(1-z)$,

$$\begin{aligned}\frac{d\sigma(e^-e^+\to jets)}{d\Omega} &= \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)\sum_a e_a^2\\ &= 3\sum_a\hat\sigma(e^-e^+\to a\overline{a}),\end{aligned} \tag{3.34}$$

and the total cross section

$$\begin{aligned}\sigma(e^-e^+\to \text{hadrons}) &= \frac{4\pi\,\alpha^2}{Q^2}\sum_a e_a^2\\ &= 3\sum_a\hat\sigma(e^-e^+\to a\overline{a})\\ &= \sigma(e^-e^+\to\mu^-\mu^+)\,3\sum_a e_a^2.\end{aligned} \tag{3.35}$$

Note that integrating over z the multiplicities of produced particles enter,

$$\int dz\, \frac{d\sigma(e^-e^+ \to hX)}{dz} = \langle n_h^{e^-e^+} \rangle \sigma(e^-e^+ \to \text{hadrons}) \tag{3.36}$$

given by

$$\langle n_h^{e^-e^+} \rangle = \frac{\sum_a e_a^2\, n_{h/a}}{\sum_a e_a^2}, \tag{3.37}$$

where $n_{h/a} = \int dz\, D_{1\,h/a}(z)$.

3.4 lepton-hadron scattering

The hadronic tensor in this case is rewritten as

$$\begin{aligned} 2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h) &= \frac{1}{(2\pi)^4} \int \frac{d^3P_X}{(2\pi)^3 2P_X^0} (2\pi)^4 \delta^4(q+P-P_X-P_h) \\ &\qquad \times \langle PS|J_\mu(0)|P_X;P_hS_h\rangle\langle P_X;P_hS_h|J_\nu(0)|PS\rangle, \\ &= \frac{1}{(2\pi)^4} \int d^4x\; e^{iq\cdot x}\, \langle PS|J_\mu(x) \sum_X |X;P_h,S_h\rangle\langle X;P_hS_h|J_\nu(0)|PS\rangle, \end{aligned}$$

which in tree approximation (Born terms) becomes

$$\begin{aligned} 2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h) &= \frac{1}{(2\pi)^4} \int d^4x\; e^{iq\cdot x}\, \langle PS| :\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x): \sum_X |X;P_hS_h\rangle \\ &\qquad \times\langle X;P_hS_h| :\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0): |PS\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4x\; e^{iq\cdot x}\, \langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk} \\ &\qquad \langle 0|\psi_k(x) \sum_X |X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\ &\quad + \frac{1}{(2\pi)^4} \int d^4x\; e^{iq\cdot x}\, \langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li} \\ &\qquad \langle 0|\overline{\psi}_j(x) \sum_X |X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk}, \\ &= \int d^4p\, d^4k\, \delta^4(p+q-k)\ \mathrm{Tr}\left(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right) + \left\{ \begin{array}{c} q \leftrightarrow -q \\ \mu \leftrightarrow \nu \end{array} \right\}, \end{aligned} \tag{3.38}$$

where we have used

$$\begin{aligned} \Phi_{ij}(p) &= \frac{1}{(2\pi)^4} \int d^4x\; e^{ip\cdot x}\, \langle PS|\overline{\psi}_j(0)\psi_i(x)|PS\rangle, \\ \Delta_{kl}(k) &= \frac{1}{(2\pi)^4} \int d^4x\; e^{ik\cdot x}\, \langle 0|\psi_k(x) \sum_X |X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle, \end{aligned}$$

and its symmetry properties (see section 2). Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. We have not considered possible target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x) \sum_X |X;P_hS_h\rangle$ $\langle X;P_hS_h|\psi_i(0)|PS\rangle$.

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements Φ are limited, i.e. p^2, $p\cdot P$ are of hadronic scale and similarly in the matrix elements Δ for k^2 and $k\cdot P_h$, one can write the delta function up to $\mathcal{O}(1/Q^2)$ as

$$\delta^4(p+q-k) \approx \delta(p^+ + q^+)\, \delta(q^- - k^-)\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T), \tag{3.39}$$

The result in leading order is then

$$2M\,\mathcal{W}_{\mu\nu} = \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \mathrm{Tr}\left([\textstyle\int dp^- \Phi(p)]\gamma_\mu [\int dk^+ \Delta(k)]\gamma_\nu \right)\Big|_{\substack{p^+ = x_B P^+ \\ k^- = P_h^-/z}}$$

$$= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} + \frac{\tilde{P}_\mu \tilde{P}_\nu}{\tilde{P}^2}\right) 2z\, I[f_1\, D_1], \tag{3.40}$$

where

$$I[f_1\, D_1] = \int d^2p_T\, d^2k_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\, f_1(x_B, \boldsymbol{p}_T) D_1(z, -z\boldsymbol{k}_T). \tag{3.41}$$

Using the contraction with the leptonic tensor,

$$L^{\mu\nu}\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} + \frac{\tilde{P}_\mu \tilde{P}_\nu}{\tilde{P}^2}\right) = \frac{4Q^2}{y^2}\left(\frac{y^2}{2} + 1 - y\right), \tag{3.42}$$

the cross section becomes

$$\frac{d\sigma(\ell H \to \ell' h X)}{dx_B\, dy\, dz\, d^2\boldsymbol{P}_{h\perp}} = \frac{4\pi\alpha^2\, s}{Q^4}\left(\frac{y^2}{2} + 1 - y\right) x_B I[f_1\, D_1], \tag{3.43}$$

or integrated over $\boldsymbol{P}_{h\perp}$,

$$\frac{d\sigma(\ell H \to \ell' h X)}{dx_B\, dy\, dz} = \frac{4\pi\alpha^2\, s}{Q^4}\left(\frac{y^2}{2} + 1 - y\right) x_B f_1(x_B)\, D_1(z). \tag{3.44}$$

The case of inclusive electroproduction is easily obtained using the result $D_1(z) = \delta(1-z)$ for quark production off a quark, giving

$$\frac{d\sigma(\ell H \to \ell' X)}{dx_B\, dy} = \frac{4\pi\alpha^2\, s}{Q^4}\left(\frac{y^2}{2} + 1 - y\right) x_B f_1(x_B). \tag{3.45}$$

This leads to the well-known result for electroproduction of hadrons h (now also including the summation over quarks and antiquarks, where the latter come from the second contribution in Eq. 3.38)

$$\frac{\frac{d\sigma(\ell H \to \ell' h X)}{dx\, dy\, dz}}{\frac{d\sigma(\ell H \to \ell' X)}{dx\, dy}} = \frac{N_h^{\ell H}(x,z)}{\sum_a e_a^2\, f_{1\, a/H}(x)} = \frac{\sum_a e_a^2\, f_{1\, a/H}(x)\, D_{1\, h/a}(z)}{\sum_a e_a^2\, f_{1\, a/H}(x)}. \tag{3.46}$$

Upon integration over z one obtains

$$\int dz\, \frac{d\sigma(\ell H \to \ell' h X)}{dx\, dy\, dz} = \langle n_h^{\ell H}(x)\rangle \frac{d\sigma(\ell H \to \ell' X)}{dx\, dy} \tag{3.47}$$

where $\langle n_h(x)\rangle$ represents the average number of produced particles as a function of x,

$$\langle n_h^{\ell H}(x)\rangle = \frac{\sum_a e_a^2\, n_{h/a}\, f_{1\, a/H}(x)}{\sum_a e_a^2\, f_{1\, a/H}(x)}, \tag{3.48}$$

where $n_{h/a} = \int dz\, D_{1\, h/a}(z)$.

Finally we note that we can write the inclusive cross section for different values of $s = Q^2/x_B\, y$ in terms of the virtual photon-quark cross section $\sigma\left(\gamma^*(Q^2)\, a \to a\right)$ as

$$\begin{aligned} \frac{d\sigma(\ell H \to \ell' X)}{dx_B\, dy\, dQ^2} &= \frac{1}{Q^2}\frac{\alpha}{\pi}\frac{1}{y}\left(\frac{y^2}{2} + 1 - y\right)\sum_a f_{1a/H}(x_B)\, \sigma\left(\gamma^*(Q^2)a \to a\right) \\ &= \frac{4\pi\,\alpha^2}{Q^4}\frac{1}{y}\left(\frac{y^2}{2} + 1 - y\right) f_{1a/H}(x_B)\, \delta\left(\frac{s}{Q^2}x_B y - 1\right), \end{aligned} \tag{3.49}$$

which is quite analogous to the situation in the Drell-Yan process.

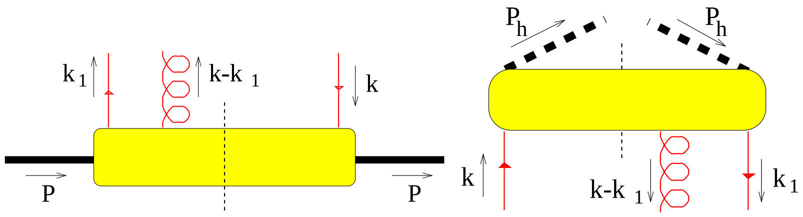


Figure 3.2: The quark-quark-gluon correlation functions

3.5 Inclusion of longitudinal gluon contributions

We will consider in this section the inclusion of diagrams with gluons connecting the soft and hard part (see fig. 3.2).

$$\Phi^{\alpha}_{A\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\; e^{i\,k_1\cdot\xi + i(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^\alpha(\eta)\psi_i(\xi)|P,S\rangle \tag{3.50}$$

(satisfying $\gamma_0\Phi^{\alpha\dagger}_A(k,k_1)\gamma_0 = \Phi^\alpha_A(k_1,k)$), and

$$\Delta^{\alpha}_{A\,ij}(k,k_1;P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\, d^4\eta\; e^{i\,k_1\cdot(\xi-\eta)+i\,k\cdot\eta}\langle 0|\psi_i(\xi)\,A^\alpha_T(\eta)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{3.51}$$

(satisfying $\gamma_0\Delta^{\alpha\dagger}_A(k,k_1)\gamma_0 = \Delta^\alpha_A(k_1,k)$). Performing the integrations $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds, using $k^+ = x\,P^+$ and $k_1^+ = y\,P^+$, the lightcone correlation functions corresponding to multiparton matrix elements, e.g.

$$\begin{aligned}\Phi^{\alpha}_{A\,ij}(x,y) &\equiv P^+\int dk^-\,d^2\boldsymbol{k}_T\;dk_1^-\,d^2\boldsymbol{k}_{1T}\;\Phi^\alpha_{A\,ij}(k,k_1;P,S)\\ &= P^+\int\frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^\alpha(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC},\end{aligned} \tag{3.52}$$

where the subscript LC refers to 'lightcone', $\xi^+ = \eta^+ = \xi_T = \eta_T = 0$. Whatever way one parametrizes the above matrix elements, it is clear that in a process the index α being $+$ requires a $+$ component of one of the vectors of the soft part (P, S), which are of order Q after expanding in $n_\pm$ as discussed in the previous chapters. These thus give the dominant contribution. We will analyse them first for lepton-hadron scattering. We obtain four contributions as given in figure 3.3. Two of them have gluons connected to the lower soft part (the hadron $\rightarrow$ quark part), the others gluons connected to the upper soft part (the quark $\rightarrow$ hadron part). Including the contribution of the handbag one has

$$\begin{aligned}2M\,\mathcal{W}_{\mu\nu} = &\int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right]\\ &-\int d^4p\,d^4k\,d^4p_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\alpha\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2+i\epsilon}\gamma_\nu\Phi^\alpha_A(p,p-p_1)\gamma_\mu\Delta(k)\right]\\ &\qquad+\mathrm{Tr}\left[\gamma_\mu\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2-i\epsilon}\gamma_\alpha\Delta(k)\gamma_\nu\Phi^\alpha_A(p-p_1,p)\right]\Bigg\}\\ &-\int d^4p\,d^4k\,d^4k_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\nu\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\gamma_\alpha\Phi(p)\gamma_\mu\Delta^\alpha_A(k-k_1,k)\right]\\ &\qquad+\mathrm{Tr}\left[\gamma_\alpha\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\gamma_\mu\Delta^\alpha_A(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}.\end{aligned} \tag{3.53}$$

[Note that we have for a quark-quark-gluon blob used momentum p_1 (or k_1) for the gluon and $p-p_1$ (or $k-k_1$) for the quark. This is easier to extend when we consider multiple gluon correlation functions.]

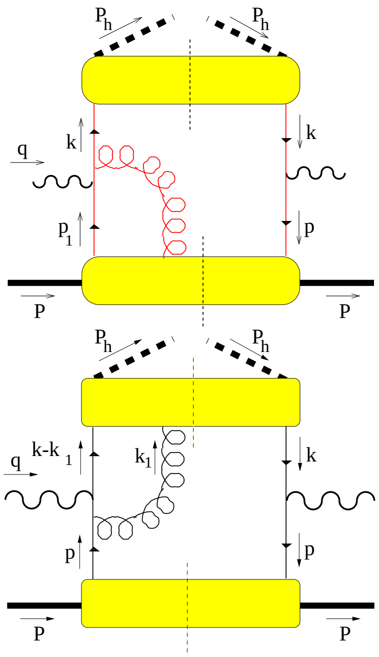


Figure 3.3: Quark-quark-gluon correlation functions in lepton-hadron scattering

In the remainder of this section we will often omit the standard integration $\int d^4p\, d^4k\, \delta^4(p+q-k)$. The momenta p_1 and k_1 connected to the soft hadronic parts are parametrized according to

$$p = \left[p^-, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_T\right], \qquad p_1 = \left[p_1^-, \frac{x_1\, Q}{\sqrt{2}}, \boldsymbol{p}_{1T}\right], \qquad p - k_1 = \left[-z_1\, \frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \ldots\right], \tag{3.54}$$

$$k = \left[\frac{Q}{\sqrt{2}}, k^+, \boldsymbol{k}_T\right], \qquad k_1 = \left[\frac{z_1\, Q}{\sqrt{2}}, k_1^+, \boldsymbol{k}_{1T}\right], \qquad k - p_1 = \left[\frac{Q}{\sqrt{2}}, -x_1\, \frac{Q}{\sqrt{2}}, \ldots\right], \tag{3.55}$$

The momentum appearing in the extra fermion propagator is $p - p_1 + q = k - p_1$ with $(k-p_1)^2 = -x_1\, Q^2$, or $k - k_1 - q = p - k_1$ with $(p-k_1)^2 = -z_1\, Q^2$. Thus one has in leading order in $1/Q$,

$$\frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 + i\epsilon} = \frac{\gamma^-}{Q\sqrt{2}} + \frac{\gamma^+}{(-x_1 + i\epsilon)\, Q\sqrt{2}} - \frac{\boldsymbol{\gamma}_T \cdot (\boldsymbol{k}_T - \boldsymbol{p}_{1T}) - m}{(-x_1 + i\epsilon)\, Q^2}, \tag{3.56}$$

$$\frac{\not{p} - \not{k}_1 + m}{(p-k_1)^2 - m^2 + i\epsilon} = \frac{\gamma^+}{Q\sqrt{2}} + \frac{\gamma^-}{(-z_1 + i\epsilon)\, Q\sqrt{2}} - \frac{\boldsymbol{\gamma}_T \cdot (\boldsymbol{p}_T - \boldsymbol{k}_{1T}) - m}{(-z_1 + i\epsilon)\, Q^2}. \tag{3.57}$$

This can be used to consider separately the contributions of transverse (A_T^α) and longitudinal (A^+) gluons.

For the transverse gluons, the trace of the first gluonic contribution becomes

$$- \int d^4p_1\ \mathrm{Tr}\left[\gamma_\alpha\, \frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 + i\epsilon}\, \gamma_\nu \Phi_A^\alpha(p, p-p_1) \gamma_\mu \Delta(k)\right]$$
$$= \int \frac{d^4p_1}{(2\pi)^4} \int d^4\xi \int d^4\eta\ e^{i\,(p-p_1)\cdot\xi + i\,p_1\cdot\eta}\ \langle P, S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\, \gamma^-}{Q\sqrt{2}}\, \gamma_\nu\, gA_T^\alpha(\eta)\, \psi(\xi)|P, S\rangle,$$

which starts off at order $1/Q$ and at this order requires leading parts from Φ_A^α (proportional to $P_+\Phi_A^\alpha P_-$) and leading parts from Δ (proportional to $P_-\Delta P_+$). As $\{\gamma^-, \gamma_T^\alpha\} = 0$ and $\gamma^- P_+ = P_-\gamma^- = 0$ only the $\gamma^- = P_+\gamma^- P_-$ part in Eq. 3.56 contributes. This term is independent of any of the components of p_1, and we thus can immediately consider the distributions $\int d^4p_1\, \Phi_A^\alpha(p, p-p_1)$, or explicitly

$$\int d^4\xi\ e^{i\,p\cdot\xi}\ \langle P, S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\, \gamma^-}{Q\sqrt{2}}\, \gamma_\nu\, gA_T^\alpha(\xi)\, \psi(\xi)|P, S\rangle. \tag{3.58}$$

This contribution will be studied in the next section. Note that it can be written in terms of the covariant derivative as

$$\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, i\,D_T^\alpha(\xi)\,\psi(\xi)|P,S\rangle$$
$$-\,p_T^\alpha\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\,\psi(\xi)|P,S\rangle. \tag{3.59}$$

In this section we consider next the contributions of longitudinal gluons (A^+). They lead to traces of the form

$$-\int d^4p_1\;\mathrm{Tr}\left[\gamma^-\,\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^+(p,p-p_1)\gamma_\mu\Delta(k)\right] \tag{3.60}$$

The first term in Eq. 3.56 does not contribute. The second term contributes at $\mathcal{O}(1)$ as the dominant contribution in Φ_A^+ is the part projected out by $\int dp_1^-\;P_+\Phi_A^+P_-$ which is of $\mathcal{O}(Q)$. Explicitly, we get for the first correction in Eq. 3.53

$$\begin{aligned}
&-\int\frac{d^4p_1}{(2\pi)^4}\int d^4\xi\int d^4\eta\; e^{i\,(p-p_1)\cdot\xi+i\,p_1\cdot\eta}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\;\frac{\gamma^+}{(x_1-i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\,gA^+(\eta)\,\psi(\xi)|P,S\rangle\\
&=\int\frac{dx_1}{2\pi}\int d^4\xi\int d\eta^-\;\frac{e^{i\,x_1\,p^+(\eta^--\xi^-)}}{x_1-i\epsilon}\,e^{i\,p\cdot\xi}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,\frac{\gamma^-\gamma^+}{2}\,\gamma_\nu\,gA^+(\eta)\,\psi(\xi)|P,S\rangle\Big|_{\substack{\eta^+=\xi^+\\ \eta_T=\xi_T}}\\
&=\int d^4\xi\int d\eta^-\;\theta(\eta^--\xi^-)\,e^{i\,p\cdot\xi}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\,igA^+(\eta^-)\,\psi(\xi)|P,S\rangle\\
&=-\int d^4\xi\; e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\,ig\int_\infty^{\xi^-}d\eta^-\;A^+(\eta^-)\,\psi(\xi)|P,S\rangle.
\end{aligned} \tag{3.61}$$

The second term in Eq. 3.53 gives

$$\begin{aligned}
&-\int d^4p_1\,\mathrm{Tr}\left[\gamma_\mu\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2-i\epsilon}\,\gamma^-\Delta(k)\gamma_\nu\Phi_A^+(p-p_1,p)\right]\\
&=\int d^4\xi\; e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,ig\int_\infty^0 d\eta^-\;A^+(\eta^-)\,\gamma_\mu\,P_-\,\Delta(k)\,\gamma_\nu\,\psi(\xi)|P,S\rangle
\end{aligned} \tag{3.62}$$

The last two terms in Eq. 3.53 give

$$\begin{aligned}
&-\int d^4k_1\Bigg\{\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma^+\Phi(p)\gamma_\mu\Delta_A^-(k_1,k)\right]\\
&\qquad+\mathrm{Tr}\left[\gamma^+\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^-(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}\\
&=\int d^4\xi\; e^{i\,k\cdot\xi}\Bigg\{\mathrm{Tr}\,\langle 0|\psi(\xi)\,a_h^\dagger a_h\,ig\int_{-\infty}^0 d\eta^+\;A^-(\eta^+)\,\overline{\psi}(0)\,\gamma_\nu\,P_+\,\Phi(p)\,\gamma_\mu|0\rangle\\
&\qquad-\mathrm{Tr}\,\langle 0|\psi(\xi)\,ig\int_{-\infty}^{\xi^+}d\eta^+\;A^-(\eta^+)\,a_h^\dagger a_h\,\overline{\psi}(0)\,\gamma_\nu\,\Phi(p)\,P_-\,\gamma_\mu|0\rangle\Bigg\}.
\end{aligned}$$

The result of multiple A^+- or A^--gluons together with the tree-level result gives in leading order in $1/Q$ (when the projectors P_+ and P_- don't matter) the exponentiated path-ordered result

$$2M\,\mathcal{W}_{\mu\nu}=\int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right] \tag{3.63}$$

with

$$\Phi_{ij}(p,P,S)=\frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\mathcal{U}(0,\infty;0_T)\,\mathcal{U}(\infty,\xi^-;\xi_T)\psi_i(\xi)|P,S\rangle, \tag{3.64}$$

$$\Delta_{ij}(k,P_h,S_h)=\frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\;\langle 0|\mathcal{U}(-\infty,\xi^+;\xi_T)\psi_i(\xi)\,a_h^\dagger a_h\,\overline{\psi}_j(0)\mathcal{U}(0,-\infty;0_T)|0\rangle. \tag{3.65}$$

Provided we assume that matrix elements containing bilocal operators $\overline{\psi}(0)\, A_T(\eta^\pm = \mp\infty, \eta_T)\, \psi(\xi)$ vanish for physical states, the above links can be connected resulting in a color gauge-invariant matrix element that must be used in the definition of the correlation functions.

Before considering the transverse gluons let us check the case of two A^+ gluons. For instance considering a gauge choice $A^- = 0$, one needs only to consider the absorption of the A^+ gluons in the 'distribution' part. Dressing the diagram leading to the first of the four terms above with another 'parallel' gluon one obtains a contribution

$$\begin{aligned}
&\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4} \int d^4\xi \int d^4\eta_1\, d^4\eta_2\; e^{i\,(p-p_1-p_2)\cdot\xi + i\,p_1\cdot\eta_1 + i\,p_2\cdot\eta_2} \\
&\qquad \times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\,\frac{\gamma^+}{(x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma^-\,\frac{\gamma^+}{(x_1+x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\, gA^+(\eta_2)\, gA^+(\eta_1)\,\psi(\xi)|P,S\rangle \\
= \;&\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_1}{2\pi}\,\frac{dx_2}{2\pi} \int d^4\xi \int d\eta_1^-\, d\eta_2^-\; \frac{e^{i\,(x_1+x_2)p^+(\eta^- - \xi^-)}\; e^{i\,x_2 p^+(\eta_2^- - \eta_1^-)}}{(x_1+x_2-i\epsilon)\,(x_2-i\epsilon)}\; e^{i\,p\cdot\xi} \\
&\qquad \times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)P_+^2\,\gamma_\nu\, gA^+(\eta_2)\, gA^+(\eta_1)\,\psi(\xi)|P,S\rangle. \qquad (3.66)
\end{aligned}$$

The integration over x_1 and x_2 gives

$$i\theta(\eta_1^- - \xi^-)\, i\theta(\eta_2^- - \eta_1^-), \qquad (3.67)$$

leading to the path ordering.

Chapter 4

Gluon fields and correlation functions

4.1 Quark-gluon correlation functions

For the analysis beyond the twist two level, it is necessary to include quark-gluon correlation functions. We define

$$\Phi^{\alpha}_{A\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\; e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S\vert\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\eta)\psi_i(\xi)\vert P,S\rangle \tag{4.1}$$

(satisfying $\gamma_0\Phi^{\alpha\dagger}_{A}(k,k_1)\gamma_0 = \Phi^{\alpha}_{A}(k_1,k)$). Performing the integrations $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds, using $k^+ = x\,P^+$ and $k_1^+ = y\,P^+$, the lightcone correlation functions corresponding to multiparton matrix elements,

$$\begin{aligned}\Phi^{\alpha}_{A\,ij}(x,y) &\equiv P^+\int dk^-\,d^2\boldsymbol{k}_T\;dk_1^-\,d^2\boldsymbol{k}_{1T}\;\Phi^{\alpha}_{A\,ij}(k,k_1;P,S)\\ &= P^+\int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S\vert\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\eta)\psi_i(\xi)\vert P,S\rangle\Big\vert_{LC},\end{aligned} \tag{4.2}$$

where the subscript LC refers to 'lightcone', $\xi^+ = \eta^+ = \xi_T = \eta_T = 0$. Up to the twist three level it is often not necessary to consider this general three-field matrix element, but it is sufficient to consider the bilocal matrix elements obtained after integration over d^4k_1,

$$\begin{aligned}\Phi^{\alpha}_{A\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\;e^{i\,k\cdot\xi}\langle P,S\vert\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\xi)\psi_i(\xi)\vert P,S\rangle\\ &= \int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k,k_1;P,S),\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\gamma_0\Phi^{\alpha\dagger}_{A}\gamma_0)_{ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\;e^{i\,k\cdot\xi}\langle P,S\vert\overline{\psi}_j(0)\,gA^{\alpha}_{T}(0)\psi_i(\xi)\vert P,S\rangle\\ &= \int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k_1,k;P,S).\end{aligned} \tag{4.4}$$

We already mention here a third possible bilocal matrix element,

$$\Phi'^{\alpha}_{A\,ij}(k;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\;e^{i\,k\cdot\xi}\langle P,S\vert\overline{\psi}_j(0)\psi_i(0)\,gA^{\alpha}_{T}(\xi)\vert P,S\rangle$$

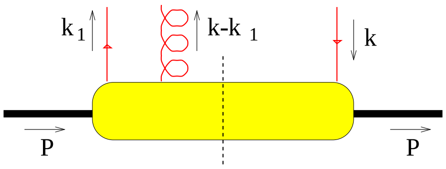


Figure 4.1: The quark-quark-gluon correlation function

$$= \int d^4k_1\ \Phi^\alpha_{A\,ij}(k+k_1,k_1;P,S). \tag{4.5}$$

One-argument lightcone correlation functions are obtained after integration over $dk^- d^2\boldsymbol{k}_T$,

$$\Phi^\alpha_{A\,ij}(x) \equiv \int dk^-\, d^2\boldsymbol{k}_T\ \Phi^\alpha_{A\,ij}(k;P,S). \tag{4.6}$$

A factor P^+ has been included in the definition of $\Phi^\alpha_A(x,y)$, such that one has for the one-argument lightcone correlation functions $\Phi^\alpha_A(x) = \int dy\, \Phi^\alpha_A(x,y)$, etc. First, as before the same constraints as before from parity and time reversal invariance can be used. Including only terms that potentially contribute at the twist three level one obtains using only parity the general form

$$\begin{aligned} F^\alpha(k,P,S) &= C_1\, \not P k_\perp^\alpha + M\, C_2\, [\gamma^\alpha, \not P] + (C_3/M)\, [\not P, \not k_\perp]\, k_\perp^\alpha + iC_4\, \epsilon^\alpha{}_{\nu\rho\sigma}\gamma^\nu\gamma_5 P^\rho k_\perp^\sigma \\ &\quad + M\, C_5\, \not P\gamma_5 S_\perp^\alpha + (C_6/M)\, \not P\gamma_5 k_\perp^\alpha\, (k\cdot S) + C_7\, [\not P, \gamma^\alpha]\gamma_5\, (k\cdot S) \\ &\quad + C_8\, [\not k_\perp, \not P]\gamma_5\, S^\alpha + C_9\, [\not S_\perp, \not P]\gamma_5\, k_\perp^\alpha + (C_{10}/M^2)\, [\not k_\perp, \not P]\gamma_5\, k_\perp^\alpha\, (k\cdot S) \\ &\quad + iM\, C_{11}\, \epsilon^\alpha{}_{\nu\rho\sigma}\gamma^\nu P^\rho S_\perp^\sigma + i(C_{12}/M)\, \epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k_\perp^\rho S_\perp^\sigma\, k_\perp^\alpha \\ &\quad + i(C_{13}/M)\, \epsilon^\alpha{}_{\nu\rho\sigma}\gamma^\nu P^\rho k_\perp^\sigma\, (k\cdot S) \end{aligned} \tag{4.7}$$

$$F^{\alpha\dagger}(k,P,S) = \gamma_0 \Big(C_1^*\, \not P k_\perp^\alpha - M\, C_2^*\, [\gamma^\alpha, \not P] + \ldots \Big) \gamma_0. \tag{4.8}$$

All amplitudes only depend on $P\cdot k$ and k^2. By choosing our conventions such that the the Dirac structures multiplying C_2, C_3, C_4, and C_{11}, C_{12}, C_{13} are antihermitean ($\Gamma^\dagger = -\gamma_0\Gamma\gamma_0$) and hermitean ($\Gamma^\dagger = \gamma_0\Gamma\gamma_0$) otherwise, all amplitudes are real when time reversal invariance applies.

Before starting with the twist-analysis we will discuss the issue of color gauge invariance. First of all, we of course need to employ correlation functions containing the covariant derivative $i\,D_\mu(\xi) = i\partial_\mu + g\,A_\mu(\xi)$ and field strength tensor $G_{\rho\sigma}(\xi) = (i/g)[D_\rho(\xi), D_\sigma(\xi)]$ instead of the A_μ fields. These are

$$\Phi^\alpha_{D\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\ e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\, iD^\alpha(\eta)\psi_i(\xi)|P,S\rangle, \tag{4.9}$$

and

$$\Phi^\alpha_{G\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\ e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\, gG^{+\alpha}(\eta)\psi_i(\xi)|P,S\rangle, \tag{4.10}$$

and as before the lightcone correlation functions

$$\Phi^\alpha_{D\,ij}(x,y) = \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\ e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\, iD^\alpha(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC}, \tag{4.11}$$

$$\Phi^\alpha_{G\,ij}(x,y) = \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\ e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\, gG^{+\alpha}(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC}. \tag{4.12}$$

Bilocal correlation functions are again obtained after integration over one momentum,

$$\begin{aligned} \Phi^\alpha_{D\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\ e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\, iD^\alpha_T(\xi)\psi_i(\xi)|P,S\rangle \\ &= \int d^4k_1\ \Phi^\alpha_{D\,ij}(k,k_1;P,S). \end{aligned} \tag{4.13}$$

and the function

$$\begin{aligned} \Phi'^\alpha_{G\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\ e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,\psi_i(0)\,G^{+\alpha}(\xi)|P,S\rangle. \\ &= \int d^4k_1\ \Phi^\alpha_{G\,ij}(k+k_1,k_1;P,S) = -ik^+\int d^4k_1\ \Phi^\alpha_{A\,ij}(k+k_1,k_1;P,S). \end{aligned} \tag{4.14}$$

The above expressions (without link) are only useful in the gauge $A^+ = 0$, in which case the relation between A^α_T and $G^{+\alpha}$ is simple, $G^{+\alpha} = \partial^+ A^\alpha_T = \partial_- A^\alpha_T$. Possible inversions are (only considering the dependence on the minus component),

$$
\begin{aligned}
A_T^\alpha(\eta^-) &= A_T^\alpha(\infty) - \int_{-\infty}^{\infty} d\zeta^-\ \theta(\zeta^- - \eta^-)\, G^{+\alpha}(\zeta^-) \\
&= A_T^\alpha(-\infty) + \int_{-\infty}^{\infty} d\zeta^-\ \theta(\eta^- - \zeta^-)\, G^{+\alpha}(\zeta^-) \\
&= \frac{A_T^\alpha(\infty) + A_T^\alpha(-\infty)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} d\zeta^-\ \epsilon(\zeta^- - \eta^-)\, G^{+\alpha}(\zeta^-)
\end{aligned}
\tag{4.15}
$$

or using the representation for the θ-function,

$$
\pm i\,\theta(\pm\xi) = \int \frac{dk}{2\pi}\,\frac{e^{ik\xi}}{k \mp i\epsilon}, \qquad i\,\epsilon(\xi) = \int \frac{dk}{2\pi}\,\mathrm{PP}\frac{e^{ik\xi}}{k},
\tag{4.16}
$$

one obtains (omitting the hadron momenta and spin vectors)

$$
\begin{aligned}
\Phi_A^\alpha(k,k_1) &= \delta(k^+ - k_1^+)\,\Phi^\alpha_{A(\infty)}(k,k_1) + \frac{i}{k^+ - k_1^+ - i\epsilon}\,\Phi_G^\alpha(k,k_1) \\
&= \delta(k^+ - k_1^+)\,\Phi^\alpha_{A(-\infty)}(k,k_1) + \frac{i}{k^+ - k_1^+ + i\epsilon}\,\Phi_G^\alpha(k,k_1) \\
&= \delta(k^+ - k_1^+)\,\frac{\Phi^\alpha_{A(\infty)}(k,k_1) + \Phi^\alpha_{A(-\infty)}(k,k_1)}{2} + \mathrm{PP}\frac{i}{k^+ - k_1^+}\,\Phi_G^\alpha(k,k_1),
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
&\delta(k^+ - k_1^+)\,\Phi^\alpha_{A(\pm\infty)\,ij}(k,k_1) \\
&\qquad \equiv \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\ e^{i\,k\cdot\xi}\,e^{i\,(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA_T^\alpha(\pm\infty,\eta^+,\eta_T)\psi_i(\xi)|P,S\rangle,
\end{aligned}
\tag{4.18}
$$

and

$$
\delta(k^+ - k_1^+)\,\left[\Phi^\alpha_{A(\infty)}(k,k_1) - \Phi^\alpha_{A(-\infty)}(k,k_1)\right] = 2\pi\,\delta(k^+ - k_1^+)\,\Phi_G^\alpha(k,k_1)
\tag{4.19}
$$

The constraints following from hermiticity, parity and time-reversal are the following,

$$
\Phi_D^{\alpha\,\dagger}(k,k_1;P,S) = \gamma_0\,\Phi_D^\alpha(k_1,k;P,S)\gamma_0,
\tag{4.20}
$$

$$
\Phi_D^\alpha(k,k_1;P,S) = \gamma_0\,\Phi_{D\alpha}(\bar{k},\bar{k}_1;\bar{P},-\bar{S})\gamma_0,
\tag{4.21}
$$

$$
\Phi_D^{\alpha\,*}(k,k_1;P,S) = (-i\gamma_5 C)\,\Phi_{D\alpha}(\bar{k},\bar{k}_1;\bar{P},\bar{S})(-i\gamma_5 C).
\tag{4.22}
$$

Similar properties hold for Φ_A and with a minus sign for the last relation (time reversal) also for Φ_G. We note the following for the boundary condition terms defined in Eq. 4.18 under time-reversal:

$$
(2\pi)\,\delta(k^+ - k_1^+)\,\Phi^{\alpha *}_{A(\infty)}(P,S;k,k_1) = (2\pi)\,\delta(\bar{k}^+ - \bar{k}_1^+)\,(-i\gamma_5 C)\,\Phi^\alpha_{A(-\infty)}(\bar{P},\bar{S};\bar{k},\bar{k}_1)\,(-i\gamma_5 C).
\tag{4.23}
$$

This is the consequence of the fact that the point $\eta^- = \infty$ is defined by $\eta\cdot n_+ = \infty$, which after time reversal transforms into the point $\eta\cdot\bar{n}_+ = -\infty$. Since the component $\bar{\eta}^-$ is not integrated over the minus sign is not removed by a change of variables as is the case in $\Phi_A(k,k_1)$ (analoguously to the case spelled out for Φ in the first paragraph of chapter 2). Thus we see that the left and right sides of Eqs 4.17 are consistent.

Integrating over $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds

$$
\begin{aligned}
\Phi_A^\alpha(x,y) &= \delta(x-y)\,\Phi^\alpha_{A(\infty)}(x) + \frac{i}{x-y-i\epsilon}\,\Phi_G^\alpha(x,y) \\
&= \delta(x-y)\,\Phi^\alpha_{A(-\infty)}(x) + \frac{i}{x-y+i\epsilon}\,\Phi_G^\alpha(x,y) \\
&= \frac{1}{2}\,\delta(x-y)\,\underbrace{\Phi^\alpha_{A(\infty)}(x) + \Phi^\alpha_{A(-\infty)}(x)}_{\equiv 2\pi\,\Phi^\alpha_{BC}(x)} + \mathrm{PP}\frac{i}{x-y}\,\Phi_G^\alpha(x,y),
\end{aligned}
\tag{4.24}
$$

where

$$\begin{aligned}
&\delta(x-y)\,\Phi^{\alpha}_{A(\pm\infty)\,ij}(x) \\
&\qquad = P^+ \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k\cdot\xi}\,e^{i\,(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\pm\infty,0,\mathbf{0}_T)\psi_i(\xi)|P,S\rangle\Big|_{LC} && (4.25)\\
&\delta(x-y)\,\underbrace{\left[\Phi^{\alpha}_{A(\infty)}(x)-\Phi^{\alpha}_{A(-\infty)}(x)\right]}_{2\pi\,\Phi^{\alpha}_{G}(x,x)} = 2\pi\,\delta(x-y)\,\Phi^{\alpha}_{G}(x,y). && (4.26)
\end{aligned}$$

We note that the constraints from hermiticity, parity and time reversal relate functions at different momenta. However, for the lightcone fractions x and y the same fraction appears right and left since they are expansions of quark momenta in hadron momenta, $k = x\,P + \ldots$ and $\bar{k} = x\,\bar{P} + \ldots$, all of which become bar-ed. Further, we note that if $\Phi_G(x,x) \neq 0$ a pole appears in Φ_A, hence the name 'gluonic pole'.

In order to define color gauge invariant functions it is necessary to include the link operator,

$$\mathcal{U}(\eta,\xi) = \mathcal{P}\exp\left(-ig\int_{\eta^-}^{\xi^-} d\zeta^\mu\,A_\mu(\zeta)\right), \qquad (4.27)$$

where we will implicitly understand that the path runs along the minus direction with $\xi^+ = \eta^+ = 0$ and $\boldsymbol{\xi}_T = \boldsymbol{\eta}_T$. Of course this means that relations hereafter need to be integrated over the minus components of the momenta. The path-ordered integral is defined as

$$\begin{aligned}
\mathcal{U}(\eta,\xi) &= \mathcal{P}\exp\left(-ig\int_0^1 ds\,\frac{d\zeta^\mu(s)}{ds}\,A_\mu(\zeta(s))\right)\\
&\equiv 1 - ig\int_0^1 ds\,\frac{d\zeta^\mu(s)}{ds}\,A_\mu(\zeta(s))\\
&\quad +(ig)^2\int_0^1 ds_1\,\frac{d\zeta^\mu(s)}{ds_1}\,A_\mu(\zeta(s_1))\int_{s_1}^1 ds_2\,\frac{d\zeta^\mu(s_2)}{ds_2}\,A_\mu(\zeta(s_2)) + \mathcal{O}(g^3), \qquad (4.28)
\end{aligned}$$

where $\zeta(s)$ is a path running from $\eta = \zeta(0)$ to $\xi = \zeta(1)$. The path ordered exponential is just the (infinite) product of infinitesimal link operators of the form

$$\mathcal{U}(\xi,\xi+d\xi) = 1 - ig\,d\xi^\mu A_\mu(\xi). \qquad (4.29)$$

From this infinitesimal form one checks that a counter-clockwise plaquette of four links, is given by

$$U(\xi,\xi+d\xi)U(\xi+d\xi,\xi+d\xi+d\eta)U(\xi+d\xi+d\eta,\xi+d\eta)U(\xi+d\eta,\xi) = 1 - ig\,d\xi^\rho d\eta^\sigma\,G_{\rho\sigma}(\xi). \qquad (4.30)$$

From the infinitesimal forms of links and plaquettes the following properties follow

$$\begin{aligned}
i\partial^+_\xi\mathcal{U}(\eta,\xi) &= \mathcal{U}(\eta,\xi)\,iD^+(\xi) && (4.31)\\
iD^\alpha_T(\eta)\mathcal{U}(\eta,\xi) &= \mathcal{U}(\eta,\xi)\,iD^\alpha_T(\xi) + \int_{\eta^-}^{\xi^-} d\zeta^-\mathcal{U}(\eta,\zeta)\,[iD^+(\zeta),iD^\alpha_T(\zeta)]\,\mathcal{U}(\zeta,\xi)\\
&= \mathcal{U}(\eta,\xi)\,iD^\alpha_T(\xi) + ig\int_{\eta^-}^{\xi^-} d\zeta^-\mathcal{U}(\eta,\zeta)\,G^{+\alpha}(\zeta)\,\mathcal{U}(\zeta,\xi), && (4.32)
\end{aligned}$$

where α = 1,2 (transverse), and hence along the link (where $\zeta_T = \xi_T$) $iD^\alpha_T(\zeta) = i\partial^\alpha_\xi + g\,A^\alpha_T(\zeta)$.

Including links we start with the gauge invariant definition of $\Phi_{ij}(x,\boldsymbol{k}_T)$,

$$\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\mathcal{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}. \qquad (4.33)$$

A note can be made at this point about the behavior of the correlation functions with a link (to be discussed in more detail below). Under time reversal the correlation function $\Phi_{ij}(x,\boldsymbol{k}_T)$ will not transform into itself, but the link will run via $\xi^- = -\infty$ instead of $\xi^- = \infty$. Only when the matrix element in Eq. 4.18 vanishes or after integration over transverse momenta time reversal can be used to constrain the parametrization of Φ and similarly other correlation functions. Multiplying the correlation function with the quark momentum we obtain,

$$k^\mu\,\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^+\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,s|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,i\partial^\mu\,\mathcal{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0}, \tag{4.34}$$

where $i\partial^\mu$ can be read as $i\overleftarrow{\partial}_0^\mu$ or $i\overrightarrow{\partial}_\xi^\mu$. Depending on the Lorentz index of the derivative, one can use the link relations given before to rewrite the derivative in terms of correlation functions containing the covariant derivative,

$$\left(\Phi_D^\alpha\right)_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^-d^2\boldsymbol{\xi}_\perp}{(2\pi)^3}\;e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,iD_T^\alpha(\xi)\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0}. \tag{4.35}$$

First of all, one of the correlation functions (with covariant derivative) is trivial. Because of the choice of link, which lies along the minus direction in the point ξ (except for the points $\xi^- = \infty$), one has for the +-component the relation

$$k^+\,\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^-\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,iD^+\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0}. \tag{4.36}$$

For the transverse component one finds

$$\begin{aligned}
k_T^\alpha\,\Phi_{ij}(x,\boldsymbol{k}_T) &= (\Phi_\partial^\alpha)_{ij}(x,\boldsymbol{k}_T)\\
&= \int \frac{d\xi^-\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\bigg\{\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,iD_T^\alpha\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0} - (\Phi_{A(\infty)}^\alpha)_{ij}(x,\boldsymbol{k}_T)\\
&\quad - \langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\int_\infty^{\xi^-} d\eta^-\,\mathcal{U}(\infty,\eta)\,g\,G^{+\alpha}(\eta)\,\mathcal{U}(\eta,\xi)\,\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0}\bigg\}.
\end{aligned} \tag{4.37}$$

Performing the $\boldsymbol{k}_T$ integration this leads to

$$\Phi_\partial^\alpha(x) = \Phi_D^\alpha(x) - \Phi_{A(\infty)}^\alpha(x) - \int dy\;\frac{i}{x-y-i\epsilon}\;\Phi_G^\alpha(x,y) = \Phi_D^\alpha(x) - \Phi_A^\alpha(x). \tag{4.38}$$

The sensitivity to background gluonic fields appears through the boundary terms, i.e. the matrix elements $\Phi_{A(\pm\infty)}^{\alpha[\Gamma]}(x)$. We have already encountered the antisymmetric combination. We define also the symmetric combination. Thus

$$\begin{aligned}
2\pi\,\Phi_{BC}^\alpha(x) &= \left[\Phi_{A(\infty)}^\alpha(x) + \Phi_{A(-\infty)}^\alpha(x)\right]\\
2\pi\,\Phi_G^\alpha(x,x) &= \left[\Phi_{A(\infty)}^\alpha(x) - \Phi_{A(-\infty)}^\alpha(x)\right]
\end{aligned}$$

where the important observation is that these two combinations have opposite behavior under time-reversal (even and odd, respectively). In later calculations we will typically see the following combination showing up,

$$\begin{aligned}
&\int dy\;\frac{x-y}{x-y-i\epsilon}\;\Phi_A(x,y) = \int dy\;\frac{i}{x-y-i\epsilon}\;\Phi_G(x,y),\\
&= \Phi_D^\alpha(x) - \Phi_\partial^\alpha(x) - \pi\,\Phi_{BC}^\alpha(x) - \pi\,\Phi_G^\alpha(x,x)
\end{aligned}$$

Considering the above as one object, $\Phi_A^{\alpha(eff)}(x)$ one needs in the parametrization T-odd functions if $\Phi_G^\alpha(x,x) \neq 0$. It is consistent with the observation that the presence of links prohibits the use of T-reversal constraints for $\Phi(x,\boldsymbol{k}_T)$.

The projections obtained for the quark-gluon corrrelation functions with transverse gluon fields are not independent from the ones defined for the quark-quark correlation functions, either. They can be connected to quark-quark correlation functions with one good and one bad quark field using the QCD equation of motion, $(i\not{D} - m)\psi(x) = 0$. From this equation it is straightforward to derive the relations

$$iD^\mu\psi + \sigma^{\mu\nu}D_\nu\psi - m\gamma^\mu\psi = 0, \tag{4.39}$$

$$i\gamma^\mu D^\nu\psi - i\gamma^\nu D^\mu\psi + im\,\sigma^{\mu\nu}\psi + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5\,iD_\rho\psi = 0. \tag{4.40}$$

or explicitly

$$iD^+\psi - i\sigma^{+-}\,iD^+\psi + i\sigma^{\alpha+}\,iD_\alpha\psi - m\,\gamma^+\psi = 0, \tag{4.41}$$
$$\gamma^+\,iD^\alpha\psi - i\epsilon_T^{\alpha\beta}\gamma^+\gamma_5\,iD_\beta\psi = im\,\sigma^{\alpha+}\psi + \gamma^\alpha\,iD^+\psi - i\epsilon_T^{\alpha\beta}\gamma_\beta\gamma_5\,iD^+\psi, \tag{4.42}$$

where $\epsilon_T^{\alpha\beta} = \epsilon^{-+\alpha\beta}$. This immediately leads to the (T-even) relations

$$\Phi_{D\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) = \epsilon_{T\alpha\beta}\,\Phi_D^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) = i\,(Mx\,e - m\,f_1) + \epsilon_{Tij}\,k_T^i S_T^j\,xh_T^\perp \tag{4.43}$$
$$\Phi_{D\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) = Mx\,h_s - m\,g_{1s}, \tag{4.44}$$
$$\begin{aligned}\Phi_D^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\,\Phi_{D\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) &= k_T^\alpha\,xf^\perp - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(xg_s^\perp - \frac{m}{M}\,h_{1s}^\perp\right)\\ &\quad -i\epsilon_T^{\alpha\beta}S_{T\beta}\,(Mx\,g_T' - m\,h_{1T})\,.\end{aligned} \tag{4.45}$$

or including the T-odd possibilities

$$\begin{aligned}\Phi_{D\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) = \epsilon_{T\alpha\beta}\,\Phi_D^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) &= i\,(Mx\,e - m\,f_1 - iMx\,h)\\ &\quad + \epsilon_{Tij}\,k_T^i S_T^j\left(xh_T^\perp + i\,\frac{m}{M}\,f_{1T}^\perp\right)\end{aligned} \tag{4.46}$$
$$\Phi_{D\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) = Mx\,h_s - m\,g_{1s} + i\,Mx\,e_s, \tag{4.47}$$
$$\begin{aligned}\Phi_D^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\,\Phi_{D\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) &= k_T^\alpha\left(xf^\perp + i\,\frac{m}{M}\,h_1^\perp\right) - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(xg_s^\perp - \frac{m}{M}\,h_{1s}^\perp - i\lambda\,x\,f_L^\perp\right)\\ &\quad - i\epsilon_T^{\alpha\beta}S_{T\beta}\,(Mx\,g_T' - m\,h_{1T} - iMx\,f_T)\,.\end{aligned} \tag{4.48}$$

These relations for Φ_D can actually be considered as defining relations for the twist three correlation functions, again including k_T-dependence. Integrating the $\boldsymbol{k}_T$-dependence the most general form for $\Phi_D^\alpha(x,y)$ actually is

$$\begin{aligned}\Phi_D^\alpha(x,y) \;=\; \frac{M}{2P^+}\Big\{ & G_D(x,y)\,i\epsilon_T^{\alpha\beta}S_{T\beta}\not{n}_+ + \tilde{G}_D(x,y)\,S_T^\alpha\,\gamma_5\not{n}_+\\ & + H_D(x,y)\,\lambda\,\gamma_5\gamma_T^\alpha\not{n}_+ + E_D(x,y)\,\gamma_T^\alpha\not{n}_+\Big\}.\end{aligned} \tag{4.49}$$

with hermiticity leading to

$$G_D^*(x,y) = -G_D(y,x), \tag{4.50}$$
$$\tilde{G}_D^*(x,y) = \tilde{G}_D(y,x), \tag{4.51}$$
$$H_D^*(x,y) = H_D(y,x), \tag{4.52}$$
$$E_D^*(x,y) = -E_D(y,x). \tag{4.53}$$

The equations of motion are then giving the relations

$$\int dy\;G_D(x,y) = G_D(x) = C(x) + ix\,f_T(x), \tag{4.54}$$
$$\int dy\;\tilde{G}_D(x,y) = \tilde{G}_D(x) = C(x) + x\,g_T(x) - \frac{m}{M}\,h_1(x), \tag{4.55}$$
$$2\int dy\;H_D(x,y) = 2\,H_D(x) = x\,h_L(x) - \frac{m}{M}\,g_1(x) + ix\,e_L(x), \tag{4.56}$$
$$2\int dy\;E_D(x,y) = 2\,E_D(x) = -x\,e(x) + \frac{m}{M}\,f_1(x) + ix\,h(x), \tag{4.57}$$

where the function $C(x)$ cannot be given in terms of quark-quark correlation functions.

From Φ_D, we can get Φ_A, in essence as $\Phi_A = \Phi_D$ - Φ_∂. For the T-even case this leads to

$$
\begin{aligned}
g\Phi_{A\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) &= \epsilon_{T\alpha\beta}\, g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T)\\
&\equiv iMx\,\tilde{e} + \epsilon_{Tij}\, k_T^i S_T^j\, x\tilde{h}_T^\perp\\
&= i\,(Mx\,e - m\,f_1) - \epsilon_{Tij}\, k_T^i S_T^j\,(h_{1T} - xh_T^\perp) \qquad (4.58)
\end{aligned}
$$

or

$$
\begin{aligned}
&g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) - g\Phi_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T)\\
&\qquad \equiv i\epsilon_T^{\alpha\beta}\, Mx\,\tilde{e} - \left(S_T^\alpha k_T^\beta - k_T^\alpha S_T^\beta\right)\, x\tilde{h}_T^\perp,\\
&\qquad = i\epsilon_T^{\alpha\beta}\,(Mx\,e - m\,f_1) + \left(S_T^\alpha k_T^\beta - k_T^\alpha S_T^\beta\right)(h_{1T} - xh_T^\perp), \qquad (4.59)
\end{aligned}
$$

$$
g\Phi_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) \equiv Mx\,\tilde{h}_s = Mx\,h_s - m\,g_{1s} - (k_T\cdot S_T)\,h_{1T} - \frac{k_T^2}{M}\,h_{1s}^\perp, \qquad (4.60)
$$

$$
\begin{aligned}
&g\Phi_A^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\, g\Phi_{A\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T)\\
&\qquad \equiv k_T^\alpha\, x\tilde{f}^\perp - i\epsilon_T^{\alpha\beta} k_{T\beta}\, x\tilde{g}_s^\perp - i\epsilon_T^{\alpha\beta} S_{T\beta}\, Mx\,\tilde{g}_T'\\
&\qquad = k_T^\alpha(xf^\perp - f_1) - i\epsilon_T^{\alpha\beta} k_{T\beta}\left(xg_s^\perp - g_{1s} - \frac{m}{M}\,h_{1s}^\perp\right) - i\epsilon_T^{\alpha\beta} S_{T\beta}\,(Mx\,g_T' - m\,h_{1T})\,, \qquad (4.61)
\end{aligned}
$$

which are useful objects as they appear as soft quark-quark-gluon parts in a diagrammatic expansion of hard scattering processes. As the operators Γ used in $\Phi_A^{\alpha[\Gamma]}$ are hermitean in the sense that $\Gamma^\dagger = \gamma_0\Gamma\gamma_0$ one has

$$
(\gamma_0\Phi_A^{\alpha\dagger}\gamma_0)^{[\Gamma]} = \left(\Phi_A^{\alpha[\Gamma]}\right)^*. \qquad (4.62)
$$

Integrated over $\boldsymbol{k}_T$ the above relations become

$$
g\Phi_{A\,\alpha}^{[\sigma^{\alpha+}]}(x) = \epsilon_{T\alpha\beta}\, g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x) \equiv i\,Mx\,\tilde{e} = i\,(Mx\,e - m\,f_1)\,, \qquad (4.63)
$$

or

$$
g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x) - g\Phi_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(x) \equiv i\epsilon_T^{\alpha\beta}\, Mx\,\tilde{e} = i\epsilon_T^{\alpha\beta}\,(Mx\,e - m\,f_1)\,, \qquad (4.64)
$$

$$
g\Phi_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x) = \lambda\, Mx\,\tilde{h}_L = \lambda\left(Mx\,h_L - m\,g_1 + 2M\,h_{1L}^{\perp(1)}\right), \qquad (4.65)
$$

$$
g\Phi_A^{\alpha[\gamma^+]}(x) - i\epsilon_T^{\alpha\beta}\, g\Phi_{A\,\beta}^{[\gamma^+\gamma_5]}(x) \equiv -i\epsilon_T^{\alpha\beta} S_{T\beta}\, Mx\,\tilde{g}_T = -i\epsilon_T^{\alpha\beta} S_{T\beta}\left(Mx\,g_T - m\,h_1 - M\,g_{1T}^{(1)}\right), \qquad (4.66)
$$

where the upper index (1) denotes

$$
f^{(1)}(x) = \int d^2\boldsymbol{k}_T\, \frac{\boldsymbol{k}_T^2}{2M^2}\, f(x,\boldsymbol{k}_T). \qquad (4.67)
$$

The tilde functions are precisely the parts vanishing for the free quark case. This was the way they have been introduced in chapter 2.

