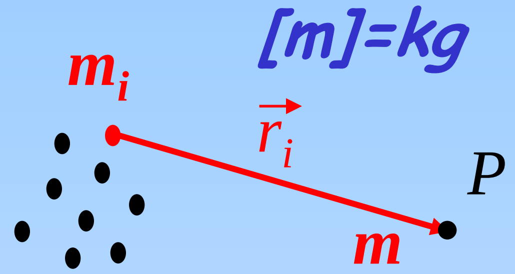
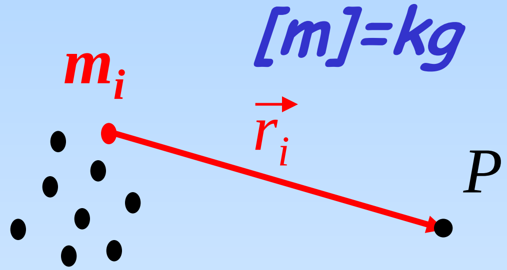
massaverdeling ⇒ gravitatieveld

Gravitatiewet:



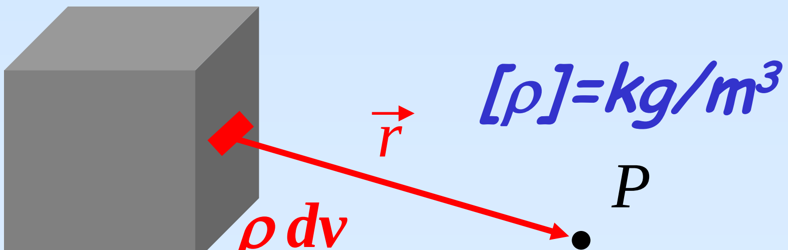
$$\vec{F} = m\vec{g}_P \equiv -G\sum_{i=1}^{N}\frac{mm_i}{r_i^2}\hat{r}_i$$

Diskreet:



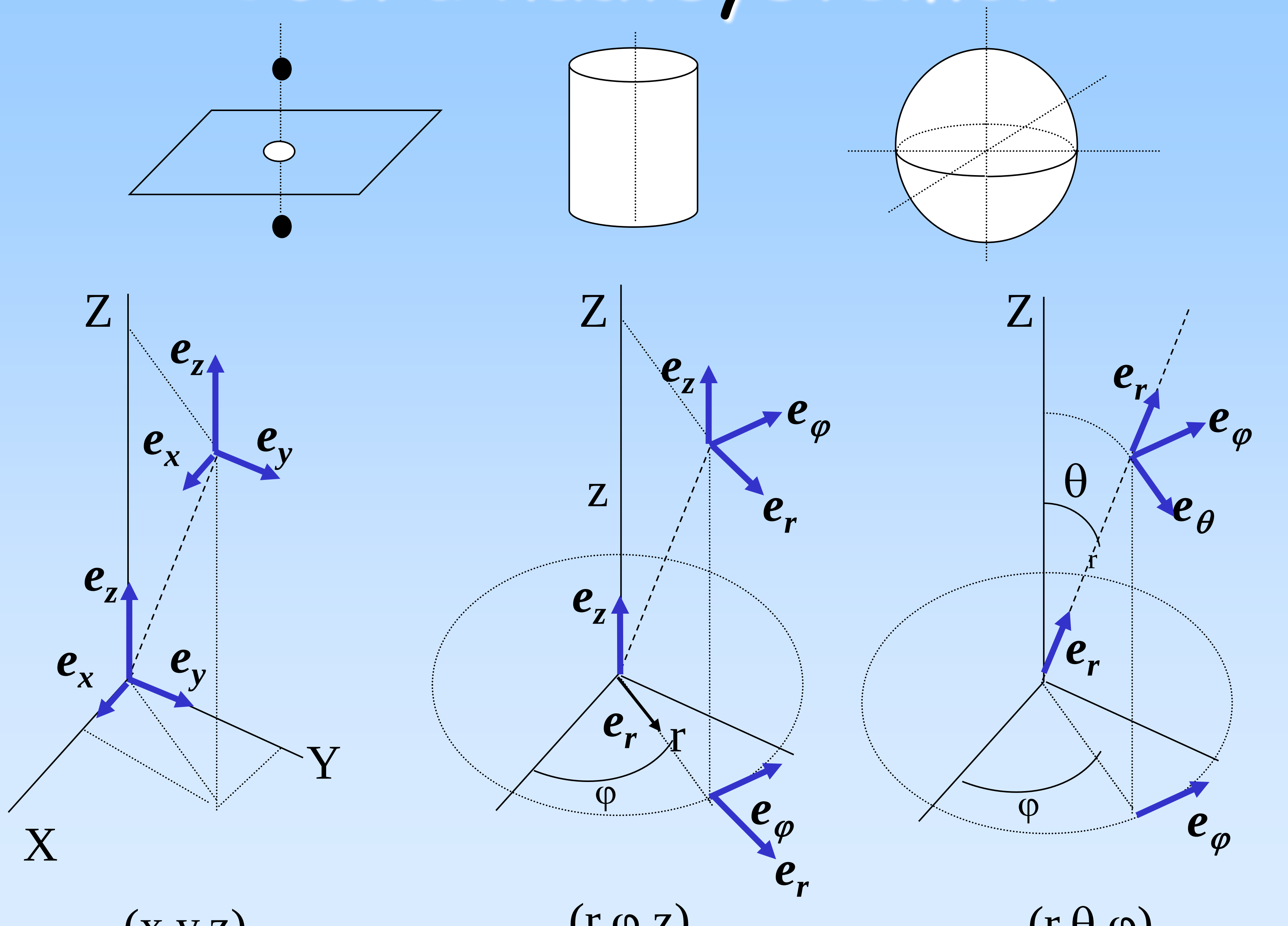
$$\vec{g}_P \equiv -G\sum_{i=1}^{N}\frac{m_i}{r_i^2}\hat{r}_i$$

