PHY3004: Nuclear and Particle Physics

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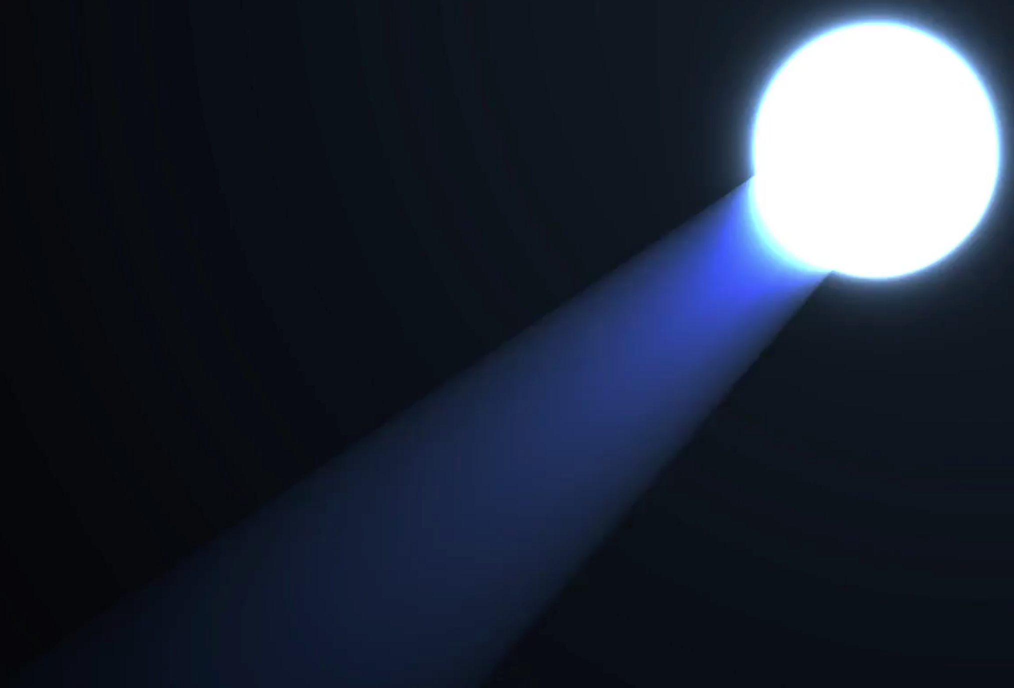
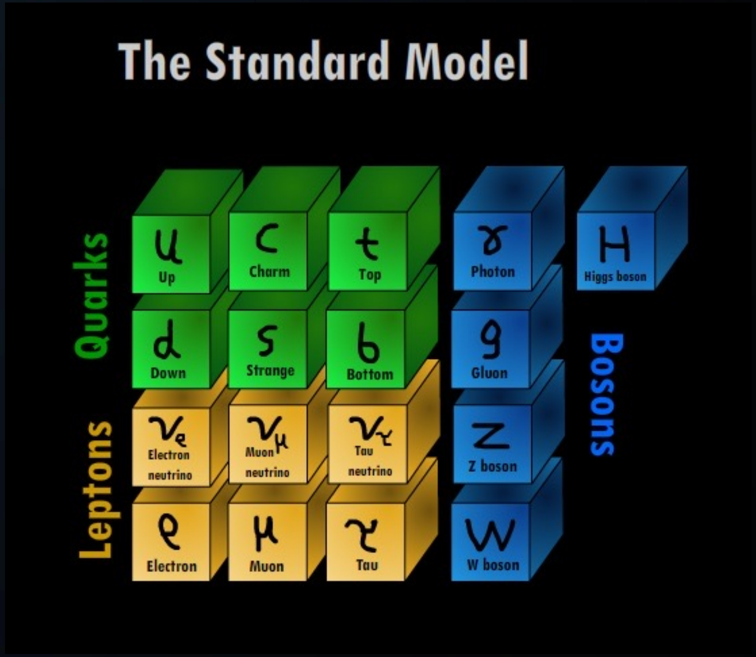


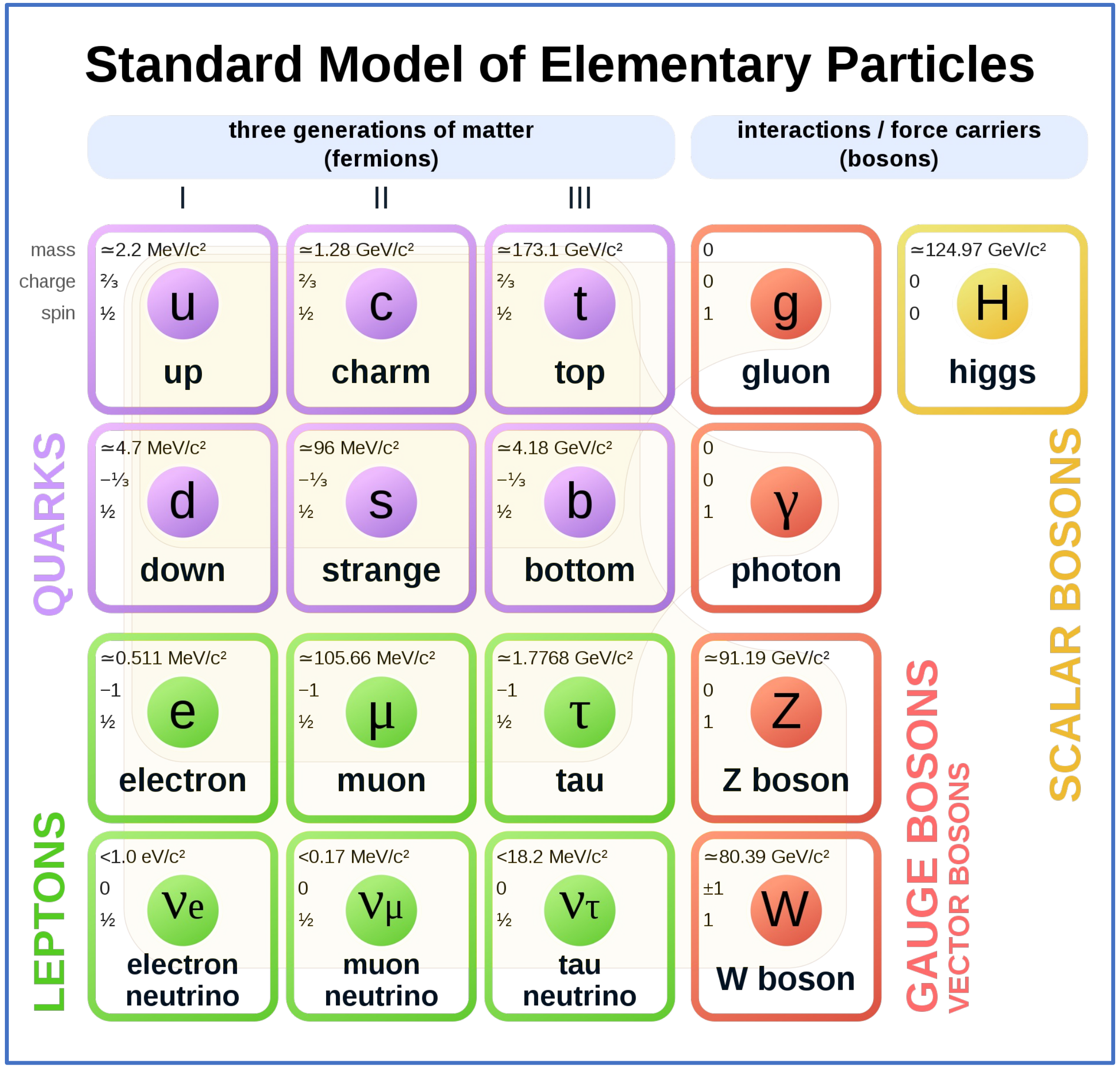


Lecture 1: “Particles”

Griffiths chapter 1

Classification of particles

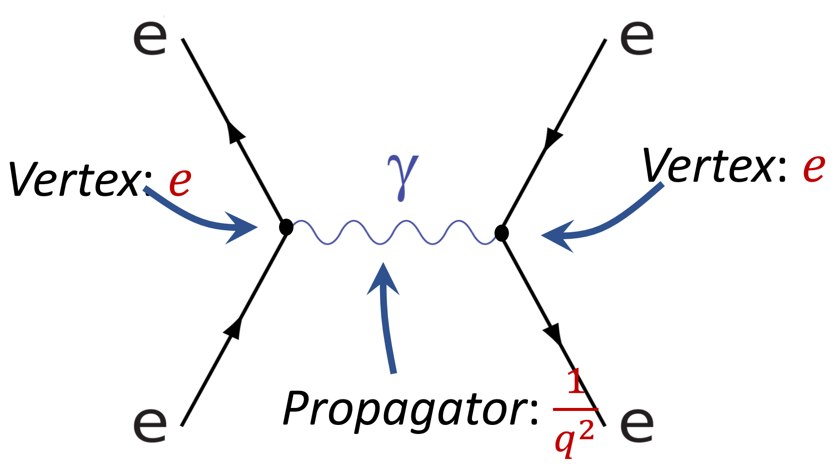
- Lepton: fundamental particle
- Hadron: consist of quarks
  - Meson: 1 quark + 1 antiquark (π^+, B_S^0, …)
  - Baryon: 3 quarks (p ,n , Λ, …)
    - Anti-baryon: 3 anti-quarks
- ***Fermion***: particle with half-integer spin.
  - Antisymmetric wave function: obeys Pauli-exclusion principle and Pauli-Dirac statistics
  - All fundamental quarks and leptons are spin-½
  - Baryons (S=$^1/_2$, $^3/_2$)
- ***Boson***: particle with integer spin
  - Symmetric wave function: Bose-Einstein statistics
  - Mesons: (S=0, 1), Higgs (S=0)
  - Force carriers: γ, W, Z, g (S=1); graviton(S=2)



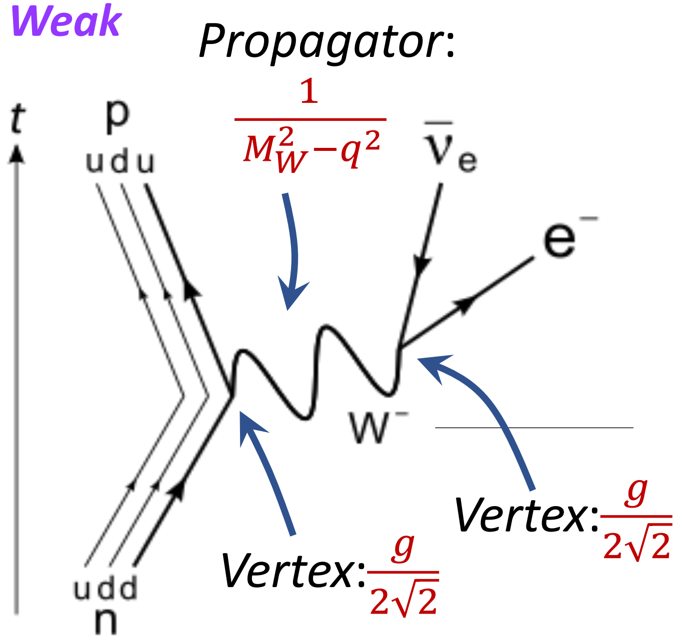
Lecture 2: “Forces”

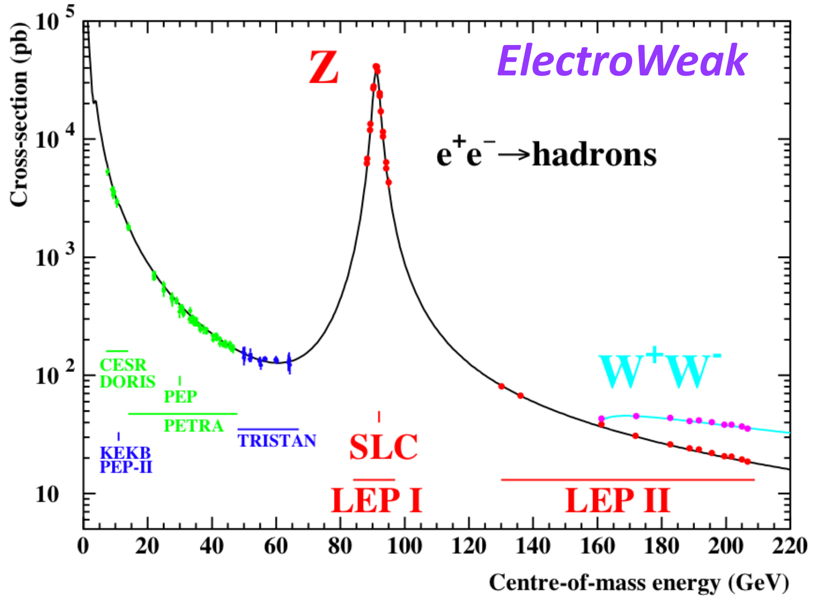
Griffiths chapter 2

EM (QED)

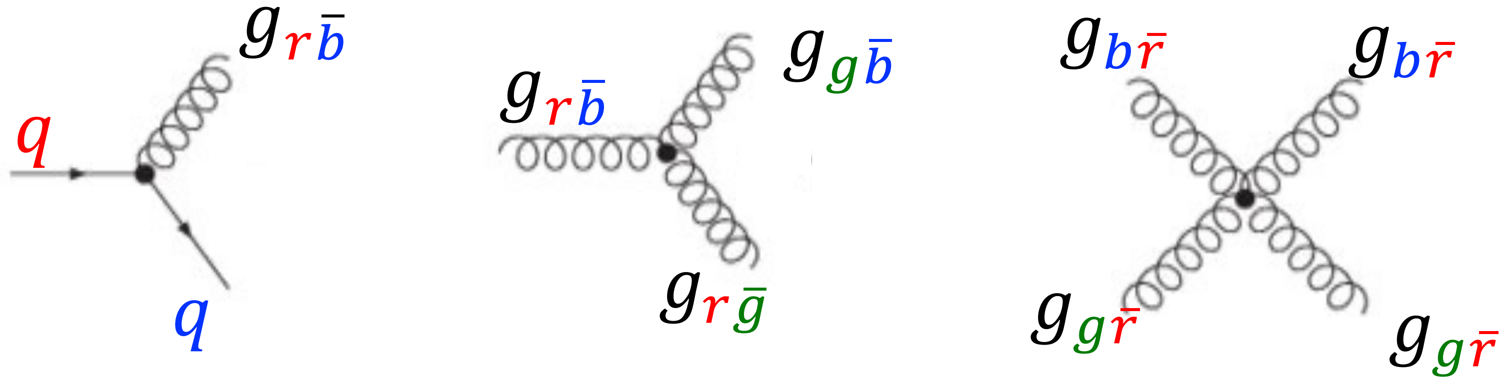


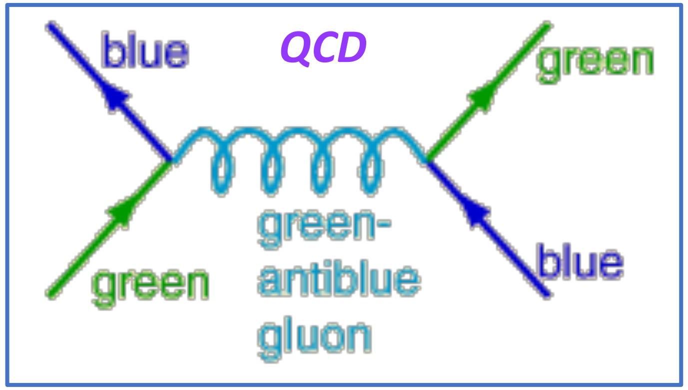
Weak





Strong (QCD)





Lecture 3: "Waves" – wave equations

Probability interpretation (Continuity equation)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Quantum Mechanics: $E \rightarrow \hat{E} = i\hbar \frac{\partial}{\partial t}$; $p \rightarrow \hat{p} = -i\hbar \vec{\nabla}$

Non-relativistic spin 0:

$$E = \frac{\vec{p}^2}{2m}$$

Schrödinger:

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi}$$

$$\psi = N e^{i(\vec{p}\vec{x} - Et)}$$

$$\rho \equiv \psi^* \psi = |N|^2$$

$$\vec{j} \equiv \frac{i\hbar}{2m}\left(\psi \vec{\nabla} \psi^* - \psi^* \vec{\nabla} \psi\right) = \frac{|N|^2}{m} \vec{p}$$

Relativistic spin 0:

$$E^2 = p^2 c^2 + m^2 c^4$$

Example: pions

Klein-Gordon:

$$-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi = -\nabla^2 \phi + \frac{m^2 c^2}{\hbar^2} \phi$$

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

$$\phi = N e^{i(\vec{p}\vec{x} - Et)}$$

$$j^\mu(\rho, \vec{j}) = i[\phi^* (\partial^\mu \phi) - \phi (\partial^\mu \phi^*)]$$

$$\rho = 2|N|^2 E$$

$$\vec{j} = 2|N|^2 \vec{p}$$

$$j^\mu = 2|N|^2 p^\mu$$

Relativistic spin- ½:

$$H = (\vec{\alpha} \cdot \vec{p} + \beta m)$$

Fundamental quarks and leptons

Dirac: (The Master in action)

$$i \frac{\partial}{\partial t} \psi = (-i\, \vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi$$

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} \quad \gamma^\mu = (\beta, \beta\vec{\alpha})$$

$$\psi = u(p) e^{i(\vec{p}\vec{x} - Et)}$$

$$u(p) = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

$$j^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi = \sum_{i=1}^{4} |\psi_i|^2$$

Relativistic spin-1:

Fundamental force carriers

Proca:

$$\boxed{\partial_\mu \partial^\mu A^\nu + m^2 A^\nu = j^\nu}$$

EM: $A^\mu = \gamma \rightarrow m = 0$

QCD: $A^\mu = g \rightarrow m = 0$

Weak: $A^\mu = W, Z \rightarrow m \neq 0$

EM: Maxwell equations for $\vec{E}$ and $\vec{B}$ fields

Lecture 3: "Waves" – gauge invariance

Lagrangians:

Spin 0 Scalar field: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

Spin ½ Dirac fermion $\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$

Spin 1 gauge boson (photon) : $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$

Euler Lagrange lead to the wave equations: $\dfrac{\partial \mathcal{L}}{\partial \phi(x)} = \partial_\mu \dfrac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi(x)\right)}$ (from field theory)

All forces result from requiring a symmetry principle: Lagrangian should stay invariant under transformations

1) QED = U(1) symmetry

$\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x)$

$A^\mu(x) \to A'^\mu(x) = A^\mu(x) - \partial^\mu \alpha(x)$

➔ 1 E.M. photon field: $A^\mu(x)$

$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi \longrightarrow \mathcal{L} = i\bar{\psi}\gamma_\mu D^\mu \psi - m\bar{\psi}\psi$

Covariant derivative: $\partial^\mu \to D^\mu \equiv \partial^\mu + iqA^\mu$

$$\mathcal{L} = \underbrace{i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi}_{\text{"free"}} \underbrace{- q\bar{\psi}\gamma_\mu \psi A^\mu}_{\text{"interaction"}}$$

2) Weak = SU(2) symmetry $\psi = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}$

$\psi(x) \to \psi'(x) = \exp\left(\frac{i}{2} g \vec{\tau} \cdot \vec{\alpha}(x)\right)\begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}$

➔ 3 weak fields: $W^{\mu+}(x)$, $W^{\mu-}(x)$, $Z^\mu(x)$

3) QCD = SU(3) symmetry $\psi = \begin{pmatrix}\psi_r \\ \psi_g \\ \psi_b\end{pmatrix}$

$\psi(x) \to \psi'(x) = \exp\left(\frac{i}{2} g_s \vec{\lambda} \cdot \vec{\alpha}(x)\right)\begin{pmatrix}\psi_r \\ \psi_g \\ \psi_b\end{pmatrix}$

➔ 8 colored gluon fields: $g^\mu(x)$

Lecture 4: “Symmetries” – Standard Model

- The Lagrangian of the Standard Model includes electromagnetic, weak and strong interactions according to the gauge field principle

- Construction of the Lagrangian: $\mathcal{L} = \mathcal{L}_{\text{free}} - \mathcal{L}_{\text{interaction}} = \mathcal{L}_{\text{Dirac}} - gJ^{\mu}A_{\mu}$
  - With g a coupling constant, J^{μ} a current ($\bar{\psi}\mathrm{O}_i\psi$) and A_{μ} a force field

A. Local $U(1)$ gauge invariance: symmetry under complex phase rotations
   - Conserved quantum number: (hyper-) charge
   - Lagrangian: $\mathcal{L} = \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - q\,\underbrace{\bar{\psi}\gamma^{\mu}\psi}_{J^{\mu}_{EM}}\,A_{\mu}$

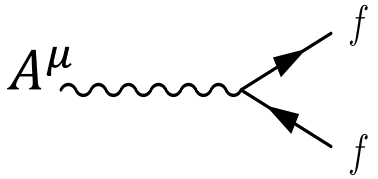
B. Local $SU(2)$ gauge invariance: symmetry under transformations in isospin doublet space.
   - Conserved quantum number: weak isospin
   - Lagrangian: $\mathcal{L} = \bar{\Psi}(i\gamma^{\mu}D_{\mu} - m)\Psi = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi - \frac{g}{2}\underbrace{\bar{\Psi}\gamma^{\mu}\vec{\tau}\Psi}_{J^{\mu}_{Weak}}\,\vec{b}_{\mu}$

C. Local $SU(3)$ gauge invariance: symmetry under transformations in colour triplet space
   - Conserved quantum number: color
   - Lagrangian: $\mathcal{L} = \bar{\Phi}(i\gamma^{\mu}D_{\mu} - m)\Phi = \bar{\Phi}(i\gamma^{\mu}\partial_{\mu} - m)\Phi - \frac{g_s}{2}\underbrace{\bar{\Phi}\gamma^{\mu}\vec{\lambda}\Phi}_{\vec{J}^{\mu}_{QCD}}\,\vec{c}_{\mu}$

Lecture 4: "Symmetries" – Standard Model

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - qJ^{\mu}_{EM}A_{\mu} \; - \frac{g}{2}J^{\mu}_{\text{Weak}}\vec{b}_{\mu} - \frac{g_s}{2}J^{\mu}_{QCD}\,\vec{c}_{\mu}$$

QED U(1) $\mathcal{L}_{\text{int}} \; = -J_{\mu}A^{\mu}$ with $J_{\mu} = q\bar{\psi}\gamma_{\mu}\psi$

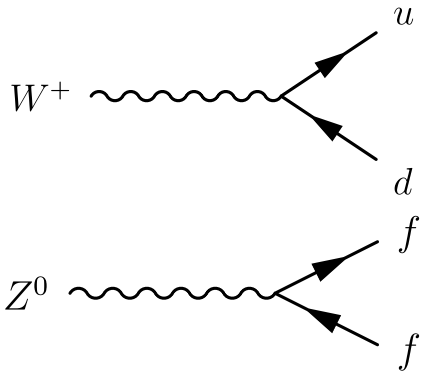


Electromagnetic interaction

Weak SU(2) : $\mathcal{L}_{\text{int}} \; = -\vec{J}_{\mu}\vec{b}^{\mu}$ with $\vec{J}_{\mu} = \frac{g}{2}\,\bar{\Psi}\,\gamma_{\mu}\vec{\tau}\,\Psi$

$W^{\pm}_{\mu} \equiv \frac{1}{\sqrt{2}}(b^1_{\mu} \mp i b^2_{\mu})$ $\qquad J^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\bar{\Psi}\gamma_{\mu}\tau^{\pm}\Psi$ $\qquad$ with $\tau^{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2)$

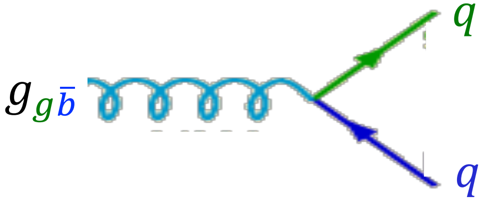
$Z_{\mu} \sim b^3_{\mu}$ $\qquad J^{3}_{\mu} = \frac{1}{2}\bar{\Psi}\gamma_{\mu}\tau^{3}\Psi$ $\qquad$ with $\tau^{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2)$



Weak interaction

Electroweak mixing SU(2)xU(1):

$$\gamma_{\mu} = \quad A_{\mu}\cos\theta_W \; + b^3_{\mu}\sin\theta_W$$
$$Z_{\mu} = -A_{\mu}\sin\theta_W \; + b^3_{\mu}\cos\theta_W$$

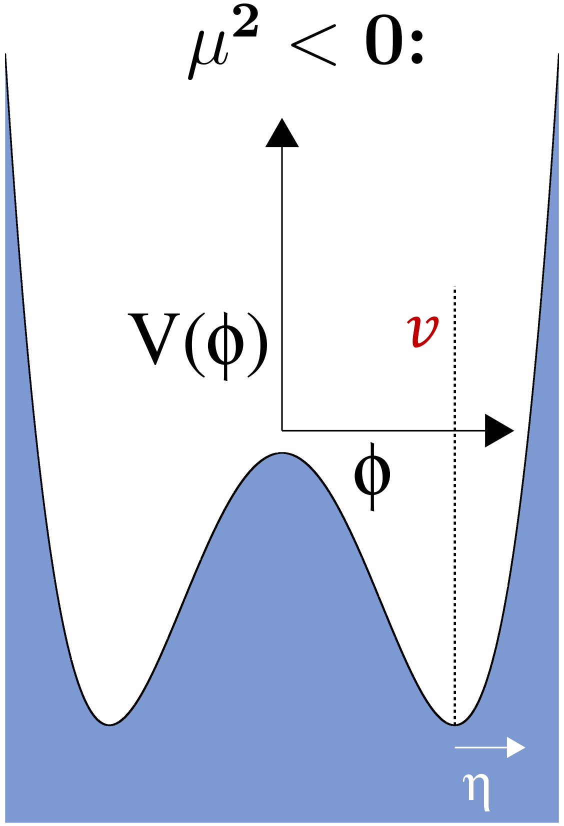


Strong interaction

Standard Model: $SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y$

Lecture 4: "Symmetries" – Symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2}_{\text{Massive Klein-Gordon term (Spin 0, mass } =\mu)} \underbrace{- \frac{1}{4}\lambda\phi^4}_{\text{Interaction term}}$$



The Lagrangian has a minimum for $\phi_0 = \sqrt{-\frac{\mu^2}{\lambda}} = v$ or $\mu^2 = -\lambda v^2$

Conclusion:

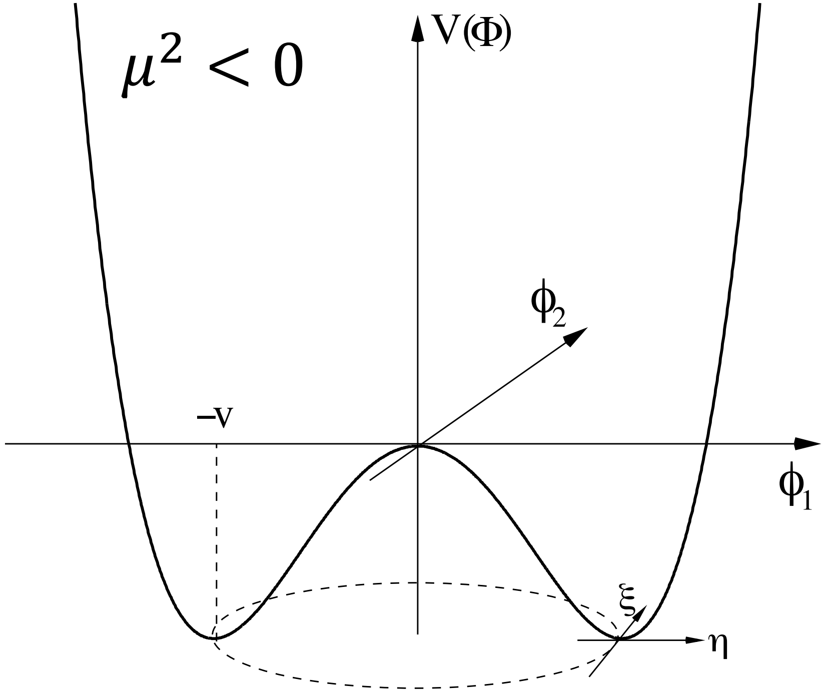
- The symmetry of the Lagrangian by adding a symmetric potential ϕ *has not been destroyed*
- The *vacuum is no longer* in a symmetric position

The real case include complex fields ϕ

Lecture 4: "Symmetries" – Symmetry breaking

$$\mathcal{L} = \underbrace{(\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 (\phi^* \phi)}_{\text{Massive Klein-Gordon term (Spin 0, mass = } \mu)} - \underbrace{\lambda\, (\phi^* \phi)^2}_{\text{Interaction term}} \quad \text{where } \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

The Lagrangian has minima for $\sqrt{\phi_1^2 + \phi_2^2} = \sqrt{\frac{-\mu^2}{\lambda}} = v$



Conclusion:

- The symmetry of the Lagrangian by adding a symmetric potential ϕ *has not been destroyed*
- The *vacuum is no longer* in a symmetric position

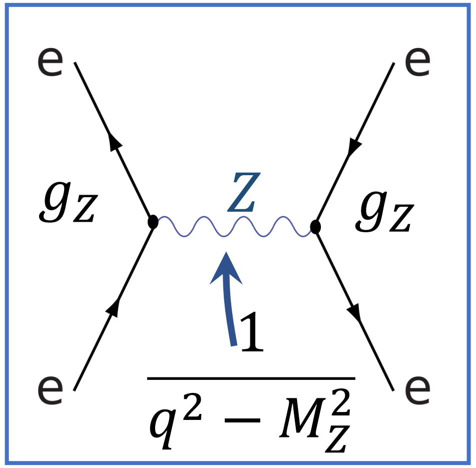
The real case includes a complex isospin doublet ϕ

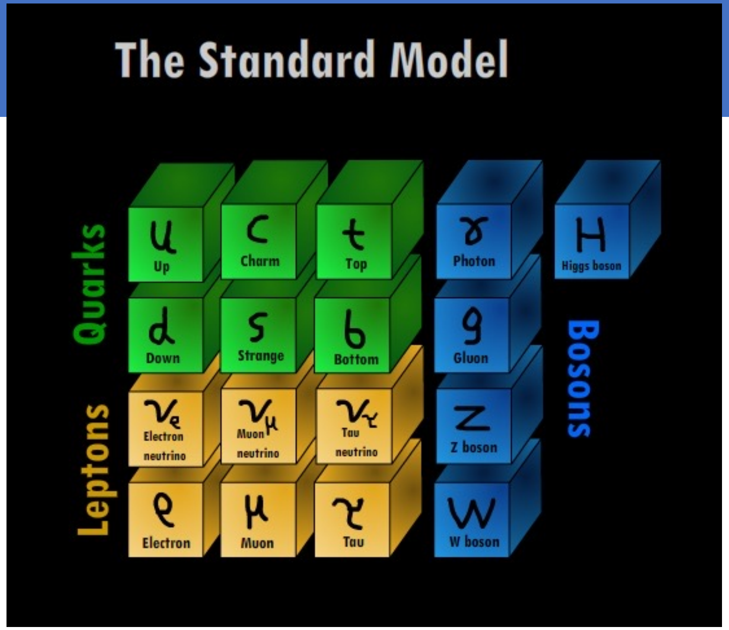
$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \qquad \phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v\end{pmatrix}$$

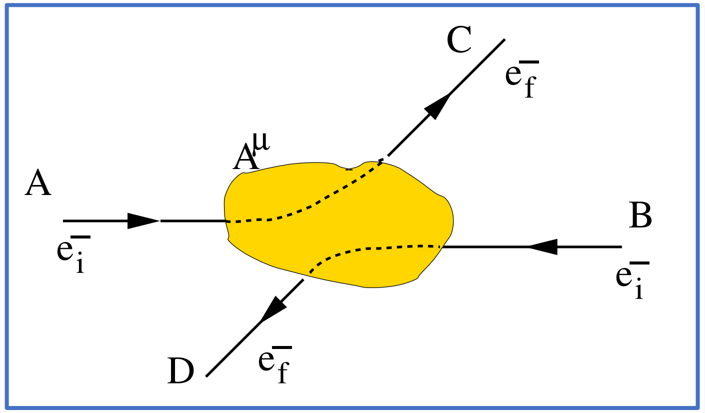
massless γ + massive W^+, W^+, Z

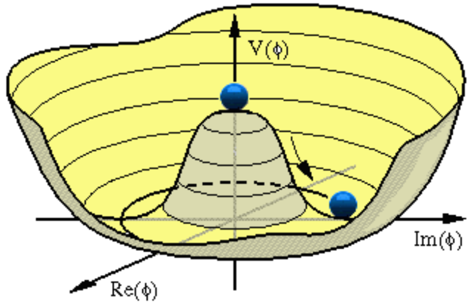
Recap: “Seeing the wood for the trees”

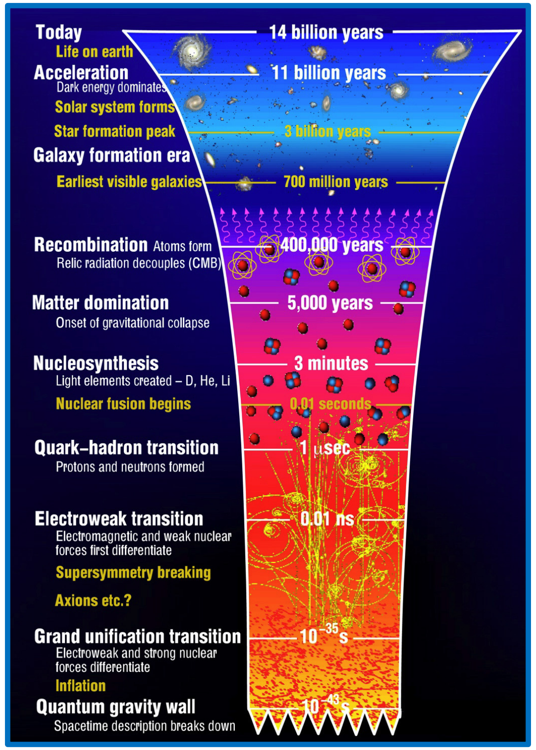
- Lecture 1: “Particles”
  - Zooming into constituents of matter
  - Skills: distinguish particle types, Spin
- Lecture 2: “Forces”
  - Exchange of quanta: EM, Weak, QCD
  - Skills: 4-vectors, Feynman diagrams
- Lecture 3: “Waves”
  - Quantum fields and gauge invariance
  - Dirac algebra, Lagrangian, co- & contra variant
- Lecture 4: “Symmetries”
  - Standard Model, Higgs, Discrete Symmetries
  - Skills: Lagrangians, Chirality & Helicity
- Lecture 5: “Scattering”
  - Cross section, decay, perturbation theory
  - Skills: Dirac-delta function, Feynman Calculus
- Lecture 6: “Detectors”
  - Energy loss mechanisms, detection technologies











Today: Scattering Theory and Feynman Calculus

Griffiths Chapter 6

Part 1 : Decay and Cross Section — Griffiths §6.1

Part 2 : Perturbation Theory and the Golden Rule — Griffiths §6.2 and PP1 Chapter 2

