PHY3004: Nuclear and Particle Physics

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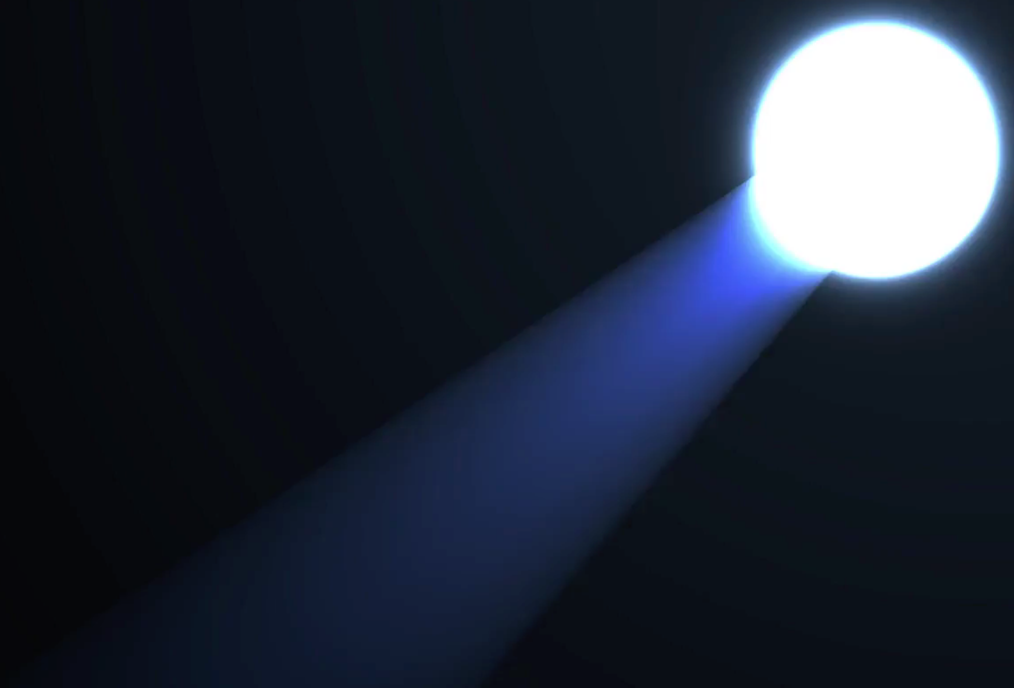
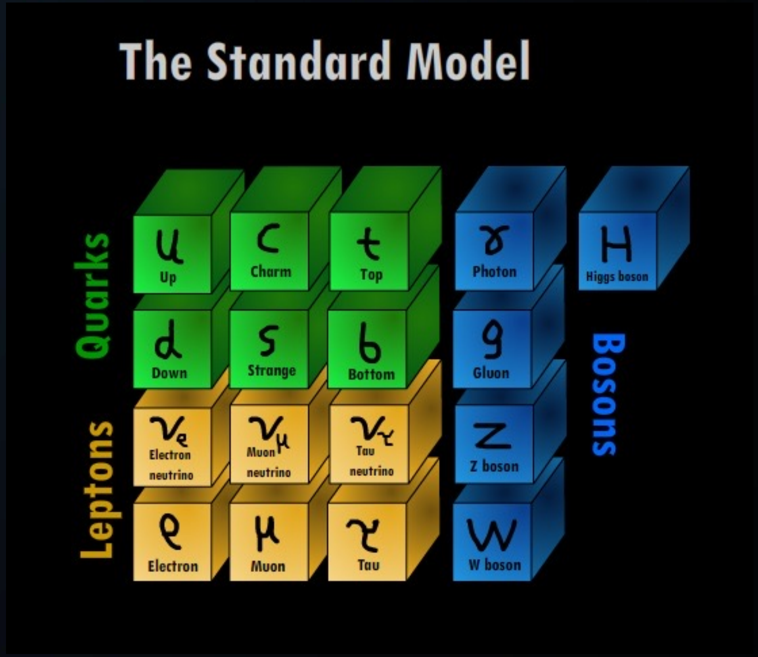


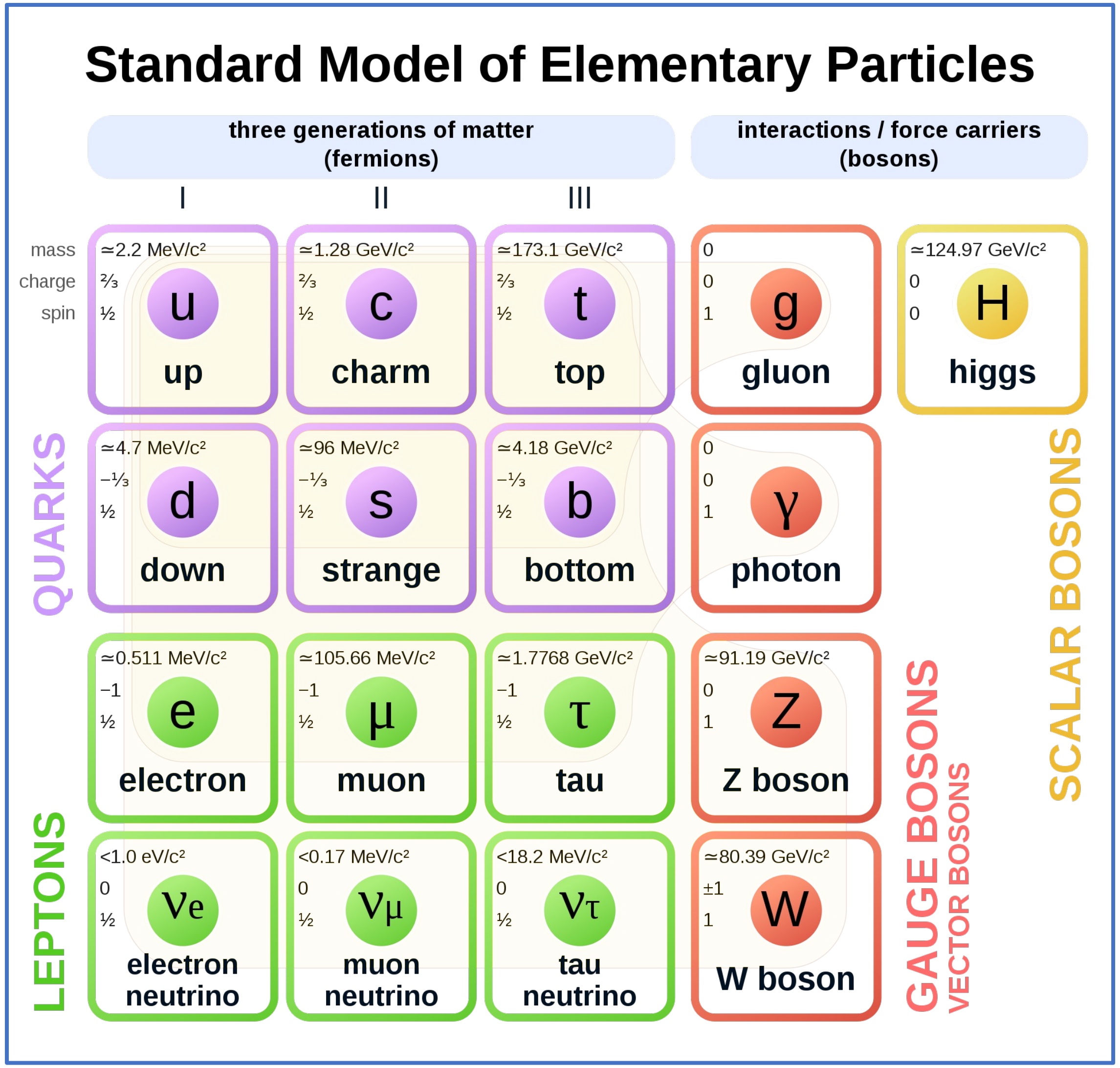


Lecture 1: “Particles”

