Gravity Friday, April 9, 2010

Problems WEEK 1 Name: ____________________________

Problem 1: The relation between Cartesian (x, y) and polar coordinates (r, θ) is given by $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$ and the inverse relations $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Calculate all elements of the transformation matrices $\Lambda^{\alpha'}{}_{\beta}$ and $\Lambda^{\mu}{}_{\nu'}$ for the transformation of Cartesian (the unprimed system) to the polar (with primes on the indices) coordinates.

(b) Given $f = x^2 + y^2 + 2xy$ and in Cartesian coordinates $\vec{V} \to (x^2 + 3y, y^2 + 3x)$, $\vec{W} \to (1, 1)$. Calculate f as function of r and θ, and determine the components of $\vec{V}$ and $\vec{W}$ for the polar basis by expressing them as functions of r and θ.

(c) Determine the components of $\tilde{\mathrm{d}}f$ in Cartesian coordinates and obtain them in polar coordinates by (i) direct calculation in polar coordinates, and (ii) by a transformation from Cartesian components.

(d) Use the metric tensor in polar coordinates to find the polar components of the 1-forms $\tilde{V}$ and $\tilde{W}$ associated with $\vec{V}$ and $\vec{W}$.

(e) Determine the polar components of $\tilde{V}$ and $\tilde{W}$ by transformation of the Cartesian components.

Problem 2: Imagine that we live on the surface of a sphere. We introduce the usual spherical coordinates(r, θ, ϕ) and have $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ as the usual orthonormal spherical basis vectors. The line-element in spherical coordinates is then given by

$$d\vec{s} = dr\ \hat{e}_r + r d\theta\ \hat{e}_\theta + r\sin\theta d\phi\ \hat{e}_\phi. \tag{1}$$

Next, we restrict ourselves to the surface of the sphere and impose the condition $r =$ constant. We choose coordinates $(x^1, x^2) = (\theta, \phi)$ on the surface. The line-element of the sphere's surface can be expressed in terms of *natural basis vectors* as

$$d\vec{s} = d\theta\vec{e}_\theta + d\phi\vec{e}_\phi, \qquad \text{with} \qquad \vec{e}_\theta = r\hat{e}_\theta,\ \vec{e}_\phi = r\sin\theta\ \hat{e}_\phi. \tag{2}$$

With this definition, the quadratic line element can be written as

$$ds^2 = (r d\theta)^2 + (r\sin\theta d\phi)^2. \tag{3}$$

(a) In terms of spherische coordinates (r, θ, ϕ), we can express the position vector as

$$\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\sin\phi\vec{j} + r\cos\theta\vec{k}. \tag{4}$$

The three natural basis vectors are given by $\vec{e}_r = \partial\vec{r}/\partial r$, $\vec{e}_\theta = \partial\vec{r}/\partial\theta$ en $\vec{e}_\phi = \partial\vec{r}/\partial\phi$. Determine these vectors and give the length of each vector. Note, that in the followin only the basis vectors on the surface, $\vec{e}_\theta$ en $\vec{e}_\phi$, are relevant.

(b) Determine the metric tensor and its inverse.

(c) Calculate all Christoffel symbols Γ^k_{ij} where the indices are elements of the set (θ, ϕ).

(d) How many independent components does the Riemann tensor have? Use the symmetry properties of this object.

(e) Calculate the components of the Riemann tensor.

(f) Calculate the components of the Ricci tensor.

(g) Calculate the scalar curvature (Ricci curvature).

(h) Calculate the components of the Einstein tensor on the surface of the spherel.

Problem 3: In this exersize we test our knowledge of covariant derivatives. We use the geometry of the surface of a two-dimensional sphere,

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

and a vector $\vec{V}$ with components $V^A = (0, 1)$. Calculate the four components of $\nabla_A v^B$ and calculate the two quantities $\nabla_\theta \nabla_\phi v^\theta$ and $\nabla_\phi \nabla_\theta v^\theta$ and determine whether these covariant derivatives commute.

Problem 4: Calculate the components of the divergence for spherical coordinates in the three-dimensional Euclidian space. Hint: for the covariant divergence of vector $\vec{V}$ one has $\nabla \cdot \vec{V}$ and for its components $V^i_{\ ;i}$.

Problem 5: Give a derivation of the following expression for the Christoffel symbols in terms of partial derivatives of the components of the metric tensor g,

$$\Gamma^{\gamma}_{\ \beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}). \tag{6}$$