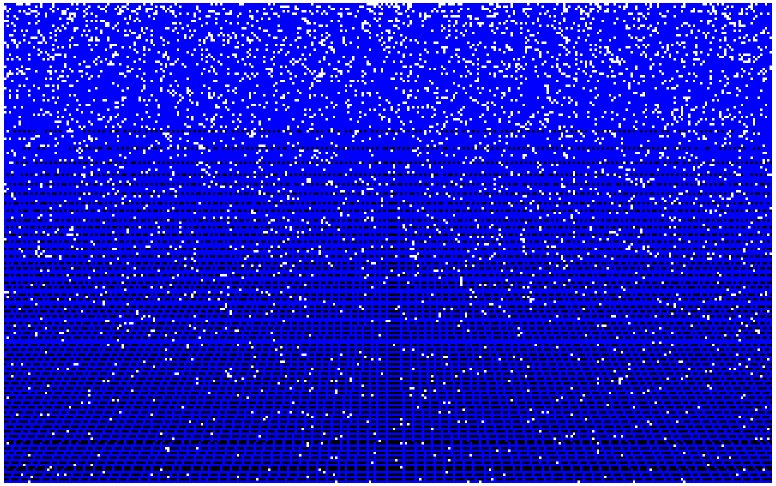
Inflation



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Limitations of standard cosmology

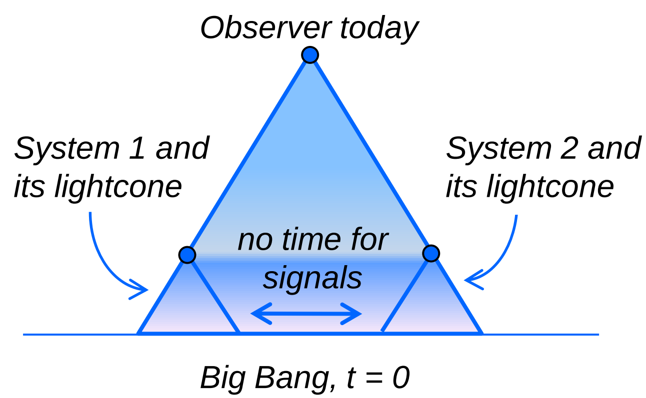
Horizon problem, flatness problem, missing exotic particles

<u>Horizon</u>: largest distance over which influences can have travelled in order to reach an observer: visible Universe of this observer

Thermal equilibrium between different parts of the Universe established by exchange of photons (radiation)

Photons decouple about 100,000 years A.B. and then the horizon was much smaller than at present

One expect that regions with about the same temperature are relatively small, but this is not the case



Inflation: one starts with a small Universe which is in thermal equilibrium and inflates this with an enormous factor. Increasingly more of this Universe is now entering our horizon.

Flatness problem and exotic particles

Experiments show: Universe now has a nearly *flat* Robertson – Walker metric

In order to explain the present flatness, the metric of the early Universe needs to resemble even more a perfectly flat RWM

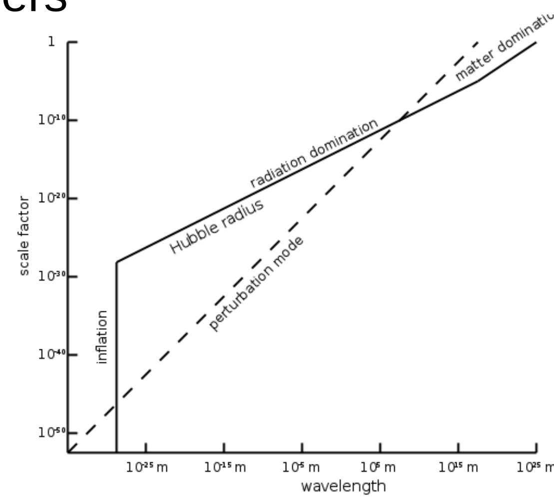
Flatness problem: which mechanism brought the earliest flatness so close to the flat RW metric?

If you assume that the Universe was always perfectly flat, then the Universe started with exactly the critical density. Why?