With the same parametrization for $\Phi_A^\alpha(x,y)$ as the one for $\Phi_D^\alpha(x,y)$ given above one obtains (including now the T-odd functions)

$$
\int dy\; G_A(x,y) = G_A(x) = C(x) + ix\, f_T(x) + i\, f_{1T}^{\perp(1)}(x), \qquad (4.68)
$$

$$
\int dy\; \tilde{G}_A(x,y) = \tilde{G}_A(x) = C(x) + x\, g_T(x) - \frac{m}{M}\, h_1(x) - g_{1T}^{(1)}(x), \qquad (4.69)
$$

$$
2\int dy\; H_A(x,y) = 2\, H_A(x) = x\, h_L(x) - \frac{m}{M}\, g_1(x) + 2\, h_{1L}^{\perp(1)}(x) + ix\, e_L(x), \qquad (4.70)
$$

$$
2\int dy\; E_A(x,y) = 2\, E_A(x) = -x\, e(x) + \frac{m}{M}\, f_1(x) + ix\, h(x) + 2i\, h_1^{\perp(1)}(x). \qquad (4.71)
$$

For the antiquarks one needs to consider matrix elements

$$
\overline{\Phi}_{A\,ij}^\alpha(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_i(\xi)A_T^\alpha(\xi)\overline{\psi}_j(0)|P,S\rangle = -\Phi_{A\,ij}^\alpha(-k,P,S), \qquad (4.72)
$$

$$
(\gamma_0\overline{\Phi}_A^{\alpha\dagger}\gamma_0)_{ij}(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_i(\xi)\,A_T^\alpha(0)\overline{\psi}_j(0)|P,S\rangle. \qquad (4.73)
$$

For them one obtains, using for the nonlocal quark-quark matrix element

$$\frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle P,S|\psi_i(\xi)\,\mathcal{U}\,i\partial^\mu\,\mathcal{U}\,\overline{\psi}_j(0)|P,S\rangle = -k^\mu\,\overline{\Phi}_{ij}(k,P,S) \tag{4.74}$$

(where $i\partial^\mu$ can be read as $i\,\overleftarrow{\partial}^\mu_\xi$ or $i\,\overrightarrow{\partial}^\mu_0$) and the equations of motion, the relations

$$\begin{aligned} g\overline{\Phi}_{A\,\alpha}^{\;[\sigma^{\alpha-}]}(x,\boldsymbol{k}_T) &= -\epsilon_{T\alpha\beta}\,g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x,\boldsymbol{k}_T)\\ &= i\left(Mx\,\overline{e}-m\,\overline{f}_1\right)-\epsilon_{Tij}\,k_T^i S_T^j\,(\overline{h}_{1T}-x\overline{h}_T^\perp) \end{aligned}\tag{4.75}$$

or

$$\begin{aligned} g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x,\boldsymbol{k}_T) &- g\overline{\Phi}_A^{\beta[i\sigma^{\alpha-}\gamma_5]}(x,\boldsymbol{k}_T)\\ &= -i\epsilon_T^{\alpha\beta}\left(Mx\,\overline{e}-m\,\overline{f}_1\right)-\left(S_T^\alpha k_T^\beta-k_T^\alpha S_T^\beta\right)(\overline{h}_{1T}-x\overline{h}_T^\perp), \end{aligned}\tag{4.76}$$

$$g\overline{\Phi}_{A\,\alpha}^{\;[i\sigma^{\alpha-}\gamma_5]}(x,\boldsymbol{k}_T) = -Mx\,\overline{h}_s+m\,\overline{g}_{1s}+(k_T\cdot S_T)\,\overline{h}_{1T}+\frac{k_T^2}{M}\,\overline{h}_{1s}^\perp, \tag{4.77}$$

$$\begin{aligned} g\overline{\Phi}_A^{\alpha[\gamma^-]}(x,\boldsymbol{k}_T) &+ i\epsilon_T^{\alpha\beta}\,g\overline{\Phi}_{A\,\beta}^{\;[\gamma^-\gamma_5]}(x,\boldsymbol{k}_T)\\ &= -k_T^\alpha(x\overline{f}^\perp-\overline{f}_1)+i\epsilon_T^{\alpha\beta}k_{T\beta}\left(x\overline{g}_s^\perp-\overline{g}_{1s}-\frac{m}{M}\,\overline{h}_{1s}^\perp\right)+i\epsilon_T^{\alpha\beta}S_{T\beta}\left(Mx\,\overline{g}_T'-m\,\overline{h}_{1T}\right). \end{aligned}\tag{4.78}$$

The $\boldsymbol{k}_T$-integrated result is

$$g\overline{\Phi}_{A\,\alpha}^{\;[\sigma^{\alpha-}]}(x) = -\epsilon_{T\alpha\beta}\,g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x) = i\left(Mx\,\overline{e}-m\,\overline{f}_1\right), \tag{4.79}$$

or

$$g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x)-g\overline{\Phi}_A^{\beta[i\sigma^{\alpha-}\gamma_5]}(x) = -i\epsilon_T^{\alpha\beta}\left(Mx\,\overline{e}-m\,\overline{f}_1\right), \tag{4.80}$$

$$g\overline{\Phi}_{A\,\alpha}^{\;[i\sigma^{\alpha-}\gamma_5]}(x) = -\lambda\left(Mx\,\overline{h}_L-m\,\overline{g}_1-2M\,\overline{h}_{1L}^{\perp(1)}\right), \tag{4.81}$$

$$g\overline{\Phi}_A^{\alpha[\gamma^-]}(x)+i\epsilon_T^{\alpha\beta}\,g\overline{\Phi}_{A\,\beta}^{\;[\gamma^-\gamma_5]}(x) = i\epsilon_T^{\alpha\beta}S_{T\beta}\left(Mx\,\overline{g}_T-m\,\overline{h}_1-M\,g_{1T}^{(1)}\right). \tag{4.82}$$

For the fragmentation part one needs to consider the matrix elements

$$\Delta^\alpha_{A\,ij}(k,k_1;P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\,d^4\eta\; e^{i\,k_1\cdot(\xi-\eta)+i\,k\cdot\eta}\langle 0|\psi_i(\xi)\,A_T^\alpha(\eta)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{4.83}$$

or after integration over k_1 the bilocal matrix elements

$$\Delta^\alpha_{A\,ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle 0|\psi_i(\xi)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{4.84}$$

$$(\gamma_0\Delta_A^{\alpha\dagger}\gamma_0)_{ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle 0|\psi_i(\xi)\,a_h^\dagger a_h\,A_T^\alpha(0)\,\overline{\psi}_j(0)|0\rangle. \tag{4.85}$$

The twist analysis for the projections

$$\begin{aligned} \Delta_A^{\alpha[\Gamma]}(z,\boldsymbol{k}_T) &= \frac{1}{4z}\int dk^+\;Tr\,(\Delta_A^\alpha\Gamma)\Big|_{k^-=P_h^-/z,\;\boldsymbol{k}_T}\\ &= \int\frac{d\xi^+d^2\xi_\perp}{4z\,(2\pi)^3}\;e^{i\,k\cdot\xi}\,Tr\,\langle 0|\psi(\xi)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\,\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=0}, \end{aligned}\tag{4.86}$$

is completely analogous to the distribution part. The equation of motion together with

$$\frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)\,i\partial^\mu\,a_h^\dagger a_h\,\overline{\psi}_j(0)|0\rangle = k^\mu\,\Delta_{ij}(k,P_h,S) \tag{4.87}$$

(where $i\partial^\mu$ can be read as $i\,\overleftarrow{\partial}^\mu_\xi$ or $i\,\overrightarrow{\partial}^\mu_0$), now give the relations

$$g\Delta_{A\,\alpha}^{[\sigma^{\alpha -}]} = -\epsilon_{\alpha\beta}\, g\Delta_A^{\alpha[i\sigma^{\beta -}\gamma_5]} = i\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right) - \epsilon_{ij}\, k_T^i S_{hT}^j \left(\frac{1}{z}\,\tilde{H}_T^\perp + i\,\frac{m}{M_h}\,\tilde{D}_{1T}^\perp\right)$$
$$= i\left(\frac{M_h}{z} E - m\, D_1 - i\,\frac{M_h}{z}\, H + i\,\frac{k_T^2}{M_h}\, H_1^\perp\right) + \epsilon_{ij}\, k_T^i S_{hT}^j \left(H_{1T} - \frac{1}{z} H_T^\perp - i\,\frac{m}{M_h}\, D_{1T}^\perp\right), \tag{4.88}$$

or

$$g\Delta_A^{\alpha[i\sigma^{\beta -}\gamma_5]} - g\Delta_A^{\beta[i\sigma^{\alpha -}\gamma_5]} = -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right) - \left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)\left(\frac{1}{z}\tilde{H}_T^\perp + i\,\frac{m}{M_h}\,\tilde{D}_{1T}^\perp\right)$$
$$= -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z} E - m\, D_1 - i\,\frac{M_h}{z}\, H + i\,\frac{k_T^2}{M_h}\, H_1^\perp\right) + \left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)\left(H_{1T} - \frac{1}{z} H_T^\perp - i\,\frac{m}{M_h}\, D_{1T}^\perp\right), \tag{4.89}$$

$$g\Delta_{A\,\alpha}^{[i\sigma^{\alpha -}\gamma_5]} = \frac{M_h}{z}\,\tilde{H}_s + i\,\frac{M_h}{z}\,\tilde{E}_s$$
$$= \frac{M_h}{z} H_s - m\, G_{1s} + i\,\frac{M_h}{z} E_s - (k_T\cdot S_{hT}) H_{1T} - \frac{k_T^2}{M_h}\, H_{1s}^\perp, \tag{4.90}$$

$$g\Delta_A^{\alpha[\gamma^-]} + i\epsilon_T^{\alpha\beta}\, g\Delta_{A\,\beta}^{[\gamma^-\gamma_5]} = k_T^\alpha\left(\frac{\tilde{D}^\perp}{z} + i\,\frac{m}{M_h}\,\tilde{H}_1^\perp\right) - \frac{\left(k_T^\alpha k_T^i + \frac{1}{2}\, k_T^2\, g_T^{\alpha i}\right)\epsilon_{T\,ij} S_{hT}^j}{M_h}\,\tilde{D}_{1T}^\perp$$
$$+ i\epsilon_T^{\alpha\beta} k_{T\beta}\left(\frac{\tilde{G}_s^\perp}{z} - i\,\lambda_h\,\frac{\tilde{D}_L^\perp}{z}\right) + i\epsilon_T^{\alpha\beta} S_{h\,\beta}\left(\frac{M_h}{z}\,\tilde{G}_T' - i\, M_h\,\frac{\tilde{D}_T}{z}\right)$$
$$= k_T^\alpha\left(\frac{1}{z} D^\perp - D_1 + i\,\frac{m}{M_h}\, H_1^\perp\right) - \frac{k_T^\alpha}{M_h}\,\epsilon_{ij} k_T^i S_{hT}^j\, D_{1T}^\perp$$
$$+ i\epsilon_T^{\alpha\beta} k_{T\beta}\left(\frac{1}{z} G_s^\perp - G_{1s} - \frac{m}{M_h}\, H_{1s}^\perp - i\,\lambda_h\,\frac{D_L^\perp}{z}\right)$$
$$+ i\epsilon_T^{\alpha\beta} S_{hT\beta}\left(\frac{M_h}{z} G_T' - m\, H_{1T} - i\, M_h\,\frac{D_T}{z}\right), \tag{4.91}$$

and $\Delta_A^{\alpha\dagger[\Gamma]} = (\Delta_A^{\alpha[\Gamma]})^*$. Note that the twist two profile functions $D_{1T}^\perp$, $H_1^\perp$ and the twist three profile functions E_L, E_T and H, that are odd under time reversal, enter as the imaginary parts in $\Delta_A^{\alpha[\Gamma]}$. Again the tilde functions are the 'interaction-dependent' distribution functions. Note that the time reveral odd functions are interaction-dependent, e.g. $D_{1T}^\perp = \tilde{D}_{1T}^\perp$, etc. We have, however, made the choices $D_T = \tilde{D}_T - z D_{1T}^{\perp(1)}$ and $H = \tilde{H} - z H_1^{\perp(1)}$ which guarantee the absence of $D_{1T}^\perp$ and $H_1^\perp$ in the integrated results. The $\boldsymbol{k}_T$-integrated results are

$$g\Delta_{A\,\alpha}^{[\sigma^{\alpha -}]}(z) = -\epsilon_{\alpha\beta}\, g\Delta_A^{\alpha[i\sigma^{\beta -}\gamma_5]}(z) = i\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right)$$
$$= i\left(\frac{M_h}{z} E - m\, D_1 - i\, M_h\left(\frac{H}{z} + 2\, H_1^{\perp\,(1)}\right)\right)$$
$$= i\left(\frac{M_h}{z} E - m\, D_1 - i\, M_h \frac{d}{dz}\left[z\, H_1^{\perp(1)}\right]\right) \tag{4.92}$$

or

$$g\Delta_A^{\alpha[i\sigma^{\beta -}\gamma_5]}(z) - g\Delta_A^{\beta[i\sigma^{\alpha -}\gamma_5]}(z) = -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right),$$
$$= -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z} E - m\, D_1 - i\, M_h\left(\frac{H}{z} + 2\, H_1^{\perp(1)}\right)\right), \tag{4.93}$$

$$g\Delta_{A\,\alpha}^{[i\sigma^{\alpha-}\gamma_5]}(z) = \lambda_h\left(\frac{M_h}{z}\,\tilde{H}_L + i\,\frac{M_h}{z}\,\tilde{E}_L\right) = \lambda_h\left(\frac{M_h}{z}H_L - m\,G_1 + i\,\frac{M_h}{z}E_L - 2M_h\,H_{1L}^{\perp\,(1)}\right), \tag{4.94}$$

$$\begin{aligned} g\Delta_A^{\alpha[\gamma^-]}(z) + i\epsilon_T^{\alpha\beta}\,g\Delta_{A\,\beta}^{[\gamma^-\gamma_5]}(z) &= i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,\tilde{G}_T + i\,\frac{M_h}{z}\,\tilde{D}_T\right) \\ &= i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,G_T - m\,H_1 - M_h\,G_{1T}^{(1)} + i\,M_h\,\left(\frac{D_T}{z} + D_{1T}^{\perp\,(1)}\right)\right) \\ &= i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,G_T - m\,H_1 - M_h\,G_{1T}^{(1)} - i\,M_h z\frac{d}{dz}\left[D_{1T}^{\perp(1)}\right]\right), \end{aligned} \tag{4.95}$$

where

$$D^{(1)}(z) = z^2\int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M_h^2}\,D_1(z,-z\boldsymbol{k}_T). \tag{4.96}$$

For the antiquark fragmentation part one needs to consider the matrix elements

$$\overline{\Delta}_{A\,ij}^{\alpha}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\ e^{-i\,k\cdot\xi}\langle 0|\overline{\psi}_j(0)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\psi_i(\xi)|0\rangle = -\Delta_{A\,ij}^{\alpha}(-k,P_h,S_h), \tag{4.97}$$

$$(\gamma_0\overline{\Delta}_A^{\alpha\dagger}\gamma_0)_{ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\ e^{-i\,k\cdot\xi}\langle 0|\overline{\psi}_j(0)\,a_h^\dagger a_h\,A_T^\alpha(0)\,\psi_i(\xi)|0\rangle. \tag{4.98}$$

Using the equation of motion together with

$$\frac{1}{(2\pi)^4}\int d^4\xi\ e^{-ik\cdot\xi}\ \langle 0|\overline{\psi}_j(0)\,i\partial^\mu\,a_h^\dagger a_h\,\psi_i(\xi)|0\rangle = -k^\mu\,\overline{\Delta}_{ij}(k,P_h,S) \tag{4.99}$$

(where $i\partial^\mu$ can be read as $i\,\overleftarrow{\partial}_0^\mu$ or $i\,\overrightarrow{\partial}_\xi^\mu$), now give the relations

$$\begin{aligned} g\overline{\Delta}_{A\,\alpha}^{[\sigma^{\alpha+}]} = \epsilon_{\alpha\beta}\,g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]} &= i\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - i\,\frac{k_T^2}{M_h}\,\overline{H}_1^\perp\right) \\ &\quad + \epsilon_{ij}\,k_T^i S_{hT}^j\,(\overline{H}_{1T} - \frac{1}{z}\,\overline{H}_T^\perp + i\,\frac{m}{M_h}\,\overline{D}_{1T}^\perp), \end{aligned} \tag{4.100}$$

or

$$\begin{aligned} g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]} - g\overline{\Delta}_A^{\beta[i\sigma^{\alpha+}\gamma_5]} &= i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - i\,\frac{k_T^2}{M_h}\,\overline{H}_1^\perp\right) \\ &\quad - \left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)\,(\overline{H}_{1T} - \frac{1}{z}\,\overline{H}_T^\perp + i\,\frac{m}{M_h}\,\overline{D}_{1T}^\perp), \end{aligned} \tag{4.101}$$

$$g\overline{\Delta}_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]} = -\frac{M_h}{z}\,\overline{H}_s + m\,\overline{G}_{1s} + i\,\frac{M_h}{z}\,\overline{E}_s + (k_T\cdot S_{hT})\overline{H}_{1T} + \frac{k_T^2}{M_h}\,\overline{H}_{1s}^\perp, \tag{4.102}$$

$$\begin{aligned} g\overline{\Delta}_A^{\alpha[\gamma^+]} - i\epsilon_T^{\alpha\beta}\,g\overline{\Delta}_{A\,\beta}^{[\gamma^+\gamma_5]} &= -k_T^\alpha\left(\frac{1}{z}\,\overline{D}^\perp - \overline{D}_1 - i\,\frac{m}{M_h}\,\overline{H}_1^\perp\right) - \frac{k_T^\alpha}{M_h}\,\epsilon_{ij}k_T^i S_{hT}^j\,\overline{D}_{1T}^\perp \\ &\quad - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(\frac{1}{z}\,\overline{G}_s^\perp - \overline{G}_{1s} - \frac{m}{M_h}\,\overline{H}_{1s}^\perp + i\,\lambda_h\,\frac{\overline{D}_L^\perp}{z}\right) \\ &\quad - i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,\overline{G}_T' - m\,\overline{H}_{1T} + i\,\frac{M_h}{z}\,\overline{D}_T\right). \end{aligned} \tag{4.103}$$

The $\boldsymbol{k}_T$-integrated results are

$$g\overline{\Delta}_{A\,\alpha}^{[\sigma^{\alpha+}]}(z) = \epsilon_{\alpha\beta}\,g\overline{\Delta}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(z) = i\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - 2i\,M_h\,\overline{H}_1^{\perp\,(1)}\right) \tag{4.104}$$

or

$$g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(z) - g\overline{\Delta}_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(z) = i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,E - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - 2i\,M_h\,\overline{H}_1^{\perp(1)}\right), \tag{4.105}$$

$$g\overline{\Delta}_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(z) = -\lambda_h\left(\frac{M_h}{z}\,\overline{H}_L - m\,\overline{G}_1 - i\,\frac{M_h}{z}\,\overline{E}_L - 2M_h\,\overline{H}_{1L}^{\perp\,(1)}\right), \tag{4.106}$$

$$g\overline{\Delta}_A^{\alpha[\gamma^+]}(z) - i\epsilon_T^{\alpha\beta}\, g\overline{\Delta}_{A\,\beta}^{[\gamma^+\gamma_5]}(z) = -i\epsilon_T^{\alpha\beta} S_{hT\beta}\left(\frac{M_h}{z}\,\overline{G}_T - m\,\overline{H}_1 - M_h\,\overline{G}_{1T}^{(1)} - i\,M_h\,\overline{D}_{1T}^{\perp(1)} + i\,\frac{M_h}{z}\,\overline{D}_T\right). \tag{4.107}$$

4.2 Gluon distribution functions

When one considers QCD corrections to the tree-level results one will also encounter gluon-gluon correlation functions, leading to gluon distributions. In a diagram one needs

$$\frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|A^\nu(0)A^\mu(\xi)|P,S\rangle, \tag{4.108}$$

but the gauge invariant object to consider is

$$\Gamma^{\mu\nu;\rho\sigma}(k;P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|F^{\mu\nu}(0)\,\mathcal{U}(0,\xi)\,F^{\rho\sigma}(\xi)|P,S\rangle. \tag{4.109}$$

The constraints following from hermiticity, parity and time reversal are

$$\Gamma^{\rho\sigma;\mu\nu\,*}(k,P,S) = \Gamma^{\mu\nu;\rho\sigma}(k,P,S) \qquad \text{[Hermiticity]} \tag{4.110}$$

$$\Gamma^{\mu\nu;\rho\sigma}(k,P,S) = \Gamma_{\mu\nu;\rho\sigma}(\bar{k},\bar{P},-\bar{S}) \qquad \text{[Parity]} \tag{4.111}$$

$$\Gamma^{\mu\nu;\rho\sigma\,*}(k,P,S) = \Gamma_{\mu\nu;\rho\sigma}(\bar{k},\bar{P},\bar{S}) \qquad \text{[Time reversal]} \tag{4.112}$$

where $\bar{k} = (k^0,-\boldsymbol{k})$.

In order to find antisymmetric structures we use the following tensor structures and relations between tensors,

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma},\\
&g^{[\mu}{}_{\rho}g^{\nu]}{}_{\sigma} \;=\; g^{\mu}{}_{[\rho}g^{\nu}{}_{\sigma]} \;=\; g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} - g^{\nu}{}_{\rho}g^{\mu}{}_{\sigma},\\
&A^{[\mu}g^{\nu][\rho}B^{\sigma]} \;=\; g^{\nu\rho}A^\mu B^\sigma + g^{\mu\sigma}A^\nu B^\rho - g^{\mu\rho}A^\nu B^\sigma - g^{\nu\sigma}A^\mu B^\rho.\\
&\epsilon^{\mu\nu\alpha\beta}\;\epsilon_{\rho\sigma\alpha\beta} \;=\; -2\,g^{[\mu}{}_{\rho}g^{\nu]}{}_{\sigma},\\
&\epsilon^{\mu\nu\alpha A}\;\epsilon^{\rho\sigma}{}_{\alpha}{}^{B} \;=\; \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta}\,g^{AB} - B^{[\mu}g^{\nu][\rho}A^{\sigma]}\\
&\epsilon^{\mu\nu AB}\;\epsilon^{\rho\sigma CD} \;=\; \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta}\,(g^{AC}g^{BD} - g^{AD}g^{BC})\\
&\qquad -C^{[\mu}g^{\nu][\rho}A^{\sigma]}\,g^{BD} + C^{[\mu}g^{\nu][\rho}B^{\sigma]}\,g^{AD} + D^{[\mu}g^{\nu][\rho}A^{\sigma]}\,g^{BC} - D^{[\mu}g^{\nu][\rho}B^{\sigma]}\,g^{AC}\\
&\qquad -C^{[\mu}D^{\nu]}\,A^{[\rho}B^{\sigma]},\\
&\epsilon^{\mu\nu A[\sigma}B^{\rho]} + \epsilon^{\rho\sigma A[\nu}B^{\mu]} \;=\; \epsilon^{\mu\nu\rho\sigma}\,g^{AB},\\
&\epsilon^{\mu\nu A[\sigma}B^{\rho]} - \epsilon^{\mu\nu B[\sigma}A^{\rho]} \;=\; \epsilon^{AB\nu\sigma}\,g^{\mu\rho} + \epsilon^{AB\mu\rho}\,g^{\nu\sigma} - \epsilon^{AB\mu\sigma}\,g^{\nu\rho} - \epsilon^{AB\nu\rho}\,g^{\mu\sigma}.\\
&\qquad = \epsilon^{AB\mu[\rho}g^{\sigma]\nu} - \epsilon^{AB\nu[\rho}g^{\sigma]\mu} \;=\; g^{\rho[\mu}\epsilon^{\nu]\sigma AB} - g^{\sigma[\mu}\epsilon^{\nu]\rho AB}.
\end{aligned}$$

For changing from matrix elements with $F_{\mu\nu}$ to matrix elements containing $\tilde{F}_{\mu\nu} \equiv -(1/2)\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, we note that

$$\begin{aligned}
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\;\epsilon^{\kappa\lambda\rho\sigma} &= (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\;\epsilon^{\kappa\lambda\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta} &= 2\,\epsilon^{\mu\nu\rho\sigma}\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\;\epsilon^{\kappa\lambda\alpha A}\;\epsilon^{\rho\sigma}{}_{\alpha}{}^{B} &= \epsilon^{\rho\sigma B[\nu}A^{\mu]},\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\;\epsilon^{\kappa\lambda AB}\epsilon^{\rho\sigma CD} &= A^{[\mu}B^{\nu]}\,\epsilon^{\rho\sigma CD},\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\;\epsilon^{\kappa\lambda A[\sigma}B^{\rho]} &= -A^{[\mu}g^{\nu][\rho}B^{\sigma]}.
\end{aligned}$$

A possible parametrization of $\Gamma^{\mu\nu;\rho\sigma}$ is

$$\begin{aligned}
\Gamma^{\mu\nu;\rho\sigma}(k;P,S) \;=\;& M^2\,X_1\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta}\\
&+ X_2\,P^{[\mu}g^{\nu][\rho}P^{\sigma]}
\end{aligned}$$

$$
\begin{aligned}
&+ X_3\, k^{[\mu} g^{\nu][\rho} k^{\sigma]} \\
&+ (X_4 + i\, X_5)\, P^{[\mu} g^{\nu][\rho} k^{\sigma]} \\
&+ (X_4 - i\, X_5)\, k^{[\mu} g^{\nu][\rho} P^{\sigma]} \\
&+ (X_6/M^2)\, P^{[\mu} k^{\nu]}\, P^{[\rho} k^{\sigma]} \\
&- 2\, M\, X_7\; \epsilon^{\mu\nu\rho\sigma}\, (k\cdot S) \\
&+ i\, M\, X_8\; \epsilon^{\mu\nu P[\sigma} S^{\rho]} \\
&+ i\, M\, X_9\; \epsilon^{\mu\nu S[\sigma} P^{\rho]} \\
&+ i\, M\, X_{10}\, \epsilon^{\mu\nu k[\sigma} S^{\rho]} \\
&+ i\, M\, X_{11}\, \epsilon^{\mu\nu S[\sigma} k^{\rho]} \\
&+ i\,(X_{12}/M)\, \epsilon^{\mu\nu P[\sigma} P^{\rho]}\, (k\cdot S) \\
&+ i\,(X_{13}/M)\, \epsilon^{\mu\nu k[\sigma} k^{\rho]}\, (k\cdot S) \\
&+ i\,(X_{14}/M)\, \epsilon^{\mu\nu P[\sigma} k^{\rho]}\, (k\cdot S) \\
&+ i\,(X_{15}/M)\, \epsilon^{\mu\nu k[\sigma} P^{\rho]}\, (k\cdot S) \\
&+ ((X_{16} + i\, X_{17})/M)\; \epsilon^{\mu\nu PS} k^{[\rho} P^{\sigma]} \\
&+ ((X_{16} - i\, X_{17})/M)\; \epsilon^{\rho\sigma PS} k^{[\mu} P^{\nu]} \\
&+ ((X_{18} + i\, X_{19})/M)\; \epsilon^{\mu\nu kS} k^{[\rho} P^{\sigma]} \\
&+ ((X_{18} - i\, X_{19})/M)\; \epsilon^{\rho\sigma kS} k^{[\mu} P^{\nu]} \\
&+ ((X_{20} + i\, X_{21})/M)\; \epsilon^{\mu\nu kP} P^{[\rho} S^{\sigma]} \\
&+ ((X_{20} - i\, X_{21})/M)\; \epsilon^{\rho\sigma kP} P^{[\mu} S^{\nu]} \\
&+ ((X_{22} + i\, X_{23})/M)\; \epsilon^{\mu\nu kP} k^{[\rho} S^{\sigma]} \\
&+ ((X_{22} - i\, X_{23})/M)\; \epsilon^{\rho\sigma kP} k^{[\mu} S^{\nu]} \\
&+ ((X_{24} + i\, X_{25})/M^3)\; \epsilon^{\mu\nu kP} k^{[\rho} P^{\sigma]}\, (k\cdot S) \\
&+ ((X_{24} - i\, X_{25})/M^3)\; \epsilon^{\rho\sigma kP} k^{[\mu} P^{\nu]}\, (k\cdot S),
\end{aligned}
\tag{4.113}
$$

with X_5, X_7, X_{16}, X_{18}, X_{20}, X_{22} and X_{24} being T-odd. The constraints from hermiticity imply that one finds real amplitudes X_i if the tensors symmetric under $\mu\nu \leftrightarrow \rho\sigma$ are multiplied with 1, while the tensors antisymmetric under $\mu\nu \leftrightarrow \rho\sigma$ are multiplied with i; the constraint from parity requires that even numbers of ϵ-tensors are combined only with vectors k and P, while odd numbers of ϵ-tensors are combined with the axial vector S and furthermore vectors k, P; finally the constraint from time-reversal requires that any ϵ-tensor appears multiplied with i and for the rest real stuff.

For the amplitudes with an odd number of ϵ-tensors, it is useful to realize that the quantity

$$
\tilde{\Gamma}^{\mu\nu;\rho\sigma}(k;P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\tilde{F}^{\mu\nu}(0)\,\mathcal{U}(0,\xi)\,F^{\rho\sigma}(\xi)|P,S\rangle. \tag{4.114}
$$

has the structure

$$
\begin{aligned}
\tilde{\Gamma}^{\mu\nu;\rho\sigma}(k;P,S) \quad = \quad & \ldots + M\, X_7\, \epsilon^{\mu\nu\alpha\beta}\; \epsilon^{\rho\sigma}{}_{\alpha\beta}\, (k\cdot S) \\
&- i\, M\, X_8\, P^{[\mu} g^{\nu][\rho} S^{\sigma]} \\
&- i\, M\, X_9\, S^{[\mu} g^{\nu][\rho} P^{\sigma]} \\
&- i\, M\, X_{10}\, k^{[\mu} g^{\nu][\rho} S^{\sigma]} \\
&- i\, M\, X_{11}\, S^{[\mu} g^{\nu][\rho} k^{\sigma]} \\
&- i\,(X_{12}/M)\, P^{[\mu} g^{\nu][\rho} P^{\sigma]}\, (k\cdot S) \\
&- i\,(X_{13}/M)\, k^{[\mu} g^{\nu][\rho} k^{\sigma]}\, (k\cdot S) \\
&- i\,(X_{14}/M)\, P^{[\mu} g^{\nu][\rho} k^{\sigma]}\, (k\cdot S) \\
&- i\,(X_{15}/M)\, k^{[\mu} g^{\nu][\rho} P^{\sigma]}\, (k\cdot S) \\
&+ ((X_{16} + i\, X_{17})/M)\, P^{[\mu} S^{\nu]} k^{[\rho} P^{\sigma]} \\
&- ((X_{16} - i\, X_{17})/M)\; \epsilon^{\mu\nu kP}\; \epsilon^{\rho\sigma PS} \\
&+ ((X_{18} + i\, X_{19})/M)\, k^{[\mu} S^{\nu]} k^{[\rho} P^{\sigma]} \\
&- ((X_{18} - i\, X_{19})/M)\; \epsilon^{\mu\nu kP}\; \epsilon^{\rho\sigma kS}
\end{aligned}
$$

$$
\begin{aligned}
&+((X_{20}+i\,X_{21})/M)\,k^{[\mu}P^{\nu]}P^{[\rho}S^{\sigma]}\\
&-((X_{20}-i\,X_{21})/M)\,\epsilon^{\mu\nu PS}\,\epsilon^{\rho\sigma kP}\\
&+((X_{22}+i\,X_{23})/M)\,k^{[\mu}P^{\nu]}k^{[\rho}S^{\sigma]}\\
&-((X_{22}-i\,X_{23})/M)\,\epsilon^{\mu\nu kS}\,\epsilon^{\rho\sigma kP}\\
&+((X_{24}+i\,X_{25})/M^3)\,k^{[\mu}P^{\nu]}k^{[\rho}P^{\sigma]}\,(k\cdot S)\\
&-((X_{24}-i\,X_{25})/M^3)\,\epsilon^{\mu\nu kP}\,\epsilon^{\rho\sigma kP}\,(k\cdot S).
\end{aligned}
\tag{4.115}
$$

In the next step we try to isolate the dominant parts by expanding the vectors in lightlike and transverse vectors, and the invariants $\sigma \equiv 2k\cdot P$ and $\tau = k^2$,

$$P = A\,n_+^\mu + \frac{M^2}{2A}\,n_-^\mu, \tag{4.116}$$

$$k = xA\,n_+^\mu + k_T^\mu + \frac{\sigma - xM^2}{2A}\,n_-^\mu, \quad \text{or} \quad k - xP = k_T^\mu + \frac{\sigma-2xM^2}{2\,P\cdot n_-}\,n_-^\mu, \tag{4.117}$$

$$S = \lambda\,\frac{A}{M}\,n_+^\mu + S_T - \lambda\,\frac{M}{2A}\,n_-^\mu, \qquad \text{or} \qquad S - \lambda\,\frac{P}{M} = S_T^\mu - \lambda\,\frac{M}{P\cdot n_-}\,n_-^\mu, \tag{4.118}$$

and we have

$$k\cdot S = \lambda\,\frac{\sigma - 2xM^2}{2M} + k_T\cdot S_T. \tag{4.119}$$

The tensors can be expressed as

$$
\begin{aligned}
g^{\mu\nu} &= n_+^{\{\mu}n_-^{\nu\}} + g_T^{\mu\nu}, &(4.120)\\
\epsilon^{\mu\nu\rho\sigma} &= -n_+^{[\mu}n_-^{\nu]}\,\epsilon_T^{\rho\sigma} + n_+^{[\mu}n_-^{\rho]}\,\epsilon_T^{\nu\sigma} - n_+^{[\mu}n_-^{\sigma]}\,\epsilon_T^{\nu\rho}\\
&\quad - n_+^{[\nu}n_-^{\rho]}\,\epsilon_T^{\mu\sigma} + n_+^{[\nu}n_-^{\sigma]}\,\epsilon_T^{\mu\rho} - n_+^{[\rho}n_-^{\sigma]}\,\epsilon_T^{\mu\nu}. &(4.121)
\end{aligned}
$$

The dominant parts are the ones containing the hightest powers of $A = P^+$, e.g. for the unpolarized part we have in order of importance,

$$\Gamma^{+\alpha;+\beta}(k,P,S) = A^2\left\{-g_T^{\alpha\beta}\left[X_2 + 2x\,X_4 + x^2\,X_3\right] + \frac{k_T^\alpha k_T^\beta}{M^2}\,X_6\right\}, \tag{4.122}$$

$$\Gamma^{+\alpha;+-}(k,P,S) = A\,k_T^\alpha\left[(X_4 + x\,X_3) - i\,X_5 + \left(\frac{\sigma-2xM^2}{2M^2}\right)X_6\right] \tag{4.123}$$

$$\Gamma^{+\alpha;\beta\gamma}(k,P,S) = A\,g_T^{\alpha[\beta}k_T^{\gamma]}\left[(X_4 + x\,X_3) + i\,X_5\right], \tag{4.124}$$

etc. Integrating over k^- we then find that the leading functions (α and β being transverse) are

$$
\begin{aligned}
\Gamma_2^{\alpha\beta}(x,\boldsymbol{k}_T) &= \int dk^-\;\Gamma^{+\alpha;+\beta}(k;P,S)\\
&= \frac{x\,P^+}{2}\Bigg(-g_T^{\alpha\beta}\,G(x,\boldsymbol{k}_T) - g_T^{\alpha\beta}\,\frac{\epsilon_T^{ij}k_{Ti}S_{Tj}}{M}\,G_T(x,\boldsymbol{k}_T)\\
&\qquad + \left(k_T^\alpha k_T^\beta + \frac{1}{2}\,g_T^{\alpha\beta}\,\boldsymbol{k}_T^2\right)\frac{H^\perp(x,\boldsymbol{k}_T)}{M^2}\\
&\qquad - i\,\epsilon_T^{\alpha\beta}\left[\lambda\,\Delta G_L(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\Delta G_T(x,\boldsymbol{k}_T)\right]\\
&\qquad - \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{2M^2}\left[\lambda\,\Delta H_L^\perp(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\Delta H_T^\perp(x,\boldsymbol{k}_T)\right]\\
&\qquad - \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}S_{Ti} + S_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{4M}\left[\Delta H_T(x,\boldsymbol{k}_T) - \Delta H_T^{\perp(1)}(x,\boldsymbol{k}_T)\right]\Bigg).
\end{aligned}
\tag{4.125}
$$

These give the leading (twist two) distribution functions, G_T, $\Delta H_s^\perp$ and ΔH_T being T-odd. Note that the tensor multiplying G_T is actually the antisymmetric version of the tensor multiplying ΔH_T. The use of the combination $\Delta H_T' \equiv \Delta H_T + \Delta H_T^{\perp(1)}$ where $\Delta H_T^{\perp(1)} = (\boldsymbol{k}_T^2/2M^2)\Delta H_T^\perp$ will become clear when we consider explicit representations. At subleading order we have

$$\begin{aligned}\Gamma_3^\alpha(x,\boldsymbol{k}_T) &= \int dk^- \;\Gamma^{+\alpha;+-}(k;P,S) \\ &= \frac{x\,M}{2}\left(i\,\epsilon_T^{\alpha\beta}\,S_{T\,\beta}\,\Delta G_{3T}'(x,\boldsymbol{k}_T) + \frac{i\,\epsilon_T^{\alpha\beta}\,k_{T\,\beta}}{M}\,\Delta G_{3s}^\perp(x,\boldsymbol{k}_T) + \frac{k_T^\alpha}{M}\,G_3^\perp(x,\boldsymbol{k}_T)\right),\end{aligned} \tag{4.126}$$

where the imaginary parts of these functions are T-odd. Furthermore

$$\begin{aligned}\Gamma_3^{\alpha;\beta\gamma}(x,\boldsymbol{k}_T) &= \int dk^- \;\Gamma^{+\alpha;\beta\gamma}(k;P,S) \\ &= \frac{x\,M}{2}\left(\frac{g_T^{\alpha[\beta}k_T^{\gamma]}}{M}\,H_3^\perp(x,\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,S_T^\alpha\,\Delta H_{3T}'(x,\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,\frac{k_T^\alpha}{M}\,\Delta H_{3s}^\perp(x,\boldsymbol{k}_T)\right),\end{aligned} \tag{4.127}$$

where the imaginary parts of these functions are T-odd.

After integration over the transverse momenta, we obtain

$$\Gamma_2^{\alpha\beta}(x) = \frac{x\,P^+}{2}\left(-g_T^{\alpha\beta}\,G(x) + i\,\epsilon_T^{\alpha\beta}\,\lambda\,\Delta G(x)\right), \tag{4.128}$$

$$\Gamma_3^\alpha(x) = \frac{x\,M}{2}\,i\,\epsilon_T^{\alpha\beta}\,S_{T\,\beta}\,\Delta G_{3T}(x), \tag{4.129}$$

$$\Gamma_3^{\alpha;\beta\gamma}(x) = \frac{x\,M}{2}\,i\,\epsilon_T^{\beta\gamma}\,S_T^\alpha\,\Delta H_{3T}(x), \tag{4.130}$$

where $\Delta G_{3T} = \Delta G_{3T}' + \Delta G_{3T}^{\perp(1)}$ and $\Delta H_{3T} = \Delta H_{3T}' + \Delta H_{3T}^{\perp(1)}$, of which the imaginary parts are T-odd.

Examples of the amplitude expansion for the various functions are

$$xG(x,\boldsymbol{k}_T) = \int \ldots \left[X_2 + 2x\,X_4 + x^2\,X_3 + \frac{\boldsymbol{k}_T^2}{2M^2}\,X_6\right], \tag{4.131}$$

$$xH^\perp(x,\boldsymbol{k}_T) = \int \ldots [X_6], \tag{4.132}$$

$$\mathcal{R}e\,xG_3^\perp(x,\boldsymbol{k}_T) = \int \ldots \left[(X_4 + x\,X_3) + \left(\frac{\sigma - 2xM^2}{2M^2}\right)X_6\right], \tag{4.133}$$

$$\mathcal{I}m\,xG_3^\perp(x,\boldsymbol{k}_T) = \int \ldots [-X_5] \tag{4.134}$$

where

$$\int \ldots = \int d\sigma d\tau\;\delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma).$$

From the matrix elements and the commutation properties of the gluon fields one immediately gets symmetry relations like $G(-x) = -G(x)$, etc. This differs from the case of quarks, where similar relations connect quark and antiquark distributions at positive and negative x-values. About the normalization, we note that for an unpolarized target

$$\int dk^+\;\Gamma_2^{\alpha\beta}(x) = -g_T^{\alpha\beta}\,\frac{(P^+)^2}{2}\int_{-\infty}^{\infty} dx\;xG(x). \tag{4.135}$$

The left-handside is equal to

$$\int d^4k\;\Gamma^{+\alpha;+\beta}(k;P,S) = \langle P|F^{+\alpha}(0)F^{+\beta}(0)|P\rangle. \tag{4.136}$$

Contracting α and β, using the support restrictions $-1 \le x \le 1$ and the symmetry relation $G(-x) = -G(x)$, one finds that

$$\int d^4k\;\Gamma^{+\alpha;+}{}_\alpha(k;P,S) = 2(P^+)^2\underbrace{\int_0^1 dx\;xG(x)}_{\epsilon_G} = \langle P|\underbrace{F^{+\alpha}(0)F^+{}_\alpha(0)}_{\theta_G^{++}(0)}|P\rangle \tag{4.137}$$

and realizing that the right-handside is only the gluonic part of the energy momentum tensor, $\int_0^1 dx\, xG(x) = \epsilon_G \le 1$, while the chosen normalization assures the complementarity with the quark part of the energy momentum tensor discussed earlier, $\epsilon_q + \epsilon_G = 1$.

4.3 Explicit spin representation

The correlator $\Gamma_2^{\alpha\beta}$ contains two transverse gluon fields and can be interpreted in terms of gluon distribution functions. To make the gluon spin explicit, it is useful to consider the explicit matrix $M^{\alpha\beta} = (2/xP^+)\Gamma_2^{\alpha\beta}$ with α and β being transverse indices. The result for M is

$$\begin{pmatrix} \begin{array}{l} G + \frac{\boldsymbol{k}_T\wedge \boldsymbol{S}_T}{M}\,G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\; H^\perp \\ \quad + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\;\Delta H_s^\perp + \frac{(k_T^1 S_T^2 + k_T^2 S_T^1)}{2M}\,\Delta H_T' \end{array} & \begin{array}{l} -i\,S_L\,\Delta G_L - i\,\frac{\boldsymbol{k}_T\cdot \boldsymbol{S}_T}{M}\,\Delta G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\; H^\perp \\ \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\;\Delta H_s^\perp - \frac{(k_T^1 S_T^1 - k_T^2 S_T^2)}{2M}\,\Delta H_T' \end{array} \\ \\ \begin{array}{l} i\,S_L\,\Delta G_L + i\,\frac{\boldsymbol{k}_T\cdot \boldsymbol{S}_T}{M}\,\Delta G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\; H^\perp \\ \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\;\Delta H_s^\perp - \frac{(k_T^1 S_T^1 - k_T^2 S_T^2)}{2M}\,\Delta H_T' \end{array} & \begin{array}{l} G + \frac{\boldsymbol{k}_T\wedge \boldsymbol{S}_T}{M}\,G_T - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\; H^\perp \\ \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\;\Delta H_s^\perp - \frac{(k_T^1 S_T^2 + k_T^2 S_T^1)}{2M}\,\Delta H_T' \end{array} \end{pmatrix} \tag{4.138}$$

where (for later convenience given also in spherical vector components and in matrix form in the nucleon spin-space)

$$\boldsymbol{k}_T\cdot \boldsymbol{S}_T = k_T^1 S_T^1 + k_T^2 S_T^2 = -g_T^{kS} = -(k_T^+ S_T^- + k_T^- S_T^+) = \begin{pmatrix} 0 & |k_T|\,e^{-i\phi} \\ |k_T|\,e^{+i\phi} & 0 \end{pmatrix}, \tag{4.139}$$

$$\boldsymbol{k}_T\wedge \boldsymbol{S}_T = k_T^1 S_T^2 - k_T^2 S_T^1 = \epsilon_T^{kS} = -i(k_T^+ S_T^- - k_T^- S_T^+) = \begin{pmatrix} 0 & -i\,|k_T|\,e^{-i\phi} \\ i\,|k_T|\,e^{+i\phi} & 0 \end{pmatrix}, \tag{4.140}$$

$$k_T^1 S_T^1 - k_T^2 S_T^2 = k_T^+ S_T^+ + k_T^- S_T^- = \begin{pmatrix} 0 & |k_T|\,e^{+i\phi} \\ |k_T|e^{-i\phi} & 0 \end{pmatrix}, \tag{4.141}$$

$$k_T^1 S_T^2 + k_T^2 S_T^1 = -i(k_T^+ S_T^+ - k_T^- S_T^-) = \begin{pmatrix} 0 & -i\,|k_T|\,e^{+i\phi} \\ i\,|k_T|\,e^{+i\phi} & 0 \end{pmatrix}. \tag{4.142}$$

We now have used transverse gluon polarizations. Instead we can use circular polarizations,

$$\begin{aligned} |+\rangle &= -\frac{1}{\sqrt{2}}\,(|x\rangle + i\,|y\rangle)\,, \\ |-\rangle &= \frac{1}{\sqrt{2}}\,(|x\rangle - i\,|y\rangle)\,, \end{aligned} \tag{4.143}$$

and we obtain

$$\begin{pmatrix} M^{++} & M^{+-} \\ \\ M^{-+} & M^{--} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(M^{11}+M^{22}) - \mathrm{Im}\,M^{12} & -\frac{1}{2}(M^{11}-M^{22}) + i\,\mathrm{Re}\,M^{12} \\ \\ -\frac{1}{2}(M^{11}-M^{22}) - i\,\mathrm{Re}\,M^{12} & \frac{1}{2}(M^{11}+M^{22}) + \mathrm{Im}\,M^{12} \end{pmatrix}. \tag{4.144}$$

For the above matrix we find for circularly polarized gluons

$$\begin{pmatrix} G + \frac{\boldsymbol{k}_T\wedge \boldsymbol{S}_T}{M}\,G_T + S_L\,\Delta G_L + \frac{\boldsymbol{k}_T\cdot \boldsymbol{S}_T}{M}\,\Delta G_T & -\frac{\boldsymbol{k}_T^2}{2M^2}\,e^{-2i\phi}\left[H^\perp + i\,\Delta H_s^\perp\right] - i\,\frac{k_T^- S_T^-}{M}\,\Delta H_T' \\ \\ -\frac{\boldsymbol{k}_T^2}{2M^2}\,e^{+2i\phi}\left[H^\perp - i\,\Delta H_s^\perp\right] + i\,\frac{k_T^+ S_T^+}{M}\,\Delta H_T' & G + \frac{\boldsymbol{k}_T\wedge \boldsymbol{S}_T}{M}\,G_T - S_L\,\Delta G_L - \frac{\boldsymbol{k}_T\cdot \boldsymbol{S}_T}{M}\,\Delta G_T \end{pmatrix} \tag{4.145}$$

Extended into a 4×4 matrix in gluon $\otimes$ nucleon spin space we obtain

$$\left(\begin{array}{cccc} G+\Delta G & \frac{|k_T| e^{-i\phi}}{M}\left[\Delta G_T - i\, G_T\right] & -e^{-2i\phi}\left[H^{\perp(1)} + i\,\Delta H_L^{\perp(1)}\right] & -i\frac{|k_T| e^{-3i\phi}}{M}\,\Delta H_T^{\perp(1)} \\ \frac{|k_T| e^{+i\phi}}{M}\left[\Delta G_T + i\, G_T\right] & G-\Delta G & -i\frac{|k_T| e^{-i\phi}}{M}\,\Delta H_T & -e^{-2i\phi}\left[H^{\perp(1)} - i\,\Delta H_L^{\perp(1)}\right] \\ -e^{+2i\phi}\left[H^{\perp(1)} - i\,\Delta H_L^{\perp(1)}\right] & i\frac{|k_T| e^{+i\phi}}{M}\,\Delta H_T & G-\Delta G & -\frac{|k_T| e^{-i\phi}}{M}\left[\Delta G_T + i\, G_T\right] \\ i\frac{|k_T| e^{+3i\phi}}{M}\,\Delta H_T^{\perp(1)} & -e^{+2i\phi}\left[H^{\perp(1)} + i\,\Delta H_L^{\perp(1)}\right] & -\frac{|k_T| e^{+i\phi}}{M}\left[\Delta G_T - i\, G_T\right] & G+\Delta G \end{array}\right) \tag{4.146}$$

4.4 Gluon fragmentation functions

For the fragmentation functions one needs the matrix element

$$\hat\Gamma^{\mu\nu;\rho\sigma}(k;P_h,S_h) = \frac{1}{(2\pi)^4}\sum_X \int d^4\xi\; e^{i\,k\cdot\xi}\langle 0|F^{\mu\nu}(\xi)|P_h,X\rangle\,\langle P_h,X|F^{\rho\sigma}(0)|0\rangle. \tag{4.147}$$

In this case no constraints arise from time reversal invariance and the most general expansion of the matrix element becomes,

$$\begin{aligned} \hat\Gamma_2^{\alpha\beta}(z,\boldsymbol{k}_T) &= \int dk^+\;\hat\Gamma^{-\alpha;-\beta}(k;P_h,S_h) \\ &= P_h^-\Bigg(-g_T^{\alpha\beta}\,\hat G(z,-z\boldsymbol{k}_T) - g_T^{\alpha\beta}\,\frac{\epsilon_T^{ij}\,k_{Ti}S_{hT\,j}}{M_h}\,\hat G_T(z,-z\boldsymbol{k}_T) \\ &\quad + \left(k_T^\alpha k_T^\beta + \frac{1}{2}\,g_T^{\alpha\beta}\,\boldsymbol{k}_T^2\right)\frac{\hat G^\perp(z,-z\boldsymbol{k}_T)}{M_h^2} \\ &\quad + i\,\epsilon_T^{\alpha\beta}\left[\lambda_h\,\Delta\hat G_L(z,-z\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_{hT}}{M_h}\Delta\hat G_T(z,-z\boldsymbol{k}_T)\right] \\ &\quad + \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{M_h^2}\left[\lambda_h\,\Delta\hat G_L^\perp(z,-z\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_{hT}}{M_h}\Delta\hat G_T^\perp(z,-z\boldsymbol{k}_T)\right] \\ &\quad + \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}S_{hT\,i} + S_{hT}^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{2M_h}\,\Delta\hat G_T^{\perp\prime}(z,-z\boldsymbol{k}_T)\Bigg). \end{aligned} \tag{4.148}$$

These give the leading (twist two) fragmentation functions. At subleading order we have

$$\begin{aligned} \hat\Gamma_3^{\alpha}(z,-z\boldsymbol{k}_T) &= \int dk^+\;\hat\Gamma_3^{-\alpha;-+}(k;P,S) \\ &= M_h\Bigg(i\,\epsilon_T^{\alpha\beta}\,S_{hT\,\beta}\,\Delta\hat G_{3T}'(z,-z\boldsymbol{k}_T) + \frac{i\,\epsilon_T^{\alpha\beta}\,k_{T\,\beta}}{M}\,\Delta\hat G_{3s}^\perp(z,-z\boldsymbol{k}_T) \\ &\quad + \frac{k_T^\alpha}{M_h}\,\hat G_3^\perp(z,-z\boldsymbol{k}_T)\Bigg). \end{aligned} \tag{4.149}$$

Furthermore

$$\begin{aligned} \hat\Gamma_3^{\alpha;\beta\gamma}(z,-z\boldsymbol{k}_T) &= \int dk^+\;\hat\Gamma^{-\alpha;\beta\gamma}(k;P,S) \\ &= M_h\Bigg(\frac{g_T^{\alpha[\beta}k_T^{\gamma]}}{M_h}\,\hat H_3^\perp(z,-z\boldsymbol{k}_T) + \frac{k_T^{[\beta}S_{hT}^{\gamma]}}{M_h}\,\frac{i\epsilon_T^{\alpha\delta}k_{T\delta}}{M_h}\,\hat H_{3T}^\perp(z,-z\boldsymbol{k}_T) \\ &\quad + i\,\epsilon_T^{\beta\gamma}\,S_{hT}^\alpha\,\Delta\hat H_{3T}'(z,-z\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,\frac{k_T^\alpha}{M_h}\,\Delta\hat H_{3s}^\perp(z,-z\boldsymbol{k}_T)\Bigg). \end{aligned} \tag{4.150}$$

After integration over the transverse momenta, we obtain

$$\hat{\Gamma}_2^{\alpha\beta}(z) = \frac{P_h^-}{z^2}\left(-g_T^{\alpha\beta}\,\hat{G}(z) + i\,\epsilon_T^{\alpha\beta}\,\lambda_h\,\Delta\hat{G}(z)\right). \tag{4.151}$$

$$\hat{\Gamma}_3^{\alpha}(z) = \frac{M_h}{z^2}\;\epsilon_T^{\alpha\beta} S_{hT\beta}\,\Delta\hat{G}_{3T}(z), \tag{4.152}$$

$$\hat{\Gamma}_3^{\alpha;\beta\gamma}(x) = \frac{M_h}{z^2}\,i\,\epsilon_T^{\beta\gamma}\,S_{hT}^{\alpha}\,\Delta\hat{H}_{3T}(z), \tag{4.153}$$

with $\Delta\hat{G}_{3T} = \Delta\hat{G}'_{3T} + \Delta\hat{G}_{3T}^{\perp(1)}$, and $\Delta\hat{H}_{3T} = \Delta\hat{H}'_{3T} + \Delta\hat{H}_{3T}^{\perp(1)}$.

Chapter 5

Drell-Yan up to $\mathcal{O}(1/Q)$

5.1 The hadron tensor

Up to $\mathcal{O}(1/Q)$ one needs to include the contributions of the handbag diagram, now calculated up to this order with in addition irreducible diagrams with one gluon coupling either to the soft part involving hadron A or the soft part involving hadron B. The expressions thus involve the quark-gluon correlation functions. The full result is neglecting $1/Q^2$ contributions given by

$$\begin{aligned}
2M\,\mathcal{W}_{\mu\nu} &= \int dk_a^-\,dk_b^+\,d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\\
&\qquad\Big\{\mathrm{Tr}\left(\Phi(k_a)\gamma_\mu\overline{\Phi}(k_b)\gamma_\nu\right)\\
&\qquad -\mathrm{Tr}\left(\gamma_\alpha\frac{\not{n}_+}{-2q^-}\gamma_\nu\Phi_A^\alpha(k_a)\gamma_\mu\overline{\Phi}(k_b)\right)-\mathrm{Tr}\left(\gamma_\mu\frac{\not{n}_+}{-2q^-}\gamma_\alpha\overline{\Phi}(k_b)\gamma_\nu\Phi_A^{\alpha\dagger}(k_a)\right)\\
&\qquad -\mathrm{Tr}\left(\gamma_\nu\frac{\not{n}_-}{(2q^+)}\gamma_\alpha\Phi(k_a)\gamma_\mu\overline{\Phi}_A^{\alpha\dagger}(k_b)\right)-\mathrm{Tr}\left(\gamma_\alpha\frac{\not{n}_-}{(2q^+)}\gamma_\mu\overline{\Phi}_A^{\alpha}(k_b)\gamma_\nu\Phi(k_a)\right)\Big\}. \qquad (5.1)
\end{aligned}$$

Here the terms with $\not{n}_\pm$ arise from fermion propagators in the hard part neglecting contributions that will appear suppressed by powers of Q^2, i.e.

$$\frac{\not{p}_1-\not{q}+m}{(p_1-q)^2}=\frac{(p_1^+-q^+)\gamma^-}{-2(p_1^+-q^+)q^-}=\frac{\gamma^-}{-2q^-}=\frac{\not{n}_+}{-2q^-}=\frac{x_A\not{P}_A}{Q^2}, \qquad (5.2)$$

$$\frac{\not{q}-\not{k}_1+m}{(q-k_1)^2}=\frac{(q^--k_1^-)\gamma^+}{2(q-k_1^-)q^+}=\frac{\gamma^+}{(2q^+)}=\frac{\not{n}_-}{(2q^+)}=-\frac{x_B\not{P}_B}{Q^2}. \qquad (5.3)$$

The full result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is [Check sign of $g_1\,\bar{g}_1$ and $1/Q$ terms]

$$\begin{aligned}
\mathcal{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\times\Big\{-g_\perp^{\mu\nu}\Big[f_1\bar{f}_1+g_{1s}\bar{g}_{1s}\Big]\\
&\quad -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp-\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}}+(\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp\\
&\quad -\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1s}^\perp\bar{h}_{1T}-\left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\,h_{1T}\bar{h}_{1T}\\
&\quad +\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\Big[-f_1\bar{f}_1+2x_A\,f^\perp\bar{f}_1+g_{1s}\bar{g}_{1s}-2x_A\,g_s^\perp\bar{g}_{1s}\\
&\qquad -\frac{M_B}{M_A}\,2x_B\,h_{1s}^\perp\bar{h}_s+\frac{m}{M_A}\,2\,h_{1s}^\perp\bar{g}_{1s}-\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp\\
&\qquad -\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1s}^\perp\bar{h}_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_A\,h_T^\perp\bar{h}_{1s}^\perp
\end{aligned}$$

$$
\begin{aligned}
&\qquad -\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T} + \boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\Bigg]\\
&+\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[f_1\bar{f}_1 - 2x_{\scriptscriptstyle B}\,f_1\bar{f}^\perp - g_{1s}\bar{g}_{1s} + 2x_{\scriptscriptstyle B}\,g_{1s}\bar{g}_s^\perp\\
&\qquad +\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_s\bar{h}_{1s}^\perp - \frac{m}{M_B}\,2\,g_{1s}\bar{h}_{1s}^\perp + \frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp\\
&\qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1s}^\perp\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1s}^\perp\bar{h}_T^\perp\\
&\qquad +\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T} - \boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\Bigg]\\
&+\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\Bigg[-2x_{\scriptscriptstyle A}\,g_T'\bar{g}_{1s} - \frac{M_B}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_s + \frac{m}{M_A}\,2\,h_{1T}\bar{g}_{1s}\\
&\qquad -\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,h_{1T}\bar{h}_{1s}^\perp + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,h_{1T}\bar{h}_{1s}^\perp - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1s}^\perp\\
&\qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\Bigg]\\
&+\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\Bigg[\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_s\bar{h}_{1T} + 2x_{\scriptscriptstyle B}\,g_{1s}\bar{g}_T' - \frac{m}{M_B}\,2\,g_{1s}\bar{h}_{1T}\\
&\qquad +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,2x_{\scriptscriptstyle B}\,h_{1s}^\perp\bar{h}_T^\perp\\
&\qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\Bigg]\Bigg\}, \qquad (5.4)
\end{aligned}
$$

where the quark distribution functions in hadron A depend on $x_{\scriptscriptstyle A}$ and $\boldsymbol{k}_{aT}^2$, $f_1(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT}^2)$ etc., while the antiquark distribution functions in hadron B depend on $x_{\scriptscriptstyle B}$ and $\boldsymbol{k}_{bT}^2$, $\bar{f}_1(x_{\scriptscriptstyle B},\boldsymbol{k}_{bT}^2)$ etc., and we use the shorthand notations

$$
\begin{aligned}
g_{1s} &= \lambda_A\,g_{1L} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,g_{1T},\\
\bar{g}_{1s} &= \lambda_B\,\bar{g}_{1L} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,\bar{g}_{1T}.
\end{aligned}
$$

In many cases it is convenient to express the tensors with respect to measured directions, i.e. $\hat{x}$ and $\hat{y}$, $\hat{y}^\mu = \epsilon_\perp^{\mu\rho}\hat{x}_\rho$. One can use

$$
\begin{aligned}
a_\perp^\mu &= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\epsilon_\perp^{\rho\sigma}x_\rho a_{\perp\sigma})\,\hat{y}^\mu\\
&= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\hat{\boldsymbol{x}}\wedge\boldsymbol{a}_\perp)\,\hat{y}^\mu\\
&= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\hat{\boldsymbol{y}}\cdot\boldsymbol{a}_\perp)\,\hat{y}^\mu\\
&= a^x\,\hat{x}^\mu + a^y\,\hat{y}^\mu, \qquad (5.5)
\end{aligned}
$$

where $\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp \equiv \epsilon_\perp^{\rho\sigma}a_{\perp\rho}b_{\perp\sigma}$. Other combinations give

$$
\begin{aligned}
&\epsilon_\perp^{\mu\rho}a_{\perp\rho} = -a^y\,\hat{x}^\mu + a^x\,\hat{y}^\mu, \qquad (5.6)\\
&a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu} = (a^xb^x - a^yb^y)\,(2\,\hat{x}^\mu\hat{x}^\nu + g_\perp^{\mu\nu})\\
&\qquad\qquad + (a^xb^y + a^yb^x)\;\hat{x}^{\{\mu}\hat{y}^{\nu\}}, \qquad (5.7)\\
&a_\perp^{[\mu}b_\perp^{\nu]} = (a^xb^y - a^yb^x)\;\hat{x}^{[\mu}\hat{y}^{\nu]}, \qquad (5.8)\\
&\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right) = -\,(a^xb^y + a^yb^x)\,(2\,\hat{x}^\mu\hat{x}^\nu + g_\perp^{\mu\nu})\\
&\qquad\qquad + (a^xb^x - a^yb^y)\;\hat{x}^{\{\mu}\hat{y}^{\nu\}}. \qquad (5.9)
\end{aligned}
$$

Table 5.1: Contractions of the lepton tensor $L_{\mu\nu}^{(DY)}$ with tensor structures appearing in the hadron tensor.