Part 3 : Feynman Calculus — Griffiths §6.3

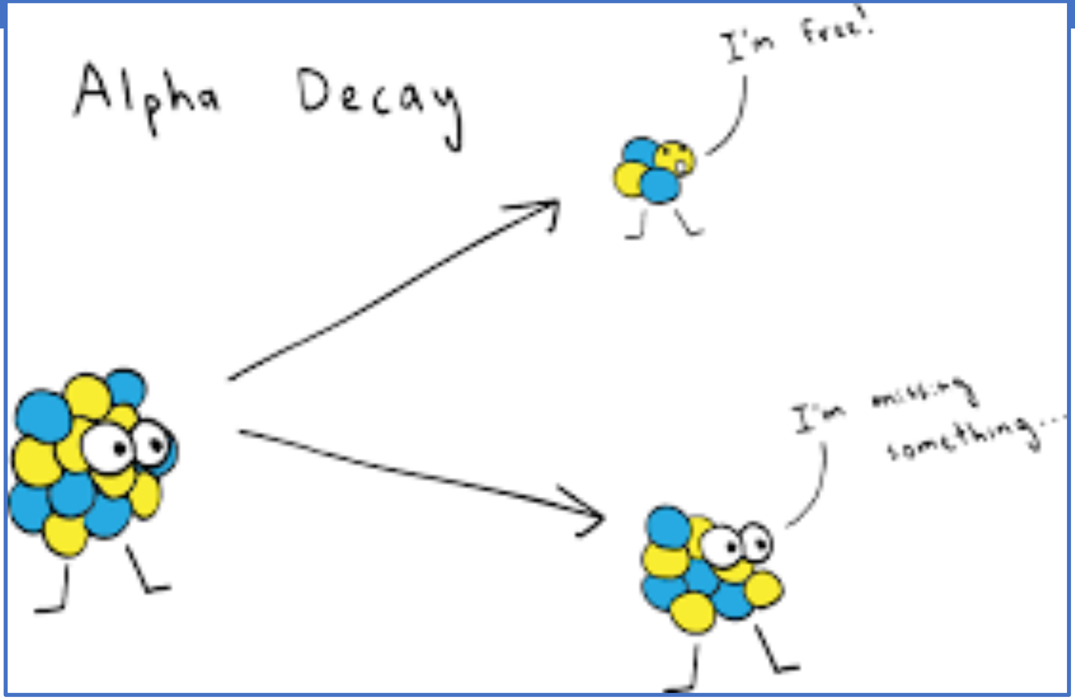
"Let's play around with physics and math"

Part 1

Decay and Cross Section

Griffiths §6.1

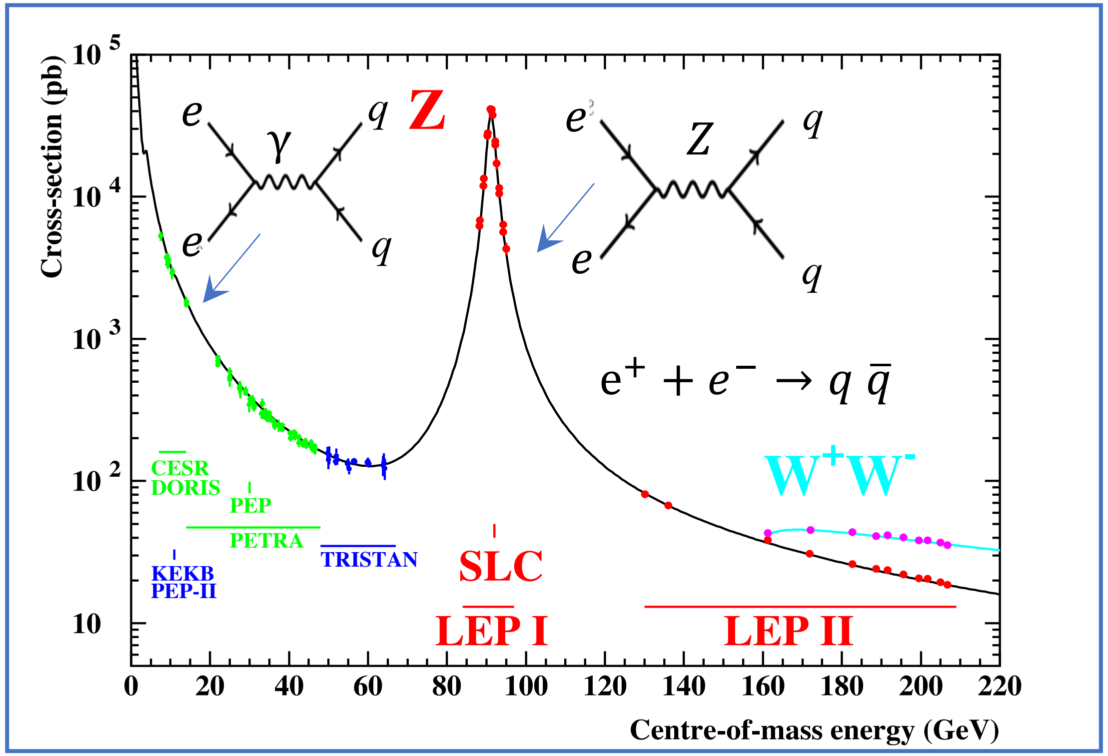
Terminology: Decays



- A quantum particle decays with equal probability per unit time
  - $dN/N = -\Gamma dt$ such that: $N = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau}$
  - $\Gamma \equiv$decay rate $\frac{1}{\Gamma} = \tau \equiv$ mean lifetime
- Often particles can decay in many quantum ways; each with its own partial decay "width" Γ_i
  - ***Total decay rate*** $\Gamma_{tot} = \sum_i \Gamma_i$ and ***lifetime*** $\tau = \frac{1}{\Gamma_{tot}}$ and ***Branching Ratio*** $BR_i = \frac{\Gamma_i}{\Gamma_{tot}}$
- The decay rate can be calculated from the Standard Model
  - Compare theory and experiment

Terminology: Cross Section

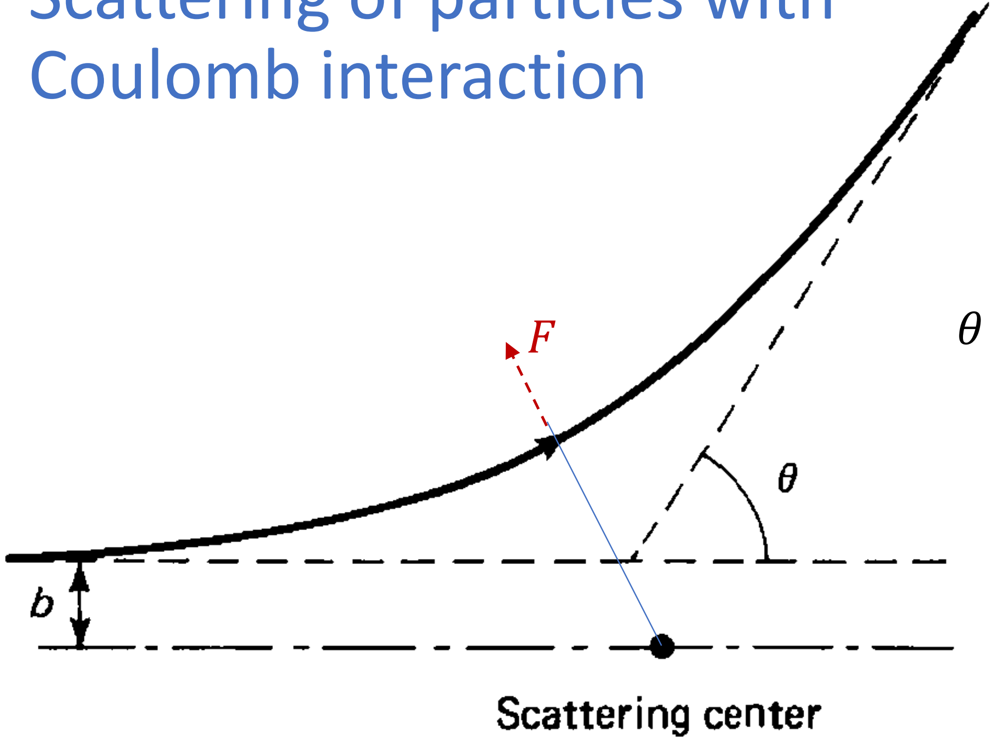
- A scattering process is measured using "cross section"; the ***effective surface*** seen by a particle colliding with a target. We use the same for collisions:
  - e.g. proton-proton colliders.
- For colliding protons many processes may happen:
  - ***Exclusive*** cross section σ_i : cross section for one specific process "i"
  - ***Inclusive*** cross section σ_{tot}: sum all possible exclusive cross sections: $\sigma_{tot} = \sum_i^N \sigma_i$
- The cross section can for example depend on the energy of the collision
- Look at the process $e^+e^- \rightarrow q\bar{q}$
  - There is a resonance at 91 GeV; the mass of the Z- boson
  - And there is a peak near 0 GeV; the photon resonance
  - "Electroweak process"



Scattering

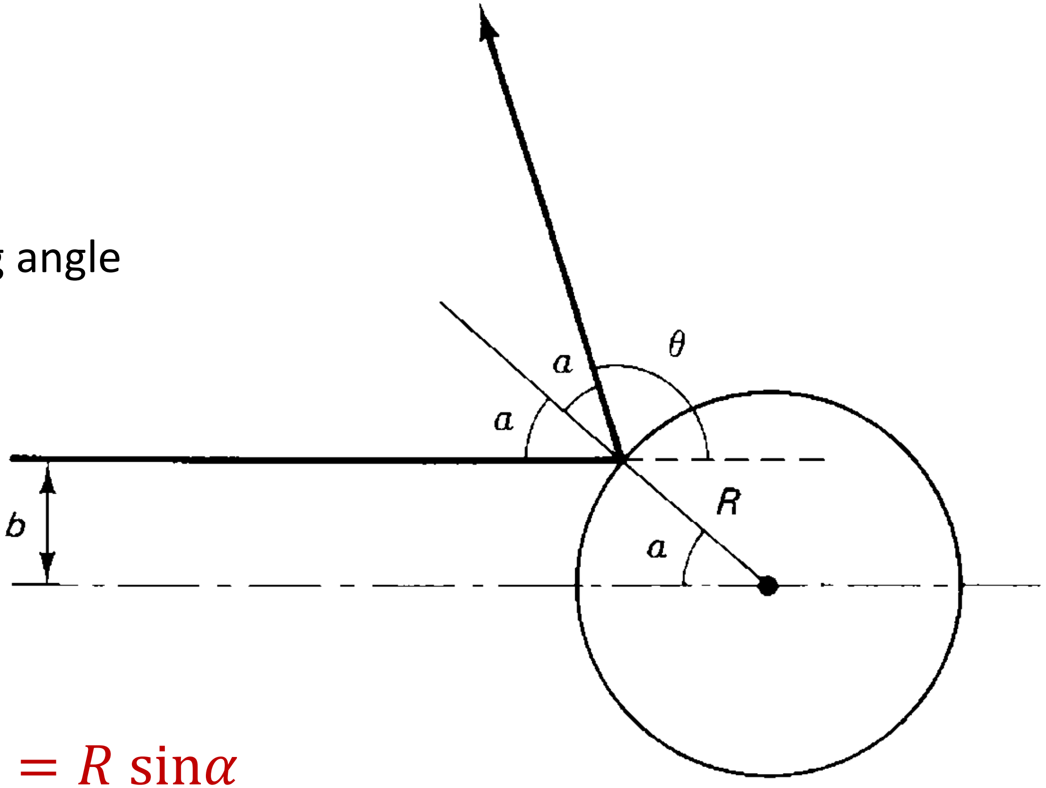
Griffiths §6.1

- Scattering of particles with Coulomb interaction



- Scattering on a “hard sphere”

θ is scattering angle



b = Impact Parameter
θ =Scattering Angle
1-to-1 relation b and θ

$b = R\,\sin\alpha$
$2\alpha + \theta = \pi$
$\sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$
$b = R\cos(\theta/2\ )$

A) Hard Sphere Scattering

Griffiths §6.1

- Calculation of hard sphere scattering

If particle goes through $d\sigma$ it will scatter through solid angle $d\Omega$:

$$\mathrm{d}\sigma = D(\theta)\mathrm{d}\Omega$$
$$\mathrm{d}\sigma = |b\,\mathrm{d}b\,\mathrm{d}\phi|$$

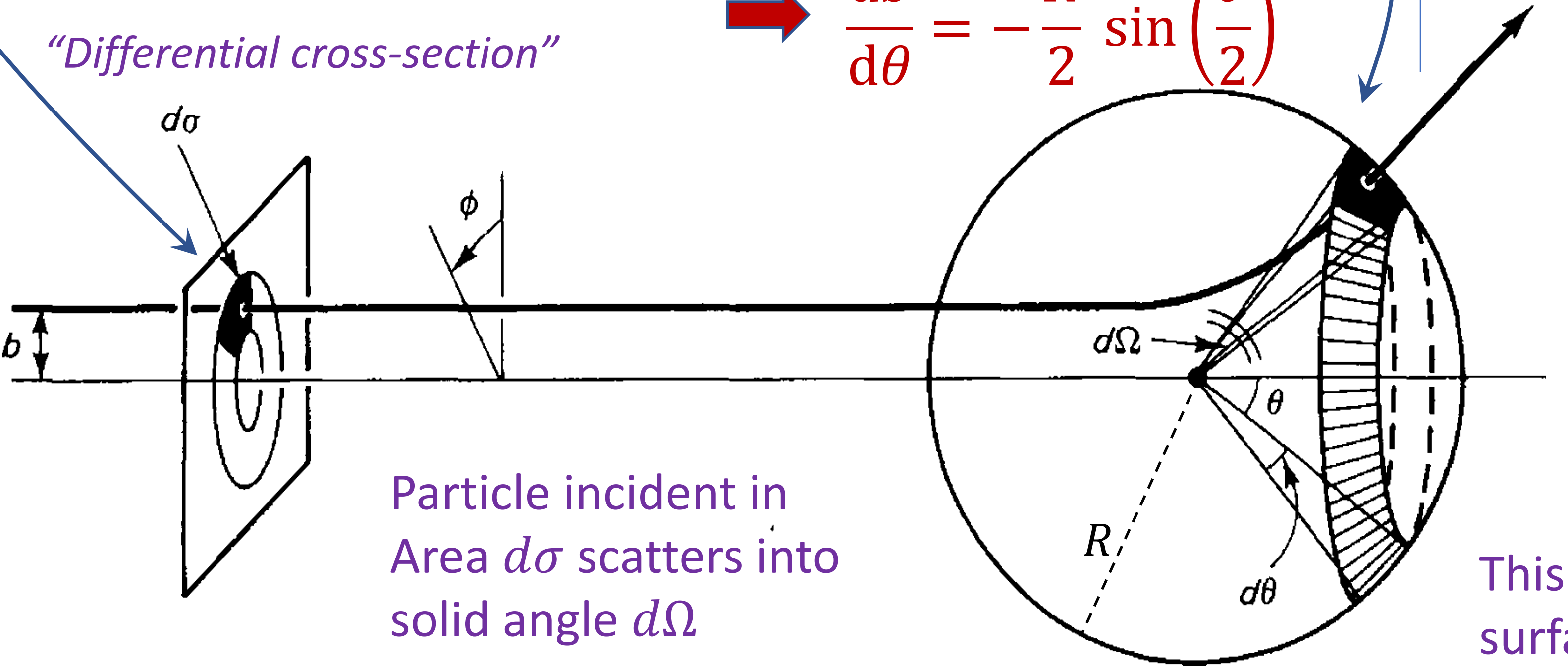
$$\mathrm{d}\Omega = |\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi|$$

$$D(\theta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{b}{\sin\theta}\left(\frac{\mathrm{d}b}{\mathrm{d}\theta}\right)\right|$$

"Differential cross-section"

Hard scattering:

$$b = R\cos(\theta/2)$$
$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$



Particle incident in Area $d\sigma$ scatters into solid angle $d\Omega$

Then:

$$D(\theta) = \frac{Rb\ \sin(\theta/2)}{2\sin\theta}$$
$$= \frac{R^2}{2}\ \frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta}$$
$$= \frac{R^2}{4} \qquad \left(\cos\alpha\sin\alpha = \tfrac{1}{2}(\sin 2\alpha)\right)$$

$$\int \mathrm{d}\Omega = \int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$$

$$\sigma = \int \mathrm{d}\sigma = \int D(\theta)\,\mathrm{d}\Omega$$
$$= \int \frac{R^2}{4}\mathrm{d}\Omega = \pi R^2$$