Griffiths chapter 1

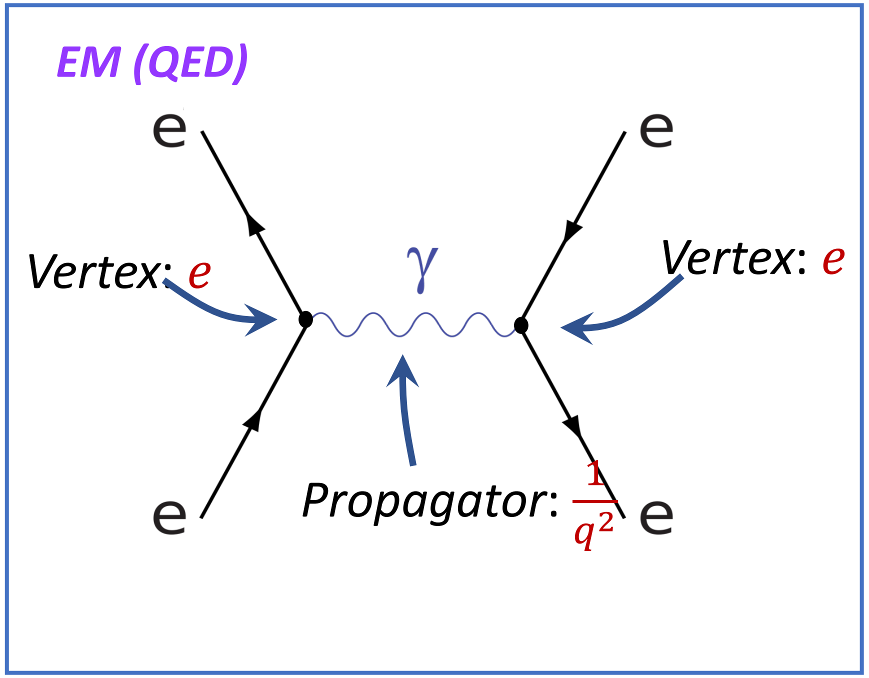
Classification of particles

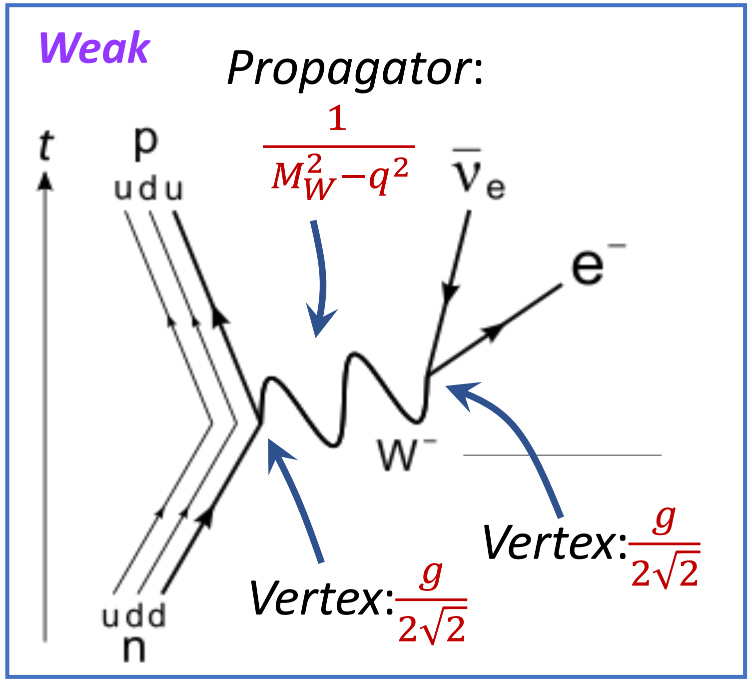
- Lepton: fundamental particle
- Hadron: consist of quarks
  - Meson: 1 quark + 1 antiquark (π^+,B_s^0, …)
  - Baryon: 3 quarks (p ,n , Λ, …)
    - Anti-baryon: 3 anti-quarks
- ***Fermion***: particle with half-integer spin.
  - Antisymmetric wave function: obeys Pauli-exclusion principle and Pauli-Dirac statistics
  - All fundamental quarks and leptons are spin-½
  - Baryons (S=$^1/_2$, $^3/_2$)
- ***Boson***: particle with integer spin
  - Symmetric wave function: Bose-Einstein statistics
  - Mesons: (S=0, 1), Higgs (S=0)
  - Force carriers: γ, W , Z, g (S=1); graviton(S=2)

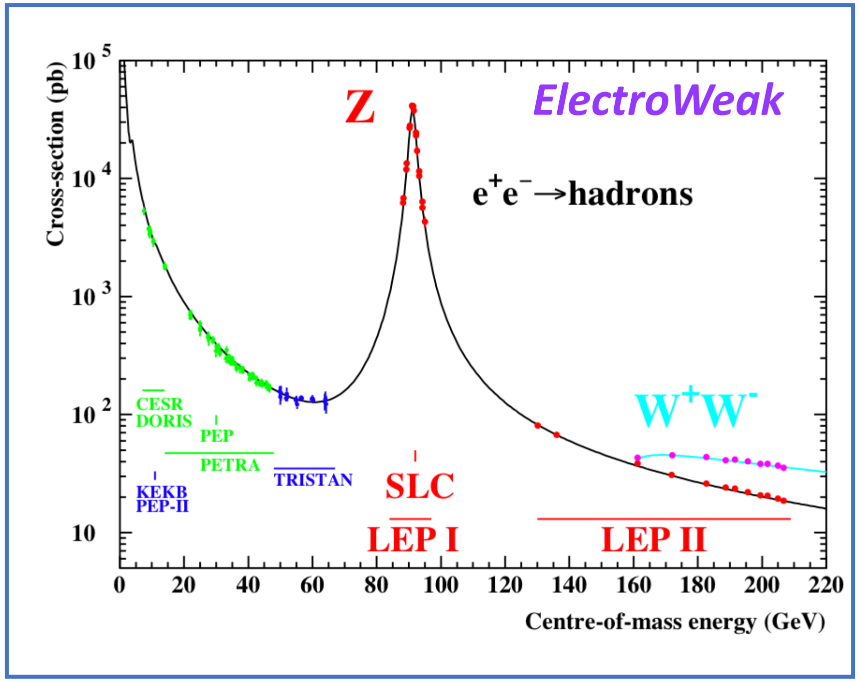


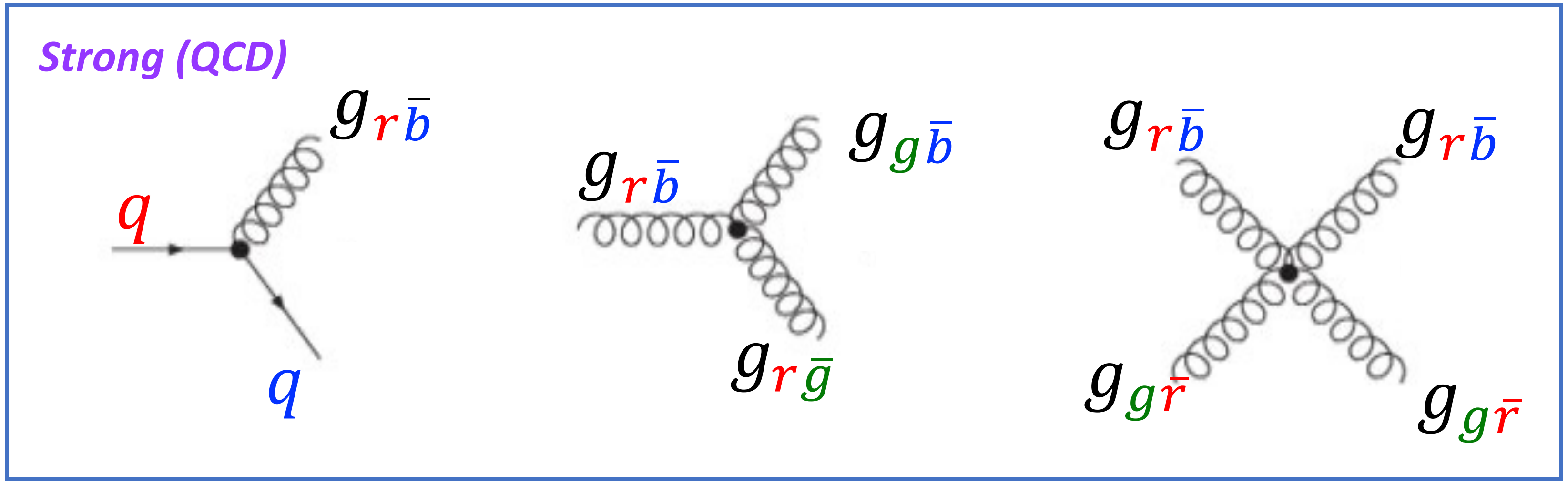
Lecture 2: “Forces”