$w^{\mu\nu}$	$L_{\mu\nu} w^{\mu\nu}$
$-g_\perp^{\mu\nu}$	$4Q^2\,\left(\frac{1}{2}-y+y^2\right)$
$a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu}$	$-4Q^2\,y\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\cos(\phi_a+\phi_b)$
$\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right)$	$4Q^2\,y\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_a+\phi_b)$
$\hat{z}^{\{\mu}a_\perp^{\nu\}}$	$-4Q^2\,(1-2y)\sqrt{y(1-y)}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$
$\hat{z}^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}$	$4Q^2\,(1-2y)\sqrt{y(1-y)}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$

Because the transverse direction is fixed by $\boldsymbol{q}_T = Q_T\,\hat{\boldsymbol{x}}$, one has specifically

$$k_{a\perp}^\mu + k_{b\perp}^\mu = Q_T\,\hat{x}^\mu, \tag{5.10}$$

$$k_{a\perp}^\mu - k_{b\perp}^\mu = \frac{\boldsymbol{k}_{a\perp}^2 - \boldsymbol{k}_{b\perp}^2}{Q_T}\,\hat{x}^\mu + 2\,\frac{\epsilon_\perp^{\rho\sigma}k_{a\perp}^\rho k_{b\perp}^\sigma}{Q_T}\,\hat{y}^\mu. \tag{5.11}$$

This allows one to pull the tensor structure outside the integration over transverse momenta.

The contractions with the lepton tensor, given in Table 5.1 use azimuthal angles defined with respect to the orthogonal momentum $\hat{\ell}$,

$$\hat{\ell}\cdot a_\perp = -|\boldsymbol{a}_\perp|\,\cos(\phi_a), \tag{5.12}$$

$$\hat{\ell}_\mu\epsilon_\perp^{\mu\nu}a_{\perp\nu} = |\boldsymbol{a}_\perp|\,\sin(\phi_a). \tag{5.13}$$

5.2 Drell-Yan after integration over transverse momenta

The hadronic tensor simplifies to

$$\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; & \frac{1}{3}\Bigg\{-g_\perp^{\mu\nu}\Big[f_1(x_A)\bar{f}_1(x_B) + \lambda_A\,\lambda_B\,g_1(x_A)\bar{g}_1(x_B)\Big] \\
& - \left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} - S_{A\perp}\cdot S_{B\perp}\,g_\perp^{\mu\nu}\right)\;h_1(x_A)\bar{h}_1(x_B) \\
& -\lambda_B\,\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\,x_A\;\left(g_T(x_A)+\tilde{g}_T(x_A)\right)\,\bar{g}_1(x_B) \\
& -\lambda_B\,\frac{M_B\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\,x_B\,h_1(x_A)\;\left(\bar{h}_L(x_B)+\bar{\tilde{h}}_L(x_B)\right) \\
& +\lambda_A\,\frac{M_A\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_A\;\left(h_L(x_A)+\tilde{h}_L(x_A)\right)\,\bar{h}_1(x_B) \\
& +\lambda_A\,\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_B\,g_1(x_A)\,\left(\bar{g}_T(x_B)+\bar{\tilde{g}}_T(x_B)\right)\Bigg\},
\end{aligned} \tag{5.14}$$

where the twist three functions are the ones given in chapter 2,

$$\tilde{g}_T(x) \;=\; g_T(x) - \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,\frac{g_{1T}(x,\boldsymbol{k}_T^2)}{x} - \frac{m}{M}\frac{h_1(x)}{x}, \tag{5.15}$$

$$\tilde{h}_L(x) \quad = \quad h_L(x) + \int d^2\boldsymbol{k}_T\, \frac{\boldsymbol{k}_T^2}{M^2}\, \frac{h_{1L}^{\perp}(x,\boldsymbol{k}_T^2)}{x} - \frac{m}{M}\frac{g_1(x)}{x}. \tag{5.16}$$

We will consider the various possibilities for unpolarized (O) and longitudinally (L) or transversely polarized (T) target hadrons.

5.2.1 Drell-Yan cross sections for unpolarized hadrons

Integrated over the transverse momenta of the produced mu pair one has for unpolarized hadrons,

$$\mathcal{W}^{\mu\nu} = \frac{1}{3}\,(-g_\perp^{\mu\nu})\; f_1(x_A)\bar{f}_1(x_B), \tag{5.17}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\boxed{\frac{d\sigma_{OO}(AB \to \mu^+\mu^- X)}{dx_A\, dx_B\, dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,\left(\frac{1}{2} - y + y^2\right)\; f_1(x_A)\,\overline{f}_1(x_B),} \tag{5.18}$$

and integrated over the muon angular distribution,

$$\boxed{\frac{d\sigma_{OO}(AB \to \mu^+\mu^- X)}{dx_A\, dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\, f_1(x_A)\,\overline{f}_1(x_B).} \tag{5.19}$$

5.2.2 Longitudinal double spin asymmetry in Drell-Yan scattering

Integrated over transverse momenta, there are no single spin asymmetries. For the case that both hadrons are longitudinally polarized, the hadronic tensor is

$$\mathcal{W}^{\mu\nu} = \frac{1}{3}\,\lambda_A\lambda_B\;(-g_\perp^{\mu\nu})\; g_1(x_A)\bar{g}_1(x_B), \tag{5.20}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\boxed{\frac{d\sigma_{LL}(\vec{A}\vec{B} \to \mu^+\mu^- X)}{dx_A\, dx_B\, dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,\left(\frac{1}{2} - y + y^2\right)\, \lambda_A\,\lambda_B\, g_1(x_A)\,\overline{g}_1(x_B),} \tag{5.21}$$

and integrated over the muon angular distribution,

$$\boxed{\frac{d\sigma_{LL}(\vec{A}\vec{B} \to \mu^+\mu^- X)}{dx_A\, dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\,\lambda_A\lambda_B\, g_1(x_A)\,\overline{g}_1(x_B).} \tag{5.22}$$

5.2.3 Transverse double spin asymmetry in Drell-Yan scattering

For the case that both hadrons are transversely polarized, the hadronic tensor is

$$\mathcal{W}^{\mu\nu} = -\frac{1}{3}\,\left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} - S_{A\perp}\cdot S_{B\perp}\, g_\perp^{\mu\nu}\right)\; h_1(x_A)\bar{h}_1(x_B) \tag{5.23}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\boxed{\frac{d\sigma_{TT}(\vec{A}\vec{B} \to \mu^+\mu^- X)}{dx_A\, dx_B\, dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,|\boldsymbol{S}_{A\perp}|\,|\boldsymbol{S}_{B\perp}|\, y(1-y)\, \cos(\phi_s^A + \phi_s^B)\; h_1(x_A)\,\overline{h}_1(x_B),} \tag{5.24}$$

and integrated over the muon angular distribution,

$$\boxed{\frac{d\sigma_{TT}(\vec{A}\vec{B} \to \mu^+\mu^- X)}{dx_A\, dx_B} = \frac{2\pi\,\alpha^2}{9\,Q^2}\,|\boldsymbol{S}_{A\perp}|\,|\boldsymbol{S}_{B\perp}|\, \cos(\phi_s^A + \phi_s^B)\, h_1(x_A)\,\overline{h}_1(x_B).} \tag{5.25}$$

5.2.4 Longitudinal-transverse double spin asymmetry in Drell-Yan scattering

For the case that one hadron is longitudinally polarized and the second transverse, the hadronic tensor is

$$\begin{aligned}\mathcal{W}^{\mu\nu} &= \frac{1}{3}\,\lambda_A \left\{ \frac{M_A\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_{\scriptscriptstyle A}\left(h_L(x_{\scriptscriptstyle A}) + \tilde{h}_L(x_{\scriptscriptstyle A})\right)\bar{h}_1(x_{\scriptscriptstyle B}) \right.\\ &\qquad \left. + \frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_{\scriptscriptstyle B}\,g_1(x_{\scriptscriptstyle A})\left(\bar{g}_T(x_{\scriptscriptstyle B}) + \bar{\tilde{g}}_T(x_{\scriptscriptstyle B})\right)\right\},\end{aligned} \tag{5.26}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\boxed{\begin{aligned}\frac{d\sigma_{LT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_{\scriptscriptstyle A}\,dx_{\scriptscriptstyle B}\,dy} &= -\frac{4\pi\,\alpha^2}{3\,Q^2}\,\lambda_A\,|\boldsymbol{S}_{B\perp}|\,(1-2y)\sqrt{y(1-y)}\;\cos(\phi_s^B)\\ &\times\left[\frac{M_A x_{\scriptscriptstyle A}}{Q}\left(h_L(x_{\scriptscriptstyle A}) + \tilde{h}_L(x_{\scriptscriptstyle A})\right)\bar{h}_1(x_{\scriptscriptstyle B}) + \frac{M_B x_{\scriptscriptstyle B}}{Q}\,g_1(x_{\scriptscriptstyle A})\left(\bar{g}_T(x_{\scriptscriptstyle B}) + \bar{\tilde{g}}_T(x_{\scriptscriptstyle B})\right)\right].\end{aligned}} \tag{5.27}$$

which vanishes upon integration over the muon angle.

5.3 Azimuthal asymmetries in Drell-Yan scattering

We will consider separately the various possibilities involving unpolarized (O), longitudinally polarized (L) and transversely polarized (T) hadrons.

5.3.1 Azimuthal asymmetries in unpolarized Drell-Yan scattering

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$\begin{aligned}\mathcal{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\times\left\{-g_\perp^{\mu\nu}\left[f_1\bar{f}_1\right]\right.\\ &\qquad\left. + \frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[-f_1\bar{f}_1 + 2x_{\scriptscriptstyle A}\,f^\perp\bar{f}_1\right] + \frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[f_1\bar{f}_1 - 2x_{\scriptscriptstyle B}\,f_1\bar{f}^\perp\right]\right\},\\ &= -g_\perp^{\mu\nu}\,I[f_1\bar{f}_1] + z^{\{\mu}\hat{x}^{\nu\}}\left(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\;x_{\scriptscriptstyle A}(f^\perp + \tilde{f}^\perp)\bar{f}_1\right]\right.\\ &\qquad\left. - \frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,f_1\,(\bar{f}^\perp + \bar{\tilde{f}}^{\,\perp})\right]\right),\end{aligned} \tag{5.28}$$

where the last expression involves integrals of the type

$$I\left[k_a^x\,f^\perp\bar{f}_1\right](x_{\scriptscriptstyle A},x_{\scriptscriptstyle B},\boldsymbol{q}_T) = \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\,k_{aT}^x\,f^\perp(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT})\bar{f}_1(x_{\scriptscriptstyle B},\boldsymbol{k}_{bT}). \tag{5.29}$$

Note that a contribution proportional to $\hat{z}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}\hat{x}_\rho$ appears, but it is multiplied with integrals of the type $I[k_a^y\,f^\perp(x_{\scriptscriptstyle A},|\boldsymbol{k}_{a\perp}|)\bar{f}_1(x_{\scriptscriptstyle B},|Q_T\,\hat{\boldsymbol{x}}-\boldsymbol{k}_{a\perp}|)]$, which vanish.

The cross section is given by

$$\frac{d\sigma_{OO}(AB\to\mu^+\mu^-X)}{dx_{\scriptscriptstyle A}\,dx_{\scriptscriptstyle B}\,dy\,d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\Bigg\{\left(\frac{1}{2}-y+y^2\right)\,I\left[f_1\bar{f}_1\right] -(1-2y)\sqrt{y(1-y)}\,\cos\phi\Bigg(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}(f^\perp+\tilde{f}^\perp)\,\bar{f}_1\right] -\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}f_1(\bar{f}^\perp+\bar{\tilde{f}}^{\perp}\right]\Bigg)\Bigg\}. \tag{5.30}$$

5.3.2 Azimuthal asymmetries in singly polarized Drell-Yan scattering

There is at tree level up to order $1/Q$ no azimuthal asymmetry in single spin asymmetries.

5.3.3 Azimuthal asymmetries in doubly polarized LL-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$\begin{aligned}
\mathcal{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\\
&\quad\times\lambda_A\lambda_B\Bigg\{-g_\perp^{\mu\nu}\Big[g_{1L}\bar{g}_{1L}\Big]-\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,h_{1L}^\perp\bar{h}_{1L}^\perp\\
&\quad+\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[g_{1L}\bar{g}_{1L}-2x_{\scriptscriptstyle A}\,g_L^\perp\bar{g}_{1L}-\frac{M_B}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\bar{h}_L+\frac{m}{M_A}\,2\,h_{1L}^\perp\bar{g}_{1L}-\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,h_{1L}^\perp\bar{h}_{1L}^\perp\right]\\
&\quad+\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[-g_{1L}\bar{g}_{1L}+2x_{\scriptscriptstyle B}\,g_{1L}\bar{g}_L^\perp+\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_L\bar{h}_{1L}^\perp-\frac{m}{M_B}\,2\,g_{1L}\bar{h}_{1L}^\perp+\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1L}^\perp\bar{h}_{1L}^\perp\right]\Bigg\},\\
&= \lambda_A\lambda_B\Bigg\{-g_\perp^{\mu\nu}I\left[g_{1L}\bar{g}_{1L}\right]-\left(2\,\hat{x}^\mu\hat{x}^\nu+g_\perp^{\mu\nu}\right)\,I\left[\frac{k_a^xk_b^x-k_a^yk_b^y}{M_AM_B}\,h_{1L}^\perp\bar{h}_{1L}^\perp\right]\\
&\quad+\hat{z}^{\{\mu}\hat{x}^{\nu\}}\Bigg(-\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1L}\right]-\frac{M_B}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp(\bar{h}_L+\bar{\tilde{h}}_L)\right]\\
&\quad+\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,g_{1L}(\bar{g}_L^\perp+\bar{\tilde{g}}_L^\perp\right]+\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1L}^\perp\right]\Bigg)\Bigg\},
\end{aligned}\tag{5.31}$$

The cross section is given by

$$\begin{aligned}
\frac{d\sigma_{LL}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_{\scriptscriptstyle A}\,dx_{\scriptscriptstyle B}\,dy\,d^2\boldsymbol{q}_T} &= \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_A\,\lambda_B\Bigg\{\left(\frac{1}{2}-y+y^2\right)\,I\left[g_{1L}\bar{g}_{1L}\right]\\
&\quad+y(1-y)\,\cos 2\phi\;I\left[\frac{k_a^xk_b^x-k_a^yk_b^y}{M_AM_B}\,h_{1L}^\perp\bar{h}_{1L}^\perp\right]\\
&\quad+(1-2y)\sqrt{y(1-y)}\,\cos\phi\Bigg(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1L}\right]\\
&\quad+\frac{M_B}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp(\bar{h}_L+\bar{\tilde{h}}_L)\right]\\
&\quad-\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,g_{1L}(\bar{g}_L^\perp+\bar{\tilde{g}}_L^\perp\right]\\
&\quad-\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1L}^\perp\right]\Bigg)\Bigg\}.
\end{aligned}\tag{5.32}$$

5.3.4 Azimuthal asymmetries in doubly polarized LT-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; & \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\; \lambda_A \left\{ -g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{bT}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, g_{1L}\,\bar g_{1T}\right]\right.\\
& -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar h_{1T}^\perp - \frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1L}^\perp\bar h_{1T}\\
& +\frac{\hat z^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1L}\,\bar g_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_L^\perp\,\bar g_{1T} - \frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\,\bar h_T\right.\\
& \qquad\left. +\frac{m}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1L}^\perp\,\bar g_{1T} - \frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar h_{1T}^\perp - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1L}^\perp\bar h_{1T}\right]\\
& +\frac{\hat z^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1L}\,\bar g_{1T} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,g_{1L}\,\bar g_T^\perp\right.\\
& \qquad +\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_L\,\bar h_{1T}^\perp - \frac{m}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,g_{1L}\,\bar h_{1T}^\perp\\
& \qquad\left. +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar h_{1T}^\perp + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1L}^\perp\bar h_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\bar h_T^\perp\right]\\
& +\frac{M_B\,\hat z^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\left[\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_L\bar h_{1T} + 2x_{\scriptscriptstyle B}\,g_{1L}\bar g_T' - \frac{m}{M_B}\,2\,g_{1L}\bar h_{1T}\right.\\
& \qquad\left.\left. +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1L}^\perp\bar h_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,h_{1L}^\perp\bar h_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\bar h_T^\perp\right]\right\}.
\end{aligned}
\tag{5.33}
$$

This can be rewritten as

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; & \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\; \lambda_A \left\{ -g_\perp^{\mu\nu}\left[g_{1L}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,\bar g_{1T}\right]\right.\\
& -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar h_{1T}^\perp - \frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1L}^\perp\bar h_{1T}\\
& -\frac{\hat z^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde g_L^\perp)\,\bar g_{1T} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar h_T+\bar{\tilde h}_T)\right]\\
& +\frac{\hat z^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle B}\,g_{1L}\,(\bar g_T^\perp+\bar{\tilde g}_T^\perp) + \frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde h_L)\,\bar h_{1T}^\perp\right.\\
& \qquad\left. -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp(\bar h_T^\perp+\bar{\tilde h}_T^\perp)\right]\\
& \left.+\frac{M_B\,\hat z^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\left[x_{\scriptscriptstyle B}\,g_{1L}\,(\bar g_T'+\bar{\tilde g}_T') + \frac{M_A}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde h_L)\,\bar h_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar h_T^\perp+\bar{\tilde h}_T^\perp)\right]\right\},
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; & \lambda_A\left\{-g_\perp^{\mu\nu}\,S_B^x\,I\left[\frac{k_b^x}{M_B}\,g_{1L}\,\bar g_{1T}\right] - \left(\hat x^{\{\mu}S_{B\perp}^{\nu\}}+S_B^x\,g_\perp^{\mu\nu}\right)\,I\left[\frac{k_a^x}{M_A}\,h_{1L}^\perp\bar h_1\right]\right.\\
& -\left((2\hat x^\mu\hat x^\nu+g_\perp^{\mu\nu})\,S_B^x - \hat x^{\{\mu}\hat y^{\nu\}}\,S_B^y\right)\\
& \qquad\times I\left[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2\,M_A\,M_B^2}\,h_{1L}^\perp\bar h_{1T}^\perp\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}\Bigg(\frac{M_B}{Q}\,x_B\,I\left[g_{1L}(\bar{g}_T+\bar{\tilde{g}}_T)\right]+\frac{M_A}{Q}\,x_A\,I\left[(h_L+\tilde{h}_L)\,\bar{h}_1\right]\\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_B\,h_{1L}^\perp\,(\bar{h}_T-\bar{h}_T^\perp+\bar{\tilde{h}}_T-\bar{\tilde{h}}_T^\perp)\right]\\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_A\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\\
&+\left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\,S_B^x-\hat{z}^{\{\mu}\hat{y}^{\nu\}}\,S_B^y\right)\Bigg(\frac{M_B}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_B\,g_{1L}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right]\\
&\qquad +\frac{M_A}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_A\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}^\perp\right]\\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_B\,h_{1L}^\perp\,(\bar{h}_T+\bar{h}_T^\perp+\bar{\tilde{h}}_T+\bar{\tilde{h}}_T^\perp)\right]\\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_A\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\Bigg\}. \qquad (5.34)
\end{aligned}
$$

The cross section is given by

$$
\boxed{
\begin{aligned}
&\frac{d\sigma_{LT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B\,dy\,d^2\boldsymbol{q}_T}=\frac{4\pi\,\alpha^2}{Q^2}\,\lambda_A\,|\boldsymbol{S}_{B\perp}|\Bigg\{\left(\frac{1}{2}-y+y^2\right)\cos(\phi-\phi_s^B)\,I\left[\frac{k_b^x}{M_B}\,g_{1L}\,\bar{g}_{1T}\right]\\
&\quad +y(1-y)\,\cos(\phi+\phi_s^B)\,I\left[\frac{k_a^x}{M_A}\,h_{1L}^\perp\bar{h}_1\right]\\
&\quad +y(1-y)\,\cos(3\phi-\phi_s^B)\,I\left[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2\,M_A\,M_B^2}\,h_{1L}^\perp\bar{h}_{1T}^\perp\right]\\
&\quad -(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^B\Bigg(\frac{M_B}{Q}\,x_B\,I\left[g_{1L}(\bar{g}_T+\bar{\tilde{g}}_T)\right]+\frac{M_A}{Q}\,x_A\,I\left[(h_L+\tilde{h}_L)\,\bar{h}_1\right]\\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_B\,h_{1L}^\perp\,(\bar{h}_T-\bar{h}_T^\perp+\bar{\tilde{h}}_T-\bar{\tilde{h}}_T^\perp)\right]\\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_A\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\\
&\quad -(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^B)\Bigg(\frac{M_B}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_B\,g_{1L}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right]\\
&\qquad +\frac{M_A}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_A\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}^\perp\right]\\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_B\,h_{1L}^\perp\,(\bar{h}_T+\bar{h}_T^\perp+\bar{\tilde{h}}_T+\bar{\tilde{h}}_T^\perp)\right]\\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_A\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\Bigg\}. \qquad (5.35)
\end{aligned}
}
$$

5.3.5 Azimuthal asymmetries in doubly polarized TT-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$
\begin{aligned}
\mathcal{W}^{\mu\nu}\ &=\ \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\ \times\Bigg\{-g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar{g}_{1T}\right]\\
&\qquad -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}} + (\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar{h}_{1T}^\perp \\
&-\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}} + (\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar{h}_{1T} - \left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\,h_{1T}\bar{h}_{1T} \\
&+\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar{g}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_T^\perp\,\bar{g}_{1T}\right. \\
&\qquad -\frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\,\bar{h}_T + \frac{m}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1T}^\perp\,\bar{g}_{1T} \\
&\qquad -\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar{h}_{1T}^\perp \\
&\qquad -\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar{h}_{1T} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\,\bar{h}_{1T}^\perp \\
&\qquad \left. -\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T} + \boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\right] \\
&+\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar{g}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,g_{1T}\,\bar{g}_T^\perp\right. \\
&\qquad +\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T\,\bar{h}_{1T}^\perp - \frac{m}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,g_{1T}\,\bar{h}_{1T}^\perp \\
&\qquad +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar{h}_{1T}^\perp \\
&\qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\bar{h}_T^\perp \\
&\qquad \left. +\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T} - \boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\right] \\
&+\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\left[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_T'\,\bar{g}_{1T} - \frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\,\bar{h}_T\right. \\
&\qquad +\frac{m}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1T}\,\bar{g}_{1T} - \frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar{h}_{1T}^\perp \\
&\qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar{h}_{1T}^\perp - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\,\bar{h}_{1T}^\perp \\
&\qquad \left. +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\right] \\
&+\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\left[\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T\,\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,g_{1T}\bar{g}_T'\right. \\
&\qquad -\frac{m}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2\,g_{1T}\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar{h}_{1T} \\
&\qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\bar{h}_T^\perp \\
&\qquad \left.\left. +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\right]\right\}, \qquad (5.36)
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; &\frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\times\left\{-g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar{g}_{1T}\right]\right. \\
&-\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}} + (\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}} + (\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp \\
& -\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}} + (\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T} - \left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\,h_{1T}\bar h_{1T} \\
& +\frac{\hat z^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(g_T^\perp+\tilde g_T^\perp)\,\bar g_{1T} \\
& \qquad -\frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar h_T+\bar{\tilde h}_T) \\
& \qquad +\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde h_T^\perp)\,\bar h_{1T}^\perp + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde h_T^\perp)\,\bar h_{1T}\Bigg] \\
& +\frac{\hat z^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle B}\,g_{1T}\,(\bar g_T^\perp+\bar{\tilde g}_T^\perp) \\
& \qquad +\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(h_T+\tilde h_T)\,\bar h_{1T}^\perp \\
& \qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp(\bar h_T^\perp+\bar{\tilde h}_T^\perp) - (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,x_{\scriptscriptstyle B}\,h_{1T}\,(\bar h_T^\perp+\bar{\tilde h}_T^\perp)\Bigg] \\
& +\frac{M_A\,\hat z^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(g_T'+\tilde g_T')\,\bar g_{1T} - \frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle B}\,h_{1T}\,(\bar h_T+\bar{\tilde h}_T) \\
& \qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde h_T^\perp)\,\bar h_{1T}^\perp - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde h_T^\perp)\bar h_{1T}\Bigg] \\
& +\frac{M_B\,\hat z^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\Bigg[\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,x_{\scriptscriptstyle A}\,(h_T+\tilde h_T)\,\bar h_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,x_{\scriptscriptstyle B}\,g_{1T}(\bar g_T'+\bar{\tilde g}_T') \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar h_T^\perp+\bar{\tilde h}_T^\perp) + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,x_{\scriptscriptstyle B}\,h_{1T}\,(\bar h_T^\perp+\bar{\tilde h}_T^\perp)\Bigg]\Bigg\}
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\;& -g_\perp^{\mu\nu}\left((\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,g_{1T}\,\bar g_{1T}\right] + (S_A^xS_B^x - S_A^yS_B^y)\,I\left[\frac{k_a^xk_b^x-k_a^yk_b^y}{2\,M_AM_B}\,g_{1T}\,\bar g_{1T}\right]\right) \\
& -\left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\left(I\left[h_1\bar h_1\right] - I\left[\frac{\boldsymbol{k}_{a\perp}^2\boldsymbol{k}_{b\perp}^2}{4\,M_A^2M_B^2}\,h_{1T}^\perp\,\bar h_{1T}^\perp\right]\right) \\
& -\left(\left(\hat x^{\{\mu}S_{A\perp}^{\nu\}} + S_A^x\,g_\perp^{\mu\nu}\right)\,S_B^x + \left(\hat y^{\{\mu}S_{A\perp}^{\nu\}} + S_A^y\,g_\perp^{\mu\nu}\right)\,S_B^y\right)\,I\left[\frac{(k_b^x)^2-(k_b^y)^2}{2\,M_B^2}\,h_1\,\bar h_{1T}^\perp\right] \\
& -\left(\left(\hat x^{\{\mu}S_{B\perp}^{\nu\}} + S_B^x\,g_\perp^{\mu\nu}\right)\,S_A^x + \left(\hat y^{\{\mu}S_{B\perp}^{\nu\}} + S_B^y\,g_\perp^{\mu\nu}\right)\,S_A^y\right)\,I\left[\frac{(k_a^x)^2-(k_a^y)^2}{2\,M_A^2}\,h_{1T}^\perp\,\bar h_1\right] \\
& -\left((2\hat x^\mu\hat x^\nu + g_\perp^{\mu\nu})\,(S_A^xS_B^x - S_A^yS_B^y) - \hat x^{\{\mu}\hat y^{\nu\}}\,(S_A^xS_B^y + S_A^yS_B^x)\right) \\
& \qquad\qquad \times I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2] - 4k_a^xk_a^yk_b^xk_b^y}{4\,M_A^2M_B^2}\,h_{1T}^\perp\,\bar h_{1T}^\perp\right] \\
& -\hat z^{\{\mu}S_{A\perp}^{\nu\}}\,S_B^x\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle A}\,(g_T+\tilde g_T)\,\bar g_{1T}\right] + \frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,h_1\,(\bar h_T+\bar{\tilde h}_T)\right]\right) \\
& +\hat z^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{A\perp\rho}\,S_B^y\,\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,h_1\,(\bar h_T^\perp+\bar{\tilde h}_T^\perp)\right] \\
& -\left(\hat z^{\{\mu}\hat x^{\nu\}}\,S_A^xS_B^x - \hat z^{\{\mu}\hat y^{\nu\}}\,S_A^yS_B^x\right)\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2M_B}\,x_{\scriptscriptstyle A}\,(g_T^\perp+\tilde g_T^\perp)\,\bar g_{1T}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad + \frac{M_B}{Q}\, I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2 M_B}\, x_{\scriptscriptstyle B}\, h_{1T}^\perp\, (\bar{h}_T + \bar{\tilde{h}}_T)\right] \\
&\qquad\qquad\qquad\qquad\qquad - \frac{M_B}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_{\scriptscriptstyle B}\, h_{1T}^\perp\, (\bar{h}_T^\perp + \bar{\tilde{h}}_T^\perp)\right]\Bigg) \\
&- \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^y S_B^y + \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^x S_B^y\right)\ \Bigg(\frac{M_A}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_{\scriptscriptstyle A}\, (g_T^\perp + \tilde{g}_T^\perp)\, \bar{g}_{1T}\right] \\
&\qquad\qquad\qquad\qquad\qquad + \frac{M_B}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_{\scriptscriptstyle B}\, h_{1T}^\perp\, (\bar{h}_T + \bar{\tilde{h}}_T)\right] \\
&\qquad\qquad\qquad\qquad\qquad - \frac{M_B}{Q}\, I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\, k_b^x}{2\,M_A^2 M_B}\, x_{\scriptscriptstyle B}\, h_{1T}^\perp\, (\bar{h}_T^\perp + \bar{\tilde{h}}_T^\perp)\right]\Bigg) \\
&+ \hat{z}^{\{\mu} S_{B\perp}^{\nu\}}\, S_A^x \left(\frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_{\scriptscriptstyle A}\, (h_T + \tilde{h}_T)\, \bar{h}_1\right] + \frac{M_B}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_{\scriptscriptstyle B}\, g_{1T}\, (\bar{g}_T + \bar{\tilde{g}}_T)\right]\right) \\
&- \hat{z}^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{B\perp\rho}\, S_A^y\, \frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_{\scriptscriptstyle A}\, (h_T^\perp + \tilde{h}_T^\perp)\bar{h}_1\right] \\
&+ \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^x S_B^x - \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^x S_B^y\right)\ \Bigg(\frac{M_B}{Q}\, I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_{\scriptscriptstyle B}\, g_{1T}\, (\bar{g}_T^\perp + \bar{\tilde{g}}_T^\perp)\right] \\
&\qquad\qquad\qquad\qquad\qquad + \frac{M_A}{Q}\, I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_{\scriptscriptstyle A}\, (h_T + \tilde{h}_T)\, \bar{h}_{1T}^\perp\right] \\
&\qquad\qquad\qquad\qquad\qquad - \frac{M_A}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_{\scriptscriptstyle A}\, (h_T^\perp + \tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\Bigg) \\
&+ \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^y S_B^y + \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^y S_B^x\right)\ \Bigg(\frac{M_B}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_{\scriptscriptstyle A}\, g_{1T}\, (\bar{g}_T^\perp + \bar{\tilde{g}}_T^\perp)\right] \\
&\qquad\qquad\qquad\qquad\qquad + \frac{M_A}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_{\scriptscriptstyle A}\, (h_T + \tilde{h}_T)\, \bar{h}_{1T}^\perp\right] \\
&\qquad\qquad\qquad\qquad\qquad - \frac{M_A}{Q}\, I\left[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_{\scriptscriptstyle A}\, (h_T^\perp + \tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\Bigg) \qquad (5.37)
\end{aligned}
$$

The cross section is given by

$$
\boxed{
\begin{aligned}
&\frac{d\sigma_{TT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_{\scriptscriptstyle A}\, dx_{\scriptscriptstyle B}\, dy\, d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\, |\boldsymbol{S}_{A\perp}|\, |\boldsymbol{S}_{B\perp}| \Bigg\{ \left(\frac{1}{2} - y + y^2\right) \cos(\phi_s^{\scriptscriptstyle A} - \phi_s^{\scriptscriptstyle B})\, I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\, g_{1T}\, \bar{g}_{1T}\right] \\
&\qquad + \left(\frac{1}{2} - y + y^2\right) \cos(\phi_s^{\scriptscriptstyle A} + \phi_s^{\scriptscriptstyle B} - 2\phi) I\left[\frac{k_a^x k_b^x - k_a^y k_b^y}{2\,M_A M_B}\, g_{1T}\, \bar{g}_{1T}\right] \\
&\qquad + y(1-y)\, \cos(\phi_s^{\scriptscriptstyle A} + \phi_s^{\scriptscriptstyle B}) \left(I\left[h_1 \bar{h}_1\right] - I\left[\frac{\boldsymbol{k}_{a\perp}^2 \boldsymbol{k}_{b\perp}^2}{4\,M_A^2 M_B^2}\, h_{1T}^\perp\, \bar{h}_{1T}^\perp\right]\right) \\
&\qquad + y(1-y)\, \cos(2\phi + \phi_s^{\scriptscriptstyle A} - \phi_s^{\scriptscriptstyle B})\, I\left[\frac{(k_b^x)^2 - (k_b^y)^2}{2\,M_B^2}\, h_1\, \bar{h}_{1T}^\perp\right] \\
&\qquad + y(1-y)\, \cos(2\phi - \phi_s^{\scriptscriptstyle A} + \phi_s^{\scriptscriptstyle B})\, I\left[\frac{(k_a^x)^2 - (k_a^y)^2}{2\,M_A^2}\, h_{1T}^\perp\, \bar{h}_1\right] \\
&\qquad + \ldots
\end{aligned}
}
$$

$$
\begin{aligned}
&\ldots \\
&-y(1-y)\,\cos(4\phi-\phi_s^{\scriptscriptstyle A}-\phi_s^{\scriptscriptstyle B})\,I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2]-4k_a^xk_a^yk_b^xk_b^y}{4\,M_A^2M_B^2}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp\right] \\
&+(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^{\scriptscriptstyle A}\,\cos(\phi_s^{\scriptscriptstyle B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle A}\,(g_T+\tilde{g}_T)\,\bar{g}_{1T}\right]\right. \\
&\qquad\qquad\left.+\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,h_1\,(\bar{h}_T+\bar{\tilde{h}}_T)\right]\right) \\
&+(1-2y)\sqrt{y(1-y)}\,\sin\phi_s^{\scriptscriptstyle A}\,\sin(\phi_s^{\scriptscriptstyle B}-\phi)\,\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,h_1\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\right] \\
&+(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^{\scriptscriptstyle A})\,\cos(\phi_s^{\scriptscriptstyle B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2M_B}\,x_{\scriptscriptstyle A}\,(g_T^\perp+\tilde{g}_T^\perp)\,\bar{g}_{1T}\right]\right. \\
&\qquad\qquad+\frac{M_B}{Q}\,I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2M_B}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar{h}_T+\bar{\tilde{h}}_T)\right] \\
&\qquad\qquad\left.-\frac{M_B}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\right]\right) \\
&+(1-2y)\sqrt{y(1-y)}\,\sin(2\phi-\phi_s^{\scriptscriptstyle A})\,\sin(\phi_s^{\scriptscriptstyle B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{\scriptscriptstyle A}\,(g_T^\perp+\tilde{g}_T^\perp)\,\bar{g}_{1T}\right]\right. \\
&\qquad\qquad+\frac{M_B}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar{h}_T+\bar{\tilde{h}}_T)\right] \\
&\qquad\qquad\left.-\frac{M_B}{Q}\,I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,k_b^x}{2\,M_A^2M_B}\,x_{\scriptscriptstyle B}\,h_{1T}^\perp\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^{\scriptscriptstyle B}\,\cos(\phi_s^{\scriptscriptstyle A}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}\,(h_T+\tilde{h}_T)\,\bar{h}_1\right]\right. \\
&\qquad\qquad\left.+\frac{M_B}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle B}\,g_{1T}\,(\bar{g}_T+\bar{\tilde{g}}_T)\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\sin\phi_s^{\scriptscriptstyle B}\,\sin(\phi_s^{\scriptscriptstyle A}-\phi)\,\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde{h}_T^\perp)\bar{h}_1\right] \\
&-(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^{\scriptscriptstyle B})\,\cos(\phi_s^{\scriptscriptstyle A}-\phi)\,\left(\frac{M_B}{Q}\,I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{\scriptscriptstyle B}\,g_{1T}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right]\right. \\
&\qquad\qquad+\frac{M_A}{Q}\,I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{\scriptscriptstyle A}\,(h_T+\tilde{h}_T)\,\bar{h}_{1T}^\perp\right] \\
&\qquad\qquad\left.-\frac{M_A}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\sin(2\phi-\phi_s^{\scriptscriptstyle B})\,\sin(\phi_s^{\scriptscriptstyle A}-\phi)\,\left(\frac{M_B}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{\scriptscriptstyle A}\,g_{1T}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right]\right. \\
&\qquad\qquad+\frac{M_A}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{\scriptscriptstyle A}\,(h_T+\tilde{h}_T)\,\bar{h}_{1T}^\perp\right] \\
&\qquad\qquad\left.\left.-\frac{M_A}{Q}\,I\left[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{\scriptscriptstyle A}\,(h_T^\perp+\tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\right)\right\}. \qquad (5.38)
\end{aligned}
$$

5.4 Convolutions and gaussian distributions

In order to study the behavior of the convolutions of distribution, it is useful to consider gaussian distributions,

$$f(x_A, \boldsymbol{k}_{aT}) = f(x_A, \boldsymbol{0}_T)\, \exp(-R_a^2 \boldsymbol{k}_{aT}^2) = f(x_A)\, \frac{R_a^2}{\pi}\, \exp(-R_a^2 \boldsymbol{k}_{aT}^2), \tag{5.39}$$

$$\bar{f}(x_B, \boldsymbol{k}_{bT}) = \bar{f}(x_B, \boldsymbol{0}_T)\, \exp(-R_b^2 \boldsymbol{k}_{bT}^2) = f(x_B)\, \underbrace{\frac{R_b^2}{\pi}\, \exp(-R_b^2 \boldsymbol{k}_{bT}^2)}_{\mathcal{P}(\boldsymbol{k}_{bT}, R_b)}, \tag{5.40}$$

In that case the convolution becomes

$$\begin{aligned} I[f\,\bar{f}] &= \int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT} + \boldsymbol{k}_{bT} - \boldsymbol{q}_T)\, f(x_A, \boldsymbol{k}_{aT})\, \bar{f}(x_B, \boldsymbol{k}_{bT}) \\ &= \frac{\pi}{R_a^2 + R_b^2}\, \exp\left(-\frac{Q_T^2\, R_a^2 R_b^2}{R_a^2 + R_b^2}\right) f(x_A, \boldsymbol{0}_T)\, \bar{f}(x_B, \boldsymbol{0}_T) \\ &= f(x_A)\, \bar{f}(x_B)\, \mathcal{P}(\boldsymbol{q}_T; R) \end{aligned} \tag{5.41}$$

with $R^2 = R_a^2 R_b^2/(R_a^2 + R_b^2)$. The other convolutions that appear in the cross sections are of the form

$$I\Big[\frac{k_a^x}{M_a}\, f\,\bar{f}\Big] = \frac{R^2}{R_a^2}\, \frac{Q_T}{M_a}\, I[f\,\bar{f}], \tag{5.42}$$

$$I\Big[\frac{\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{M_a M_b}\, f\,\bar{f}\Big] = \frac{R^2}{M_a M_b R_a^2 R_b^2}\, \left(Q_T^2\, R^2 - 1\right)\, I[f\,\bar{f}], \tag{5.43}$$

$$I\Big[\frac{2(k_a^x)^2 - \boldsymbol{k}_{aT}^2}{2M_a^2}\, f\,\bar{f}\Big] = I\Big[\frac{(k_a^x)^2 - (k_a^y)^2}{2M_a^2}\, f\,\bar{f}\Big] = \frac{R^4}{R_a^4}\, \frac{Q_T^2}{2M_a^2}\, I[f\,\bar{f}], \tag{5.44}$$

$$I\Big[\frac{2\, k_a^x k_b^x - \boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{2M_a M_b}\, f\,\bar{f}\Big] = I\Big[\frac{k_a^x k_b^x - k_a^y k_b^y}{2M_a M_b}\, f\,\bar{f}\Big] = \frac{R^4}{R_a^2\, R_b^2}\, \frac{Q_T^2}{2M_a M_b}\, I[f\,\bar{f}], \tag{5.45}$$

$$I\Big[\frac{k_a^x[(k_b^x)^2 - (k_b^y)^2]}{2M_a M_b^2}\, f\,\bar{f}\Big] = \frac{R^4}{R_a^2\, R_b^4}\, \frac{Q_T}{2M_a M_b^2}\, \left(Q_T^2 R^2 - 1\right)\, I[f\,\bar{f}], \tag{5.46}$$

$$I\Big[\frac{k_b^x k_a^y k_b^y}{M_a M_b^2}\, f\,\bar{f}\Big] = -\frac{R^4}{R_a^2\, R_b^4}\, \frac{Q_T}{2M_a M_b^2}\, I[f\,\bar{f}], \tag{5.47}$$

$$\begin{aligned} & I\Big[\frac{k_a^x[(k_b^x)^2 - (k_b^y)^2] - 2k_b^x k_a^y k_b^y}{2M_a M_b^2}\, f\,\bar{f}\Big] \\ &\quad = I\Big[\frac{4\, k_a^x (k_b^x)^2 - 2\, k_b^x\, (\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}) - k_a^x\, \boldsymbol{k}_{b\perp}^2}{2M_a M_b^2}\, f\,\bar{f}\Big] = \frac{R^6}{R_a^2\, R_b^4}\, \frac{Q_T^3}{2M_a M_b^2}\, I[f\,\bar{f}], \end{aligned} \tag{5.48}$$

$$I\Big[\frac{[(k_a^x)^2 - (k_a^y)^2]\, [(k_b^x)^2 - (k_b^y)^2] - 4k_a^x k_a^y k_b^x k_b^y}{4M_a^2 M_b^2}\, f\,\bar{f}\Big] = \frac{R^8}{R_a^4\, R_b^4}\, \frac{Q_T^4}{4M_a^2 M_b^2}\, I[f\,\bar{f}]. \tag{5.49}$$

Another way to deconvolute the results are using the following weighted quantities,

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T}{M_a}\, I\Big[\frac{k_a^x}{M_a}\, f\,\bar{f}\Big] = 2\, f^{(1)}(x_A)\, \bar{f}(x_B), \tag{5.50}$$

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T^2}{M_a M_b}\, I\Big[\frac{\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{M_a M_b}\, f\,\bar{f}\Big] = 4\, f^{(1)}(x_A)\, \bar{f}^{(1)}(x_B), \tag{5.51}$$

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T^2}{M_a^2}\, I\Big[\frac{2(k_a^x)^2 - \boldsymbol{k}_{aT}^2}{2M_a^2}\, f\,\bar{f}\Big] = 2\, f^{(2)}(x_A)\, \bar{f}(x_B), \tag{5.52}$$

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T^2}{M_a M_b}\, I\Big[\frac{2\, k_a^x k_b^x - \boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{2M_a M_b}\, f\,\bar{f}\Big] = 4\, f^{(1)}(x_A)\, \bar{f}^{(1)}(x_B), \tag{5.53}$$

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T^3}{M_a M_b^2}\, I\Big[\frac{k_a^x[(k_b^x)^2 - (k_b^y)^2]}{2M_a M_b^2}\, f\,\bar{f}\Big] = 8\, f^{(1)}(x_A)\, \bar{f}^{(2)}(x_B), \tag{5.54}$$

$$\int d^2\boldsymbol{q}_T\; \frac{Q_T^3}{M_a M_b^2}\, I\Big[\frac{k_b^x k_a^y k_b^y}{M_a M_b^2}\, f\,\bar{f}\Big] = -4\, f^{(1)}(x_A)\, \bar{f}^{(2)}(x_B), \tag{5.55}$$

$$\int d^2\boldsymbol{q}_T\ \frac{Q_T^3}{M_a M_b^2}\, I\Big[\frac{4\,k_a^x (k_b^x)^2 - 2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}) - k_a^x\,\boldsymbol{k}_{b\perp}^2}{2M_a M_b^2}\, f\,\bar{f}\Big] = 12\, f^{(1)}(x_A)\,\bar{f}^{(2)}(x_B), \tag{5.56}$$

$$\int d^2\boldsymbol{q}_T\ \frac{Q_T^4}{M_a^2 M_b^2}\, I\Big[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2] - 4k_a^x k_a^y k_b^x k_b^y}{4M_a^2 M_b^2}\, f\,\bar{f}\Big] = 24\, f^{(2)}(x_A)\,\bar{f}^{(2)}(x_B), \tag{5.57}$$

where $f^{(n)}(x_A)$ indicate the $(\boldsymbol{k}_{aT}^2/2M_a^2)^n$-moments of $f(x_A,\boldsymbol{k}_{aT}^2)$,

$$f^{(n)}(x_A) = \int d^2\boldsymbol{k}_{aT}\ \left(\frac{\boldsymbol{k}_{aT}^2}{2M_a^2}\right)^n f(x_A,\boldsymbol{k}_{aT}^2)\ =\ \frac{n!}{(2M_a^2 R_a^2)^n}\, f(x_A). \tag{5.58}$$

Chapter 7

Lepton-hadron up to $\mathcal{O}(1/Q)$

7.1 Inclusion of gluon contributions

We will consider in this section the inclusion of diagrams with gluons connecting the soft and hard part. The additional contribution is given by four diagrams. Two of them have gluons connected to the lower soft part (the hadron → quark part), the others gluons connected to the upper soft part (the quark → hadron part). Including the contribution of the handbag one has

$$\begin{aligned}
2M\,\mathcal{W}_{\mu\nu} &= \int d^4p\,d^4k\,\delta^4(p+q-k)\ \mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right] \\
&\quad - \int d^4p\,d^4k\,d^4p_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right] \\
&\qquad + \mathrm{Tr}\left[\gamma_\mu\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2-i\epsilon}\,\gamma_\alpha\Delta(k)\gamma_\nu\Phi_A^\alpha(p-p_1,p)\right]\Bigg\} \\
&\quad - \int d^4p\,d^4k\,d^4k_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma_\alpha\Phi(p)\gamma_\mu\Delta_A^\alpha(k-k_1,k)\right] \\
&\qquad + \mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^\alpha(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}
\end{aligned} \tag{7.1}$$

[Note that we have for a quark-quark-gluon blob used momentum p_1 (or k_1) for the gluon and $p-p_1$ (or $k-k_1$) for the quark. This is easier to extend when we consider multiple gluon correlation functions.] The momenta p_1 and k_1 connected to the soft hadronic parts are parametrized according to

$$p_1 = \left[p_1^-,\frac{x_1\,Q}{\sqrt{2}},\boldsymbol{p}_{1T}\right], \tag{7.2}$$

$$k_1 = \left[\frac{z_1\,Q}{\sqrt{2}},k_1^+,\boldsymbol{k}_{1T}\right], \tag{7.3}$$

The momentum appearing in the extra fermion propagator is $p-p_1+q=k-p_1$ with $(k-p_1)^2=-x_1\,Q^2$, or $k-k_1-q=p-k_1$ with $(p-k_1)^2=-z_1\,Q^2$. Thus one has in leading order in $1/Q$,

$$\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon} = \frac{\gamma^-}{Q\sqrt{2}}-\frac{\gamma^+}{(x_1-i\epsilon)\,Q\sqrt{2}}+\frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{k}_T-\boldsymbol{p}_{1T})-m}{(x_1-i\epsilon)\,Q^2}, \tag{7.4}$$

$$\frac{\not p-k_1+m}{(p-k_1)^2-m^2+i\epsilon} = \frac{\gamma^+}{Q\sqrt{2}}-\frac{\gamma^-}{(z_1-i\epsilon)\,Q\sqrt{2}}+\frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{p}_T-\boldsymbol{k}_{1T})-m}{(z_1-i\epsilon)\,Q^2}. \tag{7.5}$$

This can be used to consider separately the contributions of transverse (A_T^α) and longitudinal (A^+) gluons. For the transverse gluons, the trace of the first gluonic contribution becomes

$$-\int d^4p\,d^4k\,d^4p_1\,\delta^4(p+q-k)\,\mathrm{Tr}\left[\gamma_\alpha\,\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right]$$

$$= \int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4} \int d^4x \int d^4y\; e^{i\,(p-p_1)\cdot x + i\,p_1\cdot y}$$
$$\times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k) \frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, g A_T^\alpha(y)\,\psi(x)|P,S\rangle,$$

which starts off at order $1/Q$ and at this order requires leading parts from Φ_A^α (proportional to $P_+\Phi_A^\alpha P_-$) and leading parts from Δ (proportional to $P_-\Delta P_+$). As $\{\gamma^-, \gamma_T^\alpha\} = 0$ and $\gamma^- P_+ = P_-\gamma^- = 0$ only the $\gamma^- = P_+\gamma^- P_-$ part in Eq. 7.4 contributes. This term is independent of any of the components of p_1, and we thus can immediately consider the distributions $\int d^4p_1\, \Phi_A^\alpha(p, p-p_1)$, or explicitly

$$\int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,p\cdot x}\; \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k) \frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, g A_T^\alpha(x)\,\psi(x)|P,S\rangle. \tag{7.6}$$

This contribution will be studied in the next section. Note that it can be written in terms of the covariant derivative as

$$\int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,p\cdot x}\; \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k) \frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, i\,D_T^\alpha(x)\,\psi(x)|P,S\rangle$$
$$- \int d^4p\, d^4k\, \delta^4(p+q-k)\, p_T^\alpha \int d^4x\; e^{i\,p\cdot x}\; \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k) \frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, \psi(x)|P,S\rangle. \tag{7.7}$$

In this section we consider next the contributions of longitudinal gluons (A^+). They lead to traces of the form

$$- \int d^4p\, d^4k\, d^4p_1\, \delta^4(p+q-k)\, \mathrm{Tr}\left[\gamma^- \,\frac{\not k - \not p_1 + m}{(k-p_1)^2 - m^2 + i\epsilon}\,\gamma_\nu \Phi_A^+(p, p-p_1)\gamma_\mu \Delta(k)\right] \tag{7.8}$$

The first term in Eq. 7.4 does not contribute. The second term contributes at $\mathcal{O}(1)$ as the dominant contribution in Φ_A^+ is the part projected out by $\int dp_1^-\; P_+\Phi_A^+ P_-$ which is of $\mathcal{O}(Q)$. Explicitly, we get for the first correction in Eq. 7.1

$$- \int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4} \int d^4x \int d^4y\; e^{i\,(p-p_1)\cdot x + i\,p_1\cdot y}$$
$$\times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\gamma^- \,\frac{\gamma^+}{(x_1 - i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\, g A^+(y)\,\psi(x)|P,S\rangle$$
$$= \int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_1}{2\pi} \int d^4x \int dy^- \; \frac{e^{-i\,x_1\,p^+(x^- - y^-)}}{x_1 - i\epsilon}\; e^{i\,p\cdot x}$$
$$\times\; \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\, \frac{\gamma^-\gamma^+}{2}\,\gamma_\nu\, g A^+(y)\,\psi(x)|P,S\rangle\Big|_{y^+ = x^+,\; y_T = x_T}$$
$$= \int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x \int dy^- \;\theta(y^- - x^-)\, e^{i\,p\cdot x}$$
$$\times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\, P_+\,\gamma_\nu\, igA^+(y^-)\,\psi(x)|P,S\rangle$$
$$= -\int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,p\cdot x}$$
$$\times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\, P_+\,\gamma_\nu\, ig \int_\infty^{x^-} dy^-\; A^+(y^-)\,\psi(x)|P,S\rangle. \tag{7.9}$$

The second term in Eq. 7.1 gives

$$- \int d^4p\, d^4k\, d^4p_1\, \delta^4(p+q-k)\, \mathrm{Tr}\left[\gamma_\mu \,\frac{(\not k - \not p_1 + m)}{(k-p_1)^2 - m^2 - i\epsilon}\,\gamma^- \Delta(k)\gamma_\nu \Phi_A^+(p-p_1, p)\right]$$
$$= \int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,p\cdot x}$$
$$\times \langle P,S|\overline{\psi}(0)\, ig \int_\infty^0 dy^-\; A^+(y^-)\,\gamma_\mu\, P_-\, \Delta(k)\,\gamma_\nu\,\psi(x)|P,S\rangle \tag{7.10}$$

The last two terms in Eq. 7.1 give

$$
\begin{aligned}
-\int d^4p\,d^4k\,d^4k_1\,\delta^4(p+q-k)\Bigg\{&\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma^+\Phi(p)\gamma_\mu\Delta_A^-(k_1,k)\right]\\
&+\mathrm{Tr}\left[\gamma^+\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^-(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}\\
=\ \int d^4p\,d^4k\,\delta^4(p+q-k)\int d^4x\ e^{i\,k\cdot x}&\\
\times\Bigg\{\mathrm{Tr}\,\langle 0|\psi(x)\,a_h^\dagger a_h\,ig\int_{-\infty}^{0}&dy^+\,A^-(y^+)\,\overline{\psi}(0)\,\gamma_\nu\,P_+\,\Phi(p)\,\gamma_\mu|0\rangle\\
-\mathrm{Tr}\,\langle 0|\psi(x)\,ig\int_{-\infty}^{x^+}&dy^+\,A^-(y^+)\,a_h^\dagger a_h\,\overline{\psi}(0)\,\gamma_\nu\,\Phi(p)\,P_-\,\gamma_\mu|0\rangle\Bigg\}.
\end{aligned}
$$

The result of multiple A^+- or A^--gluons together with the tree-level result gives in leading order in $1/Q$ (when the projectors P_+ and P_- don't matter) the exponentiated path-ordered result

$$2M\,\mathcal{W}_{\mu\nu}=\int d^4p\,d^4k\,\delta^4(p+q-k)\ \mathrm{Tr}\,[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu] \tag{7.11}$$

with

$$\Phi_{ij}(p,P,S)=\frac{1}{(2\pi)^4}\int d^4x\ e^{i\,p\cdot x}\langle P,S|\overline{\psi}_j(0)\mathcal{G}(0,\infty;0_T)\,\mathcal{G}(\infty,x^-;x_T)\psi_i(x)|P,S\rangle, \tag{7.12}$$

$$\Delta_{ij}(k,P_h,S_h)=\frac{1}{(2\pi)^4}\int d^4x\ e^{ik\cdot x}\,\langle 0|\mathcal{G}(-\infty,x^+;x_T)\psi_i(x)\,a_h^\dagger a_h\,\overline{\psi}_j(0)\mathcal{G}(0,-\infty;0_T)|0\rangle. \tag{7.13}$$

Provided we assume that matrix elements containing bilocal operators $\overline{\psi}(0)\,A_T(y^\pm=\mp\infty,y_T)\,\psi(x)$ vanish for physical states, the above links can be connected resulting in a color gauge-invariant matrix element that must be used in the definition of the correlation functions.