This corresponds to the projected surface of the circle seen by the particle

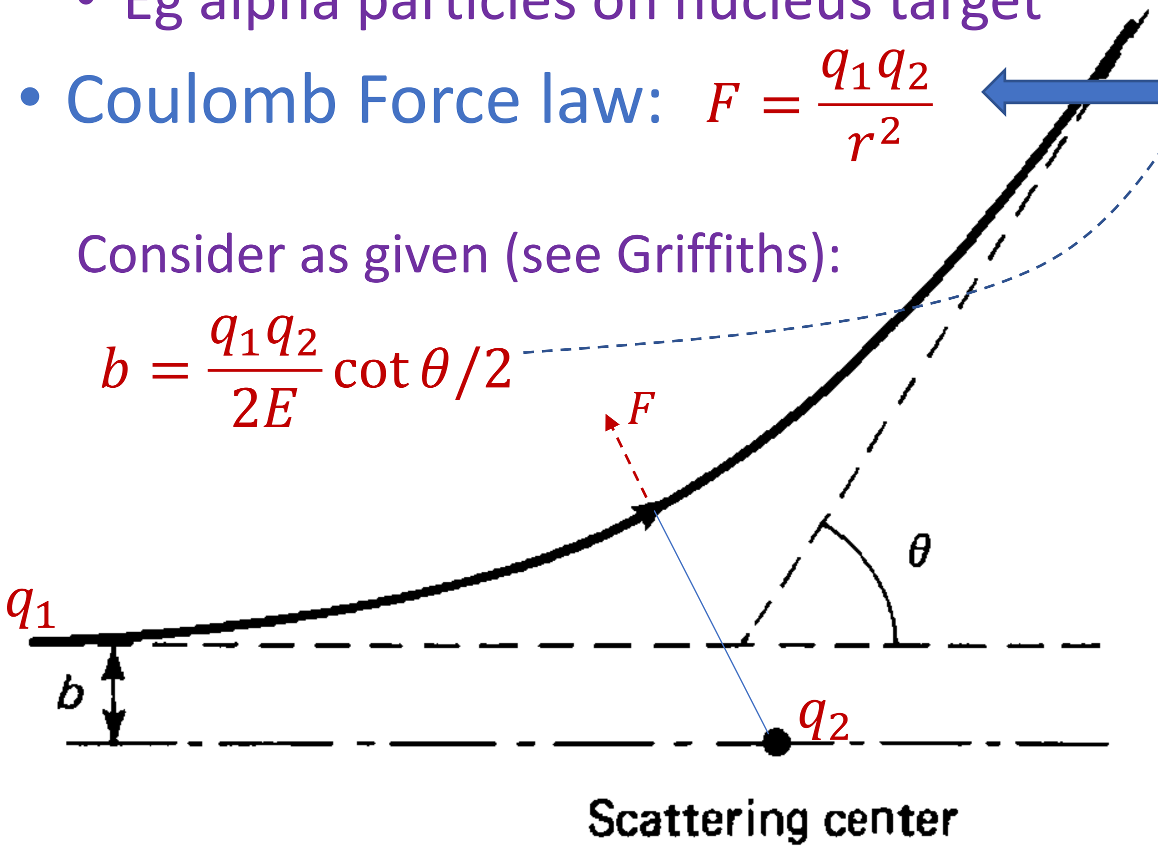
B) Rutherford Scattering (point charge)

Griffiths §6.1

- Scattering of charged particles with Coulomb force
  - Eg alpha particles on nucleus target
- Coulomb Force law: $F = \frac{q_1 q_2}{r^2}$

Consider as given (see Griffiths):

$$b = \frac{q_1 q_2}{2E} \cot\theta/2$$



$$D(\theta) = \frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right|$$

$$D(\theta) = \left(\frac{q_1 q_2}{4E\sin^2(\theta/2)}\right)^2$$

$$\sigma = \int d\sigma = \int D(\theta)\, d\Omega$$

$$\sigma = 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^\pi \frac{1}{\sin^4(\theta/2)} \sin\theta\, d\theta$$

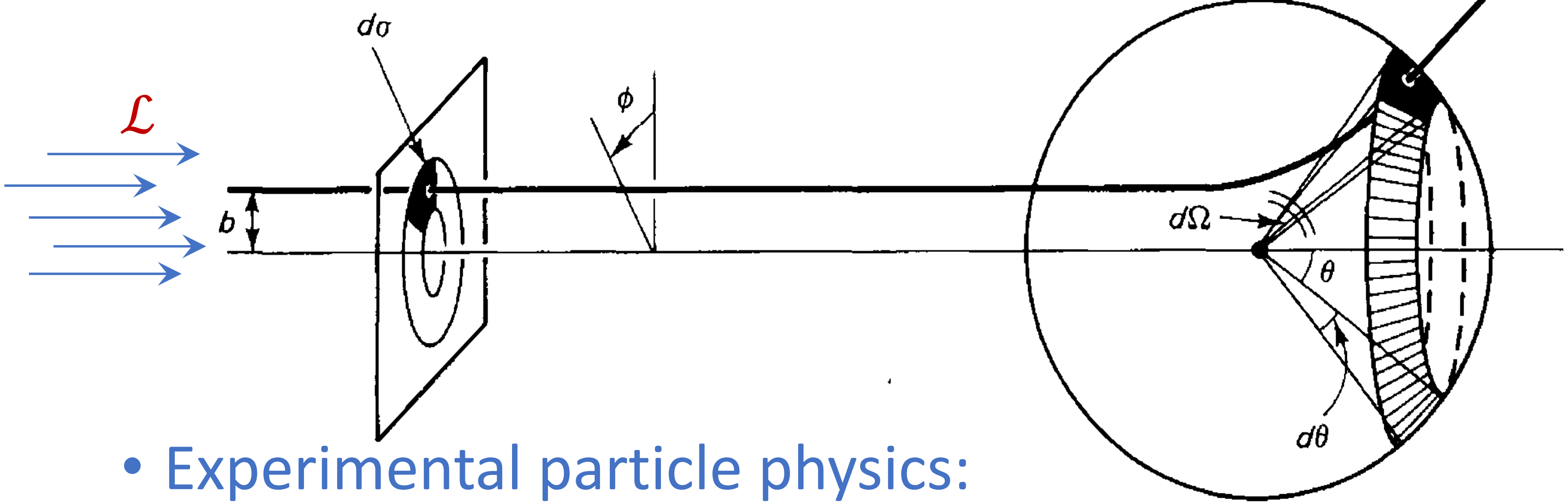
The integral turns out to be infinite!
Particle sees an infinitely large scattering surface?
Reason is that Coulomb force has infinite range.

Most "collisions" will happen at large distance, which is what Rutherford observed.

Luminosity

Griffiths §6.1

- Consider beam of particles on a target
  - Luminosity $\mathcal{L}$ is number of particles per unit time, per unit area.
  - Number of particles passing through area dσ: $dN = \mathcal{L}\,\mathrm{d}\sigma$
  - Number of particles scattering into solid angle dΩ : $dN = \mathcal{L}\,\mathrm{d}\sigma = \mathcal{L}\,D(\theta)\,\mathrm{d}\Omega$
  - By counting one can measure the *differential cross section*: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = D(\theta) = \frac{\mathrm{d}N}{\mathcal{L}\,\mathrm{d}\Omega}$
  - Alternatively the total cross section: $N = \mathcal{L}\,\sigma$



These aspects are needed when you Compare theory with experiments.

- Experimental particle physics:
  - Measure number of events N and the luminosity $\mathcal{L}$ to find cross section $\sigma = N/\mathcal{L}$
  - Compare with theoretical calculation of σ (or $\frac{d\sigma}{d\Omega}$) using e.g. Standard Model

Griffiths §6.2 and PP1 Chapter 2

Part 2

Perturbation Theory and the Golden Rule

"How to calculate the cross section σ"

Griffiths only states the "Golden Rule".
In next 7 slides we will try to understand it!
For the exam only Griffith level is required.

Fermi's Golden Rule

- To calculate decay rates and cross section in relativistic scattering we use a general formula that we cannot fully derive within the scope of these lectures
  - For a fully relativistic derivation: quantum field theory
  - We will "make it plausible" using non-relativistic single particle theory
    - see also the book of Thomsom §2.3.6 and chapter 3, or Nikhef PP1 lecture notes chapter 2

- Here is the end-result:
  - Golden Rule for decays: $1 \to 2 + 3 + 4 + \cdots n$

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \, \frac{d^4 p_j}{(2\pi)^4}$$

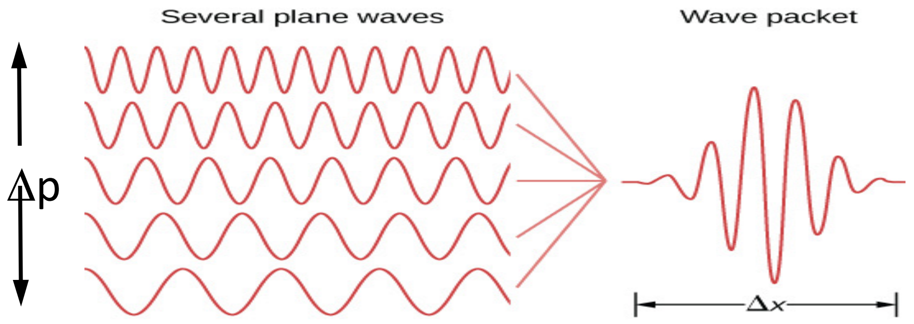
  - Golden Rule for scattering: $1 + 2 \to 3 + 4 + 5 + \cdots n$

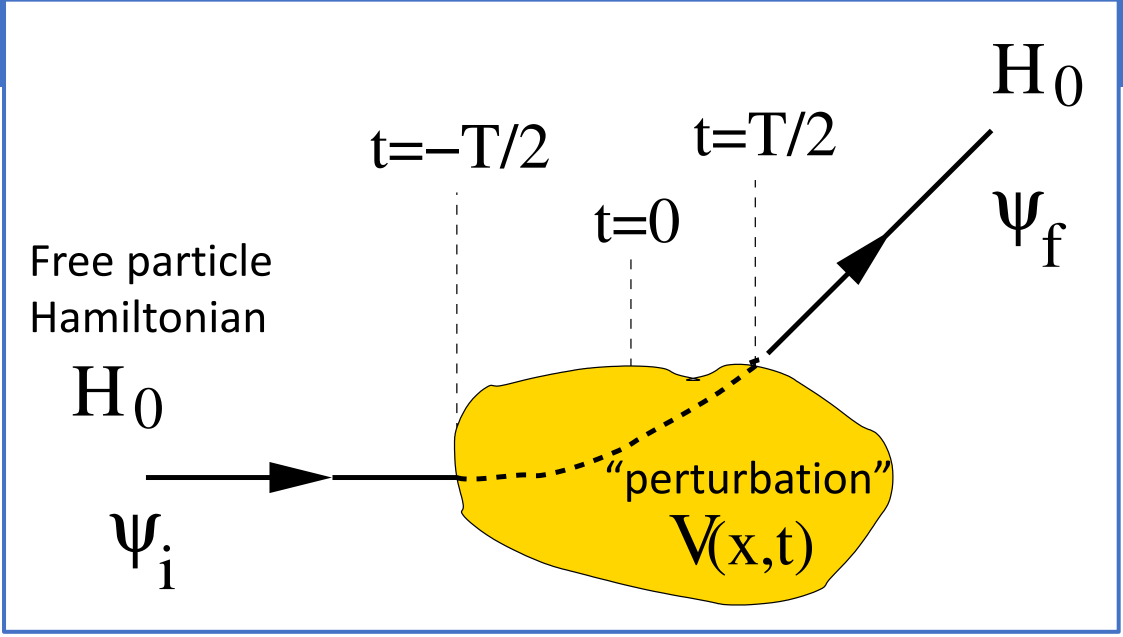
$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \, \frac{d^4 p_j}{(2\pi)^4}$$

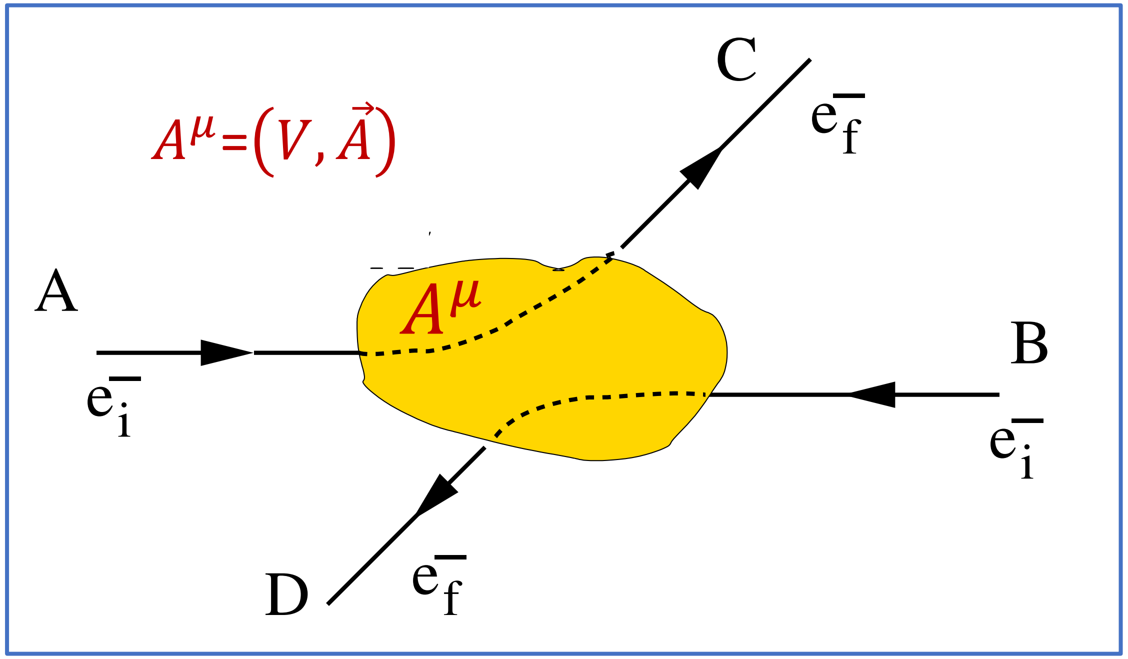
OK, let's go....

Scattering with waves

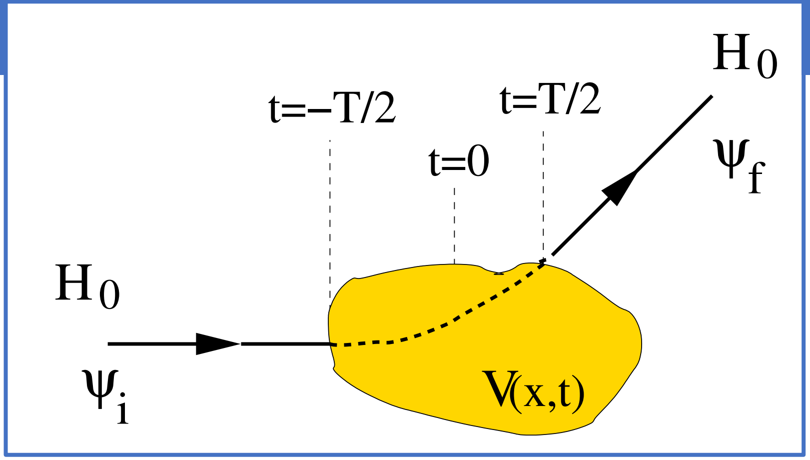
- An incoming particle is represented by a wave packet of incoming plane waves: $\psi(x) = Ne^{-ipx}$
- Example 1:
  - Calculate the scattering of these waves in an external potential
- Example 2:
  - For collisions the scattering potential A^{μ} of particle A is determined by the field of particle B and vice versa.







Perturbation Theory



- Consider the free Schrödinger equation $i\frac{\partial\psi}{\partial t} = H_0\psi$
  - H_0 is time-independent Hamiltonian $H_0\phi_m(\vec{x}) = E_m\phi_m(\vec{x})$
  - Eigenstates orthogonal: $\int \phi_m^*(\vec{x})\phi_n(\vec{x})d^3x = \delta_{mn}$
  - where $\phi_m(\vec{x})$ form orthonormal basis for any solution and $\psi_m(\vec{x},t) = \phi_m(\vec{x})e^{-iE_m t}$

(Particle = wave packet)

- Hamiltonian with time-dependent perturbation $i\frac{\partial\psi}{\partial t} = \left(H_0 + V(\vec{x},t)\right)\psi$
  - Solutions are of the form can be written as $\psi(\vec{x},t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(\vec{x})e^{-iE_n t}$
  - Substituting gives:

$$i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\phi_n(\vec{x})e^{-iE_n t} = \sum_{n=0}^{\infty} V(\vec{x},t)a_n(t)\phi_n(\vec{x})e^{-iE_n t}$$

$$i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\phi_n(\vec{x})e^{-iE_n t} + \cancel{i(-iE_n)\sum_{n=0}^{\infty} a_n(t)\,\phi_n(\vec{x})e^{-iE_n t}} = \cancel{\sum_{n=0}^{\infty} E_n a_n(t)\phi_n(\vec{x})e^{-iE_n t}} + \sum_{n=0}^{\infty} V(\vec{x},t)a_n(t)\phi_n(\vec{x})e^{-iE_n t}$$

Perturbation Theory

- Multiply the equation: $i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\phi_n(\vec{x})e^{-iE_nt} = \sum_{n=0}^{\infty}V(\vec{x},t)a_n(t)\phi_n(\vec{x})e^{-iE_nt}$

  … from the left by $\int\psi_f^* d^3x$ with $\psi_f^* = \phi_f^* e^{iE_ft}$

to find:

$$i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\underbrace{\int \mathrm{d}^3x\phi_f^*(\vec{x})\phi_n(\vec{x})}_{\delta_{fn}}\; e^{-i(E_n-E_f)t} = \sum_{n=0}^{\infty}a_n(t)\int \mathrm{d}^3x\;\phi_f^*(\vec{x})V(\vec{x},t)\phi_n(\vec{x})e^{-i(E_n-E_f)t}$$

- Using orthonormality gives: $i\frac{da_f(t)}{dt} = \sum_{n=0}^{\infty}a_n(t)\int \mathrm{d}^3x\;\phi_f^*(\vec{x})V(\vec{x},t)\phi_n(\vec{x})e^{-i(E_n-E_f)t}$

  or in short: $i\frac{\mathrm{d}a_f(t)}{\mathrm{d}t} = \sum_{n=0}^{\infty}a_n(t)V_{fn}\,e^{i\omega_{fn}t}$ with $\omega_{fn} = (E_f - E_n)$

  and the *transition matrix element:* $V_{fn} = \int d^3x\;\phi_f^*\,V(\vec{x},t)\,\phi_n(\vec{x})$

Perturbation Theory

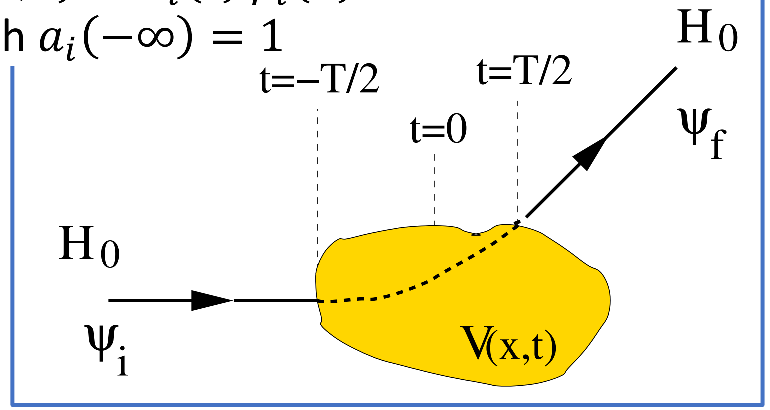
$$V_{fn} = \int d^3x \, \phi_f^* \, V(\vec{x},t) \, \phi_n(\vec{x})$$