Continu:

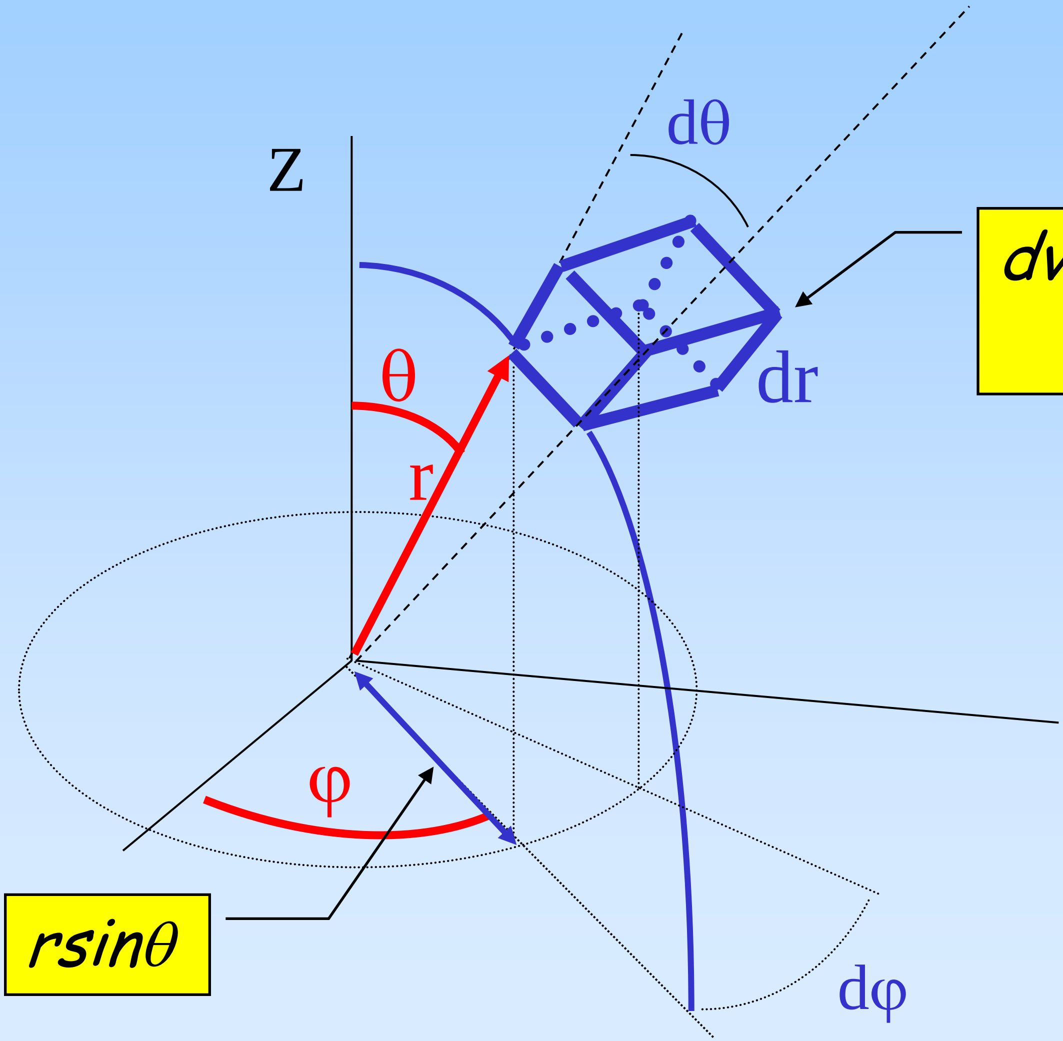


$$\vec{g}_P \equiv -G\int_{volume} dv\,\frac{\rho}{r^2}\hat{r}$$

Coördinaatsystemen

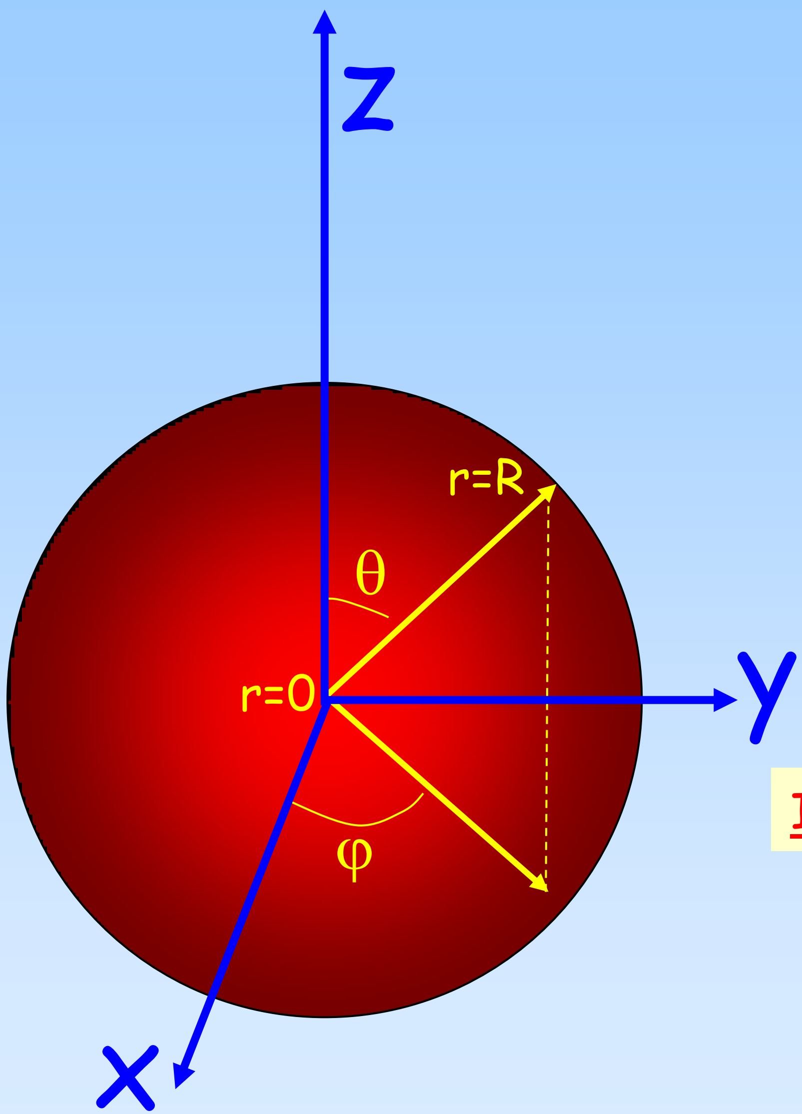


Volume integraal: bol coördinaten



$$dv = (r\,d\theta)\ (r\sin\theta\,d\varphi)\ (dr)$$
$$= r^2 \sin\theta\ d\theta\, d\varphi\, dr$$

Voorbeeld: bol inhoud



Om de bol inhoud te bepalen integreer je de functie "1" over het bol volume:

Integratie domein:

r:	[0,R]
θ:	[0,π]
φ:	[0,2π]

Integraal:

$$\int_{bol} 1dV \rightarrow \int_0^R \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta d\varphi d\theta dr$$

$$= \int_0^R \int_0^{\pi} r^2 \sin\theta \left(\varphi\Big|_0^{2\pi}\right) d\theta dr = 2\pi \int_0^R r^2 \left(-\cos\theta\Big|_0^{\pi}\right) dr$$

