I. COMPLETE GRAVITATIONAL COLLAPSE: A TOY MODEL

Complete gravitational collapse happens when the pressure is no longer capable of sustaining a star. The simplest model one can think of assumes that there is no pressure at all ($P = 0$), but of course non-zero density. The matter is then pressureless dust, with energy-momentum tensor

$$T_{\mu\nu} = \rho\, u^\mu u^n. \tag{1.1}$$

Let as assume a ball of pressureless dust that is homogeneous and isotropic, which starts out from rest with a finite radius R. Outside there is a spherically symmetric vacuum spacetime, which we assume to be asymptotically flat. By Birkhoff's theorem, the outside geometry must be Schwarzschild.

(1.1) At the surface of the dust ball, dust particles are moving on geodesics of the external Schwarzschild geometry. Using the expressions for geodesics derived in the notes, show that the motion of the surface is determined by

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{R}\frac{R-r}{r}. \tag{1.2}$$

Up to the factor $2M/R$, this is a well-known differential equation, namely that of a cycloid. Consider a point P on a wheel that is rolling in the x-direction. Let η be the angle with the vertical made by the line from the center of the wheel to P. If the radius of the wheel is $\mathcal{R}$, show that the x and y components of P depend on η through

$$\begin{aligned} x &= \mathcal{R}(\eta + \sin\eta), \\ y &= \mathcal{R}(1 + \cos\eta). \end{aligned} \tag{1.3}$$

Also show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{2\mathcal{R} - y}{y}. \tag{1.4}$$

By comparing (1.4) with (1.2), infer a parametric solution to (1.2) similar to (1.3).

(1.2) Explain why the interior geometry of the dust ball must be that of a closed ($k = +1$) FLRW Universe. The Einstein equations then reduce to two coupled differential equations for the scale factor a:

$$\begin{aligned} \left(\frac{\dot a}{a}\right)^2 &= \frac{16\pi\rho}{3} - \frac{1}{a^2}, \\ \frac{\ddot a}{a} &= -\frac{4\pi}{3}\rho, \end{aligned} \tag{1.5}$$

where the dot denotes derivation with respect to the proper time of the dust, and we recall that $P = 0$. Derive a parametric solution for a of a comoving observer in terms of the maximum value a_m, and for the proper time, using a parameter η as above. Also derive an expression for the density ρ as a function of η.

(1.3) For this model to be viable, the circumference of the star's surface should be the same as measured with the internal FLRW metric and the external Schwarzschild metric. In other words, if $\mathcal{C}$ is the equator at an arbitary moment in time, then we must have

$$\int_{\mathcal{C}} |g^{\mathrm{S}}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2} = \int_{\mathcal{C}} |g^{\mathrm{F}}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2}, \tag{1.6}$$

where g^{F} is the $k = 1$ FLRW metric with line element

$$ds^2 = -d\tau^2 + a^2(\tau)\,\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{1.7}$$

Show that it is indeed possible to have (1.6) for all times. How does it relate M and R to the initial scale factor a_m and the radial coordinate value χ_0 of the outer boundary?

(1.4) An observer A is riding on the surface of the dust ball. Derive expressions in in terms of M and R for the time it takes, according to A, for the following things to happen:

(a) the formation of an event horizon;
(b) the appearance of a singularity.

Suppose that the initial radius of the star was $R = 7 \times 10^8$ m, and that its mass is $M = 2 \times 10^{30}$ kg. (These are in fact the radius and mass of the Sun.) In that case, what are the times in (a) and (b)? *Note:* don't forget to reinstate the appropriate powers of G and c.

(1.5) An observer B is witnessing the collapse from a large distance. We have seen that the proper time t of observer B is related to the same parameter η as above by

$$\begin{aligned} t \;=\; & 2M \ln\left|\frac{(R/2M-1)^{1/2} + \tan(\eta/2)}{(R/2M-1)^{1/2} - \tan(\eta/2)}\right| \\ & + 2M(R/2M-1)^{1/2}\left[\eta + (R/4M)(\eta + \sin\eta)\right]. \end{aligned} \tag{1.8}$$

The above expression diverges as a certain value of η is approached. What is r at that value?

II. THE SCHWARZSCHILD SPACETIME IN KRUSKAL-SZEKERES COORDINATES

In Schwarzschild coordinates, the black hole metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2.1}$$

The metric appears to be singular at $r = 2M/r$, even though we know that observers can move through it, in finite proper time. The horizon is just a coordinate singularity. One set of coordinates in which the horizon is manifestly non-singular are the *Kruskal-Szekeres* coordinates.

(2.1) Define coordinates u and v by:

$$\begin{aligned} u &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M}, \\ v &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M}, \end{aligned}$$

for $r > 2M$, and

$$\begin{aligned} u &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M}, \\ v &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M}, \end{aligned}$$

for $r < 2M$. Show that the metric in these coordinates is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}(dv^2 - du^2) + r^2(u,v)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.2}$$

where $r(u,v)$ is now not a coordinate but a function of u and v determined implicitly by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2. \tag{2.3}$$

(2.2) Draw a diagram in which the horizontal axis is the u axis, the vertical axis is the v axis, and θ and ϕ are suppressed. What is the meaning of a point on such a diagram? Show that radial null geodesics (i.e., lines with $ds = 0$ and also $d\theta = d\phi = 0$) are at 45 degrees to the u and v axes.

(2.3) On the basis of Eq. (2.3), draw a few representative curves of constant r, for $r < 2M$ as well as for $r = 2M$. Also draw the curves $r = 0$ (the singularity) and $r = 2M$ (the horizon).

(2.4) Characterize the curves of constant t. Draw a few representative ones. Draw $t = -\infty$ and $t = +\infty$.

(2.5) From your conclusion about null geodesics in (2.3), draw a representative timelike geodesic (the spacetime path of a massive particle) which starts outside the horizon and falls into the black hole. Explain why any timelike geodesic which enters the horizon must end at the singularity. What do timelike geodesics look like which do *not* enter the horizon?