Classical Standard Model of Cosmology provides no answers

Modern particle physics predicts exotic particles: supersymmetric particles, monopoles, …

Inflation $\dot{a}(t) > 0,$ $\ddot{a}(t) > 0$



Friedmann equations

Previously: *flat* Robertson – Walker metric (*k = 0*). In general one hast

$$-c^2 d\tau^2 = -c^2 dt^2 + a(t)^2 d\mathbf{\Sigma}^2 \quad d\mathbf{\Sigma}^2 = \frac{dr^2}{1-kr^2} + r^2 d\mathbf{\Omega}^2, \quad \text{where } d\mathbf{\Omega}^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ give Friedmann equations

$$\Rightarrow \boxed{\begin{aligned} \left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} &= \frac{8\pi G}{3}\rho \\ 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 &= -\frac{8\pi G}{c^2} p. \end{aligned}}$$

Without cosmological constant, FV – 1 becomes

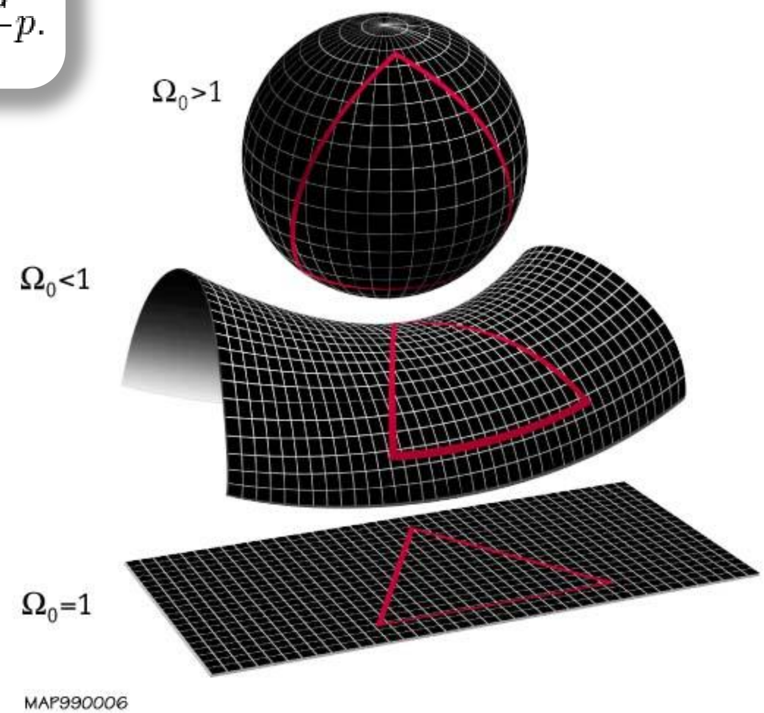
$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Critical density: for given *H* the density for which *k = 0*

$$\rho_c = \frac{3H^2}{8\pi G}$$

10^{-26} kg m^{-3}

Density / critical density: Ω



Friedmann equations

Friedmann equation 1 can be re-written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho$$

$$\rho_c = \frac{3H^2}{8\pi G}$$

$$\frac{3a^2}{8\pi G}H^2 = \rho a^2 - \frac{3kc^2}{8\pi G}$$

$$\rho_c a^2 - \rho a^2 = -\frac{3kc^2}{8\pi G}$$

$$\boxed{(\Omega^{-1} - 1)\rho a^2 = \frac{-3kc^2}{8\pi G}.}$$

On the right only constants. During expansion, density decreases ($\sim a^3$)

Since Planck era, ρa^2 decreased by a factor 10^{60}

Thus, ($\Omega^{-1} - 1$) must have increased by a factor 10^{60}

WMAP and Sloan Digital Sky Survey set Ω_0 at 1 within 1%

Then $| \Omega^{-1} - 1 | < 0.01$ and at Planck era smaller than 10^{-62}

Flatness problem: why was the initial density of the Universe so close to the critical density?

Solutions: Anthropic principle or inflation (ρa^2 rapidly increases in short time)

Dynamics of cosmological inflation

Inflation occurs when the right part of FV – 2 is positive, so for n < -1/3.