$$= \frac{4\pi}{3}\left(r^3\Big|_0^R\right) = \frac{4\pi}{3} R^3$$

Wet van Gauss

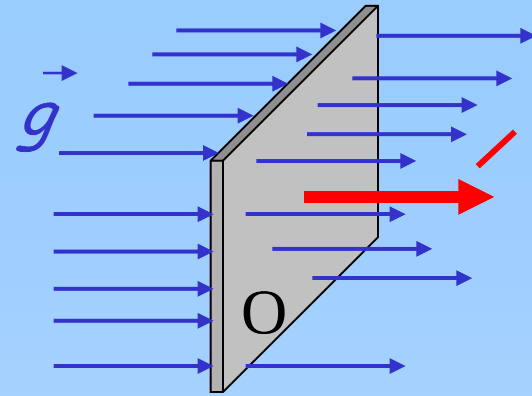
De gravitatieflux

De wet van Gauss

Voorbeeld

Flux F_g

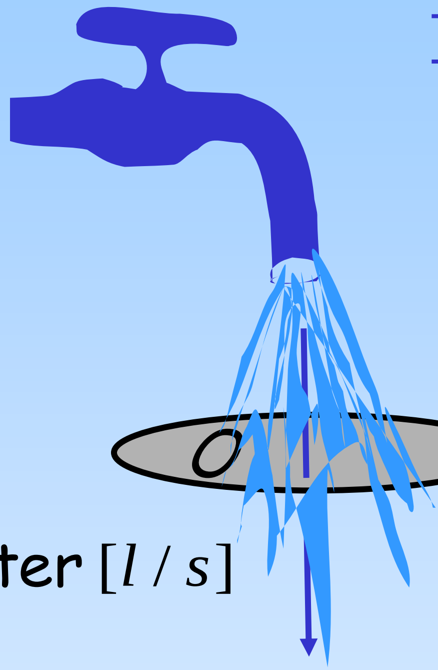
$$d\hat{o} = \hat{e}_n\, do$$



$$\angle(\vec{g}, d\hat{o}) = 0^o$$

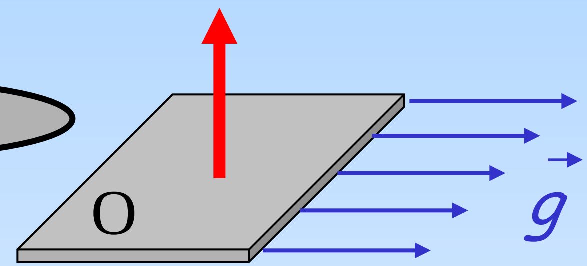
$$F_g \equiv \int_{oppervlak\, O} d\hat{o}\cdot\vec{g} \equiv \int_{oppervlak\, O} do\, \hat{e}_n\cdot\vec{g} = Og$$

Waterkraan:



"Flux":

$$\int_{oppervlak\, O} do \;\; \text{water}\, [l/s]$$

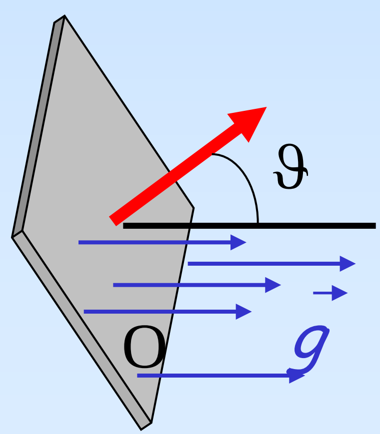


$$\angle(\vec{g}, d\hat{o}) = 90^o$$

$$F_g \equiv \int_{oppervlak\, O} d\hat{o}\cdot\vec{g} = 0$$

Verband tussen:

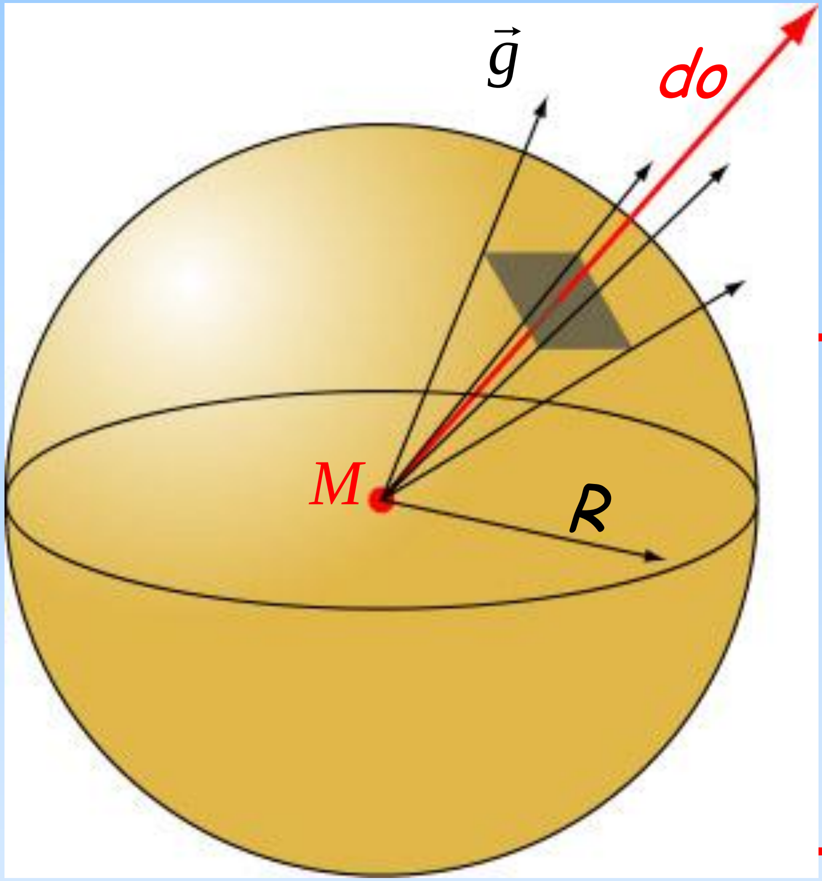
- open/dicht van de kraan
- “flux” door oppervlak O



$$\angle(\vec{g}, d\hat{o}) = \vartheta$$

$$F_g \equiv \int_{oppervlak\, O} do\, \hat{e}_n\cdot\vec{g} = Og\cos\vartheta$$

Gevolg wet van gravitatiewet



Massa M in middelpunt bol

Flux F_g door boloppervlak wordt:

$$F_g \equiv \oint_{bol} \vec{g}\cdot d\vec{o} = \oint_{bol} \frac{-GM}{R^2} do \qquad (\vec{g} // d\vec{o})$$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{-GM}{R^2} R^2 \sin\theta \, d\varphi \, d\theta \qquad (r\theta\varphi)$$

$$= -GM \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\varphi = -GM\, 4\pi = -4\pi GM$$