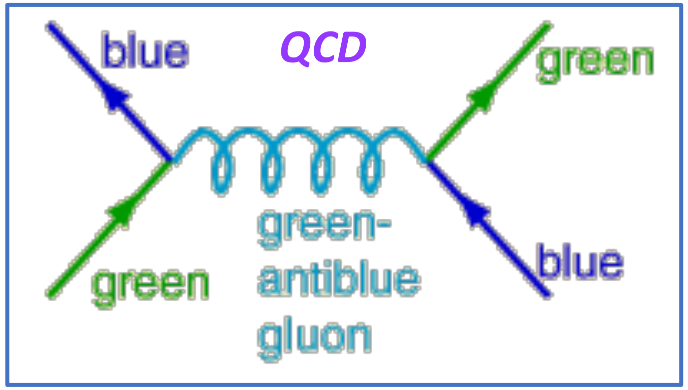
Griffiths chapter 2





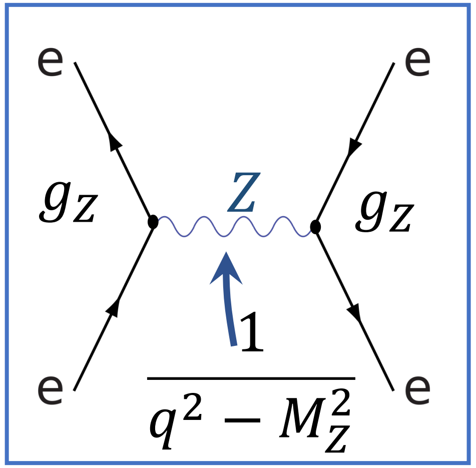


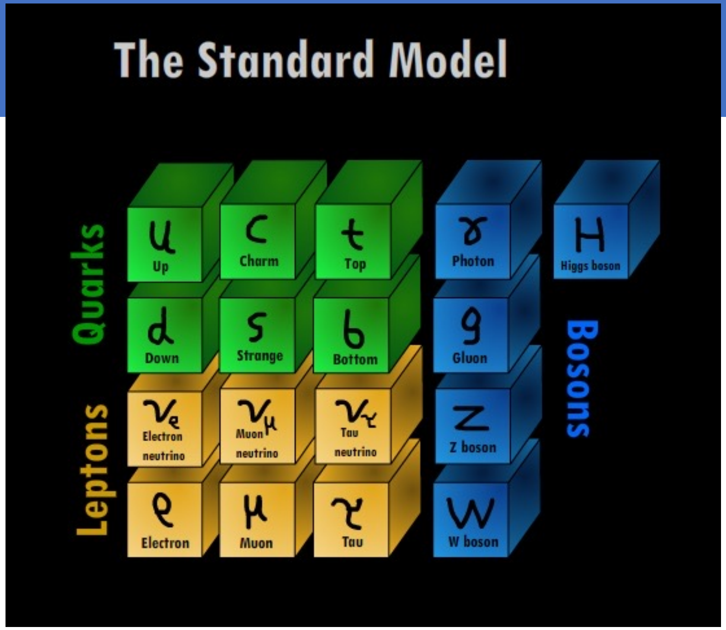


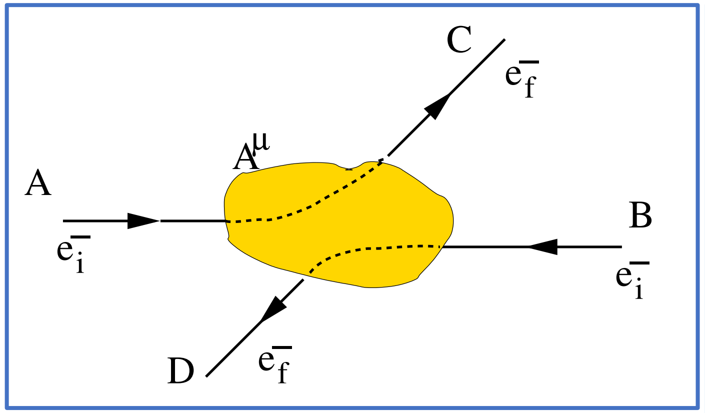


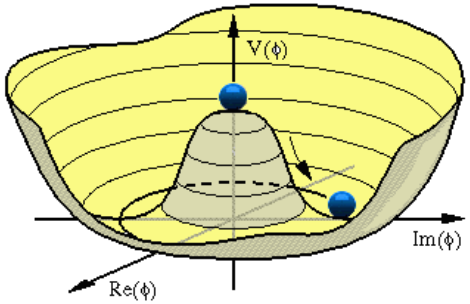
Recap: “Seeing the wood for the trees”

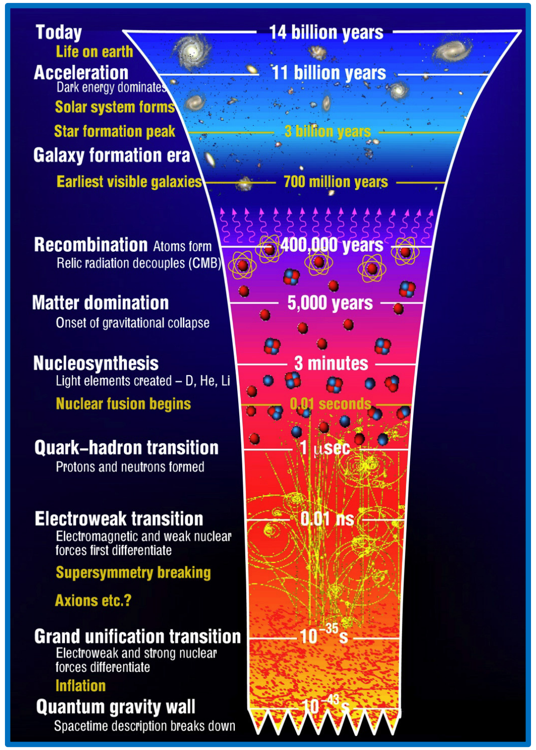
- Lecture 1: “Particles”
  - Zooming into constituents of matter
  - Skills: distinguish particle types, Spin
- Lecture 2: “Forces”
  - Exchange of quanta: EM, Weak, QCD
  - Skills: 4-vectors, Feynman diagrams
- Lecture 3: “Waves”
  - Quantum fields and gauge invariance
  - Dirac algebra, Lagrangian, co- & contra variant
- Lecture 4: “Symmetries”
  - Standard Model, Higgs, Discrete Symmetries
  - Skills: Lagrangians, Chirality & Helicity
- Lecture 5: “Scattering”
  - Cross section, decay, perturbation theory
  - Skills: Dirac-delta function, Feynman Calculus
- Lecture 6: “Detectors”
  - Energy loss mechanisms, detection technologies











Wave Equations

Contents:

1. Wave equations and Probability
   a) Wave equations for spin-0 fields
      - Schrödinger (non relativistic), Klein-Gordon (relativistic)
   b) Wave equation for spin-½ fields
      - Dirac equation (relativistic)
      - Fundamental fermions
   c) Wave equations for spin-1 fields
      - Gauge boson fields; eg. electromagnetic field
2. Gauge field theory
   a) Variational Calculus and Lagrangians
   b) Local Gauge invariance
      i. QED
      ii. Yang-Mills Theory (Weak, Strong)

Griffiths chapter 7

If you are unfamiliar with the math, just focus on the concepts.
The math requires some practice, but is less tricky then it may look.
→ Also, check the recorded videos.

Griffiths chapter 10

- Required Quantum Mechanics knowledge:
  - Angular momentum and spin: study Griffiths sections 4.2 ,4.3, In particular Pauli Matrices

Part 1
Wave Equations and Probability

1a) Spin-0
(think pions etc)

Schrödinger Equation and Probability

$$i\frac{\partial}{\partial t}(t\psi) - it\frac{\partial\psi}{\partial t} = i\psi + it\frac{\partial\psi}{\partial t} - it\frac{\partial\psi}{\partial t} = i\psi$$

- Quantization of classical non-relativistic theory:
  - Take $E = \frac{\vec{p}^2}{2m}$ and substitute energy and momentum by operators that operate on ψ:

$\vec{p} = m\vec{v}$

$E = {}^1/_2\, mv^2$

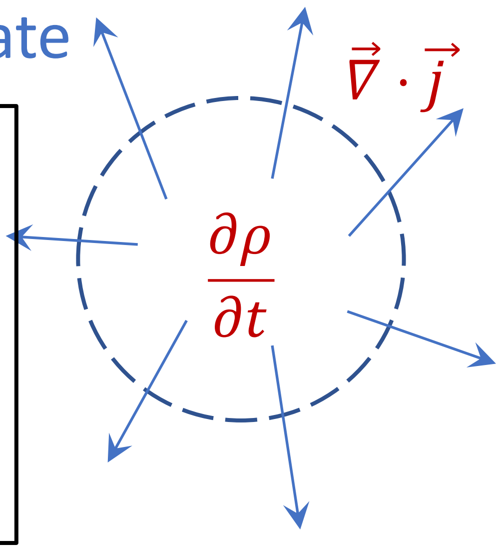
$$E \to \hat{E} = i\hbar\frac{\partial}{\partial t} \quad ; \quad p \to \hat{p} = -i\hbar\vec{\nabla}$$