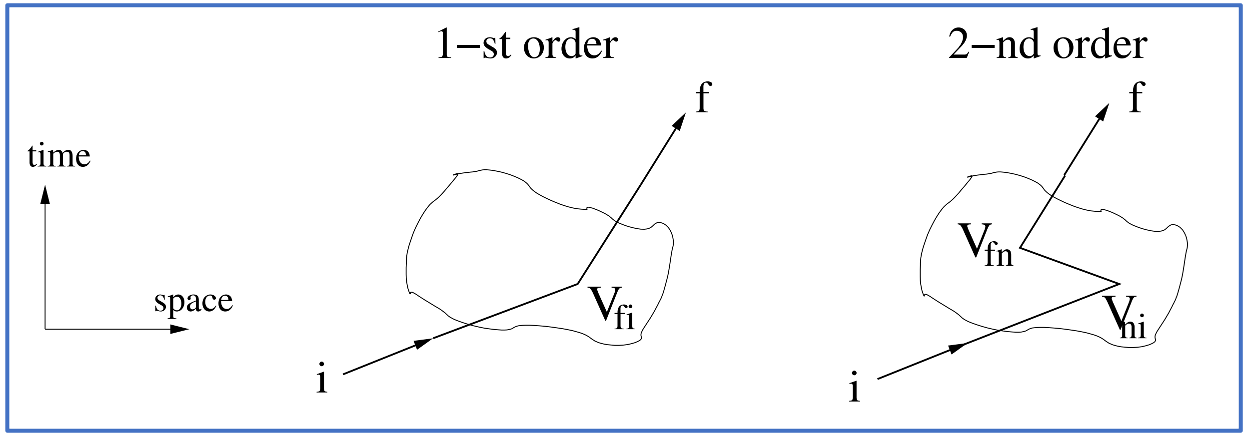
$$\omega_{fn} = (E_f - E_n)$$

- Solving differential equation: $$i \frac{\mathrm{d}a_f(t)}{\mathrm{d}t} = \sum_{n=0}^{\infty} a_n(t) V_{fn} \, e^{i\omega_{fn}t}$$
- Start with some assumption of zero-th order for a_n and then for each order o: $$i \frac{\mathrm{d}a_f^{(o+1)}(t)}{\mathrm{d}t} = \sum_{n=0}^{\infty} a_n^{(o)}(t) V_{fn} \, e^{i\omega_{fn}t}$$
- First order: assume one step interaction: $a_i(-\infty) = 1$ and $a_f(-\infty) = 0$ (for $f \neq i$) "during" interaction: $a_f^{(0)}(t) = \delta_{fi}$: $$i \frac{\mathrm{d}a_f^{(1)}(t)}{\mathrm{d}t} = V_{fi}(t) e^{i\omega_{fi}t}$$
- Perturbation theory: $$a_f^{(1)}(t) = \int_{-\infty}^{t} \frac{\mathrm{d}a_f(t')}{\mathrm{d}t} \mathrm{d}t' = -i \int_{-\infty}^{t} V_{fi}(t') e^{i\omega_{fi}t'} \mathrm{d}t' \quad \text{for } f \neq i$$

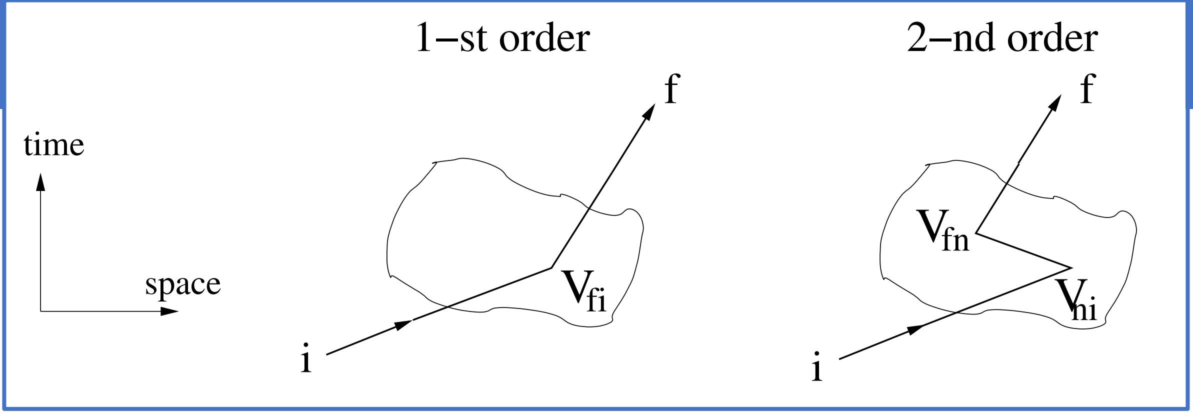
$\psi_i(\vec{x},t) = a_i(t)\phi_i(\vec{x})e^{-iE_i t}$

with $a_i(-\infty) = 1$





Perturbation Theory



- First order perturbation:

$$\omega_{fi} = (E_f - E_i)$$

$$a_f^{(1)}(t) = \int_{-\infty}^{t} \frac{\mathrm{d}a_f(t')}{\mathrm{d}t}\mathrm{d}t' = -i\int_{-\infty}^{t} V_{fi}(t')e^{i\omega_{fi}t'}\mathrm{d}t'$$

$$\left(V_{fi} = \int \mathrm{d}^3x\, \phi_f^*\, V(\vec{x},t)\, \phi_i(\vec{x}) \right)$$

Results in ("Born approximation") "*transition amplitude*" T_{fi}:

$$\boxed{T_{fi} \equiv a_f(t\to\infty) = -i\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3x\psi_f^*(\vec{x},t)V(\vec{x},t)\psi_i(\vec{x},t) = -i\int \mathrm{d}^4x\, \psi_f^*(x)V(x)\psi_i(x)}$$

Note: co-variant form!
"cheat relativity theory"

- If the potential is time independent ("static") we find:

$$T_{fi} \equiv a_f(t\to\infty) = -i\, V_{fi}\int_{-\infty}^{\infty} e^{i\omega_{fi}t}\mathrm{d}t = -2\pi V_{fi}\delta\left(E_f - E_i\right)$$

Energy conservation!

- Where we have used an implementation of the Dirac delta function:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}k$$

Golden Rule – non-relativistic

- *Transition rate* is defined as: $W_{fi} \equiv \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$ $\quad T_{fi} = -2\pi V_{fi}\delta(E_f - E_i)$
- After squaring of the delta function (not trivial, see PP1) results in transition probability per unit time: $\boxed{W_{fi} = 2\pi\,|V_{fi}|^2\delta(E_f - E_i)}$

  ($V_{fi} = \int \mathrm{d}^3x\, \phi_f^*\, V(\vec{x},t)\, \phi_i(\vec{x})$)

  - Where the delta function takes care of *energy conservation*
  - The name V_{fi} was used here since it relates to the potential V
  - We adapt the more common name for the matrix element $\mathcal{M}$
- Waves → particles: The differential cross section is: $\boxed{\sigma = \frac{W_{fi}}{\text{flux}}\Phi}$
  - Where flux represents the "density" of the number of incoming states per particle: states →particle
  - The phase space factor Φ (also 'LIPS') is the density of outgoing states (final state "realisation possibilities")

For more, see Chapter 2 of the Nikhef PP1 Lectures

- Next extend it to relativistic scattering using the matrix element $\mathcal{M}$

Golden Rule – non-relativistic

- *Transition rate* is define

$$T_{fi} = -2\pi V_{fi}\delta(E_f - E_i)$$

- After squaring of the de
  PP1) results in transitio

$$\boxed{W_{fi} = 2\pi \left|V_{fi}\right|^2 \delta(E_f - E_i)}$$

$$\left(V_{fi} = \int \mathrm{d}^3x \, \phi_f^* \, V(\vec{x},t) \, \phi_i(\vec{x}) \right)$$

  - Where the delta function … n
  - The name V_{fi} was used h … al V
  - We adapt the more comn … $\mathcal{M}$

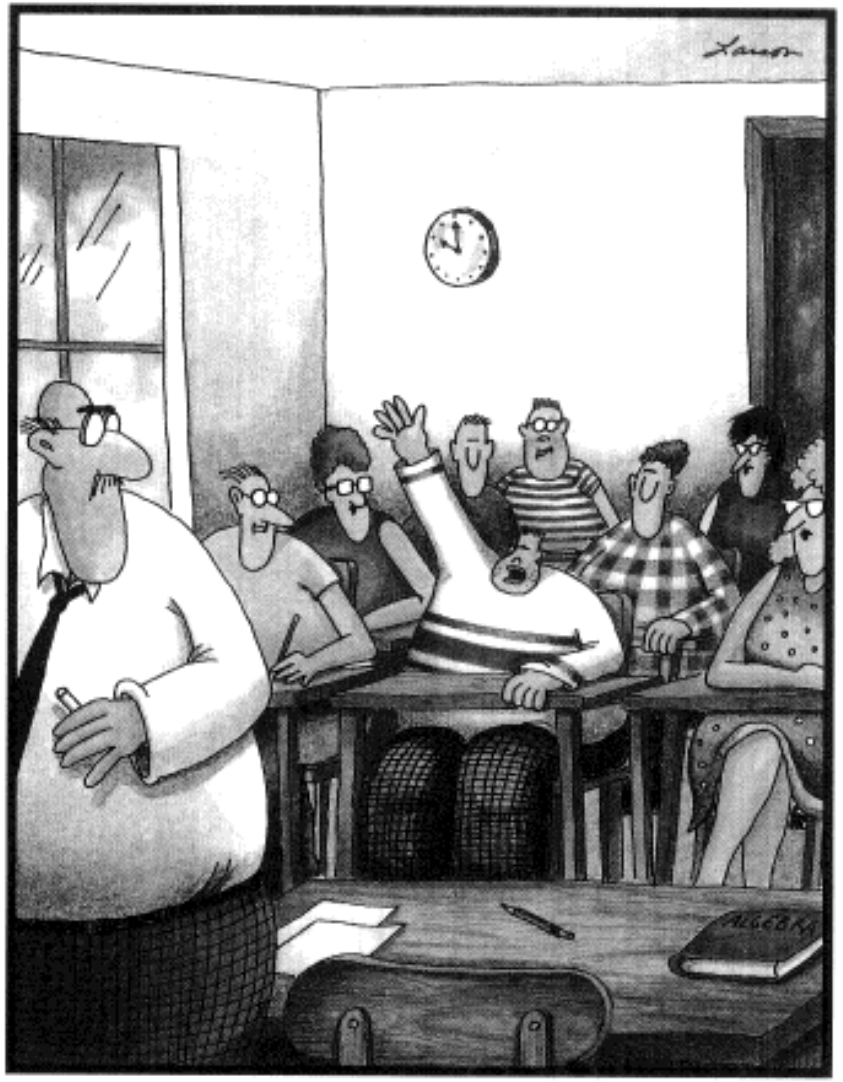
- Waves → particles: The

$$\boxed{\sigma = \frac{W_{fi}}{\text{flux}} \Phi}$$

  - Where flux represents th … ming
    states per particle: states
  - The phase space factor Φ … going
    states (final state "realisa

For more, see Chapter 2
of the Nikhef PP1 Lectures

- Next extend it to relativ … rix element $\mathcal{M}$



"Mr. Osborne, may I be excused?
My brain is full."

Fermi's Golden Rule - Relativistic

Back to Griffiths

Griffiths §6.2

- In Griffiths the relativistic Golden Rule for decay and scattering are just stated.
  - Try to gain understanding by considering each of the terms qualitatively.

Γ ≡decay rate $\frac{1}{\Gamma} = \tau$ ≡ mean lifetime

A. Golden Rule for decays: $1 \rightarrow 2 + 3 + 4 + \cdots n$

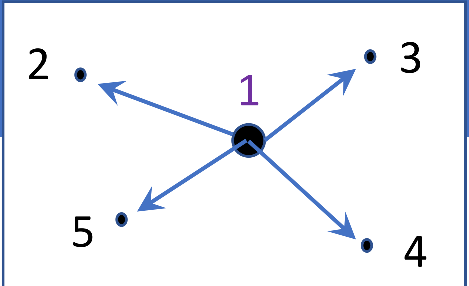
$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right) \frac{d^4 p_j}{(2\pi)^4}$$

B. Golden Rule for scattering (cross section): $1 + 2 \rightarrow 3 + 4 + 5 + \cdots n$

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right) \frac{d^4 p_j}{(2\pi)^4}$$

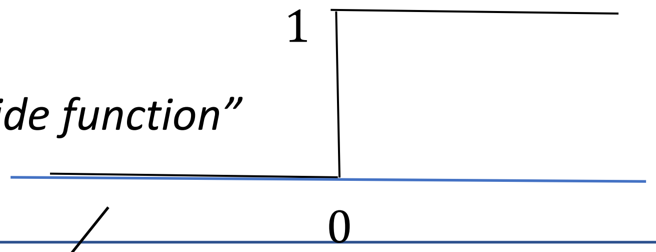
- We will discuss them in turn…

A. Decay



$\theta(p_j^0) =$

"Heavyside function"



- Golden Rule for Decay:

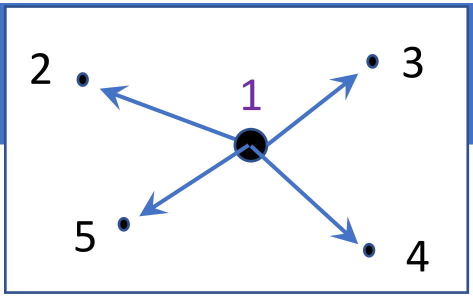
$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\, \frac{d^4 p_j}{(2\pi)^4}$$

Phase Space Φ:
Realize each possibility with equal probability

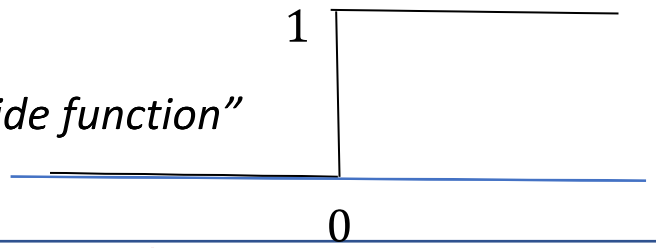
- S is a quantum factor to prevent double counting for identical particles
  - Each species with s particles in *the final state* gives factor $1/s!$
  - Eg.: decay $a \to b + b + c + c + c$ gets factor $(1/2!)\times(1/3!) = 1/12$
- $2m_1 = 2E$: density of incoming states (see lecture 3: $\rho = 2|N|^2 E$).
- $\mathcal{M}$ is the Matrix Element: contains the *dynamics* (the interesting particle physics). It is given by the Feynman rules. See later.
- $\int$ implements the integral over *all realization possibilities* to obtain the final state.
- δ is the Dirac delta function. δ^4 implements energy-momentum conservation and $\prod \delta$ assures produced particles are *on-mass shell*: $p^2 = m^2 \to E^2 - \vec{p}^2 = m^2$.
- θ step function so that only $E > 0$.
- Each δ-function comes with a factor 2π and each d^4p with $1/(2\pi)$.

underlying reason:
$\hbar = h/2\pi$

A. Decay



$\theta(p_j^0) =$
"Heavyside function"



- Decay:

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

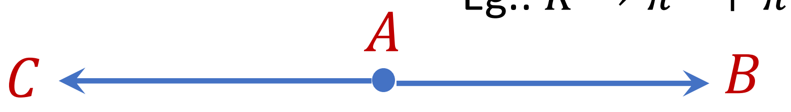
- Using the mathematical characteristics of Dirac δ functions (optional exercise, Griffiths page 205: "the $\theta(p_j^0)$-function kills the $\delta(p_j^0)$"), the second part can be shortened into:

Use: $\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} \frac{1}{2E_j} \frac{d^3\vec{p}_j}{(2\pi)^3}$$

- Consider the example $A \to B + C$

Eg.: $K \to \pi^+ + \pi^-$



$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta^4(p_1 - p_2 - p_3) \frac{d^3\vec{p}_2}{E_2} \frac{d^3\vec{p}_3}{E_3}$$

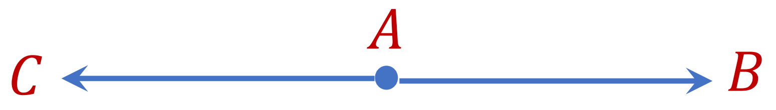
A: $p_1^\mu = (m_1, 0, 0, 0)$

B: $p_2^\mu = (E_2, p_2, 0, 0)$

C: $p_3^\mu = (E_3\, p_3, 0, 0)$

- Kinematics for the two-particle case……

Two-particle decay … calculate…



$$A:\ p_1^\mu = (m_1, 0, 0, 0)$$
$$B:\ p_2^\mu = (E_2, p_2, 0, 0)$$
$$C:\ p_3^\mu = (E_3\, p_3, 0, 0)$$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2\, \delta^4(p_1 - p_2 - p_3)\, \frac{d^3\vec{p}_2}{E_2}\, \frac{d^3\vec{p}_3}{E_3}$$

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)$$

Now: $p_1^0 = m_1$ and $\vec{p}_1 = 0$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 - \sqrt{\vec{p}_2^2 + m_2^2} - \sqrt{\vec{p}_3^2 + m_3^2}\right)}{\sqrt{\vec{p}_2^2 + m_2^2}\sqrt{\vec{p}_3^2 + m_3^2}}\, \delta^3(\vec{p}_2 + \vec{p}_3) d^3\vec{p}_2 d^3\vec{p}_3$$