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = +\frac{8\pi G}{3c^2}\rho(t),$$
$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3c^2}\Big(3P(t)+\rho(t)\Big).$$

A cosmological constant will do, but inflation works differently

Take scalar field $\Phi(t)$ which only depends on time (cosmological principle)

Langrangian – density $\mathscr{L} = -\frac{1}{2}g^{\mu\nu}\Big(\partial_\mu\Phi(t)\Big)\Big(\partial_\nu\Phi(t)\Big) - V\Big(\Phi(t)\Big)$

Note: Minkowski-metric yields the Klein Gordon equation $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \frac{m^2c^2}{\hbar^2}\psi$

Actior $S = \int d^3x\,dt\sqrt{-g}\mathscr{L}$ $\qquad g = -a^6(t)$

Euler – Lagrange equations yield equations of motion

$$\Longrightarrow \quad \ddot{\Phi}(t) + 3\frac{\dot{a}(t)}{a(t)}\dot{\Phi}(t) + c^2\partial_\Phi V\Big(\Phi(t)\Big) = 0$$

Details of evolution depend on the potential energy *V*

Dynamics of cosmological inflation

Energy – momentum tensor (T + V) for Lagrangian density (T – V)

$$\mathscr{L} = -\frac{1}{2} g^{\mu\nu}\Big(\partial_\mu \Phi(t)\Big)\Big(\partial_\nu \Phi(t)\Big) - V\Big(\Phi(t)\Big) \quad \Longrightarrow \quad T_{\mu\nu} = \Big(\partial_\mu \Phi\Big)\Big(\partial_\nu \Phi\Big) + g_{\mu\nu}\mathscr{L}$$

Insert Lagrangian and metric, and compare with T for Friedmann fluid

$$\begin{aligned} \rho(t) &= \frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t) + V\big(\Phi(t)\big) \\ P(t) &= \frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t) - V\big(\Phi(t)\big) \end{aligned}$$

Total energy density of the scalar field

Kinetic energy density of the scalar field

Potential energy density of the scalar field

During inflation: the inflaton – field is dominant

Inflation equations:

$$\begin{aligned} \ddot{\Phi}(t) + 3H(t)\dot{\Phi}(t) + c^2 \partial_\Phi V\big(\Phi(t)\big) &= 0 \\ H^2(t) &= \frac{8\pi G}{3c^2}\left(\frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t) + V\big(\Phi(t)\big)\right) \end{aligned}$$

Cosmology: choose potential energy density and determine scale factor a(t) and inflaton field

Simplified inflation equations

Assume slow evolution of the scalar field (Slow Roll Condition)

$$\frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t) \ll V\big(\Phi(t)\big) \quad \Longrightarrow \quad \begin{aligned} \rho(t) &\approx +V\big(\Phi(t)\big) \\ P(t) &\approx -V\big(\Phi(t)\big) \end{aligned}$$

Equation of state with n = -1

This leads to inflation, independent on details of inflaton field

Furthermore, assume that the kinetic energy density stays small for a long time (this prevents inflations from terminating too soon)

Simplified inflation equations (SIE)

$$\begin{aligned} \ddot{\Phi}(t) + 3H(t)\dot{\Phi}(t) + c^2\partial_\Phi V\big(\Phi(t)\big) &= 0 \\ H^2(t) = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t) + V\big(\Phi(t)\big)\right) & \end{aligned} \quad \Longrightarrow \quad \boxed{\begin{aligned} 3H(t)\dot{\Phi}(t) + c^2\partial_\Phi V\big(\Phi(t)\big) &= 0 \\ H^2(t) - \frac{8\pi G}{3c^2}V\left(\Phi(t)\right) &= 0 \end{aligned}}$$

SIE are valid when

$$\frac{\frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t)}{V\big(\Phi(t)\big)} \ll 1, \quad \text{en} \quad \frac{\ddot{\Phi}(t)}{3H(t)\dot{\Phi}(t)} \ll 1$$

INFLATION PARAMETERS

Use first SIE to re-write

$$3H(t)\dot{\Phi}(t) + c^2\partial_\Phi V\left(\Phi(t)\right) \;=\; 0 \quad \Rightarrow \quad \frac{\frac{1}{2}\frac{1}{c^2}\dot{\Phi}^2(t)}{V\left((\Phi(t)\right)} = \frac{1}{18H^2(t)}\frac{\left(\partial_\Phi V(\Phi(t))\right)^2}{V(\Phi(t))} = \underbrace{\frac{1}{6}\frac{c^4}{8\pi G}\left(\frac{\partial_\Phi V(\Phi(t))}{V(\Phi(t))}\right)^2}_{\text{Inflation parameter } \epsilon} \ll 1$$

Inflation parameter: measures slope of V
(V should be flat)