Before considering the transverse gluons let us check the case of two A^+ gluons. For instance considering a gauge choice $A^-=0$, one needs only to consider the absorption of the A^+ gluons in the 'distribution' part. Dressing the diagram leading to the first of the four terms above with another 'parallel' gluon one obtains a contribution

$$
\begin{aligned}
&\int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\int d^4x\int d^4y_1\,d^4y_2\ e^{i\,(p-p_1-p_2)\cdot x+i\,p_1\cdot y_1+i\,p_2\cdot y_2}\\
&\quad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\,\frac{\gamma^+}{(x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma^-\,\frac{\gamma^+}{(x_1+x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\,gA^+(y_2)\,gA^+(y_1)\,\psi(x)|P,S\rangle\\
&=\ \int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{dx_1}{2\pi}\,\frac{dx_2}{2\pi}\int d^4x\int dy_1^-\,dy_2^-\ \frac{e^{-i\,(x_1+x_2)p^+(x^--y_1^-)}\,e^{-i\,x_2p^+(y_1^--y_2^-)}}{(x_1+x_2-i\epsilon)\,(x_2-i\epsilon)}\,e^{i\,p\cdot x}\\
&\quad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)P_+^2\,\gamma_\nu\,gA^+(y_2)\,gA^+(y_1)\,\psi(x)|P,S\rangle.
\end{aligned} \tag{7.14}
$$

The integration over x_1 and x_2 gives

$$i\theta(y_1^--x^-)\,i\theta(y_2^--y_1^-), \tag{7.15}$$

leading to the path ordering.

Dressing a diagram with an A_T^α-gluon with a longitudinal gluon leads to one contribution of the form

$$
\begin{aligned}
&\int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\int d^4x\int d^4y_1\,d^4y_2\ e^{i\,(p-p_1-p_2)\cdot x+i\,p_1\cdot y_1+i\,p_2\cdot y_2}\\
&\quad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\,\frac{\gamma^+}{(x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma_\alpha\,\frac{\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\,gA^+(y_2)\,gA_T^\alpha(y_1)\,\psi(x)|P,S\rangle\\
&=\ \int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{dx_2}{2\pi}\int d^4x\int dy_2^-\ \frac{e^{-i\,x_2p^+(x^--y_2^-)}}{x_2-i\epsilon}\,e^{i\,p\cdot x}\\
&\quad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)P_+\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\,gA^+(y_2)\,gA_T^\alpha(x)\,\psi(x)|P,S\rangle.
\end{aligned} \tag{7.16}
$$

The x_2 integration gives $\theta(y_2^- - x^-)$, the first term of the link. Note that the $A^+(y_1)A_T^\alpha(y_2)$ contribution vanishes at $\mathcal{O}(1/Q)$ because of the nonmatching Dirac structure.

As a final note of this section, we look for the contribution that combines with Eq. 7.16 into a covariant derivative. Since the term we consider is $\mathcal{O}(g^2)$ and $\mathcal{O}(1/Q)$ we expect the $i\partial_T^\alpha$ to be in the $\mathcal{O}(g)$ contribution also at $\mathcal{O}(1/Q)$. Again considering the $A^- = 0$ gauge, the only part at $\mathcal{O}(1/Q)$ that we sofar neglected is coming from the γ_T part of the fermion propagator instead of the γ^+ in Eq. 7.9,

$$\begin{aligned}
&-\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4} \int d^4x \int d^4y\; e^{i\,(p-p_1)\cdot x + i\,p_1\cdot y} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\gamma^- \,\frac{\gamma_\alpha(k^\alpha - p_1^\alpha)}{(x_1 - i\epsilon)\,Q^2}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle \\
= \;& \int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_1}{2\pi} \int d^4x \int dy^- \;\frac{e^{-i\,x_1\,p^+(x^- - y^-)}}{x_1 - i\epsilon}\; e^{i\,p\cdot x} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y)\, i\partial_T^\alpha \psi(x)|P,S\rangle \\
&+ \int d^4p\, d^4k\, \delta^4(p+q-k)\,(k_T^\alpha - p_T^\alpha) \int \frac{dx_1}{2\pi} \int d^4x \int dy^- \;\frac{e^{-i\,x_1\,p^+(x^- - y^-)}}{x_1 - i\epsilon}\; e^{i\,p\cdot x} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle. \qquad (7.17)
\end{aligned}$$

We correctly obtain the first term in the link for the covariant derivative and the p_T^α-term in Eq. 7.7 (and for as far as we now performed the calculation, namely in $A^- = 0$ gauge the link contribution for the p_T-term arising from $g\,A_T = iD_T - i\partial_T$ in the fragmentation part).

7.2 The $\mathcal{O}(1/Q)$ contribution from transverse gluons

Up to $\mathcal{O}(1/Q)$ one needs to include the contributions of the handbag diagram, now calculated up to this order with in addition irreducible diagrams with one gluon coupling either to the soft part involving hadron H or the soft part involving hadron h. The expressions thus involve the quark-gluon correlation functions. The full expression for the symmetric and antisymmetric parts of the hadronic tensor are,

$$
\begin{aligned}
2M\,\mathcal{W}_S^{\mu\nu} \;=\;& 2z_h \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\; \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \\
&\times \Bigg\{ -g_\perp^{\mu\nu}\left[f_1 D_1 + g_{1s} G_{1s} + \frac{\epsilon_\perp^{\rho\sigma} k_{\perp\rho} S_{h\perp\sigma}}{M_h}\, f_1 D_{1T}^\perp \right] \\
&- \frac{k_\perp^{\{\mu} p_\perp^{\nu\}} + (\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)\, g_\perp^{\mu\nu}}{M M_h}\, h_{1s}^\perp H_{1s}^\perp - \frac{k_\perp^{\{\mu} S_\perp^{\nu\}} + (\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp)\, g_\perp^{\mu\nu}}{M_h}\, h_{1T} H_{1s}^\perp \\
&- \frac{p_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}}{M}\, h_{1s}^\perp H_{1T} - \left(S_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu} \right)\; h_{1T} H_{1T} \\
&- \frac{\left(k_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} p_{\perp\rho} + p_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho} \right)}{2 M M_h}\, h_{1s}^\perp H_1^\perp - \frac{\left(k_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} S_{\perp\rho} + S_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho} \right)}{2 M_h}\, h_{1T} H_1^\perp \\
&+ \frac{2\,\hat{t}^{\,\{\mu} k_\perp^{\nu\}}}{Q}\Bigg[-f_1 D_1 + f_1 \frac{D^\perp}{z_h} - g_{1s} G_{1s} + g_{1s} \frac{G_s^\perp}{z_h} \\
&\qquad + \frac{M}{M_h}\, x_B\, h_s H_{1s}^\perp - \frac{m}{M_h}\, g_{1s} H_{1s}^\perp - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp H_{1T} \\
&\qquad + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp \frac{H_T^\perp}{z_h} - \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, h_{1T} H_{1T} + \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, h_{1T} \frac{H_T^\perp}{z_h} \Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu} p_\perp^{\nu\}}}{Q}\Bigg[x_B\, f^\perp D_1 + x_B\, g_s^\perp G_{1s} + \frac{M_h}{M}\, h_{1s}^\perp \frac{H_s}{z_h} - \frac{m}{M}\, h_{1s}^\perp G_{1s} + \frac{\boldsymbol{k}_\perp^2}{M M_h}\, h_{1s}^\perp H_{1s}^\perp \\
&\qquad + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp H_{1T} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, x_B\, h_T^\perp H_{1s}^\perp + \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, x_B\, h_T^\perp H_{1T} \Bigg] \\
&+ \frac{2M\,\hat{t}^{\,\{\mu} S_\perp^{\nu\}}}{Q}\Bigg[x_B\, g_T' G_{1s} + \frac{M_h}{M}\, h_{1T} \frac{H_s}{z_h} - \frac{m}{M}\, h_{1T} G_{1s} + \frac{\boldsymbol{k}_\perp^2}{M M_h}\, h_{1T} H_{1s}^\perp \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M M_h}\, x_B\, h_T^\perp H_{1s}^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1T} H_{1T} - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, x_B\, h_T^\perp H_{1T} \Bigg] \\
&+ \frac{2M_h\,\hat{t}^{\,\{\mu} S_{h\perp}^{\nu\}}}{Q}\Bigg[\frac{M}{M_h}\, x_B\, h_s H_{1T} + g_{1s} \frac{G_T'}{z_h} - \frac{m}{M_h}\, g_{1s} H_{1T} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M M_h}\, h_{1s}^\perp H_{1T} \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M M_h}\, h_{1s}^\perp \frac{H_T^\perp}{z_h} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, h_{1T} H_{1T} - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, h_{1T} \frac{H_T^\perp}{z_h} \Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho}}{Q}\Bigg[\lambda_h f_1 \frac{D_L^\perp}{z_h} + \frac{M}{M_h}\, x_B\, h_s H_1^\perp - \frac{m}{M_h}\, g_{1s} H_1^\perp \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, f_1 D_{1T}^\perp + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, x_B\, f^\perp D_{1T}^\perp \Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu} \epsilon_\perp^{\nu\}\rho} p_{\perp\rho}}{Q}\left[\frac{M_h}{M}\, h_{1s}^\perp \frac{H}{z_h} + \frac{\boldsymbol{k}_\perp^2}{M M_h}\, h_{1s}^\perp H_1^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, x_B\, h_T^\perp H_1^\perp \right] \\
&+ \frac{2M\,\hat{t}^{\,\{\mu} \epsilon_\perp^{\nu\}\rho} S_{\perp\rho}}{Q}\left[\frac{M_h}{M}\, h_{1T} \frac{H}{z_h} + \frac{\boldsymbol{k}_\perp^2}{M M_h}\, h_{1T} H_1^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M M_h}\, x_B\, h_T^\perp H_1^\perp \right]
\end{aligned}
$$

$$+\frac{2M_h\,\hat t^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}}{Q}\left[f_1\frac{D_T}{z_h}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,f_1D_{1T}^\perp-\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M_h^2}\,x_{\scriptscriptstyle B}\,f^\perp D_{1T}^\perp\right]\Bigg\} \tag{7.18}$$

and

$$\begin{aligned}
2M\,\mathcal{W}_A^{\mu\nu} \;=\;& 2z_h\int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\\
&\times\Bigg\{i\,\epsilon_\perp^{\mu\nu}\left[f_1G_{1s}+g_{1s}D_1\right]+i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\,g_{1s}D_{1T}^\perp\\
&+i\frac{2\,\hat t^{[\mu}k_\perp^{\nu]}}{Q}\Bigg[-\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,eH_1^\perp+\frac{m}{M_h}\,f_1H_1^\perp-\lambda_h\,g_{1s}\frac{D_L^\perp}{z_h}+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\,g_{1s}D_{1T}^\perp\\
&\qquad+\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\,\frac{m}{M}\,h_{1s}^\perp D_{1T}^\perp-\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\,x_{\scriptscriptstyle B}\,g_s^\perp D_{1T}^\perp\\
&\qquad-\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\,\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,g_T'D_{1T}^\perp+\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\,\frac{m}{M_h}\,h_{1T}D_{1T}^\perp\Bigg]\\
&+i\frac{2\,\hat t^{[\mu}p_\perp^{\nu]}}{Q}\,\frac{M_h}{M}\,h_{1s}^\perp\frac{E_s}{z_h}+i\frac{2M\,\hat t^{[\mu}S_\perp^{\nu]}}{Q}\,\frac{M_h}{M}\,h_{1T}\frac{E_s}{z_h}\\
&+i\frac{2M_h\,\hat t^{[\mu}S_{h\perp}^{\nu]}}{Q}\Bigg[-g_{1s}\frac{D_T}{z_h}-\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,g_{1s}D_{1T}^\perp-\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,\frac{m}{M_h}\,h_{1s}^\perp D_{1T}^\perp\\
&\qquad+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M_h^2}\,x_{\scriptscriptstyle B}\,g_s^\perp D_{1T}^\perp+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\,\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,g_T'D_{1T}^\perp-\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\,\frac{m}{M_h}\,h_{1T}D_{1T}^\perp\Bigg]\\
&+i\frac{2\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}k_{\perp\rho}}{Q}\left[f_1\frac{G_s^\perp}{z_h}-f_1G_{1s}+g_{1s}\frac{D^\perp}{z_h}-g_{1s}D_1+\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,eH_{1s}^\perp-\frac{m}{M_h}\,f_1H_{1s}^\perp\right]\\
&+i\frac{2\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}p_{\perp\rho}}{Q}\left[x_{\scriptscriptstyle B}\,f^\perp G_{1s}+x_{\scriptscriptstyle B}\,g_s^\perp D_1+\frac{M_h}{M}\,h_{1s}^\perp\frac{E}{z_h}-\frac{m}{M}\,h_{1s}^\perp D_1\right]\\
&+i\frac{2M\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}}{Q}\left[x_{\scriptscriptstyle B}\,g_T'D_1+\frac{M_h}{M}\,h_{1T}\frac{E}{z_h}-\frac{m}{M}\,h_{1T}D_1\right]\\
&+i\frac{2M_h\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}S_{h\perp\rho}}{Q}\left[f_1\frac{G_T'}{z_h}+\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,eH_{1T}-\frac{m}{M_h}\,f_1H_{1T}\right]\Bigg\}.
\end{aligned} \tag{7.19}$$

Note that one can make the replacement $\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}=\epsilon^{\mu\nu\rho\sigma}a_{\perp\rho}\hat q_\sigma$. The quark distribution functions in hadron H depend on $x_{\scriptscriptstyle B}$ and $\boldsymbol{p}_T^2$, $f_1(x_{\scriptscriptstyle B},\boldsymbol{p}_T^2)$ etc., while the quark fragmentation functions into hadron h depend on z_h and $-z_h\boldsymbol{k}_T$ (the perpendicular momentum of hadron h with respect to the quark), $D_1(z_h,-z_h\boldsymbol{k}_T)$ etc., and we use the shorthand notations

$$g_{1s}=\lambda\,g_{1L}+\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,g_{1T},$$
$$G_{1s}=\lambda_h\,G_{1L}+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\,G_{1T}.$$

The contractions with the lepton tensor, given in Table 7.1 use azimuthal angles defined with respect to the scattering plane and the (spacelike) virtual photon momentum,

$$\hat\ell\cdot a_\perp \;=\; -\hat\ell\cdot\boldsymbol{a}_\perp \;=\; -|\boldsymbol{a}_\perp|\,\cos(\phi_a), \tag{7.20}$$

$$\hat\ell_\mu\epsilon_\perp^{\mu\nu}a_{\perp\nu} \;\equiv\; \hat\ell\wedge\boldsymbol{a}_\perp \;=\; |\boldsymbol{a}_\perp|\,\sin(\phi_a), \tag{7.21}$$

where we have used $\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp\equiv\epsilon_\perp^{\rho\sigma}a_{\perp\rho}b_{\perp\sigma}$. In many cases it is convenient to express the tensors in $\mathcal{W}^{\mu\nu}$ with respect to measured directions, i.e. $\hat x=-\hat h$ and $\hat y^\mu=\epsilon_\perp^{\mu\nu}\,\hat x_\nu$,

$$k_\perp^\mu-p_\perp^\mu \;=\; -P_{h\perp}/z \;=\; -Q_{\scriptscriptstyle T}\,\hat h^\mu, \tag{7.22}$$

Table 7.1: Contractions of the lepton tensor $L_{\mu\nu}^{(\ell H)}$ with tensor structures appearing in the hadron tensor.

$w^{\mu\nu}$	$L_{\mu\nu} w^{\mu\nu}$
$-g_\perp^{\mu\nu}$	$\frac{4Q^2}{y^2}\left(1-y+\frac{1}{2}y^2\right)$
$a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu}$	$\frac{4Q^2}{y^2}\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\cos(\phi_a+\phi_b)$
$\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right)$ $= a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} - (\epsilon_\perp^{\rho\sigma}a_{\perp\rho}b_{\perp\sigma})\,g_\perp^{\mu\nu}$ $= a_\perp^{\mu}\epsilon_\perp^{\nu\rho}b_{\perp\rho} + b_\perp^{\mu}\epsilon_\perp^{\nu\rho}a_{\perp\rho} + (a_\perp\cdot b_\perp)\,\epsilon_\perp^{\mu\nu}$	$-\frac{4Q^2}{y^2}\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_a+\phi_b)$
$\hat{t}^{\,\{\mu}a_\perp^{\nu\}}$	$-\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$
$\hat{t}^{\,\{\mu}\,\epsilon_\perp^{\nu\}\rho}\,a_{\perp\rho}$	$\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$
$i\,\epsilon_\perp^{\mu\nu}$	$\lambda_e\,\frac{4Q^2}{y^2}\,y\left(1-\frac{y}{2}\right)$
$i\,a_\perp^{[\mu}b_\perp^{\nu]}$	$\lambda_e\,\frac{4Q^2}{y^2}\,y\left(1-\frac{y}{2}\right)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_b-\phi_a)$
$i\,\hat{t}^{\,[\mu}a_\perp^{\nu]}$	$-\lambda_e\,\frac{4Q^2}{y^2}\,y\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$
$i\,\hat{t}^{\,[\mu}\,\epsilon_\perp^{\nu]\rho}a_{\perp\rho}$	$-\lambda_e\,\frac{4Q^2}{y^2}\,y\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$

$$
\begin{aligned}
a_\perp^\mu &= (\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)\,\hat{h}^\mu + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)\,\epsilon_\perp^{\mu\rho}\hat{h}_\rho, && (7.23)\\
\epsilon_\perp^{\mu\rho}\,a_{\perp\rho} &= -(\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)\,\hat{h}^\mu + (\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)\,\epsilon_\perp^{\mu\rho}\,\hat{h}_\rho, && (7.24)\\
a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu} &= \Big[2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp) - (\boldsymbol{a}_\perp\cdot\boldsymbol{b}_\perp)\Big]\;\Big(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\Big)\\
&\quad + \Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{\{\mu}\epsilon_\perp^{\nu\}\rho}\hat{h}_\rho, && (7.25)\\
\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right) &= -\Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\Big(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\Big)\\
&\quad + \Big[2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp) - (\boldsymbol{a}_\perp\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{\{\mu}\epsilon_\perp^{\nu\}\rho}\hat{h}_\rho, && (7.26)\\
a_\perp^{[\mu}b_\perp^{\nu]} &= \Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) - (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{[\mu}\epsilon_\perp^{\nu]\rho}\hat{h}_\rho = (\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp)\,\epsilon_\perp^{\mu\nu}. && (7.27)
\end{aligned}
$$

This allows one to pull the tensor structure outside the integration over transverse momenta. A useful relation is $\hat{\boldsymbol{h}}\cdot(\boldsymbol{p}_\perp+\boldsymbol{k}_\perp) = (\boldsymbol{p}_\perp^2-\boldsymbol{k}_\perp^2)/Q_T$.

In the next sections we will consider the cases of production of unpolarized and polarized leptoproduction separately, that means either one sums over all final state configurations of, say, a produced particle, or one determines from the final state configurations the spin vector S_h (characterized by λ_h and S_{hT}) that determines the production matrix. For each of the above cases we will consider separately the case of unpolarized (O) and longitudinally polarized (L) leptons and of unpolarized (O), longitudinally polarized (L) or transversely polarized initial hadron state (T).

7.3 Lepto-production integrated over transverse momenta

The hadronic tensor simplifies to

$$
\begin{aligned}
2M \int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} \;=\;& 2z_h \Big\{ -g_\perp^{\mu\nu}\Big[f_1 D_1 + \lambda\lambda_h\, g_1 G_1\Big] - \Big(S_\perp^{\{\mu}S_{h\perp}^{\nu\}} + (\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}\Big)\; h_1 H_1 \\
& + i\,\epsilon_\perp^{\mu\nu}\Big[f_1\lambda_h\,G_1 + \lambda\, g_1 D_1\Big] \\
& + \lambda_h\,\frac{2M\,\hat t^{\{\mu}S_\perp^{\nu\}}}{Q}\left[x_{\scriptscriptstyle B}\, g_T G_1 + \frac{M_h}{M}\,h_1\,\left(\frac{H_L}{z_h} + H_{1L}^{\perp(1)} - \frac{m}{M_h}\,G_1\right)\right] \\
& + \lambda\,\frac{2M_h\,\hat t^{\{\mu}S_{h\perp}^{\nu\}}}{Q}\left[\frac{M}{M_h}\,x_{\scriptscriptstyle B}\,h_L H_1 + g_1\,\left(\frac{G_T}{z_h} - G_{1T}^{(1)} - \frac{m}{M_h}\,H_1\right)\right] \\
& + \frac{2M_h\,\hat t^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}}{Q}\,h_1\,\left(\frac{H}{z_h} + H_1^{\perp(1)}\right) + \frac{2M_h\,\hat t^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}}{Q}\,f_1\,\left(\frac{D_T}{z_h} + D_{1T}^{\perp(1)}\right) \\
& + i\lambda_h\,\frac{2M\hat t^{[\mu}S_\perp^{\nu]}}{Q}\,\frac{M_h}{M}\,h_1\frac{E_L}{z_h} - i\lambda\,\frac{2M_h\,\hat t^{[\mu}S_{h\perp}^{\nu]}}{Q}\,g_1\,\left(\frac{D_T}{z_h} + D_{1T}^{\perp(1)}\right) \\
& + i\frac{2M_h\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}S_{h\perp\rho}}{Q}\left[\frac{M}{M_h}\,2x_{\scriptscriptstyle B}\,eH_1 + f_1\,\left(\frac{G_T}{z_h} - G_{1T}^{(1)} - \frac{m}{M_h}\,H_1\right)\right] \\
& + i\frac{2M\,\hat t^{[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}}{Q}\left[x_{\scriptscriptstyle B}\,g_T D_1 + \frac{M_h}{M}\,h_1\,\left(\frac{E}{z_h} - \frac{m}{M_h}\,D_1\right)\right]\Big\}, \qquad (7.28)
\end{aligned}
$$

where the quark distribution functions in hadron H depend on $x_{\scriptscriptstyle B}$, while the quark fragmentation functions into hadron h depend on z_h.

7.3.1 Unpolarized leptons and hadrons

The relevant part of the hadronic tensor is

$$
2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} = -g_\perp^{\mu\nu}\,2z\,f_1(x_{\scriptscriptstyle B})D_1(z) + \frac{2M_h\,\hat t^{\{\mu}\epsilon_\perp^{\nu\}\rho}\,S_{h\perp\rho}}{Q}\,2z\,f_1(x_{\scriptscriptstyle B})\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right). \qquad (7.29)
$$

The semi-inclusive cross section is given by

$$
\boxed{
\begin{aligned}
\frac{d\sigma_{OO}(\ell H\to \ell'\vec h X)}{dx_{\scriptscriptstyle B}\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\Big\{&\left(\frac{y^2}{2}+1-y\right)\,x_{\scriptscriptstyle B} f_1(x_{\scriptscriptstyle B})\,D_1(z) \\
&+2|\boldsymbol{S}_{h\perp}|\,(2-y)\sqrt{1-y}\,\sin(\phi_s^h)\,\frac{M_h}{Q}\,x_{\scriptscriptstyle B}\,f_1(x_{\scriptscriptstyle B})\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right)\Big\}. \qquad (7.30)
\end{aligned}}
$$

We remind that $D_{1T}^{\perp(1)}$ is related to D_T via

$$
D_{1T}^{\perp(1)}(z) = -z\int_z^1 dy\;\frac{D_T(y)}{y^3}, \qquad (7.31)
$$

or equivalently

$$
\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z) = z\frac{d}{dz}\,D_{1T}^{\perp(1)}. \qquad (7.32)
$$

7.3.2 Unpolarized leptons and longitudinally polarized hadrons

The relevant part of the hadronic tensor is

$$
\begin{aligned}
2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} \;=\;& -g_\perp^{\mu\nu}\,\lambda\,\lambda_h\,2z\,g_1(x_{\scriptscriptstyle B})G_1(z) \\
& + \frac{2M_h\,\hat t^{\{\mu}S_{h\perp}^{\nu\}}}{Q}\,\lambda\left[\frac{M}{M_h}\,2x_{\scriptscriptstyle B} z\,h_L(x_{\scriptscriptstyle B})H_1(z) + 2z\,g_1(x_{\scriptscriptstyle B})\frac{\tilde G_T(z)}{z}\right], \qquad (7.33)
\end{aligned}
$$

where

$$\frac{\tilde{G}_T(z)}{z} = \frac{G_T(z)}{z} - G_{1T}^{(1)}(z) - \frac{m}{M_h}\,H_1(z) \tag{7.34}$$

is a pure quark-quark-gluon (twist-three) matrix element. The semi-inclusive cross section is given by

$$\boxed{\begin{aligned}\frac{d\Delta\sigma_{OL}(\ell\vec{H}\to\ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\Bigg\{\lambda_h\,\left(\frac{y^2}{2}+1-y\right)\,x_B\,g_1(x_B)\,G_1(z)\\ -2|\boldsymbol{S}_{h\perp}|\,(2-y)\sqrt{1-y}\,\cos(\phi_s^h)\left[\frac{M}{Q}\,x_B^2\,h_L(x_B)H_1(z)+\frac{M_h}{Q}\,x_B\,g_1(x_B)\frac{\tilde{G}_T(z)}{z}\right]\Bigg\}\quad(7.35)\end{aligned}}$$

where $\Delta\sigma_{OL}$ indicates that only the part involving polarization is given, which can be extracted as a cross section difference, e.g. between λ = 1 and λ = -1.

7.3.3 Unpolarized leptons and transversely polarized hadrons

The relevant part of the hadronic tensor is

$$\begin{aligned}2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} &= -\left(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}\right)\,2z\,h_1(x_B)H_1(z)\\ &\quad+\frac{2M\,\hat{t}^{\,\{\mu}S_\perp^{\nu\}}}{Q}\,\lambda_h\left[2x_Bz\,g_T(x_B)G_1(z)+\frac{M_h}{M}\,4z\,h_1(x_B)\frac{\tilde{H}_L(z)}{z}\right]\\ &\quad+\frac{2M_h\,\hat{t}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}}{Q}\,2z\,h_1(x_B)\frac{\tilde{H}(z)}{z},\end{aligned} \tag{7.36}$$

where

$$\frac{\tilde{H}_L(z)}{z} = \frac{H_L(z)}{z}+\int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M_h^2}\,H_{1L}^\perp(z,-z\boldsymbol{k}_T)-\frac{m}{M_h}\,G_1(z) \tag{7.37}$$

$$\frac{\tilde{H}(z)}{z} = \frac{H(z)}{z}+\int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{M_h^2}H_1^\perp(z,-z\boldsymbol{k}_T) \tag{7.38}$$

are pure quark-quark-gluon matrix elements. The semi-inclusive cross section is given by

$$\boxed{\begin{aligned}\frac{d\Delta\sigma_{OT}(\ell\vec{H}\to\ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\Bigg\{2\,(2-y)\sqrt{1-y}\,\sin(\phi_s)\,\frac{M_h}{Q}\,x_B\,h_1(x_B)\,\frac{\tilde{H}(z)}{z}\\ -2\lambda_h\,(2-y)\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)G_1(z)+\frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{H}_L(z)}{z}\right]\\ -|\boldsymbol{S}_{h\perp}|\,(1-y)\,\cos(\phi_s+\phi_s^h)\,x_B\,h_1(x_B)H_1(z)\Bigg\}.\quad(7.39)\end{aligned}}$$

7.3.4 Polarized leptons and unpolarized hadrons

The relevant part of the hadronic tensor is

$$\begin{aligned}2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} &= i\,\epsilon_\perp^{\mu\nu}\,\lambda_h\,2z\,f_1(x_B)G_1(z)\\ &\quad+i\frac{2M_h\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{h\perp\rho}}{Q}\left[\frac{M}{M_h}\,2x_Bz\,e(x_B)H_1(z)+2z\,f_1(x_B)\frac{\tilde{G}_T(z)}{z}\right].\end{aligned} \tag{7.40}$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LO}(\vec{\ell}H \to \ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e \left\{\lambda_h\,y\,\left(1-\frac{y}{2}\right)\,x_B\,f_1(x_B)\,G_1(z) \right.$$
$$\left. -2\,|\boldsymbol{S}_{h\perp}|\,y\sqrt{1-y}\,\cos(\phi_s^h)\left[\frac{M}{Q}\,x_B^2\,e(x_B)H_1(z) + \frac{M_h}{Q}\,x_B\,f_1(x_B)\frac{\tilde{G}_T(z)}{z}\right]\right\}. \quad (7.41)$$

7.3.5 Polarized leptons and longitudinally polarized hadrons

The relevant part of the hadronic tensor is

$$2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} = i\,\epsilon_\perp^{\mu\nu}\,\lambda\,2z\,g_1(x_B)D_1 - i\frac{2M_h\,\hat{t}^{\,[\mu}S_{h\perp}^{\nu]}}{Q}\,\lambda\,2z\,g_1(x_B)\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right). \quad (7.42)$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LL}(\vec{\ell}\vec{H} \to \ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda\left\{y\,\left(1-\frac{y}{2}\right)\,x_B g_1(x_B)\,D_1(z)\right.$$
$$\left. +2\,|\boldsymbol{S}_{h\perp}|\,y\sqrt{1-y}\,\sin(\phi_s^h)\,\frac{M_h}{Q}\,x_B\,g_1(x_B)\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right)\right\}. \quad (7.43)$$

7.3.6 Polarized leptons and transversely polarized hadrons

The relevant part of the hadronic tensor is

$$\begin{aligned} 2M\int d^2P_{h\perp}\,\mathcal{W}^{\mu\nu} &= i\frac{2M_h\hat{t}^{\,[\mu}S_\perp^{\nu]}}{Q}\,\lambda_h\,2z\,h_1(x_B)\frac{E_L(z)}{z} \\ &\quad + i\frac{2M\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}}{Q}\left[2x_B z\,g_T(x_B)D_1(z) + \frac{M_h}{M}\,2z\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right], \end{aligned} \quad (7.44)$$

where

$$\frac{\tilde{E}(z)}{z} = \frac{E(z)}{z} - \frac{m}{M_h}\,D_1(z). \quad (7.45)$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LT}(\vec{\ell}\vec{H} \to \ell'\vec{h}X)}{dx_B\,dy\,dz} =$$
$$\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,|\boldsymbol{S}_\perp|\left\{-2\,y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)D_1(z) + \frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right]\right.$$
$$\left. -2\,\lambda_h\,y\sqrt{1-y}\,\sin(\phi_s)\,\frac{M_h}{Q}\,x_B\,h_1(x_B)\,\frac{\tilde{E}_L(z)}{z}\right\}. \quad (7.46)$$

7.4 Transverse momenta in lepto-production without polarization in the final state

In the case that no polarization in the final state is observed or for the case that a spin 0 particle is produced (e.g. semi-inclusive leptopionproduction, the hadronic tensor simplifies to:

$$2M\,\mathcal{W}^{\mu\nu} = \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T\;\times\left\{-g_\perp^{\mu\nu}\,2z\,f_1D_1 + i\,\epsilon_\perp^{\mu\nu}\,2z\,g_{1s}D_1\right.$$

$$
\begin{aligned}
&+\frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}p_\perp^{\nu\}}+p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2MM_h}\,2zh_{1s}^\perp H_1^\perp+\frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}S_\perp^{\nu\}}+S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2M_h}\,2zh_{1T}H_1^\perp\\
&+\frac{\hat t^{\{\mu}k_\perp^{\nu\}}}{Q}\left[-4z\,f_1D_1+4\,f_1D^\perp\right]+\frac{\hat t^{\{\mu}p_\perp^{\nu\}}}{Q}\,4xz\,f^\perp D_1\\
&+\frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat t^{\nu\}}}{Q}\left[-\frac{M}{M_h}\,4xz\,h_sH_1^\perp+\frac{m}{M_h}\,4z\,g_{1s}H_1^\perp\right]\\
&+\frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat t^{\nu\}}}{Q}\left[-\frac{M_h}{M}\,4\,h_{1s}^\perp H-\frac{\boldsymbol{k}_\perp^2}{MM_h}\,4z\,h_{1s}^\perp H_1^\perp-\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\,4xz\,h_T^\perp H_1^\perp\right]\\
&+\frac{M\,S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat t^{\nu\}}}{Q}\left[-\frac{M_h}{M}\,4\,h_{1T}H-\frac{\boldsymbol{k}_\perp^2}{MM_h}\,4z\,h_{1T}H_1^\perp+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,4xz\,h_T^\perp H_1^\perp\right]\\
&+i\frac{\hat t^{[\mu}k_\perp^{\nu]}}{Q}\left[-\frac{M}{M_h}\,4xz\,eH_1^\perp+\frac{m}{M_h}\,4z\,f_1H_1^\perp\right]\\
&+i\frac{k_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat t^{\nu]}}{Q}\left[4\,g_{1s}D^\perp-4z\,g_{1s}D_1\right]\\
&+i\frac{p_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat t^{\nu]}}{Q}\left[4xz\,g_s^\perp D_1+\frac{M_h}{M}\,4\,h_{1s}^\perp E-\frac{m}{M}\,4z\,h_{1s}^\perp D_1\right]\\
&+i\frac{M\,S_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat t^{\nu]}}{Q}\left[4xz\,g_T'D_1+\frac{M_h}{M}\,4\,h_{1T}E-\frac{m}{M}\,4z\,h_{1T}D_1\right]\Bigg\}
\end{aligned}
\tag{7.47}
$$

We will again consider all situations separately for different polarizations of the lepton and target hadrons.

7.4.1 $\boxed{\ell H\to\ell' hX}$ (unpolarized hadrons)

The hadronic tensor is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\;\times\Bigg\{-g_\perp^{\mu\nu}\,2z\,f_1D_1\\
&\qquad+\frac{\hat t^{\{\mu}k_\perp^{\nu\}}}{Q}\left[-4z\,f_1D_1+4\,f_1D^\perp\right]+\frac{\hat t^{\{\mu}p_\perp^{\nu\}}}{Q}\,4xz\,f^\perp D_1\Bigg\}\\
&= -g_\perp^{\mu\nu}\,2z\,I[f_1D_1]\\
&\qquad+\frac{\hat t^{\{\mu}\hat h^{\nu\}}}{Q}\left(4z\,I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)\,f_1\left(\frac{D^\perp}{z}-D_1\right)\right]+4xz\,I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)\,f^\perp D_1\right]\right),
\end{aligned}
\tag{7.48}
$$

where the last expression involves integrals of the type

$$
I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)\,f^\perp D_1\right](x_B,z)=\int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,f^\perp(x_B,\boldsymbol{p}_T)D_1(z,-z\boldsymbol{k}_T),
\tag{7.49}
$$

with $\boldsymbol{q}_T=-\boldsymbol{P}_{h\perp}/z=Q_T\,\hat{\boldsymbol{h}}$. Note that a contribution proportional to $\hat h_\rho\epsilon^{\rho\,\{\mu}\hat t^{\nu\}}$ appears, but it is multiplied with integrals of the type $I[(\hat{\boldsymbol{h}}\wedge\boldsymbol{p}_\perp)f_1(x_B,|\boldsymbol{p}_\perp|)D(z,z|Q_T\,\hat{\boldsymbol{h}}-\boldsymbol{p}_\perp|)]$, which vanish.

The most explicit differential cross section is immediately obtained from the first expression for the hadronic tensor and is given by

$$\boxed{\begin{aligned}&\frac{d\sigma_{OO}(\ell+H\to\ell'+h+\text{jet}+X)}{dx_{\scriptscriptstyle B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp}=\\&\qquad\frac{4\pi\alpha^2\,s}{Q^4}\left\{\left(\frac{y^2}{2}+1-y\right)\,x_{\scriptscriptstyle B}f_1\,D_1\right.\\&\qquad\qquad+2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{zQ}\,\cos(\phi_h)\;x_{\scriptscriptstyle B}f_1\,\left(\frac{D^\perp}{z}-D_1\right)\\&\qquad\qquad\left.-2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\cos(\phi_j)\,\left(x_{\scriptscriptstyle B}^2f^\perp D_1-x_{\scriptscriptstyle B}f_1D_1+x_{\scriptscriptstyle B}f_1\frac{D^\perp}{z}\right)\right\},\end{aligned}}\tag{7.50}$$

where the arguments of distribution and fragmentation functions are $f_1(x_{\scriptscriptstyle B},\boldsymbol{p}_\perp)$, $D_1(z,\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp)$, etc.

The semi-inclusive cross section where the jet direction is not determined can be found also from the (first) most general expression above, but this is cumbersome, since one must be aware that integrating over $d^2\boldsymbol{p}_\perp$, the argument of the fragmentation functions depend on $\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp$. It is easier to start with the second expression for the hadronic tensor and obtain

$$\boxed{\begin{aligned}&\frac{d\sigma_{OO}(\ell+H\to\ell'+h+X)}{dx_{\scriptscriptstyle B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}}=\\&\qquad\frac{4\pi\alpha^2\,s}{Q^4}\left\{\left(\frac{y^2}{2}+1-y\right)\,I[x_{\scriptscriptstyle B}f_1\,D_1]\right.\\&\qquad\qquad-2(2-y)\sqrt{1-y}\,\cos(\phi_h)\left(\frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_{\scriptscriptstyle B}f_1\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\right.\\&\qquad\qquad\qquad\left.\left.+\frac{M}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_{\scriptscriptstyle B}^2f^\perp D_1\Big]\right)\right\},\end{aligned}}\tag{7.51}$$

which involves the above defined convolutions over distribution and fragmentation functions.

Returning to the previous cross section, one can integrate over the transverse momenta of the produced hadrons in the jet and find

$$\boxed{\begin{aligned}&\frac{d\sigma_{OO}(\ell+H\to\ell'+h+\text{jet}+X)}{dx_{\scriptscriptstyle B}\,dy\,dz\,d^2\boldsymbol{p}_\perp}=\\&\qquad\frac{4\pi\alpha^2\,s}{Q^4}\left\{\left(\frac{y^2}{2}+1-y\right)\,x_{\scriptscriptstyle B}f_1(x_{\scriptscriptstyle B},\boldsymbol{p}_\perp)\,D_1(z)\right.\\&\qquad\qquad\left.-2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\cos(\phi_j)\;x_{\scriptscriptstyle B}^2f^\perp(x_{\scriptscriptstyle B},\boldsymbol{p}_\perp)D_1(z)\right\},\end{aligned}}\tag{7.52}$$

where the arguments of distribution and fragmentation functions are $f_1(x_{\scriptscriptstyle B},\boldsymbol{p}_\perp)$, $D_1(z)$, etc. Integrating also over the transverse momentum of the jet we obtain the result discussed earlier,

$$\frac{d\sigma_{OO}(\ell+H\to\ell'+X)}{dx_{\scriptscriptstyle B}\,dy\,dz}=\frac{4\pi\alpha^2\,s}{Q^4}\,\left(\frac{y^2}{2}+1-y\right)\,x_{\scriptscriptstyle B}f_1(x_{\scriptscriptstyle B})D_1(z).\tag{7.53}$$

7.4.2 $\boxed{\ell\vec{H}\to\ell' hX}$ (longitudinally polarized target)

The hadronic tensor is given by

$$\begin{aligned}2M\,\mathcal{W}^{\mu\nu}&=\int d^2\boldsymbol{p}_{\scriptscriptstyle T}\,d^2\boldsymbol{k}_{\scriptscriptstyle T}\,\delta^2(\boldsymbol{p}_{\scriptscriptstyle T}+\boldsymbol{q}_{\scriptscriptstyle T}-\boldsymbol{k}_{\scriptscriptstyle T})\\&\qquad\times\lambda\left\{\frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}p_\perp^{\nu\}}+p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2MM_h}\,2zh_{1L}^\perp H_1^\perp\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M}{M_h}\,4xz\,h_L H_1^\perp+\frac{m}{M_h}\,4z\,g_{1L}H_1^\perp\right]\\
&+\frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M_h}{M}\,4\,h_{1L}^\perp H-\frac{\boldsymbol{k}_\perp^2}{MM_h}\,4z\,h_{1L}^\perp H_1^\perp\right]\Bigg\}.\\
=\quad&\lambda\Bigg\{\hat{h}_\rho\epsilon_\perp^{\rho\,\{\mu}\hat{h}^{\nu\}}\,2z\,I\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)-\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,h_{1L}^\perp H_1^\perp\right]\\
&-\hat{h}_\rho\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}\Bigg(\frac{M}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,\left(x\,h_L-\frac{m}{M}\,g_{1L}\right)\,H_1^\perp\Big]\\
&+\frac{M_h}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,h_{1L}^\perp\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Big]\Bigg)\Bigg\}. \qquad (7.54)
\end{aligned}
$$

The most explicit differential cross section is given by

$$
\begin{aligned}
&\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp}=\\
&\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\Bigg\{-(1-y)\,\frac{|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{p}_\perp|}{zMM_h}\,\sin(\phi_h+\phi_j)\,x_B\,h_{1L}^\perp H_1^\perp\\
&+(1-y)\,\frac{\boldsymbol{p}_\perp^2}{MM_h}\,\sin(2\phi_j)\,x_B\,h_{1L}^\perp H_1^\perp\\
&-2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{zQ}\,\sin\phi_h\,\frac{M}{M_h}\,x_B\,\left(x_B h_L-\frac{m}{M}\,g_{1L}\right)\,H_1^\perp\\
&+2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\,\Bigg[\frac{M}{M_h}\,x_B\,\left(x_B h_L-\frac{m}{M}\,g_{1L}\right)\,H_1^\perp\\
&+\frac{M_h}{M}\,x_B h_{1L}^\perp\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Bigg]\Bigg\}, \qquad (7.55)
\end{aligned}
$$

where the arguments of distribution and fragmentation functions are $h_{1L}^\perp(x_B,\boldsymbol{p}_\perp)$, etc. and $H_1^\perp(z,\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp)$, etc. The cross section averaged over the jet angle is found from the second form for $\mathcal{W}^{\mu\nu}$,

$$
\begin{aligned}
&\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}}=\\
&\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\Bigg\{(1-y)\,\sin(2\phi_h)\,I\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)-\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,x_B h_{1L}^\perp H_1^\perp\right]\\
&+2(2-y)\sqrt{1-y}\,\sin\phi_h\,\Bigg(\frac{M}{Q}\,I\left[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B\,\left(x_B h_L-\frac{m}{M}\,g_{1L}\right)\,H_1^\perp\right]\\
&+\frac{M_h}{Q}\,I\left[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B h_{1L}^\perp\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\right]\Bigg)\Bigg\}. \qquad (7.56)
\end{aligned}
$$

Integrating over the transverse momenta of the produced hadrons in the jet one obtains

$$
\begin{aligned}
&\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp}=\\
&\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\left\{+2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\,\frac{M_h}{M}\,x_B h_{1L}^\perp(x_B,\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z}\right\}, \qquad (7.57)
\end{aligned}
$$

where $h_{1L}^{\perp}(x_B, \boldsymbol{p}_\perp)$ is a 'leading twist' distribution and the fragmentation function $\tilde{H}(z)$,

$$\frac{\tilde{H}(z)}{z} = \int d^2\boldsymbol{k}_T\;\frac{\tilde{H}(z,\boldsymbol{k}_T)}{z} = \int d^2\boldsymbol{k}_T\;\left(\frac{H}{z} + \frac{\boldsymbol{k}_T^2}{M_h^2}\,H_1^\perp\right), \tag{7.58}$$

is a 'twist three' quark-quark-gluon matrix element.

7.4.3 $\boxed{\ell\vec{H} \to \ell' h X}$ (transversely polarized target)

The hadronic tensor is given by

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} \;=\; & \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\\
&\times\Bigg\{\frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}p_\perp^{\nu\}}+p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2MM_h}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,2z h_{1T}^\perp H_1^\perp\\
&\quad+\frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}S_\perp^{\nu\}}+S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2M_h}\,2z h_{1T}H_1^\perp\\
&\quad-\frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\,4z\,\left(x\,h_T-\frac{m}{M}\,g_{1T}\right)\,H_1^\perp\\
&\quad-\frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[\frac{M_h}{M}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,4\,h_{1T}^\perp\,\left(H+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,zH_1^\perp\right)+\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\,4xz\,h_T^\perp H_1^\perp\right]\\
&\quad-\frac{M\,S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[\frac{M_h}{M}\,4\,h_{1T}\,\left(H+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,zH_1^\perp\right)-\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,4xz\,h_T^\perp H_1^\perp\right]\Bigg\}\\
=\;&\frac{1}{2}\left(\hat{h}_\rho\epsilon_\perp^{\rho\,\{\mu}S_\perp^{\nu\}}+S_{\perp\rho}\epsilon_\perp^{\rho\,\{\mu}\hat{h}_\perp^{\nu\}}\right)\,I\Big[\frac{(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{M_h}\,2z\,h_1H_1^\perp\Big]\\
&+\hat{h}_\rho\epsilon_\perp^{\rho\,\{\mu}\hat{h}^{\nu\}}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\,I\Big[\frac{4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)}{2\,M^2\,M_h}\,2z\,h_{1T}^\perp H_1^\perp\Big]\\
&+\left(2\,\hat{h}^\mu\hat{h}^\nu+g_\perp^{\mu\nu}\right)\,(\hat{\boldsymbol{h}}\wedge\boldsymbol{S}_\perp)\\
&\qquad\times\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)+\boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2\,M^2\,M_h}\,2z\,h_{1T}^\perp H_1^\perp\Big]\\
&-S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}\Bigg(\frac{M}{Q}\,I\left[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\,4z\,\left(x_Bh_T-\frac{m}{M}\,g_{1T}-x_Bh_T^\perp\right)\,H_1^\perp\right]\\
&\qquad\qquad+\frac{M_h}{Q}\,I\left[4z\,h_1\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\right]\Bigg)\\
&-h_\rho\epsilon_\perp^{\rho\,\{\mu}\hat{t}^{\nu\}}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\Bigg(\frac{M_h}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,4z\,h_{1T}^\perp\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Big]\\
&\qquad\qquad+\frac{M}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)-\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\,4z\,\left(x_Bh_T-\frac{m}{M}\,g_{1T}+x_Bh_T^\perp\right)\,H_1^\perp\Big]\Bigg)\\
&-\hat{h}^{\,\{\mu}\hat{t}^{\nu\}}\,(\hat{\boldsymbol{h}}\wedge\boldsymbol{S}_\perp)\Bigg(\frac{M_h}{Q}\,I\Big[\frac{\boldsymbol{p}_\perp^2-2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2}{2M^2}\,4z\,h_{1T}^\perp\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Big]\\
&\qquad\qquad+\frac{M}{Q}\,I\Big[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp-2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)}{2MM_h}\,4z\,\left(x_Bh_T-\frac{m}{M}\,g_{1T}+x_Bh_T^\perp\right)\,H_1^\perp\Big]\Bigg)
\end{aligned} \tag{7.59}$$

The most explicit differential cross section is given by

$$\begin{aligned}
\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\rightarrow\ell'+h+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_{\perp}} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_{\perp}|\,\Bigg\{ & (1-y)\frac{|\boldsymbol{p}_{\perp}|}{M_h}\,\sin(\phi_j+\phi_s)\,x_{B}h_1H_1^{\perp} \\
& +(1-y)\frac{|\boldsymbol{p}_{\perp}|^3}{2\,M^2\,M_h}\,\sin(3\phi_j-\phi_s)\,x_{B}h_{1T}^{\perp}H_1^{\perp} \\
& -(1-y)\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\,\sin(\phi_h+\phi_s)\,x_{B}h_1H_1^{\perp} \\
& -(1-y)\frac{\boldsymbol{p}_{\perp}^2\,|\boldsymbol{P}_{h\perp}|}{2\,M^2\,M_h}\,\sin(2\phi_j+\phi_h-\phi_s)\,x_{B}h_{1T}^{\perp}H_1^{\perp} \\
& +2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin(\phi_s)\;x_{B}h_1\,\frac{\tilde{H}}{z} \\
& +2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\frac{\boldsymbol{p}_{\perp}^2}{2\,M^2}\,\sin(2\phi_j-\phi_s)\,x_{B}h_{1T}^{\perp}\frac{\tilde{H}}{z} \\
& +2(2-y)\sqrt{1-y}\,\frac{\boldsymbol{p}_{\perp}^2}{2M_h\,Q}\,\sin(2\phi_j-\phi_s)\;x_{B}\,\left(x_{B}h_T-\frac{m}{M}\,g_{1T}-x_{B}h_T^{\perp}\right)\,H_1^{\perp} \\
& +2(2-y)\sqrt{1-y}\,\frac{\boldsymbol{p}_{\perp}^2}{2M_h\,Q}\,\sin(\phi_s)\;x_{B}\,\left(x_{B}h_T-\frac{m}{M}\,g_{1T}+x_{B}h_T^{\perp}\right)\,H_1^{\perp} \\
& -2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_{\perp}|\,|\boldsymbol{P}_{h\perp}|}{2z\,M_h\,Q}\,\sin(\phi_h+\phi_j-\phi_s)\;x_{B}\,\left(x_{B}h_T-\frac{m}{M}\,g_{1T}-x_{B}h_T^{\perp}\right)\,H_1^{\perp} \\
& -2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_{\perp}|\,|\boldsymbol{P}_{h\perp}|}{2z\,M_h\,Q}\,\sin(\phi_h-\phi_j+\phi_s)\;x_{B}\,\left(x_{B}h_T-\frac{m}{M}\,g_{1T}+x_{B}h_T^{\perp}\right)\,H_1^{\perp}\Bigg\},
\end{aligned} \tag{7.60}$$

where the distribution functions depend on x_{B} and $\boldsymbol{p}_{\perp}$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_{\perp}^2 = (\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_{\perp})^2$. We have used or can use

$$\begin{aligned}
& h_1(x_{B},\boldsymbol{p}_{T}) = h_{1T}(x_{B},\boldsymbol{p}_{T}) + \frac{\boldsymbol{p}_{T}^2}{2\,M^2}\,h_{1T}^{\perp}(x_{B},\boldsymbol{p}_{T}), \\
& x_{B}h_T(x_{B},\boldsymbol{p}_{T}) - \frac{m}{M}\,g_{1T}(x_{B},\boldsymbol{p}_{T}) - x_{B}h_T^{\perp}(x_{B},\boldsymbol{p}_{T}) = -2\,h_1(x_{B},\boldsymbol{p}_{T}) + x_{B}\tilde{h}_{1L}^{\perp}(x_{B},\boldsymbol{p}_{T}), \\
& x_{B}h_T(x_{B},\boldsymbol{p}_{T}) - \frac{m}{M}\,g_{1T}(x_{B},\boldsymbol{p}_{T}) + x_{B}h_T^{\perp}(x_{B},\boldsymbol{p}_{T}) = \\
& \qquad\qquad -\frac{\boldsymbol{p}_{T}^2}{M^2}\,h_{1T}^{\perp}(x_{B},\boldsymbol{p}_{T}) + x_{B}\left(\tilde{h}_T(x_{B},\boldsymbol{p}_{T}) + \tilde{h}_T^{\perp}(x_{B},\boldsymbol{p}_{T})\right).
\end{aligned} \tag{7.61}$$

Integrating only over the jet directions one obtains

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\left\{(1-y)\,\sin(\phi_h+\phi_s)\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B h_1 H_1^\perp\Big]\right.$$
$$+(1-y)\,\sin(3\phi_h-\phi_s)\,I\Big[\frac{4(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)-\boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2\,M^2\,M_h}\,x_B h_{1T}^\perp H_1^\perp\Big]$$
$$+2(2-y)\sqrt{1-y}\,\sin(\phi_s)\left(\frac{M_h}{Q}\,I\Big[x_B h_1\,\frac{\tilde{H}}{z}\Big]\right.$$
$$\left.+\frac{M}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\,x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}+x_B h_T^\perp\right)\,H_1^\perp\Big]\right)$$
$$+2(2-y)\sqrt{1-y}\,\sin(2\phi_h-\phi_s)\left(\frac{M_h}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2\,M^2}\,x_B h_{1T}^\perp\,\frac{\tilde{H}}{z}\Big]\right.$$
$$\left.\left.+\frac{M}{Q}\,I\Big[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp-2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2MM_h}\,x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}-x_B h_T^\perp\right)\,H_1^\perp\Big]\right)\right\}. \tag{7.62}$$

The cross section integrated over the transverse momenta of hadrons h within the jet becomes

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} =$$
$$\frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\left\{2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin\phi_s\;x_B h_1(x_B,\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z}\right.$$
$$\left.+2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\frac{\boldsymbol{p}_\perp^2}{2\,M^2}\,\sin(2\phi_j-\phi_s)\,x_B h_{1T}^\perp(x_B,\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z}\right\}, \tag{7.63}$$

where the fragmentation function is integrated over transverse momenta. Integrating further over the jet angle one finds

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\,2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin\phi_s\;x_B\,h_1(x_B)\,\frac{\tilde{H}(z)}{z}. \tag{7.64}$$

7.4.4 $\boxed{\vec{\ell}H\to\ell' hX}$ (unpolarized hadrons)

The hadronic tensor is given by

$$\begin{aligned}2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\\ &\qquad\times\left\{i\frac{\hat{t}^{\,[\mu}k_\perp^{\nu]}}{Q}\left[-\frac{M}{M_h}\,4xz\,eH_1^\perp+\frac{m}{M_h}\,4z\,f_1H_1^\perp\right]\right\}\\ &= -i\,\hat{t}^{\,[\mu}\hat{h}^{\nu]}\,\frac{M}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,\left(x\,e-\frac{m}{M}\,f_1\right)\,H_1^\perp\Big]\end{aligned} \tag{7.65}$$

The most explicit differential cross section is given by

$$\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} =$$
$$\frac{4\pi\alpha^2\,s}{Q^4}\left\{2\lambda_e y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\;\frac{M}{M_h}\,x_B\,\left(x_B e-\frac{m}{M}\,f_1\right)\,H_1^\perp\right.$$
$$\left.-2\lambda_e y\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{zQ}\,\sin\phi_h\;\frac{M}{M_h}\,x_B\,\left(x_B e-\frac{m}{M}\,f_1\right)\,H_1^\perp\right\}, \tag{7.66}$$

where the arguments of distribution and fragmentation functions are $\tilde{e}(x_B, \boldsymbol{p}_\perp)$, etc. and $H_1^\perp(z, \boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)$. Integrating over the transverse momenta of the produced hadrons in the jet one obtains zero. Integrating over the jet angle the result is

$$
\boxed{
\begin{aligned}
&\frac{d\Delta\sigma_{LO}(\vec{\ell} + H \to \ell' + h + X)}{dx_B\, dy\, dz\, d^2\boldsymbol{P}_{h\perp}} = \\
&\qquad \frac{4\pi\alpha^2\, s}{Q^4}\, 2\lambda_e y\sqrt{1-y}\, \sin\phi_h\, \frac{M}{Q}\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\, x_B\, \left(x_B e - \frac{m}{M}\, f_1\right)\, H_1^\perp\Big].
\end{aligned}
}
\tag{7.67}
$$

7.4.5 $\boxed{\vec{\ell}\vec{H} \to \ell' h X}$ (longitudinally polarized target)