Next: use $\vec{p}_3 = -\vec{p}_2$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 - \sqrt{\vec{p}_2^2 + m_2^2} - \sqrt{\vec{p}_2^2 + m_3^2}\right)}{\sqrt{\vec{p}_2^2 + m_2^2}\sqrt{\vec{p}_2^2 + m_3^2}}\, d^3\vec{p}_2$$

Next, go to spherical coordinates:

$$\vec{p}_2 \to (p, \theta, \phi)$$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 - \sqrt{p^2 + m_2^2} - \sqrt{p^2 + m_3^2}\right)}{\sqrt{p^2 + m_2^2}\sqrt{p^2 + m_3^2}}\, p^2 dp \underbrace{\int \sin\theta\, d\theta\, d\phi}_{4\pi}$$

Two-particle decay … calculate…

Griffiths pages 207 - 208

The integral over dp is not easy to calculate. Make the substitution: $u \equiv \sqrt{p^2+m_2^2} + \sqrt{p^2+m_3^2}$

Then: $\dfrac{du}{dp} = \dfrac{up}{\sqrt{p^2+m_2^2}\,\sqrt{p^2+m_3^2}}$; $pdp = \dfrac{du}{u}\sqrt{p^2+m_2^2}\sqrt{p^2+m_3^2}$

Such that we recognize:

$$\Gamma = \frac{S}{8\pi m_1}\int_{(m_2+m_3)}^{\infty} |\mathcal{M}|^2\,\delta(m_1-u)\,\frac{p}{u}\,du$$

which only has a contribution for $u = m_1$ (δ-function):

Note: $m_1 > m_2 + m_3$

Inverting the equation for u and p and putting $u = m_1$ gives (*exercise*):

$$p = |\vec{p}| = \frac{1}{2m_1}\sqrt{m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2}$$

and putting $u = m_1$ gives finally : $\boxed{\Gamma = \dfrac{S|\vec{p}|}{8\pi m_1^2}|\mathcal{M}|^2}$

Note that the δ-functions were enough to do all the integrals and put the required kinematic value for $\vec{p}$

Exercise (Optional): Kinematics relation

- Show explicitly that by inverting the equation:

$$m_1 = \sqrt{p^2 + m_2^2} + \sqrt{p^2 + m_3^2}$$

it follows that:

$$p = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

<u>Solution</u>: (put $c = 1$)

$$m_1 c = \sqrt{r^2 + m_2^2 c^2} + \sqrt{r^2 + m_3^2 c^2}. \quad \text{Square:}$$

$$m_1^2 c^2 = r^2 + m_2^2 c^2 + r^2 + m_3^2 c^2 + 2\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}$$

$$\frac{c^2}{2}(m_1^2 - m_2^2 - m_3^2) - r^2 = \sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}. \quad \text{Square again:}$$

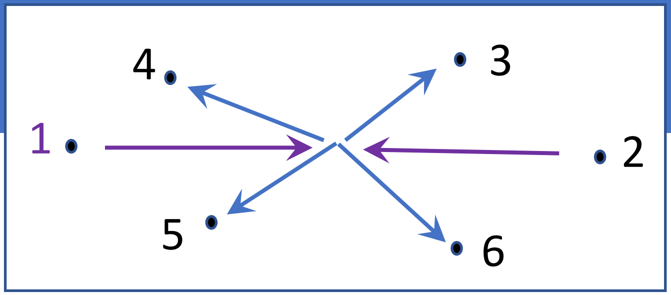
$$\frac{c^4}{4}(m_1^2 - m_2^2 - m_3^2)^2 - r^2 c^2 (m_1^2 - m_2^2 - m_3^2) + \cancel{r^4} = \cancel{r^4} + r^2 c^2 (m_2^2 + m_3^2) + m_2^2 m_3^2 c^4$$

$$\frac{c^4}{4}\left[(m_1^2 - m_2^2 - m_3^2)^2 - 4m_2^2 m_3^2\right] = r^2 c^2 (m_2^2 + m_3^2 + m_1^2 - m_2^2 - m_3^2) = r^2 m_1^2 c^2$$

$$r^2 = \frac{c^2}{4m_1^2}\left[m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 + 2m_2^2 m_3^2 - 4m_2^2 m_3^2\right]$$

$$\boxed{r = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}} \quad \checkmark$$

B. Cross Section (Scattering)



Remember NR: $\sigma = \frac{W_{fi}}{\text{flux}}\Phi$

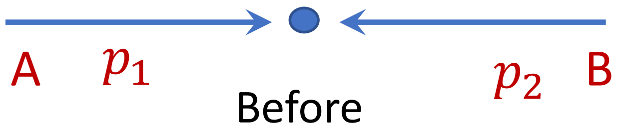
- Golden rule for cross section:

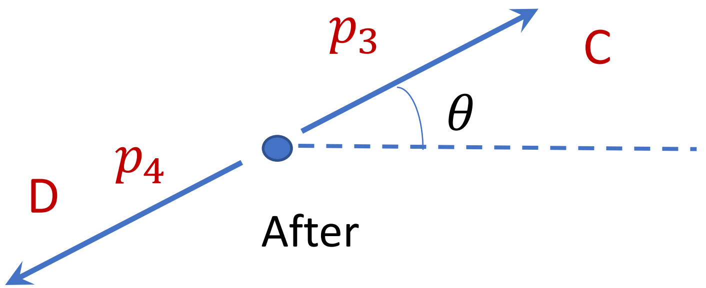
$$\underbrace{\sigma = \frac{S}{4\sqrt{(p_1\cdot p_2)^2-(m_1 m_2)^2}}}_{\text{Flux}} \int |\mathcal{M}|^2\, \underbrace{(2\pi)^4\delta^4(p_1+p_2-p_3\ldots-p_n)\times\prod_{j=3}^{n}2\pi\delta(p_j^2-m_j^2)\theta(p_j^0)\,\frac{d^4p_j}{(2\pi)^4}}_{\text{Phase Space }\Phi}$$

- Can be shortened, just as for decay, by doing the integrals over p^0 requiring on-mass relation: $p_j^0 = \sqrt{p_j^2+m_j^2} = E_j$ to find:

$$\sigma = \frac{S}{4\sqrt{(p_1\cdot p_2)^2-(m_1 m_2)^2}} \int |\mathcal{M}|^2\,(2\pi)^4\delta^4(p_1+p_2-p_3\ldots-p_n)\times\prod_{j=3}^{n}\frac{1}{2E_j}\frac{d^3\vec{p}_j}{(2\pi)^3}$$

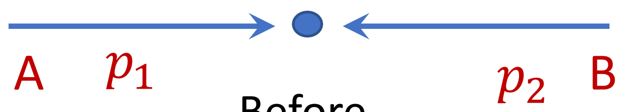
- Consider the ("2-to-2") example $A+B\rightarrow C+D$
  - Kinematics for the two-particle case……



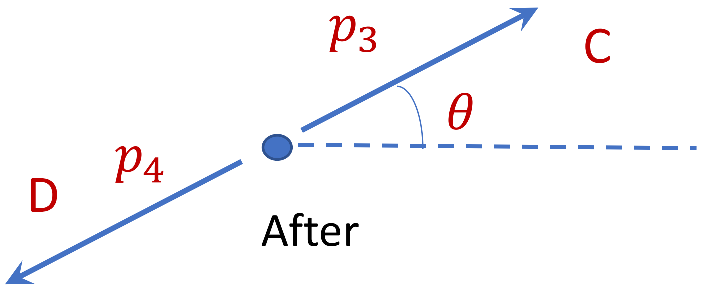


Two-to-Two cross section …

- $A + B \rightarrow C + D$



Before



After

Kinematics for the two-particle case in Center-of-Mass: $\vec{p}_2 = -\vec{p}_1$

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times \prod_{j=3}^{n} \frac{1}{2E_j} \frac{d^3\vec{p}_j}{(2\pi)^3}$$

Use: $p_1^\mu = \ (E_1, |\vec{p}_1|, 0{,}0)$ and $p_2^\mu = \ (E_2, -|\vec{p}_1|, 0{,}0)$ to see that, after kinematic calculation's – see Griffiths…

$$\sigma = \frac{S}{64\pi^2 (E_1 + E_2)|\vec{p}_1|} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \, \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4}$$

Again, split up the $\delta^4(p_1 + p_2 - p_3 - p_4)$ into $\delta(E_1 + E_2 - E_3 - E_4) \times \delta^3(\vec{p}_3 + \vec{p}_4)$ …etc… similar as decay.

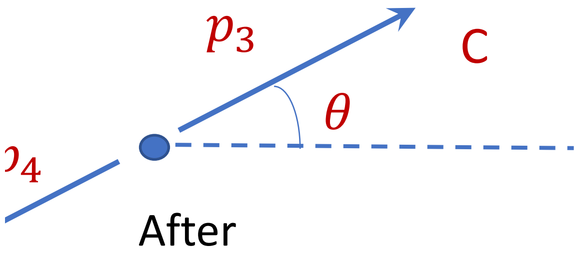
Complication: there is an angle θ in the game and we cannot carry out the integral , since $\mathcal{M}$ can depend on it. (Q: Why was there no θ in the case of decay?).

Determine the angle dependent cross section:

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

Two-to-Two cross section …

- $A + B \rightarrow C + D$



Kinematics for the two-pa

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m}}$$

Use: $p_1^\mu = (E_1, |\vec{p}_1|, 0{,}0)$

$$\sigma = \frac{S}{64\pi^2(E_1 + E_2)|\vec{p}}$$

Again, split up the $\delta^4(p_1$

Complication: there is an a

(Q: Why was there no θ in

Determine the angle depen

If you really want to know…

In the CM frame, $\mathbf{p}_2 = -\mathbf{p}_1$, so $p_1 = \left(\frac{E_1}{c}, \mathbf{p}_1\right)$, $p_2 = \left(\frac{E_2}{c}, -\mathbf{p}_1\right)$.

$$p_1 \cdot p_2 = \frac{E_1}{c}\frac{E_2}{c} - \mathbf{p}_1 \cdot (-\mathbf{p}_1) = \frac{E_1 E_2}{c^2} + \mathbf{p}_1^2$$

$$(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2 = \left(\frac{E_1 E_2}{c^2} + \mathbf{p}_1^2\right)^2 - (m_1 m_2 c^2)^2$$
$$= \frac{E_1^2 E_2^2}{c^4} + 2\frac{E_1 E_2}{c^2}\mathbf{p}_1^2 + \mathbf{p}_1^4 - m_1^2 m_2^2 c^4$$

But $m_1^2 c^2 = \frac{E_1^2}{c^2} - \mathbf{p}_1^2$, and $m_2^2 c^2 = \frac{E_2^2}{c^2} - \mathbf{p}_2^2 = \frac{E_2^2}{c^2} - \mathbf{p}_1^2$. So:

$$(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2$$
$$= \cancel{\frac{E_1^2 E_2^2}{c^4}} + 2\frac{E_1 E_2}{c^2}\mathbf{p}_1^2 + \cancel{\mathbf{p}_1^4} - \underbrace{\left(\frac{E_1^2}{c^2} - \mathbf{p}_1^2\right)\left(\frac{E_2^2}{c^2} - \mathbf{p}_1^2\right)}_{\cancel{\frac{E_1^2 E_2^2}{c^4}} - \mathbf{p}_1^2 \frac{(E_1^2 + E_2^2)}{c^2} + \cancel{\mathbf{p}_1^4}}$$
$$= \frac{1}{c^2}\mathbf{p}_1^2(E_1^2 + E_2^2 + 2E_1 E_2) = \frac{1}{c^2}\mathbf{p}_1^2(E_1 + E_2)^2.$$

$$\therefore \boxed{\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{1}{c}|\mathbf{p}_1|(E_1 + E_2).}$$

$$\frac{d^3\vec{p}_j}{(2\pi)^3}$$

culation's – see Griffiths…

) …etc… similar as decay.

since $\mathcal{M}$ can depend on it.

$$\boxed{\frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

How are you doing?

How are you doing?



How are you doing?






Part 3

Feynman Calculus

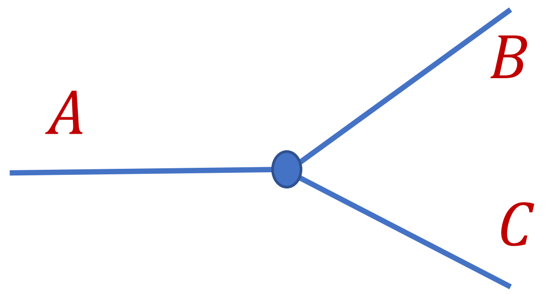
Griffiths §6.3

Or: how to find the matrix element $\mathcal{M}$
This is the *dynamics*: the interesting bit!

Feynman Rules: ABC Toy Theory

- All the "real" particle physics is in the calculation of the matrix $\mathcal{M}$.

- A full derivation of QED is not in the scope of the lectures. We give a "recipe".



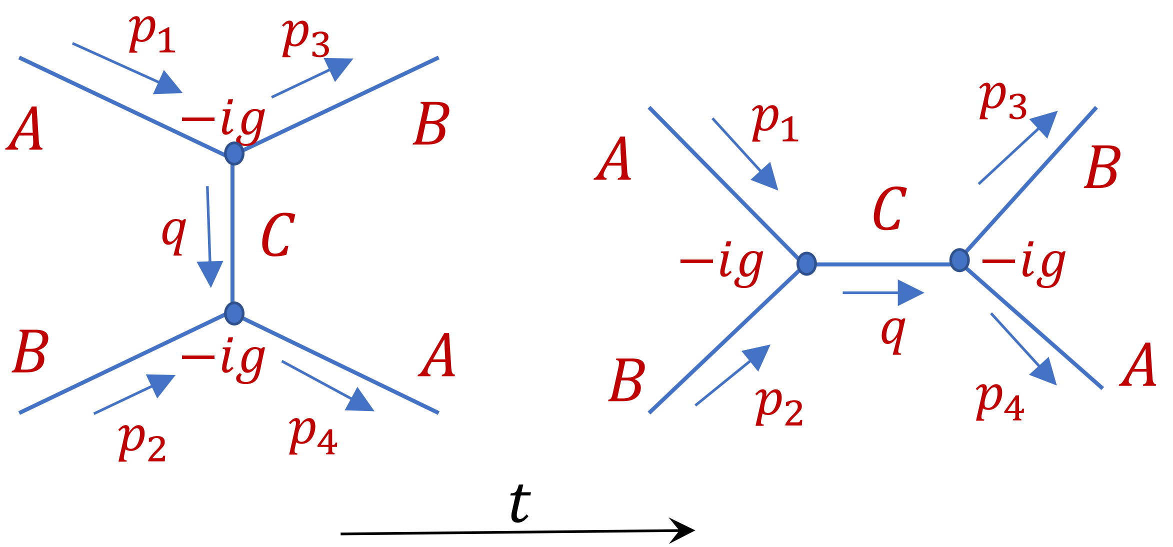
Only one fundamental vertex

- Consider ABC example theory
  - ABC model is simplest possible "theory".
  - Particles have no spin
  - Particles are their own antiparticle
    - No "arrows" needed
    - Think of π^0, K^0, η particles etc
  - No real forces, just "particles"

- For the following recipe keep perturbation theory and the golden rule in mind.

Feynman Rules: ABC Toy Theory for $A + B \rightarrow A + B$

- Recipe to find $\mathcal{M}$:
  1. Draw all the possible diagrams
  2. Label the external 4-momenta p_i^μ and put an arrow for the direction forward in time for external lines
  3. For each vertex write a factor $-ig$
  4. Propagators: for each internal line write: $\frac{i}{q_j^2 - m_j^2}$
     - Note that for an internal line: $q_j^2 \neq m_j^2$

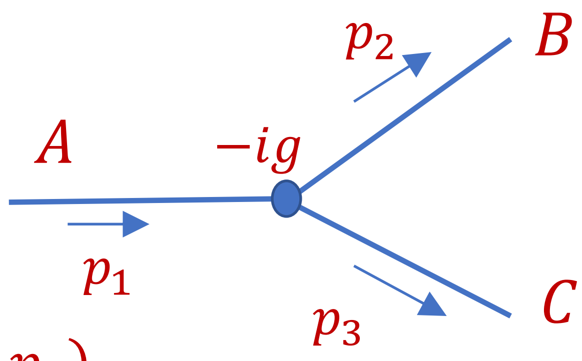


5. Conservation of energy and momentum:
   - For each vertex write a δ-function of the form: $(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)$, with a positive sign for momenta k_i going ***into*** the vertex. This δ makes sure that no momentum is "disappearing into a vertex"
6. Integrate over all internal momenta. For each internal line write a factor $\frac{1}{(2\pi)^4} d^4 q_i$
7. Result will include a delta function: $(2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n)$ reflecting overall energy and momentum conservation. *Erase* this delta function factor and multiply by i. → $\mathcal{M}$

Decay: Lifetime of $A \to B + C$

Feynman rules:

1. Diagrams: see sketch
2. Labels: see sketch
3. One vertex: $-ig$
4. Propagators: no internal lines
5. Conservation of energy and momentum: $(2\pi)^4\,\delta^4(p_1 - p_2 - p_3)$
6. Integrate: no internal momenta
7. Discard delta-function and multiply by i.