From SIE
$$-\frac{\dot{H}(t)}{H^2(t)} = \frac{1}{3}\epsilon$$

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$$

$$1 - \frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)} = \frac{1}{3}\epsilon \ll 1 \quad \rightarrow \quad \frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)} \gg 0$$

$$\Rightarrow \quad \ddot{a}(t) \gg 0$$

Then $\epsilon \ll 1$ guarantees that inflation will occur

Furthermore
$$\frac{\ddot{\Phi}(t)}{3H(t)\dot{\Phi}(t)} = -\frac{1}{9H^2(t)}\left(c^2\partial_\Phi^2 V(\Phi(t)) + 3\dot{H}(t)\right)$$
$$= \underbrace{-\frac{c^4}{24\pi G}\frac{\partial_\Phi^2 V(\Phi(t))}{V(\Phi(t))}}_{\text{parameter } \eta} + \cancel{\frac{\epsilon}{9}} \ll 1,$$

Determines the rate of change for the slope of V. We want V to remain flat for a long time. $(\eta \ll 1)$

An inflation model

Massive inflaton field: quantum field of particles with mass m

$$\mathscr{L} = -\frac{1}{2}\dot{\Phi}^2(t) - \frac{m^2}{2}\left(\frac{c}{\hbar}\right)^2\Phi^2(t)$$

Potential $V(\Phi(t))$
energy density

SIE become

$$\begin{aligned} 3H(t)\dot{\Phi}(t) + m^2\left(\frac{c^4}{\hbar^2}\right)\Phi(t) &= 0 \\ H^2(t) - \frac{8\pi G}{6}\frac{m^2}{\hbar^2}\Phi(t) &= 0 \end{aligned}$$

Two coupled DE: take rote of SIE - 2

$$H(t) = \pm\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi(t)$$

Insert in SIE – 1:

$$\pm 3\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi(t)\dot{\Phi}(t) + m^2\frac{c^4}{\hbar^2}\Phi(t) = 0 \quad \Rightarrow \quad \dot{\Phi}(t) \pm \frac{mc^4}{3\hbar}\sqrt{\frac{6}{8\pi G}} = 0$$

Solving yields

$$\Phi(t) = \Phi_0 \mp \frac{mc^4}{3\hbar}\sqrt{\frac{6}{8\pi G}}t$$

Amplitude inflaton field on t = 0

Insert inflaton field in SIE – 2. This yields

$$\begin{aligned} H(t) = \frac{\dot{a}(t)}{a(t)} &= \pm\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\left(\Phi_0 \mp \frac{mc^4}{3\hbar}\sqrt{\frac{6}{8\pi G}}t\right) \\ &= \pm\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi_0 - \frac{m^2c^4}{3\hbar^2}t. \end{aligned}$$

Expanding Universe: use + sign

Solve with $a(t) = e^{-(\kappa+\lambda t)^2}$ and this yields

$$\dot{a}(t) = \left(\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi_0 - \frac{m^2c^4}{3\hbar^2}t\right)a(t) \quad \Rightarrow \quad -2\kappa\lambda - 2\lambda^2 t = \sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi_0 - \frac{m^2c^4}{3\hbar^2}t \quad \Rightarrow \quad \kappa = \frac{\sqrt{8\pi G}}{2c^2}\Phi_0 \quad \lambda = \frac{mc^2}{\hbar\sqrt{6}}$$

An inflation model

As solutions we find

$$\Phi(t) = \Phi_0 - \frac{mc^4}{3\hbar}\sqrt{\frac{6}{8\pi G}}t,$$

$$a(t) = e^{\boxed{-\left(-\frac{\sqrt{8\pi G}}{2c^2}\Phi_0+\frac{mc^2}{\hbar\sqrt{6}}t\right)^2}}$$

Interpretation: inflaton field decays in time

Inflation parameter

$$\epsilon \equiv \frac{1}{6}\frac{3c^4}{8\pi G}\left(\frac{\partial_\Phi V(\Phi(t))}{V(\Phi(t))}\right)^2 = \frac{c^4}{12\pi G}\frac{1}{\Phi^2(t)} \ll 1.$$

$$\eta \equiv -\frac{c^4}{24\pi G}\left(\frac{\partial^2_\Phi V(\Phi(t))}{V(\Phi(t))}\right) = -\frac{c^4}{12\pi G}\frac{1}{\Phi^2(t)} = -\epsilon.$$

When one parameter is small, the other is too

Inserting the inflaton field in the expression for the inflation parameter yields

$$t \ll \frac{3\hbar}{mc^4}\sqrt{\frac{8\pi G}{6}}\Phi_0 - \frac{\hbar}{mc^2}\sqrt{6}$$