The hadronic tensor is given by

$$
\begin{aligned}
2M\, \mathcal{W}^{\mu\nu} \;=\; & \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\; \times\; \lambda\Bigg\{ i\,\epsilon_\perp^{\mu\nu}\, 2z\, g_{1L} D_1 \\
& + i\frac{k_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\Bigg[4\, g_{1L} D^\perp - 4z\, g_{1L} D_1\Bigg] \\
& + i\frac{p_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\Bigg[4xz\, g_L^\perp D_1 + \frac{M_h}{M}\, 4\, h_{1L}^\perp E - \frac{m}{M}\, 4z\, h_{1L}^\perp D_1\Bigg]\Bigg\}. \\
& \lambda\Bigg\{ i\,\epsilon_\perp^{\mu\nu}\, 2z\, I\Big[g_{1L} D_1\Big] + i\,\hat{h}_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}\Bigg(\frac{M_h}{Q}\, 4z\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\, g_{1L}\, \left(\frac{D^\perp}{z} - D_1\right)\Big] \\
& \qquad + \frac{M}{Q}\, 4z\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\, x g_L^\perp D_1\Big] + \frac{M_h}{Q}\, 4z\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\, h_{1L}^\perp\, \left(\frac{E}{z} - \frac{m}{M} D_1\right)\Big]\Bigg)\Bigg\}.
\end{aligned}
\tag{7.68}
$$

The most explicit differential cross section is given by

$$
\boxed{
\begin{aligned}
&\frac{d\Delta\sigma_{LL}(\vec{\ell} + \vec{H} \to \ell' + h + \text{jet} + X)}{dx_B\, dy\, dz\, d^2\boldsymbol{P}_{h\perp}\, d^2\boldsymbol{p}_\perp} = \\
&\qquad \frac{4\pi\alpha^2\, s}{Q^4}\, \lambda_e\lambda\Bigg\{ y\left(1 - \frac{y}{2}\right) x_B\, g_{1L}\, D_1 \\
&\qquad\qquad -2y\sqrt{1-y}\, \frac{|\boldsymbol{p}_\perp|}{Q}\, \cos(\phi_j)\, \Bigg[x_B^2\, g_L^\perp\, D_1 + x_B\, g_{1L}\, \left(\frac{D^\perp}{z} - D_1\right) \\
&\qquad\qquad\qquad + \frac{M_h}{M}\, x_B\, h_{1L}^\perp\, \left(\frac{E}{z} - \frac{m}{M_h}\, D_1\right)\Bigg] \\
&\qquad\qquad +2y\sqrt{1-y}\, \frac{|\boldsymbol{P}_{h\perp}|}{z\, Q}\, \cos(\phi_h)\, x_B\, g_{1L}\, \left(\frac{D^\perp}{z} - D_1\right)\Bigg\},
\end{aligned}
}
\tag{7.69}
$$

where the distribution functions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2 = (\boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)^2$. Integrating only over the jet angle one finds

$$
\begin{aligned}
&\frac{d\Delta\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\
&\quad\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\Bigg\{y\left(1-\frac{y}{2}\right)\,I\left[x_B\,g_{1L}\,D_1\right] \\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_h)\left(\frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B\,g_{1L}\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\right. \\
&\qquad\qquad +\frac{M}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B^2 g_L^\perp D_1\Big] \\
&\qquad\qquad \left.+\frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B h_{1L}^\perp\,\left(\frac{E}{z}-\frac{m}{M}D_1\right)\Big]\right)\Bigg\}.
\end{aligned} \tag{7.70}
$$

Integrating over the transverse momenta of the produced hadrons we obtain

$$
\begin{aligned}
&\frac{d\Delta\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \\
&\quad\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\Bigg\{y\left(1-\frac{y}{2}\right)x_B\,g_{1L}\,D_1 \\
&\qquad -2y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\cos(\phi_j)\left[x_B^2\,g_L^\perp\,D_1+\frac{M_h}{M}\,x_B\,h_{1L}^\perp\,\left(\frac{E}{z}-\frac{m}{M_h}\,D_1\right)\right]\Bigg\}
\end{aligned} \tag{7.71}
$$

with the fragmentation function integrated over $\boldsymbol{k}_T$, and thus only depending on z. Integrating also over the transverse jet momentum, one is left with the leading result for longitudinally polarized targets,

$$
\frac{d\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\,y\left(1-\frac{y}{2}\right)x_B\,g_1(x_B)\,D_1(z). \tag{7.72}
$$

7.4.6 $\boxed{\vec{\ell}\vec{H}\to\ell' hX}$ (transversely polarized target)

The hadronic tensor is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} \;=\; & \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\;\times\Bigg\{i\,\epsilon_\perp^{\mu\nu}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,2z\,g_{1T}D_1 \\
&+i\frac{k_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\left[4\,g_{1T}D^\perp-4z\,g_{1T}D_1\right] \\
&+i\frac{p_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\left[4xz\,g_T^\perp D_1+\frac{M_h}{M}\,4\,h_{1T}^\perp E-\frac{m}{M}\,4z\,h_{1T}^\perp D_1\right] \\
&+i\frac{M\,S_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\left[4xz\,g_T' D_1+\frac{M_h}{M}\,4\,h_{1T}E-\frac{m}{M}\,4z\,h_{1T}D_1\right]\Bigg\}. \\
=\; & i\,\epsilon_\perp^{\mu\nu}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\,2z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,g_{1T}D_1\Big] \\
&+i\,\hat{h}_\rho\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\left(\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\right. \\
&\qquad +\frac{M}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,xg_T^\perp D_1\Big] \\
&\qquad \left.+\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}h_{1T}^\perp\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ i\,\hat{t}^{\,[\mu}\hat{h}^{\nu]}\,(\hat{\boldsymbol{h}}\wedge \boldsymbol{S}_\perp)\left(\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\right.\\
&\qquad\qquad + \frac{M}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,xg_T^\perp D_1\Big]\\
&\qquad\qquad \left. + \frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}h_{1T}^\perp\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big]\right)\\
&+ iS_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}\left(\frac{M}{Q}\,4z\,I\Big[xg_T D_1\Big]+\frac{M_h}{Q}\,4z\,I\Big[h_1\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big]\right.\\
&\qquad\qquad \left. + \frac{M_h}{Q}\,4z\,I\Big[\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\right). \qquad (7.73)
\end{aligned}
$$

The most explicit differential cross section is given by

$$
\boxed{
\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\left\{y\left(1-\frac{y}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\,\cos(\phi_j-\phi_s)\;x_B\,g_{1T}\,D_1\right.\\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T\,D_1+\frac{M_h}{Q}\,x_B\,h_1\,\frac{\tilde{E}}{z}\right]\\
&\qquad -2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(2\phi_j-\phi_s)\left[x_B^2\,g_T^\perp\,D_1+\frac{M_h}{M}\,x_B\,h_{1T}^\perp\,\frac{\tilde{E}}{z}\right]\\
&\qquad -2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(\phi_j)\,\cos(\phi_j-\phi_s)\;x_B\,g_{1T}\,\frac{\tilde{D}^\perp}{z}\\
&\qquad \left. +2y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|\,|\boldsymbol{P}_{h\perp}|}{2zMQ}\,\cos(\phi_h)\,\cos(\phi_j-\phi_s)\;x_B\,g_{1T}\,\frac{\tilde{D}^\perp}{z}\right\}, \qquad (7.74)
\end{aligned}
}
$$

where the distribution functions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2 = (\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp)^2$. Integrating only over the jet angle one obtains

$$
\boxed{
\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\left\{y\left(1-\frac{y}{2}\right)\,\cos(\phi_h-\phi_s)\;I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,g_{1T}D_1\Big]\right.\\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\left(\frac{M}{Q}\,I\Big[x^2 g_T D_1\Big]+\frac{M_h}{Q}\,I\Big[xh_1\,\frac{\tilde{E}}{z}\Big]\right.\\
&\qquad\qquad \left. + \frac{M_h}{Q}\,I\Big[\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,xg_{1T}\,\frac{\tilde{D}^\perp}{z}\Big]\right)\\
&\qquad -2y\sqrt{1-y}\,\cos(2\phi_h-\phi_s)\left(\frac{M}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,x^2 g_T^\perp D_1\Big]\right.\\
&\qquad\qquad + \frac{M_h}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}xh_{1T}^\perp\,\frac{\tilde{E}}{z}\Big]\\
&\qquad\qquad \left.\left. + \frac{M_h}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,xg_{1T}\,\frac{\tilde{D}^\perp}{z}\Big]\right)\right\}. \qquad (7.75)
\end{aligned}
}
$$

Integrating over the transverse momenta of the produced hadrons we obtain

$$\boxed{\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+h+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \\
&\quad\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\Bigg\{y\left(1-\frac{y}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\,\cos(\phi_j-\phi_s)\,x_B\,g_{1T}\,D_1 \\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T\,D_1+\frac{M_h}{Q}\,x_B\,h_1\,\frac{\tilde{E}}{z}\right] \\
&\qquad +2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(2\phi_j+\phi_s)\left[x_B^2\,g_T^\perp\,D_1+\frac{M_h}{M}\,x_B\,h_{1T}^\perp\,\frac{\tilde{E}}{z}\right]\Bigg\}, \qquad (7.76)
\end{aligned}}$$

with the fragmentation function integrated over $\boldsymbol{k}_T$, and thus only depending on z. Integrating also over the transverse jet momentum, one is left with

$$\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = \\
&-\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\,2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)\,D_1(z)+\frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right]. \qquad (7.77)
\end{aligned}$$

7.5 Transverse momenta in lepto-production with polarization in the final state

For the case that polarization in the final state is measured, we will restrict ourselves to the leading order (parton model) results. The hadronic tensor, omitting the unpolarized contribution discussed in the previous section becomes

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T) \\
&\times\Bigg\{-g_\perp^{\mu\nu}\left[2z\,f_1D_1+2z\,g_{1s}G_{1s}+\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}\,2z\,f_1D_{1T}^\perp\right] \\
&-\frac{k_\perp^{\{\mu}p_\perp^{\nu\}}+(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)\,g_\perp^{\mu\nu}}{MM_h}\,2z\,h_{1s}^\perp H_{1s}^\perp-\frac{k_\perp^{\{\mu}S_\perp^{\nu\}}+(\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp)\,g_\perp^{\mu\nu}}{M_h}\,2z\,h_{1T}H_{1s}^\perp \\
&-\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}}{M}\,2z\,h_{1s}^\perp H_{1T}-\left(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}\right)\,2z\,h_{1T}H_{1T} \\
&+i\,\epsilon_\perp^{\mu\nu}\,2z\,f_1G_{1s}+i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\,2z\,g_{1s}D_{1T}^\perp\Bigg\}. \qquad (7.78)
\end{aligned}$$

As before we will consider the various possibilities separately.

7.5.1 $\boxed{\ell H\to\ell'\vec{h}X}$

The hadronic tensor relevant for this situation is

$$2M\,\mathcal{W}^{\mu\nu}=\int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,(-g_\perp^{\mu\nu})\,\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}\,2z\,f_1D_{1T}^\perp. \qquad (7.79)$$

The cross section is given by

$$\boxed{\begin{aligned}&\frac{d\Delta\sigma_{OO}(\ell+H\rightarrow\ell'+\vec{h}+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^{2}\boldsymbol{P}_{h\perp}\,d^{2}\boldsymbol{p}_{\perp}}=\\&\qquad\frac{4\pi\alpha^{2}\,s}{Q^{4}}\,|\boldsymbol{S}_{h\perp}|\left\{\left(1-y+\frac{1}{2}\,y^{2}\right)\,\frac{|\boldsymbol{P}_{h\perp}|}{zM_{h}}\,\sin(\phi_{h}-\phi_{s}^{h})\,x_{B}f_{1}D_{1T}^{\perp}\right.\\&\qquad\qquad\left.-\left(1-y+\frac{1}{2}\,y^{2}\right)\,\frac{|\boldsymbol{p}_{\perp}|}{M_{h}}\,\sin(\phi_{j}-\phi_{s}^{h})\,x_{B}f_{1}D_{1T}^{\perp}\right\}.\end{aligned}}\tag{7.80}$$

Averaged over the transverse momenta of the hadrons h in the jet this gives zero.

7.5.2 $\boxed{\ell\vec{H}\rightarrow\ell'\vec{h}X}$ (longitudinally polarized target)

The symmetric hadronic tensor relevant for this situation is

$$\begin{aligned}2M\,\mathcal{W}^{\mu\nu}\;=\;&\int d^{2}\boldsymbol{k}_{T}\,d^{2}\boldsymbol{p}_{T}\,\delta^{2}(\boldsymbol{p}_{T}+\boldsymbol{q}_{T}-\boldsymbol{k}_{T})\,\lambda\left\{-g_{\perp}^{\mu\nu}\,2z\,g_{1L}G_{1s}\right.\\&-\frac{k_{\perp}^{\{\mu}p_{\perp}^{\nu\}}+(\boldsymbol{k}_{\perp}\cdot\boldsymbol{p}_{\perp})\,g_{\perp}^{\mu\nu}}{MM_{h}}\,2z\,h_{1L}^{\perp}H_{1s}^{\perp}\\&\left.-\frac{p_{\perp}^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{h\perp})\,g_{\perp}^{\mu\nu}}{M}\,2z\,h_{1L}^{\perp}H_{1T}\right\}.\end{aligned}\tag{7.81}$$

We will give only the cross section averaged over the transverse momenta of the hadrons in the jet. This is

$$\boxed{\begin{aligned}&\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\rightarrow\ell'+\vec{h}+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^{2}\boldsymbol{p}_{\perp}}=\\&\qquad\frac{4\pi\alpha^{2}\,s}{Q^{4}}\,\lambda\left\{\lambda_{h}\,\left(1-y-\frac{y^{2}}{2}\right)\,x_{B}\,g_{1L}(x_{B},\boldsymbol{p}_{\perp})\,G_{1}(z)\right.\\&\qquad\qquad\left.-|\boldsymbol{S}_{h\perp}|\,(1-y)\,\frac{|\boldsymbol{p}_{\perp}|}{M}\,\cos(\phi_{j}+\phi_{s}^{h})\;x_{B}\,h_{1L}^{\perp}(x_{B},\boldsymbol{p}_{\perp})\,H_{1}(z)\right\}.\end{aligned}}\tag{7.82}$$

Integrating over the jet directions one obtains

$$\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\rightarrow\ell'+\vec{h}+X)}{dx_{B}\,dy\,dz}=\frac{4\pi\alpha^{2}\,s}{Q^{4}}\,\lambda\,\lambda_{h}\,\left(1-y-\frac{y^{2}}{2}\right)\,x_{B}\,g_{1}(x_{B})\,G_{1}(z).\tag{7.83}$$

7.5.3 $\boxed{\ell\vec{H}\rightarrow\ell'\vec{h}X}$ (transversely polarized target)

The symmetric tensor relevant in this case is

$$\begin{aligned}2M\,\mathcal{W}^{\mu\nu}\;=\;&\int d^{2}\boldsymbol{k}_{T}\,d^{2}\boldsymbol{p}_{T}\,\delta^{2}(\boldsymbol{p}_{T}+\boldsymbol{q}_{T}-\boldsymbol{k}_{T})\left\{-g_{\perp}^{\mu\nu}\,\frac{\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{\perp}}{M}\,2z\,g_{1T}G_{1s}\right.\\&-\frac{k_{\perp}^{\{\mu}p_{\perp}^{\nu\}}+(\boldsymbol{k}_{\perp}\cdot\boldsymbol{p}_{\perp})\,g_{\perp}^{\mu\nu}}{MM_{h}}\,\frac{\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{\perp}}{M}\,2z\,h_{1T}^{\perp}H_{1s}^{\perp}\\&-\frac{k_{\perp}^{\{\mu}S_{\perp}^{\nu\}}+(\boldsymbol{k}_{\perp}\cdot\boldsymbol{S}_{\perp})\,g_{\perp}^{\mu\nu}}{M_{h}}\,2z\,h_{1T}H_{1s}^{\perp}\\&-\frac{p_{\perp}^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{h\perp})\,g_{\perp}^{\mu\nu}}{M}\,\frac{\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{\perp}}{M}\,2z\,h_{1T}^{\perp}H_{1T}\\&\left.-\left(S_{\perp}^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{S}_{\perp}\cdot\boldsymbol{S}_{h\perp})\,g_{\perp}^{\mu\nu}\right)\,2z\,h_{1T}H_{1T}\right\}.\end{aligned}\tag{7.84}$$

We will give only the cross section averaged over the transverse momenta of the hadrons in the jet. This is

$$
\begin{aligned}
&\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\left\{\lambda_h\left(1-y-\frac{y^2}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\cos(\phi_j-\phi_s)\;x_B\,g_{1T}(x_B,\boldsymbol{p}_\perp)\,G_1(z)\right. \\
&\qquad -|\boldsymbol{S}_{h\perp}|\left[(1-y)\cos(\phi_s+\phi_s^h)\;x_B\,h_{1T}(x_B,\boldsymbol{p}_\perp)\,H_1(z)\right. \\
&\qquad\quad \left.\left. -(1-y)\frac{\boldsymbol{p}_\perp^2}{M^2}\cos(\phi_j+\phi_s^h)\cos(\phi_j-\phi_s)\;x_B\,h_{1T}^\perp(x_B,\boldsymbol{p}_\perp)\,H_1(z)\right]\right\}.
\end{aligned}
\tag{7.85}
$$

Integrating over the jet directions one obtains

$$
\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+\vec{h}+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\,|\boldsymbol{S}_{h\perp}|\,(1-y)\cos(\phi_s+\phi_s^h)\;x_B\,h_1(x_B)\,H_1(z) \tag{7.86}
$$

7.5.4 $\boxed{\vec{\ell}H\to\ell'\vec{h}X}$

The hadronic tensor relevant for this situation is the antisymmetric tensor

$$
2M\,\mathcal{W}^{\mu\nu} = \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\;(i\,\epsilon_\perp^{\mu\nu})\;2z\,f_1G_{1s}. \tag{7.87}
$$

The cross section is given by

$$
\begin{aligned}
&\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\to\ell'+\vec{h}+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\left\{\lambda_h\,y\left(1-\frac{1}{2}\,y\right)\;x_B\,f_1G_{1L}\right. \\
&\qquad +|\boldsymbol{S}_{h\perp}|\left[y\left(1-\frac{1}{2}\,y\right)\;\frac{|\boldsymbol{p}_\perp|}{M_h}\cos(\phi_j+\phi_s^h)\,x_B\,f_1G_{1T}\right. \\
&\qquad\quad \left.\left. -y\left(1-\frac{1}{2}\,y\right)\;\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\cos(\phi_h+\phi_s^h)\,x_B\,f_1G_{1T}\right]\right\},
\end{aligned}
\tag{7.88}
$$

where the distributions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2$. Averaged over the hadrons in the jet, one obtains

$$
\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\to\ell'+\vec{h}+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda_h\,y\left(1-\frac{1}{2}\,y\right)\;x_B\,f_1(x_B,\boldsymbol{p}_\perp)G_1(z), \tag{7.89}
$$

where $G_1(z)$ is obtained by integrating G_{1L} over the transverse momenta. A similar expression discussed earlier is obtained after integrating over the transverse momentum of the jet.

7.5.5 $\boxed{\vec{\ell}\vec{H}\to\ell'\vec{h}X}$ (longitudinally polarized target)

The antisymmetric hadronic tensor relevant for this situation is

$$
2M\,\mathcal{W}^{\mu\nu} \;=\; \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\left(i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\right)\lambda\,2z\,g_{1L}D_{1T}^\perp. \tag{7.90}
$$

An asymmetry is obtained in

$$\boxed{\begin{aligned}&\frac{d\Delta\sigma_{LL}(\vec{\ell}+\vec{H}\rightarrow\ell'+\vec{h}+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_{\perp}}=\\&\qquad\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda\left\{|\boldsymbol{S}_{h\perp}|\left[-y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{p}_{\perp}|}{M_h}\,\sin(\phi_j-\phi_s^h)\;x_{B}\,g_{1L}\,D_{1T}^{\perp}\right.\right.\\&\qquad\qquad\left.\left.+y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\,\sin(\phi_h-\phi_s^h)\;x_{B}\,g_{1L}\,D_{1T}^{\perp}\right]\right\},\qquad(7.91)\end{aligned}}$$

which vanishes upon integration over the transverse momenta of the produced hadrons.

7.5.6 $\boxed{\vec{\ell}\vec{H}\rightarrow\ell'\vec{h}X}$ (transversely polarized target)

The antisymmetric tensor relevant in this case is

$$2M\,\mathcal{W}^{\mu\nu}=\int d^2\boldsymbol{k}_{T}\,d^2\boldsymbol{p}_{T}\;\delta^2(\boldsymbol{p}_{T}+\boldsymbol{q}_{T}-\boldsymbol{k}_{T})\,\left(i\frac{k_{\perp}^{[\mu}S_{h\perp}^{\nu]}}{M_h}\right)\,\frac{\boldsymbol{p}_{\perp}\cdot\boldsymbol{S}_{\perp}}{M}\,2z\,g_{1T}D_{1T}^{\perp}. \tag{7.92}$$

An asymmetry is obtained in

$$\boxed{\begin{aligned}&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\rightarrow\ell'+\vec{h}+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_{\perp}}=\\&\qquad\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,|\boldsymbol{S}_{\perp}|\,|\boldsymbol{S}_{h\perp}|\left[-y\,\left(1-\frac{y}{2}\right)\,\frac{\boldsymbol{p}_{\perp}^2}{MM_h}\,\sin(\phi_j-\phi_s^h)\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,D_{1T}^{\perp}\right.\\&\qquad\qquad\left.+y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{p}_{\perp}|\,|\boldsymbol{P}_{h\perp}|}{zMM_h}\,\sin(\phi_h-\phi_s^h)\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,D_{1T}^{\perp}\right],\\&\qquad\qquad\qquad\qquad\qquad\qquad(7.93)\end{aligned}}$$

which again vanishes upon integration over the transverse momenta of the produced hadrons.

7.6 Convolutions and gaussian distributions

In order to study the behavior of the convolutions of distribution and fragmentation functions it is useful to consider gaussian distributions,

$$\begin{aligned}f(x,\boldsymbol{p}_{T})&=f(x,\boldsymbol{0}_{T})\,\exp(-R_H^2\boldsymbol{p}_{T}^2)\\&=f(x)\,\frac{R_H^2}{\pi}\,\exp(-R_H^2\boldsymbol{p}_{T}^2)\equiv f(x)\,\mathcal{P}(\boldsymbol{p}_{T};R_H),\end{aligned}\tag{7.94}$$

$$\begin{aligned}D(z,-z\boldsymbol{k}_{T})&=D(z,\boldsymbol{0}_{T})\,\exp(-R_h^2\boldsymbol{k}_{T}^2).\\&=D(z)\,\frac{R_h^2}{\pi\,z^2}\,\exp(-R_h^2\boldsymbol{k}_{T}^2)=\frac{D(z)}{z^2}\,\mathcal{P}(\boldsymbol{k}_{T};R_h)=D(z)\,\mathcal{P}\left(-z\boldsymbol{k}_{T};\frac{R_h}{z}\right),\end{aligned}\tag{7.95}$$

In that case the convolution becomes

$$\begin{aligned}I[f\,D]&=\int d^2\boldsymbol{p}_{T}\,d^2\boldsymbol{k}_{T}\;\delta^2(\boldsymbol{p}_{T}+\boldsymbol{q}_{T}-\boldsymbol{k}_{T})\,f(x_{B},\boldsymbol{p}_{T})\,D(z,-z\boldsymbol{k}_{T})\\&=\frac{\pi}{R_H^2+R_h^2}\,\exp\left(-\frac{Q_{T}^2\,R_H^2R_h^2}{R_H^2+R_h^2}\right)\,f(x,\boldsymbol{0}_{T})\,D(z,\boldsymbol{0}_{T})\\&=f(x_{B})\,D(z)\,\frac{\mathcal{P}(\boldsymbol{q}_{T};R)}{z^2},\end{aligned}\tag{7.96}$$

where $R^2 = R_H^2\, R_h^2/(R_H^2 + R_h^2)$. The other convolutions that appear in the cross sections are of the form

$$I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol p_T}{M}\,f\,D\Big] \;=\; \frac{R^2}{R_H^2}\,\frac{Q_T}{M}\,I[f\,D] = \frac{R^2}{R_H^2}\,\frac{Q_T}{M}\,f(x_B)\,D(z)\,\frac{\mathcal{P}(\boldsymbol q_T;R)}{z^2}, \tag{7.97}$$

$$I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol k_T}{M_h}\,f\,D\Big] \;=\; -\frac{R^2}{R_h^2}\,\frac{Q_T}{M_h}\,I[f\,D], \tag{7.98}$$

$$I\Big[\frac{\boldsymbol p_T\cdot\boldsymbol k_T}{MM_h}\,f\,D\Big] \;=\; \frac{R^2}{MM_hR_H^2R_h^2)}\left(1-Q_T^2\,R^2\right)\,I[f\,D], \tag{7.99}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2-\boldsymbol p_T^2}{2M^2}\,f\,D\Big] \;=\; \frac{R^4}{R_H^4}\,\frac{Q_T^2}{2M^2}\,I[f\,D], \tag{7.100}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)^2-\boldsymbol k_T^2}{2M_h^2}\,f\,D\Big] \;=\; \frac{R^4}{R_h^4}\,\frac{Q_T^2}{2M_h^2}\,I[f\,D], \tag{7.101}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-\boldsymbol p_T\cdot\boldsymbol k_T}{2MM_h}\,f\,D\Big] \;=\; -\frac{R^4}{R_H^2R_h^2}\,\frac{Q_T^2}{2MM_h}\,I[f\,D], \tag{7.102}$$

$$I\Big[\frac{4(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\boldsymbol p_T\cdot\boldsymbol k_T)-\boldsymbol p_T^2\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)}{2M^2M_h}\,f\,D\Big] \;=\; -\frac{R^6}{R_H^4R_h^2}\,\frac{Q_T^3}{2M^2M_h}\,I[f\,D]. \tag{7.103}$$

Another way to deconvolute the results are the following weighted quantities,

$$\int d^2\boldsymbol q_T\;\frac{Q_T}{M}\,I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol p_T}{M}\,f\,D\Big] = 2\,f^{(1)}(x)\,D(z), \tag{7.104}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T}{M_h}\,I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol k_T}{M_h}\,f\,D\Big] = 2\,f(x)\,D^{(1)}(z), \tag{7.105}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{MM_h}\,I\Big[\frac{\boldsymbol p_T\cdot\boldsymbol k_T}{MM_h}\,f\,D\Big] = -4\,f^{(1)}(x)\,D^{(1)}(z), \tag{7.106}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{M^2}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2-\boldsymbol p_T^2}{2M^2}\,f\,D\Big] = 2\,f^{(2)}(x)\,D(z), \tag{7.107}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{M_h^2}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)^2-\boldsymbol k_T^2}{2M_h^2}\,f\,D\Big] = 2\,f(x)\,D^{(2)}(z), \tag{7.108}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{MM_h}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-\boldsymbol p_T\cdot\boldsymbol k_T}{2MM_h}\,f\,D\Big] = -4\,f^{(1)}(x)\,D^{(1)}(z), \tag{7.109}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^3}{M^2M_h}\,I\Big[\frac{4(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\boldsymbol p_T\cdot\boldsymbol k_T)-\boldsymbol p_T^2\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)}{2M^2M_h}\,f\,D\Big] = -12\,f^{(2)}(x)\,D^{(1)}(z), \tag{7.110}$$

where $f^{(n)}(x)$ indicate the $(\boldsymbol k_T^2/2M^2)^n$-moments of $f(x,\boldsymbol k_T^2)$, and similarly $D^{(n)}(z)$ indicate the $(\boldsymbol k_T^2/2M_h^2)^n$-moments of $D(z,z^2\boldsymbol k_T^2)$, in terms of the Gaussian parametrizations given by

$$f^{(n)}(x) = \int d^2\boldsymbol p_T\;\left(\frac{\boldsymbol p_T^2}{2M^2}\right)^n\,f(x,\boldsymbol p_T^2) \;=\; \frac{n!}{(2M^2R_H^2)^n}\,f(x), \tag{7.111}$$

$$D^{(n)}(z) = z^2\int d^2\boldsymbol k_T\;\left(\frac{\boldsymbol k_T^2}{2M^2}\right)^n\,D(z,-z^2\boldsymbol k_T^2) \;=\; \frac{n!}{(2M_h^2R_h^2)^n}\,D(z). \tag{7.112}$$

7.7 Lepton-hadron semi-inclusive DIS including Z-exchange

[Rainer Jakob, October 9, 1997]

Differential cross section for semi-inclusive lepton hadron scattering $e + H \rightarrow e' + h + X$

$$\frac{d\sigma}{dx_B\,dy\,dz_h\,d^2\boldsymbol q_T} \;=\; \frac{\pi\alpha_{em}^2}{2\,Q^4}\,y\,z_h\,L_{\mu\nu}^{(\gamma\gamma)}\mathcal{W}^{\mu\nu}_{(\gamma\gamma)}$$

$$
\begin{aligned}
&+ \frac{\pi\alpha_w\alpha_{em}}{2\ (Q^2 - M_Z^2 + i\Gamma_Z M_Z)\ Q^2}\, y\, z_h\ L_{\mu\nu}^{(Z\gamma)}\mathcal{W}_{(Z\gamma)}^{\mu\nu} \\
&+ \frac{\pi\alpha_w\alpha_{em}}{2\ (Q^2 - M_Z^2 - i\Gamma_Z M_Z)\ Q^2}\, y\, z_h\ L_{\mu\nu}^{(\gamma Z)}\mathcal{W}_{(\gamma Z)}^{\mu\nu} \\
&+ \frac{\pi\alpha_w^2}{2\ ((Q^2 - M_Z^2)^2 + \Gamma_Z^2 M_Z^2)}\, y\, z_h\ L_{\mu\nu}^{(ZZ)}\mathcal{W}_{(ZZ)}^{\mu\nu}
\end{aligned} \tag{7.113}
$$

leptonic tensors for e^+e^-

$$
L_{\mu\nu}^{(\gamma\gamma)} = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu\, \not\ell\, \gamma_\nu\, \not\ell'\, \frac{1+\lambda\gamma_5\not s}{2}\right] = 2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2 + 2\mathrm{i}\,\lambda\,\epsilon_{\mu\nu\rho\sigma}\mathrm{q}^\rho\ell^\sigma \tag{7.114}
$$

$$
L_{\mu\nu}^{(Z\gamma)} = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu\, \not\ell(\mathrm{g}_\mathrm{V}^\ell + \mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\nu\, \not\ell'\right] = \mathrm{g}_\mathrm{A}^\ell\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2\right) + 2\mathrm{i}\mathrm{g}_\mathrm{V}^\ell\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots \tag{7.115}
$$

$$
L_{\mu\nu}^{(\gamma Z)} = \frac{1}{2}\mathrm{Tr}\left[(\mathrm{g}_\mathrm{V}^\ell + \mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\mu\, \not\ell\, \gamma_\nu\, \not\ell'\right] = \mathrm{g}_\mathrm{A}^\ell\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2\right) + 2\mathrm{i}\mathrm{g}_\mathrm{V}^\ell\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots \tag{7.116}
$$

$$
\begin{aligned}
L_{\mu\nu}^{(ZZ)} = \frac{1}{2}\mathrm{Tr}&\left[(\mathrm{g}_\mathrm{V}^\ell + \mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\mu\, \not\ell(\mathrm{g}_\mathrm{V}^\ell + \mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\nu\, \not\ell'\right] = \\
&(g_A^{\ell\,2} + g_V^{\ell\,2})\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - g_{\mu\nu}Q^2\right) + (2g_A^\ell g_V^\ell)2i\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots
\end{aligned} \tag{7.117}
$$

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hadronic tensor for semi-inclusive lepton hadron scattering $e+H \rightarrow e' + h + X$

$$2M\,\mathcal{W}^{\mu\nu}_{(\gamma\gamma)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\ \mathrm{Tr}\Big[\Phi(\mathrm{p})\gamma^\mu \Delta(\mathrm{k})\gamma^\nu\Big] \tag{7.118}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{S(\gamma\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\ d^2\boldsymbol{p}_T\ \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ & \\
- g_\perp^{\mu\nu}\Bigg[f_1 D_1 + g_{1s} G_{1s} + \frac{\epsilon_\perp^{\rho\sigma} k_{\perp\rho} S_{h\perp\sigma}}{M_h} f_1 D_{1T}^\perp - \frac{\epsilon_\perp^{\rho\sigma} p_{\perp\rho} S_{\perp\sigma}}{M} f_{1T}^\perp D_1 & \\
- \frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\ \boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp} - \boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\ \boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M M_h} f_{1T}^\perp D_{1T}^\perp \Bigg] & \\
- (S_\perp^{\{\mu} S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\ \boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}) h_{1T} H_{1T} - \frac{k_\perp^{\{\mu} p_\perp^{\nu\}} + g_\perp^{\mu\nu}\ \boldsymbol{k}_\perp\!\cdot\!\boldsymbol{p}_\perp}{M M_h} (h_{1s}^\perp H_{1s}^\perp + h_1^\perp H_1^\perp) & \\
- \frac{p_\perp^{\{\mu} S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\ \boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}}{M} h_{1s}^\perp H_{1T} - \frac{k_\perp^{\{\mu} S_\perp^{\nu\}} + g_\perp^{\mu\nu}\ \boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M_h} h_{1T} H_{1s}^\perp & \\
- \frac{k_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} p_{\perp\rho} + p_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho}}{2 M M_h} (h_{1s}^\perp H_1^\perp - h_1^\perp H_{1s}^\perp) & \\
+ \frac{p_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} S_{h\perp\rho} + S_{h\perp}^{\{\mu} \epsilon_\perp^{\nu\}\rho} p_{\perp\rho}}{2M} h_1^\perp H_{1T} - \frac{k_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} S_{\perp\rho} + S_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho}}{2M_h} h_{1T} H_1^\perp \Bigg\} &
\end{aligned} \tag{7.119}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{A(\gamma\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\ d^2\boldsymbol{p}_T\ \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ & \\
i\ \epsilon_\perp^{\mu\nu}(f_1 G_{1s} + g_{1s} D_1) - i\ \frac{p_\perp^{[\mu} S_\perp^{\nu]}}{M} f_{1T}^\perp G_{1s} + i\ \frac{k_\perp^{[\mu} S_{h\perp}^{\nu]}}{M_h} g_{1s} D_{1T}^\perp \Bigg\} &
\end{aligned} \tag{7.120}$$

$$2M\,\mathcal{W}^{\mu\nu}_{(Z\gamma)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\ \mathrm{Tr}\Big[\Phi(\mathrm{p})\,(\mathrm{g_V} + \mathrm{g_A}\gamma_5)\,\gamma^\mu \Delta(\mathrm{k})\gamma^\nu\Big] \tag{7.121}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{S(Z\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\ d^2\boldsymbol{p}_T\ \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{&\\
- g_\perp^{\mu\nu}\Bigg[g_V\bigg(f_1 D_1 + g_{1s} G_{1s} + \frac{\epsilon_\perp^{\rho\sigma} k_{\perp\rho} S_{h\perp\sigma}}{M_h} f_1 D_{1T}^\perp - \frac{\epsilon_\perp^{\rho\sigma} p_{\perp\rho} S_{\perp\sigma}}{M} f_{1T}^\perp D_1\bigg)&\\
- g_A\bigg(f_1 G_{1s} + g_{1s} D_1 + \frac{\epsilon_\perp^{\rho\sigma} k_{\perp\rho} S_{h\perp\sigma}}{M_h} g_{1s} D_{1T}^\perp - \frac{\epsilon_\perp^{\rho\sigma} p_{\perp\rho} S_{\perp\sigma}}{M} f_{1T}^\perp G_{1s}\bigg)&\\
- g_V \frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\ \boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp} - \boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\ \boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M M_h} f_{1T}^\perp D_{1T}^\perp\Bigg]&\\
- (S_\perp^{\{\mu} S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp})\, g_V h_{1T} H_{1T}&\\
- \frac{k_\perp^{\{\mu} p_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{p}_\perp}{M M_h}\left[g_V (h_{1s}^\perp H_{1s}^\perp + h_1^\perp H_1^\perp) - i\ g_A (h_{1s}^\perp H_1^\perp - h_1^\perp H_{1s}^\perp)\right]&\\
- \frac{p_\perp^{\{\mu} S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}}{M}(g_V h_{1s}^\perp H_{1T} + i\ g_A h_1^\perp H_{1T})&\\
- \frac{k_\perp^{\{\mu} S_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M_h}(g_V h_{1T} H_{1s}^\perp - i\ g_A h_{1T} H_1^\perp)&\\
- \frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} p_{\perp\rho} + p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} k_{\perp\rho}}{2M M_h}\Big[g_V\left(h_{1s}^\perp H_1^\perp - h_1^\perp H_{1s}^\perp\right) + i\ g_A\left(h_{1s}^\perp H_{1s}^\perp + h_1^\perp H_1^\perp\right)\Big]&\\
+ \frac{p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{h\perp\rho} + S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho} p_{\perp\rho}}{2M}\left(g_V h_1^\perp H_{1T} - i\ g_A h_{1s}^\perp H_{1T}\right)&\\
- \frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{\perp\rho} + S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} k_{\perp\rho}}{2M_h}\left(g_V h_{1T} H_1^\perp + i\ g_A h_{1T} H_{1s}^\perp\right)&\\
- i\ g_A \frac{S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{\perp\rho} + S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{h\perp\rho}}{2} h_{1T} H_{1T}\Bigg\}&
\end{aligned} \tag{7.122}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{A(Z\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\ d^2\boldsymbol{p}_T\ \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{&\\
+ i\ \epsilon_\perp^{\mu\nu}\Bigg[+ g_V (f_1 G_{1s} + g_{1s} D_1) - g_A (f_1 D_1 + g_{1s} G_{1s})&\\
+ g_A \frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\ \boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp} - \boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\ \boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M M_h} f_{1T}^\perp D_{1T}^\perp\Bigg]&\\
- i\ \frac{p_\perp^{[\mu} S_\perp^{\nu]}}{M}(g_V f_{1T}^\perp G_{1s} - g_A f_{1T}^\perp D_1) + i\ \frac{k_\perp^{[\mu} S_{h\perp}^{\nu]}}{M_h}(g_V g_{1s} D_{1T}^\perp - g_A f_1 D_{1T}^\perp)\Bigg\}&
\end{aligned} \tag{7.123}$$

note: for the interference of γ and Z exchange there are the simple relations for the symmetric (S) and anti-symmetric (A) parts

$$W^{\mu\nu}_{S(\gamma Z)} = \left(W^{\mu\nu}_{S(Z\gamma)}\right)^* \qquad \text{and} \qquad W^{\mu\nu}_{A(\gamma Z)} = -\left(W^{\mu\nu}_{A(Z\gamma)}\right)^* \tag{7.124}$$

$$2M\,\mathcal{W}^{\mu\nu}_{(ZZ)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T) \times \mathrm{Tr}\Big[\Phi(\mathrm{p})\,(\mathrm{g_V}+\mathrm{g_A}\gamma_5)\,\gamma^\mu\Delta(\mathrm{k})\,(\mathrm{g_V}+\mathrm{g_A}\gamma_5)\,\gamma^\nu\Big] \tag{7.125}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{S(ZZ)} = 2z_h \int d^2\boldsymbol{k}_T\; d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\Bigg\{ \\
-\,g_\perp^{\mu\nu}\Bigg[(g_V^2+g_A^2)\Bigg(f_1D_1+g_{1s}G_{1s}+\frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp D_1-\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}f_1D_{1T}^\perp \\
+\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\;-\;\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}\;\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg) \\
+\,2g_Vg_A\Bigg(f_1G_{1s}+g_{1s}D_1+\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}g_{1s}D_{1T}^\perp-\frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp G_{1s}\Bigg)\Bigg] \\
+\,(g_V^2-g_A^2)\Bigg[-(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+g_\perp^{\mu\nu}\,\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})h_{1T}H_{1T}-\frac{k_\perp^{\{\mu}p_\perp^{\nu\}}+g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\left(h_{1s}^\perp H_{1s}^\perp+h_1^\perp H_1^\perp\right) \\
-\,\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}}+g_\perp^{\mu\nu}\,\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}h_{1s}^\perp H_{1T}-\frac{k_\perp^{\{\mu}S_\perp^{\nu\}}+g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}h_{1T}H_{1s}^\perp \\
-\,\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}+p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2MM_h}\left(h_{1s}^\perp H_1^\perp-h_1^\perp H_{1s}^\perp\right) \\
+\,\frac{p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}+S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}}{2M}h_1^\perp H_{1T}-\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}+S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2M_h}h_{1T}H_1^\perp\Bigg]\Bigg\}
\end{aligned} \tag{7.126}$$

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}_{A(ZZ)} = 2z_h \int d^2\boldsymbol{k}_T\; d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\Bigg\{ \\
i\,\epsilon_\perp^{\mu\nu}\Bigg[-2g_Vg_A\Bigg(f_1D_1+g_{1s}G_{1s}-\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\;-\;\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}\;\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg) \\
+\,(g_V^2+g_A^2)\,(f_1G_{1s}+g_{1s}D_1)\Bigg] \\
+\,i\,\frac{p_\perp^{[\mu}S_\perp^{\nu]}}{M}\Big[+2g_Vg_Af_{1T}^\perp D_1-(g_V^2+g_A^2)f_{1T}^\perp G_{1s}\Big] \\
-\,i\,\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\Big[+2g_Vg_Af_1D_{1T}^\perp-(g_V^2+g_A^2)g_{1s}D_{1T}^\perp\Big]\Bigg\}
\end{aligned} \tag{7.127}$$

Chapter 8

Calculations of distribution and fragmentation functions

8.1 Quark distribution functions in the bag model

The starting point is the calculation of the lightcone correlation function for a target at rest $P = (M, \mathbf{0})$, expressed as a spatial integral,

$$\begin{aligned}
\Phi^{[\Gamma]}(x, \boldsymbol{k}_T) &= \frac{1}{2(2\pi)^3} \int da^-\, d^2\boldsymbol{a}_T\ \exp\left(-ixMa^-/\sqrt{2} + i\boldsymbol{k}_T \cdot \boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(a)\Gamma\psi(0)|P\rangle\big|_{a^+=0} \\
&= \frac{1}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T \cdot \boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(-a_z, \boldsymbol{a})\Gamma\psi(0)|P\rangle \\
&= \frac{1}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T \cdot \boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(-a_z, \boldsymbol{a})\Gamma\psi(0)|P\rangle. \qquad (8.1)
\end{aligned}$$

The $\boldsymbol{k}_T$-integrated distribution functions become

$$\Phi^{[\Gamma]}(x) = \frac{1}{2\pi\sqrt{2}} \int da\ \exp\left(ixMa\right)\ \langle P|\overline{\psi}(-a, 0, 0, a)\Gamma\psi(0)|P\rangle. \qquad (8.2)$$

In the bag model one can consider the bag as a wave packet, i.e. a superposition of plane waves centered around $\boldsymbol{P} = \mathbf{0}$. The forward scattering off a bag gives the distribution function,

$$\begin{aligned}
\Phi^{[\Gamma]}_{\rm bag}(x, \boldsymbol{k}_T) &= \frac{2M}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T \cdot \boldsymbol{a}_T\right) \\
&\qquad \int_{\rm bag} d^3r\ \langle {\rm bag}|\overline{\psi}(-a_z, \boldsymbol{r} + \boldsymbol{a})\Gamma\psi(\boldsymbol{r})|{\rm bag}\rangle. \qquad (8.3)
\end{aligned}$$

Assuming that all quarks in the bag are in the lowest eigenmode, i.e. $\psi(x) = \sum_n \psi_n(x)\, a_n$ is restricted to one mode,

$$\psi_0(t, \boldsymbol{r}) = \psi_0(\boldsymbol{r})\, e^{-i\omega t/R}, \qquad (8.4)$$

with for massless quarks ω being the solution of $j_0(\omega) = j_1(\omega)$, i.e. $\omega \approx 2.043$, one obtains

$$\Phi^{[\Gamma]}_{\rm bag}(x, \boldsymbol{k}_T) = \frac{2M}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(i\boldsymbol{k} \cdot \boldsymbol{a}\right) \int_{\rm bag} d^3r\ \overline{\psi}_0(\boldsymbol{r} + \boldsymbol{a})\Gamma\psi_0(\boldsymbol{r}), \qquad (8.5)$$

where $\boldsymbol{k} \equiv (\boldsymbol{k}_T, (xMR - \omega)/R)$. This can be rewritten as

$$\Phi^{[\Gamma]}_{\rm bag}(x, \boldsymbol{k}_T) = \frac{2M}{(2\pi)^3} \overline{\psi}_0(\boldsymbol{k}) \frac{\Gamma}{\sqrt{2}} \psi_0(\boldsymbol{k})\Big|_{k_z = (xMR-\omega)/R}, \qquad (8.6)$$

where

$$\psi_0(\boldsymbol{k}) = \int_{\rm bag} d^3r\ \exp\left(-i\boldsymbol{k} \cdot \boldsymbol{r}\right)\ \psi_0(\boldsymbol{r}). \qquad (8.7)$$

8.2 Quark distribution functions in the spectator model

To illustrate the calculation we will consider a simple nucleon-quark-diquark vertex connecting a nucleon with momentum P and mass M, a quark with momentum k and mass m and a scalar diquark. The essence of the model is that the diquark will be considered as an on-mass-shell spectator, i.e. $(P-k)^2 = M^2_{\text{diquark}}$. The quark-quark correlation function is then obtained as the product of the tree graph for $N \to q + \text{diquark}$.

The matrix element of a quark field interpolating between the nucleon state and a diquark state is

$$\langle X|\psi_i(0)|P,S\rangle = \frac{i}{k^2+m^2+i\epsilon}\,(\not k+m)_{ik}\,\Upsilon_{kl}\,U_l(P,S). \tag{8.8}$$

For a scalar diquark (with mass M_s), the vertex will be taken

$$\Upsilon^s_{ij}(P,k,P-k) = (1)_{ij}\,g_s(k^2) \tag{8.9}$$

(to be discussed later). The quark-quark correlation function for a scalar diquark becomes

$$\begin{aligned}
\Phi_{s\,ij}(k) &= \frac{1}{(2\pi)^4}\int d^4x\, e^{-ik\cdot x}\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle \\
&= \frac{1}{(2\pi)^4}\frac{(\not k+m)_{ik}}{k^2-m^2}\,\Upsilon_{kk'}U_{k'}(P,S)\,(2\pi)\,\delta\left((P-k)^2-M_s^2\right)\,\overline{U}_{l'}(P,S)(\gamma_0\Upsilon^\dagger\gamma_0)_{l'l}\,\frac{(\not k+m)_{lj}}{k^2-m^2} \\
&= \left[(\not k+m)(\not P+M)\frac{(1+\gamma_5\not S)}{2}(\not k+m)\right]_{ij}\frac{1}{(2\pi)^3}\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right) \\
&\overset{S=0}{=} \left[(\not k+m)(\not P+M)(\not k+m)\right]_{ij}\frac{1}{2(2\pi)^3}\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right),
\end{aligned} \tag{8.10}$$

which gives for an unpolarized nucleon

$$\begin{aligned}
\Phi_s(k) &= \frac{1}{2(2\pi)^3}\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right) \\
&\quad\times\Big\{\left(mM^2+m^2M-mM_s^2+(M+m)k^2\right) \\
&\qquad -\left(k^2-m^2\right)\not P+\left((M+m)^2-M_s^2+k^2-m^2\right)\not k\Big\}.
\end{aligned} \tag{8.11}$$

The result for $\Phi_s(k)$ can be used to calculate the projections

$$\Phi_s^{[\Gamma]}(x,\boldsymbol{k}_T) = \frac{1}{2}\int dk^-\,\mathrm{Tr}(\Gamma\Phi_s)\Big|_{k^+=xP^+}, \tag{8.12}$$

which depend on $x = k^+/P^+$ and $\boldsymbol{k}_T^2$. The integration over k^- can be rewritten in covariant form as

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x,\boldsymbol{k}_T) &= \int d(2k\cdot P)\,dk^2\;\delta\left(2x\,k\cdot P-x^2M^2-\boldsymbol{k}_T^2-k^2\right)\,\frac{\mathrm{Tr}(\Gamma\Phi_s)}{4\,P^+} \\
&= \int dP_s^2\,dk^2\;\delta\left(x(1-x)M^2-\boldsymbol{k}_T^2-(1-x)\,k^2-x\,P_s^2\right)\,\frac{\mathrm{Tr}(\Gamma\Phi_s)}{4\,P^+},
\end{aligned} \tag{8.13}$$

where $P_s = P-k$ and the latter form is suitable in the spectator model where one has the delta function $\delta(P_s^2-M_s^2)$ in the integrand. For the $\boldsymbol{k}_T$-integrated functions,

$$\Phi_s^{[\Gamma]}(x) = \frac{\pi}{2}\int dk^-\,d\boldsymbol{k}_T^2\,\mathrm{Tr}(\Gamma\Phi_s)\Big|_{k^+=xP^+}, \tag{8.14}$$

one has

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x) &= \pi\int dP_s^2\,dk^2\;\theta\left(x(1-x)M^2-(1-x)\,k^2-x\,P_s^2\right)\,\frac{\mathrm{Tr}(\Gamma\Phi_s)}{4\,P^+} \\
&= \pi\int_{P_s^2(min)}^{\infty} dP_s^2\int_{-\infty}^{xM^2-\frac{x}{1-x}P_s^2} dk^2\,\frac{\mathrm{Tr}(\Gamma\Phi_s)}{4\,P^+},
\end{aligned} \tag{8.15}$$

where $P_s^2(min)$ is the minimum mass contributing in the antiquark-nucleon spectral function, which is larger than $(M-m)^2$ in order to render the nucleon stable.

Writing

$$\Phi_s(k) = \tilde{\Phi}_s(k)\,\delta(P_s^2 - M_s^2), \tag{8.16}$$

we get the 'spectator model' results

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x, \boldsymbol{k}_T) &= \int dk^2\;\delta\left(x(1-x)M^2 - \boldsymbol{k}_T^2 - (1-x)\,k^2 - x\,M_s^2\right)\;\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,P^+} \\
&= \left.\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,(1-x)\,P^+}\right|_{k^2\,=\,k^2(x,\boldsymbol{k}_T^2)},
\end{aligned} \tag{8.17}$$

with

$$-k^2(x, \boldsymbol{k}_T^2) = \frac{\boldsymbol{k}_T^2}{1-x} + \frac{x}{1-x}\,M_s^2 - x\,M^2. \tag{8.18}$$

The $\boldsymbol{k}_T$-integrated result is

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x) &= \pi\int_0^\infty d\boldsymbol{k}_T^2\;\left.\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,(1-x)\,P^+}\right|_{k^2\,=\,k^2(x,\boldsymbol{k}_T^2)} \\
&= \pi\int_{\frac{x}{1-x}M_s^2 - xM^2}^\infty d(-k^2)\;\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,P^+}.
\end{aligned} \tag{8.19}$$

For the practical calculations it is convenient to introduce the quantities

$$\mu_s^2(x) = m^2(1-x) + xM_s^2 - x(1-x)M^2, \tag{8.20}$$

$$\lambda_s^2(x) = \Lambda^2(1-x) + xM_s^2 - x(1-x)M^2, \tag{8.21}$$

such that we have for the often appearing denominators,

$$m^2 - k^2 = \frac{\boldsymbol{k}_T^2 + \mu^2(x)}{1-x}, \tag{8.22}$$

$$\Lambda^2 - k^2 = \frac{\boldsymbol{k}_T^2 + \lambda^2(x)}{1-x}. \tag{8.23}$$

The function $\mu_s^2(x)$ has endpoints $\mu^2(0) = m^2$ and $\mu^2(1) = M_s^2$, acquiring a minimum at the point $x_0 = (M^2 + m^2 - M_s^2)/2M^2$ with the value

$$\mu^2(x_0) = \frac{\left((m+M_s)^2 - M^2\right)\left(M^2 - (m-M_s)^2\right)}{4\,M^2}, \tag{8.24}$$

which should be positive to avoid problems, i.e. $|M-m| \le M_s \le M+m$ and $|M-M_s| \le m \le M+M_s$, at least if $0 \le x_0 \le 1$, implying $|M_s^2 - m^2| \le M^2$. The condition for a valid application of the spectator approach for distribution functions becomes

$$M_s \ge M - m, \tag{8.25}$$

$$M_s \ge M - \Lambda. \tag{8.26}$$

The condition on the quark mass m is only relevant if the 'quark propagator' pole is kept. In many cases it is convenient to cancel it by the choice of the vertex function $g(k^2)$.

8.3 Quark-hadron vertices

The simplest way to construct valid quark-hadron vertices is to start with nonrelativistic (two-component) spinors and replace them by a (rest frame) Dirac spinor multiplied with the appropriate projection operator. Furthermore, to get the correct charge conjugation behavior, it is safest to start with the charge conjugation operator acting on positive energy spinors. Thirdly, one starts writing down everything in the hadron rest frame. Technically, the ingredients are for the (renormalized) spinors (in the rest frame),

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{8.27}$$

and for the charge conjugated u-spinors with $C = i\gamma^2\gamma^0$,

$$C\,u^{(1)} = v^{(2)}, \qquad C\,u^{(2)} = -v^{(1)}. \tag{8.28}$$

Projection operators involving these spinors are in the hadron rest frame given by

$$\sum_{\alpha=1}^{2} u^{(\alpha)}\overline{u}^{(\alpha)} = \frac{1+\gamma^0}{2} = \frac{\not{P}+M}{2M}, \tag{8.29}$$

$$\sum_{\alpha=1}^{2} v^{(\alpha)}\overline{v}^{(\alpha)} = \frac{\gamma^0-1}{2} = \frac{\not{P}-M}{2M}, \tag{8.30}$$

$$\sum_{\alpha=1}^{2} u^{(\alpha)}\overline{v}^{(\alpha)} = -\frac{(1+\gamma^0)\gamma_5}{2} = -\frac{(\not{P}+M)\gamma_5}{2M}, \tag{8.31}$$

$$\sum_{\alpha=1}^{2} v^{(\alpha)}\overline{u}^{(\alpha)} = \frac{(1-\gamma^0)\gamma_5}{2} = -\frac{(\not{P}-M)\gamma_5}{2M}. \tag{8.32}$$

For spin 1, the rest-system spin states are $\epsilon_+ = -(\epsilon_x + i\epsilon_y)/\sqrt{2}$, $\epsilon_0 = \epsilon_z$ and $\epsilon_- = (\epsilon_x - i\epsilon_y)/\sqrt{2}$, or explicitly

$$\epsilon_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \qquad \epsilon_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \epsilon_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \tag{8.33}$$

and the summation over states is

$$\sum_{\alpha} \epsilon_\mu^{(\alpha)}\epsilon_\nu^{(\alpha)*} = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}. \tag{8.34}$$

The *pion* $\rightarrow$ quark-antiquark spin-space vertex is e.g. obtained by constructing a spin zero quark-antiquark combination,

$$\begin{aligned} \Upsilon(\pi)_{ij} &= \frac{1}{\sqrt{2}}\left(u_i^{(1)}(C\,u^{(2)})_j^T - u_i^{(2)}(C\,u^{(1)})_j^T\right) \\ &= \frac{1}{\sqrt{2}}\left(u_i^{(1)}\overline{v}_j^{(1)} + u_i^{(2)}\overline{v}_j^{(2)}\right), \end{aligned} \tag{8.35}$$

i.e.

$$\Upsilon(\pi) \propto -\frac{1}{\sqrt{2}}\,\frac{(\not{P}_\pi + M_\pi)}{2M_\pi}\,\gamma_5 = \frac{1}{\sqrt{2}}\,\gamma_5\,\frac{(\not{P}_\pi - M_\pi)}{2M_\pi}, \tag{8.36}$$

$$\gamma_0\Upsilon^\dagger(\pi)\gamma_0 \propto \frac{1}{\sqrt{2}}\,\gamma_5\,\frac{(\not{P}_\pi + M_\pi)}{2M_\pi} = -\frac{1}{\sqrt{2}}\,\frac{(\not{P}_\pi - M_\pi)}{2M_\pi}\,\gamma_5. \tag{8.37}$$

The expression for $\Phi(k)$ is similar as for the nucleon with scalar diquark case with the expression between square brackets,

$$\begin{aligned} &-\frac{1}{2}\left[(\not{k}+m)\,\frac{(\not{P}+M)}{2M}\,\gamma_5\,(\not{P}-M)\,\gamma_5\,\frac{(\not{P}+M)}{2M}\,(\not{k}+m)\right] \\ &\quad = \frac{1}{2}\left[(\not{k}+m)\,(\not{P}+M)\,(\not{k}+m)\right]. \end{aligned} \tag{8.38}$$

In this expression the 'spectator-antiquark' sum is taken to be $\not{P} - M$.