Result for the amplitude: $\mathcal{M} = g$

We obtain: $\Gamma = \frac{S|\vec{p}|}{8\pi m_A^2}\,|\mathcal{M}|^2 = \frac{|\vec{p}|}{8\pi m_A^2}\,g^2$ (no identical particles: $S = 1$)

where $|\vec{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$ (see before)

So that the lifetime is: $\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\vec{p}|}$

Exercise: Pion Decay

Griffiths exercise 6.6

Calculate the lifetime of the neutral pion π^0

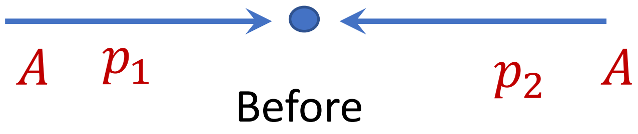
The neutral pion decays mainly via: $\pi^0 \rightarrow \gamma\gamma$. Assume that the amplitude has dimensions $[\mathrm{mass}] \times [\mathrm{velocity}]$. Griffiths: $\pi^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$

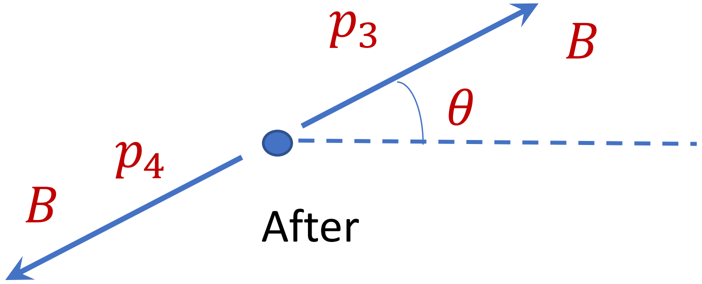
a) Motivate the reason that the amplitude should be proportional to the coupling constant: $\mathcal{M} \propto \alpha = e^2/4\pi$. Sketch a diagram of the decay.
   For dimensional reasons $\mathcal{M}$ is of the form $\mathcal{M} = \alpha m_\pi$

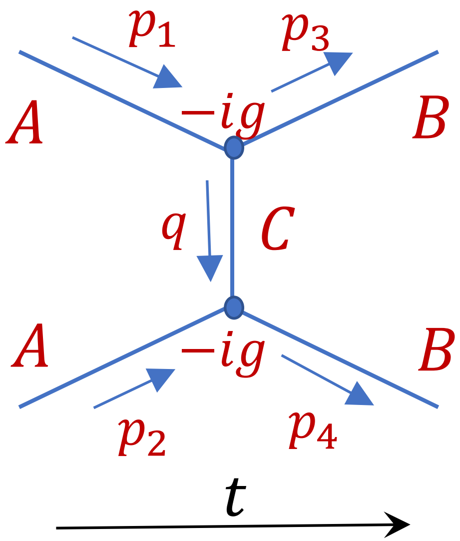
b) Use Fermi's golden rule for two-body decays to estimate the decay width Γ of the pion. What are $S, m_A, |\vec{p}|$? Express Γ in GeV.

c) Use the conversion table to calculate the lifetime of the π^0 and compare it with the experimental value. What do you think?

$A + A \to B + B$ Scattering: $\mathcal{M}$







Feynman rules:

1. Diagram: see sketch
2. Labels: see sketch
3. Two vertices: $(-ig)^2 = -g^2$
4. Propagators: one internal line: $\frac{i}{q^2 - m_C^2}$
5. Conservation of energy and momentum twice: $(2\pi)^4\ \delta^4(p_1 - p_3 - q)$ and $(2\pi)^4\ \delta^4(p_2 + q - p_4)$
6. Integrate: one integral: $\frac{1}{(2\pi)^4} d^4q$

Result so far:

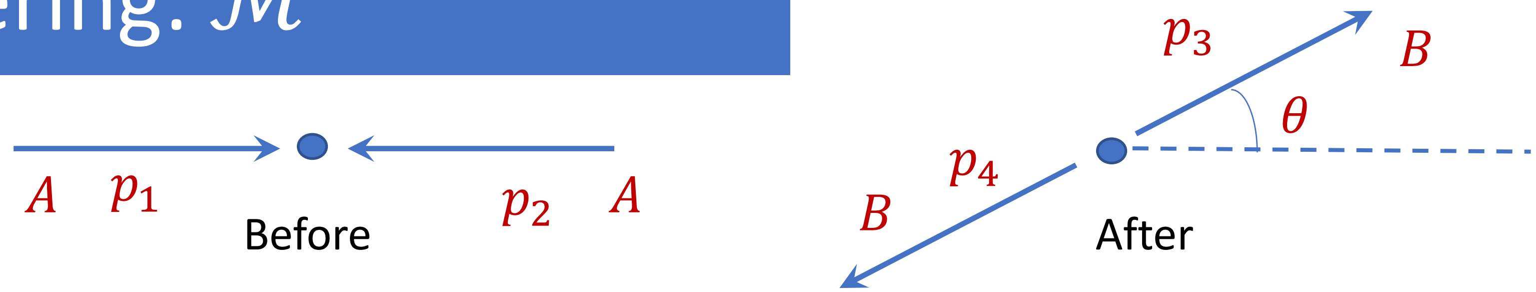
$$-(2\pi)^4 g^2 \int \frac{i}{q^2 - m_C^2} \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4) d^4q$$

Doing integral over 2nd δ^4 sets $q = p_4 - p_2$. Into first δ^4 to find

$$-g^2 \frac{i}{q^2 - m_C^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

7. Erase delta-function and multiply by i to find: $\mathcal{M} = \dfrac{g^2}{(p_4 - p_2)^2 - m_C^2}$

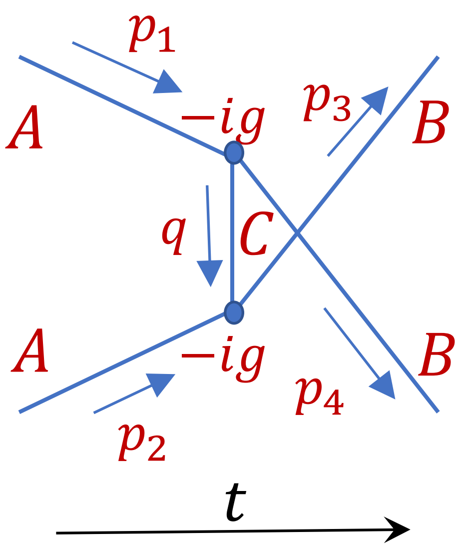
$A + A \rightarrow B + B$ Scattering: $\mathcal{M}$



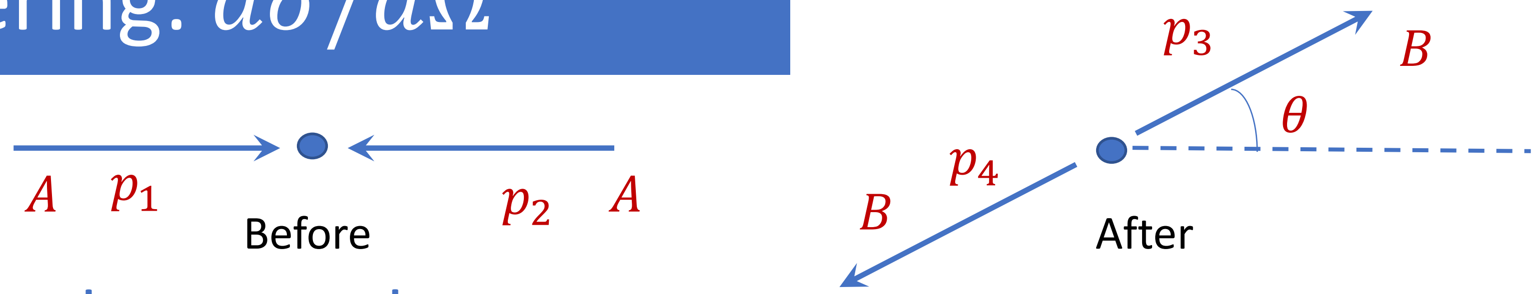
- There is a second diagram: see sketch
- Repeat the computation?
  - No, just replace: $p_3 \leftrightarrow p_4$ and fill in the end result:

$$\mathcal{M} = \frac{g^2}{(p_3 - p_2)^2 - m_C^2} + \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

  - Note: $\mathcal{M}$ does *not* depend on Lorentz frame: it is a Lorentz invariant (scalar) quantity.



$A + A \rightarrow B + B$ Scattering: $d\sigma/d\Omega$



- Look at the matrix element and assume that $m_A = m_B = m$ and $m_C = 0$ (eg. a photon): $\mathcal{M} = \dfrac{g^2}{(p_3 - p_2)^2} + \dfrac{g^2}{(p_4 - p_2)^2}$

$$
\begin{aligned}
(p_4 - p_2)^2 - m_C^2 &= p_4^2 + p_2^2 - 2p_2 \cdot p_4 \\
&= m_4^2 + m_2^2 - 2p_2 \cdot p_4 \\
&= 2m^2 - 2E_2E_4 + 2(\vec{p}_2 \cdot \vec{p}_4) \\
&= 2m^2 - 2\left(\sqrt{m^2 + \vec{p}^2}\right)\left(\sqrt{m^2 + \vec{p}^2}\right) + 2\vec{p}^2 \cos\theta \\
&= -2\vec{p}^2(1 - \cos\theta)
\end{aligned}
$$

$$(p_3 - p_2)^2 - m_C^2 = -2\vec{p}^2(1 + \cos\theta)$$

Note that for 4-vectors:
$p_i \cdot p_j = p_{i_\mu} p_j^\mu = E_iE_j - \vec{p}_i \cdot \vec{p}_j$
and that $p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$

(Invariant mass)

$$\mathcal{M} = \frac{g^2}{-2\vec{p}^2(1 - \cos\theta)} + \frac{g^2}{-2\vec{p}^2(1 + \cos\theta)} = -\frac{g^2}{2\vec{p}^2 \sin^2\theta}$$

- Plug in: $(S = 1/2)$

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi E\vec{p}^2 \sin^2\theta}\right)^2$$

Exercise: Two-to-Two Scattering

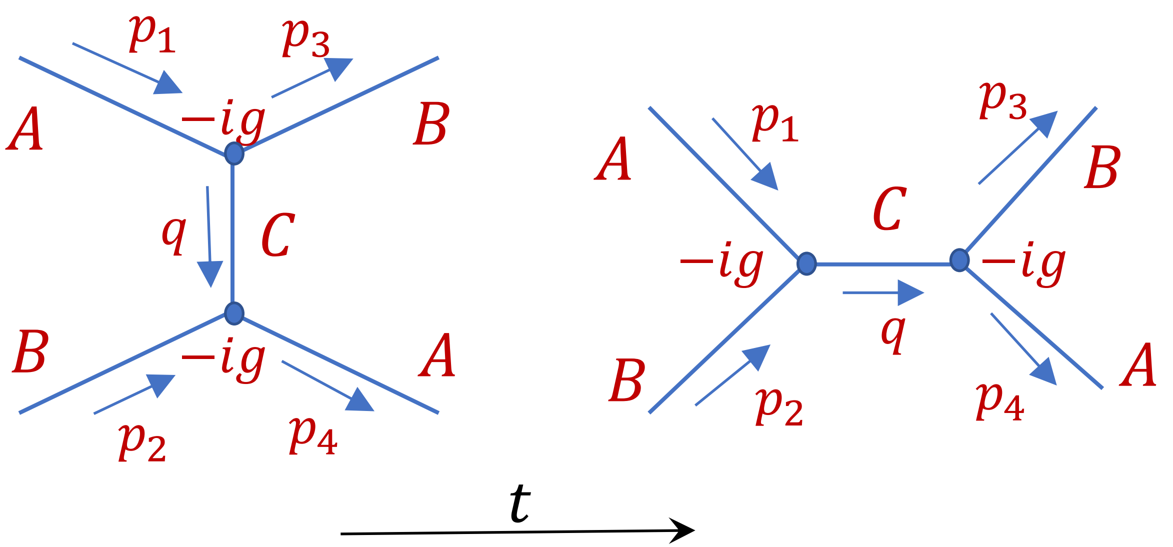
Griffiths exercise 6.15

Consider the process: $A + B \to A + B$ in the ABC theory

a) Draw the (two) lowest-order Feynman diagrams, and calculate the amplitudes
b) Find the differential cross-section in the CM frame, assuming $m_A = m_B = m\,, m_C = 0$, in terms of the (incoming) energy E and the scattering angle θ.
c) Assuming next that B is much heavier than A, calculate the differential cross-section in the lab frame.
d) For case c), find the total cross-section.

Feynman Rules: ABC Toy Theory for $A + B \to A + B$

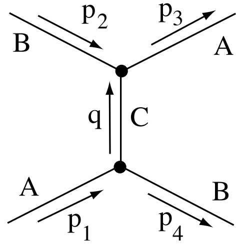
- Recipe to find $\mathcal{M}$:
  1. Draw all the possible diagrams
  2. Label the external 4-momenta p_i^μ and put an arrow for the direction forward in time for external lines
  3. For each vertex write a factor $-ig$
  4. Propagators: for each internal line write: $\frac{i}{q_j^2 - m_j^2}$
     - Note that for an internal line: $q_j^2 \neq m_j^2$



  5. Conservation of energy and momentum:
     - For each vertex write a δ-function of the form: $(2\pi)^4\,\delta^4(k_1 + k_2 + k_3)$, with a positive sign for momenta k_i going ***into*** the vertex. This δ makes sure that no momentum is "disappearing into a vertex"
  6. Integrate over all internal momenta. For each internal line write a factor $\frac{1}{(2\pi)^4} d^4 q_i$
  7. Result will include a delta function: $(2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n)$ reflecting overall energy and momentum conservation. *Erase* this delta function factor and multiply by i. → $\mathcal{M}$

Solution Exercise

(a)



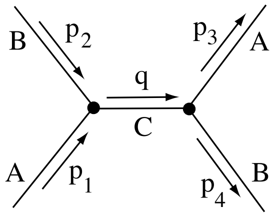


Diagram 1:

$$\int (-ig)^2 \frac{i}{q^2 - m_C^2c^2}(2\pi)^4\delta^4(p_1 - p_4 - q)(2\pi)^4\delta^4(q + p_2 - p_3)\frac{d^4q}{(2\pi)^4}$$

$$= \frac{-ig^2}{(p_1 - p_4)^2 - m_C^2c^2}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4), \quad \mathcal{M}_1 = \frac{g^2}{(p_1 - p_4)^2 - m_C^2c^2}.$$

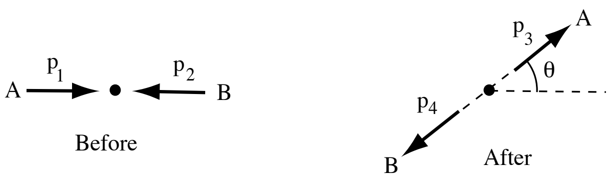
Diagram 2:

$$\int (-ig)^2 \frac{i}{q^2 - m_C^2c^2}(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(q - p_3 - p_4)\frac{d^4q}{(2\pi)^4}$$

$$= \frac{-ig^2}{(p_1 + p_2)^2 - m_C^2c^2}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4), \quad \mathcal{M}_2 = \frac{g^2}{(p_1 + p_2)^2 - m_C^2c^2}.$$

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \boxed{g^2\left[\frac{1}{(p_1 - p_4)^2 - m_C^2c^2} + \frac{1}{(p_1 + p_2)^2 - m_C^2c^2}\right]}$$

(b)



$$p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_1\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_3\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}_3\right)$$

$$(p_1 + p_2) = \left(\frac{2E}{c}, \mathbf{0}\right), \quad \text{so} \quad (p_1 + p_2)^2 = 4\frac{E^2}{c^2}$$

$$p_1 - p_4 = (0, \mathbf{p}_1 + \mathbf{p}_3), \quad \text{so}$$

$$(p_1 - p_4)^2 = -(\mathbf{p}_1 + \mathbf{p}_3)^2 = -(\mathbf{p}_1^2 + \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3)$$
$$= -(2\mathbf{p}_1^2 + 2|\mathbf{p}_1||\mathbf{p}_1|\cos\theta) = -4\mathbf{p}_1^2\cos^2\theta/2.$$

$$\mathcal{M} = \frac{g^2}{4}\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right).$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(2E)^2}\frac{|\mathbf{p}_4|}{|\mathbf{p}_1|}; \quad |\mathbf{p}_4| = |\mathbf{p}_1|,$$

$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{(2E)^2}\left(\frac{g^2}{4}\right)^2\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right)^2$$

But $\mathbf{p}_1^2 = E^2/c^2 - m_A^2c^2$, so

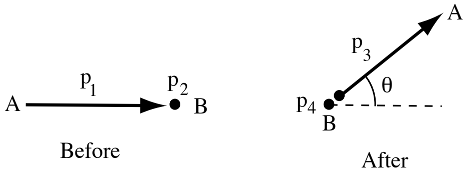
$$\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right) = \frac{c^2}{E^2} - \frac{c^2}{(E^2 - m^2c^4)\cos^2\theta/2}$$

$$= c^2\frac{[(E^2 - m^2c^4)\cos^2\theta/2 - E^2]}{E^2(E^2 - m^2c^4)\cos^2\theta/2} = c^2\frac{[E^2(\cos^2\theta/2 - 1) - m^2c^4\cos^2\theta/2]}{E^2(E^2 - m^2c^4)\cos^2\theta/2}$$

$$= -c^2\frac{(E^2\sin^2\theta/2 + m^2c^4\cos^2\theta/2)}{E^2(E^2 - m^2c^4)\cos^2\theta/2} = -c^2\frac{(E^2\tan^2\theta/2 + m^2c^4)}{E^2(E^2 - m^2c^4)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left\{\frac{g^2\hbar c^3}{64\pi}\frac{\left(E^2\tan^2\frac{\theta}{2} + m^2c^4\right)}{E^3(E^2 - m^2c^4)}\right\}^2.}$$

(c)