$([E,t] = -i\hbar \;\; , \;\; [x,p] = -i\hbar)$

  - Result is Schrödinger's equation: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi \;\; \to \;\; \frac{\partial}{\partial t}\psi = \frac{i\hbar}{2m}\nabla^2\psi$
  - Plane wave solutions: $\psi = Ne^{i(\vec{p}\vec{x}-Et)/\hbar}$ with the kinematic relation $E = p^2/2m$

$\partial\psi/\partial t = -iE\psi$ and $\nabla^2\psi = p^2\psi/2m$

- Multiply both sides Schrödinger by ψ^* and add its complex conjugate



$$\psi^*\frac{\partial}{\partial t}\psi = \psi^*\left(\frac{i\hbar}{2m}\right)\nabla^2\psi$$

$$\psi\frac{\partial}{\partial t}\psi^* = \psi\left(\frac{-i\hbar}{2m}\right)\nabla^2\psi^*$$

$+$

$$\frac{\partial}{\partial t}\underbrace{(\psi^*\psi)}_{\rho} = -\vec{\nabla}\cdot\underbrace{\left[\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi\vec{\nabla}\psi\right)\right]}_{\vec{j}}$$

*Use: $\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \psi^*\nabla^2\psi - \psi\nabla^2\psi^*$

Recognize "continuity" equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

Law of conserved currents, with:

$\rho \equiv \psi^*\psi = |N|^2$

$\vec{j} \equiv \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right) = \frac{|N|^2}{m}\vec{p}$

- Interpret: probability waves!

Relativistic: Klein-Gordon equation

- Quantization of relativistic theory

Note: $p_\mu \rightarrow -i\hbar\partial_\mu = -i\hbar(\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$

  - Start with $E^2 = p^2c^2 + m^2c^4$ and substitute again $E \rightarrow i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \rightarrow -i\hbar\vec{\nabla}$ operates on ϕ
  - Result is Klein-Gordon equation: $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi$

Use now: $\hbar = c = 1$

  - Plane wave solutions: $\phi = Ne^{i(\vec{p}\vec{x}-Et)/\hbar}$ with relativistic relation $E^2 = \vec{p}^2 + m^2$
- Use the covariant notation:

$$p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$$

$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$; $\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$ *(Exercise 5)*

$\partial_\mu\partial^\mu \equiv \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ (as usually take $c = \hbar = 1$)

$p^0 = E$ and $x^0 = t$

- Klein-Gordon in four-vector notation: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$
- Plane wave solutions: $\phi = Ne^{-i(p_\mu x^\mu)}$ (Remember this is: $\phi = Ne^{-i(Et-\vec{p}\vec{x})}$)
- Time and space coordinates are now treated fully symmetric
  - This is needed in a relativistic theory where time and space for different observes are linear combinations of each other

Klein-Gordon conserved currents

- Similar to the Schrödinger case multiply both sides by $-i\phi^*$ from left and add the expression to its complex conjugate

$$-i\phi^*\left(-\frac{\partial^2\phi}{\partial t^2}\right) = -i\phi^*(-\nabla^2\phi + m^2\phi)$$

$$i\phi\left(-\frac{\partial^2\phi^*}{\partial t^2}\right) = i\phi(-\nabla^2\phi^* + m^2\phi^*)$$

+

$$\frac{\partial}{\partial t}\underbrace{i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)}_{\rho} = \vec{\nabla}\cdot\underbrace{\left[i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*)\right]}_{-\vec{j}}$$

Again recognize "continuity" equation, the law of conserved currents:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

With now:

$$j^\mu = (\rho, \vec{j}) = i[\phi^*(\partial^\mu\phi) - \phi(\partial^\mu\phi^*)]$$

It gives for plane waves: $\phi = Ne^{-i(p_\mu x^\mu)}$

$$\rho = 2|N|^2E$$

$$\vec{j} = 2|N|^2\vec{p}$$

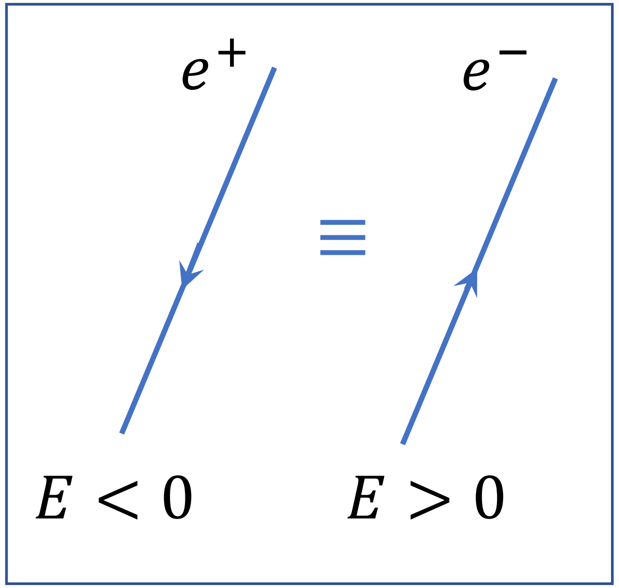
Or in 4-vector: $j^\mu = 2|N|^2p^\mu$

- The quadratic equation $E^2 = p^2 + m^2$ leads to *double solutions:* $E^2 = \cdots \Rightarrow E = \pm\cdots$
  - Positive and negative energy solutions
  - Negative solutions imply *negative probability* density ρ *!!!*
  - This bothered Dirac and therefore he looked for an equation ***linear*** in E and p …

Antiparticles

- Feynman-Stückelberg interpretation
  - Charge current of an electron with momentum $\vec{p}$ and energy E
    $j^\mu(-e) = -2e|N|^2 p^\mu = -2e|N|^2(E, \vec{p})$
  - Charge current of a positron
    $j^\mu(+e) = +2e|N|^2 p^\mu = -2e|N|^2(-E, -\vec{p})$

  The positron current with energy $-E$ and momentum $-\vec{p}$ is the same as the electron current with E and $\vec{p}$



- The negative energy *particle* solutions going backward in time describe the positive-energy *antiparticle* solutions.
  - The wave function $\phi = Ne^{-ix_\mu p^\mu}$ stays invariant for negative energy and going backwards in time
  - Consider eg. $e^{-i(-E)(-t)} = e^{-iEt}$

- A positron ***is*** an electron travelling backwards in time

Wave Equations

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1. Wave equations and Probability

Griffiths chapter 7 and PP1 chapter 1

   a) Wave equations for spin-0 fields
      - Schrödinger (non relativistic), Klein-Gordon (relativistic)
   b) Wave equation for spin-½ fields
      - Dirac equation (relativistic)
      - Fundamental fermions
   c) Wave equations for spin-1 fields
      - Gauge boson fields; eg. electromagnetic field

2. Gauge field theory

Griffiths chapter 10 and PP1 chapter 1

   a) Variational Calculus and Lagrangians
   b) Local Gauge invariance
      i. QED
      ii. Yang-Mills Theory (Weak, Strong)

- Required Quantum Mechanics knowledge:
  - Angular momentum and spin: study Griffiths sections 4.2 ,4.3, In particular Pauli Matrices

Part 1
Wave Equations and Probability

1b) Spin-½
(fundamental leptons and quarks)

Dirac Equation

Instead of $E^2 = p^2c^2 + m^2c^4$

- Dirac did not like negative probabilities and looked for a wave equation of linear form $E = i\frac{\partial}{\partial t}\psi = H\psi = (?)$, but relativistic correct.
- Try: $H = (\vec{\alpha}\cdot\vec{p} + \beta m)$ where $\vec{\alpha}\cdot\vec{p} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z$; $\vec{\alpha}?\ \ \beta?$
- We know that: $H^2\psi = E^2\psi = (\vec{p}^2 + m^2)\psi$
- Write it out: $H^2 = (\sum_i \alpha_i p_i + \beta m)(\sum_j \alpha_j p_j + \beta m)$

$$= \left(\sum_{i,j}\alpha_i\alpha_j p_i p_j + \sum_i \alpha_i\beta p_i m + \sum_j \beta\alpha_j p_j m + \beta^2 m^2\right)$$

$$= \left(\underbrace{\sum_i \alpha_i^2 p_i^2}_{=\vec{p}^2} + \underbrace{\sum_{i>j}(\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j + \sum_i(\alpha_i\beta + \beta\alpha_i)p_i m}_{=0} + \underbrace{\beta^2 m^2}_{=m^2}\right)$$

