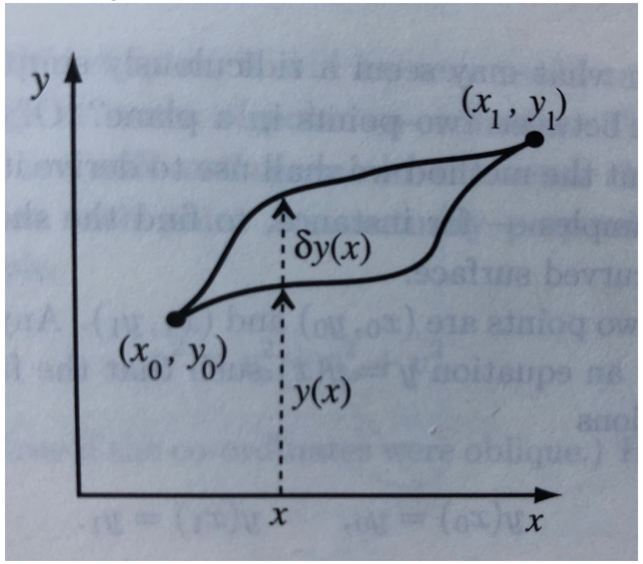
Lecture 2: Exercises

Exercises belonging to Lecture 2

Exercise-4: Variational calculus Lagrange Formalism classical



- Example of variational calculus and least action principle: what is the shortest path between two points in space?
  - Distance of two close points:

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 + y'^2}dx \quad \text{with } y' = dy/dx$$

  - Total length from (x_0, y_0) to (x_1, y_1) :

$$l = \int_{x_0}^{x_1} dl = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx = \int_{x_0}^{x_1} f(y, y')dx$$

  - Task is to find a function $y(x)$ for which l is minimal
- In general assume the path length is given by: $I = \int_{x_0}^{x_1} f(y, y')dx$
- Variational principle: shortest path is stationary: $\delta I = 0$

a) Write $\delta f(y, y') = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$ where $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

Show using partial integration that $\delta I = 0$ leads to the Hamilton Lagrange equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

b) Here for the shortest path we have $f(y') = l = \sqrt{1 + y'^2}$.

Then $\partial f/\partial y = 0$ and $\partial f/\partial y' = y'/\sqrt{1 + y'^2}$

Show that the variational principle leads to a straight line path: $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$ or that y' is a constant:

$dy/dx = a\,;\, y = ax + b$

Exercise-5: 4-Vector derivatives

a) Start with the expression for a Lorentz transformation along the x^1 axis. Write down the *inverse* transformation (i.e. express (x^0, x^1) in $(x^{0\prime}, x^{1\prime})$)

b) Use the chain rule to express the derivatives $\partial/\partial x^{0\prime}$ and $\partial/\partial x^{1\prime}$ in $\partial/\partial x^0$ and $\partial/\partial x^1$

c) Use the result to show that $(\partial/\partial x^0, -\partial/\partial x^1)$ transforms in the same way as (x^0, x^1)

d) In other words the derivative four-vectors transform inversely to the coordinate four-vectors:

$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$ and $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

Note the difference w.r.t. the minus sign!

e) Explicit 4-vectors: (ct, x, y, z) and $\left(E/c, p_x, p_y, p_z\right)$ → use next $c \equiv 1$

Co- and contravariant derivatives

Griffiths: chapter 3

- Contravariant vector:

$$x^{\mu} = (ct, \vec{x})$$

But covariant derivative:

$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

- Covariant vector:

$$x_{\mu} = (ct, -\vec{x})$$

But covariant derivative:

$$\partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

Note that the minus sign is “opposite” to the case of the coordinate four-vectors.

Exercises 6, 7, 8

6. [Griffiths exercise 2.2] "Crossing lightsabers"
   - Draw the lowest-order Feynman diagram representing Delbruck scattering: $\gamma + \gamma \rightarrow \gamma + \gamma$
   - This has no classical analogue. Explain why.
7. [Griffiths exercise 2.4]
   - Determine the invariant mass of the virtual photon in each of the lowest-order Feynman diagrams for Bhabha scattering. Assume electron and positron at rest.
8. [Griffiths exercise 2.7]
   - Examine the processes in ***the left column*** of Griffiths exercise 2.7 and state which one is possible or impossible, and why / with which interaction.

   Hint: draw the corresponding Feynman diagrams if needed.