Combining the spinors to spin 1 one obtains the $\rho \to$ quark-antiquark spin-space vertex, e.g. for the spin 0 component one has

$$\begin{aligned}
\Upsilon_{ij}(\rho; S_z = 0) &= \frac{1}{\sqrt{2}}\left(u_i^{(1)}(C\,u^{(2)})_j^T + u_i^{(2)}(C\,u^{(1)})_j^T\right) \\
&= \frac{1}{\sqrt{2}}\left(u_i^{(1)}\overline{v}_j^{(1)} - u_i^{(2)}\overline{v}_j^{(2)}\right) \\
&= \frac{1}{\sqrt{2}}\left(\left(\sigma_z u^{(1)}\right)_i \overline{v}_j^{(1)} + \left(\sigma_z u^{(2)}\right)_i \overline{v}_j^{(2)}\right), \qquad (8.39)
\end{aligned}$$

which using that $\sigma_z u = \gamma_z\gamma_5((1+\gamma_0)/2)u$ gives the vertex for a vector meson,

$$\Upsilon^\mu(\rho) \propto \frac{1}{\sqrt{2}}\gamma^\mu \frac{(\not{P}_\rho - M_\rho)}{2M_\rho}, \qquad (8.40)$$

$$\gamma_0\Upsilon^{\mu\dagger}(\rho)\gamma_0 \propto \frac{1}{\sqrt{2}}\frac{(\not{P}_\rho - M_\rho)}{2M_\rho}\gamma^\mu. \qquad (8.41)$$

The 'spin' part in $\Phi(k)$ becomes (for an unpolarized ρ-meson)

$$\begin{aligned}
&\frac{1}{2}\left[(\not{k}+m)\,\gamma_\mu \frac{(\not{P}-M)}{2M}\,(\not{P}-M)\,\frac{(\not{P}+M)}{2M}\,\gamma_\nu\,(\not{k}+m)\right]\left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2}\right) \\
&\quad = \frac{1}{2}\left[(\not{k}+m)\,(\not{P}+M)\,(\not{k}+m)\right]. \qquad (8.42)
\end{aligned}$$

In this expression the 'spectator-antiquark' sum is taken to be $-\not{P} + M$.

In order to find the baryon $\to$ quark-diquark vertex it is useful to first build a nucleon spinor either from a quark and a diquark with spin zero,

$$U_i^{(1)} = u_i^{(1)}, \qquad (8.43)$$

or to build a nucleon spinor from a quark and a diquark with spin one, e.g.

$$\begin{aligned}
U_i^{(1)} &= -\sqrt{\frac{1}{3}}\,\epsilon_0\,u_i^{(1)} + \sqrt{\frac{2}{3}}\,\epsilon_+\,u_i^{(2)} \\
&= -\sqrt{\frac{1}{3}}\,\epsilon_0\left(\sigma_z u^{(1)}\right)_i + \sqrt{\frac{1}{6}}\,\epsilon_+\left(\sigma_- u^{(1)}\right)_i \\
&= -\sqrt{\frac{1}{3}}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,u^{(1)}\right)_i, \qquad (8.44)
\end{aligned}$$

from which one obtains the vertices

$$\Upsilon^s(N) \propto 1, \qquad (8.45)$$

$$\begin{aligned}
\Upsilon^{a\,\mu}(N) &\propto \frac{1}{\sqrt{3}}\left(\gamma_\nu\gamma_5\frac{(\not{P}+M)}{2M}\right)\left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2}\right) \\
&\propto \frac{1}{\sqrt{3}}\gamma_5\left(\gamma^\mu + \frac{P^\mu}{M}\right), \qquad (8.46)
\end{aligned}$$

$$\gamma_0\Upsilon^{a\,\mu\dagger}(N)\gamma_0 \propto -\frac{1}{\sqrt{3}}\left(\gamma^\mu + \frac{P^\mu}{M}\right)\gamma_5. \qquad (8.47)$$

For obtaining the short expressions note that a projection operator $\not{P} + M$ on the 'nucleon side', i.e. left side, becomes irrelevant. From the symmetric SU(6) wave function one deduces that one (with the above factor in Υ^a) need both vertices Υ^s and Υ^a with equal relative strength. For the axial-vector diquark contribution to $\Phi(k)$ the part between brackets in the scalar diquark case is replaced by

$$\begin{aligned}
&-\frac{1}{3}\left[(\not{k}+m)\,\gamma_\mu\,\gamma_5\,\frac{(\not{P}+M)}{2M}\,(\not{P}+M)\,\frac{1+\gamma_5\,\not{S}}{2}\,\frac{(\not{P}+M)}{2M}\,\gamma_5\,\gamma_\nu\,(\not{k}+m)\right]\left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2}\right) \\
&\quad = \left[(\not{k}+m)\,(\not{P}+M)\left(\frac{1}{2} - \frac{1}{6}\gamma_5\,\not{S}\right)(\not{k}+m)\right]. \qquad (8.48)
\end{aligned}$$

In this expression the spin-sum for the axial vector diquark is taken to be $-g^{\mu\nu} + P^\mu P^\nu/M^2$.

8.4 Quark fragmentation functions in the spectator model

To illustrate this calculation we will also consider the simple nucleon-quark-diquark vertex connecting a nucleon with momentum P and mass M, a quark with momentum k and mass m and a scalar diquark. The essence again is that the diquark will be considered as an on-mass-shell spectator, i.e. $(k - P_h)^2 = M^2_{\text{diquark}}$. The quark-quark correlation function is then obtained as the product of the tree graph for $q \to N + \text{diquark}$.

For a scalar diquark (with mass M_s), the vertex is as for the distribution functions,

$$\Upsilon^s_{ij}(P, k, P - k) = (1)_{ij}\, g_s(k^2), \tag{8.49}$$

The correlation function that is the starting point for the calculation of the fragmentation functions is

$$\begin{aligned}
\Delta_{s\,ij}(k) &= \frac{1}{(2\pi)^4} \int d^4x\; e^{ik\cdot x} \langle 0|\psi_i(x) a_h^\dagger a_h \overline{\psi}_j(0)|0\rangle \\
&= \frac{1}{(2\pi)^4}\, \frac{(\not k + m)_{ik}}{k^2 - m^2}\, \Upsilon^\dagger_{kk'}\, U_{k'}(P_h, S_h)\, (2\pi)\, \delta\left((k - P_h)^2 - M_s^2\right)\, \overline{U}_{l'}(P_h, S_h)\, \gamma_0 \Upsilon_{l'l} \gamma_0\, \frac{(\not k + m)_{lj}}{k^2 - m^2} \\
&= \left[(\not k + m)(\not P_h + M_h)\frac{(1 + \gamma_5 \not S_h)}{2}(\not k + m)\right]_{ij} \frac{1}{(2\pi)^3}\, \frac{g^2(k^2)}{(k^2 - m^2)^2}\, \delta\left((k - P_h)^2 - M_s^2\right) \\
&\overset{S_h=0}{=} \left[(\not k + m)(\not P_h + M_h)(\not k + m)\right]_{ij} \frac{1}{2(2\pi)^3}\, \frac{g^2(k^2)}{(k^2 - m^2)^2}\, \delta\left((k - P_h)^2 - M_s^2\right),
\end{aligned} \tag{8.50}$$

which is the same expression as for the distribution function and gives for the production of an unpolarized nucleon

$$\begin{aligned}
\Delta_s(k) &= \frac{1}{2(2\pi)^3}\, \frac{g^2(k^2)}{(k^2 - m^2)^2}\, \delta\left((k - P_h)^2 - M_s^2\right) \\
&\quad \times \Big\{ \left(m M_h^2 + m^2 M_h - m M_s^2 + (M_h + m) k^2\right) \\
&\qquad - \left(k^2 - m^2\right) \not P_h + \left((M_h + m)^2 - M_s^2 + k^2 - m^2\right) \not k \Big\}.
\end{aligned} \tag{8.51}$$

The result for $\Delta_s(k)$ can be used to calculate the projections

$$\Delta_s^{[\Gamma]}(z, \boldsymbol{P}_{h\perp}) = \frac{1}{4z} \int dk^+\, \mathrm{Tr}(\Gamma \Delta_s)\Big|_{k^- = P_h^-/z;\; \boldsymbol{k}_T = -\boldsymbol{P}_{h\perp}/z}, \tag{8.52}$$

which depend on $z = P_h^-/k^-$ and $\boldsymbol{P}_{h\perp}^2$ or $\boldsymbol{k}_T^2$. The integration over k^+ can be rewritten in covariant form as

$$\begin{aligned}
\Delta_s^{[\Gamma]}(z, \boldsymbol{P}_{h\perp}) &= \frac{1}{2z} \int d(2k\cdot P_h)\, dk^2\; \delta\left(\frac{2\, k\cdot P_h}{z} - \frac{M_h^2}{z^2} - \boldsymbol{k}_T^2 - k^2\right)\, \frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\, P_h^-}\Big|_{\boldsymbol{k}_T = -\boldsymbol{P}_{h\perp}/z} \\
&= \frac{1}{2z} \int dP_s^2\, dk^2\; \delta\left(\frac{(1-z)\, k^2}{z} - \frac{(1-z)\, M_h^2}{z^2} - \boldsymbol{k}_T^2 - \frac{P_s^2}{z}\right)\, \frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\, P_h^-}\Big|_{\boldsymbol{k}_T = -\boldsymbol{P}_{h\perp}/z} \\
&= \frac{1}{2z}\, \Phi_s^{[\Gamma]}(1/z, -\boldsymbol{P}_{h\perp}/z),
\end{aligned} \tag{8.53}$$

where $P_s = k - P_h$ and the second form is suitable in the spectator model where one has the delta function $\delta(P_s^2 - M_s^2)$ in the integrand.

For the $\boldsymbol{P}_{h\perp}$-integrated functions,

$$\Delta_s^{[\Gamma]}(z) = \frac{\pi\, z^2}{2} \int dk^+\, d\boldsymbol{k}_T^2\, \mathrm{Tr}(\Gamma\Delta_s)\Big|_{k^- = P_h^-/z}, \tag{8.54}$$

one has

$$\begin{aligned}
\Delta_s^{[\Gamma]}(z) &= \frac{\pi\, z}{2} \int dP_s^2\, dk^2\; \theta\left(\frac{(1-z)\, k^2}{z} - \frac{(1-z)\, M_h^2}{z^2} - \frac{P_s^2}{z}\right)\, \frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\, P_h^-} \\
&= \frac{\pi\, z}{2} \int_{P_s^2(min)}^\infty dP_s^2 \int_{\frac{M_h^2}{z} + \frac{1}{1-z}\, P_s^2}^\infty dk^2\; \frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\, P_h^-} = \frac{1}{2}\, z\, \Phi_s^{[\Gamma]}(1/z),
\end{aligned} \tag{8.55}$$

where $P_s^2(min)$ is the minimum mass contributing in the antiquark-nucleon spectral function, which is larger than $(M_h - m)^2$ in order to render the hadron h stable.

Writing

$$\Delta_s(k) = \tilde{\Delta}_s(k)\,\delta(P_s^2 - M_s^2), \tag{8.56}$$

we get the 'spectator model' results

$$\begin{aligned}\Delta_s^{[\Gamma]}(z, \boldsymbol{P}_{h\perp}) &= \frac{1}{2z}\int dk^2\;\delta\left(\frac{1-z}{z}\,k^2 - \frac{(1-z)\,M_h^2}{z^2} - \boldsymbol{k}_T^2 - \frac{M_s^2}{z}\right)\left.\frac{\mathrm{Tr}(\Gamma\tilde{\Delta}_s)}{4\,P_h^-}\right|_{\boldsymbol{k}_T=-\boldsymbol{P}_{h\perp}/z} \\ &= \frac{1}{2(1-z)}\left.\frac{\mathrm{Tr}(\Gamma\tilde{\Delta}_s)}{4\,P_h^-}\right|_{k^2\,=\,k^2(z,\boldsymbol{k}_T^2)},\end{aligned} \tag{8.57}$$

with

$$k^2(z, \boldsymbol{k}_T^2) = \frac{z}{1-z}\,\boldsymbol{k}_T^2 + \frac{M_s^2}{1-z} + \frac{M_h^2}{z}. \tag{8.58}$$

The $\boldsymbol{P}_{h\perp}$-integrated result is

$$\begin{aligned}\Delta_s^{[\Gamma]}(z) &= \frac{\pi\,z^2}{2(1-z)}\int_0^\infty d\boldsymbol{k}_T^2\;\left.\frac{\mathrm{Tr}(\Gamma\tilde{\Delta}_s)}{4\,P_h^-}\right|_{k^2\,=\,k^2(z,\boldsymbol{k}_T^2)} \\ &= \frac{\pi\,z}{2}\int_{\frac{1}{1-z}\,M_s^2+\frac{1}{z}\,M_h^2}^\infty dk^2\;\frac{\mathrm{Tr}(\Gamma\tilde{\Delta}_s)}{4\,P_h^-}.\end{aligned} \tag{8.59}$$

For the practical calculations it is convenient to use the functions also encountered for the distribution functions,

$$\mu_s^2(1/z) = \frac{1-z}{z^2}\,M_h^2 + \frac{M_s^2}{z} - \frac{1-z}{z}\,m^2, \tag{8.60}$$

$$\lambda_s^2(1/z) = \frac{1-z}{z^2}\,M_h^2 + \frac{M_s^2}{z} - \frac{1-z}{z}\,\Lambda^2, \tag{8.61}$$

such that we have for the often appearing denominators,

$$k^2 - m^2 = \frac{z}{1-z}\;\left(\boldsymbol{k}_T^2 + \mu^2(1/z)\right), \tag{8.62}$$

$$k^2 - \Lambda^2 = \frac{z}{1-z}\;\left(\boldsymbol{k}_T^2 + \lambda^2(1/z)\right). \tag{8.63}$$

With the argument $1/z$ and $0 \le z \le 1$, the function $\mu_s^2(1/z)$ has endpoints $\mu^2(1/z) \to M_h^2/z$ for $z \to 0$ and $\mu^2(1) = M_s^2$, acquiring a minimum at the point $z_0 = 2M_h^2/(M_h^2 + m^2 - M_s^2)$ with the value

$$\mu^2(1/z_0) = \frac{\left((m+M_s)^2 - M_h^2\right)\,\left(M_h^2 - (m-M_s)^2\right)}{4\,M_h^2}, \tag{8.64}$$

which should be positive to avoid problems, i.e $|M_h - m| \le M_s \le M_h + m$ and $|M_h - M_s| \le m \le M_h + M_s$, at least if $0 \le z_0 \le 1$, implying $m^2 - M_s^2 \ge M_h^2$. This implies the following condition for employment of the spectator model,

$$M_s \ge m - M_h, \tag{8.65}$$

$$M_s \ge \Lambda - M_h. \tag{8.66}$$

The first condition on the quark mass is not relevant if the pole in the quark propagator is cancelled by a special choice of $g(k^2)$.

Chapter 9

Perturbative corrections

9.1 Inclusive leptoproduction

In order to illustrate the inclusion of perturbative QCD corrections, we start with inclusive lepton-hadron scattering, for which the tree level result, corresponding to $\gamma^*(q) + q(p) \to q(k)$ with $k = p + q$ in leading order in $1/Q$ is given by

$$\begin{aligned} 2M\, W^{\mu\nu}(P,q) &= \int dp^- dp^+ d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,(\not p + \not q + m)\,\gamma^\nu\right)\,\delta\left((p+q)^2 - m^2\right) \\ &\approx \int dp^- d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right) \\ &\approx \frac{1}{4}\,f_1(x_B)\ \mathrm{Tr}\left(\gamma^-\,\gamma^\mu\,\gamma^+\,\gamma^\nu\right) \ = \ -g_\perp^{\mu\nu}\,f_1(x_B) \end{aligned} \tag{9.1}$$

(Note that $g_T^{\mu\nu} = g_\perp^{\mu\nu}$ in this case as $q_T = 0$).

Perturbative corrections to this result for the nonsinglet structure functions come from the process $\gamma^*(q)+q(p) \to q(k)+G(l)$. This leads (omitting mass and vertex corrections) to the following contributions at leading order in $1/Q$,

$$\begin{aligned} 2M\, W^{\mu\nu}(P,q) &= \int dp^- d^2\boldsymbol{p}_\perp\,\theta(\mu^2 - \boldsymbol{p}_\perp^2)\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right) \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int d^4p\,d^4k\,d^4l\,\delta(l^2)\,\delta(k^2)\,\delta^4(p+q-k-l)\,d_{\alpha\beta}(l) \\ &\qquad \times\Big\{\theta\left(\boldsymbol{p}_\perp'^2 - \mu^2\right)\ \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\nu \not k_t \gamma^\beta]/\hat t^2 \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\nu \not k_t \gamma^\beta]/\hat s\hat t \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s\hat t \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s^2\Big\}, \end{aligned} \tag{9.2}$$

where $\boldsymbol{p}_\perp' = \boldsymbol{p}_\perp - \boldsymbol{l}_\perp$ and where $d_{\alpha\beta}(l)$ is the gluon summation in the final state. This depends on the choice of gauge. For this a convenient choice is the axial gauge $\hat q \cdot A = 0$, in which case one has

$$d_{\alpha\beta}(l) = -g_{\alpha\beta} + \frac{l_\alpha q_\beta + q_\alpha l_\beta}{l\cdot q} - \frac{q^2\,l_\alpha\,l_\beta}{(l\cdot q)^2}. \tag{9.3}$$

For any gluon field linked to a matrix element or constituting a final state (which means essentially on-mass-shell compared to Q^2), the above gauge choice implies a polarization summation that is equivalent to the gauge choice $n_l \cdot A = 0$, where n_l is the lightlike vector constructed from l and q,

$$n_l = \frac{x_p\sqrt{2}}{Q}\left(l + \frac{2\,l\cdot q}{Q^2}\,q\right). \tag{9.4}$$

As we have seen that gauge choice is important for a parton inpretation of the correlation functions. The theta functions cutting off or taking into account the first rung of the ladder contribution avoids double counting. Transverse momenta larger than μ^2 are not included in the soft part.

Implementing the energy-momentum conservation to eliminate integration over p^+, k^- and $\boldsymbol{l}_\perp$, using $\delta(l^2)$ to eliminate the integration over l^+, introducing

$$f_1(x,\mu^2) = \int d^2\boldsymbol{p}_\perp\, \theta(\mu^2 - \boldsymbol{p}_\perp^2)\, f_1(x,\boldsymbol{p}_\perp^2) \tag{9.5}$$

and going to components (assuming P and q to have no perpendicular components) one finds

$$\begin{aligned} 2M\, W^{\mu\nu}(P,q) &= -g_\perp^{\mu\nu}\, f_1(x_B,\mu^2) \\ &+ \frac{g^2\, C_F}{(2\pi)^3} \int dp^- dk^+ d^2\boldsymbol{p}_\perp\, d^2\boldsymbol{k}_\perp\, \frac{dl^-}{2l^-}\, \delta(k^2)\, d_{\alpha\beta}(l) \\ &\qquad \times\Big\{\theta\left(\boldsymbol{k}_\perp^2 - \mu^2\right)\ \mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t \gamma^\mu \not{k} \gamma^\nu \not{k}_t \gamma^\beta]/\hat{t}^2 \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not{k}_s \gamma^\alpha \not{k} \gamma^\nu \not{k}_t \gamma^\beta]/\hat{s}\hat{t} \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t \gamma^\mu \not{k} \gamma^\beta \not{k}_s \gamma^\nu]/\hat{s}\hat{t} \\ &\qquad\quad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not{k}_s \gamma^\alpha \not{k} \gamma^\beta \not{k}_s \gamma^\nu]/\hat{s}^2\Big\}. \end{aligned} \tag{9.6}$$

After this step we can parametrize the momenta

$$k = \left[z_k\, \frac{Q}{\sqrt{2}}, \frac{(1-z_k)(1-x_p)}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp\right], \tag{9.7}$$

$$l = \left[(1-z_k)\, \frac{Q}{\sqrt{2}}, \frac{z_k(1-x_p)}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_\perp - \boldsymbol{k}_\perp\right], \tag{9.8}$$

$$q = \left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right], \tag{9.9}$$

$$p = \left[p^-, \frac{Q}{x_p\sqrt{2}}, \boldsymbol{p}_\perp\right], \tag{9.10}$$

and rewrite the integration for the $\mathcal{O}(g^2)$ correction as

$$\frac{g^2\, C_F}{(2\pi)^3}\, \frac{Q}{2\sqrt{2}} \int \frac{dx_p}{x_p^2} \int dz_k \int d^2\boldsymbol{k}_\perp\, \delta\left(\boldsymbol{k}_\perp^2 - \frac{z_k(1-z_k)(1-x_p)}{x_p}\, Q^2\right) \int dp^-\, d^2\boldsymbol{p}_\perp\, \ldots . \tag{9.11}$$

The transverse momentum of the outgoing quark thus satisfies

$$\frac{\boldsymbol{k}_\perp^2}{Q^2} = \frac{z_k(1-z_k)(1-x_p)}{x_p}. \tag{9.12}$$

Positivity of $\boldsymbol{k}_\perp^2$ restricts the domain for z_k and x_p to the regions $0 \le z_k \le 1$ and $0 \le x_p \le 1$, while the theta function provides (in the first term) a regularization near the endpoints, for x_p not too close to unity

$$\frac{x_p}{1-x_p}\, \frac{\mu^2}{Q^2} \le z_k \le 1 - \frac{x_p}{1-x_p}\, \frac{\mu^2}{Q^2}. \tag{9.13}$$

It is useful to have explicit expressions for the vectors

$$k_s = p + q = k + l = \left[\frac{Q}{\sqrt{2}}, \frac{1-x_p}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_\perp\right], \tag{9.14}$$

$$k_t = p - l = k - q = \left[-(1-z_k)\, \frac{Q}{\sqrt{2}}, \frac{1 - z_k + x_p z_k}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp\right], \tag{9.15}$$

$$k_u = k - p = q - l = \left[z_k\, \frac{Q}{\sqrt{2}}, -\frac{x_p + z_k - x_p z_k}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp - \boldsymbol{p}_\perp\right]. \tag{9.16}$$

The Mandelstam variables for the subprocess are

$$\hat{s} = k_s^2 = \frac{1-x_p}{x_p}\, Q^2, \tag{9.17}$$

$$\hat{t} = k_t^2 = -\frac{1-z_k}{x_p}\, Q^2, \tag{9.18}$$

$$\hat{u} = k_u^2 = -\frac{z_k}{x_p}\, Q^2, \tag{9.19}$$

satisfying $\hat{s}+\hat{t}+\hat{u}+Q^2=0$ and the inverse relations are:

$$x_p = \frac{Q^2}{\hat{s}+Q^2}, \tag{9.20}$$

$$z_k = -\frac{\hat{u}}{\hat{s}+Q^2}. \tag{9.21}$$

Useful inner products are:

$$\begin{aligned} 2\,p\cdot q &= \hat{s}+Q^2 = \frac{1}{x_p}\,Q^2 = -(\hat{u}+\hat{t}), \\ 2\,k\cdot q &= -(\hat{t}+Q^2) = -\frac{x_p+z_k-1}{x_p}\,Q^2 = \hat{s}+\hat{u}, \\ 2\,l\cdot q &= -(\hat{u}+Q^2) = \frac{z_k-x_p}{x_p}\,Q^2 = \hat{s}+\hat{t}, \end{aligned} \tag{9.22}$$

The vectors $\hat{p}\propto x_p\,p\sqrt{2}/Q$, $\hat{k}\propto x_p\,k\sqrt{2}/Q$ and $\hat{l}\propto x_p\,l\sqrt{2}/Q$ are lightlike vectors, e.g. $\hat{p} \propto n_+$.

For the transverse and longitudinal structure function in

$$2M\,W^{\mu\nu}(P,q) = -2\,F_T(x_B,Q^2)\,g_\perp^{\mu\nu} + 2\,F_L(x_B,Q^2)\,\frac{\tilde{P}^\mu\tilde{P}^\nu}{\tilde{P}^2} \tag{9.23}$$

(Note $F_T = F_1 = M\,W_T = M\,W_1$ and $2\,F_L = F_2/x_B - 2\,F_1$) we then obtain the results

$$\begin{aligned} 2\,F_T(x_B,Q^2) &= f_1(x_B,\mu^2) \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1 \frac{dx_p}{x_p}\int_0^1 dz_k \int d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2 - \frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right)\frac{1}{(x_p-z_k)^2}\,f_1\left(\frac{x_B}{x_p}\right) \\ &\times\left\{\frac{(1-x_p)\,(x_p^2-2x_pz_k-2x_p^2z_k+3z_k^2+6x_pz_k^2+2x_p^2z_k^2-4z_k^3-4x_pz_k^3+2z_k^4)}{1-z_k}\,\theta\left(\boldsymbol{k}_\perp^2-\mu^2\right)\right. \\ &\left.\quad -2\,z_k\,(x_p+z_k+x_pz_k-z_k^2) + \frac{(1-z_k)\,(x_p^2+z_k^2)}{1-x_p}\right\} \\ &= f_1(x_B,\mu^2) \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1 \frac{dx_p}{x_p}\int_0^1 dz_k \int d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2 - \frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right) f_1\left(\frac{x_B}{x_p}\right)\theta\left(\boldsymbol{k}_\perp^2-\mu^2\right) \\ &\qquad\times\frac{2-2x_p+x_p^2-2z_k+4x_pz_k-2x_p^2z_k+z_k^2-2x_pz_k^2+2x_p^2z_k^2}{(1-x_p)(1-z_k)} \\ &= f_1(x_B,\mu^2) \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1 \frac{dx_p}{x_p}\int_0^1 dz_k\,\theta\left(\frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2-\mu^2\right) f_1\left(\frac{x_B}{x_p}\right) \\ &\qquad\times\left\{\frac{1+x_p^2}{(1-x_p)\,(1-z_k)} + \frac{1-z_k-2x_p+2z_kx_p-2z_kx_p^2}{1-x_p}\right\} \end{aligned} \tag{9.24}$$

$$2\,F_L(x_B,Q^2) = \frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1 \frac{dx_p}{x_p}\int_0^1 dz_k d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2 - \frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right) 4x_pz_k\,f_1\left(\frac{x_B}{x_p}\right). \tag{9.25}$$

The lower limit in the x_p integration comes from the support property of $\Phi(p)$, namely $P^+ - p^+ \geq 0$ or $p^+/P^+ = x_B/x_p \leq 1$. We note a singular part $\propto 1/(1-z_k)$ in F_T coming from the first term in the calculation. This is a collinear singularity ($\boldsymbol{k}_\perp^2 \to 0$) which is regulated by the theta function.

In order to perform the integration over z_k one notes that

$$\begin{aligned} \int_0^{1-\delta} dz_k\,\frac{f(z_k)}{1-z_k} &= \int_0^1 dz_k\,\frac{f(z_k)-f(1)}{1-z_k} - f(1)\ln(\delta) \\ &= \int_0^1 dz_k\,\frac{f(z_k)}{(1-z_k)_+} - f(1)\ln(\delta), \end{aligned} \tag{9.26}$$

or in a functional sense

$$\frac{1}{1-z_k} = \frac{1}{(1-z_k)_+} - \delta(1-z_k)\,\ln\delta. \tag{9.27}$$

One obtains

$$\begin{aligned} 2\,F_T(x_B,Q^2) &= f_1(x_B;\mu^2) + \frac{g^2\,C_F}{8\pi^2}\int_{x_B}^1 \frac{dx_p}{x_p}\,f_1\left(\frac{x_B}{x_p}\right)\left\{\frac{1+x_p^2}{1-x_p}\,\ln\left(\frac{Q^2}{\mu^2}\right)\right.\\ &\qquad \left. + \frac{1+x_p^2}{1-x_p}\,\ln\left(\frac{1-x_p}{x_p}\right) + \frac{1-2x_p-2x_p^2}{2(1-x_p)}\right\}, \qquad (9.28)\\ 2\,F_L(x_B,Q^2) &= \frac{g^2\,C_F}{8\pi^2}\int_{x_B}^1 dx_p\;2\,x_p\,f_1\left(\frac{x_B}{x_p}\right) = \frac{\alpha_s}{2\pi}\int_{x_B}^1\frac{dy}{y}\,2\,C_F\,\frac{x_B}{y}\,f_1(y). \qquad (9.29)\end{aligned}$$

The first expressions gives the scale dependence of $f_1(x;\mu^2)$, the second the perturbative result for the longitudinal structure function. The scale dependence of the structure function is determined by the singular term and requires an appropriate treatment of the singularities near $x_p = 1$.

$$f_1(x_B;Q^2) = f_1(x_B;\mu^2) + \frac{\alpha_s}{2\pi}\int_{x_B}^1 \frac{dy}{y}\,P_{qq}\left(\frac{x_B}{y}\right)\,f_1(y)\ln\left(\frac{Q^2}{\mu^2}\right), \tag{9.30}$$

where

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right]. \tag{9.31}$$

Since the singular piece only comes from the first of the four terms in the calculation, it is possible to obtain the evolution without considering the full process and only consider the gluon ladder graph contribution in the quark-quark correlation function. Implicitly here is of course the use in a hard scattering process with a large momentum scale, which defines the lightlike direction n_-. Requiring that $k^+ = xP^+$ and implementing momentum conservation, the momenta for $q(p) \to q(k) + G(l)$ can be parametrized as

$$\begin{aligned} k &= \left[p^- - \frac{x_p}{1-x_p}\,\frac{(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)^2}{2x\,P^+},\; xP^+,\; \boldsymbol{k}_\perp\right],\\ l &= \left[\frac{x_p}{1-x_p}\,\frac{(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)^2}{2x\,P^+},\; \frac{1-x_p}{x_p}\,xP^+,\; \boldsymbol{p}_\perp - \boldsymbol{k}_\perp\right],\\ p &= \left[p^-,\; \frac{x}{x_p}\,P^+,\; \boldsymbol{p}_\perp\right]. \qquad (9.32)\end{aligned}$$

The quantities k^2, $2k\cdot P$ and $(P-k)^2$ then can be expressed in $\boldsymbol{k}_T^2$ and x_p,

$$\tau = k^2 = 2k^+p^- - \frac{x_p}{1-x_p}\,(\boldsymbol{p}_\perp^2 - 2\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp) - \frac{\boldsymbol{k}_\perp^2}{1-x_p} \approx -\frac{\boldsymbol{k}_\perp^2}{1-x_p}. \tag{9.33}$$

$$\sigma = 2k\cdot P = -\frac{x_p}{1-x_p}\,\frac{\boldsymbol{k}_T^2}{x} + x\,M^2. \tag{9.34}$$

$$M_R^2 = (P-k)^2 \approx \left(\frac{x_p}{x} - 1\right)\,\frac{\boldsymbol{k}_T^2}{x}. \tag{9.35}$$

In Fig. 9.1 we show the region limited by $x \le x_p \le 1$ and $\boldsymbol{k}_T^2 \le \mu^2$, which is described in the perturbative calculation. Using the 'large' ($Q^2 \gg \boldsymbol{k}_\perp^2$) vector

$$q = \left[\frac{Q^2}{2x\,P^+},\; -x\,P^+,\; \boldsymbol{0}_\perp\right] \tag{9.36}$$

for the choice of gauge, one obtains

$$\begin{aligned}\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) &= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp)\\ &\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int dk^-\int dl^-dl^+d^2\boldsymbol{l}_\perp\,\delta(l^2)\,d_{\alpha\beta}(l)\,\frac{\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}\gamma^+\not{k}\gamma^\beta]}{(k^2)^2}\end{aligned}$$

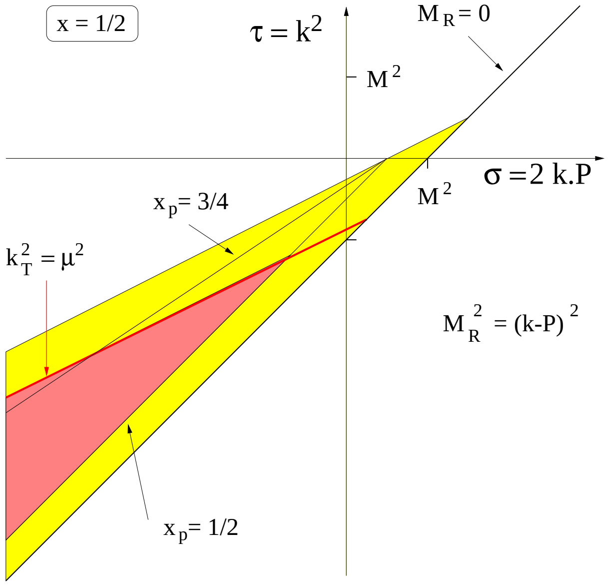


Figure 9.1: The (dark-shaded) region $\boldsymbol{k}_T^2 \geq \mu^2$ and $x \leq x_p \leq 1$ for $x = 1/2$ which is described by the ladder diagram.

$$
\begin{aligned}
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) \\
&\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int \frac{dl^+}{2l^+}\,d_{\alpha\beta}(l)\,\frac{\int dp^- d^2\boldsymbol{p}_\perp\,\text{Tr}[\Phi(p)\gamma^\alpha \not k \gamma^+ \not k \gamma^\beta]}{(k^2)^2} \\
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) \\
&\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{4}\int \frac{dx_p}{x_p^2}\,\frac{x_p}{1-x_p}\,d_{\alpha\beta}(l)\,\frac{\int dp^- d^2\boldsymbol{p}_\perp\,\text{Tr}[\Phi(p)\gamma^\alpha \not k \gamma^+ \not k \gamma^\beta]}{(k^2)^2}
\end{aligned}
$$

or

$$
\begin{aligned}
f_1(x,\boldsymbol{k}_\perp^2) &= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,f_1(x,\boldsymbol{k}_\perp^2) + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{\alpha_s\,C_F}{2\pi}\,\frac{1}{\pi\,\boldsymbol{k}_\perp^2}\int dx_p\,\frac{1-x_p}{x_p}\,f_1\left(\frac{x}{x_p}\right) \\
&\quad \times \frac{Q^4\,(1+x_p^2)\,(1-x_p)^2 + 2Q^2\,\boldsymbol{k}_\perp^2\,x_p^3\,(1-x_p) + \boldsymbol{k}_\perp^4\,x_p^4}{\left(Q^2\,(1-x_p)^2 - \boldsymbol{k}_\perp^2\,x_p^2\right)^2} + \ldots \\
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,f_1(x,\boldsymbol{k}_\perp^2) + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{\alpha_s(\mu^2)}{2\pi}\,\frac{1}{\pi\,\boldsymbol{k}_\perp^2}\int_x^1 \frac{dy}{y}\,P_{qq}\left(\frac{x}{y}\right)\,f_1(y,\mu^2) + \ldots
\end{aligned} \tag{9.37}
$$

where the last step includes the iteration of ladder graphs. The splitting function $P_{qq}(z)$ is as given above. The appropriate coefficient of the δ function $\delta(1-z)$ is easily found by using the result for $f_1(x,\boldsymbol{k}_\perp)$ for a free quark. This requires that $\int dz\,P_{qq}(z) = 0$.

9.2 Semi-inclusive leptoproduction

For 1-particle inclusive lepton-hadron scattering the tree-level result is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}(q,P,P_h) &= \int d^4p\,d^4k\,\delta^4(p+q-k)\,\text{Tr}\left(\Phi(p)\,\gamma^\mu\,\Delta(k)\,\gamma^\nu\right)\,\theta\left(\mu^2 - \boldsymbol{p}_T^2\right)\,\theta\left(\mu^2 - \boldsymbol{k}_T^2\right) \\
&= \int d^2\boldsymbol{p}_T\,\theta\left(\mu^2 - \boldsymbol{p}_T^2\right)\int d^2\boldsymbol{k}_T\,\theta\left(\mu^2 - \boldsymbol{k}_T^2\right)\,\delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \\
&\qquad \int dp^- \int dk^+\,\text{Tr}\left[\Phi(p)\,\gamma^\mu\,\Delta(k)\,\gamma^\nu\right].
\end{aligned} \tag{9.38}
$$

The natural frame to work in is the one in which the two hadrons do not have transverse components (frame II). The momenta of the photon and the hadrons can be parametrized

$$P_h = \left[\bar{z}_h\,\frac{\tilde{Q}}{\sqrt{2}}, \frac{M_h^2}{\bar{z}_h\tilde{Q}\sqrt{2}}, \boldsymbol{0}_T\right], \tag{9.39}$$

$$q = \left[\frac{\tilde{Q}}{\sqrt{2}}, -\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{q}_T\right], \tag{9.40}$$

$$P = \left[\frac{\bar{x}_B M^2}{\tilde{Q}\sqrt{2}}, \frac{1}{\bar{x}_B}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{0}_T\right]. \tag{9.41}$$

We note that $Q^2 = \tilde{Q}^2 + Q_T^2$, where $Q_T^2 = \boldsymbol{q}_T^2 = g_T^{\mu\nu} q_\mu q_\nu$. We do not assume that Q_T^2 is small at this point. Up to $\mathcal{O}(1/Q^2)$ corrections one has

$$\bar{x}_B = -\frac{q^+}{P^+} = -\frac{P_h\cdot q}{P_h\cdot P}, \tag{9.42}$$

$$\bar{z}_h = \frac{P_h^-}{q^-} = \frac{P\cdot P_h}{P\cdot q} \tag{9.43}$$

while the usual scaling variables become

$$x_B = \frac{Q^2}{2P\cdot q} = \bar{x}_B\,\frac{Q^2}{\tilde{Q}^2} = \bar{x}_B\left(1 - \frac{Q_T^2}{Q^2}\right)^{-1}, \tag{9.44}$$

$$z_h = -\frac{2P_h\cdot q}{Q^2} = \bar{z}_h\,\frac{\tilde{Q}^2}{Q^2} = \bar{z}_h\left(1 - \frac{Q_T^2}{Q^2}\right). \tag{9.45}$$

The quark momenta are parametrized as

$$k = \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}}, k^+, \boldsymbol{k}_T\right], \tag{9.46}$$

$$p = \left[p^-, \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{p}_T\right], \tag{9.47}$$

with for the tree-level calculation after implementing energy-momentum conservation $z_k = x_p = 1$. This condition must be dropped when one considers perturbative corrections, in which case one obtains the additional contribution

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}(q,P,P_h) &= \ldots \\
&+ \frac{g^2\,C_F}{(2\pi)^3}\int d^4p\,d^4k\,d^4l\;\delta^4(p+q-k-l)\,\delta(l^2)\,d_{\alpha\beta}(l) \\
&\qquad\times\big\{\theta(\boldsymbol{p}_T'^2-\mu^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t} \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t} \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\boldsymbol{k}_T'^2-\mu^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2\big\},
\end{aligned} \tag{9.48}$$

where $\boldsymbol{p}_T' = \boldsymbol{p}_T - \boldsymbol{l}_T$ and $\boldsymbol{k}_T' = \boldsymbol{k}_T + \boldsymbol{l}_T$. Implementing energy-momentum conservation we get

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu}(q,P,P_h) &= \ldots \\
&+ \frac{g^2\,C_F}{(2\pi)^3}\int dp^-\,dk^+\,d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{l}_T)\,\delta(l^2)\,d_{\alpha\beta}(l) \\
&\qquad\times\big\{\theta(\boldsymbol{p}_T'^2-\mu^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t} \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t} \\
&\qquad+\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\boldsymbol{k}_T'^2-\mu^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2\big\},
\end{aligned} \tag{9.49}$$

and the appropriate parametrization for the gluon momentum l is

$$l = \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}}, \frac{1-x_p}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{l}_T\right], \tag{9.50}$$

First we consider the Mandelstam variables starting with

$$\begin{aligned}
k_s &= p+q = k+l = \left[\frac{\tilde{Q}}{\sqrt{2}}, \frac{1-x_p}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{k}'_T\right], & (9.51)\\
k_t &= p-l = k-q = \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}}, \frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{p}'_T\right], & (9.52)\\
k_u &= p-k = l-q = \left[-z_k\,\frac{\tilde{Q}}{\sqrt{2}}, \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{p}_T-\boldsymbol{k}_T\right]. & (9.53)
\end{aligned}$$

The Mandelstam variables for the subprocess are

$$\begin{aligned}
\hat{s} &= k_s^2 = \frac{1-x_p}{x_p}\,\tilde{Q}^2 - 2\,\boldsymbol{k}_T\cdot\boldsymbol{l}_T = \frac{1-x_p}{x_p}\,Q^2 - \frac{Q_T^2}{x_p} - 2\,\boldsymbol{p}_T\cdot\boldsymbol{q}_T, & (9.54)\\
\hat{t} &= k_t^2 = -\frac{1-z_k}{x_p}\,\tilde{Q}^2 + 2\,\boldsymbol{p}_T\cdot\boldsymbol{l}_T = -(1-z_k)\,Q^2 - z_k\,Q_T^2 + 2\,\boldsymbol{k}_T\cdot\boldsymbol{q}_T, & (9.55)\\
\hat{u} &= k_u^2 = -\frac{z_k}{x_p}\,\tilde{Q}^2 = -\frac{z_k}{x_p}\,(Q^2 - Q_T^2), & (9.56)
\end{aligned}$$

satisfying $\hat{s}+\hat{t}+\hat{u}+Q^2 = 0$ and the inverse relations are:

$$\begin{aligned}
x_p &= \frac{\tilde{Q}^2}{\hat{s}+Q^2+2\,\boldsymbol{p}_T\cdot\boldsymbol{q}_T}, & (9.57)\\
z_k &= -\frac{\hat{t}+Q^2-2\,\boldsymbol{k}_T\cdot\boldsymbol{q}_T}{\tilde{Q}^2}. & (9.58)
\end{aligned}$$

The implementation of the l-integration and $\delta(l^2)$ depends on which of the four terms one considers. For the first term,

$$\text{in term 1:} \qquad \int dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \int \frac{dl^+}{2l^+}\int d^2\boldsymbol{p}'_T = \frac{1}{2}\int_{\bar{x}_B}^1 \frac{dx_p}{x_p}\,\frac{1}{1-x_p}\int d^2\boldsymbol{p}'_T, \tag{9.59}$$

(the range in x_p coming from the support of $\Phi(p)$) leading to the contribution

$$\begin{aligned}
\ldots\;\; + \int d^2\boldsymbol{p}'_T\,\theta(\boldsymbol{p}'^2_T-\mu^2)\,d^2\boldsymbol{k}_T\,\theta(\mu^2-\boldsymbol{k}_T^2)\;\delta^2(\boldsymbol{p}'_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int_{\bar{x}_B}^1 \frac{dx_p}{x_p}\,\frac{1}{1-x_p}\\
\times \int dp^-\,d^2\boldsymbol{p}_T\int dk^+\;d_{\alpha\beta}(l)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \qquad (9.60)
\end{aligned}$$

with

$$\text{in term 1:} \qquad 1-z_k = \frac{x_p}{1-x_p}\,\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} = \frac{x_p}{1-x_p}\,\frac{(\boldsymbol{p}'_T-\boldsymbol{p}_T)^2}{\tilde{Q}^2}. \tag{9.61}$$

For the fourth term we rewrite

$$\text{in term 4:} \qquad dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \int \frac{dl^-}{2l^-}\,d^2\boldsymbol{k}'_T = \frac{1}{2}\int_{\bar{z}_h}^1 dz_k\,\frac{1}{1-z_k}\int d^2\boldsymbol{k}'_T, \tag{9.62}$$

(the support in z_k coming from the support of $\Delta(k)$) and obtain

$$\begin{aligned}
\ldots\;\; + \int d^2\boldsymbol{p}_T\,\theta(\mu^2-\boldsymbol{p}_T^2)\,d^2\boldsymbol{k}'_T\,\theta(\boldsymbol{k}'^2_T-\mu^2)\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}'_T)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int_{\bar{z}_h}^1 dz_k\,\frac{1}{1-z_k}\\
\times \int dp^-\int dk^+\,d^2\boldsymbol{k}_T\;d_{\alpha\beta}(l)\,\mathrm{Tr}[\Phi(p)\gamma^\mu \not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2 \qquad (9.63)
\end{aligned}$$

with

$$\text{in term 4:}\qquad \frac{1-x_p}{x_p} = \frac{1}{x_p} - 1 = \frac{1}{1-z_k}\,\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} = \frac{1}{1-z_k}\,\frac{(\boldsymbol{k}_T' - \boldsymbol{k}_T)^2}{\tilde{Q}^2}. \qquad (9.64)$$

For the other two terms we use

$$\text{in term 2 \& 3:}\qquad \int dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \frac{1}{2}\int_{\bar{z}_h}^1 dz_k \int_{\bar{x}_B}^1 \frac{dx_p}{x_p^2}\int d^2\boldsymbol{l}_T\;\delta\left(\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} - \frac{(1-z_k)(1-x_p)}{x_p}\right), \qquad (9.65)$$

leading to the contribution

$$\begin{aligned}
\ldots\quad + \frac{g^2\,C_F}{(2\pi)^3}\int & d^2\boldsymbol{p}_T\,\theta(\mu^2 - \boldsymbol{p}_T^2)\,d^2\boldsymbol{k}_T\,\theta(\mu^2 - \boldsymbol{k}_T^2)\\
&\times \frac{1}{2}\int_{z_h}^1 dz_k \int_{\bar{x}_B}^1 \frac{dx_p}{x_p}\,\frac{1}{1-x_p}\int d^2\boldsymbol{l}_T\;\delta\left(\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} - \frac{(1-z_k)(1-x_p)}{x_p}\right),\\
&\times \int dp^- \int dk^+\;d_{\alpha\beta}(l)\Big\{\mathrm{Tr}[\Phi(p)\gamma^\mu \not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t}\\
&\qquad + \mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t}\Big\}, \qquad (9.66)
\end{aligned}$$

in which to use $\boldsymbol{l}_T = \boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T$ and consequently $\boldsymbol{p}_T' = \boldsymbol{k}_T - \boldsymbol{q}_T$ and $\boldsymbol{k}_T' = \boldsymbol{p}_T + \boldsymbol{q}_T$.

9.3 Evolution and transverse momentum dependence

In order to investigate generally the high $\boldsymbol{p}_T$-tails of the distribution and fragmentation functions, we separate the transverse momentum dependence and Lorentz structure, relevant to determine the leading or subleading character of the correlation functions from the Dirac and target spin structure. We write

$$\begin{aligned}
\Phi(x,\boldsymbol{p}_T) \;=\;& \Phi_2(x,\boldsymbol{p}_T^2)\,\gamma^- + \left(\frac{p_{T\,\alpha}}{M}\right)\Phi_2^\alpha(x,\boldsymbol{p}_T^2)\,\gamma^- + \left(\frac{p_{T\,\alpha}p_{T\,\beta} + \frac{1}{2}\,p_T^2\,g_{T\,\alpha\beta}}{M^2}\right)\Phi_2^{\alpha\beta}(x,\boldsymbol{p}_T^2)\,\gamma^- + \ldots\\
&+ \left(\frac{M}{P^+}\right)\Phi_3(x,\boldsymbol{p}_T^2) + \left(\frac{M}{P^+}\right)\left(\frac{p_{T\,\alpha}}{M}\right)\Phi_3^\alpha(x,\boldsymbol{p}_T^2) + \ldots\\
&+ \left(\frac{M}{P^+}\right)^2\Phi_4(x,\boldsymbol{p}_T^2)\,\gamma^+ + \left(\frac{M}{P^+}\right)^2\left(\frac{p_{T\,\alpha}}{M}\right)\Phi_4^\alpha(x,\boldsymbol{p}_T^2)\,\gamma^+ + \ldots\\
&+ \ldots \qquad (9.67)
\end{aligned}$$

where the quantities $\Phi_t^{\alpha\cdots}(x,\boldsymbol{p}_T^2)$ depend on the arguments x and $\boldsymbol{p}_T^2$ and get the (transverse) Lorentz structure from the Dirac gamma-matrices and the spin vectors. The integrated and $\boldsymbol{p}_T$-weighted quantities are given by

$$\Phi(x) = \int d^2p_T\;\Phi(x,\boldsymbol{p}_T) = \Phi_2(x)\,\gamma^- + \left(\frac{M}{P^+}\right)\Phi_3(x) + \left(\frac{M}{P^+}\right)^2\Phi_4(x)\,\gamma^+ \qquad (9.68)$$

$$\frac{1}{M}\,\Phi_\partial^\alpha(x) = \int d^2p_T\;\left(\frac{p_T^\alpha}{M}\right)\Phi(x,\boldsymbol{p}_T) = \Phi_2^{\alpha(1)}(x)\,\gamma^- + \left(\frac{M}{P^+}\right)\Phi_3^{\alpha(1)}(x) + \ldots \qquad (9.69)$$

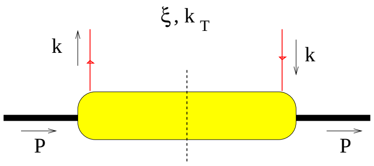
$$\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta}(x) = \int d^2p_T\;\left(\frac{p_{T\,\alpha}p_{T\,\beta} + \frac{1}{2}\,p_T^2\,g_{T\,\alpha\beta}}{M^2}\right)\Phi(x,\boldsymbol{p}_T^2) = \Phi_2^{\alpha\beta(2)}(x)\,\gamma^- + \ldots \qquad (9.70)$$

Similarly we have the (integrated) quark-quark-gluon correlation functions

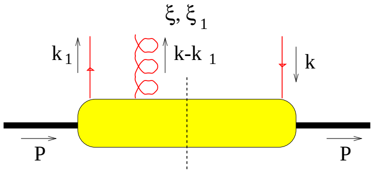
$$\Phi_A^\alpha(x,y) = \left(\frac{M}{P^+}\right)\Phi_{A,3}^\alpha(x,y)\,\gamma^- + \left(\frac{M}{P^+}\right)^2\Phi_{A,4}^\alpha(x,y) + \ldots \qquad (9.71)$$

$$\Phi_{AA}^{\alpha\beta}(x,y,z) = \left(\frac{M}{P^+}\right)^2\Phi_{AA,4}^{\alpha\beta}(x,y,z)\,\gamma^- + \ldots \qquad (9.72)$$

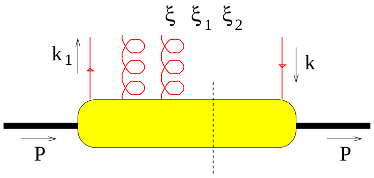
Of course, the non-integrated correlation functions can be expanded in the transverse momenta as done above. Some parts of $\Phi_A^\alpha(x,y)$ can after integration over one of the arguments be related to the (twist-three) quark-quark correlation functions $\Phi_3(x)$ via the QCD equations of motion. In the above parametrization, the functions on the right-hand side depending on $(x, \boldsymbol{p}_T^2)$ have canonical dimension -2, while functions depending on (x) are dimensionless. Graphically represented we have



The quantity $\Phi(\xi, \boldsymbol{k}_T) = \int dp^-\ \Phi(k; P, S)$ with canonical dimension -2 or after further integration the dimensionless quantities $\Phi(x)$ and $\frac{1}{M}\,\Phi_\partial^\alpha(x)$.