De essentie:

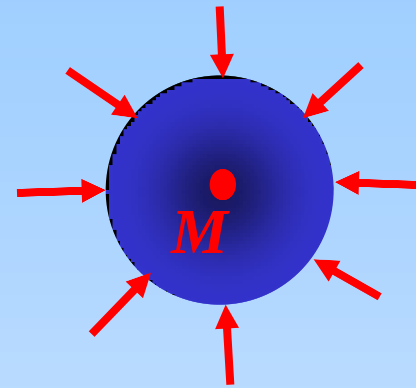
- $g \propto 1/r^2$
- boloppervlak $\propto r^2$

$F_g = -4\pi GM$ geldt voor ieder omsluitend oppervlak; niet alleen voor bol met M in middelpunt!

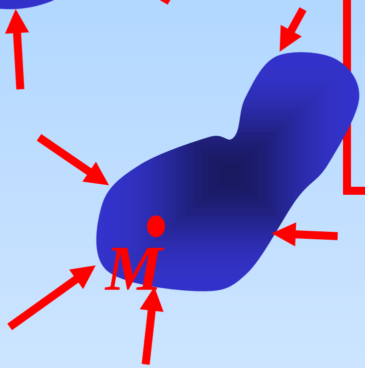
Wet van Gauss:

$$F_g \equiv \oint_{\text{oppervlak}\,O} \vec{g} \cdot d\hat{o} = -4\pi G \sum_{\text{in V}} M_i$$

Massa M omsloten door een boloppervlak

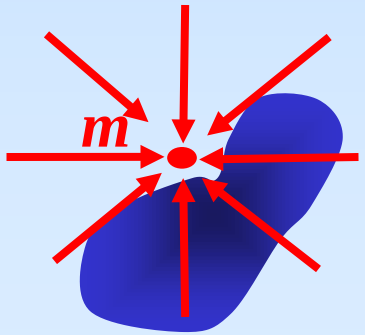


Massa M omsloten door willekeurig oppervlak



$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} F_g = -4\pi G M$$

Massa m buiten een willekeurig oppervlak

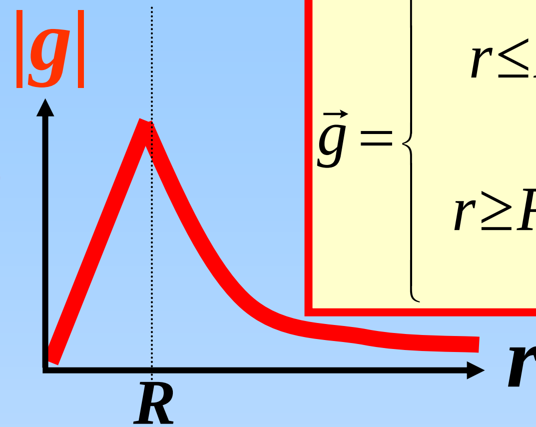


$$F_g = 0$$

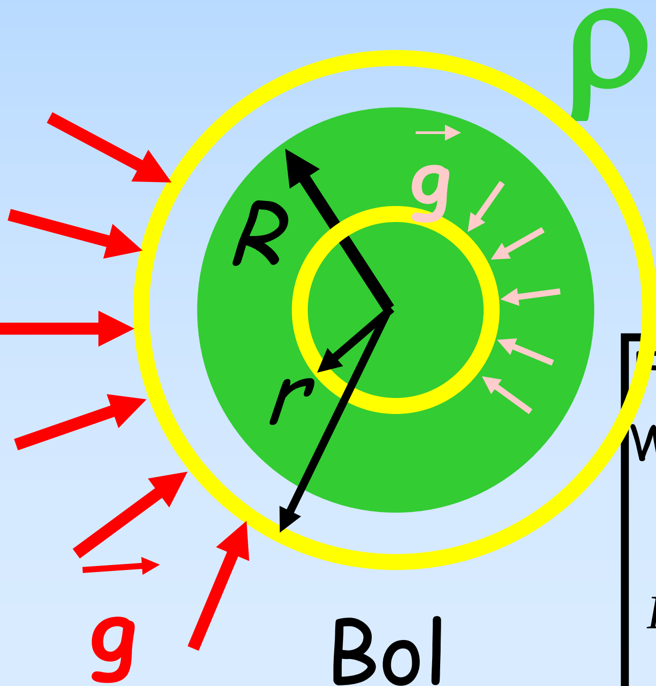
V.b. Gauss: bolvolume

Bolvolume:

- massaverdeling: ρ kg/m³
- symmetrie: g ⊥ bol, g(r)
- "Gauss box": bolletje



$$\vec{g} = \begin{cases} r \le R: \ \vec{g} = -\dfrac{4\pi G \rho \vec{r}}{3} \\ r \ge R: \ \vec{g} = -\dfrac{4\pi G \rho R^3 \hat{r}}{3r^2} \end{cases}$$