Inflation continuous to this time, and stops at $t_{\mathrm{eind}} \equiv \frac{3\hbar}{mc^4}\sqrt{\frac{8\pi G}{6}}\Phi_0$

Scale factor obeys

$$-\left(-\frac{\sqrt{8\pi G}}{2c^2}\Phi_0 + \frac{mc^2}{\hbar\sqrt{6}}t\right)^2 = -\frac{8\pi G}{4c^4}\Phi_0^2 + \sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi_0 t - \frac{m^2c^4}{\cancel{6}\hbar^2}t^2$$

$$\Rightarrow \quad a(t) = e^{-\frac{8\pi G}{4c^4}\Phi_0^2} \cdot e^{+\sqrt{\frac{8\pi G}{6}}\frac{m}{\hbar}\Phi_0 t}$$

Inflation occurs!

Specific model:

$$m \approx 10^{13}\frac{\mathrm{GeV}}{c^2} \approx 10^{-14}\mathrm{kg}, \qquad \Phi_0 \approx 10^{23}\frac{\sqrt{\mathrm{m}\cdot\mathrm{kg}}}{\mathrm{s}}$$

$$\Rightarrow \quad t \approx 10^{-35}\ \mathrm{s}, \qquad \frac{a(\mathrm{eind})}{a(\mathrm{begin})} = e^{\frac{8\pi G}{4c^4}\Phi_0^2}$$

End of inflation: reheating phase

Inflaton field decays to new particles (this results in radiation)
The potential energy density V has somewhere a deep and steep dip

Inflation parameter not small anymore: inflation breaks off

Inflation equations

$$\ddot{\Phi}(t) + (3H(t) + \Gamma)\,\dot{\Phi}(t) + \partial_\Phi V(\Phi(t)) = 0$$

Add dissipation term

$$H^2(t) = \frac{8\pi G}{3c^2}\Big(\rho_\Phi(t) + \rho_\gamma(t)\Big)$$

Add radiation

Second Friedmann equation

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3c^2}\Big(\rho_\gamma(t) + 3P_\gamma(t) + \rho_\Phi + 3P_\Phi(t)\Big)$$

$$P_\Phi = n(t)\rho_\Phi \quad \Rightarrow \quad \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3c^2}\Big(2\rho_\gamma(t) + (3n(t)+1)\rho_\Phi(t)\Big)$$

These equations tell us

- how inflatons are transferred into radiation,
- how the inflaton field decreases,
- how the scale factor evolves during this process

Give Γ, V and n(t) and everything is fixed

Re-heating equations

Eliminate the inflaton field

One has $\rho_\Phi(t) = \frac{1}{2c^2}\dot{\Phi}^2(t) + V(\Phi(t))$ $\Rightarrow$ $\dot{\rho}_\Phi(t) = \frac{1}{c^2}\dot{\Phi}(t)\ddot{\Phi}(t) + \dot{\Phi}(t)\partial_\Phi V(\Phi(t))$

Insert in $\ddot{\Phi}(t) + (3H(t) + \Gamma)\,\dot{\Phi}(t) + \partial_\Phi V(\Phi(t)) = 0$

Also use $\frac{1}{c^2}\dot{\Phi}(t) = \rho_\Phi(t) + P_\Phi(t)$

$$P_\Phi = n(t)\rho_\Phi$$

Differentiate FV – 1 and use FV – 2 to eliminate $\ddot{a}/a$

$$\boxed{\begin{aligned}\left(\frac{\dot{a}(t)}{a(t)}\right)^2 &= +\frac{8\pi G}{3c^2}\rho(t),\\ \frac{\ddot{a}(t)}{a(t)} &= -\frac{4\pi G}{3c^2}\Big(3P(t)+\rho(t)\Big).\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{\rho}_\Phi(t) &= -\Big(3H(t)+\Gamma\Big)\Big(n(t)+1\Big)\rho_\Phi(t),\\ \dot{\rho}_\gamma(t) &= -4H(t)\rho_\gamma(t) + \Gamma\Big(n(t)+1\Big)\rho_\Phi(t),\\ H^2(t) &= \frac{8\pi G}{3c^2}\Big(\rho_\Phi(t)+\rho_\gamma(t)\Big).\end{aligned}}$$

Re-heating equations are three coupled differential equations

Give Γ, V and n(t) and all is fixed

After inflation the Universe is dominated by radiation (and we can employ relativistic cosmology to describe the evolution)