- This works out if:
  - $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$
  - $\alpha_1, \alpha_2, \alpha_3, \beta$ anti-commute: ie.: $\alpha_1\alpha_2 = -\alpha_2\alpha_1$ etc
- Anti-commutator: $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$; $\{\alpha_i, \beta\} = 0$; $\beta^2 = 1$
  - Using definition: $\{A, B\} = AB + BA$:

Dirac's idea

$$\boxed{\{\alpha_i, \alpha_j\} = 2\delta_{ij}\,;\, \{\alpha_i, \beta\} = 0\,;\; \beta^2 = 1}$$

- Clearly α_i and β cannot be numbers. Let them be *matrices*!
  - In that case they operate on a wave function that is a column vector
  - The simplest case that allows the requirements are 4x4 matrices.
  - Dirac's equation becomes:

$$E\psi = H\psi = (\vec{\alpha}\vec{p} + \beta m)\psi$$

$$\boxed{\begin{array}{l}\text{Remember:}\\ E \to i\dfrac{\partial}{\partial t} \qquad \vec{p} \to -i\vec{\nabla}\end{array}}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \left[-i\underbrace{\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}}_{\vec{\alpha_i}}\cdot\vec{\nabla_i} + \underbrace{\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}}_{\beta}\cdot m\right]\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

  - It is possible making use of the Pauli spin matrices
    - $\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}$ and $\beta = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$ with $\sigma_1 = \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $\sigma_2 = \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$; $\sigma_3 = \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
    - Note that α and β are hermitian: $\alpha_i^\dagger = \alpha_i$ and $\beta^\dagger = \beta$ (Since Hamiltonian has real E eigenvalues.)

  - Remember: $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\,\mathbb{1}$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}$

  - Unfortunately we also need: $\sigma_i\beta + \beta\sigma_i = 0$ ➔ We need 4x4 matrices!

Dirac's idea

- Clearly α_i and β cannot be numbers. Let them be *matrices*!
  - In that case they operate on a wave function that is a column vector
  - The simplest case that allows the requirements are 4x4 matrices.
  - Dirac's equation becomes:

Remember:

$E \to i\frac{\partial}{\partial t} \qquad p \to -i\vec{\nabla}$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \left[-i\underbrace{\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}}_{\vec{\alpha_i}}\cdot\vec{\nabla_i} + \underbrace{\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}}_{\beta}\cdot m\right]\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

  - This is a very complicated equation!
    - What does it mean that the wave function ψ is now a 1-by-4 column vector?
    - ψ is ***not*** a 4-vector, since the indices do not represent kinematic variables, but matrix indices!
    - ψ is a called a spinor.

Covariant form of Dirac's equation

- Dirac equation: $H = E = (\vec{\alpha}\cdot\vec{p} + \beta m) \quad \Rightarrow \quad i\frac{\partial}{\partial t}\psi = \left(-i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$

- Multiply Dirac's eq. from the left by β; then it becomes:
  - $\left(\underbrace{i\beta}_{\gamma^0}\frac{\partial}{\partial t}\psi + i\,\underbrace{\beta\vec{\alpha}}_{\gamma^1,\gamma^2,\gamma^3}\cdot\vec{\nabla} - m\right)\psi = 0$ (Remember $\beta^2 = 1$)

- Introduce now the Dirac γ-matrices: $\gamma^\mu \equiv (\beta, \beta\vec{\alpha})$ (vector of four 4x4matrices!)
  - Covariant form of Dirac eq:

$\partial_0 = \frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z}$

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

Note (see def covariant derivative):
$A^\mu\partial_\mu = A^0\frac{\partial}{\partial t} + \vec{A}\cdot\vec{\nabla}$

- Realise that Dirac's equation is a set of 4 coupled differential equations.
  - Remember that ψ has 4 components!
- Requirements on $\vec{\alpha}$, β can be summarized as: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (check it)

Dirac Gamma Matrices

- There is some freedom to implement: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ in 4x4 matrices.
  - We will use the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note the indices: (confusing!)

$\mu, \nu = 0,1,2,3$ are the ***Lorentz indices in space-time***:

Dirac matrix indices: 1,2,3,4 Have to do with the row and column indices of the matrix (and spinors)

Or: $\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ and $\gamma^k = \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$ with Pauli matrices σ_k

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \ \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Note: although the gamma indices are Lorentz-indices (“space-time”), the gamma-matrices are not 4-vectors! (They are simply constants.)

Exercises: Dirac Algebra

- Dirac algebra:
  - Write the explicit form of the γ-matrices
  - Show that : $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
  - Show that : $(\gamma^0)^2 = \mathbb{1}_4$; $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{1}_4$
  - Use anti-commutation rules of α and β to show that: ${\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0$
  - Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and show: ${\gamma^5}^\dagger = \gamma^5$; $(\gamma^5)^2 = \mathbb{1}_4$; $\{\gamma^5, \gamma^\mu\} = 0$

Exercise: Solutions of free Dirac equation

See Griffiths for a derivation of the solutions

a) Show that the following plane waves are solutions to Dirac's equation

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \quad ; \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

Before KG:

$$\phi = Ne^{-i(p_\mu x^\mu)}$$

$$\psi_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x+ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \quad ; \quad \psi_4 = \begin{pmatrix} (p_x-ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

b) Write the Dirac equation for particle in rest (choose $\vec{p}=0$) and show that ψ_1 and ψ_2 are *positive energy* solutions: $E = +\left|\sqrt{\vec{p}^2+m^2}\right|$ whereas ψ_3 and ψ_4 are *negative energy* solutions: $E = -\left|\sqrt{\vec{p}^2+m^2}\right|$.

c) Consider the *helicity* operator $\vec{\sigma}\cdot\vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$ and show that ψ_1 corresponds to *positive helicity* solution and ψ_2 to *negative helicity.* Similarly for ψ_3 and ψ_4.

Spin and Helicity – hint for exercise c)

- For a given momentum p there still is a *two-fold degeneracy* with the same energy: what differentiates solutions ψ_1 from ψ_2? ➔It is spin!!
- Define the spin operator for Dirac spinors: $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, where $\vec{\sigma}$ are the three 2x2 Pauli spin matrices
- Define ***helicity*** λ as spin "up"/"down" wrt direction of motion of the particle

$$\lambda = \frac{1}{2}\vec{\Sigma}\cdot\hat{p} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} = \frac{1}{2|p|}(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)$$

- Split off the Energy and momentum part of Dirac's equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- Exercise: Try solutions ψ_1 and ψ_2 to see they are ***helicity eigenstates*** with $\lambda = +1/2$ and $\lambda = -1/2$
- Dirac wanted to solve negative energies and he found spin-½ fermions!

Antiparticles: positive and negative energy solutions

- Dirac spinor solutions $\psi_i(x^\mu) = \psi_i(t,\vec{x}) = u_i(E,\vec{p})e^{i(\vec{p}\vec{x}-Et)} = u_i(p^\mu)e^{-ip_\mu x^\mu}$
  with $i = 1,2,3,4$

- Since we work with antiparticles, instead of *negative energy particles* travelling backwards instead in time, *antiparticle solutions* can be defined

$$u_3(-E,-\vec{p})e^{i((-\vec{p})\vec{x}-(-E)t)} = v_2(E,\vec{p})e^{-i(\vec{p}\vec{x}-Et)} = v_2(p^\mu)e^{ip_\mu x^\mu}$$

$$u_4(-E,-\vec{p})e^{i((-\vec{p})\vec{x}-(-E)t)} = v_1(E,\vec{p})e^{-i(\vec{p}\vec{x}-Et)} = v_1(p^\mu)e^{ip_\mu x^\mu}$$

  - Where now the energy of the antiparticle solutions v_1 and v_2 is positive: $E > 0$

- Explicit: $u_4 = \begin{pmatrix} (p_x - ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix}$ and $u_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x + ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix}$ becomes…

Antiparticles: positive and negative energy solutions

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  - Where now the energy of the antiparticle solutions v_1 and v_2 is positive: $E > 0$

- Explicit: $v_1 = \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} p_z/(E+m) \\ (p_x + ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}$
  - Where E and $\vec{p}$ are now the energy and momentum of the antiparticle