Flux: $F_g = 4\pi r^2 g$

Wet van Gauss:

$$F_g = 4\pi G \sum_{omsloten} M \Rightarrow \begin{cases} r < R: \ 4\pi r^2 g = -4\pi G \dfrac{4}{3}\pi r^3 \rho \Leftrightarrow g = -\dfrac{4\pi G \rho r}{3} \\ r > R: \ 4\pi r^2 g = -4\pi G \dfrac{4}{3}\pi R^3 \rho \Leftrightarrow g = -\dfrac{4\pi G \rho R^3}{3r^2} \end{cases}$$

Stelling van Gauss (wiskunde)

De divergentie

De stelling van Gauss

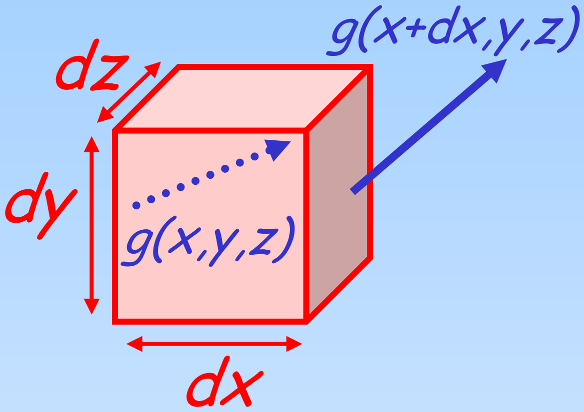
Voorbeeld

Divergentie:

$$\vec{\nabla}\cdot\vec{g} \equiv \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

Beschouw lokaal de uitdrukking (Gauss):

$$\oint_{oppervlak} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volume} \rho\, dv$$



$$\oint_{oppervlakje} \vec{g}\cdot d\vec{o} = -dxdy\left(g_z(x,y,z+dz) - g_z(x,y,z)\right) +$$

$$-dzdx\left(g_y(x,y+dy,z) - g_y(x,y,z)\right) +$$

$$-dydz\left(g_x(x+dx,y,z) - g_x(x,y,z)\right)$$

$$= -dxdydz\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}\right)$$

$$4\pi G \int_{volumetje} \rho\, dv = 4\pi G\, dxdydz\, \rho(x,y,z)$$

Compactere notatie via "divergentie":

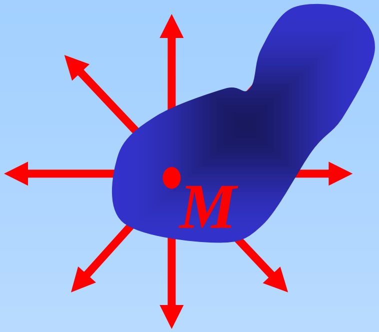
$$\vec{\nabla}\cdot\vec{g} \equiv \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

Dus:

$$\oint_{oppervlakje} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volumetje} \rho(\vec{r})\, dv \;\Leftrightarrow\; \vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G \rho(\vec{r})$$

De link: wiskunde & natuurkunde

M.b.v. gravitatiewet gevonden:



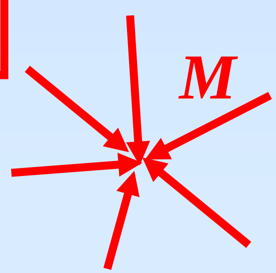
Wet van Gauss: $$F_g \equiv \oint_{oppervlak\,O} \vec{g}\cdot d\hat{o} = -4\pi G \int_{volume} \rho\, dv$$

M.b.v. Stelling van Gauss kan je “integrale” verband tussen g-veld en massaverdeling omzetten in “differentiaal verband:

$$\int_{volume} \vec{\nabla}\cdot\vec{g}\, dv = \oint_{oppervlak} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volume} \rho\, dv \;\Rightarrow\; \vec{\nabla}\cdot\vec{g} = -4\pi G\rho$$

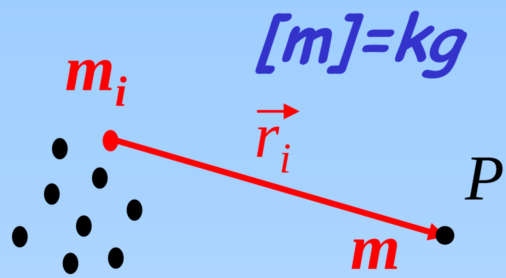
Wiskunde: Gauss

Natuurkunde: gravitatie/Gauss

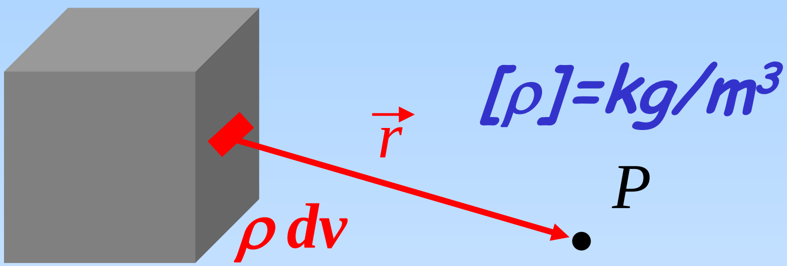


gravitatiepotentiaal

Gravitatiewet:



$$\vec{F} = m\vec{g}_P \equiv -G\sum_{i=1}^{N}\frac{mm_i}{r_i^2}\hat{r}_i = -m\nabla\Phi(\vec{r})$$