$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_B c}\right)^2 |\mathcal{M}|^2$$

$$\left.\begin{array}{ll} p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right), & p_2 = (m_B c, \mathbf{0}) \\ p_3 = \left(\frac{E}{c}, \mathbf{p}_3\right), & p_4 = (m_B c, \mathbf{0}) \end{array}\right\} \text{From these, we get:}$$

$$(p_1 - p_4) = \left(\frac{E}{c} - m_B c, \mathbf{p}_1\right); \quad (p_1 + p_2) = \left(\frac{E}{c} + m_B c, \mathbf{p}_1\right).$$

$$(p_1 - p_4)^2 - m_C^2c^2 = \left(\frac{E}{c} - m_B c\right)^2 - \mathbf{p}_1^2 - m_C^2c^2$$
$$= \left(\frac{E}{c} - m_B c\right)^2 - \left(\frac{E^2}{c^2} - m_A^2c^2\right) - m_C^2c^2 \cong m_B^2c^2.$$

$$(p_1 + p_2)^2 - m_C^2c^2 = \left(\frac{E}{c} + m_B c\right)^2 - \mathbf{p}_1^2 - m_C^2c^2$$
$$= \left(\frac{E}{c} + m_B c\right)^2 - \left(\frac{E^2}{c^2} - m_A^2c^2\right) - m_C^2c^2 \cong m_B^2c^2.$$

$$\Rightarrow \mathcal{M} = g^2\frac{2}{m_B^2c^2}. \quad \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\hbar g^2}{4\pi m_B^3c^3}\right)^2.}$$

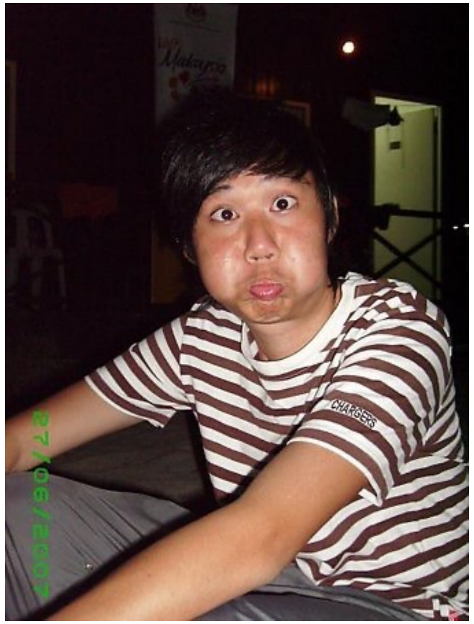
(d)

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = 4\pi\left(\frac{d\sigma}{d\Omega}\right) \Longrightarrow \boxed{\sigma = \frac{1}{\pi}\left(\frac{\hbar g^2}{2m_B^3c^3}\right)^2.}$$

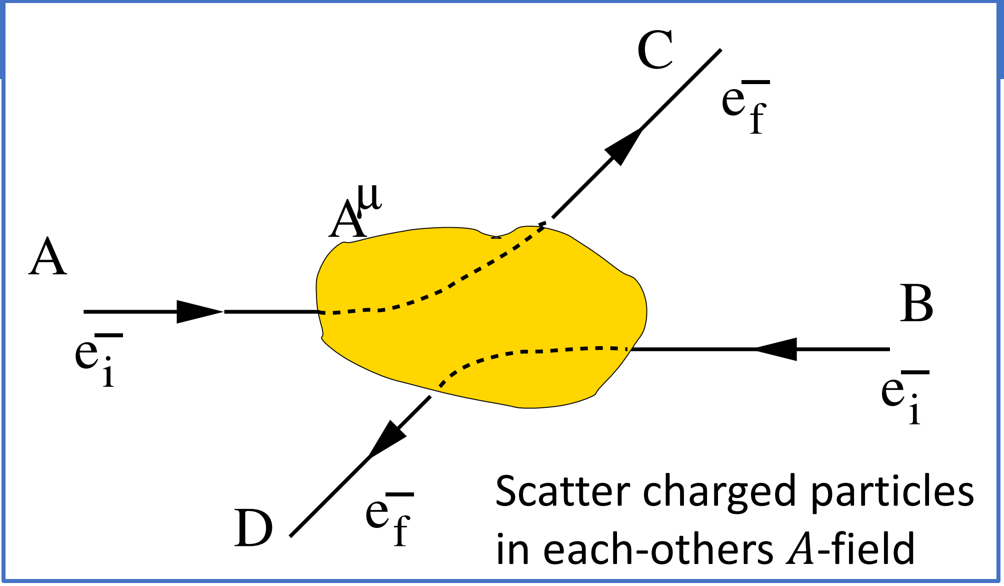
From ABC to QED

- ABC does not describe electrodynamics in the real world
  - We have charged particles and photons
  - Fermions have spin=½, forces have spin=1
    - Spin is a complication, we will leave that for a course at master level

- How to get the matrix element and Feynman rules for QED scattering?
  - This section will be quite "dense" but try to get the gist of it…



A taste of Relativistic QED Scattering



- Dirac equation in QED: $(\gamma_\mu p^\mu - m)\psi + e\gamma_\mu A^\mu \psi = 0$

  $\partial^\mu \to \partial^\mu - ieA^\mu \quad ; \quad \partial^\mu \to ip^\mu \quad ; \quad p^\mu \to p^\mu + eA^\mu$

- Perturbation theory: $(H_0 + V)\psi = E\psi$

  $H_0 = \vec{\alpha} \cdot \vec{p} + \beta m = \gamma^0 \gamma^k p^k + \gamma^0 m$

  ➔ $V = -e\gamma^0 \gamma_\mu A^\mu$

- Transition amplitude: no spin (see before): $T_{fi} = -i \int \mathrm{d}^4x \, \psi_f^*(x) V(x) \psi_i(x)$

  spin ½ (Dirac): $T_{fi} = -i \int \mathrm{d}^4x \, \psi_f^\dagger(x) V(x) \psi_i(x)$

  $= -i \int \mathrm{d}^4x \, \bar{\psi}_f(x)(-e)\gamma_\mu A^\mu(x) \psi_i(x) = -i \int j_\mu^{fi} A^\mu d^4x$

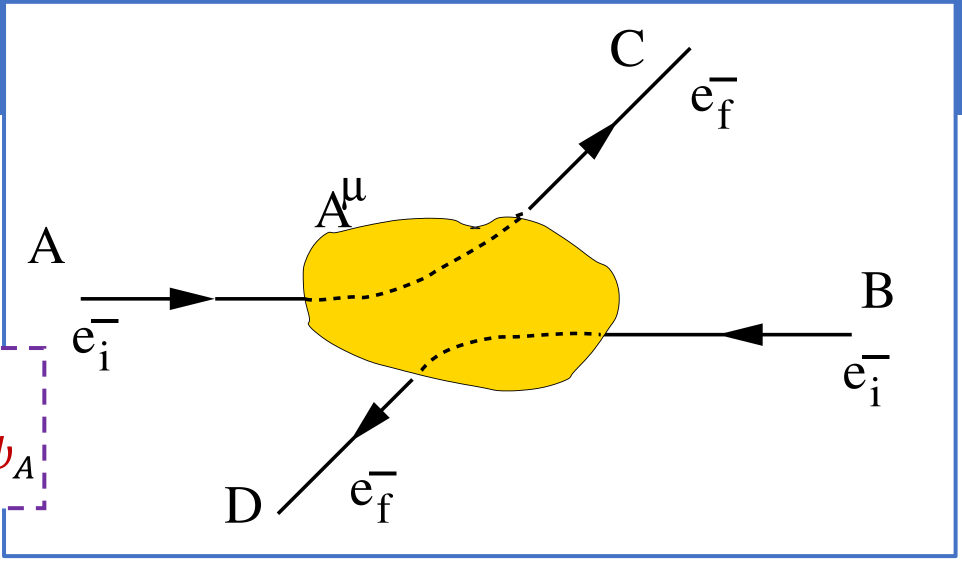
  $\boxed{j_\mu^{fi} = \text{“transition current”}}$

- Remember

  $\mathcal{L}_{QED} = \mathcal{L}_{\text{free}} - \mathcal{L}_{\text{int}} = \mathcal{L}_{\text{Dirac}} - q\bar{\psi}\gamma_\mu A^\mu \psi$ → $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$ with $J_\mu = q\bar{\psi}\gamma_\mu \psi$

- To determine A^μ insert the electromagnetic field that one particle A observes from the other particle B and vice versa.

A taste of Relativistic QED Scattering



- Particle BD scatters in the field A^μ of particle AC
- The field A^μ is obtained from Maxwell: $\partial_\nu \partial^\nu A^\mu = j^\mu_{AC}$
- Solution: $A^\mu = -\frac{1}{q^2} j^\mu_{AC}$

Remember: $j^\mu = -e\bar{\psi}\gamma^\mu\psi$
Transition current: $j^\mu_{AC} = -e\bar{\psi}_C\gamma^\mu\psi_A$

- Transition amplitude becomes:

$$T_{fi} = -i\int j^\mu_{fi} A_\mu \, d^4x = -i\int j^{(BD)}_\mu \frac{-1}{q^2} j^\mu_{(AC)} = -i\int j^{(BD)}_\mu \frac{-1}{q^2} j^\mu_{(AC)}$$

- Inserting plane wave solutions: $\psi(x) = u(p)e^{-ipx}$

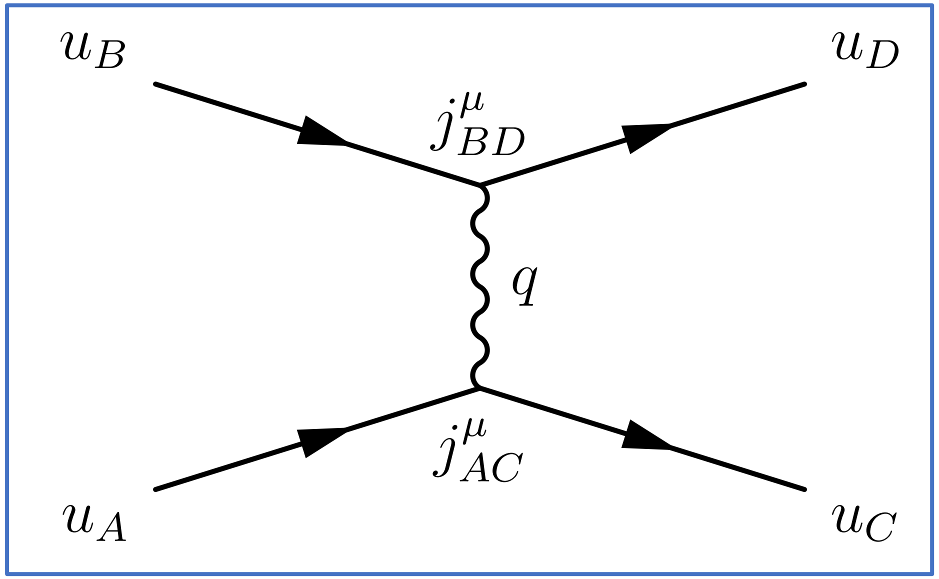
  into the current gives: $j^\mu_{(AC)} = -e\bar{u}_C\gamma^\mu u_A \, e^{i(p_C - p_A)x}$

  and: $j^\mu_{(BD)} = -e\bar{u}_D\gamma^\mu u_B \, e^{i(p_D - p_B)x}$

- Hence: $T_{fi} = -i(2\pi)^4\delta^4(p_D + p_C - p_B - p_A)\,\mathcal{M}$

  with: $-i\mathcal{M} = \underbrace{ie\,(\bar{u}_C\gamma^\mu u_A)}_{\text{vertex}} \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{\text{propagator}} \underbrace{ie(\bar{u}_D\gamma^\nu u_B)}_{\text{vertex}}$

"Matrix element"



A taste of Relativistic QED Scattering

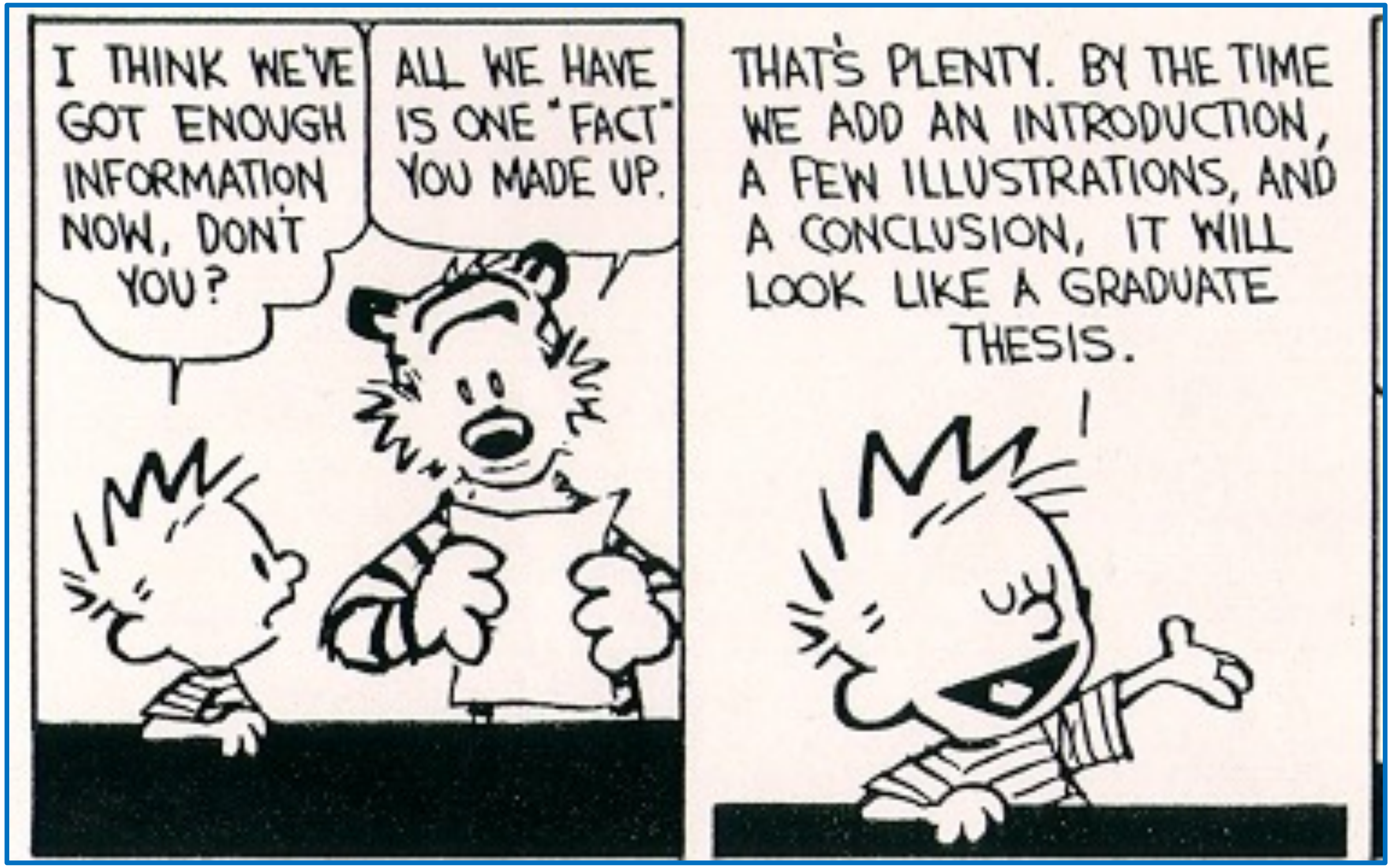
- Note that the current is of the form:

$$j^{\mu}_{fi} = \begin{pmatrix} & \bar{\psi}_f & \end{pmatrix} \begin{pmatrix} & \cdots & \\ \vdots & \gamma^{\mu} & \vdots \\ & \cdots & \end{pmatrix} \begin{pmatrix} \psi_i \end{pmatrix}$$

It is a 4-*vector* in Lorentz space (t, x, y, z) ; $\mu = 0,1,2,3$
It is a *scalar* in Dirac space (1, 2, 3, 4)

- Finally:
  - Feynman rules for QED are given in Griffiths section 7.5
  - To calculate cross sections with spin-½ particles is mathematically involved; it requires taking the square of the matrix element and summation over spin states of spinor objects.
  - We leave this fun for a topic of a master level course
  - Section 7.6, 7.7 and 7.8 of Griffiths give you an idea
    - It goes beyond the scope of this course

- Next week: "Detectors", measuring the particle processes


I THINK WE'VE GOT ENOUGH INFORMATION NOW, DON'T YOU?
ALL WE HAVE IS ONE "FACT" YOU MADE UP.
THAT'S PLENTY. BY THE TIME WE ADD AN INTRODUCTION, A FEW ILLUSTRATIONS, AND A CONCLUSION, IT WILL LOOK LIKE A GRADUATE THESIS.

Lecture 5: Exercises

Exercises belonging to Lecture 5

Exercise – 17: Pion Decay

Griffiths exercise 6.6

Calculate the lifetime of the neutral pion π^0

The neutral pion decays mainly via: $\pi^0 \rightarrow \gamma\gamma$. Assume that the amplitude has dimensions $[\mathrm{mass}]\times$ $[\mathrm{velocity}]$. Griffiths: $\pi^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$

a) Motivate the reason that the amplitude should be proportional to the coupling constant: $\mathcal{M} \propto \alpha = e^2/4\pi$. Sketch a diagram of the decay.

For dimensional reasons $\mathcal{M}$ is of the form $\mathcal{M} = \alpha m_\pi$

b) Use Fermi's golden rule for two-body decays to estimate the decay width Γ of the pion. What are $S, m_A, |\vec{p}|$? Express Γ in GeV.

c) Use the conversion table to calculate the lifetime of the π^0 and compare it with the experimental value. What do you think?

Exercise – 18: Two-to-Two Scattering

Griffiths exercise 6.15

Consider the process: $A + B \rightarrow A + B$ in the ABC theory

a) Draw the (two) lowest-order Feynman diagrams, and calculate the amplitudes
b) Find the differential cross-section in the CM frame, assuming $m_A = m_B = m\,, m_C = 0$, in terms of the (incoming) energy E and the scattering angle θ.
c) Assuming next that B is much heavier than A, calculate the differential cross-section in the lab frame.
d) For case c), find the total cross-section.