Adjoint spinors

Dirac: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

- Adjoint spinors
  - Solutions of the Dirac equation are called *spinors*
  - Current density and continuity equation require *adjoints* instead of *complex conjugates*

Remember: $(AB)^\dagger = B^\dagger A^\dagger$

$$D: \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i \sum_{k=1,2,3} \gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$$

$$D^\dagger: \; -i \frac{\partial \psi^\dagger}{\partial t}\gamma^0 - i \sum_{k=1,2,3} \frac{\partial \psi^\dagger}{\partial x^k}(-\gamma^k) - m\psi^\dagger = 0$$

$$\boxed{{\gamma^0}^\dagger = \gamma^0 \;\; ; \;\; {\gamma^k}^\dagger = -\gamma^k \;\; ; \;\; -\gamma^k\gamma^0 = \gamma^0\gamma^k}$$

- The minus sign in $(-\gamma^k)$ disturbs the Lorentz invariant form: $\psi^\dagger$ is not physical
- Restore covariance by multiplying second equation from the right by γ^0 and define:

$$\bar{\psi} = \psi^\dagger \gamma^0$$

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

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$$D^\dagger: -i \frac{\partial \psi^\dagger}{\partial t} \gamma^0 \gamma^0 - i \sum_{k=1,2,3} \frac{\partial \psi^\dagger}{\partial x^k} \left(\gamma^0 \gamma^k\right) - m\psi^\dagger \gamma^0 = 0$$

$$\boxed{{\gamma^0}^\dagger = \gamma^0 \;\; ; \;\; {\gamma^k}^\dagger = -\gamma^k \;\; ; \;\; -\gamma^k \gamma^0 = \gamma^0 \gamma^k}$$

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$$D^\dagger: \; -i \frac{\partial \bar{\psi}}{\partial t}\gamma^0 - i \sum_{k=1,2,3} \frac{\partial \bar{\psi}}{\partial x^k}\gamma^k - m\bar{\psi} = 0$$

$$\boxed{{\gamma^0}^\dagger = \gamma^0 \;\; ; \;\; {\gamma^k}^\dagger = -\gamma^k \;\; ; \;\; -\gamma^k\gamma^0 = \gamma^0\gamma^k}$$

- The minus sign in $(-\gamma^k)$ disturbs the Lorentz invariant form: $\psi^\dagger$ is not physical
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$$\bar{\psi} = \psi^\dagger \gamma^0$$

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

- Dirac spinor: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$, adjoint Dirac spinor: $\bar{\psi} = (\overline{\psi_1}, \overline{\psi_2}, \overline{\psi_3}, \overline{\psi_4})$

- Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$; adjoint Dirac equation: $i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

Dirac Current density and conserved current

- Apply a similar trick as before for Schrödinger and Klein-Gordon case:
  - Multiply adjoint Dirac eq from right by ψ and multiply Dirac eq. from left by $\bar{\psi}$

$$
\begin{array}{rl}
 & \left(i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}\right)\ \psi = 0 \\
+ & \bar{\psi}\ \left(i\gamma^\mu\partial_\mu\psi - m\psi\right) = 0 \\
\hline
 & \bar{\psi}\,\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\,\gamma^\mu\psi = 0 \\
 & \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0
\end{array}
$$

Define the 4-vec current:

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$$

Satisfies the continuity equation:

$$\partial_\mu j^\mu = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

- Probability: Zero-th component of the current:

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2 \qquad (\bar{\psi} = \psi^\dagger\gamma^0 \text{ and } {\gamma^0}^2 = 1)$$

- This always gives a *positive probability*, which was the motivation of Dirac.

Dirac in summary

- Dirac was looking for an explanation for positive and negative energy solutions by linearising Klein-Gordon equation
  - He found that his solutions described spin-½ particles
  - He predicted, based on symmetry, that for each particle there should exist an antiparticle (the negative energy solution).

- We had relativistic fields:
  - Spin-0: Klein-Gordon: e.g. pion particles
  - Spin-1/2: Dirac : e.g. quarks and leptons
  - How about forces? Spin=1

Wave Equations

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Griffiths chapter 7 and PP1 chapter 1

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   a) Variational Calculus and Lagrangians
   b) Local Gauge invariance
      i. QED
      ii. Yang-Mills Theory (Weak, Strong)

Griffiths chapter 10 and PP1 chapter 1

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  - Angular momentum and spin: study Griffiths sections 4.2 ,4.3, In particular Pauli Matrices

Part 1

Wave Equations and Probability

1c) Spin-1

The Electromagnetic Field

Griffiths §7.4

- Maxwell equations describe electric and magnetic fields induced by charges and currents: (used Heavyside-Lorentz units: $c = 1, \epsilon_0 = 1, \mu_0 = 1$)

1. Gauss' law: $\vec{\nabla} \cdot \vec{E} = \rho$
2. No magnetic charges: $\vec{\nabla} \cdot \vec{B} = 0$
3. Faraday's law of induction: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
4. Modified Ampère's law: $\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$

From 1. and 4. derive continuity

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

➔ charge conservation

This was the motivation for Maxwell to modify Ampère's law

- Define a Lorentz covariant 4-vector field $A^\mu = (V, \vec{A})$ as follows:

$\vec{B} = \vec{\nabla} \times \vec{A}$ (then automatically 2. follows)

$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$ with $V = A^0$ (then automatically 3. follows)

a) Show Maxwell equations can be summarized in covariant form:

$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ (Derive expressions for ρ and $\vec{j}$ and use: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$)

The Antisymmetric tensor $F^{\mu\nu}$

- Maxwell's equation $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ can be further shortened by introducing the antisymmetric tensor: $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- Show that Maxwell's equations become: $\partial_\mu F^{\mu\nu} = j^\nu$

- Hint: derive the expressions for charge ($q = j^0$) and current ($\vec{I} = \vec{j}$) separately.

Use the identity: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$. Remember the definitions:

$$\mathrm{A}_\mu = (A_0, -\vec{A}) \quad ; \quad \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad ; \quad g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

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Griffiths chapter 10 and PP1 chapter 1

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Part 2
Gauge Theory

2a) Variational Calculus and Lagrangians

Lagrange Formalism in field theory

Griffiths §10.2

- Relativistic Field theory: fields replace the generalized coordinates
  - Also time and space will be treated symmetric
  - Replace $L(q,\dot{q})$ for classical particles by a *Lagrange density* $\mathcal{L}\left(\phi(x^\mu), \partial_\mu\phi(x^\mu)\right)$ in terms of *fields* and *gradients* such that $L \equiv \int d^3x \mathcal{L}\left(\phi, \partial_\mu\phi\right)$

- Principle of least actions becomes:

$$S = \int_{t_1}^{t_2} d^4x\, \mathcal{L}\left(\phi(x^\mu), \partial_\mu\phi(x^\mu)\right) \quad \text{and again} \quad \delta S = 0$$

t_1, t_2 are endpoints of the path

Classical was:

$$S = \int_{t_1}^{t_2} dt\, L(q,\dot{q}) \Rightarrow \delta S = 0$$

- Euler Lagrange Equations of motion becomes:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi(x^\mu)} = \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi(x^\mu)\right)}}$$

  - Classical was:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

Lagrange Formalism in field theory

Griffiths §10.2

- Relativistic Field theory: fields replace the generalized coordinates
  - Also time and space will be treated symmetric
  - Replace $L(q,\dot{q})$ for classical particles by a *Lagrange density* $\mathcal{L}\left(\phi(x^\mu), \partial_\mu\phi(x^\mu)\right)$ in terms of *fields* and *gradients* such that $L \equiv \int d^3x \mathcal{L}(\phi, \partial_\mu\phi)$

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and again $\delta S = 0$

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$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

Exercise: Lagrangians and wave equations

Griffiths §10.2

- Scalar Field (spin 0 “pion”)

  a) Show that the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$ results in the Klein-Gordon equation: $\partial_\mu \partial^\mu \phi + m^2\phi = 0$

- Dirac Field (spin ½ Fermion)

  b) Show that the Euler-Lagrange equations for $\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$ results in the Dirac equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

- Electromagnetic field (spin 1 photon)

  c) Show that $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$ results in Maxwell’s equations: $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$

These Lagrangians are the fundamental objects in quantum field theory

Descriptions of interactions follow from symmetry principles on these objects.