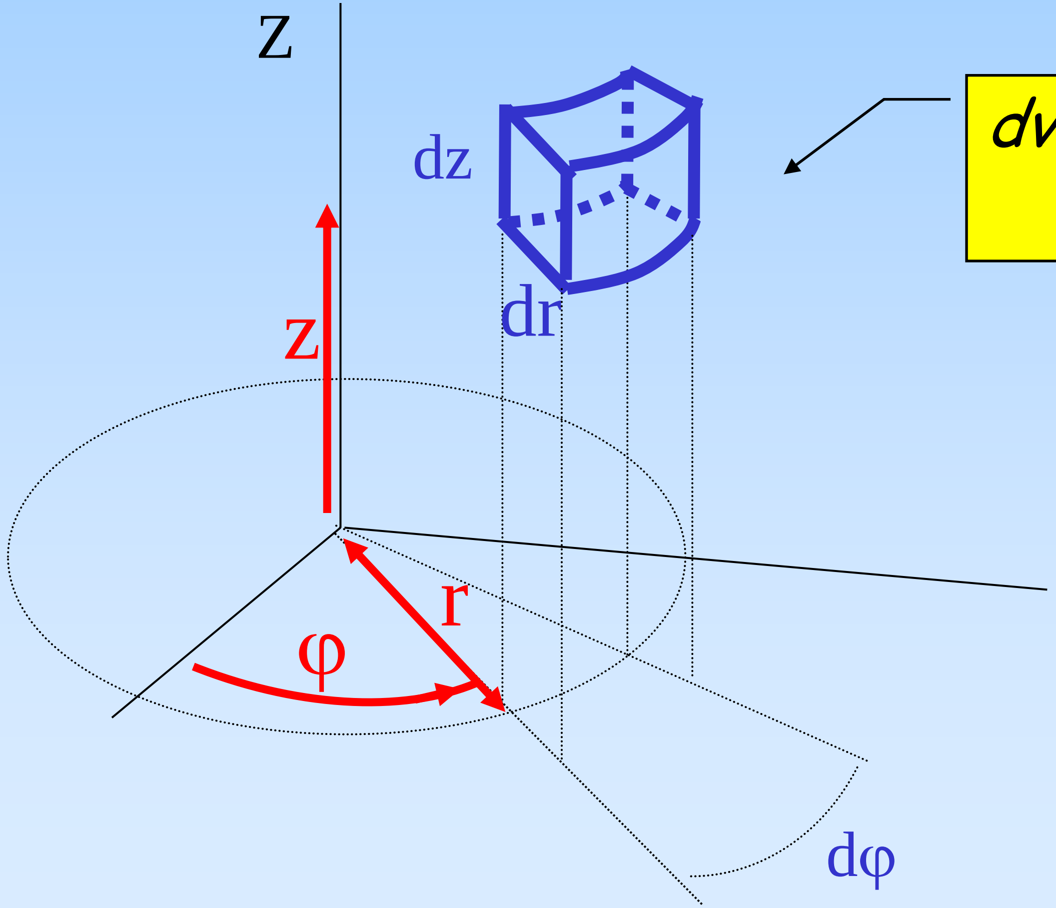
$$\vec{g}_P \equiv -G\int_{volume} dv\frac{\rho}{r^2}\hat{r} = -\nabla\Phi(\vec{r})$$

$$\vec{g}(\vec{r}) \equiv -\nabla\Phi(\vec{r})$$

$$\int_{volume}\vec{\nabla}\cdot\vec{g}\,dv = \oint_{oppervlak}\vec{g}\cdot d\vec{o} = -4\pi G\int_{volume}\rho\, dv \;\Rightarrow\; \vec{\nabla}\cdot\vec{g} = -4\pi G\rho$$

