Lecture 3: Exercises

Exercises belonging to Lecture 3

Exercise – 9: Dirac Algebra

- Dirac algebra:
  - Write the explicit form of the γ-matrices
  - Show that : $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
  - Show that : $(\gamma^0)^2 = \mathbb{1}_4$; $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{1}_4$
  - Use anti-commutation rules of α and β to show that: ${\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0$
  - Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and show: ${\gamma^5}^\dagger = \gamma^5$; $(\gamma^5)^2 = \mathbb{1}_4$; $\{\gamma^5, \gamma^\mu\} = 0$

Exercise – 10: Solutions of free Dirac equation

See Griffiths for a derivation of the solutions

a) Show that the following plane waves are solutions to Dirac's equation

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \;;\; \psi_2 = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

Before KG:
$\phi = Ne^{-i(p_\mu x^\mu)}$

$$\psi_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x + ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \;;\; \psi_4 = \begin{pmatrix} (p_x - ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

b) Write the Dirac equation for particle in rest (choose $\vec{p} = 0$) and show that ψ_1 and ψ_2 are *positive energy* solutions: $E = +\left|\sqrt{\vec{p}^2 + m^2}\right|$ whereas ψ_3 and ψ_4 are *negative energy* solutions: $E = -\left|\sqrt{\vec{p}^2 + m^2}\right|$.

c) *<u>Optional:</u>* Consider the *helicity* operator $\vec{\sigma} \cdot \vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$ and show that ψ_1 corresponds to *positive helicity* solution and ψ_2 to *negative helicity*. Similarly for ψ_3 and ψ_4.

Spin and Helicity – hint for exercise 10c)

- For a given momentum p there still is a *two-fold degeneracy*: what differentiates solutions ψ_1 from ψ_2?
- Define the spin operator for Dirac spinors: $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, where $\vec{\sigma}$ are the three 2x2 Pauli spin matrices
- Define ***helicity*** λ as spin "up"/"down" wrt direction of motion of the particle

$$\lambda = \frac{1}{2}\vec{\Sigma} \cdot \hat{p} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} = \frac{1}{2|p|}(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)$$

- Split off the Energy and momentum part of Dirac's equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- Exercise: Try solutions ψ_1 and ψ_2 to see they are ***helicity eigenstates*** with $\lambda = +1/2$ and $\lambda = -1/2$
- Dirac wanted to solve negative energies and he found spin-½ fermions!

Exercise – 11 : Lagrangians and wave equations

Griffiths §10.2

- Scalar Field (spin 0 "pion")

  a) Show that the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$ results in the Klein-Gordon equation

- Dirac Field (spin ½ Fermion)

  b) Show that the Euler-Lagrange equations for $\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$ results in the Dirac equation

- Electromagnetic field (spin 1 photon)

  c) Show that $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$ results in Maxwell's equations

These Lagrangians are the fundamental objects in quantum field theory

Descriptions of interactions follow from symmetry principles on these objects.

Exercise – 12 : Covariant Derivative

Griffiths §10.3

- We insist that the Lagrangian does not change and invent a "covariant" derivative:
  - Replace in $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ the derivative by: $\partial^\mu \to D^\mu \equiv \partial^\mu + iqA^\mu$
  - Require that the vector field A^μ transforms together with the particle wave ψ

$$\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x)$$

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) - \partial^\mu\alpha(x)$$

  - ➔ Exercise: check that the Lagrangian now is invariant!