$\Phi_A^\alpha(\xi, \xi_1)$



$\Phi_{AA}^\alpha(\xi, \xi_1, \xi_2)$

A hard part can be connected to the soft parts via an integration over the momenta of the soft part, $\int d^4k\, d^4k_1\ \ldots$ Taking the dimensions following from this integration into account we get for the simplest kernel, starting with two quarks with the relevant momentum components being ξ and k_T and leaving two quarks with non-integrated momentum components x and p_T

$$
\begin{aligned}
&\int d^4k\ \ldots H(p;k) = \\
&\quad \alpha_s \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{|\boldsymbol{p}_T|^2}\,[\gamma^+ H_{2,2}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \frac{\boldsymbol{k}_T^\alpha}{|\boldsymbol{p}_T|^3}\,[\gamma^+ H_{2,2}^{\prime\alpha}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{P^+}{M}\right) \int d\xi\, d^2k_T\ \ldots \left\{ \frac{M}{|\boldsymbol{p}_T|^3}\,[H_{3,2}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{M p_T^\alpha}{|\boldsymbol{p}_T|^2}\right) \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{M|\boldsymbol{p}_T|}\,[\gamma^+ H_{2,2}^\alpha(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \frac{k_T^\beta}{M|\boldsymbol{p}_T|^2}\,[\gamma^+ H_{2,2}^{\prime\beta\alpha}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{M p_T^\alpha}{|\boldsymbol{p}_T|^2}\right)\left(\frac{P^+}{M}\right) \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{|\boldsymbol{p}_T|^2}\,[H_{3,2}^\alpha(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{M^2 p_T^\alpha p_T^\beta - \ldots}{|\boldsymbol{p}_T|^4}\right) \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{M^2}\,[\gamma^+ H_{2,2}^{\alpha\beta}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^-] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{M}{P^+}\right) \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{M|\boldsymbol{p}_T|}\,[\gamma^+ H_{2,3}(\xi;x,\boldsymbol{k}_T^2)] + \ldots \right\} \\
&\quad + \alpha_s \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{|\boldsymbol{p}_T|^2}\,[H_{3,3}(\xi;x,\boldsymbol{p}_T^2)] + \ldots \right\} \\
&\quad + \alpha_s \left(\frac{M}{P^+}\right)^2 \int d\xi\, d^2k_T\ \ldots \left\{ \frac{1}{M^2}\,[\gamma^+ H_{2,4}(\xi;x,\boldsymbol{k}_T^2)\,\gamma^+] + \ldots \right\} + \ldots
\end{aligned}
\tag{9.73}
$$

The kernel connecting the soft quark-quark-gluon correlator to a quark-quark correlator can be analyzed to contain (we now make the $k-{}_T$ dependence implicit)

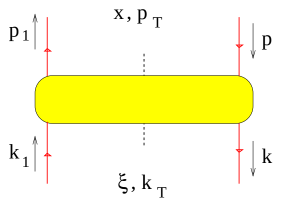
$$
\int d^4k\, d^4k_1\ \ldots H_A^\alpha(p;k,k_1) =
$$

$$
\begin{aligned}
&\alpha_s \left(\frac{P^+}{M}\right) \int d\xi\, d\xi_1 \,\ldots \frac{M}{|\boldsymbol{p}_T|^3}\, [\gamma^+ H^\alpha_{A;3,2}(\xi,\xi_1;x)\,\gamma^-] \\
&+ \alpha_s \left(\frac{P^+}{M}\right)^2 \int d\xi\, d\xi_1 \,\ldots \frac{M^2}{|\boldsymbol{p}_T|^4}\, [H^\alpha_{A;4,2}(\xi,\xi_1;x)\,\gamma^-] \\
&+ \alpha_s \left(\frac{P^+}{M}\right)\left(\frac{M p_T^\beta}{|\boldsymbol{p}_T|^2}\right) \int d\xi\, d\xi_1 \,\ldots \left\{ \frac{1}{|\boldsymbol{p}_T|^2}\, [\gamma^+ H^{\beta\alpha}_{A;3,2}(\xi,\xi_1;x)\,\gamma^-] \right. \\
&+ \alpha_s \int d\xi\, d\xi_1 \,\ldots \frac{1}{|\boldsymbol{p}_T|^2}\, [\gamma^+ H_{A;3,3}(\xi,\xi_1;x)] \\
&+ \alpha_s \left(\frac{P^+}{M}\right) \int d\xi\, d\xi_1 \,\ldots \frac{M}{|\boldsymbol{p}_T|^3}\, [H_{A;4,3}(\xi,\xi_1;x)] + \ldots
\end{aligned}
\tag{9.74}
$$

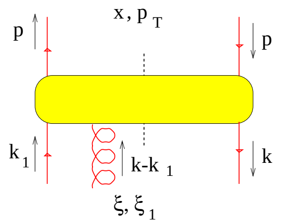
For qqGG soft contributions we need

$$
\begin{aligned}
&\int d^4k\, d^4k_1\, d^4k_2 \,\ldots H^{\alpha\beta}_{AA}(p;k,k_1,k_2) = \\
&\quad \alpha_s \left(\frac{P^+}{M}\right)^2 \int d\xi\, d\xi_1\, d\xi_2 \,\ldots \frac{M^2}{|\boldsymbol{p}_T|^4}\, [\gamma^+ H^{\alpha\beta}_{AA;4,2}(\xi,\xi_1,\xi_2;x)\,\gamma^-] \\
&\quad + \alpha_s \left(\frac{M p_T^\gamma}{|\boldsymbol{p}_T|^2}\right)\left(\frac{P^+}{M}\right)^2 \int d\xi\, d\xi_1\, d\xi_2 \,\ldots \frac{M}{|\boldsymbol{p}_T|^3}\, [\gamma^+ H^{\gamma\alpha\beta}_{AA;4,2}(\xi,\xi_1,\xi_2;x)\,\gamma^-] \\
&\quad + \alpha_s \left(\frac{P^+}{M}\right) \int d\xi\, d\xi_1\, d\xi_2 \,\ldots \frac{M}{|\boldsymbol{p}_T|^3}\, [\gamma^+ H_{AA;4,3}(\xi,\xi_1,\xi_2;x)] + \ldots
\end{aligned}
\tag{9.75}
$$

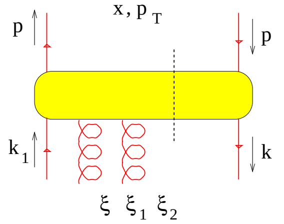
Graphically the kernels are represented by



$\int d^4k \,\ldots H(p;k)$



$\int d^4k\, d^4k_1 \,\ldots H^\alpha_A(p;k,k_1)$



$\int d^4k\, d^4k_1\, d^4k_2 \,\ldots H^{\alpha\beta}_{AA}(p;k,k_1,k_2)$

Combining the contributions we get the following corrections for the integrated result:

$$
\begin{aligned}
\Delta\Phi_2(x,\boldsymbol{p}_T^2) \;=\;& \alpha_s \frac{1}{|\boldsymbol{p}_T|^2} \int d\xi\; \Phi_2(\xi) H_{2,2}(\xi,x) \\
&+\alpha_s \frac{M}{|\boldsymbol{p}_T|^3} \int d\xi\; \Phi_2^{\alpha(1)}(\xi) H'^{\alpha}_{2,2}(\xi,x) \\
&+\alpha_s \frac{M}{|\boldsymbol{p}_T|^3} \int d\xi\; \Phi_3(\xi) H_{3,2}(\xi,x) \\
&+\alpha_s \frac{M}{|\boldsymbol{p}_T|^3} \int d\xi\, d\xi_1\; \Phi^\alpha_{A,3}(\xi,\xi_1) H^\alpha_{A;3,2}(\xi,\xi_1,x) \\
&+\ldots
\end{aligned}
\tag{9.76}
$$

For the integrated twist-3 result we get

$$\begin{aligned}\Delta\Phi_3(x,\boldsymbol{p}_T^2) &= \alpha_s\,\frac{1}{M\,|\boldsymbol{p}_T|}\int d\xi\ \Phi_2(\xi)H_{2,3}(\xi,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\ \Phi_3(\xi)H_{3,3}(\xi,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\,d\xi_1\ \Phi^{\alpha}_{A,3}(\xi,\xi_1)H^{\alpha}_{A;3,2}(\xi,\xi_1,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\ \Phi_2^{\alpha(1)}(\xi)H^{\alpha}_{2,2}(\xi,x)\\ &+\dots \end{aligned}\tag{9.77}$$

For the weighted twist-2 result we get

$$\begin{aligned}\Delta\Phi_2^{\alpha(1)}(x,\boldsymbol{p}_T^2) &= \alpha_s\,\frac{1}{M\,|\boldsymbol{p}_T|}\int d\xi\ \Phi_2(\xi)H^{\alpha}_{2,2}(\xi,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\ \Phi_3(\xi)H^{\alpha}_{3,2}(\xi,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\,d\xi_1\ \Phi^{\beta}_{A,3}(\xi,\xi_1)H^{\beta\alpha}_{A;3,2}(\xi,\xi_1,x)\\ &+\alpha_s\,\frac{1}{|\boldsymbol{p}_T|^2}\int d\xi\ \Phi_2^{\beta(1)}(\xi)H'^{\beta\alpha}_{2,2}(\xi,x)\\ &+\dots \end{aligned}\tag{9.78}$$

9.4 The unpolarized case

The quark-quark correlation function depending on $k^+ = xP^+$ and $\boldsymbol{k}_T$ is for a spin 0 or unpolarized hadron, including only leading $(M/P^+)^0$ and subleading $(M/P^+)^1$ contributions, given by

$$\begin{aligned}\Phi(x,\boldsymbol{k}_T) &= \frac{1}{2}\left\{f_1(x,\boldsymbol{k}_T) + i\,h_1^{\perp}(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M}\right\}\gamma^-\\ &+ \frac{M}{2P^+}\left\{e(x,\boldsymbol{k}_T) + f^{\perp}(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M} + i\,h(x,\boldsymbol{k}_T)\,\frac{[\gamma^-,\gamma^+]}{2}\right\}\end{aligned}\tag{9.79}$$

The $\boldsymbol{k}_T$-integrated results are

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\,f_1(x)\,\gamma^-\\ &+ \frac{M}{2P^+}\left\{e(x) + i\,h(x)\,\frac{[\gamma^-,\gamma^+]}{2}\right\},\end{aligned}\tag{9.80}$$

$$\begin{aligned}\frac{1}{M}\,\Phi^{\alpha}_{\partial}(x) &= \frac{1}{2}\,i\,h_1^{\perp(1)}(x)\,\gamma_T^{\alpha}\,\gamma^-\\ &+ \frac{M}{2P^+}\,f^{\perp(1)}(x)\,\gamma_T^{\alpha}.\end{aligned}\tag{9.81}$$

The relevant projections are (we do use the tensor $\sigma^{+-} = (i/2)[\gamma^+,\gamma^-]$ but we use $\gamma_T^{\alpha}\gamma^+$ instead of the tensor $\sigma^{\alpha+} = i\gamma_T^{\alpha}\gamma^+$),

$$\Phi^{[\gamma^+]}(x,\boldsymbol{k}_T) = f_1(x,\boldsymbol{k}_T),\tag{9.82}$$

$$\Phi^{[1]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\,e(x,\boldsymbol{k}_T),\tag{9.83}$$

$$\Phi^{[\sigma^{+-}]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\,i\,h(x,\boldsymbol{k}_T)\tag{9.84}$$

$$\Phi^{[\gamma_T^{\alpha}\gamma^+]}(x,\boldsymbol{k}_T) = -i\,k_T^{\alpha}\,h_1^{\perp}(x,\boldsymbol{k}_T),\quad\text{and}\quad \Phi_{\partial}^{\alpha[\gamma^{\beta}\gamma^+]}(x) = -i\,g_T^{\alpha\beta}\,h_1^{\perp(1)}(x),\tag{9.85}$$

$$\Phi^{[\gamma_T^{\alpha}]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\,k_T^{\alpha}\,f^{\perp}(x,\boldsymbol{k}_T).\quad\text{and}\quad \Phi_{\partial}^{\alpha[\gamma^{\beta}]}(x) = g_T^{\alpha\beta}\,\frac{M}{P^+}\,f^{\perp(1)}(x).\tag{9.86}$$

For the $\overline{\psi}(0)\, iD_T^\alpha \psi(x)$ correlation function (bilocal, i.e. integrated over d^4k_1) one obtains from the equations of motion

$$\frac{1}{M}\,\Phi_D^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) = \frac{k_T^\alpha}{M}\left(x\, f^\perp + i\,\frac{m}{M}\, h_1^\perp\right) \tag{9.87}$$
$$\frac{1}{M}\,\Phi_{D\alpha}^{[\gamma_T^\alpha\gamma^+]}(x,\boldsymbol{k}_T) = (x\, e - m\, f_1 - i\, x\, h) \tag{9.88}$$

Using the quark-quark correlations multiplied with k_T^α,

$$\frac{k_T^\alpha}{M}\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_T) = \frac{k_T^\alpha}{M}\, f_1(x,\boldsymbol{k}_T), \tag{9.89}$$
$$\frac{k_{T\alpha}}{M}\,\Phi^{[\gamma_T^\alpha\gamma^+]}(x,\boldsymbol{k}_T) = \frac{1}{M}\,\Phi_{\partial\alpha}^{[\gamma_T^\alpha\gamma^+]}(x,\boldsymbol{k}_T) = 2i\, h_1^{\perp(1)}(x,\boldsymbol{k}_T), \tag{9.90}$$

the difference $\Phi_A^\alpha = \Phi_D^\alpha - k_T^\alpha\,\Phi$ is given by

$$\frac{1}{M}\,\Phi_A^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) \;=\; \frac{k_T^\alpha}{M}\left(\underbrace{xf^\perp - f_1}_{x\tilde{f}^\perp} + i\frac{m}{M}\, h_1^\perp\right) \tag{9.91}$$
$$\frac{1}{M}\,\Phi_{A\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) \;=\; \underbrace{x\, e - \frac{m}{M}\, f_1}_{x\tilde{e}}\;\underbrace{-i\, x\, h - 2i\, h_1^{\perp(1)}}_{-ix\tilde{h}} \tag{9.92}$$

The most general form of quark-quark-gluon correlation functions integrated over all transverse momenta but not integrated over x_1 are

$$\Phi_D^\alpha(x,x_1) = \frac{M}{2P^+}\, E_D(x,x_1)\gamma_T^\alpha\,\gamma^-, \tag{9.93}$$

from which one finds that

$$\frac{1}{M}\,\Phi_D^\alpha(x) = \int dx_1\; E_D(x,x_1)\,\gamma_T^\alpha\,\gamma^- \tag{9.94}$$

(similarly for $\Phi_A(x,x_1)$). The projections are

$$\Phi_{D\alpha}^{[\gamma_T^\alpha\gamma^+]}(x,x_1) = 2\,\frac{M}{P^+}\, E_D(x,x_1) \tag{9.95}$$
$$\frac{1}{M}\,\Phi_{D\alpha}^{[\gamma_T^\alpha\gamma^+]}(x) = 2\int dx_1\; E_D(x,x_1). \tag{9.96}$$

9.5 Sample calculations

9.5.1 The fermion propagator

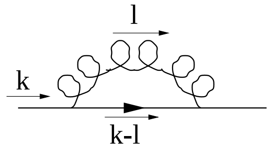
We will demonstrate perturbative calculations, in particular when one uses a lightcone gauge in detail for the example of the self-energy contribution to $\Gamma^{(2)}(p)$. Writing the (truncated) Green's function as

$$\Gamma^{(2)}(k) = -i\,(S_F)^{-1}(k) = -i[\not{k} - m_0 - \Sigma(k)], \tag{9.97}$$

one has the expansion

$$S_F(k) = \frac{1}{\not{k} - m_0} + \frac{1}{\not{k} - m_0}\,\Sigma(k)\,\frac{1}{\not{k} - m_0} + \ldots . \tag{9.98}$$

The contribution of the one-loop diagram,



to the self-energy in the lightcone gauge is given by

$$\Sigma(k) = -ig^2\,C_F \int \frac{d^4l}{(2\pi)^4}\,\frac{\gamma^\alpha(\not k - \not l + m_0)\gamma^\beta\,d_{\alpha\beta}(l)}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)}, \tag{9.99}$$

with

$$d^{\alpha\beta}(l) = g^{\alpha\beta} - \frac{l^\alpha n_-^\beta + n_-^\alpha l^\beta}{l^+}. \tag{9.100}$$

The numerator can be written as

$$\text{numerator} = -2(\not k - \not l) + 4m_0 + \gamma^+\,\frac{(\not k - \not l + m_0)\not l}{l^+} + \frac{\not l(\not k - \not l + m_0)}{l^+}\,\gamma^+.$$

Because of the expansion around the zeroth order result, it is convenient to rewrite $\not l$ as $(\not k - m_0) - (\not k - \not l - m_0)$, leading for the numerator to

$$\begin{aligned}
\text{numerator} &= -2(\not k - \not l) + 4m_0 \\
&\quad -\gamma^+\,\frac{(\not k - \not l + m_0)(\not k - m_0)}{l^+} - \frac{(\not k - m_0)(\not k - \not l + m_0)}{l^+}\,\gamma^+ \\
&\quad +\gamma^+\,\frac{(\not k - \not l + m_0)(\not k - \not l - m_0)}{l^+} + \frac{(\not k - \not l - m_0)(\not k - \not l + m_0)}{l^+}\,\gamma^+.
\end{aligned}$$

We consider the case $k_T \approx k^- \approx 0$ and $k^+ = x\,P^+$, while $l^- = \alpha\,\boldsymbol{l}_T^2/2P^+$ and $l^+ = y\,P^+$. We need the following integrals[1]

$$\begin{aligned}
\Sigma_1(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\,\frac{1}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)} \\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy \int \frac{d\alpha}{2\pi\,i}\,\frac{1}{(\alpha y - 1 + i\epsilon)(\alpha(y-x) - 1 + i\epsilon)} \\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\Theta_{11}^0(y, y-x) \\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2} \\
&= \frac{\alpha_s}{4\pi}\;\ln\Lambda^2,
\end{aligned} \tag{9.101}$$

$$\begin{aligned}
\Sigma_2(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\,\frac{k^+}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,l^+} \\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{x}{y}\int \frac{d\alpha}{2\pi\,i}\,\frac{1}{(\alpha y - 1 + i\epsilon)(\alpha(y-x) - 1 + i\epsilon)} \\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{x}{y}\,\Theta_{11}^0(y, y-x) \\
&= \Sigma_1\int \frac{dy}{y}\;x\Theta_{11}^0(y, y-x).
\end{aligned} \tag{9.102}$$

We also need the integral

$$\begin{aligned}
&-i\,\frac{g^2}{(2\pi)^4}\int d^4l\,\frac{l^+}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,k^+} \\
&\quad = \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{y}{x}\int \frac{d\alpha}{2\pi\,i}\,\frac{1}{(\alpha y - 1 + i\epsilon)(\alpha(y-x) - 1 + i\epsilon)} \\
&\quad = \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{y}{x}\,\Theta_{11}^0(y, y-x) \\
&\quad = \frac{1}{2}\,\Sigma_1.
\end{aligned} \tag{9.103}$$

[1] reminder: $\Theta_{11}^0(y, y-x) = \frac{1}{x}\,[\theta(y)\,\theta(x-y) - \theta(-y)\,\theta(y-x)]$, thus one has for the y-integration (for positive x) $\int_0^x dy$

It is useful to have the following integrals

$$\begin{aligned}
\not{L}_1(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not{l}}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)}\\
&= \frac{1}{2}\,\not{k}\,\Sigma_1 \qquad (9.104)\\
\not{L}_2(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not{k}-\not{l}-m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)}\\
&= \frac{1}{2}\,\not{k}\,\Sigma_1 - m_0\,\Sigma_1, \qquad (9.105)\\
\not{R}_1(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not{k}-\not{l}-m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,l^+}\\
&= \frac{1}{k^+}\big[\not{k}\,\underbrace{(\Sigma_2-\Sigma_1)}_{\Sigma_3} - m_0\,\Sigma_2\big], \qquad (9.106)\\
\not{R}_2(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not{k}-\not{l}+m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,l^+}\\
&= \frac{1}{k^+}\big[\not{k}\,\underbrace{(\Sigma_2-\Sigma_1)}_{\Sigma_3} + m_0\,\Sigma_2\big] = \not{R}_1(k) + \frac{2m_0}{k^+}\,\Sigma_2, \qquad (9.107)
\end{aligned}$$

where one should realize that the only nonvanishing contribution in $\not{l}$ is $\not{l} = l^+\gamma^- = \frac{l^+}{k^+}\,\not{k}$. Note that $\gamma^+\not{R}_2(k) = 2\Sigma_3 - \not{R}_1(k)\gamma^+$ and $\not{R}_2(k)\gamma^+ = 2\Sigma_3 - \gamma^+\not{R}_1(k)$. It is now straightforward to find the result

$$\begin{aligned}
\Sigma(k) &= -\not{k}\Sigma_1 + 4m_0\,\Sigma_1 - (\not{k}-m_0)\,\not{R}_2(k)\gamma^+ - \gamma^+\not{R}_2(k)\,(\not{k}-m_0)\\
&= -(\not{k}-m_0)\,\Sigma_1 + 3m_0\,\Sigma_1 - 4\Sigma_3\,(\not{k}-m_0) + (\not{k}-m_0)\,\gamma^+\not{R}_1(k) + \not{R}_1(k)\gamma^+\,(\not{k}-m_0)\\
&. \qquad (9.108)
\end{aligned}$$

Inserting in the expansion for $S_F(k)$ one obtains

$$\begin{aligned}
S_F(k) &= \frac{(1-\Sigma_1)}{(\not{k}-m_0)} + \frac{1}{(\not{k}-m_0)}\,3m_0\,\Sigma_1\,\frac{1}{(\not{k}-m_0)}\\
&\quad -\frac{1}{(\not{k}-m_0)}\,\gamma^+\not{R}_2(k) - \not{R}_2(k)\gamma^+\,\frac{1}{(\not{k}-m_0)}\\
&= \underbrace{\left(1-\not{R}_2(k)\gamma^+\right)}_{U_2^{-1}(k)}\,\frac{(1-\Sigma_1)}{(\not{k}-m_0(1+3\Sigma_1))}\,\underbrace{\left(1-\gamma^+\not{R}_2(k)\right)}_{U_1(k)}\\
&= U_2^{-1}(k)\,\frac{(1-\Sigma_1)}{(\not{k}-m)}\,U_1(k) \qquad (9.109)
\end{aligned}$$

with $m = m_0(1+3\Sigma_1)$ and up to the required (first order) precision

$$\begin{aligned}
U_1(k) &= 1-\gamma^+\not{R}_2(k) = 1 - 2\Sigma_3 + \not{R}_1(k)\gamma^+\\
&= 1 - \frac{1}{k^+}\,\Sigma_3(k)\,\gamma^+\,\not{k} - \frac{m}{k^+}\,\Sigma_2(k)\,\gamma^+ \qquad (9.110)\\
U_2(k) &= 1+\not{R}_2(k)\gamma^+ = 1 + 2\Sigma_3 - \gamma^+\not{R}_1(k)\\
&= 1 + \frac{1}{k^+}\,\Sigma_3(k)\,\not{k}\,\gamma^+ + \frac{m}{k^+}\,\Sigma_2(k)\,\gamma^+. \qquad (9.111)
\end{aligned}$$

9.6 Evolution of $f_1(x)$

The tree level contribution is given by

$$\begin{aligned}
f_1^{[\alpha_s^0]}(x) &= \frac{1}{4}\,f_1(x)\,\mathrm{Tr}\left[\gamma^-\gamma^+\right] \qquad (9.112)\\
&= f_1(x). \qquad (9.113)
\end{aligned}$$

The ladder graph gives using

$$k = \left[\frac{\alpha}{2}\,\boldsymbol{k}_T^2, x, k_T\right], \tag{9.114}$$

$$l = \left[\frac{\alpha}{2}\,\boldsymbol{k}_T^2, x-y, k_T\right], \tag{9.115}$$

$$p = [0, y, 0_T], \tag{9.116}$$

and the trace

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^-\gamma^\alpha \not k \gamma^+ \not k \gamma^\beta\right]\, d_{\alpha\beta}(l) &= -8\,\boldsymbol{k}_T^2 - 32\,\frac{(k^+)^2\,l^-}{l^+} + 16\,\frac{k^+\,\boldsymbol{k}_T\cdot\boldsymbol{l}_T}{l^+} \\
&= \boldsymbol{k}_T^2\left[-8 - 16\,\alpha\,\frac{x^2}{(x-y)} + 16\,\frac{x}{(x-y)}\right] \\
&= \boldsymbol{k}_T^2\,\frac{[8(x+y) - 16\,\alpha\,x^2]}{(x-y)}
\end{aligned} \tag{9.117}$$

the result

$$\begin{aligned}
f_1^{[\alpha_s^1,\text{ladder}]}(x) &= -i\pi\,\frac{g^2\,C_F}{(2\pi)^4}\int \frac{dy\,d\alpha}{2}\,\frac{d^2k_T}{\boldsymbol{k}_T^2}\,\frac{[2(x+y) - 4\,x^2\,\alpha]}{(x-y)\,[\alpha(x-y) - 1 + i\epsilon]\,[\alpha\,x - 1 + i\epsilon]^2}\,f_1(y) \\
&= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\left[\frac{(x+y)}{(x-y)}\,\Theta^0_{21}(x,x-y) - \frac{2\,x^2}{(x-y)}\,\Theta^1_{21}(x,x-y)\right] f_1(y) \\
&= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\left[\frac{(y^2+x^2)}{y(y-x)}\,\Theta^0_{11}(x,x-y) + \frac{x}{y}\,\delta(x)\right] f_1(y) \\
&= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\frac{(y^2+x^2)}{y(y-x)}\,\Theta^0_{11}(x,x-y) f_1(y) \\
&= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int_x^\infty \frac{dy}{y}\,\frac{(y^2+x^2)}{y(y-x)}\,f_1(y)
\end{aligned} \tag{9.118}$$

(note that the support of $f_1(y)$ is $-1 \le y \le 1$). For the self-energy graphs on the two fermion legs we need the traces

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not k}\,\Sigma(k)\right] &= \mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not k}\,\left(-\not k\Sigma_1 + \not k \not R(k)\gamma^+ + \gamma^+\not R(k)\not k\right)\right] \\
&= \mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not k}\,\left(-\not k\Sigma_1 - \not k\not k\,\frac{\gamma^+}{k^+}\,\Sigma_3 - \frac{\gamma^+}{k^+}\,\not k\not k\Sigma_3\right)\right] \\
&= -4\,\Sigma_1 - 16\,\Sigma_3 = 4(3\,\Sigma_1 - 4\,\Sigma_2),
\end{aligned} \tag{9.119}$$

$$\mathrm{Tr}\left[\gamma^-\Sigma(k)\gamma^+\,\frac{1}{\not k}\right] = 4(3\,\Sigma_1 - 4\,\Sigma_2). \tag{9.120}$$

Then (taking into account a factor 1/2)

$$f_1^{[\alpha_s^1,\text{self}]}(x) = (-\Sigma_1 - 4\Sigma_3)\,f_1(x) = (3\,\Sigma_1 - 4\,\Sigma_2)\,f_1(x) \tag{9.121}$$

$$\begin{aligned}
&= \Sigma_1\left(3 - 4\int\frac{dy}{y}\,x\Theta^0_{11}(y,y-x)\right) f_1(x) \\
&= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\left\{\frac{3}{2}\,f_1(x) - 2\,f_1(x)\int dy\,\frac{x}{x-y}\,\Theta^0_{11}(y,y-x)\right\}
\end{aligned} \tag{9.122}$$

Using

$$\int dy\,\frac{x}{x-y}\,\left[\Theta^0_{11}(y,y-x) + \Theta^0_{11}(x,x-y)\right] = 0. \tag{9.123}$$

the total result becomes

$$f_1^{[\alpha_s^1]}(x) = \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\left\{\int dy\,\left[\frac{y^2+x^2}{y(y-x)}\,f_1(y) - \frac{2x}{y-x}\,f_1(x)\right]\Theta^0_{11}(x,x-y) + \frac{3}{2}\,f_1(x)\right\}$$

The integral runs from $x \le y \le \infty$, but for the first term it is limited to $x \le y \le 1$ using the support properties of f_1. In the next step we introduce $\beta = x/y$ and find that

$$\begin{aligned} x \le y \le 1 \quad &\Longleftrightarrow \quad x \le \beta \le 1, \\ x \le y \le \infty \quad &\Longleftrightarrow \quad 0 \le \beta \le 1, \end{aligned}$$

Thus

$$\begin{aligned} f_1^{[\alpha_s^1]}(x) &= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \left\{ \int_x^1 \frac{d\beta}{\beta}\, \frac{1+\beta^2}{(1-\beta)}\, f_1\left(\frac{x}{\beta}\right) - 2\, f_1(x) \int_0^1 d\beta\, \frac{1}{1-\beta} + \frac{3}{2}\, f_1(x) \right\} \\ &= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \int_x^1 \frac{d\beta}{\beta} \left[\frac{1+\beta^2}{(1-\beta)_+} + \frac{3}{2}\, \delta(1-\beta) \right] f_1\left(\frac{x}{\beta}\right) \qquad (9.124) \\ &= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \int_x^1 \frac{d\beta}{\beta}\, P^{[f]}(\beta)\, f_1\left(\frac{x}{\beta}\right). \end{aligned}$$

9.7 Gribov-Lipatov reciprocity and evolution of D_1

For the fragmentation function D_1, the tree level contribution is given by

$$D_1^{[\alpha_s^0]}(z) = \frac{1}{4}\, D_1(z)\, \mathrm{Tr}\left[\gamma^+\gamma^-\right] = D_1(z). \qquad (9.125)$$

We note that the result can be obtained from the distribution functions by interchanging everywhere the + and − components and then later make replacements $x \to 1/z$ and $f_1(x) \to f_1(1/z) \to D_1(z)$.

The momenta for the ladder graph in the case of fragmentation can be written

$$k = \left[x, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right] \quad \longrightarrow \quad \left[\frac{1}{z}, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right], \qquad (9.126)$$

$$l = \left[x-y, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right] \qquad (9.127)$$

$$p = \left[y, 0, 0_T\right], \qquad (9.128)$$

involving also simply an interchange of the lightcone components.

We now look at the evolution equations of fragmentation functions (FF) by looking at those for distribution functions (DF) in the domain $x \ge 1$. Starting with the result for DF's obtained via $f_1^{[\alpha_s^1]}(x)$,

$$\frac{d}{d\tau}\, f(x,\tau) = \frac{\alpha_s}{2\pi}\, C_F \int_0^\infty dy \left[\frac{\mathcal{N}\left(\frac{x}{y}\right)}{1-\frac{x}{y}}\, f(y) - \frac{\left(\frac{x}{y}\right)\mathcal{N}(1)}{1-\frac{x}{y}}\, f(x) \right] \Theta_{11}(x, x-y), \qquad (9.129)$$

allowing for different numerators in order to treat the various distribution functions (f_1, g_1 and h_1). We consider $\mathcal{N}(\beta) = 2\,\beta^p$, which will make it easy to get the results for $\mathcal{N}(\beta) = 1+\beta^2$ or $\mathcal{N}(\beta) = 2\beta$. In all these cases $\mathcal{N}(1) = 2$, the numerator result coming from the fermion self energy. We find (defining $\beta = x/y$) for the DF

$$\frac{d}{d\tau}\, f(x) = \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\beta}{\beta} \left[\underbrace{\frac{2\,\beta^p}{1-\beta}\, f\left(\frac{x}{\beta}\right)}_{x\le\beta\le 1} - \underbrace{\frac{2\,\beta}{1-\beta}\, f(x)}_{0\le\beta\le 1} \right]. \qquad (9.130)$$

For the FF we employ an continuation to the region $x \ge 1$ and then a substitution $x \to 1/z$. For a quark with $k^- = xP_h^-$ to produce a hadron with P_h^- the radiated gluon can have momentum $\ell^- = (x-y)P_h^-$ with $1 \le y \le x$. This is obtained via the generalized Θ-functions if one uses for the timelike fragmenting quark a propagator prescription $1/(p^2 - m^2 - i\epsilon)$. Thus one has

$$\begin{aligned} \Theta_{11}^0(x_1,x_2) = \int \frac{d\alpha}{2\pi\, i}\, \frac{1}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)} &= \frac{\left[\theta(x_1)\,\theta(-x_2) - \theta(-x_1)\theta(x_2)\right]}{x_1 - x_2} \\ &= \frac{\theta(x_1) - \theta(x_2)}{x_1 - x_2} = \frac{\theta(-x_2) - \theta(-x_1)}{x_1 - x_2}, \end{aligned}$$

$$\Theta^0_{\underline{1}1}(x_1,x_2) = \int \frac{d\alpha}{2\pi\, i}\; \frac{1}{(\alpha x_1 - 1 - i\epsilon)(\alpha x_2 - 1 + i\epsilon)} \;=\; -\frac{[\theta(x_1)\,\theta(x_2) - \theta(-x_1)\theta(-x_2)]}{x_1 - x_2}$$
$$= -\frac{\theta(x_1) - \theta(-x_2)}{x_1 - x_2} = -\frac{\theta(x_2) - \theta(-x_1)}{x_1 - x_2},$$

and for positive x-values

$$\Theta^0_{1\underline{1}}(x, x-y) \overset{x\geq 0}{=} \frac{\theta(y-x)}{y},$$
$$\Theta^0_{\underline{1}1}(x, x-y) \overset{x\geq 0}{=} -\frac{\theta(x-y)}{y}.$$

This modification happens only for the ladder graph, not for the self-energy. With the further replacements $z \to 1/x$ and $f(x) \to f(1/z) \to D(z)$ one obtains

$$\begin{aligned}
\frac{d}{d\tau}\, f(x) &= \frac{\alpha_s}{2\pi} \int_1^\infty \frac{d\beta}{\beta}\; \underbrace{\frac{\mathcal{N}(\beta)}{\beta - 1}\, f\left(\frac{x}{\beta}\right)}_{1\leq\beta\leq x} - \int_0^1 \frac{d\beta}{\beta}\; \frac{\beta\,\mathcal{N}(1)}{1-\beta}\, f(x), \\
\frac{d}{d\tau}\, D(z) &= \frac{\alpha_s}{2\pi} \int_1^\infty \frac{d\beta}{\beta}\; \underbrace{\frac{\mathcal{N}(\beta)}{(\beta - 1)}\, D(\beta z)}_{1\leq\beta\leq 1/z} - \int_0^1 \frac{d\beta}{\beta}\; \underbrace{\frac{\beta\,\mathcal{N}(1)}{1-\beta}\, D(z)}_{0\leq\beta<1} \\
&= \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\beta}{\beta} \left[\underbrace{\frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)}\, D\left(\frac{z}{\beta}\right)}_{z\leq\beta\leq 1} - \underbrace{\frac{\beta\,\mathcal{N}(1)}{1-\beta}\, D(z)}_{0\leq\beta<1} \right] \\
&= \frac{\alpha_s}{2\pi} \int_z^1 \frac{d\beta}{\beta}\; \frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)_+}\, D\left(\frac{z}{\beta}\right)
\end{aligned} \tag{9.131}$$

We find the splitting functions

$$P^{[f]}(\beta) = \frac{\mathcal{N}(\beta)}{(1-\beta)_+}, \tag{9.132}$$

$$P^{[D]}(\beta) = \frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)_+}. \tag{9.133}$$

For a polynomial $\mathcal{N}(\beta) = 2\,\beta^p$ the moments become

$$\begin{aligned}
P^{[f]}(\beta) &= \frac{2\beta^p}{(1-\beta)_+} \\
\longrightarrow\quad A^{[f]}_n &= 2\int_0^1 d\beta\; \frac{\beta^{n-1+p} - 1}{1-\beta} = -2\sum_{j=1}^{n-1+p} \frac{1}{j} = -2\sum_{j=1}^{n} \frac{1}{j} \underbrace{- \frac{2}{n+1} \ldots - \frac{2}{n-1+p}}_{+\mathcal{A}(n)} \\
P^{[D]}(\beta) &= \frac{2\beta^{p-1}}{(\beta-1)_+} \\
\longrightarrow\quad A^{[D]}_n &= 2\int_1^\infty d\beta\; \frac{\beta^{p-n-2} - 1}{\beta - 1} \\
P^{[D]}(\beta) &= \frac{2\beta^{1-p}}{(1-\beta)_+} \\
\longrightarrow\quad A^{[D]}_n &= 2\int d\beta\; \frac{\beta^{n-p} - 1}{1-\beta} = -2\sum_{j=1}^{n-p} \frac{1}{j} = -2\sum_{j=1}^{n} \frac{1}{j} + \frac{2}{n} \underbrace{+ \frac{2}{n-1} \ldots + \frac{2}{n-p+1}}_{+\mathcal{A}(-n)}
\end{aligned}$$

Appendix A

Lightcone coordinates

We use two different notations for four-vectors,

$$a^\mu = (a^0, a^1, a^2, a^3) = (a^0, \boldsymbol{a}), \tag{A.1}$$
$$a^\mu = [a^-, a^+, a^1, a^2] = [a^-, a^+, \boldsymbol{a}_T]. \tag{A.2}$$

The metric tensor is given by $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ or $g_{++} = g_{--} = 0$ and $g_{+-} = 1$ and defines the scalar product

$$a \cdot b = g_{\mu\nu} a^\mu b^\nu \equiv g^{ab}. \tag{A.3}$$

The first vector notation corresponds to the expansion of a vector in a Cartesian basis with orthogonal vectors $\hat{t}^\mu$, $\hat{x}^\mu$, $\hat{y}^\mu$, $\hat{z}^\mu$ with t timelike ($t^2 = 1$) and the others spacelike ($\hat{x}^2 =$ -1, etc.), the second notation with an expansion in two lightlike vectors n_+^μ and n_-^μ ($n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$).

The antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is fixed by

$$\epsilon^{0123} = \epsilon^{-+12} = 1. \tag{A.4}$$

We employ the notation

$$\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma \equiv \epsilon^{abcd}. \tag{A.5}$$

A useful property is the following way of bringing a vector into the antisymmetric combination,

$$\epsilon^{\mu\nu\rho\sigma}\, g^{\alpha\beta} = \epsilon^{\alpha\nu\rho\sigma}\, g^{\mu\beta} + \epsilon^{\mu\alpha\rho\sigma}\, g^{\nu\beta} + \epsilon^{\mu\nu\alpha\sigma}\, g^{\rho\beta} + \epsilon^{\mu\nu\rho\alpha}\, g^{\sigma\beta}. \tag{A.6}$$

The following fourth rank tensor is also useful in many applications,

$$S^{\mu\nu\rho\sigma} = g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\rho}\, g^{\nu\sigma} + g^{\mu\sigma}\, g^{\nu\rho}. \tag{A.7}$$

In several cases we will also be dealing only with transverse vectors, projected out by

$$g_T^{\mu\nu} \equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu, \tag{A.8}$$

with as only nonvanishing elements $g_T^{11} = g_T^{22} = -1$ and the corresponding antisymmetric tensor with $\epsilon_T^{12} = 1$ given by

$$\epsilon_T^{\alpha\beta} = \epsilon^{-+\alpha\beta} \tag{A.9}$$

(or equivalently $g_\perp^{\mu\nu}$ and $\epsilon_\perp^{\mu\nu}$). Given a transverse vector a_T^μ, in the appropriate frame determined by two components (a^1, a^2) or determined by its length $|a_T| = \sqrt{-a_T^2}$ and an azimuthal angle ϕ_a, such that

$$a_T^\mu = (a^1, a^2) = |a_T|(\cos\phi_a, \sin\phi_a), \tag{A.10}$$

we can consider the vector

$$\bar{a}_T^\mu = \epsilon_T^{\mu\nu} a_\nu = (-a^2, a^1), \tag{A.11}$$

which has the same length, $|\boldsymbol{a}_T| = |\bar{\boldsymbol{a}}_T|$ but an azimuthal angle $\phi_{\bar{a}} = \phi_a + \pi/2$. We note that $\bar{\bar{a}}_T^\mu = -a_T^\mu$. Again we use the notation with vectors as indices, e.g.

$$g_T^{ab} \equiv g_T^{\mu\nu} a_\mu b_\nu = a_T \cdot b_T = -\boldsymbol{a}_T \cdot \boldsymbol{b}_T, \tag{A.12}$$
$$\epsilon_T^{ab} \equiv \epsilon_T^{\mu\nu} a_{T\mu} b_{T\nu} \equiv a_T \wedge b_T = a_T \cdot \bar{b}_T, \tag{A.13}$$

and we note that we can write $a_T^\mu = g_T^{a\mu}$ and $\bar a_T^\mu = g_T^{\bar a\mu} = \epsilon_T^{\mu a}$. The symmetric traceless tensor constructed from two (transverse) four vectors is denoted

$$S_T^{a\mu b\nu} = a_T^{\{\mu} b_T^{\nu\}} - (a_T\cdot b_T)\, g_T^{\mu\nu}. \tag{A.14}$$

In terms of the azimuthal angles,

$$g_T^{ab} = -|\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,\cos(\phi_b-\phi_a), \tag{A.15}$$
$$\epsilon_T^{ab} = |\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,\sin(\phi_b-\phi_a), \tag{A.16}$$
$$S_T^{abcd} = |\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,|\boldsymbol{c}_T|\,|\boldsymbol{d}_T|\,\cos(\phi_b+\phi_d-\phi_a-\phi_c). \tag{A.17}$$

For products of tensors we have the relations

$$g_T^{ij}\,g_T^{ab} = g_T^{ib}\,g_T^{ja} + \epsilon_T^{ia}\,\epsilon_T^{jb}, \tag{A.18}$$
$$\epsilon_T^{ij}\,\epsilon_T^{ab} = g_T^{ia}\,g_T^{jb} - g_T^{ib}\,g_T^{ja}, \tag{A.19}$$
$$\epsilon_T^{ab}\,g_T^{ij} = \epsilon_T^{ib}\,g_T^{aj} + \epsilon_T^{ai}\,g_T^{bj}. \tag{A.20}$$

Employing two unit-length transverse vectors $\hat x^\mu$ and $\hat y^\mu = \hat{\bar x}^\mu$, one easily derives the expansions

$$a_T^\mu = g_T^{a\mu} = -g_T^{ax}\,g_T^{x\mu} - g_T^{ay}\,g_T^{y\mu} = a^x\,\hat x^\mu + a^y\,\hat y^\mu, \tag{A.21}$$
$$\bar a_T^\mu = \epsilon_T^{\mu a} = g_T^{\bar a\mu} = g_T^{ay}\,g_T^{x\mu} - g_T^{ax}\,g_T^{y\mu} = -a^y\,\hat x^\mu + a^x\,\hat y^\mu. \tag{A.22}$$

where $a^x \equiv -(a_T\cdot\hat x) = \boldsymbol{a}_T\cdot\hat{\boldsymbol{x}}$ and $a^y \equiv -(a_T\cdot \hat y) = \hat x \wedge a_T$. For second rank tensors one obtains

$$\begin{aligned} a_T^{\{\mu} b_T^{\nu\}} - (a_T\cdot b_T) &= S_T^{a\mu b\nu} = S_T^{axbx}\,S_T^{x\mu x\nu} + S_T^{axby}\,S_T^{x\mu y\nu} \\ &= (a^x b^x - a^y b^y)\,(2\hat x^\mu \hat x^\nu + g_T^{\mu\nu}) + (a^x b^y + a^x b^y)\,\hat x^{\{\mu}\hat y^{\nu\}}, \end{aligned} \tag{A.23}$$
$$a_T^{[\mu} b_T^{\nu]} = g_T^{a[\mu} g_T^{\nu]b} = \epsilon_T^{ab}\,\epsilon_T^{\mu\nu} = (a^x b^y - a^y b^x)\,\epsilon_T^{\mu\nu}, \tag{A.24}$$
$$\begin{aligned} \frac{1}{2}\left(a_T^{\{\mu}\bar b_T^{\nu\}} + b_T^{\{\mu}\bar a_T^{\nu\}}\right) &= -S_T^{axby}\,S_T^{x\mu x\nu} + S_T^{axbx}\,S_T^{x\mu y\nu} \\ &= -(a^x b^y + a^x b^y)\,(2\hat x^\mu \hat x^\nu + g_T^{\mu\nu}) + (a^x b^x - a^y b^y)\,\hat x^{\{\mu}\hat y^{\nu\}}, \end{aligned} \tag{A.25}$$
$$\frac{1}{2}\left(a_T^{\{\mu}\bar b_T^{\nu\}} - b_T^{\{\mu}\bar a_T^{\nu\}}\right) = \epsilon_T^{ab}\,g_T^{\mu\nu} = (a^x b^y - a^y b^x)\,g_T^{\mu\nu}, \tag{A.26}$$

Finally we want to give the following useful way of transforming a tensor involving external momenta q_T into quark transverse momenta k_T and p_T. Having $p_T + k_T = q_T$ one sees that

$$\begin{aligned} S_T^{q\alpha q\beta} &= S_T^{p\alpha p\beta} + S_T^{k\alpha k\beta} + 2\,S_T^{p\alpha k\beta}. \\ &= S_T^{p\alpha p\beta} + S_T^{k\alpha k\beta} + 2\,S_T^{\rho\alpha\sigma\beta} p_{T\rho} k_{T\sigma}. \end{aligned} \tag{A.27}$$

Appendix B

Frames in leptoproduction

B.1 The inclusive case

Consider $\ell(k) + H(P) \to \ell'(k') + X$ in which the momentum transfer is $q = k - k'$. Using lightcone coordinates the momenta k, q and P satisfying $q^2 = -Q^2$, $2\,P\cdot q = Q^2/x$ and $P\cdot q = y\,P\cdot k$ are given by

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, \frac{y\,\boldsymbol{\ell}_T^2}{Q\sqrt{2}}, \underbrace{\frac{1}{y}\,\boldsymbol{q}_T + Q\,\frac{\sqrt{1-y}}{y}\,\hat{\ell}_T}_{\boldsymbol{\ell}_T}\Big], \tag{B.1}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, \frac{Q_T^2 - Q^2}{Q\sqrt{2}}, \boldsymbol{q}_T\Big], \tag{B.2}$$

$$P = \Big[\Big(\frac{xM}{Q\sqrt{2}}\Big), \frac{Q}{x\sqrt{2}}, \boldsymbol{0}_T\Big], \tag{B.3}$$

where $P^- = M^2/2P^+$ will be neglected in the rest. Here we have allowed for an arbitrary transverse momentum in q being $\boldsymbol{q}_T = Q_T\,\hat{q}_T$. We note that the parton momenta satisfying $p^2 = (p+q)^2 = 0$ (up to mass effects and with $|\boldsymbol{p}_T| \ll Q$) and the final state remnant momenta are given by

$$p = xP + p_T = \Big[0, \frac{Q}{\sqrt{2}}, (\boldsymbol{p}_T)\Big], \tag{B.4}$$

$$p' = p + q = \Big[\frac{Q}{\sqrt{2}}, \frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\Big], \tag{B.5}$$

$$P' = P - p = \Big[0, \frac{(1-x)}{x}\,\frac{Q}{\sqrt{2}}, \boldsymbol{0}_T\Big], \tag{B.6}$$

$$P' + p' = \Big[\frac{Q}{\sqrt{2}}, \frac{xQ_T^2 + (1-x)Q^2}{xQ\sqrt{2}}, \boldsymbol{q}_T\Big]. \tag{B.7}$$

Note that the hadronic mass is given by

$$W^2 = \frac{(1-x)}{x}\,Q^2 \qquad \text{or} \qquad x = \frac{Q^2}{W^2 + Q^2}. \tag{B.8}$$

Specific frames are:
(1) $Q_T = 0$.
In this frame one has for the momenta

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, \frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{1-y}}{y}\,\hat{\ell}_T\Big], \tag{B.9}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{0}_T\Big], \tag{B.10}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \boldsymbol{0}_T\Big], \tag{B.11}$$

$$k' = k - q = \Big[\frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, \frac{1}{y}\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big], \tag{B.12}$$

$$p' = xP + q = \Big[\frac{Q}{\sqrt{2}}, 0, \mathbf{0}_T\Big]. \tag{B.13}$$

(2) $\boldsymbol{q}_T = -Q\sqrt{1-y}\,\hat{\ell}_T$ (HERA frame).
In this frame the electron and target hadron are parallel and one has

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, 0, \mathbf{0}_T\Big], \tag{B.14}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, -y\,\frac{Q}{\sqrt{2}}, -Q\sqrt{1-y}\,\hat{\ell}_T\Big], \tag{B.15}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.16}$$

$$k' = k - q = \Big[\frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, y\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big], \tag{B.17}$$

$$p' = xP + q = \Big[\frac{Q}{\sqrt{2}}, (1-y)\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big]. \tag{B.18}$$

(3) $Q_T = Q$.
In this frame one has in essence that q is transverse, implying for the hadronic momenta

$$q = \Big[\frac{Q}{\sqrt{2}}, 0, Q\,\hat{q}_T\Big], \tag{B.19}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big]. \tag{B.20}$$

(4) $Q_T \downarrow Q$ (Bjorken frame).
First boosting the original momenta by multiplying the plus components with a factor $\sqrt{Q_T^2 - Q^2}/Q$ and the minus components with the inverse factor one obtains

$$q = \Big[\sqrt{\frac{Q_T^2 - Q^2}{2}}, \sqrt{\frac{Q_T^2 - Q^2}{2}}, Q\,\hat{q}_T\Big], \tag{B.21}$$

$$P = \Big[0, \frac{Q^2}{x\sqrt{2(Q_T^2 - Q^2)}}, \mathbf{0}_T\Big]. \tag{B.22}$$

In the limit $Q_T \downarrow Q$ this frame has a purely transverse momentum exchange and an infinite momentum for the target hadron.

B.2 Diffractive scattering

For diffractive scattering one can proceed in steps by starting with the hadron momentum P, taking out a momentum $x_P P$, from which one takes a parton momentum $\beta x_P P$. One has

$$P = \Big[0, \frac{Q}{\beta x_P\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.23}$$

$$k_P = x_P\,P = \Big[0, \frac{Q}{\beta\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.24}$$

$$p = \beta\,k_P = \beta x_P\,P = \Big[0, \frac{Q}{\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.25}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.26}$$

$$p' = p + q = \Big[\frac{Q}{\sqrt{2}}, 0, \mathbf{0}_T\Big]. \tag{B.27}$$

$$k'_P = k_P - p = \left[0, \frac{(1-\beta)}{\beta}\,\frac{Q}{\sqrt{2}}, \mathbf{0}_T\right], \tag{B.28}$$

$$P' = P - k_P = \left[0, \frac{1-x_P}{x_P}\,\frac{Q}{\beta\sqrt{2}}, \mathbf{0}_T\right], \tag{B.29}$$

Note that

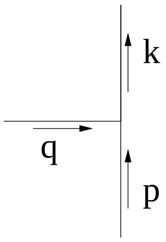
$$W^2 = (p' + p'_P + P')^2 = \frac{1-\beta x_P}{\beta x_P}\,Q^2 \qquad \text{or} \qquad \beta x_P = \frac{Q^2}{Q^2+W^2}, \tag{B.30}$$

$$M_X^2 = (p' + k'_P)^2 = \frac{(1-\beta)}{\beta}\,Q^2 \qquad \text{or} \qquad \beta = \frac{Q^2}{Q^2+M_X^2}. \tag{B.31}$$

Appendix C

Kinematics

C.1 Single parton case: absorption of hard momentum



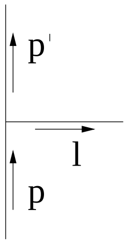
$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$

$$k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

$p^2 = k^2 = 0$

C.2 Single-parton distribution: branching



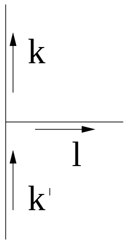
$$p \approx \left[0,\ \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad \boldsymbol{p}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\,\tilde{Q}^2$$

$$p' \approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad p'^2 = -\frac{1}{(1-x_p)}\,\boldsymbol{p}_T^2$$

$p^2 = l^2 = 0$

C.3 Single-parton fragmentation: branching



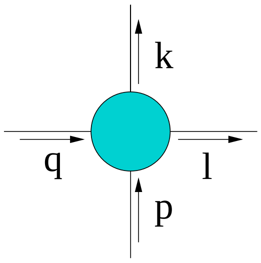
$$k \approx \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T\right] \qquad \boldsymbol{k}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\,\tilde{Q}^2$$

$$k' \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T\right] \qquad k'^2 = \frac{z_k}{(1-z_k)}\,\boldsymbol{k}_T^2$$

$k^2 = l^2 = 0$

C.4 Single-parton case: absorption and branching



$$p \approx \left[0,\ \frac{1}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \qquad q^2 = -Q^2 = -(\tilde{Q}^2 + Q_T^2)$$

$$k \approx \left[z_k\frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right] \qquad (\boldsymbol{q}_T^2 = Q_T^2)$$

$$p^2 = k^2 = l^2 = 0 \qquad l \approx \left[(1-z_k)\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{q}_T\right] \qquad Q_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\tilde{Q}^2$$

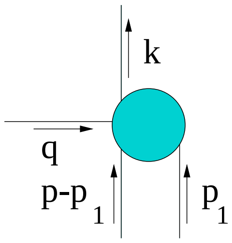
$$\begin{aligned} p_s &= p + q = k + l \\ &\approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \\ p_t &= k - p = q + l \\ &\approx \left[z_k\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{1}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right] \\ p_u &= p - l = k - q \\ &\approx \left[-(1-z_k)\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \end{aligned}$$

$$\hat{s} = p_s^2 = \frac{z_k(1-x_p)}{x_p}\tilde{Q}^2 = \frac{z_k}{(1-z_k)}Q_T^2$$

$$\hat{t} = p_t^2 \ = \ -\frac{z_k}{x_p}\tilde{Q}^2 \ = \ -\frac{z_k}{(1-x_p)(1-z_k)}Q_T^2$$

$$\hat{u} = p_u^2 \ = \ -\frac{(1-z_k)}{x_p}\tilde{Q}^2 \ = \ -\frac{1}{(1-x_p)}Q_T^2$$

C.5 Multi-parton distribution: absorption



$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ x_1\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$

$$k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

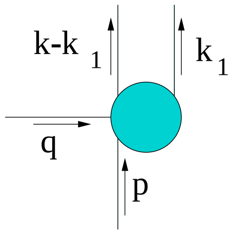
$$p_1^2 = p^2 = p\cdot p_1 = k^2 = 0$$

In general ($p = \sum p_i$):

$$k - p_i \approx \left[\frac{Q}{\sqrt{2}},\ -x_i\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (k-p_i)^2 = -x_i\,Q^2$$

$$q + p_i \approx \left[\frac{Q}{\sqrt{2}},\ -(1-x_i)\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (q+p_i)^2 = -(1-x_i)\,Q^2$$

C.6 Multi-parton fragmentation: absorption



$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$

$$k_1 \approx \left[z_1\,\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

$$k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

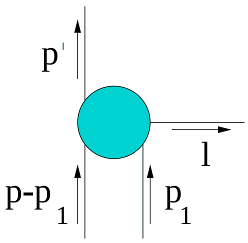
$$k_1^2 = k^2 = k \cdot k_1 = p^2 = 0$$

In general ($k = \sum k_i$):

$$k_i - p \approx \left[z_i\,\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (k_i - p)^2 = -z_i\,Q^2$$

$$k_i - q \approx \left[-(1-z_i)\,\frac{Q}{\sqrt{2}},\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (q - k_i)^2 = -(1-z_i)\,Q^2$$

C.7 Multi-parton distribution: branching



$$p \approx \left[0,\ \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ \frac{x_1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad \boldsymbol{p}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\,\tilde{Q}^2$$

$$p_1^2 = p^2 = p \cdot p_1 = l^2 = 0 \qquad p' \approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad p'^2 = -\frac{1}{1-x_p}\,\boldsymbol{p}_T^2$$

In general ($p = \sum p_i$):

$$l - p_i \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{((1-x_i)-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad (l - p_i)^2 = -\frac{x_i}{(1-x_p)}\,\boldsymbol{p}_T^2$$

$$p' - p_i \approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(x_p - x_i)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad (p' - p_i)^2 = -\frac{(1-x_i)}{(1-x_p)}\,\boldsymbol{p}_T^2$$

Equivalently one can also use the following parametrization (where l defines the minus direction)

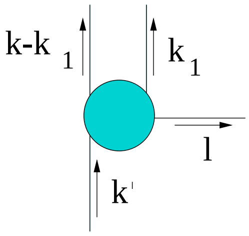
$$p \approx \left[0,\ \frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ x_1\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad l - p_i \approx \left[\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ -x_i\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ -\mathbf{0}_T\right]$$

$$l \approx \left[\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ 0,\ \mathbf{0}_T\right] \qquad p' - p_i \approx \left[-\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ (1-x_i)\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p' \approx \left[-\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ \frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

C.8 Multi-parton fragmentation: branching



$$k \approx \left[z_k \, \frac{\tilde{Q}}{\sqrt{2}}, \; 0, \; \mathbf{0}_T \right]$$

$$k_1 \approx \left[z_1 \, z_k \, \frac{\tilde{Q}}{\sqrt{2}}, \; 0, \; \mathbf{0}_T \right]$$

$$l \approx \left[(1-z_k) \, \frac{\tilde{Q}}{\sqrt{2}}, \; \frac{(1-x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}}, \; \boldsymbol{k}_T \right] \qquad \boldsymbol{k}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p} \, \tilde{Q}^2$$

$$k_1^2 = k^2 = k \cdot k_1 = l^2 = 0 \qquad k' \approx \left[\frac{\tilde{Q}}{\sqrt{2}}, \; \frac{(1-x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}}, \; \boldsymbol{k}_T \right] \qquad k'^2 = \frac{z_k}{(1-z_k)} \, \boldsymbol{k}_T^2$$

In general ($k = \sum k_i$):

$$k_i + l \approx \left[(1-(1-z_i)z_k) \, \frac{\tilde{Q}}{\sqrt{2}}, \; \frac{(1-x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}}, \; \boldsymbol{k}_T \right] \qquad (l+k_i)^2 = -\frac{z_i \, z_k}{(1-z_k)} \, \boldsymbol{k}_T^2$$

$$k_i - k' \approx \left[-(1-z_i \, z_k) \, \frac{\tilde{Q}}{\sqrt{2}}, \; -\frac{(1-x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}}, \; -\boldsymbol{k}_T \right] \qquad (k_i - k')^2 = -\frac{(1-z_i) \, z_k}{(1-z_k)} \, \boldsymbol{k}_T^2$$

Appendix D

Useful formulae

D.1 Combining denominators

Feynman trick:

$$\frac{1}{AB} = \int_0^1 dx \; \frac{1}{(xA + (1-x)B)^2}. \tag{D.1}$$

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \; \delta\left(\sum x_i - 1\right) \; \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n}. \tag{D.2}$$

D.2 Some indefinite integrals

$$(\mathrm{D}=2) \qquad \int dp \; \sqrt{p^2 \pm m^2} = \frac{1}{2} p\sqrt{p^2 \pm M^2} + \frac{1}{2} M^2 \; \ln\left(p + \sqrt{p^2 \pm M^2}\right) \tag{D.3}$$

$$(\mathrm{D}=0) \qquad \int dp \; \frac{1}{\sqrt{p^2 \pm M^2}} = \ln\left(p + \sqrt{p^2 \pm M^2}\right) \tag{D.4}$$

$$(\mathrm{D}=0) \qquad \frac{\pi}{2} \int dp \; \frac{p}{p^2 \pm M^2} = \pi \; \ln\left(p^2 \pm M^2\right) \tag{D.5}$$

$$(\mathrm{D}=\text{-}1) \qquad \int dp \; \frac{1}{p^2 + M^2} = \frac{1}{M} \; \tan^{-1}\left(\frac{p}{M}\right), \tag{D.6}$$

$$(\mathrm{D}=\text{-}1) \qquad \int dp \; \frac{1}{p^2 - M^2} = \frac{1}{M} \; \ln\left|\frac{p-M}{p+M}\right|, \tag{D.7}$$

D.3 Special functions

Beta function

$$\begin{aligned} B(\mu, \nu) &= \int_0^1 dx \; x^{\mu-1}(1-x)^{\nu-1} && \text{(D.8)} \\ &= \int_0^\infty dy \; y^{\nu-1}(1+y)^{-\mu-\nu} && \text{(D.9)} \\ &= 2\int_0^{\pi/2} d\theta \; \sin^{2\mu-1}\theta \, \cos^{2\nu-1}\theta && \text{(D.10)} \\ &= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. && \text{(D.11)} \end{aligned}$$

The Gamma function is defined for $\mathrm{Re}(z) > 0$ as

$$\Gamma(z) = \int_0^\infty dx \; x^{z-1} \, \exp(-x) \tag{D.12}$$

and satisfies $\Gamma(n) = (n-1)!$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$ and the relation

$$z\,\Gamma(z) = \Gamma(z+1). \tag{D.13}$$

The function has simple poles with residue $(-)^n(1/n!)$ at the points $z = -n$. Near negative integers the expansion can be obtained from

$$\begin{aligned} \Gamma(1+\epsilon) &= 1 + \int_0^\infty dx\; (\epsilon\,\ln x + \frac{1}{2}\,\epsilon^2\,\ln^2 x + \ldots)\exp(-x) && \text{(D.14)}\\ &= 1 - \epsilon\,\gamma_E + \epsilon^2\left(\frac{1}{2}\,\gamma_E^2 + \frac{1}{12}\,\pi^2\right) + \ldots, && \text{(D.15)} \end{aligned}$$

where $\gamma_E \approx 0.5772$ is Euler's constant. One has

$$\begin{aligned} \Gamma(z) &\stackrel{z\approx 0}{=} \frac{1}{z} - \gamma_E + \mathcal{O}(z), && \text{(D.16)}\\ \Gamma(z) &\stackrel{z\approx -n}{=} \frac{(-)^n}{n!(z+n)} - \gamma_E + 1 + \ldots + \frac{1}{n} + \mathcal{O}(z+n), && \text{(D.17)} \end{aligned}$$

