I. GRAVITATIONAL WAVES FROM BINARY BLACK HOLES

Two black holes (or two neutron stars, or a neutron star and a black hole) with masses m_1 and m_2 are in a circular orbit around the common center of mass (CM), with angular frequency ω. Assume the distance R between the two is large enough that the black holes can be viewed as point particles and that the effect of orbital energy loss through GW emission is negligible. With respect to an observer at a distance r, the system is oriented such at the line of sight makes an angle ι with the normal to the orbital plane.

(1.1) Pick a coordinate system (x, y, z) such that the direction to the observer is along the z axis, and the CM of the system is at the origin $(0, 0, 0)$. Without loss of generality we may assume that the orbital plane is oriented in such a way that its intersection with the (x, y) plane is in the x axis. Write an expression for the positions $\mathbf{x}_1$, $\mathbf{x}_2$ of the black holes as a function of time.

(1.2) In the TT gauge, a gravitational wave propagating in the z-direction corresponds to a metric perturbation

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \tag{1.1}$$

and in the quadrupole approximation one has

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}(\ddot{M}_{11} - \ddot{M}_{22}), \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}, \end{aligned} \tag{1.2}$$

where in each case the RHS is evaluated at the retarded time $t - r/c$. Compute the relevant moments M_{ij} and their double time derivatives $\ddot{M}_{ij}$.

(1.3) Substitute the results into (1.2) and evaluate at the retarded time t_{ret} to find the gravitational wave polarizations.

(1.4) Explain why the radiation is at *twice* the orbital frequency.

(1.5) If the black holes are sufficiently far apart, we can use the Newtonian centripetal force to express the separation R in terms of ω and the component masses m_1, m_2. Define the chirp mass

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}, \tag{1.3}$$

and write h_+, $h_\times$ in terms of it.

(1.6) What do the polarizations look like when $\iota = 0$ (i.e., the system is seen face-on) and when $\iota = \pi/2$ (edge-on)?

(1.7) What is the total power emitted in gravitational waves for the Earth-Sun system? What is the total power emitted by a system consisting of two black holes with masses $m_1 = m_2 = 10\,M_\odot$ at radii of $20G(m_1 + m_2)/c^2$ (i.e., quite close but still comforably far from the last stable orbit)? (The mass of the Sun is 2×10^{30} kg and that of the Earth, 6×10^{24} kg.)