I. COSMOLOGY

Assume the most general homogeneous and isotropic spacetime with spatially flat geometry, i.e.,

$$ds^2 = -d\tau^2 + a^2(\tau)\,(dx^2 + dy^2 + dz^2). \tag{1.1}$$

(1.1) Compute the Christoffel symbols. Plug these into the expression for the Ricci tensor:

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \partial_\nu \Gamma^\nu_{\mu\rho} - \partial_\mu \Gamma^\nu_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu}. \tag{1.2}$$

Contract to get the Ricci scalar $R = R^\mu{}_\mu$. Construct the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.3}$$

(1.2) Write the components of the energy-momentum tensor for a perfect fluid,

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\,(g_{\mu\nu} + u_\mu u_\nu). \tag{1.4}$$

(1.3) Show that the Einstein equations *with cosmological constant*,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.5}$$

reduce to

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3}, \tag{1.6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{1.7}$$

(1.4) Now set $\Lambda = 0$. Combine Eqns. (1.6) and (1.7) to find

$$\dot{\rho} = -3(\rho + P)\,\frac{\dot{a}}{a}. \tag{1.8}$$

Hence this equation follows from the Einstein equations. However, show that it already follows from energy-momentum conservation,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.9}$$

(1.5) Assume pressureless dust ($P = 0$). Integrate Eq. (1.8) to find ρ as a function of a, up to an integration factor ρ_0. Substitute the solution into Eq. (1.6) - still assuming $\Lambda = 0$ - and solve for $a(\tau)$. Check that your result is consistent with Eq. (1.7).

(1.6) Now assume radiation ($P = \rho/3$). Again integrate Eq. (1.8) to find $\rho(a)$. Explain why at an early stage in the evolution of the Universe, radiation would have been the dominant factor in the dynamics of the Universe. Then again solve Eq. (1.6) for $a(\tau)$. Discuss the difference with the solution for pressureless dust.

(1.7) Discuss qualitatively what would have been different in (1.5) and (1.6) had you set $\Lambda > 0$ or $\Lambda < 0$.