D.4 Dimensional regularization

Minkowski space integrals can easily be turned into Euclidean integrals using

$$\begin{aligned} &i\,k_E^0 = k^0, && \text{(D.18)}\\ &i\,d^n k_E = d^n k && \text{(D.19)}\\ &-k_E^2 = k^2. && \text{(D.20)} \end{aligned}$$

Basically n-dimensional Euclidean integrals are performed via the angular decomposition and integration

$$\begin{aligned} &\int d^n x = \int_0^\infty dr\; r^{n-1}\int_0^\pi d\theta_{n-1}\;\sin^{n-2}\theta_{n-1}\int_0^\pi d\theta_{n-2}\;\sin^{n-3}\theta_{n-2}\ldots\int_0^{2\pi} d\theta_1, && \text{(D.21)}\\ &\int_0^\pi d\theta\;\sin^m\theta = \sqrt{\pi}\,\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}. && \text{(D.22)} \end{aligned}$$

Radial and one-dimensional integrals are

$$\int d^n x\; f(r) = \frac{2\,\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int dr\; r^{n-1}\,f(r), \tag{D.23}$$

$$\int dx\;\frac{x^\beta}{(x^2+m^2)^\alpha} = \frac{1}{2}\,\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha - \frac{\beta+1}{2}\right)}{\Gamma(\alpha)\,(m^2)^{\alpha-(\beta+1)/2}}. \tag{D.24}$$

A basic integral is

$$\begin{aligned} \int d^n k\,\frac{1}{(k^2 - M^2 + i\epsilon)^s} &= i\int d^n k_E\,\frac{1}{(-k_E^2 - M^2)^s} && \text{(D.25)}\\ &= i\,(-)^s\,\pi^{n/2}\,\frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)}\,(M^2)^{n/2-s} && \text{(D.26)}\\ &= \frac{i\,(-)^s\,\pi^{n/2}}{\left(s-\frac{n}{2}\right)}\,\frac{\Gamma\left(s-\frac{n}{2}\right)}{\Gamma(s)}\,(M^2)^{n/2-s} && \text{(D.27)}\\ &\approx \frac{i\,(-)^s\,\pi^s}{\left(s-\frac{n}{2}\right)}\,\frac{1}{\Gamma(s)}\left[1 - \left(s - \frac{n}{2}\right)[\gamma_E + \ln(\pi\,M^2)] + \ldots\right], && \text{(D.28)} \end{aligned}$$

where the last expansion is near $n = 2s$, or with $n = 2s - \epsilon$,

$$\int d^{2s-\epsilon}k\,\frac{1}{(k^2 - M^2 + i\epsilon)^s} \approx \frac{i\,(-)^s\,\pi^s}{\Gamma(s)}\left[\frac{2}{\epsilon} - \gamma_E - \ln(\pi\,M^2) + \mathcal{O}(\epsilon)\right]. \tag{D.29}$$

Note that one needs to be careful with factors like

$$M^\epsilon = e^{\epsilon \ln M} = 1 + \frac{\epsilon}{2} \ln M^2 + \mathcal{O}(\epsilon),$$
$$\frac{1}{(2\pi)^n} = \frac{1}{(2\pi)^{2s-\epsilon}} = \frac{1}{(2\pi)^s} \left[1 + \frac{\epsilon}{2} \ln(4\pi^2) + \mathcal{O}(\epsilon)\right].$$

Thus one has

$$M^\epsilon \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - M^2 + i\epsilon)^s} \overset{n=2s-\epsilon}{\approx} \frac{i\,(-)^s}{(4\pi)^s \Gamma(s)} \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) + \mathcal{O}(\epsilon)\right]. \tag{D.30}$$

Including numerators we have for instance

$$\int d^n k \, \frac{1}{((k-p)^2 - M^2 + i\epsilon)^s} = i\,(-)^s \, \pi^{n/2} \, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)} \, M^{n-2s} \tag{D.31}$$

$$\int d^n k \, \frac{k_\mu}{((k-p)^2 - M^2 + i\epsilon)^s} = -ip_\mu \,(-)^s \, \pi^{n/2} \, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)} \, M^{n-2s} \tag{D.32}$$

$$\int d^n k \, \frac{k_\mu k_\nu}{((k-p)^2 - M^2 + i\epsilon)^s} = i \left[p_\mu p_\nu - \frac{1}{2} \frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma\left(s - \frac{n}{2}\right)} \, M^2 \, g_{\mu\nu}\right] \times (-)^s \, \pi^{n/2} \, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)} \, M^{n-2s}. \tag{D.33}$$

Actually the above integrals can be obtained by realizing that the proper averaging requires

$$k_\mu k_\nu \longrightarrow \frac{1}{n} \, k^2 \, g_{\mu\nu},$$
$$k_\mu k_\nu k_\rho k_\sigma \longrightarrow \frac{1}{n(n+2)} \, (k^2)^2 \, (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}).$$

This leads to

$$\int d^n k \, \frac{1}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s \, \pi^{n/2} \, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)} \, M^{n-2s}$$
$$\int d^n k \, \frac{k^2}{(k^2 - M^2 + i\epsilon)^s} = -i\,(-)^s \, \pi^{n/2} \, \frac{n}{2} \, \frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma(s)} \, M^{n-2s-2}$$
$$\int d^n k \, \frac{k_\mu k_\nu}{(k^2 - M^2 + i\epsilon)^s} = -i\,(-)^s \, \pi^{n/2} \, \frac{1}{2} \, g_{\mu\nu} \, \frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma(s)} \, M^{n-2s-2}$$
$$\int d^n k \, \frac{(k^2)^2}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s \, \pi^{n/2} \, \frac{n(n+2)}{4} \, \frac{\Gamma\left(s - \frac{n}{2} - 2\right)}{\Gamma(s)} \, M^{n-2s-4}$$
$$\int d^n k \, \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s \, \pi^{n/2} \, \frac{1}{4} \, (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \, \frac{\Gamma\left(s - \frac{n}{2} - 2\right)}{\Gamma(s)} \, M^{n-2s-4}$$

Finally because $g_{\mu\nu} g^{\mu\nu} = g^\mu_\mu = n$ one has in n dimensions

$$\gamma^\mu \gamma_\mu = n, \tag{D.34}$$
$$\gamma_\mu \gamma^\rho \gamma^\mu = -(n-2)\, \gamma^\rho, \tag{D.35}$$
$$\gamma_\mu \gamma^\rho \gamma^\sigma \gamma^\mu = 4\, g^{\rho\sigma} - (4-n)\, \gamma^\rho \gamma^\sigma, \tag{D.36}$$
$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = -2\, \gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\, \gamma^\nu \gamma^\rho \gamma^\sigma. \tag{D.37}$$

D.5 Some useful relations involving distributions

The integral representation of the step functions are

$$\theta(\pm x) = \pm \int \frac{d\alpha}{2\pi\, i} \, \frac{e^{i\alpha x}}{\alpha \mp i\epsilon}, \tag{D.38}$$

$$\epsilon(\pm x) = \mathrm{P} \int \frac{d\alpha}{\pi\, i} \, \frac{e^{i\alpha x}}{\alpha} = \theta(x) - \theta(-x). \tag{D.39}$$

The derivatives satisfy $\theta'(x) = \delta(x)$ and $\epsilon'(x) = 2\,\delta(x)$.
Other useful relations are

$$\frac{1}{x - i\epsilon} = P\,\frac{1}{x} + i\pi\,\delta(x), \tag{D.40}$$

$$\ln(x - i\epsilon) = \ln|x| - i\pi\,\theta(-x). \tag{D.41}$$

In order to treat endpoint singularities in $\int dx\; f(x)/(1-x)$ near $x = 1$ one usually employs the "+" prescription,

$$(F(x))_+ \equiv \lim_{\beta\to 0}\left[F(x)\,\theta(1-\beta-x) + \delta(1-\beta-x)\int_0^{1-\beta} dy\,F(y)\right], \tag{D.42}$$

satisfying $\int_0^1 dx (F(x))_+ = 0$. Note that for an integral not running between 0 and 1 one has

$$\begin{aligned}
\int_x^1 dz\;\frac{G(z)}{(1-z)_+} &= \int_x^{1-\beta} dz\;\frac{G(z)}{(1-z)} - G(1-\beta)\int_0^{1-\beta} dz\;\frac{1}{1-z}\\
&= \int_x^1 dz\;\frac{G(z)-G(1)}{1-z} - G(1)\int_0^x dz\;\frac{1}{1-z}\\
&= \int_x^1 dz\;\frac{G(z)-G(1)}{1-z} + G(1)\,\ln(1-x)
\end{aligned} \tag{D.43}$$

One thus has regularized via the endpoint of the integration leading to

$$\frac{1}{1-x} = -\ln\beta\;\delta(1-x) + \frac{1}{(1-x)_+} + \mathcal{O}(\beta). \tag{D.44}$$

One can also use

$$\frac{1}{(1-x)^{1-\epsilon}} = -\frac{1}{\epsilon}\,\delta(1-x) + \frac{1}{(1-x)_+} + \mathcal{O}(\epsilon).$$

To see how this works, consider for a small ϵ

$$\begin{aligned}
\int_x^1 dx\,\frac{G(z)}{(1-z)^{1-\epsilon}} &= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)^{1-\epsilon}} + G(1)\int_x^1 dx\,\frac{1}{(1-z)^{1-\epsilon}}\\
&= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)} + G(1)\int_0^{1-x} dy\;y^{\epsilon-1}\\
&= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)} - G(1)\,\frac{1}{\epsilon} + G(1)\,\ln(1-x).
\end{aligned}$$

Another useful "+" function is

$$\int_x^1 dz\;G(z)\left(\frac{\ln(1-z)}{1-z}\right)_+ = \int_x^1 dz\;\frac{(G(z)-G(1))\ln(1-z)}{1-z} + \frac{1}{2}\,G(1)\,\ln^2(1-x). \tag{D.45}$$

D.6 Theta functions

Often it is useful to attack loop-integrals via lightcone variables leading to specific integrals of the type

$$\Theta^m_{n_1 n_2 \ldots}(x_1, x_2, \ldots) = \int \frac{d\alpha}{2\pi\,i}\;\frac{\alpha^m}{(\alpha x_1 - 1 + i\epsilon)^{n_1}(\alpha x_2 - 1 + i\epsilon)^{n_2}\ldots}. \tag{D.46}$$

In these integrals m can be reduced via

$$\Theta^m_{111\ldots}(x_1, x_2, x_3, \ldots) = \frac{1}{(x_1 - x_2)}\left[\Theta^{(m-1)}_{11\ldots}(x_2, x_3, \ldots) - \Theta^{(m-1)}_{11\ldots}(x_1, x_3, \ldots)\right]. \tag{D.47}$$

Proof:

$$\begin{aligned}
\Theta^m_{111}(x_1, x_2, x_3) &= \int \frac{d\alpha}{2\pi\,i}\;\frac{\alpha}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_3 - 1 + i\epsilon)}\\
&= \frac{1}{(x_1 - x_2)}\int \frac{d\alpha}{2\pi\,i}\;\frac{(\alpha x_1 - 1) - (\alpha x_2 - 1)}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_3 - 1 + i\epsilon)}
\end{aligned}$$

The lower index can be lowered together with an upper index via

$$\Theta^{m}_{n_1\ldots}(x_1,\ldots) = -(n_1-1)\frac{d}{dx_1}\,\Theta^{(m-1)}_{(n_1-1)\ldots}(x_1,\ldots). \tag{D.48}$$

Proof:

$$\begin{aligned}\Theta^{1}_{21}(x_1,x_2) &= \int\frac{d\alpha}{2\pi\, i}\,\frac{\alpha}{(\alpha x_1-1+i\epsilon)^2(\alpha x_2-1+i\epsilon)}\\ &= -\frac{d}{dx_1}\int\frac{d\alpha}{2\pi\, i}\,\frac{1}{(\alpha x_1-1+i\epsilon)(\alpha x_2-1+i\epsilon)}\end{aligned}$$

The reduction of $\Theta^0_{111\ldots}$ to $\Theta_{11\ldots}$ is achieved via

$$\Theta^0_{111\ldots}(x_1,x_2,x_3,\ldots) = \frac{x_2\,\Theta^0_{11\ldots}(x_2,x_3) - x_1\,\Theta^0_{11\ldots}(x_1,x_3)}{x_1-x_2}. \tag{D.49}$$

Proof:

$$\begin{aligned}\Theta^0_{111\ldots}(x_1,x_2,x_3) &= \int\frac{d\alpha}{2\pi\, i}\,\frac{1}{(\alpha x_1-1+i\epsilon)(\alpha x_2-1+i\epsilon)(\alpha x_1-1+i\epsilon)}\\ &= \int\frac{d\alpha}{2\pi\, i}\,\frac{\alpha x_1-(\alpha x_1-1)}{(\alpha x_1-1+i\epsilon)(\alpha x_2-1+i\epsilon)(\alpha x_1-1+i\epsilon)}\\ &= x_1\,\Theta^1_{111\ldots}(x_1,x_2,x_3) - \Theta^0_{11}(x_2,x_3)\end{aligned}$$

and using the previous relation.

Finally we can reduce $n=2$ (for $m=0$) via

$$\int_{x_0}^{x_1} dy\;\Theta^0_{21}(y,x_2) = (x_1-x_0)\,\Theta^0_{111}(x_0,x_1,x_2) \tag{D.50}$$

Proof:

$$\begin{aligned}\int_{x_0}^{x_1} dy\;\Theta^0_{21}(y,x_2) &= \int_{x_0}^{x_1} dy\int\frac{d\alpha}{2\pi\, i}\,\frac{1}{(\alpha y-1+i\epsilon)^2(\alpha x_2-1+i\epsilon)}\\ &= \int\frac{d\alpha}{2\pi\, i}\,\frac{1}{\alpha}\int_{\alpha x_0}^{\alpha x_1} d(\alpha y)\;\frac{1}{(\alpha y-1+i\epsilon)^2(\alpha x_2-1+i\epsilon)}\end{aligned}$$

and then using that

$$\frac{1}{(\alpha x_0-1)} - \frac{1}{(\alpha x_1-1)} = \frac{\alpha(x_1-x_0)}{(\alpha x_0-1)(\alpha x_1-1)}$$

Further reduction for $n=2$ and $m=0$ then is possible, giving

$$\Theta^0_{21}(x_1,x_2) = -\frac{d}{dx_1}\left[x_1\,\Theta^0_{11}(x_1,x_2)\right]. \tag{D.51}$$

The only integral to be calculated actually is $\Theta^0_{11}(x_1,x_2)$, which is easily done via a contour integration in the complex plane. The explicit results for the simplest functions are then

$$\Theta^0_{11}(x_1,x_2) = \frac{\theta(x_1)\theta(-x_2)-\theta(-x_1)\theta(x_2)}{x_1-x_2} = \frac{\theta(x_1)-\theta(x_2)}{x_1-x_2}, \tag{D.52}$$

$$\Theta^0_2(x) = \delta(x), \tag{D.53}$$

$$\Theta^0_{111}(x_1,x_2,x_3) = \frac{x_2}{(x_1-x_2)}\,\Theta^0_{11}(x_2,x_3) - \frac{x_1}{(x_1-x_2)}\,\Theta^0_{11}(x_1,x_3), \tag{D.54}$$

$$\Theta^0_{21}(x_1,x_2) = \frac{x_2}{(x_1-x_2)}\,\Theta^0_{11}(x_1,x_2) - \frac{x_1}{(x_1-x_2)}\,\delta(x_1), \tag{D.55}$$

$$\Theta^1_{21}(x_1,x_2) = \frac{1}{(x_1-x_2)}\,\Theta^0_{11}(x_1,x_2) - \frac{1}{(x_1-x_2)}\,\delta(x_1), \tag{D.56}$$

$$\Theta^1_{111}(x_1,x_2,x_3) = \frac{1}{(x_1-x_2)}\left[\Theta^0_{11}(x_2,x_3) - \Theta^0_{11}(x_1,x_3)\right]. \tag{D.57}$$

We also will use the following equality for Θ^0_{11} functions, seen using a principal value prescription,

$$\int dy\ \frac{x}{x-y}\left[\Theta^0_{11}(y,y-x)+\Theta^0_{11}(x,x-y)\right]=0. \tag{D.58}$$

The singularities in the two terms come from $y\uparrow x$ (first term) and $y\downarrow x$ (right term) respectively.
Notes on the $i\epsilon$ prescription:

$$\Theta^0_{11}(x_1,x_2)=\int\frac{d\alpha}{2\pi\,i}\ \frac{1}{(\alpha x_1-1+i\epsilon)(\alpha x_2-1+i\epsilon)}=\frac{1}{x_1-x_2}\left[\theta(x_1)\,\theta(-x_2)-\theta(-x_1)\theta(x_2)\right]$$
$$\Theta^0_{\underline{1}1}(x_1,x_2)=\int\frac{d\alpha}{2\pi\,i}\ \frac{1}{(\alpha x_1-1-i\epsilon)(\alpha x_2-1+i\epsilon)}=\frac{1}{x_1-x_2}\left[\theta(x_1)\,\theta(x_2)-\theta(-x_1)\theta(-x_2)\right]$$

D.7 IR and UV singularities

In a $D=0$ divergent integral one can separate the singularities e.g. as follows

$$\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\quad=\quad\int d^2k_T\left[\frac{M^2}{\boldsymbol{k}_T^2(\boldsymbol{k}_T^2+M^2)}+\frac{1}{\boldsymbol{k}_T^2+M^2}\right]. \tag{D.59}$$

Regularized one sees that

$$M^\epsilon\int\frac{d^{2-\epsilon}k_T}{\boldsymbol{k}_T^2}\quad=\quad\underbrace{M^\epsilon\int d^{2-\epsilon}k_T\ \frac{M^2}{\boldsymbol{k}_T^2(\boldsymbol{k}_T^2+M^2)}}_{\substack{-2\pi/\epsilon\\ \text{(IR-divergence)}}}+\underbrace{M^\epsilon\int d^{2-\epsilon}k_T\ \frac{1}{\boldsymbol{k}_T^2+M^2}}_{\substack{2\pi/\epsilon\\ \text{(UV-divergence)}}} \tag{D.60}$$

Appendix E

Spin

E.1 Density matrices: definition and example for spin 1/2

In many applications in quantum mechanics one does not have a pure state to begin with. An impure state is described with a density operator

$$\rho = \sum_i |i\rangle\, p_i\, \langle i|, \tag{E.1}$$

where $|i\rangle$ are pure states, not necessarily orthogonal, and p_i are the probabilities (e.g. a beam of spin 1/2 electrons with 50 % spin along the z-axis, 25 % spin along x-axis and 25 % spin along y-axis). We do know

$$0 \le p_i \le 1 \qquad \text{and} \qquad \sum_i p_i = 1.$$

It is straightforward to obtain the following properties:

- $\mathrm{Tr}\,\rho = \sum_i p_i = 1$,
- $\langle A\rangle = \sum_i p_i \langle i|A|i\rangle = \mathrm{Tr}\,(\rho A) = \mathrm{Tr}\,(A\rho)$ and $\mathrm{Tr}\,(\rho^2) \le 1$,
- ρ is a positive definite, self-adjoint ($\rho^\dagger = \rho$) operator,
- For a pure state $\rho^2 = \rho \Longleftrightarrow \mathrm{Tr}\,(\rho^2) = 1$. In that case ρ is a projection operator.

E.2 Spherical tensor operators

More general, for spin s one can use the spherical tensor operators R^L_M, defined as

$$\begin{aligned}
\langle s, m|R^L_M|s, m'\rangle = \left(R^L_M\right)_{mm'} &= \sqrt{2L+1}\; C^{s\ L\ s}_{m'Mm} \\
&= (-)^L \sqrt{2L+1}\; C^{L\ s\ s}_{Mm'm} \\
&= (-)^{s-m'} \sqrt{2s+1}\; C^{s\ \ s\ L}_{m-m'M} \\
&= (-)^{s-m} \sqrt{(2s+1)(2L+1)} \begin{pmatrix} s & s & L \\ m & -m' & -M \end{pmatrix}.
\end{aligned} \tag{E.2}$$

The last expression involves the so-called $3j$-symbol. The properties of these tensor operators are:

- $\mathrm{Tr}\ R^L_M = (2s+1)\,\delta_{L0}\,\delta_{M0}$,
- $\mathrm{Tr}\ (R^{L'}_{M'} R^{L\,\dagger}_M) = (2s+1)\,\delta_{LL'}\,\delta_{MM'}$,
- R^L_M are real $(2s+1)\times(2s+1)$ matrices,
- $R^{L\,\dagger}_M = (-)^M R^L_{-M}$.

Expanding the density matrix ρ,

$$\rho \equiv \frac{1}{2s+1}\sum_{L=0}^{2s}\sum_{M=-L}^{L} \rho_M^{L\,*} R_M^L = \frac{1}{2s+1}\sum_{L=0}^{2s}\sum_{M=-L}^{L} \rho_M^L R_M^{L\,\dagger}, \tag{E.3}$$

the quantities ρ_M^L are (complex) numbers satisfying

- $\rho_M^L = \text{Tr}\,(\rho R_M^L)$,
- $\rho_M^{L\,*} = (-)^M \rho_{-M}^L$, i.e. ρ_0^L real (because ρ is self-adjoint),
- $\rho_0^0 = 1$ (because Tr $\rho = 1$)
- $\sum_{L,M} |\rho_M^L|^2 \leq 2s+1$ (because Tr $\rho^2 \leq 1$).

E.3 Spin 1/2

Besides the tensor operators a Cartesian set is used, the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{E.4}$$

The explicit tensor operators and their relation to the Cartesian operators are

$$R_0^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}, \tag{E.5}$$

$$R_1^1 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix} = -\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y), \tag{E.6}$$

$$R_0^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \tag{E.7}$$

$$R_{-1}^1 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y), \tag{E.8}$$

The R_m^1 are precisely the spherical components of the spin vector $\boldsymbol{\sigma}$. The explicit form of the density matrix is

$$\rho = \frac{1}{2}(\mathbf{1} + \boldsymbol{\sigma}\cdot\boldsymbol{P}) = \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - i\,P_y \\ P_x + i\,P_y & 1-P_z \end{pmatrix} \tag{E.9}$$

(E.10)

$$= \frac{1}{2}\sum_{L=0}^{1}\sum_{M=-L}^{L} \rho_M^L R_M^{L\,\dagger} = \frac{1}{2}\begin{pmatrix} 1+\rho_0^1 & \rho_{-1}^1\sqrt{2} \\ -\rho_1^1\sqrt{2} & 1-\rho_0^1 \end{pmatrix}. \tag{E.11}$$

The vector $\boldsymbol{P}$ is called the polarization vector or also the spin vector of a state. For a pure state $|\boldsymbol{P}| = 1$, for an unpolarized state $|\boldsymbol{P}| = 0$ corresponding with $\rho = \frac{1}{2}\mathbf{1}$. The numbers ρ_m^1 are just the spherical components of the polarization vector,

$$\rho_1^1 = -\frac{1}{\sqrt{2}}(P_x + iP_y), \tag{E.12}$$

$$\rho_0^1 = P_z, \tag{E.13}$$

$$\rho_{-1}^1 = \frac{1}{\sqrt{2}}(P_x - iP_y). \tag{E.14}$$

One has $\text{Tr}\,\rho^2 = \frac{1}{2}(1+\boldsymbol{P}^2) \leq 1$. The degree of polarization is $0 \leq |\boldsymbol{P}| \leq 1$.

We note that any matrix $\tilde{M}$ in the spin-space can be transformed into a function depending on the polarization vector $\boldsymbol{P}$, which we write as

$$(P^i) = (S_T^1, S_T^2, S_L). \tag{E.15}$$

Explicitly we have

$$M(S) = \mathrm{Tr}\,\left(\rho(S)\,\tilde{M}\right). \tag{E.16}$$

where the density matrix is written as

$$\rho(S) = \frac{1}{2}\left(\begin{array}{cc} 1+S_L & S_T^1 - i\,S_T^2 \\ S_T^1 + i\,S_T^2 & 1-S_L \end{array}\right) = \left(\begin{array}{cc} \frac{1}{2}+\frac{1}{2}\,S_L & \frac{1}{\sqrt{2}}\,S_T^- \\ -\frac{1}{\sqrt{2}}\,S_T^+ & \frac{1}{2}-\frac{1}{2}\,S_L \end{array}\right). \tag{E.17}$$

The equivalence is

$$M(S) = M_O + S_L\,M_L + S_T^1\,M_T^1 + S_T^2\,M_T^2 = M_O + S_L\,M_L - S_T^+\,M_T^- - S_T^-\,M_T^+ \tag{E.18}$$

$$\tilde{M}_{mm'} = \left(\begin{array}{cc} M_O+M_L & M_T^-\sqrt{2} \\ -M_T^+\sqrt{2} & M_O-M_L \end{array}\right) = \left(\begin{array}{cc} M_O+M_L & M_T^1 - i\,M_T^2 \\ M_T^1 + i\,M_T^2 & M_O-M_L \end{array}\right). \tag{E.19}$$

if the matrix $\tilde{M}$ is written on the basis of eigenstates of σ_z.

The parameters in the density matrix in Eq. E.17 can be given an explicit probabilistic interpretation. Introducing $p_m(\theta,\phi)$ as the probabilities to have spin-component m along the direction specified by θ and ϕ, we have

$$P_z = S_L = p_{1/2}(\hat{z}) - p_{-1/2}(\hat{z}) \tag{E.20}$$

$$P_x = S_T^1 = p_{1/2}(\hat{x}) - p_{-1/2}(\hat{x}) \tag{E.21}$$

$$P_y = S_T^2 = p_{1/2}(\hat{y}) - p_{-1/2}(\hat{y}), \tag{E.22}$$

showing all these parameters to lie in the interval $[-1,1]$.

E.4 Spin 1

A Cartesian set transforming like a vector is given by the matrices

$$\Sigma_x = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right), \quad \Sigma_y = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{array}\right), \quad \Sigma_z = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right). \tag{E.23}$$

A set of tensor operators of rank two (symmetric and traceless) is the set $\Sigma_{ij} = \frac{1}{2}\,\Sigma_{\{i}\Sigma_{j\}} - \frac{2}{3}\,\delta_{ij}\,\mathbf{1}$,

$$\Sigma_{xx} = \frac{1}{6}\left(\begin{array}{ccc} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{array}\right), \quad \Sigma_{yy} = \frac{1}{6}\left(\begin{array}{ccc} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{array}\right), \quad \Sigma_{zz} = \frac{1}{3}\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{array}\right) \tag{E.24}$$

$$\Sigma_{xy} = \frac{1}{2}\left(\begin{array}{ccc} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{array}\right), \; \Sigma_{xz} = \frac{1}{2\sqrt{2}}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right), \; \Sigma_{yz} = \frac{1}{2\sqrt{2}}\left(\begin{array}{ccc} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{array}\right). \tag{E.25}$$

The explicit tensor operators and their relation to the Cartesian operators are

$$R_0^0 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) = \mathbf{1} \tag{E.26}$$

$$R_1^1 = \sqrt{\tfrac{3}{2}}\left(\begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right) = -\sqrt{\frac{3}{4}}\,(\Sigma_x + i\,\Sigma_y), \tag{E.27}$$

$$R_0^1 = \sqrt{\tfrac{3}{2}}\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right) = \sqrt{\frac{3}{2}}\,\Sigma_z \tag{E.28}$$

$$R^1_{-1} = \sqrt{\tfrac{3}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\left(\Sigma_x - i\,\Sigma_y\right), \tag{E.29}$$

$$R^2_2 = \sqrt{3}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\left(\Sigma_{xx} - \Sigma_{yy} + 2i\,\Sigma_{xy}\right) \tag{E.30}$$

$$R^2_1 = \sqrt{\tfrac{3}{2}}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = -\sqrt{3}\left(\Sigma_{xz} + i\,\Sigma_{yz}\right) \tag{E.31}$$

$$R^2_0 = \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sqrt{\frac{9}{2}}\,\Sigma_{zz} \tag{E.32}$$

$$R^2_{-1} = \sqrt{\tfrac{3}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \sqrt{3}\left(\Sigma_{xz} - i\,\Sigma_{yz}\right) \tag{E.33}$$

$$R^2_{-2} = \sqrt{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\left(\Sigma_{xx} - \Sigma_{yy} - 2i\,\Sigma_{xy}\right) \tag{E.34}$$

The explicit form of the density matrix can be written in one of the following forms,

$$\rho = \frac{1}{3}\left(\mathbf{1} + \frac{3}{2}\,\Sigma_i P_i + 3\,\Sigma_{ij} T_{ij}\right) \tag{E.35}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{P_z}{2} + \frac{T_{zz}}{2} & \frac{P_x - i\,P_y}{2\sqrt{2}} + \frac{T_{xz} - i\,T_{yz}}{\sqrt{2}} & \frac{T_{xx} - T_{yy} - 2i\,T_{xy}}{2} \\ \frac{P_x + i\,P_y}{2\sqrt{2}} + \frac{T_{xz} + i\,T_{yz}}{\sqrt{2}} & \frac{1}{3} - T_{zz} & \frac{P_x - i\,P_y}{2\sqrt{2}} - \frac{T_{xz} - i\,T_{yz}}{\sqrt{2}} \\ \frac{T_{xx} - T_{yy} + 2i\,T_{xy}}{2} & \frac{P_x + i\,P_y}{2\sqrt{2}} - \frac{T_{xz} + i\,T_{yz}}{\sqrt{2}} & \frac{1}{3} - \frac{P_z}{2} + \frac{T_{zz}}{2} \end{pmatrix} \tag{E.36}$$

$$= \frac{1}{3}\sum_{L=0}^{2}\sum_{M=-L}^{L} \rho^L_M {R^L_M}^\dagger \tag{E.37}$$

$$= \frac{1}{3}\begin{pmatrix} 1 + \rho^1_0\sqrt{\frac{3}{2}} + \rho^2_0\sqrt{\frac{1}{2}} & \rho^1_{-1}\sqrt{\frac{3}{2}} + \rho^2_{-1}\sqrt{\frac{3}{2}} & \rho^2_{-2}\sqrt{3} \\ -\rho^1_1\sqrt{\frac{3}{2}} - \rho^2_1\sqrt{\frac{3}{2}} & \rho^2_0\sqrt{2} & \rho^1_{-1}\sqrt{\frac{3}{2}} - \rho^2_{-1}\sqrt{\frac{3}{2}} \\ \rho^2_2\sqrt{3} & -\rho^1_1\sqrt{\frac{3}{2}} + \rho^2_1\sqrt{\frac{3}{2}} & 1 - \rho^1_0\sqrt{\frac{3}{2}} + \rho^2_0\sqrt{\frac{1}{2}} \end{pmatrix}, \tag{E.38}$$

where T is a traceless, symmetric tensor. One identifies

$$\rho^1_1 = -\sqrt{\frac{3}{4}}\left(P_x + iP_y\right), \tag{E.39}$$

$$\rho^1_0 = \sqrt{\frac{3}{2}}\,P_z, \tag{E.40}$$

$$\rho^1_{-1} = \sqrt{\frac{3}{4}}\left(P_x - iP_y\right), \tag{E.41}$$

$$\rho^2_2 = \sqrt{\frac{3}{4}}\left(T_{xx} - T_{yy} + 2i\,T_{xy}\right) \tag{E.42}$$

$$\rho^2_1 = -\sqrt{3}\left(T_{xz} + i\,T_{yz}\right), \tag{E.43}$$

$$\rho^2_0 = \sqrt{\frac{9}{2}}\,T_{zz}, \tag{E.44}$$

$$\rho^2_{-1} = \sqrt{3}\,(T_{xz} - i\,T_{yz})\,, \tag{E.45}$$

$$\rho^2_{-2} = \sqrt{\frac{3}{4}}\,(T_{xx} - T_{yy} - 2i\,T_{xy}) \tag{E.46}$$

One has $\mathrm{Tr}\,\rho^2 = \frac{1}{3}(1 + \frac{3}{2}\,P^iP^i + 3\,T^{ij}T^{ij}) \le 1$. The degree of polarization is defined as $0 \le [\frac{3}{4}\,P_iP_i + \frac{3}{2}\,T_{ij}T_{ij}]^{1/2} \le 1$.

Again one can establish the equivalence between a spin-dependent function $M(S)$ and a matrix $\tilde{M}$ in spin-space. If

$$(P^i) = (S_T^1, S_T^2, S_L) \tag{E.47}$$

$$(T^{ij}) = \frac{1}{2}\begin{pmatrix} S_{TT}^{11} - \frac{2}{3}\,S_{LL} & S_{TT}^{12} & S_{LT}^1 \\ S_{TT}^{21} & S_{TT}^{22} - \frac{2}{3}\,S_{LL} & S_{LT}^2 \\ S_{LT}^1 & S_{LT}^2 & \frac{4}{3}\,S_{LL} \end{pmatrix}, \tag{E.48}$$

with $S_{TT}^{22} = -S_{TT}^{11}$ and $S_{TT}^{12} = S_{TT}^{21}$. Explicitly we have

$$M(S) = \mathrm{Tr}\,\left(\rho(S)\,\tilde{M}\right). \tag{E.49}$$

where the density matrix is written as

$$\rho(S) = \begin{pmatrix} \frac{1}{3} + \frac{1}{2}\,S_L + \frac{1}{3}\,S_{LL} & \frac{1}{2}\,(S_T^- + S_{LT}^-) & \frac{1}{\sqrt{2}}\,S_{TT}^{1-} \\ -\frac{1}{2}\,(S_T^+ + S_{LT}^+) & \frac{1}{3} - \frac{2}{3}\,S_{LL} & \frac{1}{2}\,(S_T^- - S_{LT}^-) \\ -\frac{1}{\sqrt{2}}\,S_{TT}^{1+} & -\frac{1}{2}\,(S_T^+ - S_{LT}^+) & \frac{1}{3} - \frac{1}{2}\,S_L + \frac{1}{3}\,S_{LL} \end{pmatrix}. \tag{E.50}$$

Note that $\frac{1}{\sqrt{2}}\,S_{TT}^{1-} = \frac{1}{2}(S_{TT}^{11} - i\,S_{TT}^{12}) = \frac{1}{4}(S_{TT}^{11} - S_{TT}^{22} - 2i\,S_{TT}^{12})$. The equivalence is

$$M(S) = M_O + S_L\,M_L + 2\,S_{LL}M_{LL} + S_T^i\,M_T^i + S_{LT}^i\,M_{LT}^i + S_{TT}^{1i}\,M_{TT}^{1i} \tag{E.51}$$

$$\tilde{M}_{mm'} = \begin{pmatrix} M_O + M_L + M_{LL} & M_T^- + M_{LT}^- & M_{TT}^{1-}\sqrt{2} \\ -(M_T^+ + M_{LT}^+) & M_O - 2\,M_{LL} & M_T^- - M_{LT}^- \\ -M_{TT}^{1+}\sqrt{2} & -(M_T^+ - M_{LT}^+) & M_O - M_L + M_{LL} \end{pmatrix} \tag{E.52}$$

if the matrix $\tilde{M}$ is written on the basis of eigenstates of Σ_z.

The parameters in the density matrix in Eq. E.50 can be given an explicit probabilistic interpretation. Introducing $p_m(\theta,\phi)$ as the probabilities to have spin-component m along the direction specified by θ and ϕ, we see from the diagonal elements

$$S_{LL} = \frac{1}{2}\left[p_1(\hat{z}) + p_{-1}(\hat{z})\right] - p_0(\hat{z}), \tag{E.53}$$

$$S_L = p_1(\hat{z}) - p_{-1}(\hat{z}), \tag{E.54}$$

implying $-1 \le S_{LL} \le 1/2$ and $-1 \le S_L \le 1$.

E.5 Reaction parameters

In general one can express the initial state density matrix as

$$\rho_i \equiv \frac{1}{2s_i + 1}\sum_{\ell,m}(\rho_i)^\ell_m R^{\ell\,\dagger}_m, \tag{E.55}$$

and obtain from that, given a transition matrix T, the final state density matrix

$$\rho_f = \frac{\mathcal{T}\,\rho_i\,\mathcal{T}^\dagger}{\mathrm{Tr}\,(\mathcal{T}\,\rho_i\,\mathcal{T}^\dagger)} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\,\mathcal{T}\,R^\ell_m{}^\dagger\,\mathcal{T}^\dagger}{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathcal{T}\,R^\ell_m{}^\dagger\,\mathcal{T}^\dagger\right)} \tag{E.56}$$

$$= \frac{1}{2s_f+1}\sum_{\ell',m'}(\rho_f)^{\ell'}_{m'}R^{\ell'}_{m'}{}^\dagger, \tag{E.57}$$

with

$$(\rho_f)^{\ell'}_{m'} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathcal{T}\,R^\ell_m{}^\dagger\,\mathcal{T}^\dagger\,R^{\ell'}_{m'}\right)}{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathcal{T}\,R^\ell_m{}^\dagger\,\mathcal{T}^\dagger\right)}. \tag{E.58}$$

Defining spin transfer parameters

$$\langle\ell',m'|\ell,m\rangle \equiv \frac{\mathrm{Tr}\left(\mathcal{T}\,R^\ell_m{}^\dagger\,\mathcal{T}^\dagger\,R^{\ell'}_{m'}\right)}{\mathrm{Tr}\,(\mathcal{T}\,\mathcal{T}^\dagger)}, \tag{E.59}$$

one gets

$$(\rho_f)^{\ell'}_{m'} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\;\langle\ell',m'|\ell,m\rangle}{\sum_{\ell,m}(\rho_i)^\ell_m\;\langle 0,0|\ell,m\rangle}. \tag{E.60}$$

To illustrate the reaction parameters, let us consider the reaction parameters for a process with one spin 1/2 particle in initial and one spin 1/2 particle in the final state. in that case it is actually more common to use the Euclidean vector notation instead of the tensor operators. With the 2×2 scattering matrix given by $\mathcal{T}$ and

$$\rho_i = \frac{1}{2}\,(\mathbf{1}+\boldsymbol{\sigma}\cdot\boldsymbol{P}_{\mathrm{in}})\,,$$

one finds

$$\rho_f = \frac{\mathcal{T}\,\rho_i\,\mathcal{T}^\dagger}{\mathrm{Tr}\,(\mathcal{T}\,\rho_i\,\mathcal{T}^\dagger)} = \frac{\mathcal{T}\,\mathcal{T}^\dagger + P^i_{\mathrm{in}}\,\mathcal{T}\,\sigma^i\,\mathcal{T}^\dagger}{\mathrm{Tr}\left(\mathcal{T}\,\mathcal{T}^\dagger + P^i_{\mathrm{in}}\,\mathcal{T}\,\sigma^i\,\mathcal{T}^\dagger\right)} \tag{E.61}$$

or introducing the C^{ij} spin transfer parameters,

$$C^{ij} = \frac{\mathrm{Tr}\left(\mathcal{T}\,\sigma^i\,\mathcal{T}^\dagger\,\sigma^j\right)}{\mathrm{Tr}\,(\mathcal{T}\,\mathcal{T}^\dagger)}, \tag{E.62}$$

with arguments $i,j = 0,1,2,3$ using also the matrix $\sigma^0 \equiv \mathbf{1}$ one obtains:

$$\rho_f = \frac{1}{2}\,(\mathbf{1}+\boldsymbol{\sigma}\cdot\boldsymbol{P}_{\mathrm{out}})\,,$$

with

$$P^i_{\mathrm{out}} = \frac{C^{0i}+P^j_{\mathrm{in}}\,C^{ji}}{C^{00}+P^k_{\mathrm{in}}\,C^{k0}} \tag{E.63}$$

($C^{00}=1$). Note that the final state density matrix ρ_f can be used as the input density matrix for a decay process, in this way enabling polarimetry.

E.6 Spin vectors

In situations such as deep inelastic scattering one likes to work with spin vectors in the cross sections just as with the momenta, e.g. one considers the hadronic tensor $W^{\mu\nu}(q,P,S)$, where q is the virtual photon momentum, P is the target momentum and S the spin vector. We will discuss the meaning of such spin vectors in initial and final state.

We start with a hadron in the initial state with momentum P, which is specified by a *density matrix*, for an ensemble of initial states indicated with α,

$$\begin{aligned}(\rho_i)_{\alpha\beta}(P,S) &= \sum_i \langle P;\alpha|i\rangle\, p_i\, \langle i|P;\beta\rangle \\ &\overset{\text{H rest frame}}{\equiv} \frac{1}{2}\Big(\delta_{\alpha\beta} + \boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{S}\Big)\end{aligned} \tag{E.64}$$

Note that this in a general frame generalizes to a vector $S^\mu(P)$ which satisfies $P\cdot S = 0$ and $-1 \le S^2 \le 0$.

Then, let's assume that a particle h is produced in the final state with momentum P_h, which is analyzed via its decay products, e.g. a Λ decaying into $p\pi^-$. In that case the probability of finding a specific final state configuration (f) is contained in the *decay matrix*

$$\begin{aligned}R^{(\text{decay})}_{\alpha'\beta'}(f) &\propto \sum_{\lambda_f} \mathcal{T}^*_{\beta'\to\lambda_f}(f)\,\mathcal{T}_{\alpha'\to\lambda_f}(f) \\ &= \sum_{\lambda_f} \langle P_h;\alpha'|\mathcal{T}^\dagger|f;\lambda_f\rangle\,\langle f;\lambda_f|\mathcal{T}|P_h;\beta'\rangle.\end{aligned} \tag{E.65}$$

In this matrix one has summed over all polarizations λ_f in the decay channel (in the example proton polarizations). The decay matrix depends on the phase space (f) of the decay channel (in the example $\theta^{cm}_{p\pi^+}$). This and other examples are given below. In general we may write

$$R^{(\text{decay})}_{\alpha'\beta'}(P_h,f) \overset{\text{h rest frame}}{\equiv} w(f)\,\Big(\delta_{\alpha'\beta'} + \boldsymbol{\sigma}_{\alpha'\beta'}\cdot\boldsymbol{A}_h(f)\Big), \tag{E.66}$$

normalized to $\sum_f R_{\alpha'\beta'}(f) = \delta_{\alpha'\beta'}$ if $\sum_f w(f) = 1$. Note that the summation over f is just symbolic for all kinematic variables (usually angles) appearing in the decay. Sometimes one only integrates over a particular subset in f, in which case the righthandside is multiplied with a function depending on the remaining variables. This defines A_h as the analyzing power of the decay channel. Note that A_h depends on P_h and f, i.e. $A_h(P_h,f)$. In general it satisfies $P_h\cdot A_h = 0$. For a decaying particle h with a polarization state determined by a density matrix $\rho(P_h,S_h)$ defined in the same way as ρ_{in}, with $P_h\cdot S_h = 0$ and $-1\le S_h^2 \le 0$, the probability of a final state is given by

$$W(h\to f) = \text{Tr}\left(\rho(P_h,S_h)\,R^{(\text{decay})}(P_h,f)\right) = w(f)\,[1 - S_h\cdot A_h(P_h,f)]\,. \tag{E.67}$$

Starting with some (general) initial state, the result for a semi-inclusive measurement in which the decay products of hadron h are detected needs the production matrix,

$$\begin{aligned}R^{(\text{prod})}_{\alpha\beta;\alpha'\beta'}(P;P_h) &= \mathcal{T}^*_{\beta\to\beta'}(P,P_h)\,\mathcal{T}_{\alpha\to\alpha'}(P,P_h) \\ &= \langle P;\beta|\mathcal{T}^\dagger|P_h;\beta'\rangle\,\langle P_h;\alpha'|\mathcal{T}|P;\alpha\rangle\end{aligned} \tag{E.68}$$

Equivalently S- and S_h-dependent matrix elements can be introduced,

$$R^{(\text{prod})}_{\alpha'\beta'}(P,S;P_h) = \rho_{\beta\alpha}(P,S)\,R^{(\text{prod})}_{\alpha\beta;\alpha'\beta'}(P,P_h), \tag{E.69}$$

$$R^{(\text{prod})}_{\alpha\beta}(P;P_h,S_h) = R^{(\text{prod})}_{\alpha\beta;\alpha'\beta'}(P,P_h)\,(2S_h+1)\,\rho_{\beta'\alpha'}(P_h,S_h), \tag{E.70}$$

$$R^{(\text{prod})}(P,S;P_h,S_h) = \rho_{\beta\alpha}(P,S)\,R^{(\text{prod})}_{\alpha\beta;\alpha'\beta'}(P,P_h)\,(2S_h+1)\rho_{\beta'\alpha'}(P_h,S_h), \tag{E.71}$$

in which the spin vectors (thus) can only appear linearly. Note that because of the definition of the decay matrix it is convenient to absorb a multiplicity factor in the final state spin-dependence. If this is not done, it will explicitly appear in the following expressions.

The probability to produce the final state configuration f via the decay of h is

$$\begin{aligned}W(i\to h\to f) &= (\rho_{\text{in}})_{\beta\alpha}(P,S)\,R^{(\text{prod})}_{\alpha\beta,\alpha'\beta'}(P;P_h)\,R^{(\text{decay})}_{\beta'\alpha'}(P_h,f) \\ &= R^{(\text{prod})}_{\alpha'\beta'}(P,S;P_h)\,R_{\beta'\alpha'}(P_h,f).\end{aligned} \tag{E.72}$$

Using the parametrization of $R^{(\text{decay})}$ in terms of the analyzing power one has

$$W(i\to h\to f) = R^{(\text{prod})}\left(P,S;P_h,A_h(P_h,f)\right)\,w(f), \tag{E.73}$$

while summing (or integrating) over the decay products of h one is not able to measure the polarization of h, and finds

$$\sum_f W(i \to h \to f) = \sum_{\alpha'} R^{(\mathrm{prod})}_{\alpha'\alpha'}(P,S;P_h) = R^{(\mathrm{prod})}(P,S;P_h,0). \tag{E.74}$$

E.7 Examples of analyzing power in decays

E.7.1 Λ decay

The decay amplitude for $\Lambda \to N\pi$ is given by

$$\mathcal{T}_{m\to\lambda}(s,\Omega) = \sqrt{\frac{1}{2\pi}}\, A_\lambda\, D^{(1/2)*}_{m\lambda}(\Omega). \tag{E.75}$$

Explicitly,

$$\begin{aligned}
\mathcal{T}_{1/2\to 1/2}(s,\Omega) &= \sqrt{\frac{1}{2\pi}}\, A_+(s)\, \cos(\theta/2)\\
\mathcal{T}_{1/2\to -1/2}(s,\Omega) &= -\sqrt{\frac{1}{2\pi}}\, A_-(s)\, \sin(\theta/2)\, e^{i\,\phi}\\
\mathcal{T}_{-1/2\to 1/2}(s,\Omega) &= \sqrt{\frac{1}{2\pi}}\, A_+(s)\, \sin(\theta/2)\, e^{-i\,\phi}\\
\mathcal{T}_{-1/2\to -1/2}(s,\Omega) &= \sqrt{\frac{1}{2\pi}}\, A_-(s)\, \cos(\theta/2)
\end{aligned}$$

For a spin up or down Λ one then finds respectively

$$\begin{aligned}
W_+(\theta,\phi) &= \frac{1}{2\pi}\left(|A_+|^2\,\cos^2(\theta/2) + |A_-|^2\,\sin^2(\theta/2)\right)\\
&= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\,\cos\theta \qquad \text{(E.76)}\\
W_-(\theta,\phi) &= \frac{1}{2\pi}\left(|A_+|^2\,\sin^2(\theta/2) + |A_-|^2\,\cos^2(\theta/2)\right)\\
&= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) - \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\,\cos\theta. \qquad \text{(E.77)}
\end{aligned}$$

For more general use we calculate the so-called decay matrix (summed over the final state helicities λ),

$$R_{mn}(s,\Omega) \propto \sum_\lambda \mathcal{T}^*_{m\to\lambda}(s,\Omega)\,\mathcal{T}_{n\to\lambda}(s,\Omega),$$

which is equal to

$$\begin{aligned}
R(\theta,\phi) &= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\,\phi}\\ \sin\theta\, e^{i\,\phi} & -\cos\theta\end{pmatrix}\\
&= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right)\,\mathbf{1} + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}^{\mathrm{cm}}, \qquad \text{(E.78)}
\end{aligned}$$

appropriately normalized giving

$$R(\theta,\phi) = \frac{1}{4\pi}\left(\mathbf{1} + \alpha\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}^{\mathrm{cm}}\right) \tag{E.79}$$

where $\alpha = 0.642$ for the decay $\Lambda \to p\pi^-$. Thus one has

$$\boldsymbol{A}_h\left(\Omega^{\mathrm{cm}}_{p\pi^-}\right) \overset{\Lambda\ \text{rest-frame}}{=} \alpha\,\hat{\boldsymbol{p}}^{\mathrm{cm}}. \tag{E.80}$$

Given a specific polarization for the decaying hadron ($h = \Lambda$) via a density matrix $\rho(S_h)$ one finds a CM distribution

$$\begin{aligned}
W(S_h;\theta,\phi) &= \mathrm{Tr}\left[\rho(S_h)\,R(\theta,\phi)\right]\\
&= 1 + \boldsymbol{S}_h\cdot A_h(\theta,\phi)\\
&= 1 + \alpha\,(S_L\,\cos\theta + S^1_T\,\sin\theta\,\cos\phi + S^2_T\,\sin\theta\,\sin\phi). \qquad \text{(E.81)}
\end{aligned}$$

In covariant form (with $P_\Lambda = P_p + P_\pi$) one obtains for the analyzing power,

$$\begin{aligned} A^\mu_\Lambda(P_p, P_\pi) &= \alpha \left[\frac{M_\Lambda^2 - M_p^2 + M_\pi^2}{M_\Lambda \sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\, P_p^\mu - \frac{M_\Lambda^2 + M_p^2 - M_\pi^2}{M_\Lambda \sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\, P_\pi^\mu \right], \\ &= 0.642 \left[1.710\, \frac{P_p^\mu}{M_\Lambda} - 9.38\, \frac{P_\pi^\mu}{M_\Lambda} \right] \end{aligned} \tag{E.82}$$

with $\Delta(M_\Lambda^2, M_p^2, M_\pi^2) = \left(M_\Lambda^2 - (M_p + M_\pi)^2\right)\left(M_\Lambda^2 - (M_p - M_\pi)^2\right) = 4M_\Lambda^2\, |\boldsymbol{p}^{\rm cm}|^2$. This illustrates how the analyzing power in the final state is determined from momenta of the decay products. For instance the helicity $A_{\Lambda\, L} = M_\Lambda\, A_\Lambda^- / P_\Lambda^-$ is given by

$$A_{\Lambda\, L} = \alpha \left[\frac{M_\Lambda^2 - M_p^2 + M_\pi^2}{\sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\, \frac{z_p}{z_\Lambda} - \frac{M_\Lambda^2 + M_p^2 - M_\pi^2}{\sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\, \frac{z_\pi}{z_\Lambda} \right], \tag{E.83}$$

which takes the values $A_{\Lambda\, L} = \alpha$ for the maximal value of z_p, namely $(z_p)_{\rm max} = 0.936\, z_\Lambda$ and the value $A_{\Lambda\, L} = -\alpha$ for $(z_p)_{\rm min} = 0.756\, z_\Lambda$. Furthermore we have

$$A^\mu_{\Lambda T} = A^\mu_{\Lambda\perp} - A_{\Lambda L}\, \frac{P^\mu_{\Lambda\perp}}{M_\Lambda}. \tag{E.84}$$

E.7.2 ρ decay

The decay amplitude for $\rho \to \pi\pi$ is given by

$$\mathcal{T}_m(s,\Omega) = \sqrt{\frac{3}{4\pi}}\, A(s)\, D^{(1)*}_{m0}(\Omega). \tag{E.85}$$

Explicitly,

$$\begin{aligned} \mathcal{T}_1(s,\Omega) &= -\sqrt{\frac{3}{8\pi}}\, A\, \sin\theta\, e^{i\,\phi} \\ \mathcal{T}_0(s,\Omega) &= \sqrt{\frac{3}{4\pi}}\, A\, \cos\theta \\ \mathcal{T}_{-1}(s,\Omega) &= \sqrt{\frac{3}{8\pi}}\, A\, \sin\theta\, e^{-i\,\phi} \end{aligned}$$

Next we calculate the decay matrix,

$$R_{mn}(s,\Omega) \propto \mathcal{T}_m^*(s,\Omega)\, \mathcal{T}_n(s,\Omega),$$

which is equal to

$$R(\theta,\phi) \propto \frac{|A(s)|^2}{4\pi} \begin{pmatrix} \frac{3}{2}\sin^2\theta & -\frac{3}{2\sqrt{2}}\sin 2\theta\, e^{-i\,\phi} & -\frac{3}{2}\sin^2\theta\, e^{-2i\,\phi} \\ -\frac{3}{2\sqrt{2}}\sin 2\theta\, e^{i\,\phi} & 3\cos^2\theta & \frac{3}{2\sqrt{2}}\sin 2\theta\, e^{-i\,\phi} \\ -\frac{3}{2}\sin^2\theta\, e^{-2i\,\phi} & \frac{3}{2\sqrt{2}}\sin 2\theta\, e^{i\,\phi} & \frac{3}{2}\sin^2\theta \end{pmatrix}. \tag{E.86}$$

After normalization this gives

$$R(\theta,\phi) = \frac{1}{4\pi}\left(\mathbf{1} + 3\,\Sigma_{ij}\, A_{ij}\right), \tag{E.87}$$

with the tensor

$$(A_h)_{ij}\,(\Omega^{\rm cm}_{\pi\pi}) \overset{\rho\ \text{rest-frame}}{=} \frac{1}{3}\,\delta_{ij} - \hat{p}_i^{\rm cm}\hat{p}_j^{\rm cm}. \tag{E.88}$$

and the vector part being zero.

Given a specific polarization for the decaying hadron ($h = \rho$) via a density matrix $\rho(S_h)$ one finds a CM distribution

$$\begin{aligned}
W(S_h;\theta,\phi) &= \mathrm{Tr}\left[\rho(S_h)\,R(\theta,\phi)\right] \\
&= \frac{1}{4\pi}\left[1 + 3\,(T_h)_{ij}(A_h)_{ij}(\theta,\phi)\right] \\
&= \frac{3}{8\pi}\left(\frac{2}{3} - \frac{2}{3}\,S_{LL}\,(3\cos^2\theta - 1) - S_{LT}^1\,\sin 2\theta\,\cos\phi - S_{LT}^2\,\sin 2\theta\,\sin\phi\right. \\
&\qquad \left. - S_{TT}^{11}\,\sin^2\theta\,\cos 2\phi - S_{TT}^{12}\,\sin^2\theta\,\sin 2\phi\right).
\end{aligned} \tag{E.89}$$

In covariant form (with $P_h = P_1 + P_2$) we have

$$\hat{p}_{\rm cm}^\mu = \frac{P_1^\mu - P_2^\mu}{\sqrt{M_\rho^2 - 4M_\pi^2}}, \tag{E.90}$$

giving

$$\begin{aligned}
A_h^{\mu\nu} &= \frac{1}{M_\rho^2 - 4M_\pi^2}\left[\frac{1}{2}\,P_h^{\{\mu}P_h^{\nu\}} - P_1^{\{\mu}P_1^{\nu\}} - P_2^{\{\mu}P_2^{\nu\}}\right] - \frac{1}{3}\left(g^{\mu\nu} - \frac{P_h^\mu P_h^\nu}{M_\rho^2}\right) \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,P_h^{\{\mu}P_h^{\nu\}} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[P_1^{\{\mu}P_1^{\nu\}} + P_2^{\{\mu}P_2^{\nu\}}\right] - \frac{1}{3}\,g^{\mu\nu} \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,S^{\mu\alpha\nu\beta}\,P_{h\,\alpha}P_{h\,\beta} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[S^{\mu\alpha\nu\beta}\,P_{1\,\alpha}P_{1\,\beta} + S^{\mu\alpha\nu\beta}\,P_{2\,\alpha}P_{2\,\beta}\right] \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,S^{\mu P_h \nu P_h} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[S^{\mu P_1 \nu P_1} + S^{\mu P_2 \nu P_2}\right].
\end{aligned} \tag{E.91}$$

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