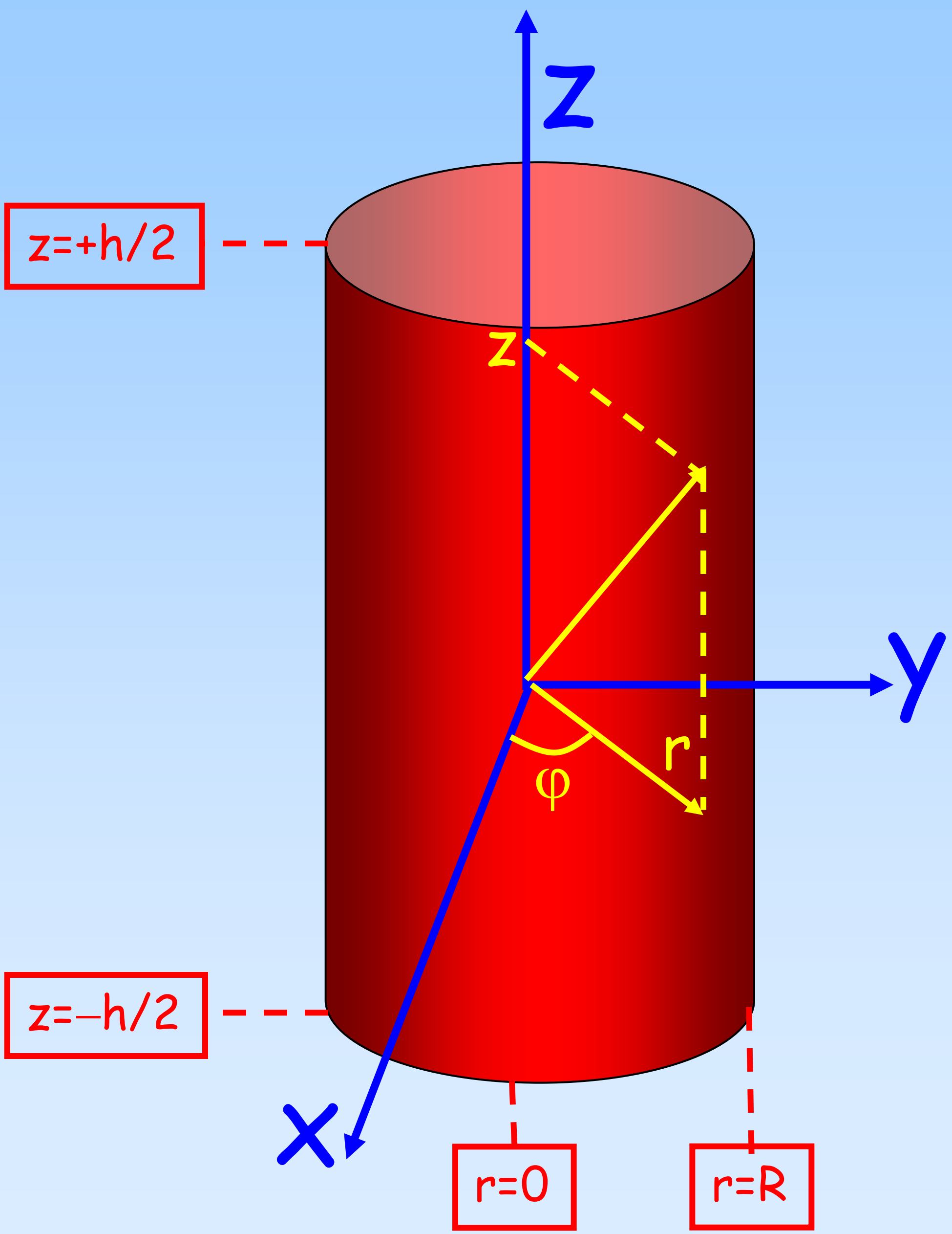
$$-\nabla\cdot\vec{g}(\vec{r}) = \nabla\cdot\nabla\Phi(\vec{r}) = \nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

Volume integraal: cilindercoördinaten



$dv = (dz)\ (r d\varphi)\ dr$
$= r\, dz dr d\varphi$

Voorbeeld: cilinder inhoud



Om de cilinder inhoud te bepalen integreer je de functie “1” over het cilinder volume:

Integratie domein:

z:	[−h/2,+h/2]
r:	[0,R]
φ:	[0,2π]

Integraal:

$$\int_{cilinder} 1 dV \rightarrow \int_{-h/2}^{+h/2}\int_{0}^{2\pi}\int_{0}^{R} r dr d\varphi dz$$

$$= \int_{-h/2}^{+h/2}\int_{0}^{2\pi} \frac{1}{2} r^2 \Big|_0^R d\varphi dz$$

$$= \int_{-h/2}^{+h/2}\int_{0}^{2\pi} \frac{1}{2} R^2 d\varphi dz = \pi R^2 h$$

V.b.: hoeveel m^3 H_2O ongeveer op aarde?

Straal aarde: $\approx 6.400 \times 10^6$ m
Gemiddelde H_2O laag: $\approx 10^3$ m

$\Rightarrow$ integratie domein:

r:	[$Ri \equiv 6.399 \times 10^6$ m, $Ro \equiv 6.400 \times 10^6$ m]
θ:	$[0,\pi]$
φ:	$[0,2\pi]$

$$\int_{bolschil} 1 dV \rightarrow \int_{Ri}^{Ro}\int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta d\varphi d\theta dr$$

$$= \int_{Ri}^{Ro}\int_0^{\pi} r^2 \sin\theta \left(\varphi\Big|_0^{2\pi}\right) d\theta dr = 2\pi \int_{Ri}^{Ro} r^2 \left(-\cos\theta\Big|_0^{\pi}\right) dr$$

$$= \frac{4\pi}{3}\left(r^3\Big|_{Ri}^{Ro}\right) = \frac{4\pi}{3}\left(Ro^3 - Ri^3\right) \rightarrow 5.15 \times 10^{17} m^3$$

Natuurlijk zelfde als volume van een 10^3 m dikke bolschil bij r= 6.400×10^6 m:

$$H_2O \approx 4\pi(6.400 \times 10^6)^2 \times 10^3 \approx 5.15 \times 10^{17}\ m^3$$