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Griffiths chapter 10 and PP1 chapter 1

   a) Variational Calculus and Lagrangians
   b) Local Gauge invariance
      i. QED
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Part 2
Gauge Theory

2b) Local Gauge Invariance
i) QED

The Gauge Principle: Interactions from symmetries

Griffiths §10.3

- *global* gauge invariance: the phase of the wave function is not observable: Changing the wave function $\psi(x) \rightarrow \psi'(x) = e^{i\alpha}\psi(x)$ should not change the Lagrangian for an electron
  - Look at Dirac Lagrangian: $\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi$
  - It should not change for $\psi \rightarrow \psi'$ and $\bar{\psi} \rightarrow \overline{\psi'} = \psi'^{\dagger}\gamma^{0}$; $\overline{\psi'} = e^{-i\alpha}\bar{\psi}$ ➔ OK. $\mathcal{L}' = \mathcal{L}$

$e^{-i\alpha}\bar{\psi}e^{i\alpha}\psi = \bar{\psi}\psi$

- *local* gauge invariance: invariance under changing phases in space and time
  - An electron wave function can have a different phases at different places and times
    - $\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$ and $\bar{\psi}(x) \rightarrow \overline{\psi'}(x) = e^{-i\alpha(x)}\bar{\psi}(x)$
  - Check this for the Dirac Lagrangian

$\mathcal{L}' \sim ie^{-i\alpha(x)}\bar{\psi}e^{i\alpha(x)}(\partial_{\mu}\psi + i\partial_{\mu}(\alpha)\psi)$

Problem in the term: $\partial_{\mu}\psi(x) \rightarrow \partial_{\mu}\psi'(x) = e^{i\alpha(x)}\left(\partial_{\mu}\psi(x) + \underbrace{i\partial_{\mu}\alpha(x)\psi(x)}_{\text{trouble}}\right)$

Hence: $\mathcal{L} \rightarrow \mathcal{L}' + i\partial_{\mu}\alpha(x)\ \bar{\psi}(x)\,\gamma^{\mu}\,\psi(x)$

Note: $x \equiv x^{\mu}$

- It seems that the Lagrangian will change, but this is not allowed!

Covariant Derivative

Griffiths §10.3

- We insist that the Lagrangian does not change and invent a “covariant” derivative:
  - Replace in $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ the derivative by: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$
  - Require that the vector field A^μ transforms together with the particle wave ψ

$$\psi(x) \rightarrow \psi'(x) = \mathrm{e}^{i\alpha(x)}\psi(x)$$
$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \frac{1}{q}\partial^\mu\alpha(x)$$

  - ➔ Exercise: check that the Lagrangian $i\bar{\psi}\gamma_\mu D^\mu\psi - m\bar{\psi}\psi$ now is invariant!

- What have we done?
  - We ***insist*** the electron can have a local phase factor $\alpha(x)$ without changing the physics
  - We then ***must*** at the same time introduce a photon field $A^\mu(x)$, which couples to charge!

➔ ***Gauge invariance implies interactions!***

- Remember gauge transformations EM field: $A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu\lambda$ is same photon
  - λ is coupled to the phase of the wave function of the electrons

- The same principle can also be used for weak and strong interactions: implement other symmetries

Quantum Electrodynamics (QED)

- The free Dirac Lagrangian is: $\mathcal{L}_{\text{free}} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$

- Introducing electromagnetism implies: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$

- Resulting in: $\mathcal{L}_{EM} = i\bar{\psi}\gamma_\mu\, D^\mu\psi - m\bar{\psi}\psi$

$$\mathcal{L}_{EM} = i\bar{\psi}\gamma_\mu\, \partial^\mu\psi - m\bar{\psi}\psi \; - q\bar{\psi}\gamma_\mu A^\mu\psi$$

$\mathcal{L}_{EM} = \mathcal{L}_{\text{free}} - \mathcal{L}_{\text{int}}$ with $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$ and $J_\mu = q\bar{\psi}\gamma_\mu\psi$

- Remember that the Dirac probability current was $J_\mu = \bar{\psi}\gamma_\mu\psi$ such that we now have a charge current: $J_\mu = q\bar{\psi}\gamma_\mu\psi$

- The system is described as free Lagrangian plus an interaction Lagrangian of the form: "current × field" $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$

Part 2
Gauge Theory

2b) Local Gauge Invariance
ii) Yang-Mills theories* (Weak, Strong)

* Note: this is a more technical part: focus on the concept involved; the precise mathematics is less important for now

Yang Mills Theories

- QED is called a U(1) symmetry. It means that a 1-dimensional unitary transformation (the phase factor $\mathrm{e}^{i\alpha(x)}$) does not change the physics.
  - The unitary symmetry couples to the charge quantum number

- Let us require that the weak interaction can not differentiate between rotations in the space of "up-down": Isospin.

- Rewrite $\mathcal{L} = \bar{u}(i\gamma^\mu\partial_\mu - m)u + \bar{d}(i\gamma^\mu\partial_\mu - m)d$ where u (isospin up) and d (isospin down) are a doublet of spinor waves as follows:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\, I\, \partial_\mu - I\, m)\psi \quad \text{with } \psi = \begin{pmatrix} u \\ d \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - We think of the "up" and "down" directions in weak isospin space

SU2 Gauge Invariance

EM was: $\psi(x) \rightarrow \psi'(x) = G_{EM}\psi(x) = e^{i\alpha(x)}\psi(x)$

- We require gauge invariance: $\psi(x) \rightarrow \psi'(x) = G(x)\psi(x)$ with $G(x) = \exp\left(\frac{i}{2}\vec{\tau} \cdot \vec{\alpha}(x)\right)$
  - $\vec{\tau} = \tau_1, \tau_2, \tau_3$ are the Pauli Matrices
  - This is now a rotation in isospin space generated by 2x2 Pauli matrices!

- Just like QED there is the problem that the Lagrangian does not automatically stay invariant (just write it out), because: $\partial_\mu\psi(x) \rightarrow \partial_\mu\psi'(x) = G(x)(\partial_\mu\psi) + \underbrace{(\partial_\mu G)\psi}_{\text{trouble}}$

- To solve this a corresponding covariant derivative must be introduced to keep the Lagrangian invariant: $I\partial_\mu \rightarrow D_\mu = I\partial_\mu + igB_\mu$ $\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  - g is the coupling constant that replaces charge q in QED and B_μ is now a new vector force field that replaces A_μ of QED.
  - The object B_μ is now a 2x2 matrix: $B_\mu = \frac{1}{2}\vec{\tau} \cdot \vec{b}_\mu = \frac{1}{2}\tau_1^a b_\mu^a = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

    $\vec{b}_\mu = (b_1, b_2, b_3)$ are now three new gauge fields
  - We need 3 instead of one, because there are three generators of 2x2 rotations

- We now get the desired behaviour if : $D_\mu\psi(x) \rightarrow D'_\mu\psi'(x) = G(x)(D_\mu\psi)$

Gauge transformation for B_μ field – (for experts)

- We get the desired behaviour if: $D_\mu\psi(x) \rightarrow D'_\mu\psi'(x) = G(x)\left(D_\mu\psi\right)$
- The left side of this equation is: $$D'_\mu\psi'(x) = \left(\partial_\mu + igB'_\mu\right)\psi'$$ $$= G\left(\partial_\mu\psi\right) + \left(\partial_\mu G\right)\psi + ig\,B'_\mu\,(G\psi)$$
- While the right hand side is: $G\left(D_\mu\psi\right) = G\left(\partial_\mu\psi\right) + \;ig\;G\;B_\mu\;\psi$
- So the required transformation of the field is: $igB'_\mu(G\psi) = igG\left(B_\mu\psi\right) - \left(\partial_\mu G\right)\psi$
- Multiply the equation by G^{-1} on the right (and omitting ψ): $B'_\mu = GB_\mu G^{-1} + \frac{i}{g}\left(\partial_\mu G\right)G^{-1}$
- Compare this to the case of electromagnetism where $G_{em} = e^{i\alpha(x)}$gives:

$$A'_\mu = G_{em}AG_{em}^{-1} + \frac{i}{g}\left(\partial_\mu G_{em}\right)G_{em}^{-1} = A_\mu - \frac{1}{q}\partial_\mu\alpha$$

… which is exactly what we had before.

Interpretation: weak Interaction

- We try to describe an interaction with a symmetry between two states:
  - "up" and "down" states with invariance under SU2 rotations
- To do this requires the existence of three force fields, related to the gauge field: $\vec{B}_\mu$
  - What are they?
  - They must be three massless bosons, similar to the photon, that couple to "up" and "own" states.
    - They are the W^-, Z^0, W^+ bosons.
    - How come they have a mass (unlike the photon?) → Higgs mechanism
- Again the interaction Lagrangian will be of the form "current × field:" $\vec{J}_\mu \vec{b}^\mu$, where the current is now: $J_\mu = \frac{g}{2}\bar{\psi}\gamma_\mu\vec{\tau}\psi$ (for EM it was: $J_\mu = q\bar{\psi}\gamma_\mu\psi$)
- The "up" and "down" states are $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ and $\psi = \begin{pmatrix} \nu \\ e \end{pmatrix}$ and we describe the *weak interaction.*
- How about the *strong interaction*?

The strong interaction

- The "charge" of the strong interaction is "colour"
- The wave function of a quark has three components:
  - $\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$; Require a symmetry generated by 3x3 rotations in 3-dim color space: SU(3)
- There are 8 generator matrices λ_i and as a consequence there are 8 vector fields needed to keep the Lagrangian invariant
  - There exist 8 gluons, related to:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The Standard Model

- The Standard Model implements *local gauge invariance* at the same time to
  - Electromagnetism (U(1) symmetry transformations) → 1 photon
  - Weak interaction (SU(2) symmetry transformations) → 3 weak bosons
  - Strong interaction (SU(3) symmetry transformations) → 8 gluons

- The SM gauge group is $SU(3) \otimes SU(2) \otimes U(1)$

- For an exact symmetry the force particles should be massless.
  - $SU(3)$ is exact → massless gluons
  - $SU(2) \otimes U(1)$ is an approximate (ie "broken") symmetry.
    - It is broken in the Higgs mechanism such that there remains one massless boson (photon) and three massive particles (W^-, Z^0, W^+).

Lecture 3: Discussion Topics

Discussions Topics belonging to
Lecture 3

Topic-7: Lorentz and Dirac indices

Griffiths §4.2 – §4.3

- Explain the difference between Lorentz indices and Dirac indices
- Is γ^μ a four-vector? Why (not)?
- Is $j^\mu = \bar{\psi}\gamma^\mu\psi$ a four-vector? Why (not)?

Topic-8: Helicity and Chirality

Griffiths §4.2 – §4.3

- Explain the difference between helicity and chirality
  - How is each one defined?
- Which of the two is Lorentz invariant?
- Which one of the two do we refer to when talking about the left or right *handedness* of a particle?

Topic-8: Helicity vs Chirality – background information

a) Write out the chirality operator γ^5 in the Dirac-Pauli representation. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

b) The helicity operator is defined as $\lambda = \frac{1}{2}\vec{\Sigma}\cdot\hat{p}$. Show that helicity operator and the chirality operator have the same effect on a spinor solution, i.e.

$$\gamma^5\psi = \gamma^5 \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} \approx \lambda \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} = \lambda\psi \qquad \text{with: } \chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix};\ \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$$

in the relativistic limit where $E \gg m$

c) Show explicitly that for a Dirac spinor:

$\bar{\psi}\gamma^\mu\psi = \overline{\psi_L}\gamma^\mu\psi_L + \overline{\psi_R}\gamma^\mu\psi_R$ making use of $\psi = \psi_L + \psi_R$ and the projection operators: $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_R = \frac{1}{2}(1+\gamma^5)\psi$

d) Explain why the weak interaction is called left-handed.

Wolfgang Pauli

"I cannot believe God is a weak left-hander."

Topic-9: Maxwell's equation

- Maxwell's equations can be described relativistically with the 4-vector field A^{μ}.
- Show how you get E and B fields from A^{μ}
- Explain the concept of gauge invariance.
- Is the A^{μ} field physical or not?
- The photon is a spin-1 quantum, but why can it not have a spin-0 component?

Topic-9: The photon field and gauge invariance

- Field A^μ is just introduced as a mathematical tool
  - Gauge freedom: you are free to choose any A^μ as long as $\vec{E}$ and $\vec{B}$ fields don't change:

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda \qquad \begin{aligned} V \rightarrow V' &= V + \frac{\partial \lambda}{\partial t} \\ \vec{A} \rightarrow \overrightarrow{A'} &= \vec{A} - \vec{\nabla}\lambda \end{aligned}$$

- Choose the Lorentz gauge condition: $\partial_\mu A^\mu = 0$
- Maxwell equation in Lorentz gauge becomes:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu \quad \rightarrow \quad \partial_\mu \partial^\mu A^\nu = j^\nu$$

  - Very similar to Klein-Gordon equation $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$
    - But now 4-equations → 4 polarizations states of the photon field??
- Photon field solutions: $A^\mu(x) = N\varepsilon^\mu(p)e^{-ip_\nu x^\nu}$
  - A gauge transformation implies: $\varepsilon^\mu \rightarrow \varepsilon'^\mu = \varepsilon^\mu + ap^\mu$
  - Different polarization vectors which differ by multiple of p^μ describe same photon
    - Only 3 degrees of freedom remain → 3 polarization states: spin: -1, 0, 1 → choose $\varepsilon^0 = 0$
- Mass of the photon is zero:
  - Thus $p^\mu p_\mu = 0 \rightarrow \varepsilon^\mu p_\mu \rightarrow \vec{\varepsilon} \cdot \vec{p} = 0$
  - Now only two transverse polarization states remain: Chose $\vec{p} = (0,0,p) \rightarrow \vec{\varepsilon}^1 = (1,0,0)$ and $\vec{\varepsilon}^2 = (0,1,0)$

Lecture 3: Exercises

Exercises belonging to Lecture 3

Exercise – 9: Dirac Algebra

- Dirac algebra:
  - Write the explicit form of the γ-matrices
  - Show that : $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
  - Show that : $(\gamma^0)^2 = \mathbb{1}_4$; $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{1}_4$
  - Use anti-commutation rules of α and β to show that: ${\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0$
  - Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and show: ${\gamma^5}^\dagger = \gamma^5$; $(\gamma^5)^2 = \mathbb{1}_4$; $\{\gamma^5, \gamma^\mu\} = 0$

Exercise – 10: Solutions of free Dirac equation

See Griffiths for a derivation of the solutions

a) Show that the following plane waves are solutions to Dirac's equation

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \;;\; \psi_2 = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

Before KG:
$\phi = Ne^{-i(p_\mu x^\mu)}$

$$\psi_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x + ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \;;\; \psi_4 = \begin{pmatrix} (p_x - ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

b) Write the Dirac equation for particle in rest (choose $\vec{p} = 0$) and show that ψ_1 and ψ_2 are *positive energy* solutions: $E = +\left|\sqrt{\vec{p}^2 + m^2}\right|$ whereas ψ_3 and ψ_4 are *negative energy* solutions: $E = -\left|\sqrt{\vec{p}^2 + m^2}\right|$.

c) *<u>Optional:</u>* Consider the *helicity* operator $\vec{\sigma} \cdot \vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$ and show that ψ_1 corresponds to *positive helicity* solution and ψ_2 to *negative helicity*. Similarly for ψ_3 and ψ_4.

Spin and Helicity – hint for exercise 10c)

- For a given momentum p there still is a *two-fold degeneracy*: what differentiates solutions ψ_1 from ψ_2?
- Define the spin operator for Dirac spinors: $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, where $\vec{\sigma}$ are the three 2x2 Pauli spin matrices
- Define ***helicity*** λ as spin "up"/"down" wrt direction of motion of the particle

$$\lambda = \frac{1}{2}\vec{\Sigma} \cdot \hat{p} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} = \frac{1}{2|p|}(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)$$

- Split off the Energy and momentum part of Dirac's equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m\right] \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- Exercise: Try solutions ψ_1 and ψ_2 to see they are ***helicity eigenstates*** with $\lambda = +1/2$ and $\lambda = -1/2$
- Dirac wanted to solve negative energies and he found spin-½ fermions!

Exercise – 11 : Lagrangians and wave equations

Griffiths §10.2

- Scalar Field (spin 0 "pion")

  a) Show that the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$ results in the Klein-Gordon equation

- Dirac Field (spin ½ Fermion)

  b) Show that the Euler-Lagrange equations for $\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ results in the Dirac equation

- Electromagnetic field (spin 1 photon)

  c) Show that $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$ results in Maxwell's equations

These Lagrangians are the fundamental objects in quantum field theory

Descriptions of interactions follow from symmetry principles on these objects.

Exercise – 12 : Covariant Derivative

Griffiths §10.3

- We insist that the Lagrangian does not change and invent a "covariant" derivative:
  - Replace in $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ the derivative by: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$
  - Require that the vector field A^μ transforms together with the particle wave ψ

$$\psi(x) \rightarrow \psi'(x) = \mathrm{e}^{iq\alpha(x)}\psi(x)$$

$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \partial^\mu\alpha(x)$$

  - ➔ Exercise: check that the Lagrangian now is invariant!